

Price Modelling and Asset Valuation in Carbon Emission and Electricity Markets



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to my parents

SABINE & GOTTHART

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Abstract

This thesis is concerned with the mathematical analysis of electricity and carbon emission markets.

We introduce a novel, versatile and tractable stochastic framework for the joint price formation of electricity spot prices and allowance certificates. In the proposed framework electricity and allowance prices are explained as functions of specific fundamental factors, such as the demand for electricity and the prices of the fuels used for its production. As a result, the proposed model very clearly captures the complex dependency of the modelled prices on the aforementioned fundamental factors. The allowance price is obtained as the solution to a coupled forward-backward stochastic differential equation. We provide a rigorous proof of the existence and uniqueness of a solution to this equation and analyse its behaviour using asymptotic techniques. The essence of the model for the electricity price is a carefully chosen and explicitly constructed function representing the supply curve in the electricity market.

The model we propose accommodates most regulatory features that are commonly found in implementations of emissions trading systems and we analyse in detail the impact these features have on the prices of allowance certificates. Thereby we reveal a weakness in existing regulatory frameworks, which, in rare cases, can lead to allowance prices that do not conform with the conditions imposed by the regulator.

We illustrate the applicability of our model to the pricing of derivative contracts, in particular clean spread options and numerically illustrate its ability to “see” relationships between the fundamental variables and the option contract, which are usually unobserved by other commonly used models in the literature.

The results we obtain constitute flexible tools that help to efficiently evaluate the financial impact current or future implementations of emissions trading systems have on participants in these markets.

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Chapter 1

Introduction

This chapter contains an overview of relevant background information, the principal goals of this thesis and its structure. We explain the working principle of carbon emission and electricity markets, emphasizing important aspects of them. Intuitively and using real world data we motivate the work that constitutes the thesis and end with an outline of its structure and some helpful information on the notation adopted throughout.

1.1 Carbon Emission and Electricity Markets: Background and Working Principles

The world's changing climate and in particular global warming have been recognised by policy-makers around the world as key 21st century problems requiring innovative and effective solutions to curb the negative effects on ecosystems. Global warming is widely believed to be the result of a greenhouse effect, caused by increases in atmospheric gases such as carbon dioxide, methane, ozone and water vapour. Since the onset of the industrial revolution the increased demand for energy and the accompanying burning of fossil fuels have led to a sharp increase in emissions, thereby substantially contributing to global warming. This has resulted in the pressing need to incentivise the use of clean technologies for the production of energy.

We initially review the mechanisms that have been suggested by policy makers as possible answers to the quest for effective means to reduce emissions. Thereby we pay special attention to emissions trading, the mechanism this thesis focuses on. Subsequently we turn to electricity markets, electricity being our example of choice for a good whose production relies predominantly on the burning of fossil fuels, therefore offering itself in an exemplary way for the academic study of emission markets. In

particular we are interested in the relationship between the financial market in which electricity and emissions are traded and the real economy in which firms producing electricity interact. Again we discuss only a selection of the features that distinguish electricity markets, emphasizing those that are most relevant for the purposes of this thesis.

1.1.1 Carbon Emission Markets

Forced to address the issue of global warming, 37 countries ratified the Kyoto Protocol on 11 December 1997 in Kyoto, Japan. Under this agreement, binding limits, expressed in assigned amount units (AAU) and measured in tonnes of CO₂ equivalent greenhouse gas (GHG),¹ are imposed on the emissions of participating countries. To meet their obligations, countries may — amongst other options — draw upon any of the following three market-based mechanisms:

1. The *Clean Development Mechanism* (CDM), defined in Article 12 of the Kyoto Protocol, allows countries to implement emission-reduction projects in developing countries. For this they receive certified emission reduction (CER) credits, each worth one tonne of CO₂ equivalent, which can be used for meeting Kyoto targets.
2. The *Joint Implementation* (JI) mechanism, defined in Article 6 of the Kyoto Protocol, allows countries to earn emission reduction units (ERU), each worth one tonne of CO₂ equivalent, from establishing emission-reduction projects in other Kyoto countries. Like CERs, these units can be used to meet Kyoto targets.
3. *Emissions Trading* (or *Cap-and-Trade*) as defined in Article 17 of the Kyoto Protocol, allows market participants that have surplus AAUs, CERs or ERUs, to sell their excess capacity to other participants. This creates the so-called carbon market.

The Kyoto Protocol merely constitutes a global framework that encourages participating countries to put in place platforms on which CERs, ERUs and AAUs can be traded. Subject to broad provisions, the market design of any local implementation of an emissions trading system is left to the hosting countries.

¹The CO₂ equivalent of a given greenhouse gas denotes the amount of CO₂ that has the same global warming potential over a specified timescale.

We illustrate the working principle of emissions trading at the example of the European Union Emissions Trading Scheme (EU ETS). The market design features found in the EU ETS are commonly also found in other successful implementations of such trading schemes, for example the SO₂ and NO_x trading programmes in the US. We choose the former however, because of its scale and current importance.

1.1.1.1 Emissions Trading in the EU ETS

The limit on emissions during one *compliance period*, also referred to as the cap on emissions, is realised through an *initial allocation* of *allowance certificates* — each worth one EU Allowance Unit (EUA) and permitting its holder to emit one tonne of CO₂ equivalent² GHGs — by the government to firms in the market. At the end of each compliance period, firms must offset their accumulated emissions by submitting an adequate number of certificates. If they fail to do so (the event of *non-compliance*), they must pay a monetary *penalty* for each unit of excess emissions. Throughout a compliance period allowances are traded actively and this leads to the formation of a price, which represents the cost of carbon.

In practice, an emissions trading scheme consists of multiple compliance periods, each with its own distinct cap and penalty. Subsequent periods are joined by connecting mechanisms, which regulate the transition from one compliance period to the next. The key mechanisms go by the names of *banking*, *borrowing* and *withdrawal*. The banking mechanism allows market participants to carry allowance certificates, allocated for compliance at the end of the current period, forward to the next compliance period. Similarly, borrowing enables firms to use next period's certificates for compliance at the end of the current trading period. The withdrawal mechanism constitutes additional punishment for non-compliance: it prescribes that in addition to the monetary penalty payment one allowance certificate from next period's allocation is withdrawn for each unit of excess emissions at the end of the current period.

Since the coming into force of the Linking Directive, the EU has been accepting credits from Clean Development- and Joint Implementation Mechanisms for compliance in its trading scheme. Because one EUA is equivalent to an AAU, the base unit of the Kyoto Protocol, CERs, ERUs and EUAs can all be traded within the same system straightforwardly. In practice, this takes mostly place on platforms such as the European Energy Exchange (EEX), where EUA and CER spot and future contracts and option contracts written on them are traded actively.

²For simplicity we write CO₂ from now on whenever we mean CO₂ equivalent GHGs.

Emission reduction as part of a trading scheme occurs in two ways. The immediate consequence is to shift production within the existing fleet of resources to pollution friendlier ones — an effect we refer to as *load shifting*. The cost of carbon also makes it attractive for firms to invest in *long-term abatement measures*, if the cost of reducing their emissions by one unit lies below the value of an allowance certificate. Even if a firm has sufficient allowances to cover its emissions it should make use of all available emission-reduction measures whose marginal abatement cost (MAC) lies below the value of the allowance certificate. It can then sell spare certificates to companies whose MAC is above the market price of allowances and make a profit. For this reason it has been argued that cap-and-trade schemes provide emission reduction at the lowest cost to society. However, there is also evidence which suggests that the implied cost of carbon to make long-term investment in renewables such as solar cells worthwhile is \$196 per tonne of CO₂ — far above current allowance prices (cf. [eco, 2009]).

1.1.1.2 Stylized Facts

The period from 2005 to 2007 corresponds to the first phase of the EU ETS. This phase allowed the banking of certificates, implying that permits for compliance at the end of 2005 and 2006 were allowed to be carried forward to the following year. The year 2008 marked the beginning of a new phase unconnected to the previous one. These first years during which the EU ETS was operational serve well to illustrate the sensitivity of allowance prices to the initial allocation and the effect of implementations of connecting mechanisms.

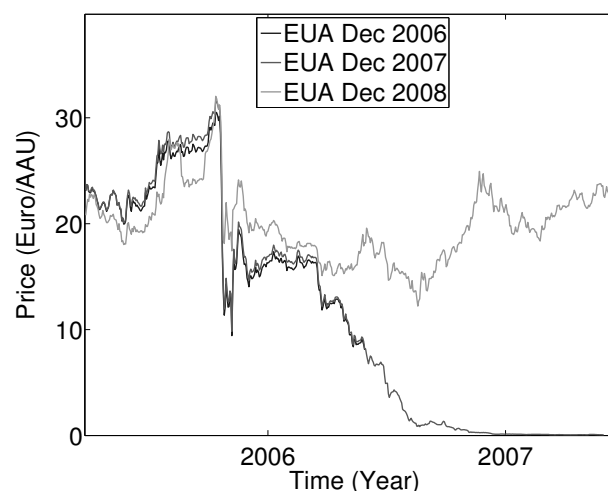


Figure 1.1: Price paths of EUA futures contracts. Data source: [EEX, 2012].

Figure 1.1 plots the prices of EUA futures contracts with delivery dates at the end of 2006, 2007 and 2008 throughout the two years 2006 and 2007. Initially the scarcity of allowances implied by the cap was difficult to estimate and hence futures prices with all delivery dates moved in loose lockstep. In the middle of 2006, however, it became apparent that, given emission levels to date, the allocation of certificates for the first trading period had been far too generous. As a result the prices of futures contracts with delivery dates within this period dropped sharply and continued to tumble. Due to the effect of the banking mechanism this affected both contracts with delivery date at the end of 2006 and 2007 equally. In 2007 it had become clear that emissions would not reach the cap that had been imposed by regulators; as a result the price of allowance certificates fell to zero. Because 2008 was the beginning of a new trading period, with a newly calculated cap and not connected to the previous one, the 2008 futures contract remained unaffected.

1.1.2 Electricity Markets

The generation of electricity relies on a wide spectrum of technologies. Renewables, such as wind and solar, but also nuclear and hydropower all lend themselves to the electricity generation process. Nevertheless, the most important production method remains the burning of fossil fuels and their resulting conversion to electricity. In general a large amount of CO₂ emissions are released during this process. However, depending on the fossil fuel used the emission intensity of the generation process can vary substantially. If one is interested in the effect emissions trading has on the electricity market, it is therefore important to keep track of the different processes which contribute to the supply of electricity at any time. This question is closely related to understanding the mechanism of formation of the electricity spot price.

1.1.2.1 Price Setting Mechanism

Since the onset of electricity market deregulation in 1998 and the resulting formation of competitive markets, electricity has become part of the financial products establishment. In doing so, it does however still distinguish itself in multiple ways from its peers.

One of the key characteristics of electricity is its *non-storability*. As a consequence it is produced immediately before being consumed, a feature that closely relates production costs to price formation. In fact, prices are usually determined following economic equilibrium arguments by constructing the demand and the supply curve

and finding the unique point at which the two intersect, determining the market price. This is done on a daily basis for each hour of the following day in the so-called day-ahead auctions organised by exchanges, for example in Europe by the EEX.

Because of the nature of the price setting mechanism employed, the construction of the supply curve constitutes a very important part in the price formation process. In practice, electricity producers submit day-ahead *bids* to a central *market administrator*, whose task it is to allocate the production of electricity amongst them. Typically, firms' bids have the form of price-quantity pairs, with each pair comprising the amount of electricity the firm is willing to produce, and the price at which the firm is willing to sell this quantity. An example would be a generator submitting bids (600MW,100€), (200MW,120€), (200MW,200€). This generator offers to sell its first 600MW for the specified hour at a price of 100€, the next 200MW at a price of 120€ and a further 200MW at a price of 200€. The market administrator aggregates the bids for each price level and arranges them according to the *merit order*, a rule by which cheaper production units are called upon before more expensive ones in the electricity generation process. This procedure results in the supply curve, also known as the *bid stack*. The auction price is now the market clearing price for each hour. It clearly reflects the costs of production of the last technology in the merit order that is needed to meet demand, which is why it is commonly said that electricity is sold at marginal cost. Throughout this thesis we refer to the day-ahead auction price as the electricity *spot price*.

Although the auction based trading volume has increased substantially over the last few years, for example at the EEX from 49 terawatt hours in 2003 to 279 terawatt hours in 2010 (cf. [EEX, 2012]), a large share of electricity in Europe is still traded over-the-counter or on a forward basis. However, we believe that in a competitive market with rational agents the day-ahead auction price also serves as the key reference point for real-time and over-the-counter prices (cf. [Ockenfels et al., 2008]).

1.1.2.2 The Impact of Cap-and-Trade

We briefly illustrate how the introduction of a cap-and-trade system impacts electricity markets. For this purpose we distinguish between *business-as-usual* (BAU), the scenario of a market that is not covered by emissions trading, and a market in which a cap-and-trade system is operating.

The aim of emissions trading is to incentivise the use of cleaner technologies. This goal is achieved by levying a cost on emissions. From the point of view of producers the costs of polluting become internalized. Assuming a firm is in possession of a

certain number of allowance certificates it faces a choice: it may remain idle and sell unused certificates to the market or choose to produce and to utilize its certificates for compliance purposes. With the latter choice the forgone profit constitutes an opportunity cost, which must be taken into account as an increase in production costs. Therefore, the price at which the firm is submitting its bids increases by an amount equal to the *marginal emission rate* of the plant (measured in tonnes of CO₂ per MWh) multiplied by the allowance price.

Although not a requirement for any of the models proposed in this thesis, in the absence of emissions trading, pollution-intensive fuels have historically gathered on the lower-price end of the bid stack because they are cheaper to use, whereas environmentally friendlier technologies tend to be more expensive and are concentrated at the higher-price end.³ The rationale behind cap-and-trade is that for sufficiently high carbon costs, pollution-intensive technologies become more expensive than environmentally friendlier ones. Hence, to preserve the merit order, the market administrator now calls upon environmentally friendly technologies before pollution-intensive ones. The bid stack has been *rearranged* and this leads to cleaner production of electricity.

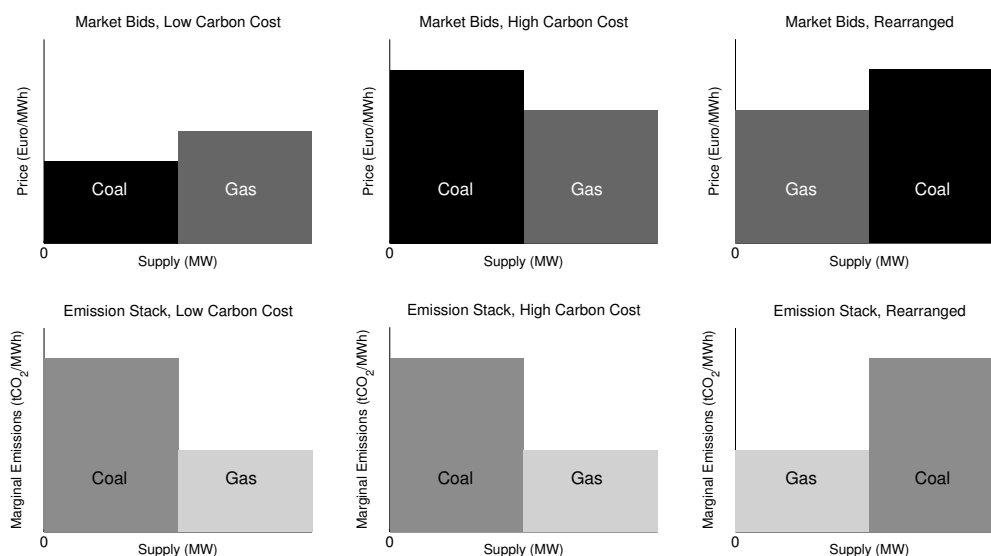


Figure 1.2: A schematic of the rearrangement of the bid and the emission stacks as the cost of carbon increases.

Example 1. Consider a simple market with one coal and one gas generator, who bid at only one price level. We illustrate the influence of emissions trading on the bid

³An exception to this rule of thumb constitute must-run bids, which are always placed on the left end of the bid stack and may, for example, contain bids from nuclear generators that do not emit.

stack in Figure 1.2. Initially the cost of carbon is low and bids from coal generators are cheaper than those from gas generators. Accordingly, coal bids come first in the bid stack and the marginal emissions corresponding to bids further to the left in the stack are relatively higher than those corresponding to bids on the right. As emissions become more costly the bid levels from coal and gas generators increase, with coal bids doing so more than gas bids. This results in the market administrator rearranging the bid stack and placing gas first. The result of the higher allowance price are lower emissions as intended.

1.1.2.3 Stylized Facts

Due to the non-storability of electricity and the complex nature of the price setting mechanism, the typical electricity spot price time series is very distinct in nature. We illustrate some of the key features of electricity prices at the example of the Pennsylvania, New Jersey, Maryland market region (PJM) in the US.

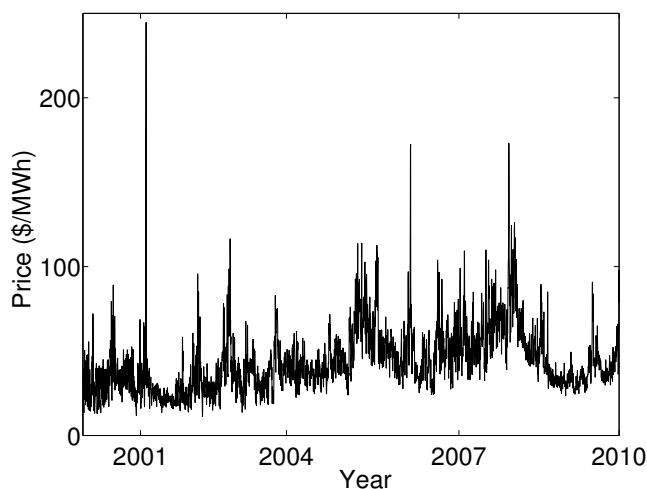


Figure 1.3: Daily electricity spot price in PJM. Data source: [PJM, 2012].

Figure 1.3 plots daily PJM spot price data for the period from 2001 to 2010. We immediately observe the high *volatility* and the striking occurrence of sudden *spikes* in the data. These features are still partly unexplained although it is likely that unexpected outages, errors in the predictions of load and network transmission failures — all phenomena which result in demand being very close to the market capacity — contribute to the spikey nature. The *cyclical* nature of prices is widely attributed to fluctuations in temperature, which, through their effect on air-conditioning and heating needs, influence consumers’ demand for electricity.

Although not visible in the above time series, another unexplained peculiarity of electricity prices is the fact that frequently they are observed to be *negative*. For example, in the PJM market region, taking real-time and spot prices into account, over 100,000 occurrences of negative prices were observed in 2003 alone (as pointed out by René Carmona during a personal communication, 2012)! Also in this case no definite explanations are available at present, although possible suspects include must-run bids; to guarantee that they are sold generators may be submitting them at a negative price.

1.2 Motivation

In mathematical finance and most famously in its application to equity markets, it has become tradition, when modelling spot prices, to exogenously specify some stochastic process belonging to the class of semimartingales and calibrate it to historical data. The analysis of the working principles of carbon emission and electricity markets given above has shown that there are intricate relationships between the cost structure of energy producers, fundamental economic variables, such as demand and the level of accumulated emissions, and the prices of both electricity and emission permits. As we have seen these relationships result in distinct time series for prices with numerous features that are difficult to capture satisfactorily by prescribing the price evolution directly by some stochastic process. Therefore the need arises for models that capture the relationships between prices and fundamental variables to their full extent in accurate, efficient and mathematically convenient ways. A possible answer has been provided by the *structural* modelling framework, which recently has begun to gain traction in the literature⁴ and which the models advocated in this thesis fit in.

The structural approach takes the following stance: instead of directly prescribing the price evolution by some stochastic process, the fundamental factors which are known to drive prices form the starting point. A considerable amount of effort is then spent to model the price setting mechanism and to relate the exogenous fundamental factors to the formation of the price we set out to describe. In doing so the methodology may draw upon any of the traditional approaches in the literature, from equilibrium modelling to arbitrage-free pricing. For example, as we will see in Chapter 2, employing the arbitrage-free pricing paradigm we obtain allowance prices

⁴Because of the broad range of topics covered in this thesis, we do not provide a literature review at this point. Instead each chapter begins with a review of the literature that is relevant for its understanding and places it in context.

as a function α of time, the level of the cumulative emissions and a number of fundamental factors such as the demand for electricity and fuel prices. Therefore, the price A_t of an emission permit at time t , given a level of cumulative emissions E_t and observations of the fundamental factors X_t is given by

$$A_t = \alpha(t, E_t, X_t).$$

Similarly, in Chapters 3 and 5, through an explicit construction, we obtain the electricity bid stack b as a function of the same fundamental factors. Under certain assumptions on the nature of demand in the market, the electricity spot price P_t is then given by

$$P_t = b(X_t).$$

1.2.1 Putting Structural Models to Use

Some of the advantages of structural models are obvious. Since prices in the structural framework are constructed as functions of underlying variables, the relationships between them become very clear. Most importantly, correlations can often be calculated easily and accurately. A relevant example in electricity markets are the often-observed correlations between fuels, demand and electricity prices.

A second, less obvious, reason to promote structural models is their ability to serve as price predictors if no liquid market has yet been established for the financial product in question. As an example we mention power plants in the US. There, cap-and-trade systems have been considered as possible options to fight increasing levels of pollution for several years. However, currently they are only operating in a few small geographic regions. Because of the risk of cap-and-trade systems being introduced, plants need to factor the cost of emissions into their long-term financial planning. Given data availability for the fundamental factors X , the structural models for allowance prices, which we introduce in Chapters 3 and 5 are ideally suited for this purpose.

An example, which combines the two advantages of structural models advocated above is the valuation of power plants in the spirit of the theory of real options, in markets which expect to be affected by emission regulation in the future. Strings of spread options (options on the spread between the price of one megawatt hour of electricity and the cost of the amount of fuel needed to produce such a megawatt hour) with maturities covering a given period are most frequently used to value the optionality of a power plant, which can be run when it is profitable to do so (namely when the price of electricity is greater than the cost of producing it), and shut down

otherwise. In a nutshell, if an economic agent takes control on day t , of a gas power plant for a period $[T_1, T_2]$, then for every day $\tau \in [T_1, T_2]$ of this period, he or she can decide to run the power plant when $P_\tau > hS_\tau + k$ and book a profit $P_\tau - hS_\tau - k$ for each unit of power produced and shut the plant down if⁵ $P_\tau \leq hS_\tau + k$. Here P_τ denotes the price at which one unit (1 MWh) of power can be sold on day τ , S_τ the price of one unit of the fuel utilised by the plant (typically one MMBtu), h the efficiency or heat rate of the plant (i.e. the number of units of fuel needed to produce one unit of electricity) and k the daily (fixed) costs of operations and maintenance of the plant. So in this somewhat oversimplified analysis of the optionality of the plant, the value at time t of the control of the plant operation on day τ can be expressed as $e^{-r(\tau-t)}\mathbb{E}[(P_\tau - hS_\tau - k)^+|\mathcal{F}_t]$ where as usual, the exponent $+$ stands for the positive part, r for the constant interest rate used as discount factor to compute present values of future cash flows, and \mathcal{F}_t denotes the information available on day t when the conditional expectation is computed. So the operational control (for example as afforded by a tolling contract) of the plant over the period $[T_1, T_2]$ could be valued on day t as

$$V_t^{pp} = \sum_{\tau=T_1}^{T_2} e^{-r(\tau-t)}\mathbb{E}[(P_\tau - hS_\tau - k)^+|\mathcal{F}_t].$$

This rather simplistic way of valuing a power generation asset had far-reaching implications in the developments of the energy markets and is the main reason why spread options are of the utmost importance. However, such a valuation procedure is flawed in the presence of emission regulation as the costs of production have to include the costs specific to the regulation. To be more specific, the day- τ potential profit $(P_\tau - hS_\tau - k)^+$ of the spark spread has to be modified to $(P_\tau - hS_\tau - eA_\tau - k)^+$ in order to accommodate the cost of the regulation. Here A_τ is the price of one allowance certificate worth one tonne of CO₂ equivalent, and e is the marginal emission rate of the plant, namely the number of tonnes of CO₂ emitted by the plant during the production of one unit of electricity. Such an option is often called a *clean spread* to emphasize the fact that the externality is being paid for and the real option approach to power plant valuation leads to the following *clean price* for the control of the plant over the period $[T_1, T_2]$ in the presence of regulation:

$$V_t^{cpp} = \sum_{\tau=T_1}^{T_2} e^{-r(\tau-t)}\mathbb{E}[(P_\tau - hS_\tau - eA_\tau - k)^+|\mathcal{F}_t]. \quad (1.1)$$

⁵Note that ignoring constraints such as ramp-up rates and start-up costs, this scheduling is automatically induced when generators bid at the level of their production costs in the day-ahead auction for power.

In order to price such cross-commodity derivatives, a joint model is clearly required for fuel prices, electricity prices and carbon allowance prices. Moreover, in the absence of a history of emission regulation in the market under consideration it is important that allowance prices are inferred from underlying fundamental factors. We introduce a model suitable for this task in Chapter 5.

1.2.2 A Look at the Data

The relationships we alluded to above, namely between the price of electricity and emission permits and costs of production and fundamental factors can easily be observed in data.

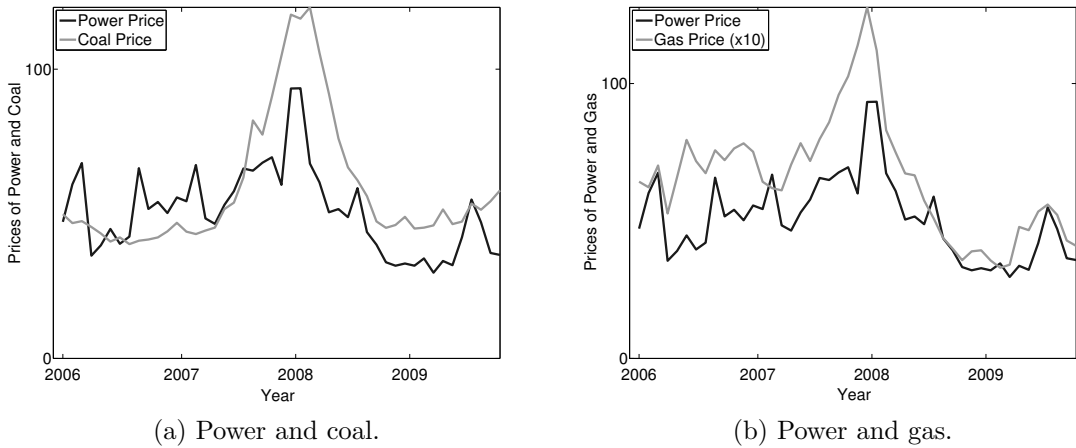


Figure 1.4: Monthly (averaged) power spot in the PJM market region and fuel prices. Data source: [PJM, 2012].

Figure 1.4 plots monthly averaged power spot prices in the PJM market region and coal and gas prices in Figure 1.4a and Figure 1.4b respectively. Over a three year period we can clearly observe the positive correlation between the cost of the factors of production (coal and gas) and the derived good (electricity).

Figure 1.5 illustrates the dependence between allowance and electricity prices. The price drop in allowances seen here is the same as the one observed in Figure 1.1 and due to the overly generous allocation of allowances at the beginning of the trading period. The new information in Figure 1.5 is the subsequent drop in the prices of power futures. This interdependence of allowance and electricity prices is explained by the results in Chapters 3 and 5, which show how fluctuations in the allowance price are directly translated to proportional changes in the costs of production of power

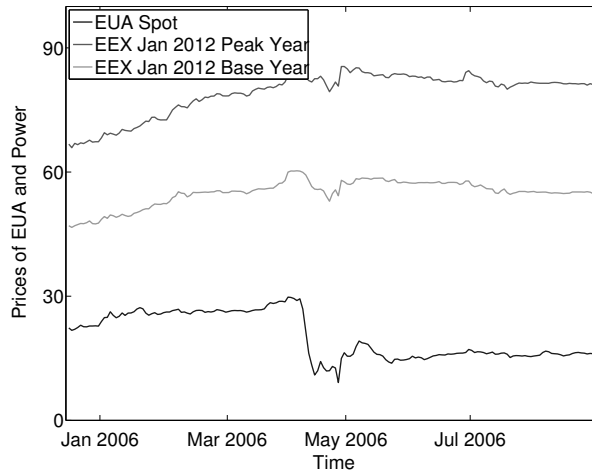


Figure 1.5: EUA spot and EEX power futures prices. Data source: [EEX, 2012].

plants. In other words electricity producers charge consumers for the allowances that they must purchase in the market.

1.3 A Brief Overview of the Thesis

To give the reader an overview, we briefly summarize the individual chapters and highlight the contributions of the thesis.

1.3.1 Structure

Chapter 2. This chapter is dedicated to deriving the equations that describe the prices of allowance certificates and derivative contracts written on them. In a fairly general framework we introduce the mathematical setting for the thesis and the stochastic factors which characterize the emission and the electricity market. We discuss the link between the empirical measure and measures that can be used to define viable pricing systems in the context of the arbitrage-free pricing paradigm. Further, we analyze the conditions under which our market is complete — a property which allows the unambiguous valuation of derivative contracts — and explain what we mean by the unique pricing measure, even in the absence of market completeness. The regulations governing emission markets, discussed in §1.1.1.1, wield their influence for the first time when we show how they immediately allow us to deduce the value of the allowance certificate at compliance time. Using this insight we proceed to characterize the price process of the allowance certificate as the solution of a coupled forward-backward stochastic differential equation (FBSDE). A considerable amount

of effort is spent in the sequel to prove the existence and uniqueness of a solution to this equation. Having completed our analysis of the allowance pricing problem, we discuss the valuation of derivative contracts in the market. For a general payoff condition, we show that their value must again satisfy a FBSDE, albeit this time the equation is decoupled. The proofs of existence and uniqueness of a solution to the derivative pricing problem are given and turn out to be considerably easier than those for the allowance pricing problem.

Chapter 3. We introduce a first concrete joint model for allowance and electricity prices. Taking an exogenously given stochastic process for the evolution of demand for electricity as a starting point, we show how it directly translates into electricity prices via the bid stack and into a measure for the accumulation of emissions in the market via the marginal emission stack. This is first done in the business-as-usual setting. Subsequently we include the feedback from the emission regulation to the actions of market participants. This is realised via the idea of load shifting or the rearrangement of the bid stack introduced in §1.1.2.2. Having collected all the necessary ingredients we formulate the allowance pricing problem in the setting of this chapter. Proving some regularity properties of the coefficients we relate the proposed model to the theoretical framework introduced in Chapter 2. Subsequently we explain in detail the working principle of connecting mechanisms, namely of banking, borrowing and withdrawal, and examine their impact on the allowance price. Further, again referring to the theoretical framework introduced in Chapter 2, we illustrate the pricing of derivatives in the setting of the chapter at the example of a European call option. The last section of the chapter is devoted to the numerical illustration of the proposed model. In the case of specific functional forms for the bid and the emission stack and for a feasible parameter set we solve the FBSDE which represents the allowance price and analyze the dependency of prices on the two underlying variables, the demand for electricity and the cumulative emissions. This is done in the setting of one compliance period and that of multiple compliance periods. Our numerical scheme also allows us to test the effectiveness of the proposed model of a cap-and-trade scheme. By simulating the state variables in the market we show how stricter penalization of non-compliance immediately leads to increases in abatement.

Chapter 4. This chapter contains a brief asymptotic analysis of the allowance pricing problem in the setting of Chapter 3. In particular, we demonstrate how the behaviour of the allowance price near compliance time and with the cumulative emissions taking values close to the cap is governed by a much simpler equation than that introduced in Chapter 3. Moreover, a dedicated analysis of critical boundary

layers strongly suggests that the solution to the type of equation that characterises the allowance price possesses, to date unproven, smoothness properties.

Chapter 5. We generalize the results of Chapter 3. In particular, in addition to its dependency on demand, we allow the equilibrium electricity price under business-as-usual to be a function of stochastic fuel prices. As a result the merit order is now stochastic and we propose a considerably more complex model for the bid stack, which allows different technologies to swap places in the stack. From the information contained in the bid stack we again deduce a measure for the accumulation of emissions in the market, from which the formulation of the pricing equation for allowance certificates in this extended setting follows. As before, we prove some regularity properties of the coefficients, which allows us to relate the proposed model to the theoretical framework introduced in Chapter 2. The inclusion of stochastic fuel prices as driving factors of electricity and allowance prices results in the proposed model being ideally suited to the important task of the valuation of clean spread options as discussed in §1.2.1. We introduce clean versions of traditional spark and dark spread options and show how their price processes satisfy a version of the derivative pricing FBSDE introduced in Chapter 2. Similarly to the structure of Chapter 3, the last part deals with the numerical illustration of the model, for the case of a concrete bid and emission stack. Because of the inclusion of a stochastic merit order, the bid stack model introduced in this chapter deserves some attention in its own right and we discuss it in some detail highlighting its versatility. A series of dedicated case studies at the end of the chapter demonstrates the impact our model has on the prices of clean spread options and the application of these contracts to the valuation of power plants.

Chapter 6. We summarise the results presented in this thesis, link the contributions of the individual chapters together and discuss possible ways to continue our work.

1.3.2 Contributions

The contributions of this thesis are threefold. The main inspiration for the thesis was the development and the implementation of realistic joint structural models for electricity and allowance prices. We believe that this constitutes the first rigorous proposal of such a joint model, especially taking into account the important feedback from the financial market in which allowances are traded to the real economy in which power plants produce electricity. Although we aligned our modelling framework with the regulations governing the EU ETS, features such as the existence of a cap and a

penalty, the implementation of connecting mechanisms and the capturing of the above mentioned feedback effect, equally apply to many other emissions trading systems around the world and we believe that the models proposed in this thesis will indeed be equally relevant for countries like Japan, Australia and the US.

The second contribution is technical. The proof, presented in Chapter 2, of the existence and uniqueness of a solution to the FBSDE describing the price of an allowance certificate is a non-trivial extension of existing results (as detailed in the chapter specific introduction to Chapter 2). In particular, it allows one to considerably relax the conditions imposed on the coefficients of the forward equations of the FBSDE and accommodates the case of a random terminal condition.

Third, our asymptotic analysis of the allowance pricing problem provides an improved understanding of the nature of the solution to the type of equation describing the allowance price. In addition, short of a rigorous proof, it lends strong support to the conjecture that not only does there exist a solution to this type of equation, as proven in Chapter 2, but also that this solution possesses good regularity properties.

1.3.3 Publications

We end this summary with a list of the papers that, to date, originated from or were inspired by the research presented in this thesis:

1. Howison, S. and Schwarz, D. (2012). Risk-neutral pricing of financial instruments in emissions markets: a structural approach. *SIAM Journal on Financial Mathematics*. To appear.
2. Carmona, R., Coulon, M., and Schwarz, D. (2012a). Electricity price modeling and asset valuation: a multi-fuel structural approach. *Mathematics and Financial Economics*. To appear.
3. Carmona, R., Coulon, M., and Schwarz, D. (2012b). The valuation of clean spread options: Linking electricity, emissions and fuels. *Quantitative Finance*. To appear.
4. Dewynne, J., Howison, S., and Schwarz, D. (2012). Asymptotic expansions for nonlinear options and emission market models. Working paper, University of Oxford, Oxford-Man Institute.

1.4 A Word on Notation

Throughout the thesis we use Roman and Greek capital letters to denote stochastic processes. For better readability of the text we abstain from printing the time subscript of processes unless they are set as equations or its inclusion is necessary for the unambiguous interpretation of the statement in question. We use Roman and Greek lower case letters for variables, parameters and functions. An exception is represented by the capital letter C (with various subscripts), which always represents arbitrary constants.

We slightly abuse notation by letting the same letter denote functions with the same meaning but possibly different technical definitions in different chapters. For example, throughout the thesis μ_e is used to represent the rate at which the aggregate of all agents in the market emits CO₂; in Chapter 3 it is a function of two variables, whereas in the setting of Chapter 5 it becomes a function of four variables.

Lastly, unless otherwise stated, all equalities and inequalities between random variables are to be understood in an almost sure sense.

Chapter 2

Risk-Neutral Pricing of Financial Instruments in Emission Markets: A Structural Framework

This chapter introduces the mathematical framework in which we will embed the concrete models that we propose in Chapters 3 and 5. We show how the arbitrage-free price of an allowance certificate appears very naturally as the solution to a forward-backward stochastic differential equation and prove the existence and uniqueness of a solution to this equation. Further, we show that also derivative contracts in our market are characterized as solutions to a FBSDE, albeit a much simpler one, and again we prove the existence and uniqueness of a solution to this equation.

In §2.1 we characterize the market in which we will operate throughout this thesis. We choose Brownian motion to model the stochasticity in our market. In this context we value allowance certificates using the theory of arbitrage-free pricing. The origins of this pricing paradigm go back to the monumental work [Black and Scholes, 1973], which for the first time introduced the famed option pricing formula named after its discoverers Black and Scholes in the case of underlyings that follow geometric Brownian motion. Later the theory was extended in [Harrison and Pliska, 1981] and [Delbaen and Schachermayer, 1994] to underlyings belonging to the class of semi-martingales. These texts also include criteria for the completeness of the market, establishing when every derivative can be replicated with a portfolio consisting of a riskless bond and the risky underlying assets. At the end of §2.1 we show how the regulations governing emissions trading allow us to unambiguously characterize the value of the allowance certificate at compliance time. The binary nature of this terminal condition was introduced for the first time in [Carmona et al., 2009a] in a single period setting using equilibrium arguments. Later, the same terminal condition

appeared in [Carmona and Hinz, 2011], where also a simple multi-period framework is considered.

Using standard results on the representation of martingales as stochastic integrals §2.2 derives the forward-backward stochastic differential equation that describes the price of allowance certificates in the structural setting, explains why it is non-standard and why for our purposes it is important to extend currently available results concerning the existence and uniqueness of solutions to FBSDEs of this type.

In §2.3 we deliver the proofs for whose importance we argued at the end of the previous section. The theory of forward-backward stochastic differential equations has been studied for almost two decades now. We refer to the monograph [Ma and Yong, 1999] for references on early work and to the introduction of the paper [Ma et al., 2011] for a more recent overview. The key issues in the theory of FBSDEs until now are first, the regularity of the coefficients required to extend solutions valid on a small time interval (cf. [Antonelli, 1993]) to an arbitrary long time horizon and second, the need for the non-degeneracy (or uniform ellipticity) of the forward equations. The latter condition has been considerably relaxed in the recent work [Ma et al., 2011], wherein, in a one-dimensional setting, the forward diffusion is allowed to degenerate. However, the arguments in the paper still rely on the uniform Lipschitz continuity of the coefficients of the FBSDE, in particular of the terminal condition. This is the difficulty entailed by the current regulation of emission markets: because of the binary nature of the allowance certificate at compliance time the terminal condition exhibits a gradient singularity and does not satisfy a Lipschitz condition! The first successful attempts to resolve this problem were made in [Carmona et al., 2012c]. The results in that work, however, are based on a transformation which imposes very strict conditions on the forward components of the FBSDE (in fact, the forward equation corresponding to the process X in our setting is a simple Brownian motion) — far too strict to allow the formulation of any realistic model. The work [Carmona and Delarue, 2011] extends the results from [Carmona et al., 2012c] although it still requires the coefficients of the forward diffusions to be uniformly Lipschitz continuous and the terminal condition to be deterministic, prohibiting, as we will show, the modelling of markets with multiple compliance periods. The new results that we present in §2.3 are important because they considerably relax the regularity assumptions imposed on the forward diffusion (X is only assumed to possess a strong solution in a suitable space) and because they permit the terminal condition of the FBSDE to be random. Therefore, our results cover the concrete models we

propose in Chapters 3 and 5, both in the setting of markets with single and multiple compliance periods.

In §2.4 we discuss the valuation of derivative contracts in our setting. Again, we show how the price of any derivative in our market appears very naturally as the solution to a FBSDE, although one that is in some sense considerably simpler than the equation describing the price of the allowance certificate. We proceed to prove the existence and uniqueness of a solution to this equation, drawing heavily upon the results we introduced in §2.3. To the best of our knowledge, this is the first rigorous framework, which permits the valuation of derivative contracts written on the allowance certificate in the structural modelling framework.

2.1 Market Setup

This section introduces the stochastic model of the emission market and illustrates the relationships between the state variables. For now, we only specify a minimal market structure and work with general coefficients. Later on, in Chapters 3 and 5, we introduce concrete models, which will be shown to satisfy the assumptions we make in the present section.¹

2.1.1 Stochastic Market Model

We consider a finite time horizon $[0, T]$, where $T \in [0, \infty)$. We identify this time interval with the duration of a single compliance period (usually assumed to be one year) in an emission market. Our initial setup is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\Omega = C([0, T], \mathbb{R}^d)$. Specifically, Ω is the space of all continuous functions $\hat{W} := (\hat{W}^1, \dots, \hat{W}^d)$ taking values in \mathbb{R}^d for some $d \in \mathbb{N}$. The measure \mathbb{P} is assumed fixed so that the process \hat{W} has the law of a d -dimensional Brownian motion. In particular, \mathbb{P} represents the *physical measure* that would be observed from empirical studies of data. We let $\mathcal{F}^{\hat{W}}$ denote the filtration generated by the paths of \hat{W} and \mathcal{F} its augmentation by the \mathbb{P} -null sets, that is,

$$\mathcal{F}_t := \sigma \left(\mathcal{F}_t^{\hat{W}} \cup \mathcal{N} \right), \quad \text{for } t \in [0, T],$$

where

$$\mathcal{N} := \left\{ F \subseteq \Omega; \exists G \in \mathcal{F}_T^{\hat{W}} \text{ with } F \subseteq G, \mathbb{P}(G) = 0 \right\}.$$

¹Whenever this is not the case we point out possible modifications of the models, which would lead to the coefficients being of a suitably regular nature.

Then the filtration \mathcal{F} satisfies the “usual conditions”: \mathcal{F}_0 contains all the null sets of \mathbb{P} and \mathcal{F} is right-continuous. Moreover, since the Brownian filtration $\mathcal{F}^{\hat{W}}$ is left-continuous, \mathcal{F} is in fact continuous.

In the following we describe our market setup. Thereby we will frequently refer to [Yan, 1998] for definitions and important results. We consider a market, in which there exist d observable factors. For ease of notation we introduce the index set $I = \{1, \dots, d\}$, which will be used throughout the text. We refer to the observable factors as the *fundamental factors* in our model and assume that they are represented by a continuous, \mathcal{F} -adapted process $X := (X^1, \dots, X^d)$, taking values in \mathbb{R}^d . For now, the reader may think of each component X^i , $i \in I$, as describing a quantity such as the demand for goods, whose production causes emissions, or the price of a good used in this pollution-intensive production process. Further, we let $R \subset I$ and assume that the prices of all tradable assets in the market are represented by $\{X^i : i \in R\}$.

With these definitions we shall consider a small investor who is allowed to trade continuously in the risky assets and in a risk-free asset, which pays zero interest rate. At this stage the zero interest rate assumption is purely made for notational convenience.² We shall make the standard assumption that the investor trades in a frictionless market. In particular, she does not face any transaction costs or influence market prices. Moreover, no restrictions on short selling are in place. Further, the investor may engage in *allowable* strategies (cf. [Yan, 1998, Definition 2.3]), that is, strategies for which there exists a constant $C_0 > 0$, such that the value of the investor’s portfolio, when trading in the riskless and the risky assets, is bounded below by $-C_0 \sum_{i \in R} X_t^i$, for $t \in [0, T]$.

We assume that our market is *fair* (cf. [Yan, 1998, Definition 2.1]), that is, the set \widehat{MM} of measures equivalent to the physical measure and under which all traded assets are martingales is non-empty. Specifically,

$$\widehat{MM} := \{Q \sim \mathbb{P}; \text{ for } i \in R, X^i \text{ are } Q\text{-martingales}\} \neq \emptyset.$$

The above assumption immediately rules out the existence of so-called arbitrage opportunities. In particular, there now exists no allowable self-financing strategy with initial wealth zero and terminal wealth that is non-negative almost surely and strictly positive with strictly positive probability. Note that the reverse implication does not always hold. In fact, it requires the slightly stronger assumption of no free lunch

²All results presented in this chapter are valid in the case of a non-negative interest rate $r \geq 0$. In fact, when we introduce concrete models in Chapter 3 and Chapter 5 we will include an interest rate term in our formulation.

with vanishing risk (cf. [Delbaen and Schachermayer, 1994, Yan, 1998]). When the implication is true in both directions the result is known as the First Fundamental Theorem of Asset Pricing.

Each martingale measure now constitutes a possible pricing system and the value of any contingent claim under this system is calculated as the conditional expectation of the payoff of the claim under the associated measure. We distinguish between two classes of claims. First, we say that a claim is *replicable* if there exists a self-financing trading strategy, whose associated wealth process is always non-negative and whose terminal value is equal to the payoff of the claim. Then the price of such a claim is independent of the choice of martingale measure. In particular, it is always equal to the initial cost of setting up the unique replicating portfolio (cf. [Yan, 1998]). Second, there are *non-replicable* claims, for which the choice of martingale measure does matter and hence the value of the claim is not determined uniquely. In this case the academic literature offers a range of strategies to determine a price: super-replication pricing (cf. [El Karoui and Quenez, 1995, Föllmer and Kramkov, 1997]), finding the minimal entropy martingale measure (cf. [El Karoui and Rouge, 2001]) or the utility indifference price (cf. [Henderson, 2002]). We do not focus on how to mathematically determine a unique price for non-replicable claims and simply make the following assumption.

Assumption 1. The trader chooses a *pricing measure* $\mathbb{Q} \in \widehat{MM}$.

With the above assumption in place we may speak, in both cases, that is of replicable and non-replicable claims, of a unique price. We shall also speak of the unique pricing measure \mathbb{Q} , bearing in mind that in the case of a replicable claim \mathbb{Q} may be chosen arbitrarily from the set of equivalent martingale measures.

As a special case we mention markets that possess the property of *completeness*. In a complete market every contingent claim is replicable and Assumption 1 becomes redundant. Specifically, the set of all martingale measures \widehat{MM} is a singleton, a result which is known as the Second Fundamental Theorem of Asset Pricing. In order for the market introduced above to be complete it is necessary that $R \equiv I$ and that some additional technical assumptions hold. For the precise nature of these assumptions we refer the reader to [Yan, 1998, §5].

To shed some further light on the relationship between the physical measure \mathbb{P} and the pricing measure \mathbb{Q} , we introduce the concept of *market price of risk*. Our assumption of the existence of a pricing measure is equivalent to the existence of a progressively measurable, \mathcal{F} -adapted process $\Theta := (\Theta^1, \dots, \Theta^d)$, taking values in \mathbb{R}^d

and referred to as a market price of risk process. Θ has the property that it satisfies the square-integrability condition

$$\mathbb{P} \left(\left[\int_0^T \|\Theta_t\|^2 dt \right] < \infty \right) = 1, \quad (2.1)$$

and that, with the definition

$$Z_t := \exp \left(\int_0^t \langle \Theta_u, d\hat{W}_u \rangle - \frac{1}{2} \int_0^t \|\Theta_u\|^2 du \right), \quad \text{for } t \in [0, T],$$

$\mathbb{E}[Z_T] = 1$, to wit, the process Z is a martingale.³ The pricing measure \mathbb{Q} is now related to the physical measure \mathbb{P} through the relationship

$$\mathbb{Q}(F) = \int_F Z_T d\mathbb{P}(\omega), \quad \text{for } F \in \mathcal{F}_T. \quad (2.2)$$

In light of our earlier comment regarding market completeness we point out that if the market is complete there exists a unique market price of risk. If, however, the market is not complete, there exists one market price of risk for each martingale measure $\mathbb{Q} \in \widehat{MM}$ and Assumption 1 implies that the trader chooses one of them.

We now define a process $W := (W^1, \dots, W^d)$ by

$$W_t^i := \hat{W}_t^i - \int_0^t \Theta_u^i du, \quad \text{for } t \in [0, T], \quad i \in I.$$

Girsanov's theorem (cf. [Karatzas and Shreve, 2005, Theorem 5.1]) allows us to conclude that the process W is a d -dimensional standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. Further, we assume the existence of two measurable functions $\mu : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$, with $\mu_i = 0$ for $i \in R$ (reflecting the zero interest rate assumption), and $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d}$, such that, under the pricing measure \mathbb{Q} and for $x_0 \in \mathbb{R}^d$, the process X is a strong solution to the stochastic integral equation

$$X_t = x_0 + \int_0^t \mu(u, X_u) du + \int_0^t \sigma(X_u) dW_u, \quad \text{for } t \in [0, T]. \quad (2.3)$$

For now, we simply assume that μ and σ are suitably regular for X to be a strong solution to (2.3). Later, in §2.3, where we are concerned with the existence and uniqueness of solutions to the equations that are part of our model, we will impose additional conditions on μ and σ .

³Our assumption that the pricing measure is equivalent to \mathbb{P} (i.e. $\mathbb{Q} \in \widehat{MM}$) already implies that Z is a true martingale. However, we draw attention to the fact that, in general, a sufficient condition for this to be the case is the well known Novikov condition (cf. [Karatzas and Shreve, 2005, Corollary 5.13]), namely that

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T \|\Theta_t\|^2 dt \right) \right] < \infty.$$

2.1.2 The Allowance Certificate at Compliance Time

Let us assume for now that fluctuations in the allowance certificates' price are represented by some continuous, \mathcal{F} -adapted process A (measured in € per tonne of CO₂), which will be defined rigorously below. From the previous discussion of the working principle of emission markets it is clear that in a one-period market the value of the allowance certificate at the time of compliance can only take two values: it is equal to the penalty, in the event of non-compliance, or is equal to zero, in the event of compliance. If we assume that there exists a continuous, \mathcal{F} -adapted process E denoting the level of the emissions accumulated up to the present time (measured in tonnes of CO₂) and if we denote by $\gamma \geq 0$ the level of the cap (measured in tonnes of CO₂), then the former event is characterised by $\{E_T \geq \gamma\}$. Therefore, at compliance time the value of the allowance certificate is prescribed by the condition

$$A_T = \pi I_{[\gamma, \infty)}(E_T), \quad (2.4)$$

where $\pi \geq 0$ denotes the penalty (measured in € per tonne of CO₂) prevalent in the emission market under consideration.

Clearly condition (2.4) only holds in a one-period market. Importantly, as we will show in Chapter 3, the consideration of multiple trading periods does not change the binary nature of the terminal condition; it merely affects the level of the cap and the value of the allowance certificate in both the event of compliance and non-compliance. In fact, in the case of multiple trading periods the value of the allowance certificate at compliance time is prescribed by the condition

$$A_T = \tilde{A}_T + \pi I_{[\tilde{\gamma}, \infty)}(E_T), \quad (2.5)$$

for $\tilde{\gamma} \geq 0$ and a bounded, non-negative, \mathcal{F}_T -measurable random variable \tilde{A}_T , with upper bound $\bar{A} \geq 0$. As we will see later \tilde{A}_T represents the value of the following period allowance certificate at the end of the current trading period. In particular, it may depend on E_T . Note that (2.5) becomes (2.4) if we set $\tilde{A}_T = 0$ and $\tilde{\gamma} = \gamma$. Therefore, from now on throughout this chapter we work with the terminal condition corresponding to the multi-period setting bearing in mind that it contains the single-period one as a special case.

The models that we deal with throughout this thesis will all be shown to have the property that the accumulation of emissions throughout the compliance period can be perfectly explained by the observation of the d factors represented by X and the allowance price as represented by A . In particular, E depends on X and A in

a Markovian way, that is, its dynamics can be written as a function of X and A . In mathematical terms we assume that E is square integrable and has absolutely continuous paths. The latter assumption guarantees the existence of a measurable function $\mu_e : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$, such that, for $e_0 \in \mathbb{R}$, the process E satisfies the ordinary integral equation

$$E_t = e_0 + \int_0^t \mu_e(X_u, A_u) \, du, \quad \text{for } t \in [0, T]. \quad (2.6)$$

The fact that the allowance price has an effect on the emissions rate allows us to capture the quintessential purpose of any cap-and-trade system, namely that of inducing abatement: higher allowance prices lead to cleaner resources in the market replacing emission-intensive ones. We will see that the mathematical implication of this feature is that the equation describing our pricing problem becomes nonlinear.

From (2.4) and (2.5) it is clear that the cumulative emissions process E plays an important role in determining the value of the allowance certificate at the end of the compliance period. In fact, for given values of \tilde{A}_T , the value of the certificate is a deterministic function of E_T . This motivates us to consider the terminal conditions (2.4) and (2.5) as payoffs of an option contract. Then it is clear, that in the proposed framework the allowance certificate itself is a contract written on the cumulative emissions process E and hence can be thought of as a derivative on the exogenously given, fundamental factors X .

2.2 Derivation of the Allowance Pricing Equation

Under Assumption 1 the price of any contingent claim is given by the conditional expectation of its payoff under this measure. For notational simplicity, we introduce a function $\phi : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ and define it to represent the terminal value of the allowance certificate, that is

$$\phi(\tilde{a}, e) := \tilde{a} + \pi I_{[\tilde{\gamma}, \infty)}(e), \quad \text{for } (\tilde{a}, e) \in [0, \overline{\mathcal{A}}] \times \mathbb{R}. \quad (2.7)$$

Occasionally, when we want to emphasize the dependence on the parameters π and $\tilde{\gamma}$ we shall write $\phi(\cdot, \cdot; \pi, \tilde{\gamma})$. Note that in the setting of a single compliance period the function $\phi(0, \cdot; \pi, \gamma)$ is used to determine the terminal value of the allowance certificate, whereas in the case of multiple compliance periods $\phi(\cdot, \cdot; \pi, \tilde{\gamma})$ is used. Because of its importance later on we point out that ϕ is a linear function of \tilde{a} , but only right-continuous as a function of e with the discontinuity occurring at $\tilde{\gamma}$.

Then at any time up to compliance the price of the allowance certificate is given by the continuous version of the uniformly integrable, non-negative martingale $\mathbb{E}^{\mathbb{Q}}[\phi(\tilde{A}_T, E_T)|\mathcal{F}_t]$, that is

$$A_t = \mathbb{E}^{\mathbb{Q}}[\phi(\tilde{A}_T, E_T)|\mathcal{F}_t], \quad \text{for } t \in [0, T],$$

so that the price process is indeed continuous and adapted to the filtration \mathcal{F} .

From an application of the Martingale Representation Theorem for Brownian motion there exists a progressively measurable process $Z^a := (Z^{a,1}, \dots, Z^{a,d})$, satisfying

$$\mathbb{E}^{\mathbb{Q}} \left[\int_0^T \|Z_t^a\|^2 dt \right] < \infty$$

and such that

$$A_t = \mathbb{E}^{\mathbb{Q}} \left[\phi(\tilde{A}_T, E_T) \right] + \int_0^t \langle Z_u^a, dW_u \rangle, \quad \text{for } t \in [0, T]. \quad (2.8)$$

Combining (2.3), (2.6) and (2.8) and applying Itô's Lemma, we find that the pricing problem is summarised by the following FBSDE: for $t \in [0, T]$,

$$\begin{cases} X_t = x_0 + \int_0^t \mu(u, X_u) du + \int_0^t \sigma(X_u) dW_u \\ E_t = e_0 + \int_0^t \mu_e(X_u, A_u) du \\ A_t = \phi(\tilde{A}_T, E_T) - \int_t^T \langle Z_u^a, dW_u \rangle. \end{cases} \quad (2.9)$$

The FBSDE (2.9) is non-standard in two ways. First, the market design of the kind of emissions trading system that we set out to model immediately leads to a *gradient singularity* in the terminal condition ϕ . Further, the nature of our models, which, in the spirit of the structural approach, explain the dynamics of the cumulative emissions process as a function of the underlying fundamental factors X and the allowance price A , imply the *degeneracy* of the forward evolving process E . As a result, two ubiquitous assumptions made in the literature when proving the existence and uniqueness of solutions to FBSDEs for arbitrary $T \geq 0$, namely that of uniform ellipticity and the Lipschitz continuity of the terminal condition (cf. [Delarue, 2002] and the references found therein), are clearly violated. As we pointed out in the introduction, very little progress has been made to date to address the question of existence and uniqueness of a solution to systems of the type of (2.9). In fact the only two papers of relevance that are known to us are [Carmona et al., 2012c] and

[Carmona and Delarue, 2011]. However, the assumptions made therein on the coefficients, in particular the uniform Lipschitz continuity of μ and σ and the deterministic nature of the terminal condition, turn out to be too restrictive for the models we propose in Chapters 3 and 5. Therefore, we dedicate the next section to the proof of the existence and uniqueness of a solution to the FBSDE (2.9) under less strict assumptions, which turn out to be suitable for our purposes.

2.3 Existence and Uniqueness of a Solution to the Allowance FBSDE

In this section we prove the existence of unique processes⁴ $((X, E), A, Z^a)$ that satisfy (2.9). To ease the presentation throughout this section we introduce the notation \mathcal{H}^0 to denote the collection of all \mathbb{R}^n -valued, progressively measurable (with respect to \mathcal{F}) stochastic processes.⁵ With this definition, we further introduce the subsets \mathcal{H}^2 and \mathcal{S}^2 defined by

$$\mathcal{H}^2 := \{Y \in \mathcal{H}^0 : \|Y\|_2 < \infty\}, \quad \text{where } \|Y\|_2 := \left(\mathbb{E}^{\mathbb{Q}} \left[\int_0^T \|Y_t\|^2 dt \right] \right)^{\frac{1}{2}}$$

and

$$\mathcal{S}^2 := \{Y \in \mathcal{H}^0 : \|Y\|_{2,*} < \infty\}, \quad \text{where } \|Y\|_{2,*} := \left(\mathbb{E}^{\mathbb{Q}} \left[\sup_{t \in [0, T]} \|Y_t\|^2 \right] \right)^{\frac{1}{2}}.$$

Then it is well known that \mathcal{H}^2 equipped with the norm $\|\cdot\|_2$ and \mathcal{S}^2 equipped with the norm $\|\cdot\|_{2,*}$ are Banach spaces.

From now on we make the following assumption on the regularity of the coefficients of the FBSDE (2.9).

Assumption 2. The coefficients μ , σ , μ_e of the FBSDE (2.9) are said to satisfy Assumption 2 if the following properties hold:

(A2.1) The functions μ and σ are such that the integral equation (2.3) admits a unique, strong solution $X \in \mathcal{S}^2$.

(A2.2) The function μ_e satisfies the following:

⁴In the following we occasionally place parentheses around a collection of stochastic processes to emphasize that they play a similar role, either mathematically or conceptually.

⁵We note that throughout this thesis, whenever we state that a particular process is an element of \mathcal{H}^2 or \mathcal{S}^2 , we do not specify the dimensionality of the process, that is, the value of n . However, we trust that this will always be clear from the context.

- For some $L_1 \geq 1$, the linear growth condition,

$$|\mu_e(x, a)| \leq L_1(1 + |x| + |a|), \quad \text{for } (x, a) \in \mathbb{R}^d \times \mathbb{R}.$$

- For $x \in \mathbb{R}^d$, the function $\mu_e(x, \cdot)$ is strictly decreasing and satisfies, for $l_1, l_2 > 0$ and $1/L_1 \leq l_1 \leq l_2 \leq L_1$,

$$l_1 \leq -\frac{\mu_e(x, a) - \mu_e(x, a')}{a - a'} \leq l_2, \quad \text{for } a \neq a'.$$

Since $l_2 \leq L_1$ (A2.2) immediately implies that μ_e is L_1 -Lipschitz continuous as a function of a , uniformly in x .

Remark 1. Sufficient conditions for Assumption (A2.1) are for μ and σ to be globally Lipschitz continuous and to exhibit no more than linear growth in the space variable (cf. [Karatzas and Shreve, 2005, Theorem 5.2.9]). However, we emphasise that these are by no means necessary and we will later make use of SDEs, which do not satisfy these conditions but still admit a unique strong solution in the space \mathcal{S}^2 .

Remark 2. The strict monotonicity condition in Assumption (A2.2) is important in order for us to be able to obtain the existence result presented in this chapter. In particular, it can be seen as an analogue to the strict convexity requirement on the primitive of μ_e in the theory of hyperbolic conservation laws (cf. [LeFloch, 2002]).

The important result that we now prove throughout this section is summarised in the following theorem.

Theorem 1 (Existence and uniqueness of the allowance price FBSDE). *With Assumption 2 in place and given that \tilde{A}_T is a bounded and non-negative, \mathcal{F}_T -measurable random variable, there exist unique processes $((X, E), A, Z^a) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$ satisfying the FBSDE (2.9) and the “relaxed” terminal condition*

$$\tilde{A}_T + \pi I_{[\tilde{\gamma}, \infty)}(E_T) \leq A_T \leq \tilde{A}_T + \pi I_{[\tilde{\gamma}, \infty)}(E_T). \quad (2.10)$$

Remark 3. The assumptions underlying the above theorem are still satisfied in the case of a non-zero interest rate and the theorem can trivially be extended to that case. We will take advantage of this fact in Chapters 3 and 5, where the interest rate will be reintroduced.

The result in Theorem 1 is unusual in so far as it says that in order to guarantee the existence of a solution, we have to weaken the usual definition of what it means for a triple of processes to satisfy the FBSDE (2.9). Specifically, in the event that the

cumulative emissions process hits the cap we cannot expect the allowance price in our model to satisfy the terminal condition ϕ , as one would expect given the market design of the emissions trading systems that we are modelling. Moreover, it has been shown in [Carmona et al., 2012c] and [Carmona and Delarue, 2011] that the degeneracy of the forward evolving process E together with the gradient singularity of the terminal condition ϕ may conspire and cause the emissions process to exhibit a point mass at the cap, making the “relaxed” terminal condition (2.10) meaningful.

There is an intuitive economic interpretation of the terminal condition (2.10). In the event that the cumulative emissions end up directly at the cap, there is no incentive for any market participants to either buy or sell allowance certificates; the market is in a state of equilibrium and no trading occurs. In this case no financially meaningful price exists and the result in Theorem 1 reflects this fact.

We prove Theorem 1 in several steps. First, noticing that X is decoupled, we may solve the associated stochastic integral equation separately from (E, A, Z^a) . This allows us to consider equation (2.9) as a one-dimensional FBSDE with random drift coefficient. Second, we replace ϕ with a function that is Lipschitz continuous in the e -direction and therefore smoothes the discontinuity exhibited by ϕ . We show that, in this case, existence and uniqueness of a solution are guaranteed by [Ma et al., 2011]. The main challenge then is to prove the existence and uniqueness of a solution to the unsmoothed problem. We achieve this by approximating the discontinuous terminal condition in an appropriate way and by making use of compactness and limiting arguments.

To explain how we interpret (2.9) as a one-dimensional FBSDE with random drift coefficient, let us define a random field $\hat{\mu}_e$ by

$$\hat{\mu}_e(t, \omega, a) := \mu_e(X_t, a), \quad \text{for } (t, \omega, a) \in [0, T] \times \Omega \times \mathbb{R}. \quad (2.11)$$

Note that the random field $\hat{\mu}_e$ inherits the properties of μ_e and X . In particular, we point out that $\hat{\mu}_e(0, \omega, a)$ is \mathcal{F}_0 -measurable and hence deterministic. Since E is assumed to satisfy (2.6), it follows that it equally satisfies

$$E_t := e_0 + \int_0^t \hat{\mu}_e(u, \omega, A_u) du, \quad \text{for } t \in [0, T], \quad (2.12)$$

so that (2.9) and (2.12) immediately imply that (E, A, Z^a) satisfies the following FBSDE with random drift coefficient: for $t \in [0, T]$,

$$\begin{cases} E_t = e_0 + \int_0^t \hat{\mu}_e(u, \omega, A_u) du \\ A_t = \phi(\tilde{A}_T, E_T) - \int_t^T \langle Z_u^a, dW_u \rangle. \end{cases} \quad (2.13)$$

From now on, we work with equation (2.13) bearing in mind that a solution to this FBSDE allows us to construct the full solution $((X, E), A, Z^a)$ to the original problem (2.9).

2.3.1 Existence — Regularized Terminal Condition

Following through with the second step towards our goal of proving Theorem 1 we show the existence and uniqueness of a solution to equation (2.13) in the case when the terminal condition ϕ is regularized in an appropriate way. For this purpose, we assume the existence of a function $\hat{\phi} : [0, \bar{\mathcal{A}}] \times \mathbb{R} \mapsto \mathbb{R}$ on which we impose the following conditions.

Assumption 3. The terminal condition $\hat{\phi}$ is said to satisfy Assumption 3 if the following properties hold for all $\tilde{a} \in [0, \bar{\mathcal{A}}]$:

(A3.1) There exists some $L_2 > 0$, such that the function $\hat{\phi}(\tilde{a}, \cdot)$ is L_2 -Lipschitz continuous, that is,

$$|\hat{\phi}(\tilde{a}, e) - \hat{\phi}(\tilde{a}, e')| \leq L_2 |e - e'|, \quad \text{for } e, e' \in \mathbb{R}.$$

(A3.2) The function $\hat{\phi}(\tilde{a}, \cdot)$ is non-decreasing.

(A3.3) $\hat{\phi}(\tilde{a}, \cdot)$ has the same bounds as ϕ , that is

$$\inf_{e \in \mathbb{R}} \hat{\phi}(\tilde{a}, e) = \inf_{e \in \mathbb{R}} \phi(\tilde{a}, e) \quad \text{and} \quad \sup_{e \in \mathbb{R}} \hat{\phi}(\tilde{a}, e) = \sup_{e \in \mathbb{R}} \phi(\tilde{a}, e).$$

Remark 4. Note that by the definition of ϕ in (2.7), we may consider regularizations of the unsmoothed terminal condition that satisfy Assumption 3 and can be written in the form $\hat{\phi}(\tilde{a}, e) = \tilde{a} + \pi J(e)$, where $J : \mathbb{R} \mapsto [0, 1]$ is an approximation of the indicator function $I_{[\tilde{\gamma}, \infty)}(\cdot)$, chosen so that (A3.1) and (A3.2) hold. In fact, in the following we will only consider regularizations of this form.

We denote a solution to the FBSDE (2.13) with terminal condition $\hat{\phi}$ by $(E^{\hat{\phi}}, A^{\hat{\phi}}, Z^{\hat{\phi}})$. In the following we will occasionally be interested in this solution, when the FBSDE is set on a time interval $[t_0, T]$, for $t_0 \in [0, T)$, with an initial condition $e_{t_0} \in \mathbb{R}$ supplied at time t_0 . In this case we denote a corresponding solution by $(E^{\hat{\phi}, t_0, e_{t_0}}, A^{\hat{\phi}, t_0, e_{t_0}}, Z^{\hat{\phi}, t_0, e_{t_0}})$. Clearly, it satisfies, for $t \in [t_0, T]$,

$$\begin{cases} E_t^{\hat{\phi}, t_0, e_{t_0}} = e_{t_0} + \int_{t_0}^t \hat{\mu}_e(u, \omega, A_u^{\hat{\phi}, t_0, e_{t_0}}) \, du \\ A_t^{\hat{\phi}, t_0, e_{t_0}} = \hat{\phi}(\tilde{A}_T, E_T^{\hat{\phi}, t_0, e_{t_0}}) - \int_t^T \langle Z_u^{\hat{\phi}, t_0, e_{t_0}}, dW_u \rangle. \end{cases} \quad (2.14)$$

The main result in this section then is the following.

Proposition 1 (Existence and uniqueness regularized FBSDE). *Under the conditions postulated in Assumption 2 and Assumption 3 and given that \tilde{A}_T is a bounded and non-negative \mathcal{F}_T -measurable random variable, the following holds for $t_0 \in [0, T)$ and $e_{t_0} \in \mathbb{R}$:*

(i) *The FBSDE (2.14) admits a unique solution $(E^{\hat{\phi}, t_0, e_{t_0}}, A^{\hat{\phi}, t_0, e_{t_0}}, Z^{\hat{\phi}, t_0, e_{t_0}}) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$.*

(ii) *There exists a unique, progressively measurable (with respect to \mathcal{F}) random field $\alpha^{\hat{\phi}} : [0, T] \times \Omega \times \mathbb{R} \mapsto \mathbb{R}$, satisfying*

$$\alpha^{\hat{\phi}}(t, \omega, E_t^{\hat{\phi}, t_0, e_{t_0}}) = A_t^{\hat{\phi}, t_0, e_{t_0}} = \mathbb{E}^{\mathbb{Q}}[\hat{\phi}(\tilde{A}_T, E_T) | \mathcal{F}_t], \quad \text{for } (t, \omega) \in [t_0, T] \times \Omega$$

and

$$0 \leq \frac{\alpha^{\hat{\phi}}(t, \omega, e) - \alpha^{\hat{\phi}}(t, \omega, e')}{e - e'} \leq L_2, \quad \text{for } (t, \omega) \in [t_0, T] \times \Omega, \quad e \neq e'.$$

Occasionally, when we want to emphasize the dependency of the random field $\alpha^{\hat{\phi}}$ on the value of the random variable \tilde{A}_T and the parameters π and $\tilde{\gamma}$, we write $\alpha^{\hat{\phi}}(\cdot, \cdot, \cdot; \tilde{A}_T, \pi, \tilde{\gamma})$.

Proof. In the present regularized setting the proof follows directly from [Ma et al., 2011]. We first notice that our assumptions imply that also [Ma et al., 2011, Assumption 2.1] holds. In particular, by its definition, the process $\hat{\mu}_e(\cdot, \cdot, e)$ is \mathcal{F} -progressively measurable and the random function $\hat{\phi}(\tilde{A}_T, e)$ is \mathcal{F}_T -measurable, both for fixed $e \in \mathbb{R}$. Further, since $X \in \mathcal{S}^2$ by (A2.1) and under the linear growth assumption in (A2.2), the process $\mu_e(t, \omega, 0)$ is square integrable; moreover \tilde{A}_T is bounded, so that the integrability condition I_0 in [Ma et al., 2011, Assumption 2.1] is satisfied. By Assumption (A2.2) and Assumption 3 respectively, $\hat{\mu}_e$ and $\hat{\phi}$ are both Lipschitz continuous in the spatial variable, uniformly in time and randomness and with a common Lipschitz constant $L := \max(L_1, L_2)$.

(i) The existence of a unique solution $(E^{\hat{\phi}, t_0, e_{t_0}}, A^{\hat{\phi}, t_0, e_{t_0}}, Z^{\hat{\phi}, t_0, e_{t_0}}) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$ to the FBSDE (2.14) is now implied by [Ma et al., 2011, Theorem 6.1 (i) and Theorem 7.1 (ii) and Remark 2.4 (ii)].

(ii) Similarly, the existence of the random field $\alpha^{\hat{\phi}}$ follows from [Ma et al., 2011, Theorem 6.1 (iii) and Theorem 7.1 (i)] and uniqueness follows from [Ma et al., 2011, Remark 2.4 (ii)]. The bounds on the slope of $\alpha^{\hat{\phi}}$ follow from

[Ma et al., 2011, Theorem 5.6 (ii)] with $\lambda = \epsilon = 0$ and [Ma et al., 2011, Theorem 6.1 (iii) and Theorem 7.1 (i)]. In particular, in [Ma et al., 2011, Theorem 5.6 (ii)] λ is identified with the lower and C_2 with the upper bound. Since in our setting $\lambda = 0$ and $C_2 = L_2$, the bounds in the proposition follow.

□

Remark 5. We point out that the results in [Ma et al., 2011] are derived under the assumption that the Brownian motion W is one-dimensional (i.e. $d = 1$). A careful study of the proofs in the paper however reveals that this assumption is unnecessary and that none of the individual results rely on it. As pointed out in the paper, it is however critically important that the process evolving forward in time (i.e. E) is \mathbb{R} -valued. At the time of writing of this thesis [Ma et al., 2011] is still a working paper and we expect the assumption of W being one-dimensional to be dropped in the final version. In fact, this is already the case in the more recent working paper [Ma et al., 2012], which shows that the random field $\alpha^{\hat{\phi}}$, that characterizes the solution of the FBSDE, is also a Sobolev-type solution to a backward stochastic partial differential equation.

Our generalization of the above existence result in §2.3.2 to the case of a discontinuous terminal condition, heavily relies on a new estimate — independent of L_2 — for the slope of the random field $\alpha^{\hat{\phi}}$. In order to readily obtain this estimate, we will need the fact that in [Ma et al., 2011] the slope of $\alpha^{\hat{\phi}}$ is bounded by the unique solutions $C^1([0, T]) \ni \bar{a}, \underline{a} : [0, T] \mapsto \mathbb{R}$ to the two *dominating ODEs*,

$$\begin{aligned}\bar{a}(t) &= \bar{h} + \int_t^T \bar{F}(u, \bar{a}(u)) \, du \\ \underline{a}(t) &= \underline{h} + \int_t^T \underline{F}(u, \underline{a}(u)) \, du,\end{aligned}\tag{2.15}$$

where

$$\begin{aligned}\bar{h} &:= \operatorname{ess\,sup}_{\omega \in \Omega} (\operatorname{ess\,sup}_{\substack{e, e' \in \mathbb{R} \\ e \neq e'}} h(\tilde{A}_T, e, e')), \\ \underline{h} &:= \operatorname{ess\,inf}_{\omega \in \Omega} (\operatorname{ess\,inf}_{\substack{e, e' \in \mathbb{R} \\ e \neq e'}} h(\tilde{A}_T, e, e'))\end{aligned}$$

and

$$\begin{aligned}\bar{F}(t, y) &:= \operatorname{ess\,sup}_{\omega \in \Omega} (\operatorname{ess\,sup}_{\substack{a, a' \in \mathbb{R} \\ a \neq a'}} F(t, \omega, a, a', y)), \\ \underline{F}(t, y) &:= \operatorname{ess\,inf}_{\omega \in \Omega} (\operatorname{ess\,inf}_{\substack{a, a' \in \mathbb{R} \\ a \neq a'}} F(t, \omega, a, a', y))\end{aligned}\tag{2.16}$$

and

$$h(\tilde{a}, e, e') := \frac{\hat{\phi}(\tilde{a}, e) - \hat{\phi}(\tilde{a}, e')}{e - e'}, \quad \text{for } \tilde{a} \in [0, \bar{\mathcal{A}}], e \neq e'$$

and

$$F(t, \omega, a, a', y) := \frac{\hat{\mu}_e(t, \omega, a) - \hat{\mu}_e(t, \omega, a')}{a - a'} y^2, \quad \text{for } a \neq a', y \in \mathbb{R}.$$

There are two different usages of “essential supremum” in the above definitions. First, the interior essential supremum is that of an uncountable family of random variables. Second, the exterior essential supremum is that of a single random variable. For the precise definition of the two different concepts, the interested reader may consult [Föllmer and Schied, 2010, Appendix 5].

Note that h is the slope of a line segment joining the points $(e, \hat{\phi}(\tilde{a}, e))$ and $(e', \hat{\phi}(\tilde{a}, e'))$. Since $\hat{\phi}$ is non-decreasing, $0 \leq \underline{h} \leq \bar{h}$; in fact \bar{h} is the smallest Lipschitz constant of $\hat{\phi}$, that is $\bar{h} \leq L_2 \leq L$. Similarly, for fixed $(t, y) \in [0, T] \times \mathbb{R}$, F is the slope of a line segment joining the points $(a, \hat{\mu}_e(t, \omega, a))$ and $(a', \hat{\mu}_e(t, \omega, a'))$ multiplied by a non-negative constant. Since $\hat{\mu}_e(t, \omega, \cdot)$ is strictly decreasing, $\underline{F} \leq \bar{F} \leq 0$.

2.3.2 Existence — Discontinuous Terminal Condition

In this section we prove the existence of a solution to the FBSDE (2.13) with discontinuous terminal condition ϕ . Our strategy is to approximate ϕ using a monotone sequence of functions and to construct a limiting argument.

Let $(\hat{\phi}_n)_{n \geq 1}$ be a pointwise decreasing sequence of functions, $\hat{\phi}_n : [0, \bar{\mathcal{A}}] \times \mathbb{R} \mapsto \mathbb{R}$ all satisfying Assumption 3 and such that

$$\hat{\phi}_n \rightarrow \phi \quad \text{pointwise.} \quad (2.17)$$

Example 2. As an example we explicitly construct a sequence $(\hat{\phi}_n)_{n \geq 1}$, satisfying (2.17). For $n \geq 1$ and $\tilde{a} \in [0, \bar{\mathcal{A}}]$, let $\hat{\phi}_n$ be given by

$$\hat{\phi}_n(\tilde{a}, e) := \begin{cases} \tilde{a}, & \text{if } -\infty < e < \tilde{\gamma} - \frac{1}{n} \\ n\pi(e - \tilde{\gamma}) + \tilde{a} + \pi, & \text{if } \tilde{\gamma} - \frac{1}{n} \leq e < \tilde{\gamma} \\ \tilde{a} + \pi, & \text{if } \tilde{\gamma} \leq e < \infty. \end{cases}$$

Then $|\hat{\phi}_n(e) - \phi(e)| = \pi|n(e - \tilde{\gamma}) + 1|I_{(\tilde{\gamma}-1/n, \tilde{\gamma})}(e) \leq \pi I_{(\tilde{\gamma}-1/n, \tilde{\gamma})}(e) \rightarrow 0$, as $n \rightarrow \infty$ implying pointwise convergence. Moreover, the convergence also holds in L^p , for

$p \geq 1$, as the following argument shows:

$$\begin{aligned} \|\hat{\phi}_n - \phi\|_{L^p} &= \left(\int_0^{\bar{\mathcal{A}}} \int_{\tilde{\gamma} - \frac{1}{n}}^{\tilde{\gamma}} \pi^p |n(e - \tilde{\gamma}) + 1|^p \, de \, d\tilde{a} \right)^{\frac{1}{p}} \\ &\leq \left(\frac{\pi^p \bar{\mathcal{A}}}{n} \right)^{\frac{1}{p}} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Here we remind the reader that $\bar{\mathcal{A}}$ denotes the upper bound of the random variable \tilde{A}_T .

By Proposition 1, for $n \geq 1$, we may associate with each $\hat{\phi}_n$ a unique triple $(E^{n,t_0,e_{t_0}}, A^{n,t_0,e_{t_0}}, Z^{n,t_0,e_{t_0}}) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$ solving the FBSDE (2.14) with $\hat{\phi}_n$ as terminal condition. Moreover, there exist unique random fields $\alpha_n : [0, T] \times \Omega \times \mathbb{R} \mapsto \mathbb{R}$, Lipschitz continuous in the spatial variable, such that $A_t^{n,t_0,e_{t_0}} = \alpha_n(t, \omega, E_t^{n,t_0,e_{t_0}})$, for $(t, \omega) \in [t_0, T] \times \Omega$.

The following uniform estimate will play an important role and eventually allow us to pass to the limit through the sequence of approximating terminal conditions.

Lemma 1 (Uniform Lipschitz estimate). *There exists a constant $C_1(L_1) > 0$, independent of L_2 , such that for all $n \geq 1$,*

$$0 \leq \frac{\alpha_n(t, \omega, e) - \alpha_n(t, \omega, e')}{e - e'} \leq \frac{C_1}{T - t}, \quad \text{for } (t, \omega) \in [0, T] \times \Omega, \, e \neq e'.$$

Therefore, given $\delta \in (0, T]$, for $(t, \omega) \in [0, T - \delta] \times \Omega$ and $n \geq 1$, the random fields $\alpha_n(t, \omega, \cdot)$ are Lipschitz continuous, uniformly in time, randomness and n .

Proof. From [Ma et al., 2011, Theorem 6.1(iii) and Theorem 7.1(i)] we know that any bounds for the solutions of the dominating ODEs also constitute bounds for the slope of the random field α_n , $n \geq 1$. We specifically want to obtain bounds independent of the regularity of the terminal condition $\hat{\phi}_n$ and therefore of n .

For the lower bound we note that, for $\tilde{a} \in [0, \bar{\mathcal{A}}]$, each $\hat{\phi}_n(\tilde{a}, \cdot)$ is increasing and bounded. Therefore,

$$\underline{h}_n := \operatorname{ess\,inf}_{\omega \in \Omega} \left(\operatorname{ess\,inf}_{\substack{e, e' \in \mathbb{R} \\ e \neq e'}} \frac{\hat{\phi}_n(\tilde{A}_T, e) - \hat{\phi}_n(\tilde{A}_T, e')}{e - e'} \right) = 0 = \underline{h}, \quad \text{for } n \geq 1.$$

By the same argument as the one used in Proposition 1 we conclude that zero is indeed a lower bound for the slope of α_n , for all $n \geq 1$.

In the case of the upper bound, independence of the regularity of $\hat{\phi}_n$ means finding an upper bound for the solution of the first of the two dominating ODEs (2.15) with

a generic — possibly unbounded — terminal condition, say $h^* \geq 0$. Recall that $a^* : [0, T] \mapsto \mathbb{R}$ is the unique solution to the dominating ODE if it solves

$$a^*(t) = h^* + \int_t^T \bar{F}(u, a^*(u)) \, du, \quad \text{for } t \in [0, T].$$

Using definition (2.16) of \bar{F} , an easy direct computation shows that, assuming it does not vanish, a^* is given by

$$a^*(t) = \left(\frac{1}{h^*} - \int_t^T F^*(u) \, du \right)^{-1}, \quad \text{for } t \in [0, T], \quad (2.18)$$

where

$$F^*(t) := \operatorname{ess\,sup}_{\omega \in \Omega} \left(\operatorname{ess\,sup}_{\substack{a, a' \in \mathbb{R} \\ a \neq a'}} \frac{\hat{\mu}_e(t, \omega, a) - \hat{\mu}_e(t, \omega, a')}{a - a'} \right).$$

By the strict monotonicity of $\hat{\mu}_e$ in the spatial dimension, F^* is always strictly negative and therefore the right-hand side in (2.18) is always strictly positive, so that a^* , as given above, is indeed the unique solution. We now derive an upper bound for a^* . From (2.18) it follows that

$$\begin{aligned} a^*(t) &\leq \left(\int_t^T |F^*(u)| \, du \right)^{-1} \\ &\leq \left(\inf_{u \in [t, T]} |F^*(u)| (T - t) \right)^{-1}, \quad \text{for } t \in [0, T], \end{aligned}$$

where the infimum exists and is finite by Assumption (A2.2). This immediately leads to the following estimate:

$$a^*(t) \leq \frac{C_1}{T - t}, \quad \text{for } t \in [0, T]$$

and with $C_1 = \sup_{u \in [t, T]} |F^*(u)|^{-1}$. The statement in the Lemma follows. \square

Having obtained the uniform Lipschitz estimate in Lemma 1, we are in a position to prove the convergence of the random fields α_n .

Proposition 2 (Convergence of random fields). *There exists a random field α , such that*

$$\alpha_n \rightarrow \alpha \quad \text{pointwise,}$$

with the following properties:

- (i) *The range of α is contained in $[0, \bar{\phi}]$, where $\bar{\phi} := \bar{\mathcal{A}} + \pi$.*

(ii) The function $\alpha(t, \omega, \cdot)$ is increasing, for $(t, \omega) \in [0, T] \times \Omega$.

(iii) α is increasing in π , decreasing in $\tilde{\gamma}$ and increasing in \tilde{A}_T in the almost sure partial ordering.

(iv) The function $\alpha(t, \omega, \cdot)$ is $C_1/(T-t)$ -Lipschitz continuous, for $(t, \omega) \in [0, T] \times \Omega$.

Proof. From Proposition 1 we know that $\alpha_n(t_0, \omega, e_{t_0}) = \mathbb{E}^{\mathbb{Q}}[\hat{\phi}_n(\tilde{A}_T, E_T^{n, t_0, e_{t_0}}) | \mathcal{F}_{t_0}]$. Recall that since \tilde{A}_T was assumed to be non-negative and bounded above by \bar{A} , the unsmoothed terminal condition $\phi(\tilde{A}_T, E_T) := \tilde{A}_T + \pi I_{[\tilde{\gamma}, \infty)}(E_T)$ is bounded by $[0, \bar{\phi}]$. Assumption (A3.3) guarantees that the same bound applies to the smoothed terminal condition. Therefore, the sequence $(\hat{\phi}_n)_{n \geq 1}$ can be uniformly bounded by $[0, \bar{\phi}]$. Since t_0 and e_{t_0} were chosen arbitrarily, it follows that, for all $n \geq 1$, also the random field α_n is uniformly bounded by $[0, \bar{\phi}]$. The fact that the sequence $(\hat{\phi}_n)_{n \geq 1}$ is non-increasing allows us to deduce from the comparison theorem for FBSDEs [Ma et al., 2011, Theorem 8.6]⁶ that the same applies to $(\alpha_n)_{n \geq 1}$; hence, as a bounded decreasing sequence it converges to a limit α pointwise.

(i) The fact that α can only take values in $[0, \bar{\phi}]$ follows immediately from the fact that each α_n is $[0, \bar{\phi}]$ -valued and the definition of pointwise convergence.

(ii) From Proposition 1 we know that for $(t, \omega) \in [0, T] \times \Omega$, each $\alpha_n(t, \omega, \cdot)$ is an increasing function. Therefore, also the limit α is increasing.

(iii) An application of the comparison theorem for FBSDEs [Ma et al., 2011, Theorem 8.6] together with the fact that, for $(t, \omega, e) \in [0, T] \times \Omega \times \mathbb{R}$, each terminal condition $\hat{\phi}_n(\cdot, e; \cdot, \cdot)$ is increasing in π , decreasing in $\tilde{\gamma}$ and increasing in \tilde{A}_T (in the almost sure partial ordering) reveals that the same monotonicity properties also hold for each $\alpha_n(t, \omega, e; \cdot, \cdot, \cdot)$. We conclude that the same is true for the limiting random field α .

(iv) We introduce the notation \mathcal{K}_b to denote the collection of all bounded, real-valued random fields $u : [0, T] \times \Omega \times \mathbb{R} \mapsto \mathbb{R}$, for which $u(t, \omega, \cdot)$ is a continuous function of e , for $(t, \omega) \in [0, T] \times \Omega$. Further, we introduce the subset $\mathcal{K}_b^{0,1}$ defined by

$$\mathcal{K}_b^{0,1} := \left\{ u \in \mathcal{K}_b : \|u\|_{\mathcal{K}_b^{0,1}} < \infty \right\},$$

⁶Theorem 8.6 applies under the same conditions as the existence statement in Proposition 1.

where

$$\|u\|_{\mathcal{K}_b^{0,1}} := \mathbb{E}^{\mathbb{Q}} \left[\int_0^T \|u\|_{C_b^{0,1}}(t, \omega) dt \right]$$

and

$$\|u\|_{C_b^{0,1}}(t, \omega) := \operatorname{ess\,sup}_{e \in \mathbb{R}} |u(t, \omega, e)| + \operatorname{ess\,sup}_{\substack{e, e' \in \mathbb{R} \\ e \neq e'}} \frac{|u(t, \omega, e) - u(t, \omega, e')|}{|e - e'|}.$$

Then $\mathcal{K}_b^{0,1}$ equipped with the norm $\|\cdot\|_{\mathcal{K}_b^{0,1}}$ is a Banach space.

We now consider $\{\alpha_n : n \geq 1\}$ as a subset of $\mathcal{K}_b^{0,1}$. Clearly, this set is bounded. Moreover, the uniform estimate in Lemma 1 implies that, for $\delta \in (0, T]$, it is equicontinuous in the spatial variable on $[0, T - \delta] \times \Omega \times \mathbb{R}$, independently of the regularity of the terminal conditions $\hat{\phi}_n$, $n \geq 1$. The Arzelà–Ascoli theorem now guarantees the compactness of $\{\alpha_n : n \geq 1\}$ and the existence of a subsequence $(\alpha_{\psi(n)})_{n \geq 1}$, which, on $(t, \omega, e) \in [0, T - \delta] \times \Omega \times \mathbb{R}$, converges uniformly to $\alpha \in \mathcal{K}_b^{0,1}$. The limit α is Lipschitz continuous and by Lemma 1 it is $C_1/(T - t)$ -Lipschitz continuous. It is worth pointing out that, since uniform convergence implies pointwise convergence, the two limits in the proposition coincide on $[0, T] \times \Omega \times \mathbb{R}$ (except possibly on a null set). □

Next, we turn to the convergence of the approximating sequence of emissions processes $(E^n)_{n \geq 1}$.

Proposition 3 (Convergence of emissions process). *There exists a stochastic process E , such that*

$$E^n \rightarrow E \quad \textit{pointwise}.$$

Further,

(i) *with α as defined in Proposition 2, the stochastic integral equation*

$$\tilde{E}_t = e_0 + \int_0^t \hat{\mu}_e(u, \omega, \alpha(u, \omega, \tilde{E}_u)) du, \quad \textit{for } t \in [0, T], \quad (2.19)$$

has a strong solution.

(ii) *The limit E is in \mathcal{S}^2 ; specifically,*

$$E^n \rightarrow E \quad \textit{in } \mathcal{S}^2.$$

Proof. For each $n \geq 1$ the process E^n satisfies the stochastic integral equation

$$E_t^n = e_0 + \int_0^t \hat{\mu}_e(u, \omega, \alpha_n(u, \omega, E_u^n)) \, du, \quad \text{for } t \in [0, T].$$

Since, for $(t, \omega) \in [0, T] \times \Omega$, the function $\hat{\mu}_e(t, \omega, \cdot)$ is strictly decreasing and the sequence $(\alpha_n)_{n \geq 1}$ is decreasing it follows from the comparison theorem for semimartingales (cf. [Protter, 2005, Theorem 54]), that the sequence $(E^n)_{n \geq 1}$ is increasing. Therefore, as a monotone sequence $(E^n)_{n \geq 1}$ converges to an extended $\mathbb{R} \cup \{\infty\}$ -valued limit, say E .

- (i) By the linear growth assumption postulated in (A2.2), the fact that $X \in \mathcal{S}^2$, the boundedness of α and the Lipschitz continuity in the spatial variable of $\hat{\mu}_e$ and α (on $[0, T] \times \Omega \times \mathbb{R}$ in the case of the latter) it follows that the stochastic integral equation (2.19) has a strong solution \tilde{E} (cf. [Kunita, 1990, Theorem 3.4.6]).
- (ii) We now prove that E^n converges to \tilde{E} (as characterised by (2.19)) in the norm of \mathcal{S}^2 . From the same assumptions as used in (i) to argue the existence of \tilde{E} , we may deduce that \tilde{E} cannot explode or oscillate as t tends to T . Therefore, by the continuity of the solution, \tilde{E} can be extended to the closed time interval $[0, T]$. Further, appealing to the comparison theorem for semimartingales once more, we deduce that $E_t^n \leq \tilde{E}_t$, for $t \in [0, T)$ and $n \geq 1$. Letting t tend to T on both sides of the inequality we conclude that the same is true on the closed interval $[0, T]$. We now use Itô's formula and the fact that $2qs \leq q^2 + s^2$, for $q, s \in \mathbb{R}$, to deduce that, for $t \in [0, T]$,

$$\begin{aligned} (E_t^n - \tilde{E}_t)^2 &= 2 \int_0^t (E_u^n - \tilde{E}_u) \left(\hat{\mu}_e(u, \omega, \alpha_n(u, \omega, E_u^n)) - \hat{\mu}_e(u, \omega, \alpha(u, \omega, \tilde{E}_u)) \right) \, du \\ &\leq \int_0^t (E_u^n - \tilde{E}_u)^2 \, du + \int_0^t \left(\hat{\mu}_e(u, \omega, \alpha_n(u, \omega, E_u^n)) - \hat{\mu}_e(u, \omega, \alpha(u, \omega, \tilde{E}_u)) \right)^2 \, du. \end{aligned}$$

Taking expectations and using the fact that $(\alpha_n)_{n \geq 1}$ is a decreasing sequence and the monotonicity properties of $\hat{\mu}_e$, in particular that $\hat{\mu}_e(t, \omega, \cdot)$ is decreasing, we obtain, for $t \in [0, T]$,

$$\mathbb{E}^{\mathbb{Q}} \left[(E_t^n - \tilde{E}_t)^2 \right] \leq \int_0^t \mathbb{E}^{\mathbb{Q}} \left[(E_u^n - \tilde{E}_u)^2 \right] \, du + \epsilon_n,$$

where $\epsilon_n := \int_0^T \mathbb{E}^{\mathbb{Q}} \left[\left(\hat{\mu}_e(u, \omega, \alpha(u, \omega, E_u^n)) - \hat{\mu}_e(u, \omega, \alpha(u, \omega, \tilde{E}_u)) \right)^2 \right] \, du$. An application of Gronwall's inequality yields

$$\mathbb{E}^{\mathbb{Q}} \left[(E_t^n - \tilde{E}_t)^2 \right] \leq \epsilon_n e^t, \quad \text{for } t \in [0, T].$$

Since $(E_n)_{n \geq 1}$ is an increasing sequence and $\alpha(t, \omega, \cdot)$ is increasing and $\hat{\mu}_e(t, \omega, \cdot)$ is strictly decreasing, we may apply the Dominated Convergence theorem to deduce that $\epsilon_n \rightarrow 0$. Therefore,

$$\mathbb{E}^{\mathbb{Q}} \left[(E_t^n - \tilde{E}_t)^2 \right] \rightarrow 0. \quad (2.20)$$

Going through the same arguments again as above, but this time taking the supremum before taking the expectation we find that

$$\mathbb{E}^{\mathbb{Q}} \left[\sup_{t \in [0, T]} (E_t^n - \tilde{E}_t)^2 \right] \leq \int_0^T \mathbb{E}^{\mathbb{Q}} \left[(E_u^n - \tilde{E}_u)^2 \right] du + \epsilon_n.$$

From the Dominated Convergence theorem and from (2.20) it follows that the first term on the right-hand side of the inequality tends to zero and by the same arguments as before also ϵ_n tends to zero. Since the left hand side is non-negative, we obtain

$$\mathbb{E}^{\mathbb{Q}} \left[\sup_{t \in [0, T]} (E_t^n - \tilde{E}_t)^2 \right] \rightarrow 0.$$

It follows that $E^n \rightarrow \tilde{E}$ in \mathcal{S}^2 and hence that $E = \tilde{E}$.

□

The following two lemmas, which we derive in the regularized setting, will be needed to characterize the allowance price at compliance time. In particular, they allow us to observe the behaviour of the random fields α_n , $n \geq 1$, close to the terminal time T .

Lemma 2 (Estimate I of boundary behaviour). *For all $n \geq 1$, the following estimate holds almost surely:*

$$\mathbb{E}^{\mathbb{Q}} \left[\tilde{A}_T | \mathcal{F}_t \right] \leq \alpha_n(t, \omega, e) \leq \mathbb{E}^{\mathbb{Q}} \left[\tilde{A}_T | \mathcal{F}_t \right] + \pi, \quad \text{for } (t, \omega, e) \in [0, T] \times \Omega \times \mathbb{R}.$$

Proof. Using the fact that we can write $\hat{\phi}_n(\tilde{a}, e) = \tilde{a} + \pi J_n(e)$, for $(\tilde{a}, e) \in [0, \bar{\mathcal{A}}] \times \mathbb{R}$, (see Remark 4), from the linearity of the conditional expectation, the non-negativity of π and the $[0, 1]$ -valuedness of each J_n , we deduce that for each $n \geq 1$ and any $(t_0, \omega, e_{t_0}) \in [0, T] \times \Omega \times \mathbb{R}$

$$\alpha_n(t_0, \omega, e_{t_0}) = \mathbb{E}^{\mathbb{Q}} \left[\tilde{A}_T | \mathcal{F}_{t_0} \right] + \mathbb{E}^{\mathbb{Q}} \left[\pi J_n(E_T^{n, t_0, e_{t_0}}) | \mathcal{F}_{t_0} \right] \leq \mathbb{E}^{\mathbb{Q}} \left[\tilde{A}_T | \mathcal{F}_{t_0} \right] + \pi.$$

Since t_0 and e_{t_0} were chosen arbitrarily, the upper bound follows. The proof of the lower bound is now obvious. □

Lemma 3 (Estimate II of boundary behaviour). *For $q \in \{1, 2\}$, there exist integrable random variables $C_2(\omega; q), C_3(\omega; q) > 0$, independent of L_2 , such that, for all $n \geq 1$, any $(t, \omega, e) \in [0, T) \times \Omega \times \mathbb{R}$ and $\hat{e} \in \mathbb{R}$, the following estimates hold almost surely:*

$$\begin{aligned}\alpha_n(t, \omega, e) &\geq \mathbb{E}^{\mathbb{Q}} \left[\hat{\phi}_n(\tilde{A}_T, \hat{e}) | \mathcal{F}_t \right] - \pi C_2(\omega; q) \left(\frac{T-t}{e-\hat{e}} \right)^q, & \text{if } e > \hat{e}, \\ \alpha_n(t, \omega, e) &\leq \mathbb{E}^{\mathbb{Q}} \left[\hat{\phi}_n(\tilde{A}_T, \hat{e}) | \mathcal{F}_t \right] + \pi C_3(\omega; q) \left(\frac{T-t}{\hat{e}-e} \right)^q, & \text{if } e < \hat{e}.\end{aligned}$$

Proof. For notational convenience, we define the operator $\mathbb{E}_{t_0}^{\mathbb{Q}}[\cdot] := \mathbb{E}^{\mathbb{Q}}[\cdot | \mathcal{F}_{t_0}]$. We first prove the lower bound. For $n \geq 1$, let us define the sets $\Omega_{l,-}^n, \Omega_{l,+}^n \subset \Omega$ by

$$\Omega_{l,-}^n := \left\{ \omega \in \Omega : E_T^{n,t_0,e_{t_0}} \in (-\infty, \hat{e}) \right\} \quad \text{and} \quad \Omega_{l,+}^n := \left\{ \omega \in \Omega : E_T^{n,t_0,e_{t_0}} \in [\hat{e}, \infty) \right\}.$$

Using the fact that we can write $\hat{\phi}_n(\tilde{a}, e) = \tilde{a} + \pi J_n(e)$, for $(\tilde{a}, e) \in [0, \bar{\mathcal{A}}] \times \mathbb{R}$, (see Remark 4) and from the linearity of the conditional expectation, the law of total probability, the monotonicity and $[0, 1]$ -valuedness of each J_n and the non-negativity of π , we deduce that for each $n \geq 1$ and $(t_0, \omega, e_{t_0}) \in [0, T] \times \Omega \times \mathbb{R}$

$$\begin{aligned}\alpha_n(t_0, \omega, e_{t_0}) &= \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\hat{\phi}_n(\tilde{A}_T, E_T^{n,t_0,e_{t_0}}) \right] \\ &= \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\tilde{A}_T \right] + \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\pi J_n(E_T^{n,t_0,e_{t_0}}) | \Omega_{l,-}^n \right] \mathbb{E}_{t_0}^{\mathbb{Q}} \left[I_{\Omega_{l,-}^n}(\omega) \right] \\ &\quad + \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\pi J_n(E_T^{n,t_0,e_{t_0}}) | \Omega_{l,+}^n \right] \mathbb{E}_{t_0}^{\mathbb{Q}} \left[I_{\Omega_{l,+}^n}(\omega) \right] \\ &\geq \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\tilde{A}_T \right] + \pi J_n(\hat{e}) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[I_{\Omega_{l,+}^n}(\omega) \right] \\ &\geq \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\tilde{A}_T \right] + \pi J_n(\hat{e}) - \pi \mathbb{E}_{t_0}^{\mathbb{Q}} \left[I_{\Omega_{l,-}^n}(\omega) \right].\end{aligned}$$

We now derive a uniform upper bound for the second conditional expectation in the last line above. Consider for this purpose, for $n \geq 1$, the sets $\Omega_{l,-}^n$:

$$\begin{aligned}\Omega_{l,-}^n &:= \left\{ \omega \in \Omega : E_T^{n,t_0,e_{t_0}} \in (-\infty, \hat{e}) \right\} \\ &= \left\{ \omega \in \Omega : e_{t_0} + \int_{t_0}^T \hat{\mu}_e(t, \omega, A_t^{n,t_0,e_{t_0}}) dt < \hat{e} \right\} \\ &\subseteq \left\{ \omega \in \Omega : \int_{t_0}^T |\hat{\mu}_e(t, \omega, \bar{\phi})| dt > e_{t_0} - \hat{e} \right\} \\ &\subseteq \left\{ \omega \in \Omega : \sup_{t \in [t_0, T]} |\hat{\mu}_e(t, \omega, \bar{\phi})| > \frac{e_{t_0} - \hat{e}}{T - t_0} \right\} \\ &=: \hat{\Omega}_{l,-}^n,\end{aligned} \tag{2.21}$$

where in the last estimated expression we have used the fact that $t_0 \in [0, T]$. Moreover, the supremum is well-defined by (A2.1) and (A2.2). It follows that $I_{\Omega_{i,-}^n}(\omega) \leq I_{\hat{\Omega}_{i,-}^n}(\omega)$, for all $\omega \in \Omega$ and $n \geq 1$. This allows us to bound the conditional expectation $\mathbb{E}^{\mathbb{Q}}[I_{\Omega_{i,-}^n}(\omega)|\mathcal{F}_{t_0}]$ above. We note that since X was assumed to be in \mathcal{S}^2 , it follows from the definition of $\hat{\mu}_e$ in (2.11) and the linear growth assumption in (A2.2) that the supremum in (2.21) is guaranteed to have finite moments of first and second order. Then, arguing as in the proof of Chebyshev's inequality, we deduce that for $e_{t_0} > \hat{e}$ and $q \in \{1, 2\}$ we have the almost sure bound

$$\mathbb{E}^{\mathbb{Q}}[I_{\Omega_{i,-}^n}(\omega)|\mathcal{F}_{t_0}] \leq \mathbb{E}^{\mathbb{Q}}[I_{\hat{\Omega}_{i,-}^n}(\omega)|\mathcal{F}_{t_0}] \leq C_2(\omega; q) \left(\frac{T - t_0}{e_{t_0} - \hat{e}} \right)^q.$$

Since t_0 and e_{t_0} were chosen arbitrarily, the lower bound in the proposition follows immediately.

For the upper bound we argue analogously. For $n \geq 1$ and $\hat{e} \in \mathbb{R}$, we define the sets $\Omega_{u,-}^n, \Omega_{u,+}^n \subset \Omega$ by

$$\Omega_{u,-}^n := \left\{ \omega \in \Omega : E_T^{n,t_0,e_{t_0}} \in (-\infty, \hat{e}) \right\} \quad \text{and} \quad \Omega_{u,+}^n := \left\{ \omega \in \Omega : E_T^{n,t_0,e_{t_0}} \in (\hat{e}, \infty) \right\}.$$

Then, in the same way as before,

$$\begin{aligned} \alpha_n(t_0, \omega, e_{t_0}) &= \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\hat{\phi}_n(\tilde{A}_T, E_T^{n,t_0,e_{t_0}}) \right] \\ &\leq \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\tilde{A}_T \right] + \pi J_n(\hat{e}) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[I_{\Omega_{u,-}^n}(\omega) \right] \\ &\quad + \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\pi J_n(E_T^{n,t_0,e_{t_0}}) \right] \mathbb{E}_{t_0}^{\mathbb{Q}} \left[I_{\Omega_{u,+}^n}(\omega) \right] \\ &= \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\tilde{A}_T \right] + \pi J_n(\hat{e}) \\ &\quad + \pi \left(\mathbb{E}_{t_0}^{\mathbb{Q}} \left[J_n(E_T^{n,t_0,e_{t_0}}) \right] - J_n(\hat{e}) \right) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[I_{\Omega_{u,+}^n}(\omega) \right] \\ &\leq \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\tilde{A}_T \right] + \pi J_n(\hat{e}) + \pi \mathbb{E}_{t_0}^{\mathbb{Q}} \left[I_{\Omega_{u,+}^n}(\omega) \right]. \end{aligned}$$

Again we derive a uniform upper bound for the second conditional expectation in the last line above. Consider, for $n \geq 1$, the sets $\Omega_{u,+}^n$, then

$$\begin{aligned} \Omega_{u,+}^n &:= \left\{ \omega \in \Omega : E_T^{n,t_0,e_{t_0}} \in (\hat{e}, \infty) \right\} \\ &\subseteq \left\{ \omega \in \Omega : \sup_{t \in [t_0, T]} |\hat{\mu}_e(t, \omega, 0)| > \frac{\hat{e} - e_{t_0}}{T - t_0} \right\} \\ &=: \hat{\Omega}_{u,+}^n, \end{aligned}$$

where again we assumed that $t_0 \in [0, T)$. Since now $I_{\Omega_{u,+}^n}(\omega) \leq I_{\hat{\Omega}_{u,+}^n}(\omega)$, for all $\omega \in \Omega$ and $n \geq 1$, by the same arguments as before we derive the almost sure bound

$$\mathbb{E}^{\mathbb{Q}}[I_{\Omega_{u,+}^n}(\omega)|\mathcal{F}_{t_0}] \leq \mathbb{E}^{\mathbb{Q}}[I_{\hat{\Omega}_{u,+}^n}(\omega)|\mathcal{F}_{t_0}] \leq C_3(\omega; q) \left(\frac{T - t_0}{\hat{e} - e_{t_0}} \right)^q.$$

Since t_0 and e_{t_0} were chosen arbitrarily, the upper bound in the proposition follows immediately. \square

It remains to prove the convergence of the solution to the backward equation in (2.14).

Proposition 4 (Convergence of allowance price process). *Setting $A_t := \alpha(t, \omega, E_t)$, for $(t, \omega) \in [0, T) \times \Omega$, it holds that*

$$A^n \rightarrow A, \quad \text{pointwise.}$$

Moreover,

- (i) *the solution can be extended to the closed time interval, that is $A_T := \lim_{t \rightarrow T} A_t$ exists. The extended process A is a $[0, \bar{A}]$ -valued martingale in \mathcal{S}^2 ; hence it has a representation of the form*

$$A_t = A_0 + \int_0^t \langle Z_u^a, dW_u \rangle, \quad \text{for } t \in [0, T]$$

and for some $Z^a \in \mathcal{H}^2$.

- (ii) *At compliance time T it holds that*

$$\tilde{A}_T + \pi I_{(\tilde{\gamma}, \infty)}(E_T) \leq A_T \leq \tilde{A}_T + \pi I_{[\tilde{\gamma}, \infty)}(E_T).$$

Proof. By Proposition 2 (iv) and by the decrease of the sequence $(\alpha_n)_{n \geq 1}$ and the increase of the sequence $(E^n)_{n \geq 1}$, we may write, for $(t, \omega) \in [0, T) \times \Omega$,

$$\begin{aligned} \alpha(t, \omega, E_t) &= \lim_{n \rightarrow \infty} \alpha(t, \omega, E_t^n) = \liminf_{n \rightarrow \infty} \alpha(t, \omega, E_t^n) \\ &\leq \liminf_{n \rightarrow \infty} \alpha_n(t, \omega, E_t^n) \leq \limsup_{n \rightarrow \infty} \alpha_n(t, \omega, E_t^n) \\ &\leq \limsup_{n \rightarrow \infty} \alpha_n(t, \omega, E_t) = \lim_{n \rightarrow \infty} \alpha_n(t, \omega, E_t) = \alpha(t, \omega, E_t). \end{aligned}$$

Since from Proposition 1 we know that $A_t^n = \alpha_n(t, \omega, E_t^n)$ the above says that, for $(t, \omega) \in [0, T) \times \Omega$,

$$A_t \leq \lim_{n \rightarrow \infty} A_t^n \leq A_t,$$

which implies the pointwise convergence.

- (i) Since each $(A_t^n)_{t \in [0, T]}$ is a $[0, \bar{\mathcal{A}}]$ -valued martingale, by the Dominated Convergence Theorem, also the limit $(A_t)_{t \in [0, T]}$ is a $[0, \bar{\mathcal{A}}]$ -valued martingale. Moreover, by Doob's Martingale Convergence Theorem the solution A may be extended to the closed time interval $[0, T]$ and the resulting process $(A_t)_{t \in [0, T]}$ is again a martingale. As a bounded martingale A is an element of \mathcal{S}^2 and from the Martingale Representation theorem we know that there exists $Z^a \in \mathcal{H}^2$, such that

$$A_t = A_0 + \int_0^t \langle Z_u^a, dW_u \rangle, \quad \text{for } t \in [0, T]. \quad (2.22)$$

- (ii) We consider the estimates in Lemma 3. Taking the limit on the left and the right-hand side we notice that both estimates are also satisfied by α and ϕ . Now consider the sets

$$\Omega_{e,+} := \{\omega \in \Omega : E_T = \lim_{t \rightarrow T} E_t > \tilde{\gamma}\} \quad \text{and} \quad \Omega_{e,-} := \{\omega \in \Omega : E_T = \lim_{t \rightarrow T} E_t < \tilde{\gamma}\}.$$

Then we may deduce that $E_t > \tilde{\gamma}$, for t close to T and $\omega \in \Omega_{e,+}$. From the first estimate in Lemma 3 (with $\hat{e} := \tilde{\gamma}$) it follows that

$$\lim_{t \rightarrow T} \alpha(t, \omega, E_t) \geq \tilde{A}_T + \pi, \quad \text{for } \omega \in \Omega_{e,+}.$$

Similarly, defining $\delta(\omega) := 1/2(\tilde{\gamma} - E_T)$, we find that $\delta > 0$ on $\Omega_{e,-}$ and we deduce that $E_t < \tilde{\gamma} - \delta$, for t close to T and $\omega \in \Omega_{e,-}$. From the second estimate in Lemma 3 (with $\hat{e} := \tilde{\gamma} - \delta$) it follows that

$$\lim_{t \rightarrow T} \alpha(t, \omega, E_t) \leq \tilde{A}_T, \quad \text{for } \omega \in \Omega_{e,-}.$$

Further, Lemma 2 tells us that $\tilde{A}_T \leq \lim_{t \rightarrow T} \alpha(t, \omega, E_t) \leq \tilde{A}_T + \pi$ almost surely. Proposition 4 (ii) now follows. □

We finally obtain the existence of a solution to the FBSDE (2.9) in the sense of Theorem 1.

Proof of Theorem 1 (Existence). Assumption (A2.1) guarantees the existence of a (unique) forward equation $X \in \mathcal{S}^2$ satisfying the unsmoothed FBSDE (2.9). The existence of $(E, A, Z^a) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$ follows from Propositions 3 and 4 respectively. □

2.3.3 Uniqueness

This section addresses the question whether the solution to the FBSDE (2.9) is in fact unique. Thereby we complete the proof of Theorem 1.

The following Lemma concerning the martingale property of a class of stochastic integrals will prove useful below.

Lemma 4. *Let $\hat{E} \in \mathcal{S}^2$ and $\hat{Z} \in \mathcal{H}^2$ and let B denote a scalar valued, standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, then the stochastic integral*

$$\int_0^t \hat{E}_u \hat{Z}_u dB_u, \quad \text{for } t \in [0, T], \quad (2.23)$$

is a true martingale.

Our strategy is to show that the absolute value of (2.23) is dominated by an integrable, non-negative random variable and hence uniformly integrable. Then standard results from stochastic analysis allow us to conclude that (2.23) is in fact a martingale.

Proof. We define a non-negative random variable $C_4(\omega)$ to be given by

$$C_4(\omega) := \sup_{t \in [0, T]} \left| \int_0^t \hat{E}_u \hat{Z}_u dB_u \right|.$$

Clearly, $C_4(\omega)$ dominates the absolute value of (2.23). It remains to show that it is integrable. From an application of first, the Burkholder-Davis-Gundy inequality and second, the Hölder inequality, we deduce that for some positive constant $C_5 > 0$,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[\sup_{t \in [0, T]} \left| \int_0^t \hat{E}_u \hat{Z}_u dB_u \right| \right] &\leq C_5 \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |\hat{E}_u \hat{Z}_u|^2 du \right)^{\frac{1}{2}} \right] \\ &\leq C_5 \mathbb{E}^{\mathbb{Q}} \left[\sup_{u \in [0, T]} |\hat{E}_u| \left(\int_0^T |\hat{Z}_u|^2 du \right)^{\frac{1}{2}} \right] \\ &\leq C_5 \left(\mathbb{E}^{\mathbb{Q}} \left[\sup_{u \in [0, T]} |\hat{E}_u|^2 \right] \right)^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{Q}} \left[\int_0^T |\hat{Z}_u|^2 du \right] \right)^{\frac{1}{2}} \\ &< \infty, \end{aligned}$$

since $\hat{E} \in \mathcal{S}^2$ and $\hat{Z} \in \mathcal{H}^2$. Hence $C_4(\omega)$ is indeed integrable.

The absolute value of the stochastic integral (2.23) is now bounded above (uniformly in time) by the non-negative, integrable random variable $C_4(\omega)$. Therefore, (2.23) is uniformly integrable and of class D (cf. [Revuz and Yor, 2001, Chapter 4, Definition 1.6]). Hence it is a martingale. \square

We now turn to the proof of uniqueness of a solution.

Proposition 5 (Uniqueness). *With Assumption 2 in place and given that \tilde{A}_T is a bounded and non-negative, \mathcal{F}_T -measurable random variable, there exists at most one solution $(E, A, Z^a) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$ satisfying the FBSDE (2.13) and the “relaxed” terminal condition*

$$\tilde{A}_T + \pi I_{[\tilde{\gamma}, \infty)}(E_T) \leq A_T \leq \tilde{A}_T + \pi I_{[\tilde{\gamma}, \infty)}(E_T).$$

Proof. Suppose (E, A, Z^a) and $(E', A', Z^{a'})$ are two solutions to (2.13) and define, for $t \in [0, T]$,

$$\delta E_t := E_t - E'_t, \quad \delta A_t := A_t - A'_t, \quad \delta Z_t^a := Z_t^a - Z_t^{a'}.$$

By an application of Itô’s product rule to $\delta E_t \delta A_t$ and noting that $\delta E_0 = 0$, we find

$$\delta E_T \delta A_T = \int_0^T \delta A_t (\hat{\mu}_e(t, \omega, A_t) - \hat{\mu}_e(t, \omega, A'_t)) dt + \int_0^T \delta E_t \langle \delta Z_t^a, dW_t \rangle.$$

Taking the expectation on both sides and making use of Lemma 4, we obtain

$$\mathbb{E}^{\mathbb{Q}} [\delta E_T \delta A_T] = \mathbb{E}^{\mathbb{Q}} \left[\int_0^T \delta A_t (\hat{\mu}_e(t, \omega, A_t) - \hat{\mu}_e(t, \omega, A'_t)) dt \right]. \quad (2.24)$$

Let us introduce the following partition of the sample space: $\Omega = \Omega_{\tilde{\gamma}} \cup \Omega_{\tilde{\gamma}} \cup \Omega_{\tilde{\gamma}, \tilde{\gamma}+} \cup \Omega_{\tilde{\gamma}, \tilde{\gamma}-} \cup \Omega_{\tilde{\gamma}+, \tilde{\gamma}} \cup \Omega_{\tilde{\gamma}-, \tilde{\gamma}}$, where

$$\begin{aligned} \Omega_{\tilde{\gamma}} &:= \{\omega \in \Omega : E_T \neq \tilde{\gamma} \text{ and } E'_T \neq \tilde{\gamma}\}, \\ \Omega_{\tilde{\gamma}} &:= \{\omega \in \Omega : E_T = \tilde{\gamma} \text{ and } E'_T = \tilde{\gamma}\}, \\ \Omega_{\tilde{\gamma}, \tilde{\gamma}+} &:= \{\omega \in \Omega : E_T = \tilde{\gamma} \text{ and } E'_T > \tilde{\gamma}\}, \\ \Omega_{\tilde{\gamma}-, \tilde{\gamma}} &:= \{\omega \in \Omega : E_T < \tilde{\gamma} \text{ and } E'_T = \tilde{\gamma}\} \end{aligned}$$

and $\Omega_{\tilde{\gamma}, \tilde{\gamma}-}$ and $\Omega_{\tilde{\gamma}+, \tilde{\gamma}}$ are defined analogously.

With the above partition we find that

$$\delta E_T \delta A_T = (E_T - E'_T)(\phi(\tilde{A}_T, E_T) - \phi(\tilde{A}_T, E'_T)), \quad \text{for } \omega \in \Omega_{\tilde{\gamma}},$$

which, since $\phi(\tilde{A}_T, \cdot)$ is an increasing function, is always non-negative. Clearly, $\delta E_T \delta A_T = 0$, for $\omega \in \Omega_{\tilde{\gamma}}$. Further, for $\omega \in \Omega_{\tilde{\gamma}, \tilde{\gamma}+}$,

$$(\tilde{\gamma} - E'_T)(\tilde{A}_T + \pi - \phi(\tilde{A}_T, E'_T)) \leq \delta E_T \delta A_T \leq (\tilde{\gamma} - E'_T)(\tilde{A}_T - \phi(\tilde{A}_T, E'_T)).$$

Since for $\omega \in \Omega_{\tilde{\gamma}, \tilde{\gamma}+}$, we have that $\phi(\tilde{A}_T, E'_T) = \tilde{A}_T + \pi$ it follows that also in this case $\delta E_T \delta A_T$ is non-negative. Similarly, one shows that the same is true for $\omega \in \Omega_{\tilde{\gamma}, \tilde{\gamma}-}$.

By symmetry it is also true for $\omega \in \Omega_{\tilde{\gamma}+, \tilde{\gamma}} \cup \Omega_{\tilde{\gamma}-, \tilde{\gamma}}$, so that we may conclude that the left hand side in (2.24) is non-negative. Further, by (A2.2), the function $\hat{\mu}_e(t, \omega, \cdot)$ is strictly decreasing, for $(t, \omega) \in [0, T] \times \Omega$; hence the integrand in (2.24) is always non-positive. This implies that $\delta A = 0 dt \otimes d\mathbb{Q}$ almost everywhere and by the continuity of the processes that $A = A'$. From the definition of E in (2.12) and that of A in (2.22), we immediately deduce that also $E = E'$ and $Z^a = Z^{a'}$. \square

Proof of Theorem 1 (Uniqueness). The uniqueness of a solution to the FBSDE (2.9) now follows from Proposition 5 and the fact that by Assumption (A2.1) the solution X to the forward equation (2.3) is unique. \square

2.4 Valuation of Derivatives

Earlier we already commented that in our setting the allowance certificate is a derivative on the fundamental factors X in the market. The reader may wonder why we then devote an additional section to the valuation of derivatives. The allowance certificate has one very special property: it is the sole derivative in our framework, which, through its coupling, via the drift μ_e of the cumulative emissions process, affects the evolution of the process it is written on. This justifies a separate analysis of “other” derivatives. Throughout the thesis we will occasionally use the term *generic derivative* to refer to a derivative other than the allowance certificate, when the meaning is not clear from the context.

We assume the existence of a continuous, \mathcal{F} -adapted process V , to be specified rigorously below, which represents the value of a generic derivative in our market. For a measurable function $\psi : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ we assume that the payoff of the derivative is given at time $\tau \in [0, T]$ by

$$V_\tau := \psi(X_\tau, E_\tau, A_\tau).$$

2.4.1 Derivation of the Pricing Equation

Analogously to the valuation of the allowance certificate, the price of the generic derivative at any time before expiry is given by the continuous version of the uniformly integrable, non-negative martingale $\mathbb{E}^\mathbb{Q}[\psi(X_\tau, E_\tau, A_\tau)|\mathcal{F}_t]$, that is

$$V_t := \mathbb{E}^\mathbb{Q}[\psi(X_\tau, E_\tau, A_\tau)|\mathcal{F}_t], \quad \text{for } t \in [0, \tau].$$

The Martingale Representation Theorem again implies the existence of a progressively measurable process $Z^v := (Z^{v,1}, \dots, Z^{v,d})$, satisfying

$$\mathbb{E}^{\mathbb{Q}} \left[\int_0^{\tau} \|Z_t^v\|^2 dt \right] < \infty$$

and such that

$$V_t = \mathbb{E}^{\mathbb{Q}} [\psi(X_{\tau}, E_{\tau}, A_{\tau})] + \int_0^t \langle Z_u^v, dW_u \rangle, \quad \text{for } t \in [0, \tau]. \quad (2.25)$$

From (2.9), set on the time interval $[0, \tau]$ with $\alpha(\tau, \omega, E_{\tau})$ as terminal condition, it now follows that the derivative pricing problem takes the form of the following FBSDE: for $t \in [0, \tau]$,

$$\begin{cases} X_t = x_0 + \int_0^t \mu(u, X_u) du + \int_0^t \sigma(X_u) dW_u \\ E_t = e_0 + \int_0^t \mu_e(X_u, A_u) du \\ A_t = \alpha(\tau, \omega, E_{\tau}) - \int_t^{\tau} \langle Z_u^a, dW_u \rangle \\ V_t = \psi(X_{\tau}, E_{\tau}, A_{\tau}) - \int_t^{\tau} \langle Z_u^v, dW_u \rangle. \end{cases} \quad (2.26)$$

We note that the backward equation for V is decoupled in the sense that none of the coefficients of the forward equations (X, E) are functions of V . This is a reflection of the fact that the price of no derivative other than the allowance certificate itself has an impact on the actions of firms in the market and in particular on the rate at which they emit.

2.4.2 Existence and Uniqueness of a Solution to the Derivatives Pricing Problem

For generic derivatives this section provides the analogue result to Theorem 1, namely the proof of existence and uniqueness of a solution to the equation describing the price of a generic derivative, that is to the FBSDE (2.26). Because we may draw upon our previous results from §2.3 and because of the decoupled nature of the backward equation for V , the structure of (2.26) is considerably simpler compared to the allowance price FBSDE (2.9) and the arguments in this section will be straightforward and brief.

Assumption 4. The payoff function ψ is said to satisfy Assumption 4 if the following linear growth property holds:

$$|\psi(x, e, a)| \leq L_3(1 + |x| + |e| + |a|), \quad \text{for } (x, e, a) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}.$$

The main result of this section is the following theorem.

Theorem 2. *With Assumptions 2 and 4 in place, there exist unique processes $((X, E), A, V, (Z^a, Z^v)) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2$ satisfying the FBSDE (2.26).*

Remark 6. We point out, as we did for Theorem 1, that the above theorem still remains valid in the case of a non-zero interest rate.

The proof of Theorem 2 becomes straightforward once we notice that due to the decoupled nature of the equation for V , the FBSDE (2.26) can simply be regarded as the conditional expectation of the random variable $\psi(X_\tau, E_\tau, A_\tau)$.

Proof of Theorem 2. Theorem 1 guarantees that both forward stochastic integral equations and the backward equation for A in (2.26) have unique strong solutions in \mathcal{S}^2 . Further, Theorem 1 also implies that the random variables X_τ , E_τ and A_τ are elements of L^2 . By the at most linear growth of ψ , guaranteed by Assumption 4, also $\psi(X_\tau, E_\tau, A_\tau) \in L^2$. Hence the conditional expectation $\mathbb{E}^\mathbb{Q}[\psi(X_\tau, E_\tau, A_\tau)|\mathcal{F}_t]$ is well-defined and unique almost surely. Moreover, the Martingale Representation Theorem implies the existence of a unique integrand $Z^v \in \mathcal{H}^2$ in the martingale representation of $\mathbb{E}^\mathbb{Q}[\psi(X_\tau, E_\tau, A_\tau)|\mathcal{F}_t]$, which proves the existence of a unique backward component in (2.26). \square

2.5 Conclusion

This chapter introduced, in a general setting, our model of the economy and in particular of the emission market. From our assumptions about the economy we derived, both, in the case of an emission market with a single compliance period and in that of a market with multiple compliance periods, the equation describing the arbitrage-free price of allowance certificates and showed that, in fact, the certificate is a contract written on cumulative emissions and thereby a derivative on the fundamental factors in the market. Moreover, we proved that this pricing problem is well-defined, that is, there exists a unique solution to the (coupled) forward-backward stochastic differential equation representing it. Thereby, we extended current results in the literature concerning the existence and uniqueness of a solution to FBSDEs, which do

not satisfy the ubiquitous conditions of uniform ellipticity and of Lipschitz continuity of the coefficients. As a by-product of this proof we revealed an interesting consequence of the market design of existing emissions trading systems, namely that, in the considered modelling setup, in the event that the cumulative emissions reach the cap precisely, the allowance price does not match the price prescribed by the regulator, as captured by our need to relax the terminal condition of the allowance price FBSDE. Further, we gained a thorough understanding of the relationship between allowances and important variables and parameters in the market. In particular, we proved that the allowance certificate is increasing in the cumulative emissions variable and we proved its precise monotonic dependency on the the cap on emissions and the monetary penalty, that is applied to emissions exceeding the cap. Similarly, also in the framework of arbitrage-free pricing, we derived a (decoupled) forward-backward stochastic differential equation describing the price of a generic derivative in the market and also in this case proved the existence and uniqueness of a solution to this equation.

Equipped with these foundational results we now begin to discuss concrete models of emission and electricity markets.

Chapter 3

A Model with Stochastic Demand

In this chapter we introduce a simple joint model for electricity and emissions trading. In the structural spirit we explain electricity prices as a function of demand for electricity and the cost of the emission. The prices of allowance certificates are determined by demand for electricity and the level of the cumulative emissions to date. We analyse in detail the differences between emission markets with a single compliance period and those with multiple compliance periods. Throughout our analysis of emission markets we focus on the trading of AAUs only. The joining of multiple markets using CERs and ERUs in the present setting is left to future research and addressed from a different point of view for example in [Carmona and Fehr, 2011]. The material in this chapter gave rise to and can be found in the paper [Howison and Schwarz, 2012].

The emphasis of this chapter is on the introduction of a new model. Therefore, at times we may be less rigorous in our presentation than we were in Chapter 2. For example, in §3.1.1 we do not mention explicitly that the stochastic processes we introduce are adapted to the filtration \mathcal{F} . However, this should still be clear to the reader as each process in this chapter corresponds to a process in Chapter 2 and this correspondence will always be made clear.

In §3.1 we introduce our model of the bid stack and therefore of the electricity spot price. The bid stack was first introduced in [Barlow, 2002] under the assumption that the bids of firms that produce electricity are constant. Subsequently, the idea was developed further, to allow bids to be driven by stochastic factors, notably by stochastic fuel prices (cf. [Hinz, 2003, Aïd et al., 2009, Coulon and Howison, 2009]). For a detailed overview of different approaches to modelling the electricity bid stack, we refer the reader to [Carmona and Coulon, 2012]. In the absence of the implementation of a cap-and-trade scheme, under business-as-usual, our setting is closest to [Barlow, 2002]. However, an important feature of our model, distinguishing it from

all of the before mentioned models of the bid stack is to include, as part of the production costs, the costs firms face due to the existence of an emission regulation and to allow these costs to be stochastic and endogenous.

Our model of the bid stack permits us to deduce which generators are active at any point in time. With this information the demand process is translated, via the bid stack, into the rate at which the market emits, which in turn is translated into an emissions process. As noted above, the bid stack both influences and is influenced by the allowance price. This feature allows us to capture the response of the market to a stricter emission regulation, as it very clearly explains the impact of higher allowance prices on the market emission rate.

§3.2 is dedicated to the formulation of the allowance pricing problem based on the model introduced in §2.2, but in the current market setting. This is first done for an emission market with a single compliance period and subsequently for multi-period markets. In the latter case we explain in detail how our model takes different inter-temporal connecting mechanisms, such as borrowing, banking and withdrawal, into account and the impact they have on the allowance price.

The first academic treatment of emission markets can be traced back to [Dales, 1968, Montgomery, 1972]. Early models of allowance trading in discrete and in continuous time were proposed in [Cronshaw and Kruse, 1996, Rubin, 1996, Schennach, 2000, Leiby and Rubin, 2001, Maeda, 2004, Tietenberg, 2006]. More recently, emission markets have been treated from two different angles. On the one hand are full equilibrium models that derive the price processes of allowances and goods (the production of which causes pollution) from the preferences of individual firms and additional sources of uncertainty. These have proved insightful but rather cumbersome in their complexity (cf. [Carmona et al., 2009a, Carmona et al., 2009b]). On the other hand are reduced-form models, which specify the allowance price evolution directly in the form of a process that has the correct terminal properties and the correct bounds; then the parameters in the model are calibrated to market data. In these models, the event of non-compliance is described exogenously and no causal explanation is given for the accumulation of emissions in the economy. Within this class of models one can distinguish between models that ignore the feedback from the allowance price to the rate at which firms emit (cf. [Chesney and Taschini, 2008, Grüll and Kiesel, 2009, Carmona and Hinz, 2011]) and those that take this feedback effect into account through an exogenously specified abatement function (cf. [Borovkov et al., 2010, Carmona et al., 2012c, Hinz and Novikov, 2010]). The novelty of our approach is to include the aforementioned feedback from the cost of

the emission to the market emission rate in a realistic setting, based on the concept of the electricity bid stack. Therefore, we obtain a joint model for electricity and allowance prices taking into account the complex relationships between firms in the market and the spot prices of electricity and allowances.

Over the past few years a liquid market for futures and options on emission permits has developed. This leads us to turn our attention to the pricing of such derivatives (at the example of a European call option) in §3.3. The interrelation between spot and derivative contracts in the EU ETS has been analyzed for the first time in [Uhrig-Homburg and Wagner, 2008]. In a reduced-form framework the risk-neutral valuation of options on the allowance certificate has been discussed in [Carmona and Hinz, 2011] and [Chesney and Taschini, 2008]. Neither of these papers includes the important feedback from the cost of carbon to the rate at which the market emits. This feedback was for the first time included in [Carmona et al., 2012c] in the context of a highly stylized and oversimplified modelling framework. Based on the model introduced in §2.4 we illustrate the pricing of derivative contracts in our bid stack based structural setting with the example of a European call option written on the allowance certificate.

A dedicated numerical case study in §3.4 highlights the features of our model. Exploiting a connection between FBSDEs and partial differential equations (PDEs) we solve the equations representing the prices of allowances and derivatives numerically using a finite difference scheme for the corresponding PDEs.

3.1 From Electricity Markets to Carbon Emissions

In this section we develop our approach to modelling the interaction between electricity and emission markets. We introduce the state variables and the key parameters that are later shown to drive the price formation of allowance certificates and the electricity spot price. An important part is played by the merit order — a rule by which available resources with the lowest marginal costs of production are called upon first to supply electricity. We introduce the electricity bid stack, which is modelled as a continuous map from the supply of electricity to its marginal price and analogously define the emission stack as a continuous map to the marginal emissions caused by the production of the last unit. Using an equilibrium assumption, we relate supply to demand; we show how this allows us to deduce which technologies are used to meet demand at any point in time and the total market emission rate this production schedule implies. Lastly, we illustrate the impact the introduction of a cost of carbon

has on the bid and emission stacks. In the context of an emissions trading scheme the merit order assumption very naturally leads to load shifting, the reallocation of energy production from emission-intensive to pollution friendly resources.

3.1.1 Market Set-up

Initially, the finite time interval $[0, T]$ corresponds to one compliance period (assumed to be the length of one year); later we will consider multi-period markets.

Agents in our market demand a good, the production of which causes emissions; as discussed in the introduction, we take this good to be electricity. Firms can produce electricity using different technologies that vary in their costs of production and their emissions intensity. We assume the market is subject to an emissions trading scheme, as follows. Each registered firm receives an initial allocation of allowances, which can be used to offset its cumulative emissions at the end of the compliance period. If a firm is unable to submit a sufficient number of certificates, its excess emissions are subject to the payment of a monetary penalty.

Analogous to the idea of a representative agent, we ignore the aggregation problem and instead take the point of view of the whole market. Our goal then becomes to determine the arbitrage-free price of emission permits as a function of the aggregate forces that act in the market. As was shown in §2.1.2, this price directly and crucially depends on the cumulative emissions during the compliance period and the factors that explain the accumulation.

The actions of consumers in the market result in *demand for electricity*. We assume that this demand is represented by an exogenously given stochastic process D (measured in MW). Firms respond by generating electricity. In particular, at any time throughout the time interval $[0, T]$, the amount of *electricity supplied* by the aggregate of all firms is represented by a stochastic process Ξ . We assume that the production level of the market is always non-negative and below a constant maximum production capacity $\bar{\xi} \geq 0$. Therefore, Ξ is bounded:

$$0 \leq \Xi_t \leq \bar{\xi}, \quad \text{for } t \in [0, T].$$

Moreover, we assume that there are always sufficient resources in the market to meet demand, so that also D is bounded:

$$0 \leq D_t \leq \bar{\xi}, \quad \text{for } t \in [0, T].$$

The demand process is assumed to be perfectly inelastic, as is frequently justifiable in electricity markets (cf. [Carmona et al., 2009a, Coulon and Howison, 2009]),

and demand and supply are related by a Walrasian equilibrium assumption (cf. [Walras and Jaffé, 1952]). This concept is realised by the *market administrator*, who ensures that aggregate demand for and aggregate supply of energy are matched on a daily basis, namely that

$$D_t = \Xi_t, \quad \text{for } t \in [0, T]. \quad (3.1)$$

The production of electricity causes CO₂ emissions, in a way that we describe more precisely in §3.1.2 and §3.1.3. In agreement with Chapter 2, cumulative emissions are described by a stochastic process denoted E and this quantity is measured in tonnes of CO₂. Moreover, since emission intensive production resources are finite, E is bounded as will be shown below; i.e. there exists a constant $\bar{e} \geq 0$, such that

$$0 \leq E_t \leq \bar{e}, \quad \text{for } t \in [0, T].$$

Also in accordance with Chapter 2, the regulator decides on an acceptable maximum level of cumulative emissions during the compliance period (the cap) and issues a corresponding amount of allowance certificates, $\gamma \geq 0$, measured in tonnes of CO₂. At the end of the compliance period, cumulative emissions in the market are offset against the initial allocation of allowances. Certificates that are not used for this purpose expire worthless in the case of the single-period setup, whereas unaccounted emissions are subject to monetary penalty payment at a rate $\pi \geq 0$ per tonne. Thus, an amount $(E_T - \gamma)^+$ of emissions is penalised.

To clarify how the setup introduced above is viewed in light of the notation introduced in Chapter 2 we point out that throughout this chapter we assume that $d = 1$, so that the process X representing the fundamental variables in the market is scalar valued. Moreover, we identify X with the demand process D , i.e. $X = D$. Further, as announced, we introduce a non-negative interest rate by assuming the existence of a riskless money market account with constant risk-free rate $r \geq 0$.

3.1.2 The Bid and Emission Stack

We turn to the modelling of the bid stack as a common price setting mechanism in electricity markets and demonstrate its direct connection to the accumulation of emissions via the emission stack. These concepts are introduced in the business-as-usual setting, i.e. under the assumption that no cap-and-trade system is operating in the market. The model we introduce is a much simplified and stylized representation of reality. This is done with a very clear purpose in mind, namely to be able to

articulate the effects of an implementation of a cap-and-trade scheme as clearly as possible in the next section in 3.1.3.

Electricity producers submit day-ahead bids to the central market administrator (as introduced above), whose task it is to allocate the production of electricity amongst them. Key to our analysis is the following assumption, which summarises the actions of the central market administrator.

Assumption 5. The market administrator ensures that resources are used according to the **merit order**. This means that the cheapest production technologies are called upon first to satisfy a given demand and hence electricity is supplied at the lowest possible price.

Recall that a bid represents the amount of electricity a single generator is willing to supply at a specific price. Therefore, firms' bids reflect the costliness of production technologies. In practice bid levels are mostly determined by fixed and variable costs. Examples of these costs are rent and salaries, in the case of fixed costs, and, in the case of variable costs, the costs of production factors as determined for example by the prices of fuels and the price of emissions if the plant is situated in a region covered by a cap-and-trade scheme. Obviously any of these costs play an integral part in determining the merit order arrangement in Assumption 5. In particular, if the cost factors are modelled stochastically this is reflected in the stochastic nature of the merit order arrangement (cf. Chapter 5). We make the following simplifying assumption (cf. [Barlow, 2002]).

Assumption 6. Under business-as-usual firms' bids and hence the merit order are constant.

Although the assumption may seem drastic, we point out that for now, we are only interested in the relative position of the different technologies in the bid stack. Therefore, fluctuations in variable costs only become important if they induce merit order changes. From historic data observations this is only the case in the long-run. Moreover this assumption will allow us to focus exclusively on the impact of emissions trading on variable costs and hence on the merit order in §3.1.3.

The increasing map from market supply of electricity to marginal price, which is a direct result of Assumption 5, forms the *bid stack*. As explained in the introduction, the bid stack is, strictly speaking, an increasing simple function. In practice, however, it consists of sufficiently many steps to be approximated by a continuous function. This leads us to the following definition.

Definition 1. The **business-as-usual bid stack** is given by the bounded function

$$b^{\text{BAU}}(\xi) : [0, \bar{\xi}] \mapsto \mathbb{R},$$

where $b^{\text{BAU}} \in C^1(0, \bar{\xi})$ and $db^{\text{BAU}}/d\xi > 0$.

Here and throughout the rest of the paper, the variable $\xi \in [0, \bar{\xi}]$ represents the supply of electricity (measured in MW). Correspondingly, $b^{\text{BAU}}(\cdot)$ denotes the bid level of the marginal production unit (measured in € per MWh).

Remark 7. As pointed out in the introduction, emission-intensive technologies tend to be cheaper than environmentally friendly ones as a means to produce electricity. Therefore, we find that bids associated with a small level of electricity supply stem mostly from emission-intensive generators, while bids at the right end of the interval $[0, \bar{\xi}]$ stem mostly from environmentally friendly ones (as remarked earlier an exception to this rule are nuclear plants, which do not cause any CO₂ emissions and are generally placed at the very low-price end of the bid stack). In between exists a spectrum in which a mixture of technologies contribute to bids. This assumption has been confirmed (cf. [Coulon and Howison, 2009]) by analysing the correlation between production costs and bid levels.

Analogous to the bid stack we construct an emission stack, by constructing a map from the supply of electricity to the marginal emissions associated with the supply of the last unit.

Definition 2. The **marginal emission stack** is given by the bounded function

$$e(\xi) : [0, \bar{\xi}] \mapsto \mathbb{R}_{++},$$

where $e \in C^1(0, \bar{\xi})$.

With the above definition, $e(\cdot)$ associates with a specific supply of electricity the emission rate of the marginal unit (measured in tonnes of CO₂ per MWh).

Proposition 6. *The business-as-usual market emission rate μ_e^{BAU} is given by*

$$\mu_e^{\text{BAU}}(\xi) := \kappa \int_0^\xi e(u) \, du, \quad \text{for } \xi \in [0, \bar{\xi}],$$

where $\kappa > 0$.

Proof. By the merit order assumption the generation capacity associated with the interval $[0, \xi]$ is used to generate an amount ξ of electricity. The market emission rate per hour is then obtained by integrating over the marginal emission stack up to this level of supply. We rescale this rate using the strictly positive constant κ in order to obtain μ_e^{BAU} , the market emission rate measured on the same time scale which also characterises t . \square

Note that whereas the rate e is measured per hour, μ_e^{BAU} is rescaled to the same timescale that characterises t . If, for example, t is measured in years, κ would take the value of the number of hours per year and μ_e^{BAU} would now correspond to the market emission rate per year, given an allowance price and a level of demand.

3.1.3 Load Shifting — a Short-Term Abatement Measure

We now analyse the effects of emissions trading on the business-as-usual economy. As explained above, emissions trading puts a price on carbon and thereby increases the production costs of firms. In particular, it makes it more expensive for firms that rely on emission-intensive technologies to produce. For each unit of CO₂ that these firms emit in excess of their initial allocation, they must buy an allowance contract in order to avoid penalisation; the cost of carbon is a real cost. Alternatively, if a firm owns more allowances than it requires, it can sell spare ones in the market. In this case, the cost of carbon represents an opportunity cost.

We ignore the possibility that firms might invest in long-term abatement projects and only focus on the direct impact on the bid stack. We assume that, in order to maintain their profit margin, firms pass the emissions-related increase in production costs on to consumers. Because the cost of carbon is represented by the price of an allowance certificate, the variable costs and consequently the business-as-usual bids of each firm increase by an amount equal to the allowance price multiplied by the marginal emission rate of that firm. On an aggregate level this means that, for a given allowance price $a \geq 0$ the bid stack now becomes the function g , defined as follows:

$$g(\xi, a) := b^{\text{BAU}}(\xi) + a \times e(\xi), \quad \text{for } (\xi, a) \in [0, \bar{\xi}] \times \mathbb{R}_+. \quad (3.2)$$

For $a = 0$, (3.2) is equivalent to the business-as-usual bid stack. For positive certificate prices emissions trading may cause the mapping $\xi \mapsto g(a, \cdot)$ to lose its monotonicity. In particular, we observe that bids associated with large marginal emission rates become relatively more expensive, as the cost of carbon makes it relatively more costly for firms relying on dirty fuels, such as coal, to produce.

By the merit order assumption the market administrator calls upon generators in increasing order of their bid levels. We define the set of active generation units \mathcal{G} at a given allowance and electricity price $p \in \mathbb{R}$ by

$$\mathcal{G}(a, p) := \{ \xi \in [0, \bar{\xi}] : g(a, \xi) \leq p \}, \quad \text{for } (a, p) \in \mathbb{R}_+ \times \mathbb{R}. \quad (3.3)$$

By the definition of sublevel set, $\mathbb{R} \ni p \mapsto \lambda(\mathcal{G}(a, \cdot))$, for $a \in \mathbb{R}_+$ and where λ denotes the Lebesgue measure, is strictly increasing; under the following assumption, it is also continuous and therefore has a continuous inverse.

Assumption 7. The set of points on which $\partial g / \partial \xi = 0$ has Lebesgue measure zero, that is, for $a \in \mathbb{R}_+$,

$$\lambda \left(\left\{ \xi \in [0, \bar{\xi}] : \frac{db^{\text{BAU}}}{d\xi}(\xi) + a \frac{de}{d\xi}(\xi) = 0 \right\} \right) = 0.$$

Using (3.3), for observed values of the allowance price, the *market bid stack* b is now defined by

$$b(\xi, a) := \lambda(\mathcal{G}(a, \cdot))^{-1}(\xi), \quad \text{for } (\xi, a) \in [0, \bar{\xi}] \times \mathbb{R}_+. \quad (3.4)$$

Given our Walrasian equilibrium assumption (3.1) we may evaluate (3.4) at the current value of the allowance certificate and the level of demand and obtain the price at which electricity is supplied to the market. Therefore, we immediately obtain a price process for electricity P , given by

$$P_t := b(\Xi_t, A_t), \quad \text{for } t \in [0, T].$$

Whereas under business-as-usual an amount $\xi \in [0, \bar{\xi}]$ of electricity is supplied using the generation capacity $[0, \xi]$ (considered as a subset of the domain of the emission stack e), emissions trading may shift this interval further to the right, or, depending on the shape of the marginal emission stack, split it up into multiple sets — an effect we refer to as load shifting. We make the impact of load shifting on the market emission rate μ_e precise in the next proposition.

Proposition 7. *In the presence of cap-and-trade and given an allowance price a and a supply level ξ , the market emission rate μ_e is given by*

$$\mu_e(\xi, a) = \kappa \int_{\mathcal{G}_p(a, \xi)} e(u) \, du, \quad \text{for } (\xi, a) \in [0, \bar{\xi}] \times \mathbb{R}_+, \quad (3.5)$$

where $\mathcal{G}_p(a, \xi) := \mathcal{G}(a, b(\xi, a))$.

Proof. Immediate from the discussion above. □

We note that the business-as-usual market emission rate is of course a special case of (3.5), which is obtained by setting $a = 0$, in which case $\mathcal{G}_p(0, \xi) = [0, \xi]$.

Remark 8. As described earlier, in reality the bid and marginal emission stack are step functions, whose finitely many constant values correspond to firms' bids and their corresponding marginal emissions. To model the impact of a positive allowance price on the bid stack in this case, one would add the cost of carbon to bids as usual, then the resulting step function is *rearranged* in increasing order. Because of the discrete nature of the problem, the rearrangement induces a permutation ν on the bids, which is then applied to the marginal emission stack. The market emission rate is now obtained by integrating the rearranged emission stack over the interval $[0, \xi]$. We prefer to work with the continuous approximation of the bid and marginal emission stack. In this case the permutation ν cannot be defined explicitly and we identify active firms with the set \mathcal{G}_p .

In the following lemma we prove some technical properties of μ_e , which show that the model we propose for the market emission rate makes intuitive sense and leads to a suitably regular function.

Lemma 5. *The market emission rate μ_e satisfies the following:*

(L.1) *For $a \in \mathbb{R}_+$, the map $\xi \mapsto \mu_e(\cdot, a)$ is*

- (i) *strictly increasing and*
- (ii) *Lipschitz continuous.*

(L.2) *For $\xi \in [0, \bar{\xi}]$, the map $a \mapsto \mu_e(\xi, \cdot)$ is*

- (i) *non-increasing and*
- (ii) *Lipschitz continuous*

(L.3) *μ_e is bounded.*

Proof.

(L.1) (i) By Assumption 7 and the definition of sublevel set, for $0 \leq \xi < \xi' \leq \bar{\xi}$ and $a \in \mathbb{R}_+$, $\mathcal{G}_p(a, \xi) \subset \mathcal{G}_p(a, \xi')$. Since $e(\xi) > 0$ on $[0, \bar{\xi}]$ the result follows.

- (ii) For $0 \leq \xi < \xi' \leq \bar{\xi}$ and $a \in \mathbb{R}_+$ and with the definition $\Delta^\xi \mathcal{G}_p(a, \xi, \xi') := \mathcal{G}_p(a, \xi') \setminus \mathcal{G}_p(a, \xi)$,

$$\begin{aligned} \mu_e(\xi', a) - \mu_e(\xi, a) &= \kappa \int_{\Delta^\xi \mathcal{G}_p(a, \xi, \xi')} e(u) \, du \\ &\leq \lambda(\Delta^\xi \mathcal{G}_p(a, \xi, \xi')) \kappa \max_{u \in [0, \bar{\xi}]} e(u) \\ &= (\xi' - \xi) \kappa \max_{u \in [0, \bar{\xi}]} e(u). \end{aligned}$$

The case $\xi' < \xi$ is treated similarly.

- (L.2) (i) For $0 \leq a < a' < \infty$ and $\xi \in [0, \bar{\xi}]$ and with the definition $\Delta^a \mathcal{G}_p(a, a', \xi) := \mathcal{G}_p(a, \xi) \setminus \mathcal{G}_p(a', \xi)$,

$$\mu_e(\xi, a) - \mu_e(\xi, a') = \kappa \int_{\Delta^a \mathcal{G}_p(a, a', \xi)} e(u) \, du - \kappa \int_{\Delta^a \mathcal{G}_p(a', a, \xi)} e(u) \, du.$$

Since $\lambda(\Delta^a \mathcal{G}_p(a, a', \xi)) = \lambda(\Delta^a \mathcal{G}_p(a', a, \xi))$, the result follows from the observation that, for a given $\xi \in [0, \bar{\xi}]$,

$$\begin{aligned} e(u) &= (g(u, a') - g(u, a))(a' - a)^{-1} \\ &> (b(\xi, a') - b(\xi, a))(a' - a)^{-1}, \quad \text{for } u \in \Delta^a \mathcal{G}_p(a, a', \xi) \end{aligned}$$

and

$$\begin{aligned} e(u) &= (g(u, a') - g(u, a))(a' - a)^{-1} \\ &< (b(\xi, a') - b(\xi, a))(a' - a)^{-1}, \quad \text{for } u \in \Delta^a \mathcal{G}_p(a', a, \xi). \end{aligned}$$

and that $\mu_e(\xi, a) - \mu_e(\xi, a') = 0$ if $\Delta^a \mathcal{G}_p(a, a', \xi) = \Delta^a \mathcal{G}_p(a', a, \xi) = \emptyset$.

- (ii) From above we know that for $0 \leq a < a' < \infty$ and $\xi \in [0, \bar{\xi}]$,

$$\mu_e(\xi, a) - \mu_e(\xi, a') \leq C_6 \lambda(\Delta^a \mathcal{G}_p(a, a', \xi)),$$

for some constant $C_6 \geq 0$. It is also clear that $\Delta^a \mathcal{G}_p(a, a', \xi)$ (and similarly $\Delta^a \mathcal{G}_p(a', a, \xi)$) can be written as the union of a finite number of intervals. As a increases to a' , there are three possibilities: (i) existing intervals grow or shrink; (ii) new intervals appear or existing ones disappear (by Assumption 7 this always happens at a point) (iii) the intervals remain unchanged. Differentiating the level curves $g(u, a) = b(\xi, a)$ with respect to a , for given $(\xi, a) \in [0, \bar{\xi}] \times \mathbb{R}_+$ and $u \in [0, \bar{\xi}]$ such that the equality holds, we find that

$$\frac{\partial u}{\partial a} = - \left(\frac{\partial g}{\partial a} - \frac{\partial b}{\partial a} \right) \left(\frac{\partial g}{\partial u} \right)^{-1}.$$

By Assumption 7 the right hand side is bounded by a constant, say $C_7 \geq 0$.¹ Therefore, as a changes, in each case (i) – (iii), the endpoints of the intervals defining $\Delta^a \mathcal{G}_p(a', a, \xi)$ do not move faster than $C_7(a' - a)$. Therefore, also $\lambda(\Delta^a \mathcal{G}_p(a, a', \xi)) \leq C_7(a' - a)$. The case $a > a'$ is treated similarly and the result follows.

(L.3) Boundedness of μ_e follows from the boundedness of the function e and the fact that $\mathcal{G}_p(a, \xi) \subseteq [0, \bar{\xi}]$, for all $(a, \xi) \in \mathbb{R}_+ \times [0, \bar{\xi}]$.

□

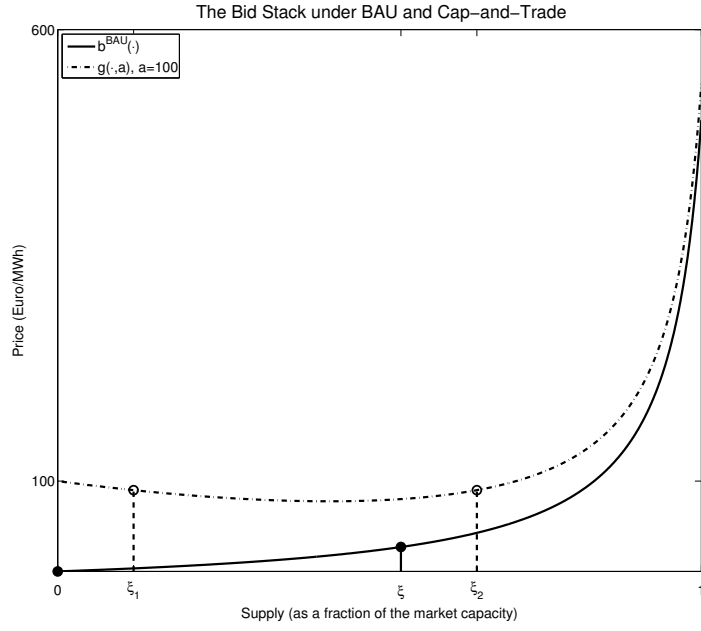
Remark 9. We acknowledge immediately that μ_e as defined above does not satisfy the strict monotonicity criteria required by Assumption (A2.2) for the existence proof of the pricing FBSDE in Chapter 2. In particular μ_e as defined above is only non-increasing as a function of a , but not strictly decreasing. We choose to ignore this defect for the following reasons: first, we found that the lack of strict monotonicity did not have an adverse effect on the numerical scheme that we implemented and used for the numerical analysis presented in §3.4; second, it would easily be possible to redefine the market emission rate to satisfy the strict monotonicity criteria by multiplying μ_e with an “abatement function” mapping $\mathbb{R} \rightarrow [1 - \delta, 1]$, for some small $\delta > 0$, and strictly decreasing as a function of a . An example of a suitable function would be a reflected, scaled and shifted hyperbolic tangent. This procedure could also be justified intuitively by arguing that it takes into account the effects of firms investing in long-term abatement projects.

Figure 3.1 illustrates the effect of load shifting and the resulting reduction in the market emission rate under the assumption that under BAU dirtier production technologies are placed further to the left in the bid stack than cleaner ones (see Remark 7).

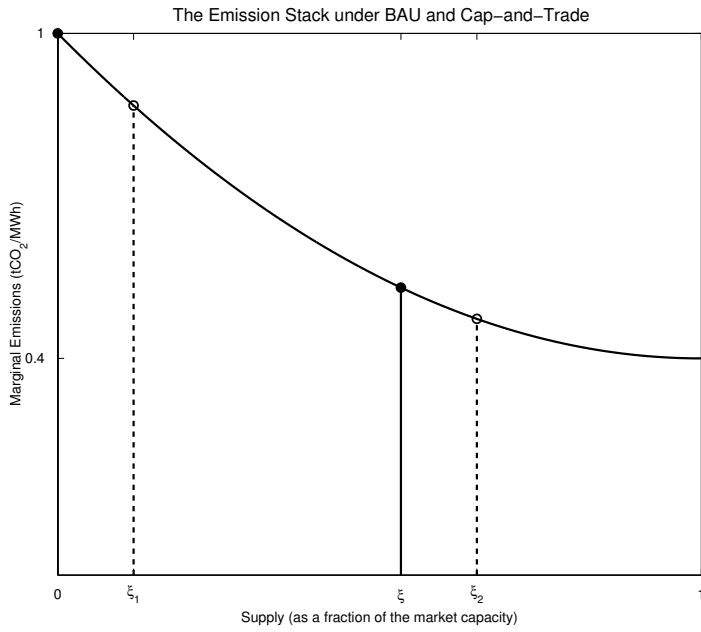
3.2 The Allowance Pricing Problem

In this section we formulate the problem of determining the arbitrage-free price of the allowance certificate in the present modelling context. We do this initially in the setting of an emission market with one compliance period; subsequently we generalise the model to deal with markets that consist of multiple consecutive compliance periods

¹Throughout the rest of the proof we allow C_7 to change from occurrence to occurrence.



(a) Bid stack b^{BAU} and the function g .



(b) Emission stack e .

Figure 3.1: Under business-as-usual conditions, the bid stack b^{BAU} implies that resources associated with the interval $[0, \xi]$ are used to supply electricity. Therefore, the market emission rate is obtained by integrating the emission stack from 0 to ξ . Under the influence of a cap-and-trade scheme the function g leads to resources being shifted to the interval $[\xi_1, \xi_2]$. The market emission rate is now given by the (smaller) integral over the emission stack from ξ_1 to ξ_2 .

and examine the impact that connecting mechanisms, namely banking, borrowing and withdrawal have on the certificate price.

The formulation of the pricing model follows immediately from the results in Chapter 2. As mentioned before, we set $d = 1$ and identify the now scalar valued process X , representing the fundamental factors, with the demand process D , that is, $X = D$. It follows that D evolves according to an Itô diffusion; specifically, for measurable functions $\mu_d : [0, \bar{\xi}] \mapsto \mathbb{R}$ and $\sigma_d : [0, \bar{\xi}] \mapsto \mathbb{R}$, satisfying Assumption (A2.1), and under the measure \mathbb{Q} , the demand process satisfies the stochastic integral equation

$$D_t = d_0 + \int_0^t \mu_d(D_u) du + \int_0^t \sigma_d(D_u) dW_u, \quad \text{for } t \in [0, T], \quad (3.6)$$

where $d_0 \in (0, \bar{\xi})$. The assumption that demand is perfectly inelastic is reflected in the fact that both coefficients are functions of demand only. In practice demand for electricity exhibits seasonal periodicity, an attribute that would cause μ_d to depend on time explicitly. Because the current chapter primarily serves to analyze the impact of emissions trading in a stylized setting, we choose to ignore this feature for now.

Cumulative emissions are measured from the beginning of the compliance period when $t = 0$. Subsequently, given our Walrasian equilibrium assumption (3.1), they are determined by evaluating the market emission rate μ_e , derived in Proposition 7, at the stochastic values of demand and the allowance price and integrating up to the current time. Consequently, in the current setting, the stochastic integral equation (2.6) now reads

$$E_t = e_0 + \int_0^t \mu_e(D_u, A_u) du, \quad \text{for } t \in [0, T]. \quad (3.7)$$

Notice that the concrete model we propose in this chapter implies that μ_e is non-negative. Hence the process E is non-decreasing, which makes intuitive sense considering that it represents a cumulative quantity.

3.2.1 Single Compliance Period

In the setting of a single compliance period, the value of the allowance certificate at compliance time is determined by the terminal condition ϕ with $\tilde{A}_T = 0$, that is,

$$A_T = \phi(0, E_T) = \pi I_{[\gamma, \infty)}(E_T). \quad (3.8)$$

The following proposition now summarizes the main result in this section.

Proposition 8. *Assuming that demand for electricity is the only fundamental factor in the economy and that the market emissions rate is given by (3.5), the price process A of an allowance certificate in an emission market with one compliance period is described by the unique solution $((D, E), A, Z^a) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$ to the following FBSDE: for $t \in [0, T]$,*

$$\begin{cases} D_t = d_0 + \int_0^t \mu_d(D_u) \, du + \int_0^t \sigma_d(D_u) \, dW_u \\ E_t = e_0 + \int_0^t \mu_e(D_u, A_u) \, du \\ A_t = \pi I_{[\gamma, \infty)}(E_T) - r \int_t^T A_u \, du - \int_t^T e^{ru} Z_u^a \, dW_u. \end{cases} \quad (3.9)$$

Proof. The problem formulation is obvious from the discussion above. Existence and uniqueness of a solution to equation (3.9) in the relevant spaces follow immediately from Theorem 1 and Remark 3 with $d = 1$, $X = D$, $\tilde{A}_T = 0$ and from Lemma 5 (and strictly speaking using a modification of μ_e as suggested in Remark 9 to satisfy the strict monotonicity assumption that is necessary for Theorem 1). \square

At this point we outline a connection between the solution to certain Markovian FBSDEs and PDEs. In particular, it is sometimes the case that the backward component of a FBSDE can be characterized as the solution of a PDE. This is an extension of the well known Feynman-Kac formula (cf. [Ma and Yong, 1999]). Crucial for this connection to work is the regularity of the solution to the PDE. In fact, in the setting of [Ma and Yong, 1999], the solution must be a classical solution.

To illustrate this link between FBSDEs and PDEs in our setting, we conjecture that there exists a deterministic function² $\alpha : [0, T] \times [0, \bar{\xi}] \times [0, \bar{e}] \mapsto \mathbb{R}$, suitably regular on $[0, T)$ to be a classical solution to the PDE

$$\begin{aligned} \mathcal{N}\alpha &= 0, & \text{on } U_T, \\ \alpha &= \pi I_{[\gamma, \infty)}(e), & \text{on } \{t = T\} \times U, \end{aligned} \quad (3.10)$$

where $U := (0, \bar{\xi}) \times (0, \bar{e})$, $U_T := [0, T) \times U$ and

$$\mathcal{N} := \frac{\partial}{\partial t} + \frac{1}{2} \sigma_d^2(d) \frac{\partial^2}{\partial d^2} + \mu_d(d) \frac{\partial}{\partial d} + \mu_e(d, \cdot) \frac{\partial}{\partial e} - r.$$

Under the imposed regularity assumption a simple application of Itô's formula then shows that $A_t = \alpha(t, D_t, E_t)$, for $t \in [0, T)$, and therefore that the PDE (3.10) indeed

²By a slight abuse of notation we use α at this stage, although the same notation was used in Chapter 2 to denote the random field corresponding to the allowance FBSDE with random drift coefficient. We trust that the intended meaning is clear from the context.

characterizes the solution to the backward equation of the FBSDE (3.9). Notice that μ_e takes α as its second argument; hence the equation belongs to the class of semilinear PDEs. This feature together with the lack of viscosity in the e -direction causes the PDE to be highly non-standard and currently we are not aware of any results in the literature that guarantee the value function α to be sufficiently regular for the above argument to be rigorous. However, an asymptotic analysis in Chapter 4 lends strong support to this conjecture. Therefore, we introduce the link at this point, as it will become important for the numerical solution of the FBSDE in §3.4.

In addition to the terminal condition suitable boundary conditions have to be supplied. These depend on the specification of the coefficients of the PDE and we postpone the issue to section §3.4, where the numerical solution of the problem is discussed in detail.

Remark 10. We emphasize that when stating the PDE (3.10) and when pointing out its connection with the FBSDE (3.9), we ignored the fact that in order to obtain the existence and uniqueness of a solution to the FBSDE (3.9) in Theorem 1, we had to relax the enforcement of the binary terminal condition at the point where the cumulative emissions variable equals the cap. The reason why we feel that our proceeding is justified is that our main usage of equation (3.10) is the numerical approximation of the solution to the FBSDE (3.9). For this purpose the terminal condition is evaluated at discrete points of the domain only and we may simply choose our grid in a way that avoids the evaluation of the terminal condition at the cap.

3.2.2 Multiple Compliance Periods

We now consider the pricing problem in an emission market with two compliance periods. In principle, the results presented in this section can easily be extended to an arbitrary number of periods. To ease the presentation, however, we choose to present the canonical case. Taking $0 = T_0 < T_1 < T_2 = T$, we consider the two compliance periods $[0, T_1]$ and $[T_1, T]$. For simplicity we assume that each period corresponds to one year and we introduce the set $J = \{1, 2\}$ to index the periods. As previously, the process D represents the aggregate demand for electricity. For $j \in J$, the processes $(E_t^j)_{t \in [T_{j-1}, T_j]}$, measure the cumulative emissions within the first and the second compliance period. Therefore E_t^1 represents the emissions accumulated over the time interval $[0, t]$ and E_t^2 represents the emissions accumulated over the time interval $[T_1, t]$. Similarly, $(A_t^j)_{t \in [T_{j-1}, T_j]}$ represents the price of an allowance certificate

for compliance at time T_j . Each year, the regulator issues a number $\gamma_j \geq 0$ of allowance certificates and sets the penalty $\pi_j \geq 0$.

Demand for electricity is given at time $t = 0$ and thereafter evolves continuously throughout the trading period $[0, T]$. Cumulative emissions are measured from the beginning of each compliance period. Lastly, we note that each process A^j corresponds to a different vintage of allowance certificates. If we disregard mechanisms that connect compliance periods, a certificate issued during the first period is for compliance at time T_1 only. However, we now wish to consider mechanisms that connect compliance periods and permit allowances to be transferred between periods. In this case both vintages of certificates have a more complex dependence. In particular, the second period allowance price depends on cumulative emissions not only during the second period, but also on the ones during the previous period, as we describe below. The connecting mechanisms are now expressed through the terminal conditions at time T_1 and T_2 ; for now, we do not determine them explicitly and instead represent them by the terminal condition $\phi(\tilde{A}_{T_j}, E_{T_j}^j; \pi_j, \tilde{\gamma}_j)$ with generic bounded, non-negative and \mathcal{F}_{T_j} -measurable random variables \tilde{A}_{T_j} and parameters $\tilde{\gamma}_j \geq 0$, for $j \in J$.

Corollary 1. *Assuming that demand for electricity is the only fundamental factor in the economy, that the market emissions rate is given by (3.5) and in the setting of an emission market with two consecutive compliance periods, linked by some connecting mechanism(s), the price processes A^j , $j \in J$, of the first and second period allowance certificate are described by the unique solutions $((D, E^j), A^j, Z^{a,j}) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$ to the following FBSDEs: for $j \in J$ and $t \in [T_{j-1}, T_j]$,*

$$\begin{cases} D_t = d_{T_{j-1}} + \int_{T_{j-1}}^t \mu_d(D_u) \, du + \int_{T_{j-1}}^t \sigma_d(D_u) \, dW_u \\ E_t^j = e_{T_{j-1}} + \int_{T_{j-1}}^t \mu_e(D_u, A_u^j) \, du \\ A_t^j = \phi(\tilde{A}_{T_j}, E_{T_j}^j; \pi_j, \tilde{\gamma}_j) - r \int_t^{T_j} A_u \, du - \int_t^{T_j} e^{ru} Z_u^{a,j} \, dW_u, \end{cases} \quad (3.11)$$

where $d_{T_{j-1}} := D_{T_{j-1}}$ and $e_{T_{j-1}} \in \mathbb{R}$.

Proof. Again the problem formulation follows immediately from the discussion above. With the same proviso concerning the monotonicity of μ_e , existence and uniqueness of solutions $((D, E^j), A^j, Z^{a,j})$ to the equations (3.11) in the relevant spaces are guaranteed by Theorem 1 and Remark 3 with d_0 replaced with $d_{T_{j-1}}$ and e_0 replaced with $e_{T_{j-1}}$. \square

As in §3.2.1, we conjecture the existence of suitably regular functions $\alpha_j : [T_{j-1}, T_j] \times [0, \bar{\xi}] \times [0, \bar{e}] \mapsto \mathbb{R}$, such that $A_t^j = \alpha_j(t, D_t, E_t^j)$, for $t \in [T_{j-1}, T_j)$, and

$$\begin{aligned} \mathcal{N}\alpha_j &= 0, & \text{on } U_{T_j}, \\ \alpha_j &= \phi, & \text{on } \{t = T_j\} \times U, \end{aligned} \tag{3.12}$$

where $U_{T_j} := [T_{j-1}, T_j) \times U$.

3.2.2.1 Connecting Mechanisms: Banking and Withdrawal

Banking and withdrawal are two mechanisms that connect compliance periods and are implemented in most emission markets. Both affect the supply of certificates during the second compliance period. This leads us to let the parameter $\tilde{\gamma}_2$ denote the aggregate supply of certificates during the second compliance period. The implementation of banking offers an additional incentive for reducing emissions, since it specifies that spare allowance certificates, for compliance at the end of the first period, become perfect substitutes for certificates issued during the second compliance period. This means that in the event of compliance, a number $(\gamma_1 - E_{T_1}^1)$ of certificates with price $A_{T_1}^1$ are exchanged for certificates valid during the next compliance period, with price $A_{T_1}^2$.

This incentive to reduce emissions is strengthened by the withdrawal mechanism, which constitutes additional punishment for firms that exceed their emission limit. Under this mechanism excess emissions at the end of the first compliance period are not only penalised at the rate π_1 , but moreover a corresponding number of certificates are withdrawn from the subsequent allocation. Whereas any amount of certificates can be banked, at most next period's allocation can be withdrawn from the market. Therefore, in the event of non-compliance, a number $\min(E_{T_1}^1 - \gamma_1, \gamma_2)$ of certificates with price $A_{T_1}^2$ are subtracted from γ_2 . In the event that the entire allocation of the second period has been withdrawn and there remain unaccounted for emissions (at the end of the first period), we specify that these are penalised at the combined rate of the first period penalty π_1 and — to compensate for the lack of certificates that can be withdrawn — an additional penalty. For simplicity we choose this penalty to be equal to $A_{T_1}^2$.

These features imply that during the second period the aggregate supply of certificates now stems from two sources. First, the regulator issues a number of permits γ_2 at the beginning of the period. Second, as explained above, a number of certificates are banked or withdrawn. The aggregate supply of certificates during the second

period is then given by

$$\tilde{\gamma}_2 = (\gamma_2 + \gamma_1 - E_{T_1}^1)^+. \quad (3.13)$$

Of course $\tilde{\gamma}_2$ as specified above is a random variable. However, since it is \mathcal{F}_{T_1} -measurable we may consider it to be deterministic for the pricing problem of the second period certificate, which is set on the time interval $[T_1, T]$ and for which $\tilde{\gamma}_2$ will ultimately play an important role.

Figure 3.2 illustrates the banking and withdrawal mechanisms in the two-period market under consideration. In Figure 3.2a compliance at $t = T_1$ leads to the banking of a number $(\gamma_1 - E_{T_1}^1)$ of certificates. The market is in compliance at $t = T_2$ because of this additional supply of certificates. In Figure 3.2b non-compliance at $t = T_1$ leads to the withdrawal of a number $(E_{T_1}^1 - \gamma_1)$ of certificates. This leads to non-compliance at $t = T_2$ because of the decreased supply of certificates during the second compliance period, even though cumulative emissions during the period $[T_1, T]$ are below the second-period cap.

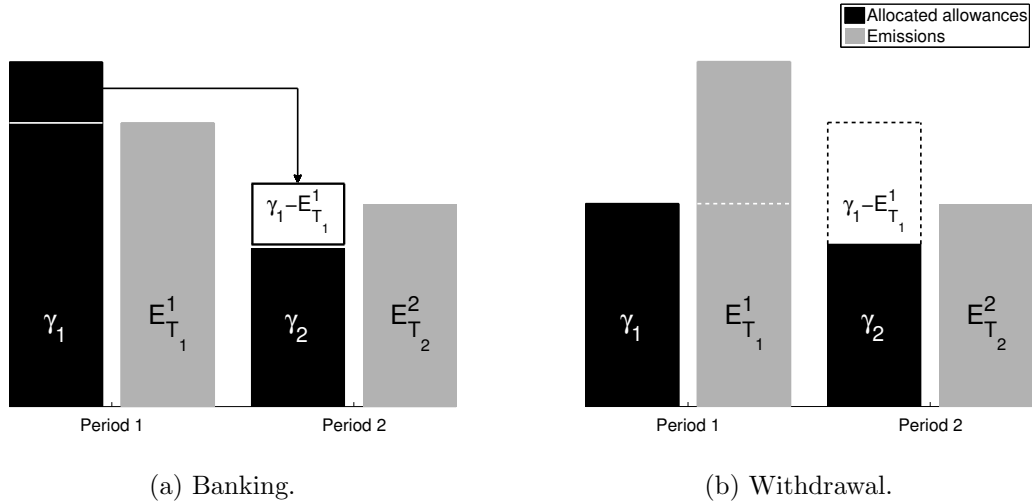


Figure 3.2: Compliance period connecting mechanisms in an emission market with two periods.

The terminal condition at $t = T_1$ for the allowance price in an emission market with two compliance periods connected by the mechanisms of banking and withdrawal now follows. Banking implies that in the event of compliance, that is if $E_{T_1}^1 < \gamma_1$, the value of the first period allowance at time T_1 equals the value of the second period allowance at time T_1 . In the event of non-compliance at time T_1 with $\gamma_1 \leq E_{T_1}^1 < \gamma_1 + \gamma_2$, the penalisation of excess emissions and the withdrawal of certificates lead

to the first period allowance certificate taking the value of the sum of the price of the second period certificate and π_1 . In the event of non-compliance at time T_1 with $E_{T_1}^1 \geq \gamma_1 + \gamma_2$, the double penalisation rule implies that the value of the first period allowance certificate also equals the sum of π_1 and the price of the second period certificate. Therefore,

$$A_{T_1}^1 = \phi(A_{T_1}^2, E_{T_1}^1; \pi_1, \gamma_1) = A_{T_1}^2 + \pi_1 I_{[\gamma_1, \infty)}(E_{T_1}^1). \quad (3.14)$$

At time T_2 , the terminal condition for the allowance price is now the same as in the one period case with the exception that the aggregate supply of certificates $\tilde{\gamma}_2 := (\gamma_2 + \gamma_1 - E_{T_1}^1)^+$, see (3.13), is used:

$$A_{T_2}^2 = \phi(0, E_{T_2}^2; \pi_2, \tilde{\gamma}_2) = \pi_2 I_{[\tilde{\gamma}_2, \infty)}(E_{T_2}^2). \quad (3.15)$$

Note that because the terminal condition at time $t = T_2$ uses the aggregate supply of certificates $\tilde{\gamma}_2$ it is path-dependent. In particular, it depends on $E_{T_1}^1$ as mentioned earlier. In the context of pricing futures contracts on allowance certificates in a two-period market a similar terminal condition was introduced in [Carmona and Hinz, 2011].

3.2.2.2 Connecting Mechanisms: Banking, Borrowing and Withdrawal

In addition to banking and withdrawal, consecutive compliance periods may also be connected by the borrowing mechanism. The effect of banking and withdrawal at time T_1 is to increase the value of the first period allowance certificate from zero to $A_{T_1}^2$ in the event of compliance (due to the banking mechanism) and from π_1 to $(\pi_1 + A_{T_1}^2)$ in the event of non-compliance (due to the withdrawal mechanism). In contrast, the borrowing mechanism decreases the probability with which non-compliance occurs.

In an emission market in which borrowing is allowed, firms may bring forward certificates from the second allocation γ_2 and use them for compliance at time T_1 . This does not affect the aggregate supply of certificates during the first compliance period, but whereas previously non-compliance occurred when $E_{T_1}^1 \geq \gamma_1$, this is no longer the case, as certificates from the second period may be borrowed to supplement the aggregate supply during the first period. Therefore non-compliance now only occurs if $E_{T_1}^1 \geq \gamma_1 + \gamma_2$, in which case the entire allocation of the second period must be borrowed and additional units of excess emissions are penalised at the combined rate of π_1 and $A_{T_1}^2$.

The borrowing mechanism is illustrated in Figure 3.3. Compliance at time T_1 is only possible by borrowing a number $(E_{T_1}^1 - \gamma_1)$ of certificates from the second

compliance period. As a result these certificates are deducted from γ_2 . In our example this leads to non-compliance at time T_2 .

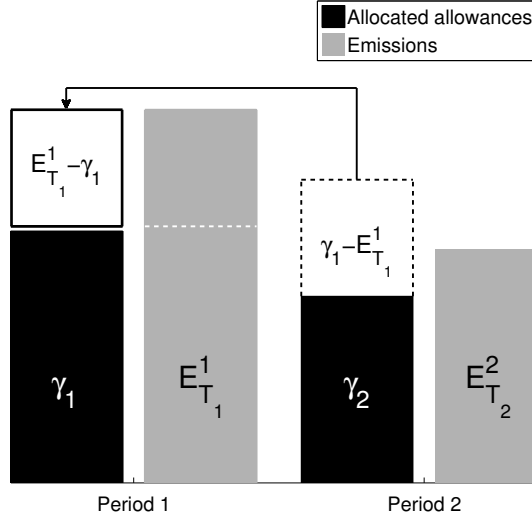


Figure 3.3: Borrowing mechanism in an emission market with two periods.

Whereas $A_{T_2}^2$ continues to be given by (3.15), the terminal condition at time $t = T_1$ in an emission market, which connects subsequent compliance periods with the banking, borrowing and withdrawal mechanisms, now follows and is given by

$$A_{T_1}^1 = \phi(A_{T_1}^2, E_{T_1}^1; \pi_1, \gamma_1 + \gamma_2) = A_{T_1}^2 + \pi_1 I_{[\gamma_1 + \gamma_2, \infty)}(E_{T_1}^1). \quad (3.16)$$

3.3 Carbon Derivatives

In the setting of the market introduced at the beginning of this chapter, we now illustrate the pricing of derivatives. For this purpose we work in the emission market of §3.2.1 with one compliance period. Our example of choice is a *European call* with maturity $\tau \in [0, T]$ and strike $k \geq 0$. We let the stochastic process V , as introduced in §2.4, describe fluctuations in the price of the option. By the definition of a call option, at maturity τ the value of the contract is given by

$$V_\tau := (A_\tau - k)^+.$$

The following proposition summarises the pricing problem for the call option.

Proposition 9. *Assuming that demand for electricity is the only fundamental factor in the economy and that the market emissions rate is given by (3.5), the price process*

V of a European call option written on the allowance certificate is described by the unique solution $((D, E), A, V, (Z^a, Z^v)) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2$ to the following FBSDE: for $t \in [0, \tau]$,

$$\begin{cases} D_t = d_0 + \int_0^t \mu_d(D_u) \, du + \int_0^t \sigma_d(D_u) \, dW_u \\ E_t = e_0 + \int_0^t \mu_e(D_u, A_u) \, du \\ A_t = \alpha(\tau, \omega, E_\tau) - r \int_t^\tau A_u \, du - \int_t^\tau e^{ru} Z_u^a \, dW_u \\ V_t = (A_\tau - k)^+ - r \int_t^\tau V_u \, du - \int_t^\tau e^{ru} Z_u^v \, dW_u. \end{cases}$$

Proof. The existence of a unique solution in the named spaces follows immediately from Theorem 2 and Remark 6 with $d = 1$, $X = D$ and with $\psi(x, e, a) := (a - k)^+$. \square

Recall that in §3.2.1 we conjectured the existence of a deterministic function α such that $A_t = \alpha(t, D_t, E_t)$, for $t \in [0, T)$, and where α was characterised as the solution to a semilinear PDE. Based on this assumption, we now conjecture in the case of the call option that there exists a deterministic function $v : [0, \tau] \times [0, \bar{\xi}] \times [0, \bar{e}] \mapsto \mathbb{R}$, suitably regular on $[0, \tau)$ to be a classical solution to the PDE

$$\begin{aligned} \mathcal{L}v &= 0, & \text{on } U_\tau, \\ v &= (\alpha(\tau, d, e) - k)^+, & \text{on } \{t = \tau\} \times U, \end{aligned} \tag{3.17}$$

where $U_\tau := [0, \tau) \times U$ and

$$\mathcal{L} := \frac{\partial}{\partial t} + \frac{1}{2} \sigma_d^2(d) \frac{\partial^2}{\partial d^2} + \mu_d(d) \frac{\partial}{\partial d} + \mu_e(d, \alpha(t, d, e)) \frac{\partial}{\partial e} - r$$

and where we postpone the specification of relevant boundary conditions to §3.4.4. Under the imposed regularity assumption on v a simple application of Itô's formula then shows that $V_t = v(t, D_t, E_t)$, for $t \in [0, \tau)$.

Because the allowance price (representing the cost of carbon) has an impact on the rate at which firms emit, but the call contract does not the partial differential equation (3.10), characterising α , is nonlinear, whereas (3.17) is linear.

3.4 Numerical Case Study

This section is dedicated to the numerical illustration of the model. We illustrate the dependence of allowance prices on demand and the cumulative emissions and compare

prices in the setting of a single-period market to those implied by multiperiod markets. Moreover, we show the effects different connecting mechanisms have on the allowance price. Since the justification of any emissions trading system is the reduction of emissions, we simulate the state variables in our market and illustrate the impact load shifting has on the mean cumulative emissions. Further, we demonstrate the dependence structure of the European call option on the allowance certificate.

3.4.1 A Worked Example

We begin by specifying the functions and parameters in the model.

3.4.1.1 Functional Form of the Bid and the Emission Stack

We take the business-as-usual bid stack to be of the form

$$b^{\text{BAU}}(\xi) := \underline{b} + \left(\frac{\bar{b} - \underline{b}}{\bar{\xi}^{\theta_1}} \right) \xi^{\theta_1}, \quad \text{for } \xi \in [0, \bar{\xi}],$$

where $-\infty < \underline{b} < \bar{b} < \infty$ and $\theta_1 > 2$. With this choice b^{BAU} is strictly convex and strictly increasing on its domain of definition. The parameters \underline{b} and \bar{b} correspond to the minimum and maximum prices of electricity the model can produce. Because the range of allowed bids and typically observed market prices in many auction based electricity markets is well known, these are relatively easy to infer in practice. The parameter θ_1 controls the steepness of the stack and in particular how quickly the marginal costs of generators increase.

Similarly, we take the marginal emission stack to be of the form

$$e(\xi) := \bar{e} - \left(\frac{\bar{e} - \underline{e}}{\bar{\xi}^{\theta_2}} \right) \xi^{\theta_2}, \quad \text{for } \xi \in [0, \bar{\xi}],$$

where $0 \leq \underline{e} < \bar{e} < \infty$ and $\theta_2 \in [0, 1)$. With this definition also e is strictly convex and decreasing on its domain of definition. The parameters \underline{e} and \bar{e} correspond to the minimum and maximum marginal emission rate in the market. In a market with coal and gas generators only and under the reasonable assumption that coal is the more emission intensive technology than gas, \bar{e} would represent the marginal emission rate of coal and \underline{e} that of gas. The parameter θ_2 controls the fuel mix in the market. The smaller the value of θ_2 , the smaller the proportion of the market capacity that is served by the pollution-intensive technology.

Clearly b^{BAU} and e satisfy the assumptions in Definitions 1 and 2. Further, since a linear combination of strictly convex functions is also strictly convex, so is the

function g defined by (3.2). Therefore, also Assumption 7 is satisfied. Moreover, we note that for this choice of bid and emission stack the set \mathcal{G}_p , as defined in Proposition 7, is always of the form $[\xi_1, \xi_2]$ for $0 \leq \xi_1 \leq \xi_2 \leq \bar{\xi}$.

3.4.1.2 The Demand Process

We specify that under \mathbb{Q} the process D satisfies the stochastic integral equation

$$D_t = d_0 - \int_0^t \eta (D_u - \bar{D}) \, du + \int_0^t \sqrt{2\eta\bar{\sigma}_d D_u (\bar{\xi} - D_u)} \, dW_u, \quad \text{for } t \in [0, T], \quad (3.18)$$

where $\bar{D}, \eta, \bar{\sigma}_d > 0$. With this definition (3.18) is a Jacobi diffusion process. Such an equation possesses a strong solution (cf. [Karlin and Taylor, 1981]); its key properties are that it has a linear, mean-reverting drift component and that it degenerates on the boundary. Moreover, subject to $\bar{D} \in (0, \bar{\xi})$ and $\min(\bar{D}, \bar{\xi} - \bar{D}) \geq \bar{\xi}\bar{\sigma}_d$, the process remains within the interval $(0, \bar{\xi})$; its stationary distribution is a beta distribution and its mean is given by \bar{D} (cf. [Forman and Sørensen, 2008]). Since the solution to (3.18) is bounded (given the right choice of parameters) it is clear that $D \in \mathcal{S}^2$, as required by Assumption (A2.1).

3.4.1.3 Choice of Parameters

Tables 3.1, 3.2, 3.3 summarise the parameter values used for the numerical study that follows. We note that they do not correspond to a particular example of an electricity market. But they can be considered representative of a medium sized market whose fuel mix predominantly consists of coal and gas generators.

Table 3.1 specifies the parameters for the bid and the emission stack. Using (3.7) now with $A := 0$ and $D := \bar{\xi}$, and with the assumption that there are 24×365 production hours in the year, we find that $\bar{e} = 1.6519 \times 10^8$.

\bar{b}	\underline{b}	θ_1	\bar{e}	\underline{e}	θ_2	κ	$\bar{\xi}$
200	0	10	1.2	0.4	0.4	8760	30000

Table 3.1: Parameters for the bid and emission stack.

The parameters relating to demand are given in Table 3.2.

Calculating the cumulative emissions now for $A := 0$ and demand at its mean level $D := \bar{D}$, we find that $E_T = 1.2961 \times 10^8$. This leads us to choose the cap slightly below this level, in order to incentivise a reduction in emissions. The parameters

η	\bar{D}	$\bar{\sigma}_d$	d_0	e_0	r
10	21000	0.05	21000	0	0.05

Table 3.2: Parameters for the demand and the emission process and the risk-free rate.

characterising the emissions trading scheme are given in Table 3.3. We note that here time is measured in years.

γ	π	T
1.17×10^8	100	1

Table 3.3: Parameters characterising the emissions trading scheme.

3.4.2 The Allowance Price Value Function

Now that we have specified all coefficients and parameters of the model explicitly, we present the necessary boundary conditions for the allowance price valuation partial differential equation and discuss its solution. First, this is done in the setting of an emission market with one compliance period, second we consider two periods connected by either banking and withdrawal or banking, borrowing and withdrawal.

With regards to the problems (3.10) (in the case of one period) and (3.12) (in the case of two periods) the question arises, at which points of the boundary we need to specify boundary conditions in addition to the terminal condition and what conditions make sense given the original stochastic problems (3.9) and (3.11). The former question is answered by considering the Fichera function f (cf. [Oleinik and Radkevich, 1973]). Defining $n := (n_d, n_e)$ to be the inward normal vector to the boundary, Fichera's function for the operator \mathcal{N} (and \mathcal{L}) reads

$$f(t, d, e) := \left(\mu_d(d) - \frac{1}{2} \frac{\partial}{\partial d} \sigma_d^2(d) \right) n_d + \mu_e(d, \alpha(t, d, e)) n_e, \quad \text{on } \partial U_T \text{ (or } \partial U_{T_i}).$$

In the case when the coefficients μ_d and σ_d are of the form prescribed in (3.18) we find that

$$f(t, d, e) = \eta \left((\bar{D} - \bar{\sigma}_d \bar{\xi}) + (2\bar{\sigma}_d - 1) d \right) n_d + \mu_e(d, \alpha(t, d, e)) n_e, \quad \text{on } \partial U_T \text{ (or } \partial U_{T_i}).$$

At points of the boundary, where $f \geq 0$ information is outward flowing and no boundary conditions have to be specified; at those points where $f < 0$ the information

is inward flowing and boundary conditions are necessary. Considering the parts of the boundary corresponding to $d = 0$ and $d = \bar{\xi}$, we find that $f \geq 0$ if and only if $\min(\bar{D}, \bar{\xi} - \bar{D}) \geq \bar{\xi}\bar{\sigma}_d$, which is the same condition prescribed in §3.4.1.2 to guarantee that the Jacobi diffusion stays within the interval $(0, \bar{\xi})$. At points of the boundary corresponding to $e = 0$, we find that $f \geq 0$ always. On the part of the boundary on which $e = \bar{e}$, $f < 0$ except at the point $(d, e) = (0, \bar{e})$, where $f = 0$, an ambiguity which could be resolved by smoothing the domain. Therefore, no boundary conditions are necessary except when $e = \bar{e}$. The nature of the condition at this part of the boundary depends on whether we consider a market with one compliance period or multiple periods and we specify it in the relevant sections below.

Given values for the demand for electricity and the cumulative emissions, the valuation equation representing the allowance pricing problem determines the arbitrage-free price of an allowance certificate. We illustrate this dependence by solving the partial differential equation numerically, using the finite difference scheme explained in Appendix A.1.

3.4.2.1 One Compliance Period

The boundary condition at $e = \bar{e}$ takes the form

$$\alpha(t, d, e) = e^{-r(T-t)}\pi, \quad [0, T) \times (0, \bar{\xi}) \times \{e = \bar{e}\}. \quad (3.19)$$

The condition (3.19) follows from the fact that, as soon as the cumulative emissions surpass the cap, every additional tonne of CO₂ is penalised at a rate π at time T .

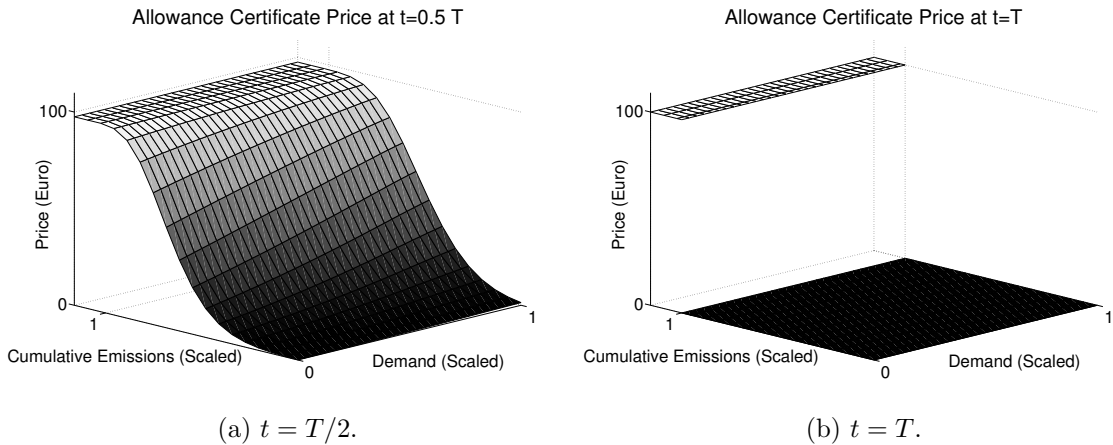


Figure 3.4: The two plots show the price of an allowance certificate, in an emission market with one compliance period, at different times up to expiry ($\pi = 100\text{€}$).

The numerical results are displayed in Figure 3.4. At time $t = T/2$, the allowance price depends on the cumulative emissions to date and the current level of demand, as shown in Figure 3.4a. For each fixed level of emissions $e \in [0, \bar{e}]$, $\alpha(T/2, \cdot, e)$ is increasing in d . This makes intuitive sense, since for higher levels of demand, the corresponding market emission rate is greater and consequently it is more likely that the cap will be reached. Similarly, fixing $d \in [0, \bar{\xi}]$ results in $\alpha(T/2, d, \cdot)$ being an increasing function of e . In particular, we can think of the current level of cumulative emissions determining an interval for the allowance price and the demand for electricity setting the exact price within this interval. Further, we notice that the allowance price equals the discounted penalty, if cumulative emissions exceed the cap. At the end of the compliance period, α is given by the terminal condition (3.8). Figure 3.4b reflects the digital nature of the price at this time and its independence of d .

3.4.2.2 Multiple Compliance Periods — Banking and Withdrawal

We illustrate the framework introduced in this section for the case of two consecutive compliance periods. We determine the prices of the first and second period allowance certificate as functions of demand and cumulative emissions. For $j = 2$, we solve the partial differential equation (3.12) with the terminal condition $\phi = \phi(0, e; \pi_2, \tilde{\gamma}_2)$ (following from (3.15)) and the boundary condition at $e = \bar{e}$ taking the form

$$\alpha_2(t, d, e) = e^{-r(T_2-t)}\pi_2, \quad [T_1, T_2) \times (0, \bar{\xi}) \times \{e = \bar{e}\}. \quad (3.20)$$

The problem is equivalent to the one-period pricing problem with the exception that the aggregate supply of certificates $\tilde{\gamma}_2$ depends on $E_{T_1}^1$. To overcome this difficulty we introduce a new variable $e_{T_1} \geq 0$ representing the level of the cumulative emissions at the end of the first compliance period. The price of the second period allowance certificate depends not only on the current values of $(t, d, e) \in [0, T] \times [0, \bar{\xi}] \times [0, \bar{e}]$, but also on e_{T_1} , i.e. $\alpha_2 = \alpha_2(t, d, e; e_{T_1})$. We then solve (3.12) for $j = 1$ with the terminal condition $\phi = \phi(\alpha_2(T_1, d, 0; e), e; \pi_1, \gamma_1)$ (following from (3.14)) and the previously conjectured representation $A_{T_1}^2 = \alpha_2(T_1, d, 0; e_{T_1})$. The boundary condition at $e = \bar{e}$ now reads

$$\alpha_1(t, d, e) = e^{-r(T_1-t)} (\pi_1 + e^{-r(T_2-T_1)}\pi_2), \quad [0, T_1) \times (0, \bar{\xi}) \times \{e = \bar{e}\}. \quad (3.21)$$

Figure 3.5a plots the value of the allowance certificate at time $t = T_1/2$. The effects of banking and withdrawal become very clear as the value of the certificate exceeds the penalty for a sufficiently high level of cumulative emissions. At the end of the compliance period α is given by the terminal condition (3.14), as shown in Figure

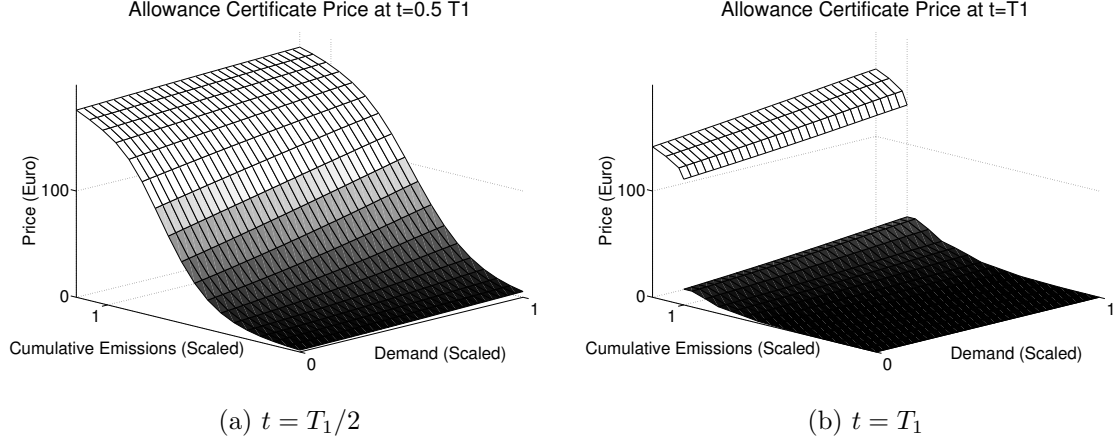


Figure 3.5: The two plots show the value of the first-period allowance certificate A^1 at different times up to compliance, in an emission market with two compliance periods, which are connected by the banking and withdrawal mechanisms ($\pi^1 = 100\text{€}$).

3.5b. Concerning the price behaviour of the second period certificate, we note that it mirrors the one-period model with the initial allocation γ replaced by $\tilde{\gamma}_2$.

3.4.2.3 Multiple Compliance Periods — Borrowing, Banking and Withdrawal

As in the previous section, we analyse the prices of allowance certificates in the two-period setting. The only difference is that now, in addition to banking and withdrawal, also the borrowing mechanism is implemented. During the second compliance period the problem is equivalent to the market that only uses the banking and withdrawal mechanisms as described in §3.2.2.1. This is the case, because in both markets the effect on the aggregate supply of certificates during the second period is the same. Suppose the market is in compliance at time $t = T_1$. Then, a number $(\gamma_1 - E_{T_1}^1)$ of certificates are banked to the second period and added to γ_2 , independently of whether borrowing is allowed or not. Otherwise, if the market is not in compliance at time $t = T_1$ a number $\min(E_{T_1}^1 - \gamma_1, \gamma_2)$ of certificates are withdrawn from γ_2 (if borrowing is not allowed) or the same number are brought forward and hence also deducted from γ_2 (if borrowing is allowed). Therefore, for $j = 2$ we solve the partial differential equation (3.12) with terminal condition $\phi = \phi(0, e; \pi_2, \tilde{\gamma}_2)$ (following from (3.15)) and, as before, obtain $\alpha_2 = \alpha_2(t, d, e; e_{T_1})$, for $(t, d, e) \in [0, T] \times [0, \bar{\xi}] \times [0, \bar{e}]$ and $e_{T_1} \geq 0$. Subsequently, we solve (3.12) for $j = 1$ and with the terminal condition $\phi = \phi(\alpha_2(T_1, d, 0; e), e; \pi_1, \gamma_1 + \gamma_2)$ (following from (3.16)), where, once again, we use

π	0	25	50	75	100	150	200
$\hat{E}_T (1 \times 10^8)$	1.32	1.23	1.20	1.18	1.17	1.16	1.15
$\hat{\sigma}_{\hat{E}} (1 \times 10^3)$	5.91	7.30	6.20	5.53	5.20	4.56	4.36

Table 3.4: Monte Carlo estimate of the mean cumulative emissions \hat{E}_T and the corresponding standard error $\hat{\sigma}_{\hat{E}}$.

Figure 3.7 plots the results of the simulation of 10^6 paths. Under business-as-usual the cumulative emissions are expected to exceed the cap. As emissions trading is introduced the market reacts by abating but initially (at a penalty level of $\pi = 25$) the cumulative emissions still exceed the cap. Further increases in the penalty (close to $\pi = 100$) lead to sufficient load shifting in order for the market to reach a state of compliance. More aggressive regulation now only leads to small reductions in the cumulative emissions; i.e. our analysis confirms the well known stylized fact that emissions trading cannot incentivise firms to reduce cumulative emissions far below the cap.

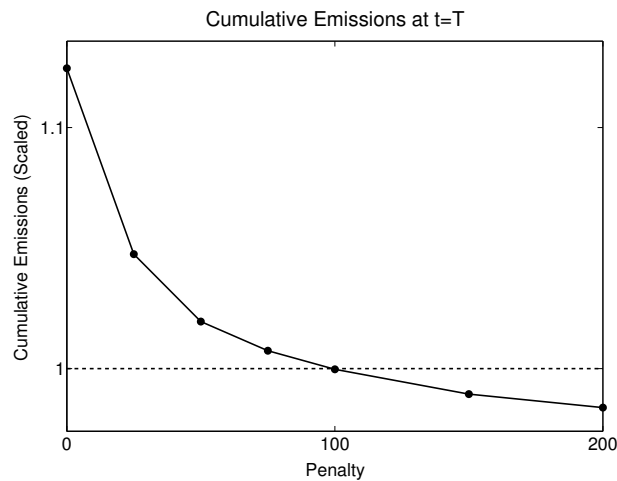


Figure 3.7: Mean of the cumulative emissions E_T for different values of the penalty π . The level of the cap is represented by the dashed line.

3.4.4 A Call Option on Emissions

For the numerical solution of the call option, we specify

$$v(t, d, e) = e^{-r(T-t)} (\pi - e^{r(T-\tau)} k)^+, \quad \text{on } [0, \tau) \times (0, \bar{\xi}) \times \{e = \bar{e}\}. \quad (3.22)$$

The condition (3.22) follows from recalling that on $[0, T) \times (0, \bar{\xi}) \times \{e = \bar{e}\}$ the value of the allowance certificate α is given by $\alpha = e^{-r(T-t)}\pi$. For the same reasons as put forward in §3.2.1 boundary conditions at $d = 0, \bar{\xi}$ and $e = 0$ are not necessary.

Because the partial differential equation (3.17) requires the allowance price as an input, it is necessary to solve (3.17) and (3.10) in parallel in order to obtain the value of the option. The numerical scheme that determines the price of the call as a function of the demand for electricity and the cumulative emissions is explained in Appendix A.1.1.3 and the resulting value surface is plotted in Figure 3.8.

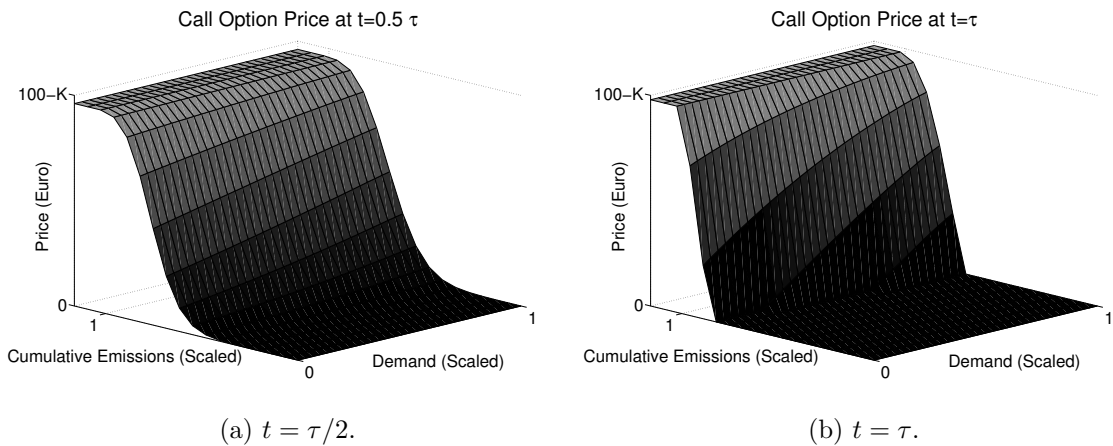


Figure 3.8: The price of a call option, with strike k , on the allowance certificate, in an emission market with one compliance period, at different times up to expiry.

3.5 Conclusion

This chapter introduced a concrete joint model for electricity spot prices and emission markets in a simple setting, with demand for electricity as the only exogenously given stochastic factor and under the assumption that firms' business-as-usual bids and also the marginal emission rates of individual generators are constant. This allowed us to concentrate on the important feedback from the cap-and-trade scheme to the allocation of resources in the market and, in particular, to illustrate how emissions trading incentivises the usage of cleaner resources instead of pollution-intensive ones. We stated the pricing FBSDE for the allowance certificate in the case of single and multi-period emission markets and in detail analysed connecting mechanisms between consecutive compliance periods. Lastly, we exemplified our model's capabilities to price derivative contracts at the example of a European call option written on the allowance certificate.

Chapter 4

Asymptotic Analysis of the Allowance Pricing Problem

This chapter analyses the allowance price asymptotically in order to gain a better understanding of its behaviour and the technical properties of the solution representing it. In particular, we provide solid evidence (short of a rigorous proof) that the conjecture we made in §3.2.1 concerning the regularity of the solution to the allowance pricing problem is in fact true. The content of this chapter forms part of the paper [Dewynne et al., 2012].

The setting throughout this chapter is the same as in Chapter 3. Specifically, we assume that demand for electricity is the only fundamental variable in the market. In order to be able to employ the asymptotic techniques that we shall use we take as a starting point the partial differential equation formulation of the allowance pricing problem. To recap, we assume the existence of a deterministic function $\alpha : [0, T] \times [0, \bar{\xi}] \times [0, \bar{e}] \mapsto \mathbb{R}$, suitably regular on $[0, T]$ to be a classical solution to the PDE

$$\begin{aligned} \frac{\partial \alpha}{\partial t} + \frac{1}{2} \sigma_d(d)^2 \frac{\partial^2 \alpha}{\partial d^2} + \mu_d(d) \frac{\partial \alpha}{\partial d} + \mu_e(d, \alpha) \frac{\partial \alpha}{\partial e} - r\alpha &= 0, & \text{on } U_T, \\ \alpha &= I_{[\gamma, \infty)}(e), & \text{on } \{t = T\} \times U, \end{aligned} \tag{4.1}$$

where $U := (0, \bar{\xi}) \times (0, \bar{e})$, $U_T := [0, T] \times U$.

In §4.1 we show that close to maturity and near the cap, to leading order, the allowance price satisfies a hyperbolic conservation law PDE. A further boundary layer analysis in §4.2 reveals that in regions where the smoothness of the leading order solution is not clear a priori, specific properties of our model result in the solution being smoothed.

Our analysis throughout this chapter is formal. In particular, we assume that all the derivatives we require exist. Further, to ease the notation we shall assume that

$\pi = 1$ and to simplify the presentation we only state the domains on which definitions are valid when the intended meaning would not be clear otherwise.

4.1 Valuation Equation Close to Maturity and near the Cap

Since we are interested in the solution of (4.1) close to compliance time and near the cap we change variables, defining

$$\epsilon\tau := T - t \quad \text{and} \quad \delta\tilde{e} := e - \gamma.$$

By a slight abuse of notation we also denote the solution to the rescaled problem by $\alpha = \alpha(\tau, d, \tilde{e})$. In order to obtain a distinguished limit in the asymptotic analysis later on it is necessary to choose $\delta = \epsilon$, so that α now satisfies the following PDE:

$$\begin{aligned} -\frac{1}{\epsilon} \frac{\partial \alpha}{\partial \tau} + \frac{1}{2} \sigma_d(d)^2 \frac{\partial^2 \alpha}{\partial d^2} + \mu_d(d) \frac{\partial \alpha}{\partial d} + \frac{1}{\epsilon} \mu_e(d, \alpha) \frac{\partial \alpha}{\partial \tilde{e}} - r\alpha &= 0, \\ \alpha(0, d, \tilde{e}) &= I_{[0, \infty)}(\tilde{e}). \end{aligned} \tag{4.2}$$

We formally pose a Poincaré expansion for α , writing

$$\alpha = \alpha_0 + \epsilon\alpha_1 + \mathcal{O}(\epsilon^2). \tag{4.3}$$

Since $\alpha - \alpha_0$ is of order epsilon, we may expand μ_e near α_0 , obtaining

$$\mu_e(d, \alpha) = \mu_e(d, \alpha_0) + \epsilon\alpha_1 \frac{\partial \mu_e}{\partial \alpha}(d, \alpha_0) + \mathcal{O}(\epsilon^2).$$

Using (4.3) and the expansion for μ_e in (4.2) and collecting terms of the same order we find that to leading order the PDE (4.2) becomes

$$\begin{aligned} \frac{\partial \alpha_0}{\partial \tau} - \mu_e(d, \alpha_0) \frac{\partial \alpha_0}{\partial \tilde{e}} &= 0, \\ \alpha_0(0, d, \tilde{e}) &= I_{[0, \infty)}(\tilde{e}). \end{aligned} \tag{4.4}$$

The PDE (4.4) is a first order nonlinear hyperbolic conservation law with Riemann initial data; its characteristic equations are

$$\begin{aligned} (\dot{\tau}(s), \dot{d}(s), \dot{\tilde{e}}(s)) &= (1, 0, -\mu_e(d, \alpha_0)), \\ \dot{\alpha}_0(\tau(s), d(s), \tilde{e}(s)) &= 0, \end{aligned} \tag{4.5}$$

where the dot denotes the derivative with respect to s and clearly the dependence on d is purely parametric. We want to identify with $s = 0$ values of α_0 on the boundary.

Therefore, we choose $\tau = s$. Then, solving (4.5) we find that, for $d(0) := d_0 \in (0, \bar{\xi})$ and $\tilde{e}(0) := \tilde{e}_0 \in (-\gamma, \bar{e} - \gamma)$, the characteristic projections are of the form

$$(\tau(s), d(s), \tilde{e}(s)) = \begin{cases} (s, d_0, -\mu_e(d_0, 0)s + \tilde{e}_0), & \text{if } \tilde{e}_0 < 0, \\ (s, d_0, -\mu_e(d_0, 1)s + \tilde{e}_0), & \text{if } \tilde{e}_0 > 0. \end{cases} \quad (4.6)$$

Moreover,

$$\alpha_0(\tau(s), d(s), \tilde{e}(s)) = I_{[0, \infty)}(\tilde{e}_0).$$

From (4.6) we obtain that

$$d\tilde{e}/d\tau = \begin{cases} -\mu_e(d_0, 0), & \text{if } \tilde{e}_0 < 0, \\ -\mu_e(d_0, 1), & \text{if } \tilde{e}_0 > 0, \end{cases}$$

which now easily allows us to plot the characteristic projections in the $\tilde{e}\tau$ plane. The result is presented in Figure 4.1. We point out that Figure 4.1 is of course a

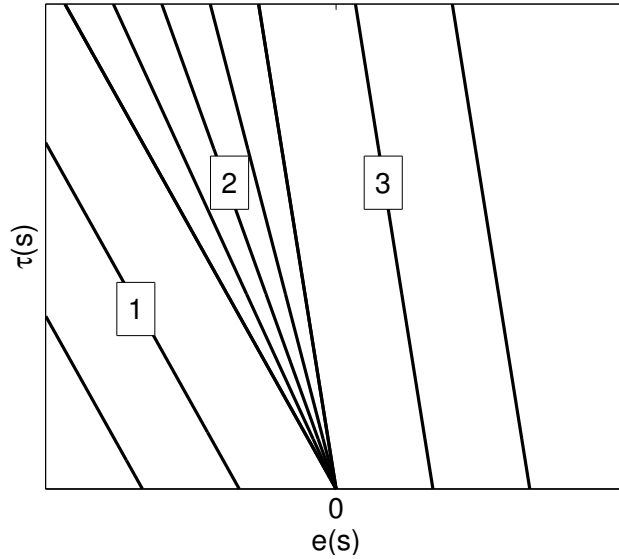


Figure 4.1: Characteristic projections associated with (4.4) in the $\tilde{e}\tau$ plane for fixed $d_0 \in (0, \bar{\xi})$; $e := \tilde{e}$.

simplified picture, as it disregards the d -dependence. It is, however, straightforward to understand the behaviour of the characteristic projections in this direction. Recall that μ_e is strictly increasing as a function of d . Therefore, for any $\tilde{e}_0 \in (-\gamma, \bar{e} - \gamma)$, the gradient of the characteristic curves emanating from the line parallel to the d -axis and passing through \tilde{e}_0 decreases as a function of d_0 .

The method of characteristics fully determines the solution in region $\boxed{1}$ (where it equals zero) and region $\boxed{3}$ (where it equals one). Inside the expansion fan, in region

[2], we aim to find a similarity solution, that is, we want to show that there exists a function $f : [0, \bar{\xi}] \times \mathbb{R} \mapsto \mathbb{R}$, such that

$$\alpha_0(\tau, d, \tilde{e}) = f(d, \eta(\tau, \tilde{e})), \quad (4.7)$$

where $\eta(\tau, \tilde{e}) := \tilde{e}/\tau$. Substituting (4.7) into (4.4) we find that $\eta(\tau, \tilde{e}) = -\mu_e(d, f(d, \eta(\tau, \tilde{e})))$. Differentiating both sides with respect to η yields

$$\partial_\eta f = -1/\partial_a \mu_e(d, \eta(\tau, \tilde{e})), \quad (4.8)$$

which is always strictly positive since $\partial_a \mu_e$ is always strictly negative. This makes intuitive sense, as η is increasing as a function of the rescaled emissions variable \tilde{e} and increasing as a function of the rescaled and reversed time variable τ (because \tilde{e} is negative inside the expansion fan). Therefore, (4.8) says that at a fixed time and for a fixed level of demand the allowance price is increasing as a function of the emissions and for fixed levels of demand and the emissions it increases as the time to compliance is increased. We conclude that the solution to (4.4) is given by

$$\alpha_0(\tau, d, \tilde{e}) = \begin{cases} 0, & \text{if } \tilde{e} < -\tau\mu_e(d, 0), \\ \mu_e(d, \cdot)^{-1}\left(-\frac{\tilde{e}}{\tau}\right), & \text{if } -\tau\mu_e(d, 0) < \tilde{e} < -\tau\mu_e(d, 1), \\ 1, & \text{if } \tilde{e} > -\tau\mu_e(d, 1). \end{cases}$$

The form of this solution is well known in the context of hyperbolic conservation laws with Riemann initial data (cf. [LeFloch, 2002]); in particular the solution is said to undergo a rarefaction wave (or ‘expansion fan’) between the two constant values zero and one.

4.2 Boundary Layer Analysis

It is clear that the leading order solution α_0 is smooth everywhere except across the sides of the expansion fan. This is where we undertake a boundary layer analysis in order to gain a better understanding of the behaviour of the solution in these two regions.

4.2.1 Layer between [1] and [2].

In this case, the transition that we are interested in takes place across the surface $\eta = -\mu_e(d, 0)$. Our aim is to analyse the solution near this surface with the help of an appropriate change of variable.

Let us set¹

$$\nu_1 \underline{y} := \tilde{e} + \underline{\mu}_{00}(d)\tau,$$

where $\underline{\mu}_{00}(d) := \mu_e(d, 0)$ and, as before, by a slight abuse of notation, we continue to write $\alpha = \alpha(\tau, d, \underline{y})$. Near the considered transition we expect the solution to be small. Therefore, we stretch it by defining

$$\nu_1 \underline{\alpha}(\tau, d, \underline{y}) := \alpha(\tau, d, \underline{y}).$$

Moreover, because α is small in the boundary layer, we expand μ_e near zero, giving

$$\mu_e(d, \nu_1 \underline{\alpha}) = \underline{\mu}_{00}(d) + \nu_1 \underline{\alpha} \underline{\mu}_{01}(d) + \mathcal{O}(\nu_1^2), \quad (4.9)$$

where $\underline{\mu}_{01}(d) := \frac{\partial \mu_e}{\partial a}(d, 0)$.

This time, in order to obtain a distinguished limit, we choose $\nu_1 = \epsilon^{1/2}$. Then $\underline{\alpha}$ satisfies the equation

$$\begin{aligned} \frac{1}{\epsilon} \frac{\partial \underline{\alpha}}{\partial \tau} &= \frac{1}{2} \sigma_d(d)^2 \left(\frac{\partial}{\partial d} + \frac{1}{\epsilon^{1/2}} \underline{\mu}'_{00}(d) \tau \frac{\partial}{\partial \underline{y}} \right)^2 \underline{\alpha} \\ &\quad + \mu_d(d) \left(\frac{\partial}{\partial d} + \frac{1}{\epsilon^{1/2}} \underline{\mu}'_{00}(d) \tau \frac{\partial}{\partial \underline{y}} \right) \underline{\alpha} + \frac{1}{\epsilon} \underline{\mu}_{01}(d) \underline{\alpha} \frac{\partial \underline{\alpha}}{\partial \underline{y}} - r \underline{\alpha}, \end{aligned} \quad (4.10)$$

with initial condition $\underline{\alpha}(0, d, \underline{y}) = \underline{y} I_{[0, \infty)}(\underline{y})$.

We formally pose a Poincaré expansion for $\underline{\alpha}$, writing

$$\underline{\alpha} = \underline{\alpha}_0 + \epsilon^{1/2} \underline{\alpha}_1 + \mathcal{O}(\epsilon).$$

Substituting this power series into (4.10), we discover that, to leading order, the solution is now characterised by

$$\begin{aligned} \frac{\partial \underline{\alpha}_0}{\partial \tau} &= \frac{1}{2} \sigma_d(d)^2 \underline{\mu}'_{00}(d)^2 \tau^2 \frac{\partial^2 \underline{\alpha}_0}{\partial \underline{y}^2} + \underline{\mu}_{01}(d) \underline{\alpha}_0 \frac{\partial \underline{\alpha}_0}{\partial \underline{y}}, \\ \underline{\alpha}_0(0, d, \underline{y}) &= \underline{y} I_{[0, \infty)}(\underline{y}). \end{aligned} \quad (4.11)$$

Because of its uniformly parabolic nature it is well known that (for sufficiently regular coefficients) equation (4.11) possesses a smooth solution away from the initial condition (cf. [Krylov, 1996]). Recall that our global leading order solution α_0 satisfies (4.4), a hyperbolic conservation law with Riemann type initial data. It is a well known fact that solutions to equations of this type are only Lipschitz continuous across the sides of the expansion fan (cf. [LeFloch, 2002]). The reason we observe a smoothing effect inside our boundary layer is due to the diffusion in the d -direction in (4.1), which, because the surface defining the edge of the expansion fan varies in d , also (partly) acts in the e -direction.

¹I am grateful to Sam Howison for pointing out this change of variable to me.

4.2.1.1 Small Time Analysis

The short τ behaviour of (4.11) is governed by a balance of the left hand side and the first term on the right hand side. Therefore, we seek a similarity solution of the form

$$\underline{\alpha}_0 \sim \tau^{\frac{3}{2}} g(\underline{z}),$$

where $\underline{z} := \underline{y}/\tau^{3/2}$. Substituting this into the equation governing the short-time behaviour yields

$$\zeta^2 g_{\underline{z}\underline{z}} = 3(g - \underline{z}g_{\underline{z}}),$$

where $\zeta^2(d) := \sigma_d(d)^2 \underline{\mu}'_{00}(d)^2$. A simple change of variable of the form

$$\underline{z} =: \frac{\zeta}{\sqrt{3}} \underline{\zeta}$$

leads to the ordinary differential equation

$$g_{\underline{\zeta}\underline{\zeta}} + \underline{\zeta}g_{\underline{\zeta}} - g = 0, \quad (4.12)$$

where by a slight abuse of notation we continue to denote the solution by $g = g(\underline{\zeta})$. Solving (4.12), we obtain

$$\begin{aligned} g &= \frac{\zeta}{\sqrt{6\pi}} \left(\underline{\zeta} \int_{-\infty}^{\underline{\zeta}} e^{-\frac{u^2}{2}} du + e^{-\frac{\underline{\zeta}^2}{2}} \right), \\ g &\sim \frac{\zeta}{\sqrt{3}} \underline{\zeta}, & \text{as } \underline{\zeta} \rightarrow \infty, \\ g &\rightarrow 0, & \text{as } \underline{\zeta} \rightarrow -\infty, \end{aligned}$$

which gives the short-time behaviour of (4.11) explicitly.

4.2.1.2 Matching of Inner and Outer Variables

For large values of \underline{y} the behaviour of (4.11) is governed by a balance of the left hand side and the second term on the right hand side. A similarity solution is given by

$$\underline{\alpha}_0 = -\frac{1}{\underline{\mu}_{01}(d)} \frac{\underline{y}}{\tau}.$$

Using the expansion (4.9), we find that

$$\underline{\mu}_{00}(d) + \underline{\mu}_{01}(d)\nu_1\underline{\alpha}_0 \sim -\frac{\tilde{\epsilon}}{\tau} = \underline{\mu}_{00}(d) - \frac{\nu_1\underline{y}}{\tau}.$$

Therefore, for small $\tilde{\epsilon}$,

$$\underline{\alpha}_0 \sim -\frac{1}{\underline{\mu}_{01}(d)} \frac{\underline{y}}{\tau},$$

so that the inner limit of the outer solution matches the outer (large \underline{y}) limit of the inner solution. Similarly it is shown that as $\underline{y} \rightarrow -\infty$ it holds that $\underline{\alpha}_0 \rightarrow 0$.

4.2.2 Layer between 2 and 3.

In this case, the transition that we are interested in takes place across the surface $\eta = -\mu_e(d, 1)$. This motivates the following change of variable:

$$\nu_2 \bar{y} := \tilde{e} + \bar{\mu}_{00}(d)\tau,$$

where $\bar{\mu}_{00}(d) := \mu_e(d, 1)$ and, as before, we continue to write $\alpha = \alpha(\tau, d, \bar{y})$. Near the considered transition we expect the solution to be close to one. We stretch it by defining

$$\nu_2 \bar{\alpha}(\tau, d, \bar{y}) := 1 - \alpha(\tau, d, \bar{y}).$$

Because α is close to one in the boundary layer, we expand μ_e near one, giving

$$\mu_e(d, 1 - \nu_2 \bar{\alpha}) = \bar{\mu}_{00}(d) - \nu_2 \bar{\alpha} \bar{\mu}'_{01}(d) + \mathcal{O}(\nu_2),$$

where $\bar{\mu}'_{01}(d) := \frac{\partial \mu_e}{\partial a}(d, 1)$.

We choose $\nu_2 = \epsilon^{1/2}$ and find that $\bar{\alpha}$ satisfies

$$\begin{aligned} \frac{1}{\epsilon} \frac{\partial \bar{\alpha}}{\partial \tau} &= \frac{1}{2} \sigma_d(d)^2 \left(\frac{\partial}{\partial d} + \frac{1}{\epsilon^{1/2}} \bar{\mu}'_{00}(d) \tau \frac{\partial}{\partial \bar{y}} \right)^2 \bar{\alpha} \\ &\quad + \mu_d(d) \left(\frac{\partial}{\partial d} + \frac{1}{\epsilon^{1/2}} \bar{\mu}'_{00}(d) \tau \frac{\partial}{\partial \bar{y}} \right) \bar{\alpha} - \frac{1}{\epsilon} \bar{\mu}_{01}(d) \bar{\alpha} \frac{\partial \bar{\alpha}}{\partial \bar{y}} + r \left(\frac{1}{\epsilon^{1/2}} - \bar{\alpha} \right), \end{aligned} \quad (4.13)$$

with $\bar{\alpha}(0, d, \bar{y}) = \bar{y} I_{[0, \infty)}(\bar{y})$.

We formally pose a Poincaré expansion for $\bar{\alpha}$, writing

$$\bar{\alpha} = \bar{\alpha}_0 + \epsilon^{1/2} \bar{\alpha}_1 + \mathcal{O}(\epsilon).$$

Substituting this power series into (4.13), we discover that, to leading order, the solution is now characterised by

$$\begin{aligned} \frac{\partial \bar{\alpha}_0}{\partial \tau} &= \frac{1}{2} \sigma_d(d)^2 \bar{\mu}'_{00}(d)^2 \tau^2 \frac{\partial^2 \bar{\alpha}_0}{\partial \bar{y}^2} - \bar{\mu}_{01}(d) \bar{\alpha}_0 \frac{\partial \bar{\alpha}_0}{\partial \bar{y}}, \\ \bar{\alpha}_0(0, d, \bar{y}) &= \bar{y} I_{[0, \infty)}(\bar{y}). \end{aligned} \quad (4.14)$$

The same comment that we made at the end of §4.2.1 also applies to (4.14). We observe diffusive effects across the right side of the expansion fan and these effects are credited to the diffusion terms in the d -direction in the original problem formulation (4.1).

In exactly the same manner as in §4.2.1.1 and §4.2.1.2 one can find similarity solutions to the equations governing (4.14) in small time and for large (and small) \bar{y} . We do not repeat the analysis here.

4.3 Conclusion

This brief chapter served two key purposes. First, we showed that close to maturity the behaviour of the allowance price certificate is governed by a hyperbolic conservation law partial differential equation. A typical feature of these types of equations (given the right monotonicities of their convection coefficient and terminal condition) is that the solution undergoes an expansion fan and is constant on either side of the fan. A boundary layer analysis revealed that the particular nature of our problem setup and in particular the effects of demand for electricity exhibit a smoothing effect across the sides of the fan, where the solution is usually merely Lipschitz continuous. This observation lends strong support to our conjecture in §3.2.1 that the solution to the allowance pricing problem is in fact smooth.

Although we restricted our analysis in the present chapter to one fundamental variable in the market, namely demand for electricity, we point out that the same results can in principle also be obtained in a higher dimensional setting.

Having gained a detailed understanding of our structural approach to modelling electricity and emission markets it is time to turn our attention to models which are increasingly realistic and therefore lend themselves to new applications.

Chapter 5

A Model with Stochastic Fuels

This chapter forms a natural, albeit not straightforward, extension of the model introduced in Chapter 3. Again we introduce a joint model for electricity and emissions trading in the spirit of Chapter 3. However, whereas in our earlier model the only stochastic cost factor that was taken into account by firms was the cost of the emission, we now include stochastic fuel prices as part of variable costs and hence as drivers of firms' bids. In this generalized market setting — in the structural spirit — we explain electricity prices as a function of demand for electricity, the prices of fuels and the cost of the emission. The prices of allowance certificates are determined by demand for electricity, the prices of fuels and the current level of the cumulative emissions to date. The main thrust of the present chapter then is to provide new and versatile tools to capture the complex dependence structure between demand for electricity, the prices of fuels and the allowance price. In particular, these tools open up a new and important application, namely the pricing of clean spread options, that is, options on the spread between the spot price of one unit of electricity and the entailed costs of production faced by a power plant. These contracts are critically important because of their usage in the theory of real options to value power plants over long time horizons (cf. [Eydeland and Wolyniec, 2003]). Moreover, because our model contains the prices of allowance certificates as an endogenous variable it permits plants to factor in the likely future cost of emissions, even in markets where emissions trading has not been implemented yet. Some of the material in this chapter forms part of the two papers [Carmona et al., 2012b] and [Carmona et al., 2012a].

As in Chapter 3, also the emphasis of the present chapter is on the introduction of a new model. Therefore, at times and in particular when we introduce the stochastic market setting in §5.1 we may be not entirely rigorous in our presentation. However, the correspondence of the objects we introduce with Chapter 2 will always be made clear and all necessary mathematical details can be found therein.

In §5.1 we introduce our model for the bid stack. The choice of an appropriate function for this price setting mechanism, which captures the subtle dependence of the electricity price upon the level of supply and the production costs, is far from trivial. Various studies have analyzed the strong links between these time series (cf. [De Jong and Schneider, 2009, Koenig, 2011]) and if the bid curve is chosen appropriately, observed stylized facts of historical data can be matched very well. Different approaches have been considered in the literature, as reviewed recently by [Carmona et al., 2012a]. In the absence of emissions trading the ones closest to our setting are [Pirrong and Jermakyan, 2008, Coulon and Howison, 2009, Aïd et al., 2012]. However, as previously in Chapter 3, an important feature of our model is the inclusion of the cost of the emission as an endogenous variable. This allows us to capture the dependence of the allowance price on fluctuations in fuel prices and extraordinary cyclicalities in demand as witnessed, for example, during recessions.

Analogously to Chapter 3, our construction of the market bid stack in §5.1 allows us to infer the rate at which the market emits, which in turn is translated into a process describing the cumulative emissions in the market.

In §5.2 we formulate the allowance pricing problem. In particular, we present the model introduced in §2.2 in the context of our current market setting.

In §5.3 we dedicate ourselves to clean spread options. These types of options are used in commodity and energy markets to encapsulate the profitability of a production process by comparing the price of a refined product to the costs of production including, but not limited to, the prices of the inputs to the production process. When the output commodity is electric power, such spread options are called spark spreads if the electricity is produced from natural gas, and dark spreads if the electricity is produced from coal. Since both processes are the sources of greenhouse gas emissions (in higher quantities for the latter than the former) it is natural in this case to include the price of allowance certificates as part of the costs entering the production process and in this case the option contracts are known as clean dark and spark spreads. As mentioned above, the inclusion of the cost of emissions may even be desirable in a market in which no cap-and-trade system is operating yet, but where the establishment of such a system is likely in the future.

In industry, Margrabe's classical spread option formula (cf. [Margrabe, 1978]) is still used widely and has been extended by various authors (see [Carmona and Durrleman, 2003] for an overview) including to the three commodity case, as required for the pricing of clean spreads (cf. [Alos et al., 2011]). Closed

form formulae for spread options in the structural setting (without factoring in the cost of the emission) are obtained in [Carmona et al., 2012a] and [Aid et al., 2012], whereby in the case of the latter the closed form comes at the expense of a fixed merit order. The model we present in §5.3 follows naturally from §2.4 and lends itself to the above mentioned applications because of its ability to capture the complex relationships between the fundamental variables in the market and the prices of electricity and allowances.

The purpose of §5.4 is twofold. First, we give a concrete example of a two-fuel bid stack model and the corresponding market emission rate, first introduced in §5.1, and the processes describing the fundamental variables in the market in order to make them amenable to numerical computations. Subsequently, we thoroughly illustrate the dependence of the now explicitly specified bid stack and market emission rate on demand for electricity and the prices of fuels and the allowance certificate. Second, with a dedicated series of case studies, we illustrate the effects that our model has when applied to the aforementioned task of pricing clean spread options and valuing power plants.

5.1 From Electricity Markets to Carbon Emissions

In this section we develop our extended approach to modelling the interaction between electricity and emission markets. We introduce the state variables and the key parameters that influence the price formation of allowance certificates and the electricity spot price. An important part is played by the fuel bid curves, which represent the bids of all generators using a common production technology. Based on the definition of the fuel bid curves we construct the electricity market bid stack and, using an equilibrium assumption which relates supply to demand, we derive the market emission rate.

5.1.1 Market Set-Up

In Chapter 3, we analysed in detail different market design features of emission markets, for example the difference between single and multi-period emission markets. In the current chapter it is not our goal to explore additional design features of the emission market, but instead we are concerned with an increasingly realistic extension of our approach to modelling the bid stack and the market emission rate, as introduced in Chapter 3. Therefore, we focus on a single compliance period only and represent its duration (assumed to be one year) by the finite time interval $[0, T]$.

The market we operate in is similar to the one introduced in §3.1.1: in response to demand from consumers, firms generate electricity using a variety of production technologies. Thereby, their actions are subject to the regulations associated with the implementation of a cap-and-trade scheme. Here is the key difference to our analysis in Chapter 3: previously, firms' bids were only a function of the price of the emissions and hence constant in the absence of emissions trading. Now we allow firms' variable costs and therefore their bids to depend on fuel prices. As a result, even under business-as-usual, bids are no longer constant. Specifically, we consider coal and gas powered plants, so that business-as-usual bids are driven by the spot price of either coal or gas. In principle, the results presented in this chapter could easily be formulated in the setting of an arbitrary number of technologies (cf. [Carmona et al., 2012a] and [Carmona et al., 2012b]). For the benefit of concreteness and a more accessible presentation, however, we restrict our analysis to just coal and gas as production technologies. In this setting, we aim to determine the market prices of electricity and allowance certificates as a function of the aggregate forces in the market, that is, of demand for electricity, cumulative emissions and the prices of coal and gas.

To simplify the notation throughout the text we introduce the set $J := \{c, g\}$ to index the two technologies in the market.¹ In line with earlier notation, we assume that demand for electricity is represented by a stochastic process D (measured in MW) and that D is bounded:

$$0 \leq D_t \leq \bar{\xi}, \quad \text{for } t \in [0, T],$$

where $\bar{\xi} \geq 0$ denotes the market capacity. The aggregate amount of electricity supplied by plants relying on technology $j \in J$ is represented by stochastic processes Ξ^j (measured in MW) and we assume that these processes are bounded: for $j \in J$ and $t \in [0, T]$,

$$0 \leq \Xi_t^j \leq \bar{\xi}_j$$

where $\bar{\xi}_j \geq 0$ denotes the capacity of all firms relying on technology j . It follows that

$$\bar{\xi} = \bar{\xi}_c + \bar{\xi}_g.$$

Moreover, the market administrator ensures that demand and supply are matched on a daily basis, so that

$$D_t = \Xi_t^c + \Xi_t^g, \quad \text{for } t \in [0, T]. \quad (5.1)$$

¹Although J was used to index different compliance periods in a multi-period emission market in Chapter 3, we redefine it at this point and, throughout this chapter, use it exclusively to represent the different production technologies.

Further, we assume that the prices of coal and gas are observable in the market and that their fluctuations are described by stochastic processes S^j , $j \in J$.

As before, cumulative emissions are described by a bounded stochastic process E (measured in tonnes of CO₂); specifically,

$$0 \leq E_t \leq \bar{e}, \quad \text{for } t \in [0, T],$$

where $\bar{e} \geq 0$.

The cap imposed on emissions is denoted by a constant $\gamma \in [0, \bar{e}]$ (measured in tonnes of CO₂) and the penalty to be paid in the event of non-compliance by $\pi \geq 0$. In light of the notation introduced in Chapter 2 we point out that throughout this chapter $d = 3$, so that the process X representing the fundamental variables takes values in \mathbb{R}^3 . In particular, X is identified with the demand process and the processes describing the prices of fuels, i.e. $X = (D, S^c, S^g)$. Lastly, as before, $r \geq 0$ represents the constant risk-free rate.

5.1.2 Price Setting in Electricity Markets: A Stochastic Bid Stack Model

In our setting heterogeneous firms generate and supply electricity. In this process heterogeneity amongst firms is induced in two ways: first, the market of electricity providers is segmented into coal and gas driven power plants. Therefore, the costs of production of plants that rely on different fuel types may differ considerably and are strongly linked to the price of the fuel used in the power generation process. Second, power plants exhibit variation in their heat and emission rates — measures of their fuel and emission efficiency. Therefore, also the costs of production of plants using the same technology can vary, for example, as old generators may have higher heat rates (and therefore be less efficient) than new units.

Each firm now decides at which price to bid its electricity in the day-ahead auction. The important relationships between bids and costs and the cost structure of the individual firm are summarised in the following assumption:

Assumption 8. Bids are driven by production costs. Furthermore,

(A8.1) costs depend on fuel and emission prices and firm-specific heat and emission rates only.

(A8.2) Firms' marginal costs are strictly increasing as a function of supply.

Bids are submitted to the central market administrator and we still take for granted that Assumption 5 concerning the merit order arrangement is in force. It is not our aim, however, to provide a mathematical model that explains the aggregation of individual bids or captures observed features such as strategic bidding. Instead, for each technology, we approximate the aggregate of generators using this technology with a continuum of generators and construct a corresponding bid curve that satisfies Assumption 8. The construction of this bid curve is done in several steps. First, we define the heat rate (measured in MMBtu per MWh) of a continuum of firms.

Definition 3. The **heat rate** (the number of units of fuel needed to produce one unit of electricity) of the continuum of firms relying on fuel type $j \in J$ is given by

$$h_j(\xi) : [0, \bar{\xi}_j] \mapsto \mathbb{R}_+,$$

where $h_j \in C^1(0, \bar{\xi}_j)$ and $dh_j/d\xi > \epsilon$, for some $\epsilon > 0$.

Second, we define the marginal emission rate (measured in tonnes of CO₂ per MWh) of a continuum of firms.

Definition 4. The **marginal emission rate** (the number of units of CO₂ emitted during the production of one unit of electricity) of the continuum of firms relying on fuel type $j \in J$ is given by the bounded function

$$e_j(\xi) : [0, \bar{\xi}_j] \mapsto \mathbb{R}_+,$$

where $e_j \in C^1(0, \bar{\xi}_i)$ and $de_j/d\xi > 0$.

Third, we assume that coal and gas generators' bid levels are linear in the allowance and fuel price. The resulting *fuel bid curves* are denoted by b_j , $j \in J$, and they are given by

$$b_j(\xi, a, s) := e_j(\xi)a + h_j(\xi)s, \quad \text{for } (\xi, a, s) \in [0, \bar{\xi}_j] \times \mathbb{R}_+ \times \mathbb{R}_{++}. \quad (5.2)$$

Note that with this definition, for $j \in J$, the bid curve b_j satisfies Assumption 8.

In order to simplify the notation below, we introduce the generalized (left-continuous) inverse of the function $\xi \mapsto b_j(\cdot, a, s)$, for $j \in J$ and $(a, s) \in \mathbb{R}_+ \times \mathbb{R}_{++}$, and denote it by \tilde{b}_j . In particular, adhering to the standard convention $\sup \emptyset = -\infty$, this is given by

$$\tilde{b}_j(p, a, s) := 0 \vee \sup \{ \xi \in [0, \bar{\xi}_j] : b_j(\xi, a, s) < p \}, \quad (5.3)$$

for $(p, a, s) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_{++}$. To further ease the notation we define, for $j \in J$,

$$\underline{b}_j(a, s) := b_j(0, a, s) \quad \text{and} \quad \bar{b}_j(a, s) := b_j(\bar{\xi}_j, a, s).$$

We easily see now that, for $(a, s) \in \mathbb{R}_+ \times \mathbb{R}_{++}$,

$$\tilde{b}_j(p, a, s) = \begin{cases} 0, & \text{if } p \in (-\infty, \underline{b}_j(a, s)], \\ \bar{\xi}_j, & \text{if } p \in (\bar{b}_j(a, s), \infty) \end{cases}$$

and

$$0 < \tilde{b}_j(p, a, s) \leq \bar{\xi}_j, \quad \text{if } p \in (\underline{b}_j(a, s), \bar{b}_j(a, s)].$$

For electricity prices below $\underline{b}_j(a, s)$, no capacity from the j th technology will be available. Similarly, once all resources from a technology are exhausted, increases in the electricity price will not lead to further production units being brought online. So defined, the inverse function \tilde{b}_j maps a given price of electricity and the prices of fuel j and the emission to the amount of electricity supplied by generators relying on this fuel type.

Drawing upon the merit order assumption we construct the market bid stack b in the following proposition.

Proposition 10. *Setting $\mathbf{s} := (s_c, s_g)$ and with b_j , $j \in J$, as defined above, the market bid stack b is given by the left-continuous function*

$$b(\xi, a, \mathbf{s}) := \min_{j \in J} \underline{b}_j(a, s_j) \vee \sup \left\{ p \in \mathbb{R} : \tilde{b}_c(p, a, s_c) + \tilde{b}_g(p, a, s_g) < \xi \right\}, \quad (5.4)$$

for $(\xi, a, \mathbf{s}) \in [0, \bar{\xi}] \times \mathbb{R}_+ \times \mathbb{R}_{++}^2$ and with the convention $\sup \emptyset = -\infty$.

Proof. We define \tilde{b} by

$$\tilde{b}(p, a, \mathbf{s}) := \tilde{b}_c(p, a, s_c) + \tilde{b}_g(p, a, s_g), \quad \text{for } (p, a, \mathbf{s}) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_{++}^2.$$

Then, by the definition of \tilde{b}_j , for fixed $(a, s) \in \mathbb{R}_+ \times \mathbb{R}_{++}$, the function $\tilde{b}(\cdot, a, \mathbf{s})$ is a non-decreasing map from the electricity price to the corresponding amount of electricity generated by the market. For each fixed set of fuel and emission prices $(a, s) \in \mathbb{R}_+ \times \mathbb{R}_{++}$ the function b is now defined as the generalized (left-continuous) inverse of the map $p \mapsto \tilde{b}(\cdot, a, \mathbf{s})$. As the left-continuous inverse of a non-decreasing map, $b(\cdot, a, \mathbf{s})$ is strictly increasing, for $(a, s) \in \mathbb{R}_+ \times \mathbb{R}_{++}$. Therefore, the definition of b as market bid stack makes sense and we immediately obtain the result in the proposition. \square

We emphasize the different roles played by the first variable (supply) and all subsequent variables (factors driving bid levels) of the bid stack function b . Due to the inelasticity assumption, the level of demand fully determines the quantity of electricity that is being supplied; the role played by the variables a , s_c and s_g is to determine the merit order arrangement and the price at which firms bid.

Remark 11. The importance of the above construction of the bid stack as a function of, amongst other variables, coal and gas prices and in particular the fact that the two technologies are allowed to swap places in the merit order (caused by changes in fuel or emission prices), is supported by prominent developments in recent gas price data: new shale gas discoveries have led to a dramatic drop in US natural gas prices in recent years. In particular, prices have dropped from a high of over \$13 in 2008 to under \$3 in January 2012. This decrease in variable costs has immediately been translated into lower bids from gas generators and resulted in gas being placed before coal in the merit order.

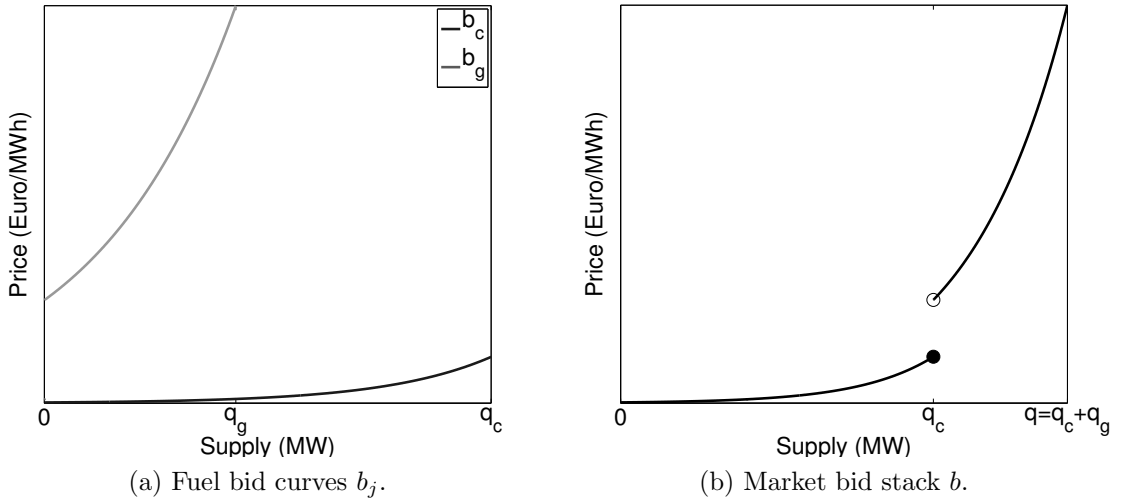


Figure 5.1: A schematic of a pair of coal and gas bid curves (b_c and b_g) and the resulting market bid stack b ; $q := \bar{\xi}$.

At any point in time the market electricity price is now given by evaluating the bid stack function b at the stochastic values of demand, the allowance price and the fuel prices, that is,

$$P_t = b(D_t, A_t, S_t^c, S_t^g), \quad \text{for } t \in [0, T].$$

In the following lemma we prove a technical property of the market bid stack.

Lemma 6. *The market bid stack b , as defined in (5.4) is at most linearly growing. In particular, setting $\mathbf{s} := (s_c, s_g)$ as before, there exists a constant $C > 0$ such that*

$$b(\xi, a, \mathbf{s}) \leq C(1 + |a| + |s_c| + |s_g|), \quad \text{for } (a, \mathbf{s}) \in \mathbb{R}_+ \times \mathbb{R}_{++}^2.$$

Proof. Since, for $(a, \mathbf{s}) \in \mathbb{R}_+ \times \mathbb{R}_{++}$, the function $b(\cdot, a, \mathbf{s})$ is strictly increasing and only defined on $[0, \bar{\xi}]$, it is sufficient to prove that $b(\bar{\xi}, a, \mathbf{s})$ is linearly growing in a and \mathbf{s} .

Notice first that, for $j \in J$ and some $C_j > 0$, it holds that

$$\begin{aligned} \tilde{b}_j(p, a, s_j) &\geq 0 \vee \sup \{ \xi \in [0, \bar{\xi}_j] : C_j(1 + |\xi| + |a| + |s_j|) < p \} \\ &= \left(0 \vee \frac{p}{C_j} - 1 - |a| - |s_j| \right) \wedge \bar{\xi}_j, \end{aligned}$$

for $(p, a, s_j) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_{++}$. Therefore, for some constants $C'_j > 0$, $j \in J$, we find that

$$\begin{aligned} b(\bar{\xi}, a, \mathbf{s}) &\leq \sup \left\{ p \in \mathbb{R} : \left(0 \vee \frac{p}{C_c} - 1 - |a| - |s_c| \right) \wedge \bar{\xi}_c \right. \\ &\quad \left. + \left(0 \vee \frac{p}{C_g} - 1 - |a| - |s_g| \right) \wedge \bar{\xi}_g < \bar{\xi} \right\}. \quad (5.5) \end{aligned}$$

Considering the two cases of one of the two summands in (5.5) being equal to either its minimum or maximum and the case of both summands taking values in between their respective minimum and maximum, one sees easily that the supremum is always at most linearly growing in a and \mathbf{s} . \square

5.1.3 Deriving CO₂ Emissions: The Market Emission Rate

In analogue to §3.1.3, we now show how the concept of the electricity bid stack, as introduced in §5.1.2, allows us to deduce the market emission rate μ_e . In the present setting the emission rate of the entire market is obtained by summing the emission rates of coal and gas generators. In doing so, it is important to only take those generators into account that are scheduled to run in the current market environment, that is, for the current level of supply and current allowance and fuel prices.

Proposition 11. *With the definition of the electricity bid stack in (5.4) and with the definition $\mathbf{s} := (s_c, s_g)$, the market emission rate μ_e , for a given level of supply, allowance price and fuel prices, is given by*

$$\mu_e(\xi, a, \mathbf{s}) := \int_0^{\tilde{b}_c(b(\xi, a, \mathbf{s}), a, s_c)} e_c(u) \, du + \int_0^{\tilde{b}_g(b(\xi, a, \mathbf{s}), a, s_g)} e_g(u) \, du, \quad (5.6)$$

for $(\xi, a, \mathbf{s}) \in [0, \bar{\xi}] \times \mathbb{R}_+ \times \mathbb{R}_{++}^2$ and with \tilde{b}_j , $j \in J$, as defined by (5.3).

Proof. The marginal emission rate of the continuum of generators relying on technology j is given by e_j , for $j \in J$. By definition, for $j \in J$, the function \tilde{b}_j maps given prices of electricity, fuel j and the allowance certificate to the amount of electricity supplied by generators using technology j . Since the market price of electricity is given by b , the upper limit of integration follows. \square

The following lemma proves some technical properties of the market emission rate.

Lemma 7. *The market emission rate μ_e , as defined by (5.6), satisfies the following:*

(L.1) For $(\xi, \mathbf{s}) \in [0, \bar{\xi}] \times \mathbb{R}_{++}^2$, the map $a \mapsto \mu_e(\xi, \cdot, \mathbf{s})$ is

- (i) non-increasing and
- (ii) Lipschitz continuous.

(L.2) μ_e is bounded.

Proof. We note that the generalized inverses of the fuel bid curves, \tilde{b}_j , $j \in J$, and also the market bid stack b are not everywhere continuously differentiable. However, they are continuously differentiable on appropriate subsets that partition the domain. Therefore, when we differentiate these functions below it must be born in mind that this is always only done on a subset of the domain, where this operation is permitted. Further, to ease the notation throughout the proof we define the functions $\xi_j(\xi, a, \mathbf{s}) := \tilde{b}_j(b(\xi, a, \mathbf{s}), a, s_j)$, for $j \in J$ and $(\xi, a, \mathbf{s}) \in [0, \bar{\xi}] \times \mathbb{R}_+ \times \mathbb{R}_{++}^2$.

(L.1) (i) Applying Leibniz's integral rule to (5.6), we obtain

$$\frac{\partial \mu_e}{\partial a}(\xi, a, \mathbf{s}) = e_c(\xi_c) f_c(\xi, a, \mathbf{s}) + e_g(\xi_g) f_g(\xi, a, \mathbf{s}), \quad (5.7)$$

for $(\xi, a, \mathbf{s}) \in [0, \bar{\xi}] \times \mathbb{R}_+ \times \mathbb{R}_{++}^2$, where, for $j \in J$,

$$f_j(\xi, a, \mathbf{s}) := \frac{\partial \tilde{b}_j}{\partial p}(b(\xi, a, \mathbf{s}), a, s_j) \frac{\partial b}{\partial a}(\xi, a, \mathbf{s}) + \frac{\partial \tilde{b}_j}{\partial a}(b(\xi, a, \mathbf{s}), a, s_j).$$

If, for $(\xi, a, \mathbf{s}) \in [0, \bar{\xi}] \times \mathbb{R}_+ \times \mathbb{R}_{++}^2$, it holds that, for some $j \in J$, $0 < \tilde{b}_j < \bar{\xi}_j$, then for the same $j \in J$, we find that

$$\frac{\partial \tilde{b}_j}{\partial p}(b(\xi, a, \mathbf{s}), a, s_j) = \left(\frac{\partial b_j}{\partial u}(\xi_j, a, s_j) \right)^{-1}$$

and

$$\frac{\partial \tilde{b}_j}{\partial a}(b(\xi, a, \mathbf{s}), a, s_j) = -e_j(\xi_j) \left(\frac{\partial b_j}{\partial u}(\xi_j, a, s_j) \right)^{-1},$$

where $b_j := b_j(u, a, s)$ for $(u, a, s) \in [0, \bar{\xi}_j] \times \mathbb{R}_+ \times \mathbb{R}_{++}$. Moreover, if, for some $j \in J$, either $\tilde{b}_j = 0$ or $\tilde{b}_j = \bar{\xi}_j$, then $f_j = 0$ for the same $j \in J$.

Further, if, for $(\xi, a, \mathbf{s}) \in [0, \bar{\xi}] \times \mathbb{R}_+ \times \mathbb{R}_{++}^2$, it holds that, for all $j \in J$, $0 < \tilde{b}_j < \bar{\xi}_j$, then (after some lengthy calculations) we find that

$$\frac{\partial b}{\partial a}(\xi, a, \mathbf{s}) = \frac{e_c(\xi_c)\partial_u b_g(\xi_g, a, s_g) + e_g(\xi_g)\partial_u b_c(\xi_c, a, s_c)}{\partial_u b_c(\xi_c, a, s_c) + \partial_u b_g(\xi_g, a, s_g)}.$$

Moreover, it is clear that if $0 < \tilde{b}_j < \bar{\xi}_j$ and either $\tilde{b}_{J \setminus j} = 0$ or $\tilde{b}_{J \setminus j} = \bar{\xi}_{J \setminus j}$, then $\partial_a b = \partial_a b_j$.

It now follows that, for $j \in J$,

$$f_j(\xi, a, \mathbf{s}) = \begin{cases} 0, & \text{if } b(\xi, a, \mathbf{s}) \in \mathcal{B}_1(a, \mathbf{s}), \\ \frac{e_{J \setminus j}(\xi_{J \setminus j}) - e_j(\xi_j)}{\partial_u b_c(\xi_c, a, s_c) + \partial_u b_g(\xi_g, a, s_g)}, & \text{if } b(\xi, a, \mathbf{s}) \in \mathcal{B}_2(a, \mathbf{s}), \\ 0, & \text{if } b(\xi, a, \mathbf{s}) \in \mathcal{B}_3(a, \mathbf{s}), \end{cases} \quad (5.8)$$

where

$$\begin{aligned} \mathcal{B}_1(a, \mathbf{s}) &:= (-\infty, \max_{j \in J}(\underline{b}_j(a, s_j))), \\ \mathcal{B}_2(a, \mathbf{s}) &:= (\max_{j \in J}(\underline{b}_j(a, s_j)), \min_{j \in J}(\bar{b}_j(a, s_j))), \\ \mathcal{B}_3(a, \mathbf{s}) &:= (\min_{j \in J}(\bar{b}_j(a, s_j)), \infty). \end{aligned}$$

Therefore, from (5.7) and (5.8), we obtain

$$\frac{\partial \mu_e}{\partial a}(\xi, a, \mathbf{s}) = \begin{cases} 0, & \text{if } b(\xi, a, \mathbf{s}) \in \mathcal{B}_1(a, \mathbf{s}), \\ -\frac{(e_c(\xi_c) - e_g(\xi_g))^2}{\partial_u b_c(\xi_c, a, s_c) + \partial_u b_g(\xi_g, a, s_g)}, & \text{if } b(\xi, a, \mathbf{s}) \in \mathcal{B}_2(a, \mathbf{s}), \\ 0, & \text{if } b(\xi, a, \mathbf{s}) \in \mathcal{B}_3(a, \mathbf{s}), \end{cases} \quad (5.9)$$

which is clearly non-positive.

- (ii) The Lipschitz continuity follows directly from (5.9) and Definition 4 once we observe that by Definition 3 and the definition of the fuel bid curves in (5.2), the denominator is strictly positive and bounded away from zero.

(L.2) Boundedness of the function μ_e follows from the two facts that, for $j \in J$, e_j is bounded and $\tilde{b}_j \leq \bar{\xi}_j$, for all $(\xi, a, \mathbf{s}) \in [0, \bar{\xi}] \times \mathbb{R}_+ \times \mathbb{R}_{++}$.

□

Remark 12. As in Chapter 3, we acknowledge immediately that μ_e as defined above does not satisfy the strict monotonicity criteria required by Assumption (A2.2) for the existence proof of the pricing FBSDE in Chapter 2. Our suggested remedy here is the same as before and we refer the reader to Remark 9.

5.2 The Allowance Pricing Problem

In this section we formulate the problem of determining the arbitrage-free price of the allowance certificate in the modelling context of the current chapter. As mentioned before, we do this only in the case of a single compliance period and focus on the extension to stochastic fuel prices.

The formulation of the pricing model follows immediately from the results in Chapter 2. As mentioned before, we set $d = 3$ and identify the process X , representing the fundamental factors, with the demand process and the processes representing the prices of the fuels, that is, $X = (D, S^c, S^g)$. It follows that D , S^c and S^g evolve according to Itô diffusions; specifically, for measurable functions $\mu_d : [0, T] \times [0, \bar{\xi}] \mapsto \mathbb{R}$ and $\sigma_d : [0, \bar{\xi}] \mapsto \mathbb{R}$, satisfying Assumption (A2.1), and under the measure \mathbb{Q} , the demand process satisfies the stochastic integral equation

$$D_t = d_0 + \int_0^t \mu_d(u, D_u) du + \int_0^t \sigma_d(D_u) dW_u^1, \quad \text{for } t \in [0, T], \quad (5.10)$$

where $d_0 \in (0, \bar{\xi})$. Similarly, for $j \in J$, we assume the existence of measurable functions $\mu_j : \mathbb{R}_{++} \mapsto \mathbb{R}$ and $\sigma_j : \mathbb{R}_{++} \mapsto \mathbb{R}$, satisfying Assumption (A2.1), such that, under the measure \mathbb{Q} , the process S^j satisfies the stochastic integral equation

$$S_t^j = s_0^j + \int_0^t \mu_j(S_u^j) du + \int_0^t \sigma_j(S_u^j) dW_u^j, \quad \text{for } t \in [0, T], \quad (5.11)$$

where $s_0^j \in \mathbb{R}_{++}$ and $\langle W^2, W^3 \rangle_t = \int_0^t \rho du$, for some constant $\rho \in [-1, 1]$.

Cumulative emissions are measured from the beginning of the compliance period when $t = 0$. Subsequently, given our Walrasian equilibrium assumption (3.1), they are determined by evaluating the market emission rate μ_e , derived in Proposition 11, at the stochastic values of demand and the allowance price and integrating up to the current time. Consequently, in the current setting, the stochastic integral equation (2.6) now reads

$$E_t = e_0 + \int_0^t \mu_e(D_u, A_u, S_u^c, S_u^g) du, \quad \text{for } t \in [0, T]. \quad (5.12)$$

Notice that also the concrete model we propose in this chapter implies that μ_e is non-negative. Hence again, the process E is non-decreasing, as it should be given that it represents a cumulative quantity.

Since we are working in the setting of a single compliance period, the value of the allowance certificate at compliance time is determined by the terminal condition ϕ with $\tilde{A}_T = 0$, that is,

$$A_T = \phi(0, E_T) = \pi I_{[\gamma, \infty)}(E_T). \quad (5.13)$$

The following proposition now summarizes the main result in this section.

Proposition 12. *Assuming that demand for electricity and the prices of coal and gas are the only fundamental factors in the economy and that the market emission rate is given by (5.6), the price process A of an allowance certificate in an emission market with one compliance period is described by the unique solution $((D, S^c, S^g, E), A, Z^a) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$ to the following FBSDE: for $t \in [0, T]$,*

$$\left\{ \begin{array}{l} D_t = d_0 + \int_0^t \mu_d(u, D_u) du + \int_0^t \sigma_d(D_u) dW_u^1 \\ S_t^c = s_0^c + \int_0^t \mu_c(S_u^c) du + \int_0^t \sigma_c(S_u^c) dW_u^2 \\ S_t^g = s_0^g + \int_0^t \mu_g(S_u^g) du + \int_0^t \sigma_g(S_u^g) dW_u^3 \\ E_t = e_0 + \int_0^t \mu_e(D_u, S_u^c, S_u^g, A_u) du \\ A_t = \pi I_{[\gamma, \infty)}(E_T) - r \int_t^T A_u du - \int_t^T \langle e^{ru} Z_u^a, dW_u \rangle. \end{array} \right. \quad (5.14)$$

Proof. The problem formulation is obvious from the discussion above. Existence and uniqueness of a solution to equation (5.14) in the relevant spaces follow immediately from Theorem 1 and Remark 3 with $d = 3$, $X = (D, S^c, S^g)$, $\tilde{A}_T = 0$ and from Lemma 7 (and strictly speaking, also in this case, using a modification of μ_e as suggested in Remark 9 to satisfy the strict monotonicity assumption that is necessary for Theorem 1). \square

As we did in Chapter 3, we conjecture that there exists a deterministic function $\alpha : [0, T] \times [0, \bar{\xi}] \times \mathbb{R}_{++}^2 \times [0, \bar{e}] \mapsto \mathbb{R}$, suitably regular on $[0, T)$ to be a classical solution to the PDE

$$\begin{aligned} \mathcal{N}\alpha &= 0, & \text{on } U_T, \\ \alpha &= \pi I_{[\gamma, \infty)}(e), & \text{on } \{t = T\} \times U, \end{aligned} \quad (5.15)$$

where $U := (0, \bar{\xi}) \times \mathbb{R}_{++}^2 \times (0, \bar{e})$, $U_T := [0, T) \times U$ and

$$\begin{aligned} \mathcal{N} := & \frac{\partial}{\partial t} + \frac{1}{2} \sigma_d^2(d) \frac{\partial^2}{\partial d^2} + \frac{1}{2} \sigma_c^2(s_c) \frac{\partial^2}{\partial s_c^2} + \rho \sigma_c(s_c) \sigma_g(s_g) \frac{\partial^2}{\partial s_c \partial s_g} + \frac{1}{2} \sigma_g^2(s_g) \frac{\partial^2}{\partial s_g^2} \\ & + \mu_d(t, d) \frac{\partial}{\partial d} + \mu_c(s_c) \frac{\partial}{\partial s_c} + \mu_g(s_g) \frac{\partial}{\partial s_g} + \mu_e(d, \cdot) \frac{\partial}{\partial e} - r. \end{aligned}$$

Under the imposed regularity assumption, a simple application of Itô's formula then shows that $A_t = \alpha(t, D_t, S_t^c, S_t^g, E_t)$, for $t \in [0, T)$, and therefore that the PDE (5.15) indeed characterizes the solution to the backward equation of the FBSDE (5.14). As

mentioned in Chapter 3, we are not aware of any results in the literature that would guarantee the value function α to be sufficiently regular for the above argument to be rigorous, but we conjecture it with some confidence in light of the asymptotic results in Chapter 4.

In addition to the terminal condition suitable boundary conditions have to be supplied. These depend on the specification of the coefficients of the PDE and we postpone the issue to the appendix §A.2, where the numerical solution of the problem is discussed in detail.

5.3 Clean Spread Option Valuation

In this section we revisit the pricing of derivatives in the structural setting. The extension of our market to include stochastic fuel prices now allows us to consider more involved contracts than the simple European call option in Chapter 3. In particular, we discuss clean spread options. These contracts are important because of their usage in the theory of real options to value power plants. With this application in mind, we are concerned with spread options whose payoff is defined to be the positive part of the difference between the market spot price of electricity and the costs of production faced by a particular power plant to generate one MWh. We assume that the latter are composed of constant fixed costs $k \geq 0$ and variable fuel and emission costs. If coal (gas) is the fuel featured in the payoff then the option is known as a *clean dark (spark) spread*. For $j \in J$, we denote by $\tilde{h}_j > 0$ and $\tilde{e}_j > 0$ the heat and the marginal emission rates, which define the option contract. Then dark and spark spread options with maturity $\tau \in [0, T]$ are defined by the payoffs

$$\left(P_\tau - \tilde{h}_c S_\tau^c - \tilde{e}_c A_\tau - k\right)^+ \quad \text{and} \quad \left(P_\tau - \tilde{h}_g S_\tau^g - \tilde{e}_g A_\tau - k\right)^+, \quad (5.16)$$

respectively.

Since, as we mentioned, spread options are frequently traded to either value or hedge physical assets (generating units) the heat and emission rates that feature in the option payoff are usually in line with the efficiency of power plants in the market. Therefore, we impose the restrictions,

$$\tilde{h}_j \in [h_j(0), h_j(\bar{\xi}_j)] \quad \text{and} \quad \tilde{e}_j \in [e_j(0), e_j(\bar{\xi}_j)].$$

Remark 13. While spread option contracts are often written on forwards, we consider spread options on spot prices, as required by the applications of valuing and hedging physical assets, which motivate our analysis.

The following proposition summarises the clean spread pricing problem for the example of a clean dark spread contract.

Proposition 13. *Assuming that demand for electricity and the prices of coal and gas are the only fundamental factors in the economy and that the market emission rate is given by (11), the price process V^c of a clean dark spread option is described by the unique solution $((D, E, S^c, S^g), A, V, (Z^a, Z^v)) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2$ to the following FBSDE: for $t \in [0, \tau]$,*

$$\left\{ \begin{array}{l} D_t = d_0 + \int_0^t \mu_d(u, D_u) \, du + \int_0^t \sigma_d(D_u) \, dW_u^1 \\ S_t^c = s_0^c + \int_0^t \mu_c(S_u^c) \, du + \int_0^t \sigma_c(S_u^c) \, dW_u^2 \\ S_t^g = s_0^g + \int_0^t \mu_g(S_u^g) \, du + \int_0^t \sigma_g(S_u^g) \, dW_u^3 \\ E_t = e_0 + \int_0^t \mu_e(D_u, A_u, S_u^c, S_u^g) \, du \\ A_t = \alpha(\tau, \omega, E_\tau) - r \int_t^\tau A_u \, du - \int_t^\tau e^{ru} Z_u^a \, dW_u \\ V_t = \left(b(D_\tau, A_\tau, S_\tau^c, S_\tau^g) - \tilde{h}_c S_\tau^c - \tilde{e}_c A_\tau - k \right)^+ - r \int_t^\tau V_u \, du - \int_t^\tau e^{ru} Z_u^v \, dW_u. \end{array} \right.$$

Proof. The existence of a unique solution in the named spaces follows immediately from Theorem 2 and Remark 6 with $d = 3$, $X = (D, S^c, S^g)$ and with $\psi(x, e, a) = \psi((d, s_c, s_g), e, a) := (b(d, a, s_c, s_g) - \tilde{h}_c s_c - \tilde{e}_c a - k)^+$ and from Lemma 6. \square

As we did in the case of the European call option in §3.3, we now make a connection between the FBSDE and a candidate PDE. Recall that we conjectured the existence of a deterministic function α such that $A_t = \alpha(t, D_t, S_t^c, S_t^g, E_t)$, for $t \in [0, T)$, where α was characterised as the solution to a semilinear PDE. Based on this assumption, we now conjecture in the case of the spread option that there exists a deterministic function $v : [0, \tau] \times [0, \bar{\xi}] \times \mathbb{R}_{++}^2 \times [0, \bar{e}] \mapsto \mathbb{R}$, suitably regular on $[0, \tau)$ to be a classical solution to the PDE

$$\begin{aligned} \mathcal{L}v &= 0, && \text{on } U_\tau, \\ v &= \left(b(d, a, s_c, s_g) - \tilde{h}_c s_c - \tilde{e}_c \alpha(\tau, d, s_c, s_g, e) - k \right)^+, && \text{on } \{t = \tau\} \times U, \end{aligned}$$

where $U_\tau := [0, \tau) \times U$ and

$$\begin{aligned} \mathcal{L} := & \frac{\partial}{\partial t} + \frac{1}{2} \sigma_d^2(d) \frac{\partial^2}{\partial d^2} + \frac{1}{2} \sigma_c^2(s_c) \frac{\partial^2}{\partial s_c^2} + \rho \sigma_c(s_c) \sigma_g(s_g) \frac{\partial^2}{\partial s_c \partial s_g} + \frac{1}{2} \sigma_g^2(s_g) \frac{\partial^2}{\partial s_g^2} \\ & + \mu_d(t, d) \frac{\partial}{\partial d} + \mu_c(s_c) \frac{\partial}{\partial s_c} + \mu_g(s_g) \frac{\partial}{\partial s_g} + \mu_e(d, \alpha(t, d, s_c, s_g, e)) \frac{\partial}{\partial e} - r \end{aligned}$$

and where we postpone the specification of relevant boundary conditions to §A.2. Under the imposed regularity assumption on v a simple application of Itô's formula then shows that $V_t = v(t, D_t, S_t^c, S_t^g, E_t)$, for $t \in [0, \tau)$.

5.4 Numerical Case Study

This section is dedicated to the numerical illustration of the model. We introduce concrete functional forms for the fuel bid curves and the marginal emission rate and deduce the implied market bid stack and emission rate. Further, we specify the processes describing the evolution of demand for electricity and the prices of coal and gas. The section culminates in a series of dedicated case studies, which analyse in detail the impact of the proposed structural models for electricity and allowance prices on the prices of clean spread options and the valuation of power plants.

5.4.1 A Worked Example

We begin by specifying the functions and parameters in the model.

5.4.1.1 Exponential Heat and Emission Rates

We take the variation of heat and emission rates throughout each continuum of firms to be of exponential form. This choice allows us to obtain the market bid stack b in closed form. Specifically, we define, for $j \in J$,

$$\begin{aligned} e_j(\xi) &:= \hat{e}_j \exp(m_j \xi) \\ h_j(\xi) &:= \hat{h}_j \exp(m_j \xi), \end{aligned} \quad \text{for } \xi \in [0, \bar{\xi}_j], \quad (5.17)$$

where $\hat{e}_j, \hat{h}_j, m_j > 0$ are strictly positive constants. The marginal emission rate and the heat rate of each technology vary throughout the continuum of generators, to reflect differences in efficiencies within the fleet of coal and gas generators. We notice immediately though that with the above definition the ratio h_j/e_j is fixed. This constraint of the model, however, is reasonable as less efficient plants with higher heat rates tend to have correspondingly higher emission rates.

5.4.1.2 The Market Bid Stack

With the heat and emission rates being of the form given by (5.17) the fuel bid curves b_j , $j \in J$, read

$$b_j(\xi, a, s) := (\hat{e}_j a + \hat{h}_j s) \exp(m_j \xi), \quad \text{for } (\xi, a, s) \in [0, \bar{\xi}_j] \times \mathbb{R}_+ \times \mathbb{R}_{++}$$

and, as before, for a given electricity, allowance and fuel price, the level of supply from technology $j \in J$ is given by the generalized inverse of the fuel bid curve, introduced in §5.3 and denoted by \tilde{b}_j . For $(a, s) \in \mathbb{R}_+ \times \mathbb{R}_{++}$, this can now be written explicitly in the form

$$\tilde{b}_j(p, a, s) = \begin{cases} 0, & \text{if } p \in (-\infty, \underline{b}_j(a, s)], \\ \frac{1}{m_j} \log \left(\frac{p}{\hat{e}_j a + \hat{h}_j s_j} \right), & \text{if } p \in (\underline{b}_j(a, s), \bar{b}_j(a, s)], \\ \bar{\xi}_j, & \text{if } p \in (\bar{b}_j(a, s), \infty). \end{cases} \quad (5.18)$$

These functions define the market bid stack, whose definition we restate here for the sake of completeness and easy reference:

$$b(\xi, a, \mathbf{s}) := \min_{j \in J} \underline{b}_j(a, s_j) \vee \sup \left\{ p \in \mathbb{R} : \tilde{b}_c(p, a, s_c) + \tilde{b}_g(p, a, s_g) < \xi \right\}, \quad (5.19)$$

for $(\xi, a, s) \in [0, \bar{\xi}_j] \times \mathbb{R}_+ \times \mathbb{R}_{++}$.

Notice that for given $(a, s) \in \mathbb{R}_+ \times \mathbb{R}_{++}$, depending on the value of $p \in \mathbb{R}$, the function \tilde{b}_j , $j \in J$, can be equal to one of three different expressions, as shown in (5.18). Therefore, there are nine different permutations of the sum of the two inverse functions inside the supremum in (5.19). The advantage of our choice of an exponential function for the heat and the marginal emission rates is that for each permutation this sum and also its inverse can be calculated explicitly. To ease the intuitive interpretation of this statement, we point out that each permutation corresponds to either coal or gas or both technologies determining the marginal bid and therefore the market price of electricity. Then the explicit calculation of the market bid stack is straightforward if only one technology determines the marginal bid with the other technology being either used to full capacity or lying idle. If both technologies determine the marginal bid the market bid stack can still be calculated in closed form because the sum of two inverses of the exponential function is invertible and can be written in closed form. We summarise this discussion in the following proposition.

Proposition 14. *Given that heat and emission rates of the continuum of coal and gas generators are of exponential form as given in (5.17) and with the definitions*

$$b_{cg}(\xi, a, \mathbf{s}) := \left(\hat{e}_c a + \hat{h}_c s_c \right)^{\beta_c} \left(\hat{e}_g a + \hat{h}_g s_g \right)^{\beta_g} \exp(\gamma \xi), \quad \text{for } (\xi, a, \mathbf{s}) \in [0, \bar{\xi}] \times \mathbb{R}_+ \times \mathbb{R}_{++}^2,$$

where

$$\beta_c := \frac{m_g}{m_c + m_g}, \quad \beta_g := 1 - \beta_c = \frac{m_c}{m_c + m_g}, \quad \gamma := \frac{m_c m_g}{m_c + m_g},$$

and

$$j_- := \operatorname{argmin}_{j \in J} \{\bar{\xi}_j\} \quad \text{and} \quad j_+ := \operatorname{argmax}_{j \in J} \{\bar{\xi}_j\},$$

the market bid stack b , defined in (5.19), is given explicitly by

$$b(\xi, a, \mathbf{s}) = b_{\text{low}}(\xi, a, \mathbf{s}) \mathbb{I}_{[0, \bar{\xi}_{j_-}]}(\xi) + b_{\text{mid}}(\xi, a, \mathbf{s}) \mathbb{I}_{(\bar{\xi}_{j_-}, \bar{\xi}_{j_+}]}(\xi) + b_{\text{high}}(\xi, a, \mathbf{s}) \mathbb{I}_{(\bar{\xi}_{j_+}, \bar{\xi}]}(\xi),$$

where, for $(\xi, a, \mathbf{s}) \in [0, \bar{\xi}] \times \mathbb{R}_+ \times \mathbb{R}_{++}^2$,

$$\begin{aligned} b_{\text{low}}(\xi, a, \mathbf{s}) := & b_c(\xi, a, s_c) \mathbb{I}_{\{b_c(\xi, a, s_c) < \underline{b}_g(a, s_g)\}} \\ & + b_g(\xi, a, s_g) \mathbb{I}_{\{b_g(\xi, a, s_g) < \underline{b}_c(a, s_c)\}} \\ & + b_{cg}(\xi, a, \mathbf{s}) \mathbb{I}_{\{b_c(\xi, a, s_c) \geq \underline{b}_g(a, s_g), b_g(\xi, a, s_g) \geq \underline{b}_c(a, s_c)\}}, \end{aligned}$$

$$\begin{aligned} b_{\text{mid}}(\xi, a, \mathbf{s}) := & b_{j_+}(\xi, a, s_{j_+}) \mathbb{I}_{\{b_{j_+}(\xi, a, s_{j_+}) < \underline{b}_{j_-}(a, s_{j_-})\}} \\ & + b_{j_+}(\xi - \bar{\xi}_{j_-}, s_{j_+}) \mathbb{I}_{\{b_{j_+}(\xi - \bar{\xi}_{j_-}, a, s_{j_+}) > \bar{b}_{j_-}(a, s_{j_-})\}} \\ & + b_{cg}(\xi, a, \mathbf{s}) \mathbb{I}_{\{b_{j_+}(\xi, a, s_{j_+}) \geq \underline{b}_{j_-}(a, s_{j_-}), b_{j_+}(\xi - \bar{\xi}_{j_-}, a, s_{j_+}) \leq \bar{b}_{j_-}(a, s_{j_-})\}}, \end{aligned}$$

$$\begin{aligned} b_{\text{high}}(\xi, a, \mathbf{s}) := & b_c(\xi - \bar{\xi}_g, a, s_c) \mathbb{I}_{\{b_c(\xi - \bar{\xi}_g, a, s_c) > \bar{b}_g(a, s_g)\}} \\ & + b_g(\xi - \bar{\xi}_c, a, s_g) \mathbb{I}_{\{b_g(\xi - \bar{\xi}_c, a, s_g) > \bar{b}_c(a, s_c)\}} \\ & + b_{cg}(\xi, a, \mathbf{s}) \mathbb{I}_{\{b_c(\xi - \bar{\xi}_g, a, s_c) \leq \bar{b}_g(a, s_g), b_g(\xi - \bar{\xi}_c, a, s_g) \leq \bar{b}_c(a, s_c)\}}. \end{aligned}$$

Proof. We prove the expression for b_{low} only. The expressions for b_{mid} and b_{high} are proven similarly.

Suppose $\xi \in [0, \bar{\xi}_{j_-}]$ and that for a fixed $(a, \mathbf{s}) \in \mathbb{R}_+ \times \mathbb{R}_{++}^2$ coal is the technology at the margin, that is, $b_c(\xi, a, s_c) < \underline{b}_g(a, s_g)$. Then $\tilde{b}_c(p, a, s_c) + \tilde{b}_g(p, a, s_g) = m_c^{-1} \log(p/(\hat{e}_c a + \hat{h}_c s_c)^{-1}) + 0$, so that $b_{\text{low}}(\xi, a, \mathbf{s}) = b_c(\xi, a, s_c)$. The same argument shows that if gas is the technology at the margin, that is, $b_g(\xi, a, s_g) < \underline{b}_c(a, s_c)$, then $b_{\text{low}}(\xi, a, \mathbf{s}) = b_g(\xi, a, s_g)$. For the case when both coal and gas are at the margin, $\tilde{b}_c(p, a, s_c) + \tilde{b}_g(p, a, s_g) = m_c^{-1} \log(p/(\hat{e}_c a + \hat{h}_c s_c)^{-1}) + m_g^{-1} \log(p/(\hat{e}_g a + \hat{h}_g s_g)^{-1})$. In this case $b_{\text{low}}(\xi, a, \mathbf{s}) = (\hat{e}_c a + \hat{h}_c s_c)^{\beta_c} (\hat{e}_g a + \hat{h}_g s_g)^{\beta_g} \exp(\gamma \xi)$. \square

Remark 14. The result in Proposition 14 can in principle be extended to the general case of an arbitrary number of fuel types. However, the level of complexity of the formulas increases rapidly, which is why we refer the interested reader to the papers [Carmona et al., 2012a] and [Carmona et al., 2012b], where such an extension is considered and where it is also shown that the above two-fuel bid stack model leads to closed-form formulae for the prices of spread options in the absence of emissions trading, that is in the case of business-as-usual.

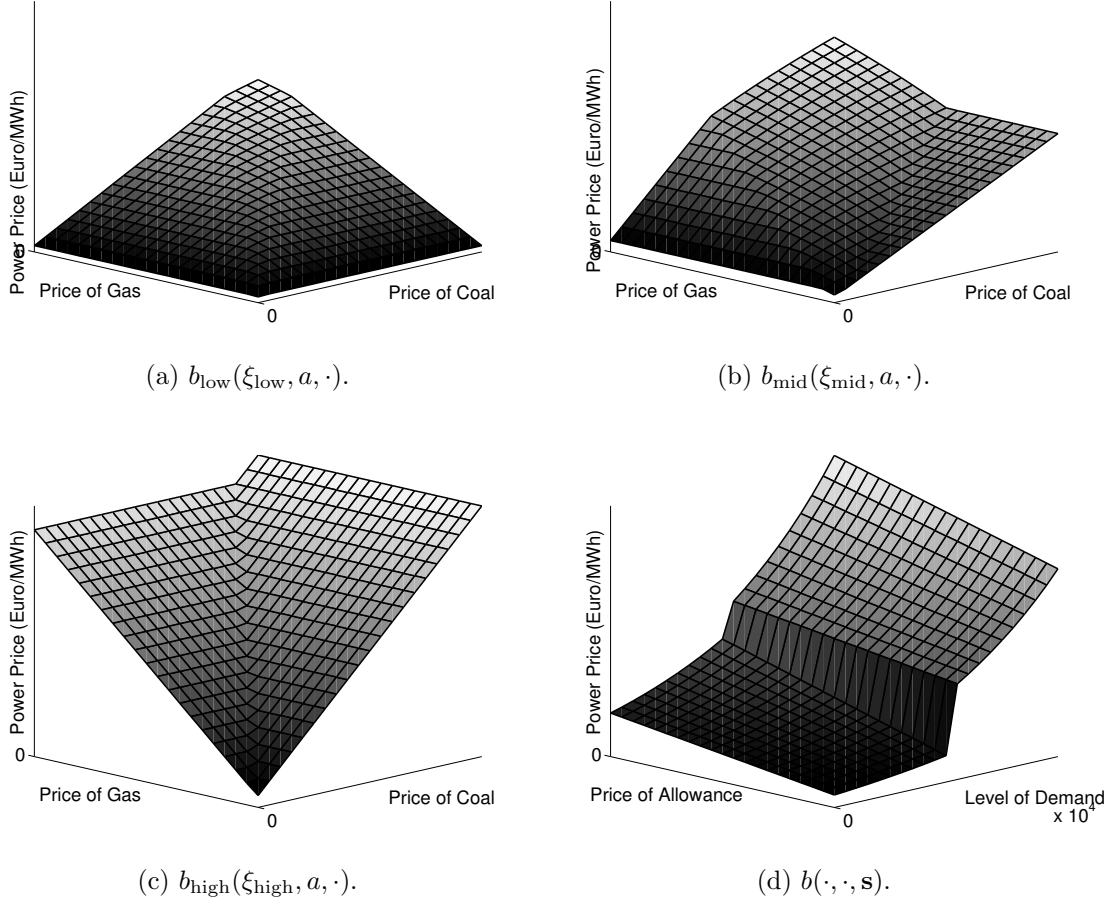


Figure 5.2: Illustration of the dependence of the market bid stack on fuel prices and demand and the allowance price; $\bar{\xi}_c > \bar{\xi}_g$ and $b_c = b_g$ on $(\xi, a, \mathbf{s}) \in [0, \bar{\xi}_g] \times \mathbb{R}_+ \times \mathbb{R}_{++}^2$.

We illustrate the dependence of the market bid stack on the fundamental variables in Figure 5.2. For this purpose we chose $a > 0$ and coal to be the dominant technology, that is, $\bar{\xi}_c > \bar{\xi}_g$. Otherwise the parameters for both fuel bid curves were chosen identically to ease the interpretation of the plots. Figure 5.2a plots the market bid stack b in the case of a low supply level $\xi_{\text{low}} \in [0, \bar{\xi}_g]$. First, notice that the power price never equals zero due to the cost of the emission being positive and included as part of the variable costs in our model. Second, this low level of supply can be fully met by either coal or gas by itself. Therefore, given our parameter choice, the surface is symmetric in the prices of coal and gas. As an example, consider the scenario of a low coal price and a high gas price; in this case the marginal bid is determined by coal alone and vice versa in the case of a high coal price and a low gas price. Moreover, around the diagonal we notice a small region, where the two fuel bid curves overlap

sufficiently for the marginal bid level to be determined by both technologies.

Figure 5.2b plots the market bid stack b in the case of a medium supply level $\xi_{\text{mid}} \in (\bar{\xi}_g, \bar{\xi}_c]$. This supply level can be delivered either solely by coal generators or by a mix of both technologies. Therefore, the price is always set by coal, except in a wide region around the diagonal where the marginal bid is determined by both technologies.

Figure 5.2c plots the market bid stack b in the case of the supply level being equal to the market capacity, that is $\xi_{\text{high}} = \bar{\xi}$. To offer this level of supply both technologies have to generate their respective capacities. Therefore, the price of electricity is always determined by the more expensive technology, as determined by the prices of coal and gas. Notice that the asymmetry of the surface is due to the fact that $\bar{\xi}_c > \bar{\xi}_g$.

In Figure 5.2d we fixed the prices of coal and gas with $s_g > s_c$. The difference between the fuel prices is large enough to separate the two fuel bid curves completely. Consequently, the price of electricity is first set by coal, then, as the level of supply increases, it jumps as the capacity of coal is exhausted and gas generators now determine the marginal bid. As a function of the allowance price the electricity price increases linearly for all $\xi \in [0, \bar{\xi}]$. In particular, the effect of the allowance price on the coal and gas fuel bid curves is the same, because, for simplicity, we chose their emission rates to coincide.

5.4.1.3 The Market Emission Rate

We now discuss the market emission rate μ_e , as defined in (5.6), in light of our concrete choice of functional forms (5.17) for heat and emission rates. The integrals in equation (5.6) can now be calculated explicitly and the market emission rate is of the form

$$\begin{aligned} \mu_e(\xi, a, \mathbf{s}) &:= \hat{e}_c \int_0^{\tilde{b}_c(b(\xi, a, \mathbf{s}), a, s_c)} e^{m_c u} \, du + \hat{e}_g \int_0^{\tilde{b}_g(b(\xi, a, \mathbf{s}), a, s_g)} e^{m_g u} \, du \\ &= \frac{\hat{e}_c}{m_c} \left[e^{m_c \tilde{b}_c(b(\xi, a, \mathbf{s}), a, s_c)} - 1 \right] + \frac{\hat{e}_g}{m_g} \left[e^{m_g \tilde{b}_g(b(\xi, a, \mathbf{s}), a, s_g)} - 1 \right]. \end{aligned} \quad (5.20)$$

We illustrate the dependence of the market emission rate, as defined by (5.20), on the fundamental variables in Figure 5.3. As above, we chose $a > 0$ and coal to be the dominant technology, that is, $\bar{\xi}_c > \bar{\xi}_g$. Further, to reflect the fact that coal is more pollution intensive than gas, $\hat{e}_c > \hat{e}_g$. Figure 5.3a plots the market emission rate μ_e in the case of a low supply level $\xi_{\text{low}} \in [0, \bar{\xi}_g]$. For low coal prices and relatively more expensive gas prices the entire supply stems from coal generators. Hence the market emission rate is higher than if coal prices are relatively more expensive than

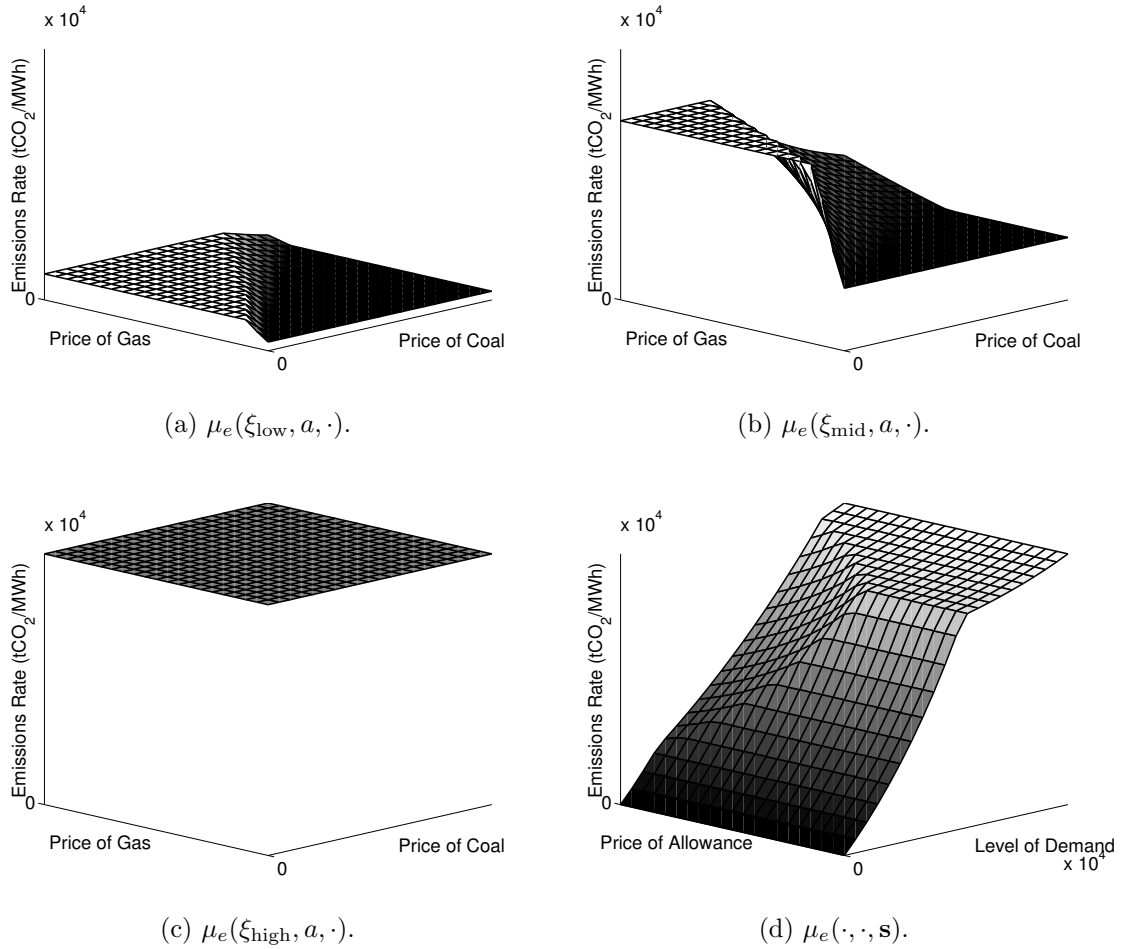


Figure 5.3: Illustration of the dependence of the market emission rate on fuel prices and demand and the allowance price; $\bar{\xi}_c > \bar{\xi}_g$ and $\hat{e}_c > \hat{e}_g$.

gas prices, in which case the entire supply stems from gas generators. Further, around the diagonal we again observe a small region, where both technologies contribute to supply. Therefore, the market emission rate lies between that of coal and gas.

Figure 5.3b plots the market emission rate μ_e in the case of a medium supply level $\xi_{\text{mid}} \in (\bar{\xi}_g, \bar{\xi}_c]$. For prices of coal and gas close to zero, the positive allowance price and the fact that $\hat{e}_c > \hat{e}_g$ imply that the bids from coal generators are still relatively more expensive than those from gas generators. Gas generators are used to full capacity and any increase in the price of coal does not lead to further load shifting. If on the other hand gas is relatively more expensive than coal, the entire supply stems from coal generators and sufficient increases in the price of coal cause load to shift from coal generators to gas generators.

Figure 5.3c plots the market bid stack b in the case of the supply level being

equal to the market capacity, that is $\xi_{\text{high}} = \bar{\xi}$. At this level of supply all generation resources in the market are used and no load shifting is possible.

In Figure 5.3d the prices of coal and gas are fixed with $s_g > s_c$. The difference between the fuel prices is large enough to separate the two fuel bid curves completely if $a = 0$. To explain the impact of the allowance price, consider a supply level close to half the market capacity. Initially, the entire supply stems from coal generators. Sufficient increases in the allowance price, however, lead to coal bids becoming relatively more expensive. Hence a mix of both technologies contributes to supply and the market emission rate decreases.

5.4.1.4 The Demand Process

As in §3.4.1.2, we specify that under \mathbb{Q} the fluctuations in demand are described by a Jacobi diffusion process. The only difference to our previous analysis is that here we allow demand to exhibit seasonal variation. Therefore, D is given by

$$D_t = d_0 - \int_0^t \eta (D_u - \bar{D}(t)) \, du + \int_0^t \sqrt{2\eta\bar{\sigma}_d D_u (\bar{\xi} - D_u)} \, dW_u^1, \quad \text{for } t \in [0, T], \quad (5.21)$$

where $\bar{D} : [0, T] \mapsto (0, \bar{\xi})$ and $\eta, \bar{\sigma}_d > 0$. The conditions for the solution D to remain within the interval $(0, \bar{\xi})$ now read $\inf_{t \in [0, T]} (\min(\bar{D}(t), \bar{\xi} - \bar{D}(t))) \geq \bar{\xi}\bar{\sigma}_d$. To capture the seasonal character of demand, we choose a function $\bar{D}(\cdot)$ of the form:

$$\bar{D}(t) := \varphi_0 + \varphi_1 \sin(2\pi\vartheta t), \quad \text{for } t \in [0, T],$$

where $\varphi_0, \varphi_1, \vartheta > 0$ are positive constants.

5.4.1.5 The Fuel Price Processes

We assume that the prices of coal S^c and gas S^g follow correlated exponential OU processes under the measure \mathbb{Q} , that is, for $j \in J$,

$$S_t^j = s_0^j - \int_0^t \eta_j \left(\log S_u^j - \bar{s}_j - \frac{\bar{\sigma}_j^2}{2\eta_j} \right) S_u^j \, du + \int_0^t \bar{\sigma}_j S_u^j \, dW_u^j, \quad \text{for } t \in [0, T]$$

and $\langle W^2, W^3 \rangle_t = \int_0^t \rho \, du$. The parameters $s_0^j, \eta_j, \bar{s}_j, \bar{\sigma}_j > 0$ are positive constants and the constant $\rho \in [-1, 1]$. With this definition, for $j \in J$, $S^j > 0$ and $S^j \in \mathcal{S}^2$.

\hat{h}_c	\hat{e}_c	m_c	$\bar{\xi}_c$	\hat{h}_g	\hat{e}_g	m_g	$\bar{\xi}_g$	$\bar{\xi}$
3	0.9	0.00005	12000	7	0.4	0.00003	18000	30000

Table 5.1: Parameters relating to the bid stack and the emission stack.

η	φ_0	φ_1	ϑ	$\bar{\sigma}_d$	d_0
50	21000	3000	1	0.1	21000

Table 5.2: Parameters relating to the demand process.

5.4.1.6 Choice of Parameters

The tables in this section summarise the parameters used for the implementation of our model that follows below. We refer to the parameters specified in Tables 5.1 – 5.5 as the “base case” and indicate whenever we depart from this choice. Note that as in Chapter 3 our parameter choices do not correspond to a particular electricity market, but that all values are within a realistic realm.

Table 5.1 summarises the parameters specifying the bid curves. We consider a medium sized electricity market served by coal and gas generators and with gas being the dominant technology. For the marginal emission rates, Table 5.1 implies that $e_c \in [0.9, 1.64]$ and $e_g \in [0.4, 0.69]$ (both measured in tCO₂ per MWh), so that the most pollution intensive gas plant is still cleaner than the least pollution intensive coal plant. For the heat rates, we observe that $h_c \in [3, 5.5]$ and $h_g \in [7, 12]$ (both measured in MMBtu per MWh). Using (5.20) now with $\xi = \bar{\xi}$ and with the assumption that there are 8760 production hours in the year, we find the upper bound for the cumulative emission process. In particular, $\bar{e} = 2.13e + 08$.

Table 5.2 contains the parameters specifying the demand process D . We model periodicities on an annual time scale and the chosen rate of mean-reversion assumes that demand reverts to its (time dependent) mean over the course of one week.

In Table 5.3 we summarise the parameters that specify the behavior of the prices of coal and gas. Both are chosen to be slowly mean-reverting, at least in comparison to demand. To ease the analysis, we assume that all parameters are identical for the two fuels.

Table 5.4 defines the cap-and-trade scheme that we assume to be in place. The duration of the compliance period T is measured in years and we set the cap at 70% of the upper bound \bar{e} for the cumulative emissions, in order to incentivise a reduction

η_c	\bar{s}_c	$\hat{\sigma}_c$	s_0^c	η_g	\bar{s}_g	$\hat{\sigma}_g$	s_0^g	ρ
1.5	2	0.5	exp(2)	1.5	2	0.5	exp(2)	0.3

Table 5.3: Parameters relating to the fuel price processes.

π	Γ	T	r
100	1.4e+08	1	0.05

Table 5.4: Parameters relating to the cap-and-trade scheme.

in emissions. This choice of parameters results in A_0 being approximately equal to $\pi/2$, a value for which there is significant initial overlap between gas and coal bids in the stack. Furthermore, the parameters imply a bid stack structure such that at mean levels of coal and gas prices, the most expensive coal bid is relatively cheaper than the cheapest gas bid in the case of business-as-usual, while the cheapest coal bid is relatively more expensive than the most expensive gas bid if the price of the emission is equal to the penalty.

Finally, in Table 5.5 we specify the four spread option contracts used in the base case scenario to represent high and low efficiency coal plants, and high and low efficiency gas plants.²

5.4.2 Case Studies

We now consider several case studies to investigate some of the features that distinguish the joint structural model for electricity and allowance prices that we propose. This section is supposed to provide the reader with an intuition for the behaviour of the model and we emphasize that the obtained results are, of course, highly dependent on the choice of parameters in §5.4.1.6.

When clean spread contracts are used to value power plants, as explained in §1.2.1, the heat and emission rates, \tilde{h}_j and \tilde{e}_j , $j \in J$, that define the option contract, correspond to the efficiency of the power plant that is to be valued. Therefore, throughout the case studies, we shall often refer to clean spread options with low heat and emission rates defining the option as contracts with high efficiency parameters

²Note that low efficiency means dirtier and corresponds to large values of h_j and e_j , for $j \in J$, and vice versa.

\tilde{h}_c		\tilde{h}_g		\tilde{e}_c		\tilde{e}_g	
high eff.	low eff.	high eff.	low eff.	high eff.	low eff.	high eff.	low eff.
3.5	5	7.5	11.5	1.05	1.5	0.43	0.66

Table 5.5: Parameters relating to the spread options.

and to those with high heat and emission rates defining the option as contracts with low efficiency parameters (see also Table 5.5).

5.4.2.1 Seasonal Effects and Impact of the Emission Market

The first two effects we want to study are seasonal effects due to the seasonality of power demand and the impact of an increasingly strict emission regulation on the prices of clean spread options. In Figure 5.4 we display the prices of clean dark and spark spread options, for high and low efficiency parameters, as a function of spread maturity and for different levels of the cap γ . The three levels of the cap, ranging from $2e + 08$ tonnes of CO_2 to $1.2e + 08$ tonnes of CO_2 , imply initial allowance prices of 5€, 52€ and 94€.

We first comment on the seasonal effects that are observed throughout Figures 5.4a – 5.4d. As can be seen spread prices exhibit a great deal of seasonality depending on the expiry date of the contract. This is of course caused by the seasonal fluctuations of demand for electricity. Since the electricity spot price is strictly increasing as a function of demand, a higher level of demand at maturity makes the spread option more valuable. Therefore, the price of contracts which expire on days when a high demand load is likely, is relatively more expensive than the price of contracts which expire on days when demand can be expected to be low.

Second, we explain the impact of a tightening of the emission regulation, or, in other words, increases in the allowance price. We begin by stating the obvious, namely that high efficiency contracts are generally more valuable than low efficiency ones. Further, given our parameter choice, for low allowance prices, coal bids can be expected to be cheaper than gas bids. Hence we observe that, in this case, dark spread options are more valuable than spark spread options. As the cap is tightened the allowance price increases and initially coal and gas bids approach similar levels. Further tightening of the cap results in coal bids becoming more expensive than gas bids. On peak demand days the electricity price is now set by low efficiency coal plants. The effect of these subsequent increases in the allowance price on the

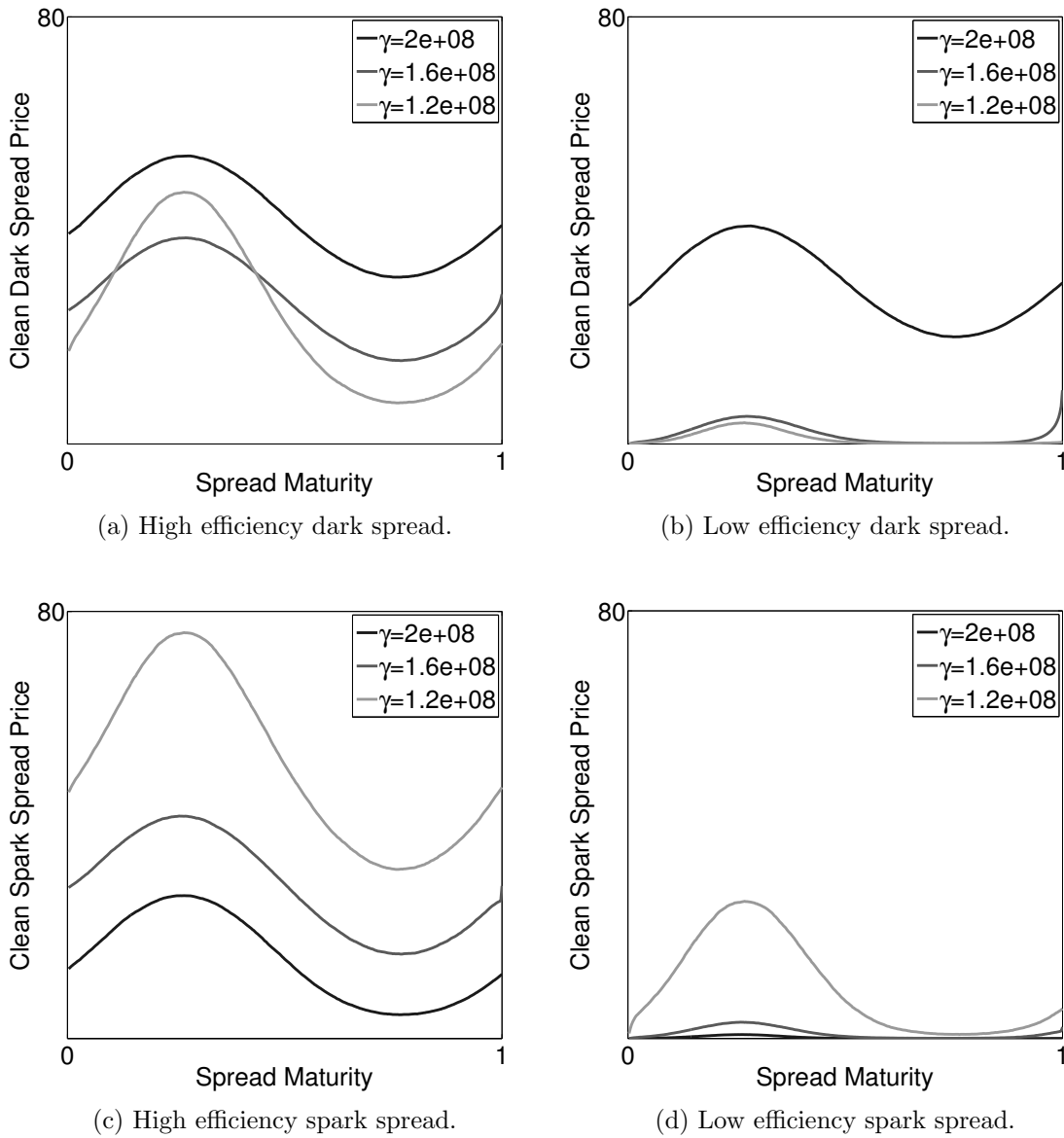


Figure 5.4: High and low efficiency clean dark and spark spread option prices as a function of spread maturity and for different levels of the cap.

spread option now depends on whether the production costs which define the option contract increase by more or by less than the electricity price. Looking at Figure 5.4a on maturity dates that coincide with peak demand levels, we observe that — as expected — a first tightening of the cap decreases the value of the dark spread contract. A further tightening of the cap, however, results in low efficiency coal plants setting the electricity price. The production costs of these plants are higher than those factored into the high efficiency dark spread contract. As a result the value of the

dark spread now increases. This effect can only be observed when demand is high, as otherwise the electricity price is set by high efficiency coal or even gas plants and the positive difference between the production costs of the price setting plants and those that define the contract first vanishes and then becomes negative, resulting in a decrease in the value of the option contract. The results in Figure 5.4b are now easy to understand. The production costs factored into the low efficiency dark spread contract are close to those of the least efficient coal plants in the market. Therefore, even on peak demand days, the production costs of the plant setting the electricity price are likely to be lower than those that feature in the option contract. Hence the value of the low efficiency dark spread decreases as the cap is tightened. Figures 5.4c and 5.4d bear no surprises: given our parameter choice, the electricity price is rarely set by highly efficient gas plants. Therefore, as the allowance price increases, the costs of production of any price setting power plant (low efficient gas or coal plant) increase by more than those that define the spark spread contract, whose value hence increases.

5.4.2.2 Impact of Fuel Price Changes

Our base case parameter set assumes that the initial conditions of the processes representing the prices of coal and gas are equal to the respective long term mean price levels (see Table 5.3). We now analyse the effect different orderings of the initial fuel prices have on the value of power plants when valued as a string of spread options as detailed in (1.1).³ Note, however, that since $\eta_c = \eta_g = 1.5$ (implying a typical mean-reversion time of 8 months), the simulated fuel price distributions will again be centred near their mean-reversion levels by the end of the trading period. Hence, in this case study, we capture a temporary and not a permanent shift in fuel prices.

As an example we choose to vary the initial gas price s_0^g in Figure 5.5 and consider levels of the cap equal to either $1.8e + 08$ tonnes of CO_2 or $1e + 08$ tonnes of CO_2 . As the initial gas price increases from values below the long term mean level to values above the long term mean level, it induces changes in the initial merit order.

In Figure 5.5a we consider the effects on high efficiency coal plants. In both cases, that of a lenient and that of a strict cap, the power plant value increases as a function of the initial gas price. The difference in the slope of the two lines is explained by

³Given the record low prices of under \$2 recently witnessed in the US natural gas market (due primarily to shale gas discoveries), it is natural to ask how such fuel price variations affect our spread option results.

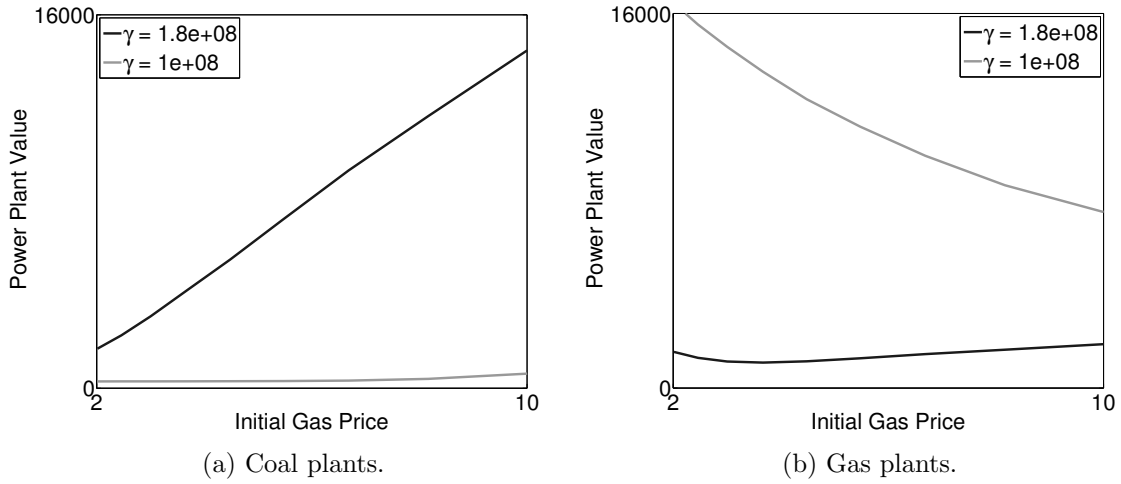


Figure 5.5: Value of high efficiency coal and gas power plants as a function of the initial gas price and for different levels of the cap.

considering the initial merit order arrangements corresponding to the two cap levels if the initial fuel price is chosen as in the base case, i.e. $s_0^g \approx 7.4\text{€}$. If the allowance price is low, coal is placed first in the merit order and gas second. Any increase in the initial gas price (above the base case parameter choice) immediately translates into higher electricity prices and hence into higher valuations of the coal plant, whose production costs are unaffected. Similarly, decreases in the initial gas price (below the base case parameter choice) decrease the electricity price while not affecting the production costs of the plant. This immediately translates into lower valuations for the coal plant. If the allowance price is high, gas is placed first and coal second in the initial merit order. Increases in the initial gas price within the range considered in Figure 5.5a are not sufficient, however, to lead to a swap of the two technologies in the merit order and the market price of electricity is never set solely by gas generators. Therefore, the sensitivity of the market price of electricity (and therefore of the coal plant, since its production costs are unaffected) to changes in the initial gas price in the case of a strict cap is considerably smaller than in the case of the lenient cap. In fact, as the initial gas price decreases coal and gas bids separate completely; the market price of electricity is determined solely by coal bids and the plant value becomes independent of s_0^g . This phenomenon is reflected in the flattening out of the grey line in Figure 5.5a.

In Figure 5.5b we consider high efficiency gas plants. This time the electricity price and the production costs of the gas plant are both affected by changes in the initial gas price. In the case of a lenient cap any increase in the initial gas price

(above the base case parameter choice) increases the bid levels of the price setting gas plants. Given our parameter choice, demand is on average high enough for these plants to be less efficient than the valued gas plant. Therefore, the electricity price increases relatively more than the production costs of the gas plant; as a result the plant value increases. Initial decreases in the initial gas price (below the base case parameter choice) decrease the electricity price at least as much as the production costs of the plant. Hence also the plant value decreases. Further decreases in the initial gas price eventually cause even the most expensive gas bids to become cheaper than coal bids. The electricity price is now set solely by coal and hence independent of gas prices. Since the production costs of the plant continue to decrease, the value of the plant increases as s_0^g becomes very small. In the case of a strict cap the initial merit order is reversed with gas being placed first and coal second. Any increase in s_0^g (within the range considered in Figure 5.5b) has either no effect on the electricity price or only a moderate one as coal and gas bids reach similar levels. Therefore, the production costs of the plant increase by at least as much as the electricity price and hence the plant value decreases.

The effects on low efficiency plants are very similar and we do not consider them separately here.

5.4.2.3 Comparison with Reduced-Form

We now present two comparisons illustrating the difference between clean spread prices that are implied by the structural model we propose and clean spread prices implied by one of two reduced-form models. In both cases the electricity price is given by the bid stack model introduced in §5.4.1.2. The comparisons are chosen to analyse the effects of two features that distinguish the structural allowance price model we present, namely short-term abatement through load-shifting and the binary nature of the allowance price at compliance time. For our analysis we focus on clean dark spread options only, since the results for clean spark spreads bear no additional surprises. Further, to simplify the analysis we assume that demand is deterministic, that is $\bar{\sigma}_d = 0$ in (5.21).⁴

We first compare clean spread prices when the allowance price A is given by the structural model to those when the allowance price is given by a geometric Brownian motion. That is we assume the existence of a process A^{ln} , which, under the measure

⁴This assumption is justifiable considering the very high mean-reversion levels that are commonly observed in electricity demand.

σ_a	ρ_{ac}	ρ_{ag}	σ_e	ρ_{ec}	ρ_{eg}
0.6	-0.2	0.4	0.006	-0.2	0.2

Table 5.6: Parameters for the comparison between structural and reduced-form models.

\mathbb{Q} , satisfies

$$A_t^{\text{ln}} = a_0 + \int_0^t r A_u^{\text{ln}} du + \int_0^t \sigma_a A_u^{\text{ln}} dB_u^1, \quad \text{for } t \in [0, \tau], \quad (5.22)$$

where $a_0 := A_0$ and B^1 is a scalar valued Brownian motion. The volatility parameter $\sigma_a > 0$ and the correlation parameters $\rho_{ac}, \rho_{ag} \in \mathbb{R}$ were chosen to approximately match the volatility and correlation structure that we observed from simulations of the structural model; their precise values are given in Table 5.6.

Given the stochastic integral equation (5.22) it is well known that A_τ^{ln} has a lognormal distribution. Therefore, its support is the entire positive half line and the allowance price at compliance time is unbounded as opposed to the binary terminal value produced by the structural model. This is precisely the essence of our first comparison, to analyse the effects of the binary nature of the terminal condition that features in our structural model. The results are presented in Figure 5.6.

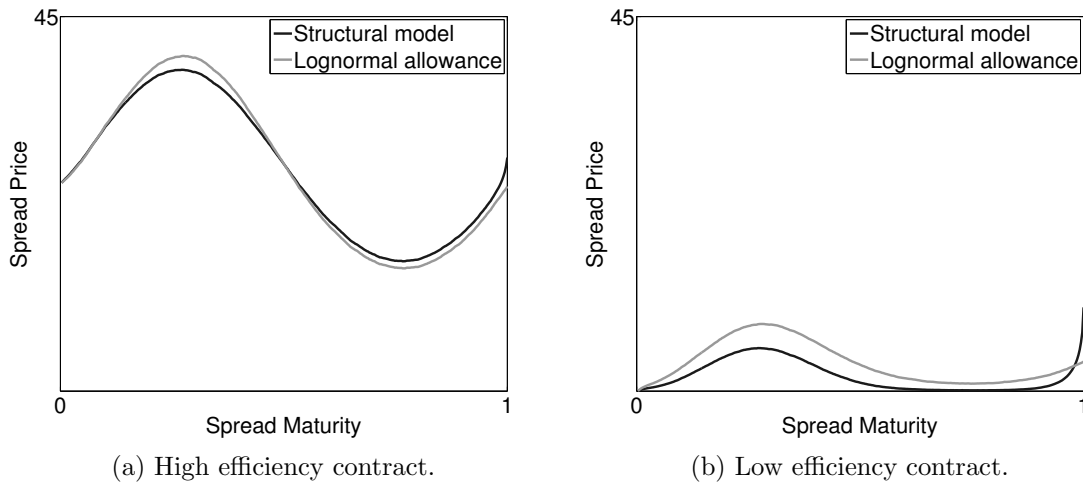


Figure 5.6: Comparison between high and low efficiency clean dark spread option prices as implied by the structural model and a geometric Brownian motion model for the allowance price.

In Figure 5.6a we observe that for early maturity dates the lognormal allowance

model implies a relatively more expensive spread price when compared to the structural model. Further, the relationship is reversed for maturity dates later on in the trading period. There are two key effects that explain this behaviour. First, the load shifting mechanism, taken into account by the structural model, induces a mean-reversion effect on electricity bids. For example, as coal prices increase, so do the bids of coal generators. This leads to a substitution away from coal towards gas as a technology used for the generation of electricity and hence to a reduction in allowance prices. Because allowance prices are factored into bids, the bids of coal and gas generators become cheaper, partly offsetting the earlier increase in bid levels due to the increase in the price of coal. Therefore, the overall effect of the load shifting mechanism is to keep the bid curves of coal and gas generators closer together. This causes clean spread options to become relatively cheaper compared to the case when the allowance price is simply given by geometric Brownian motion and does not respond to changes in the electricity bid stack structure. The second effect, is the impact of the binary terminal condition. In the case of the structural model the clean dark spread contract considerably gains in value in the event that the allowance price converges to zero at the end of the compliance period. The paths associated with this event cause the option price implied by the structural model to become relatively more expensive than the price implied by the reduced form model.

In Figure 5.6b we observe the same two effects as in Figure 5.6a. However, it is clear that in the case of the low efficiency contract, which we are considering here, the load shifting induced mean-reversion effect has a stronger impact on the option price than in the case of the high efficiency contract considered in Figure 5.6a.

For the second comparison we assume that there exist processes $(E^{\text{ln}}, A^{\text{dig}})$ representing cumulative emissions and the allowance price and that under the measure \mathbb{Q} they satisfy, for $t \in [0, \tau]$,

$$\begin{cases} E_t^{\text{ln}} = e_0 + \int_0^t \tilde{\mu}_e E_u^{\text{ln}} du + \int_0^t \sigma_e E_u^{\text{ln}} dB_u^1 \\ A_t^{\text{dig}} = \pi I_{[\gamma, \infty)}(E_\tau^{\text{ln}}) - r \int_t^\tau A_u^{\text{dig}} du - \int_t^\tau e^{ru} Z_u^{\text{dig}} dB_u^2, \end{cases}$$

where B^2 is a scalar valued Brownian motion and Z^{dig} some square integrable process. The drift $\tilde{\mu}_e > 0$ is chosen such that $A_0^{\text{dig}} = A_0$ and, as before, the volatility parameter $\sigma_e > 0$ and the correlation parameters $\rho_{ec}, \rho_{eg} \in \mathbb{R}$ were chosen to match the volatility and correlation structure observed from simulations of the structural model; their values are specified in Table 5.6. In this case the allowance price is modelled as a digital option on a lognormally distributed cumulative emission process.

Since the terminal conditions of A and A^{dig} are the same, we chose this comparison to understand the impact the feedback mechanism from the allowance price to the market emission rate, as expressed via the coupling of the FBSDE (5.14), has on clean spread option prices. The numerical results are plotted in Figure 5.7.

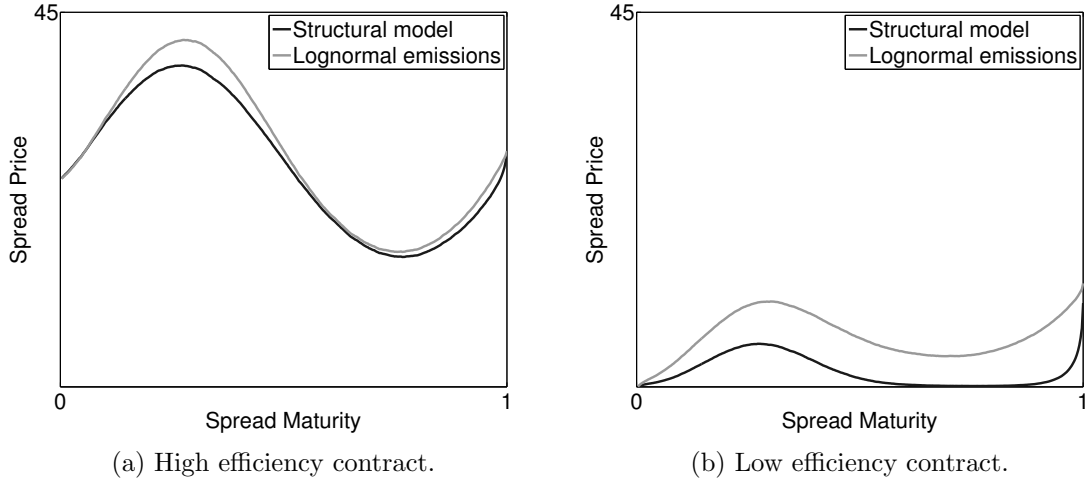


Figure 5.7: Comparison between high and low efficiency clean dark spread option prices when the allowance price is given by the structural model and as a digital option on a lognormal cumulative emission process.

In Figure 5.7 we observe that clean dark spread prices in the case when the allowance price is modelled as a digital option on a geometric Brownian motion emission process are now always more expensive than those implied by the structural model, except for the two maturity dates $\tau = 0$ and $\tau = 1$, where they are equal by definition. As in the case of the previous comparison in Figure 5.6 the reduced-form model overprices the clean spread contract due to the negligence of the effects of load shifting on the allowance price. Therefore, the results in Figure 5.7 are explained in the same way as in the case of the allowance price itself following geometric Brownian motion with the exception that at compliance time the two price paths converge because both models imply the same terminal values for the allowance price.

5.5 Conclusion

This chapter introduced a joint model for electricity spot prices and emission markets in a setting, with demand for electricity and the prices of coal and gas as exogenously given stochastic factors. The proposed electricity bid stack model allows the two considered technologies to swap places in the merit order. Taking into account the

effect of the stochastic merit order (and the level of demand) on the rate at which the market emits we stated the pricing FBSDE for the allowance certificate and that for clean spread options written on the spread between the spot price of electricity and the costs of production, including the cost of the emission. The benefits of the proposed model were exemplified with the help of a series of dedicated numerical studies for the example of the pricing of clean spread options and their usage for the valuation of power plants.

Chapter 6

Conclusion

We end this thesis with a review of its key findings and outline possible directions for the continuation of this work.

6.1 This Thesis

In this thesis, we rigorously established a novel, versatile and tractable stochastic framework, which explains the joint price formation of electricity and allowance certificates. Naturally, there exists a close link between the market for a produced good and the market in which a negative externality that arises from the production of this good is traded. In particular, it is the sole rationale behind emission markets that the financial market in which allowance certificates are traded has the effect of incentivizing the utilisation of cleaner production technologies in the real market in which the refined good is traded. Because we explained the prices of electricity and allowances as functions of certain fundamental factors, we capture this feedback and also the dependency of the prices on the fundamental factors in a very clear way. Further, we developed two specific models that fit this framework; the first one using only demand for electricity as a fundamental factor and the second one using in addition the prices of coal and gas.

The mathematical contribution of this thesis is the derivation and the mathematical analysis of the models which describe the prices of the allowance certificate and electricity. In the case of the allowance certificate the proposed models took the form of a coupled forward-backward stochastic differential equation and we dedicated a considerable amount of effort to proving the existence and uniqueness of a solution to this type of equation. Further, we analysed its behaviour using asymptotic techniques and revealed interesting qualitative properties of the solution. In the case of

the electricity spot price the models we proposed were given by carefully, explicitly constructed functions.

From a practical point of view, the framework we proposed accommodates most regulatory features that are frequently found in implementations of emissions trading systems and we analysed in detail the impact these features have on the prices of allowance certificates. Thereby we revealed a weakness in existing regulatory frameworks: if the cumulative emissions in the market end up exactly at the cap, the allowance price does not necessarily conform with the conditions imposed by the regulator! Further, we applied our framework to the pricing of derivative contracts, in particular clean spread options. Using numerical tools that were developed for this thesis, we illustrated some of the relationships between the fundamental variables and the prices of the spread option contracts.

We emphasize that in the models we proposed the allowance price is an endogenous variable. Therefore, our models also apply to regions that are not currently covered by emissions trading, but where the implementation of such a scheme is likely in the future. This is of great value when our model is used as an effective tool for the valuation of generation assets in electricity markets. Power plants are routinely valued over long time horizons and for this purpose possible future scenarios, such as an increase in production costs due to emission regulation must be taken into account.

6.2 Future Work

The framework introduced in this thesis assumes that the feedback from the emission market to the power sector is instantaneous but not permanent. Especially over long time horizons it becomes necessary to consider the effects of irreversible abatement measures in the power sector, for example, a certain percentage of coal fired power plants being converted to gas plants due to new government legislation. Or alternatively, the modernisation of plants by the private sector, which leads to a lower heat and hence emission rate. This opens up new and interesting questions. For example, at what levels of the allowance price is it feasible for a plant to invest in permanent abatement measures, given that the benefits of this investment will only pay off in the long-term?

Throughout this thesis we assumed that we were modelling one (isolated) emission market. In reality, however, the Linking Directive allows multiple markets to be interconnected by the Clean Development and Joint Implementation mechanisms.

How these mechanisms impact the allowance price structure and their efficacy in reducing emissions on a global scale is an important task to understand and to model quantitatively.

Mathematically, the models we proposed in this thesis and in particular the $(4+1)$ -dimensional model introduced in Chapter 5 exhaust the capability of currently available numerical techniques. The encountered problem here is the curse of dimensionality, when solving high dimensional partial differential equations. Therefore, our models should provide a strong incentive for research that directly deals with the numerical solution of the forward-backward stochastic differential equation in a way, which circumvents the rapid increase in complexity experienced in the case of numerical solutions of high-dimensional partial differential equations. We point out that the specific structure of our problem, in particular, the decoupledness of the process X , which, in the case of the two-fuel model, is responsible for the higher dimensionality, may help.

Lastly, our asymptotic analysis of the allowance pricing model suggested that the solution to the partial differential equation, which we conjectured to describe the allowance price, may in fact be smooth. A rigorous proof of this result would be interesting and also provide an alternative proof (to the one we presented in this thesis) for the existence of a solution to a class of forward-backward stochastic differential equations.

Appendix A

Numerical Procedures and Error Analysis

The appendix overviews the numerical schemes used throughout this thesis.

A.1 Numerical Schemes for the (2+1)-Dimensional Model

This appendix explains the numerical schemes that were used to produce the results in §3.4 and demonstrates their efficacy.

A.1.1 Numerical Solution of the Allowance and Option Pricing Problem

We comment on the numerical schemes employed to solve the nonlinear partial differential equations satisfied by the allowance price in the case of a single and multiple compliance periods (§3.2.1 and §3.2.2) and the linear partial differential equation satisfied by the call option (§3.3).

A.1.1.1 One Compliance Period

We discretize the domain \bar{U}_T by choosing mesh widths $\Delta d > 0$, $\Delta e > 0$ and a time step $\Delta t > 0$. For indexes $i, l, k \in \mathbb{N} \cup \{0\}$, the discrete mesh points (d_i, e_l, t_k) are then defined by

$$\begin{aligned}d_i &:= i\Delta d, \\e_l &:= l\Delta e, \\t_k &:= k\Delta t.\end{aligned}$$

The finite difference scheme we employ produces approximations $\alpha_{i,l}^k$, which are assumed to converge to the true solution α as the mesh width tends to zero.

Since the partial differential equation (3.10) is posed backwards in time with a terminal condition, we choose a backward finite difference for the time derivative in order to work with an explicit scheme.

In the e -direction we are approximating a conservation law PDE with discontinuous terminal condition.¹ The first derivative in the e -direction, relating to the nonlinear part of the partial differential equation, is discretised against the drift direction using a one-sided upwind difference. Because characteristic information is propagating in the direction of decreasing e , this one-sided difference is also used to calculate the value of the approximation on the part of the boundary corresponding to $e = 0$.

In the d -direction the equation is parabolic everywhere except on the boundary, where it degenerates. Hence we use central differences to discretize the first and second order derivative. At the boundaries corresponding to $d = 0$ and $d = \bar{\xi}$, where the second derivative vanishes and no boundary conditions need to be specified, we again use a one-sided difference in our numerical scheme.

With smooth boundary data, on a smooth domain and with a strictly decreasing (in a) coefficient μ_e , the scheme described above can be expected to exhibit first order convergence. In our setting, we expect the discontinuous terminal condition and the fact that μ_e is merely decreasing in a , to have adverse effects on the convergence rate.

We analyse the convergence of our numerical scheme in the supremum norm and the 1-norm at $t = 0$; i.e. we calculate

$$Err_l^\infty := \frac{\|\alpha_{h_l} - \alpha_{h_{l+1}}\|_\infty}{\|\alpha_{h_l}\|_\infty} \quad \text{and} \quad Err_l^1 := \frac{\|\alpha_{h_l} - \alpha_{h_{l+1}}\|_1}{\|\alpha_{h_l}\|_1},$$

where α_{h_l} represents the approximate solution given the vector of mesh parameters h_l at refinement level l and

$$\|\alpha_{i,j}^k\|_1 := \sum_{i,j} |\alpha_{i,j}^k| \Delta d \Delta e.$$

The parameters that define the different mesh widths are displayed in Table A.1; here we carefully paid attention that our choice honours the Courant-Friedrichs-Lewy condition for the convergence of explicit schemes (cf. [LeVeque, 1990]). As mentioned above we expect the main contributions to the error to stem from the hyperbolic part of the equation. Therefore, we choose a very fine grid in the e -direction for our analysis.

¹For an in depth discussion of numerical schemes for these types of equations see [LeVeque, 1990].

	h_1	h_2	h_3	h_4	h_5
$\bar{\xi}/\Delta d$	6	12	24	48	96
$\bar{e}/\Delta e$	100	200	400	800	1600
$1/\Delta t$	110	440	1760	7040	28160

Table A.1: Parameters for the convergence analysis of the numerical scheme.

Table A.2 displays the results from our convergence study. Note that the error decays much faster in the 1-norm. This is not at all surprising, as we expect the error from the discontinuous terminal condition to propagate in the direction of decreasing e . This leads to a significant error on a small part of the grid, which is picked up by the infinity norm, whereas the approximation converges much faster everywhere else, as shown by our analysis of the error in the 1-norm.

l	1	2	3	4
Err_l^∞	0.0746	0.0355	0.0227	0.0105
Err_l^1	0.0066	0.0020	0.0013	0.0006

Table A.2: Numerical error in the supremum and the 1-norm.

Figure A.1 plots the error Err_l^∞ as a function of the mesh width. From the slope of the line of best fit through the error points, we estimate the convergence rate of our scheme to be 0.9131 in the infinity norm.

A.1.1.2 Multiple Compliance Periods

To deal with the path-dependency that enters the pricing problem in an emission market with multiple compliance periods — in the case of banking and withdrawal — through the terminal conditions (3.14) and (3.15), we need to introduce an additional variable corresponding to the cumulative emissions at time T_1 , that is, representing the random variable $E_{T_1}^1$, used in the definition of the aggregate supply of certificates during the second period $\tilde{\gamma}_2$, as defined in (3.13). We denote this variable by $e_1 \in [0, \bar{e}]$.

The problem is then solved backwards beginning with the second compliance period, for which we solve the partial differential equation (3.12) with corresponding boundary conditions and the terminal condition (3.15), taking into account the extra dimension introduced by the variable e_1 . We then store the values of $\alpha_2(T_1, d, 0, e_1)$

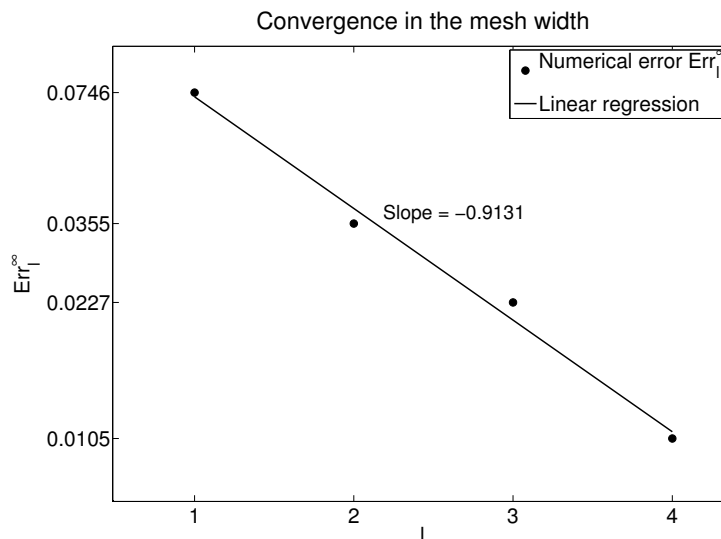


Figure A.1: Illustration of the convergence of the finite difference scheme used to solve equation (3.10).

calculated on the discrete mesh points associated with our chosen discretization of the now enlarged grid $(0, \bar{\xi}) \times (0, \bar{e}) \times (0, \bar{e})$.

Subsequently, we solve the partial differential equation corresponding to the first compliance period, using the stored values of α_2 for the evaluation of the terminal condition.

We compute the numerical approximation to α_1 and α_2 using the scheme described in A.1.1.1. For a market in which banking, borrowing and withdrawal are implemented, the terminal condition at the end of the first period is modified in the obvious way.

We note that compared with the single-period problem, the additional dimension of the multi-period problem (due to the variable e_1), increases the complexity in two ways. First, the partial differential equation (3.12) must be solved for a sufficiently large number of values of e_1 . Second, we need to store $\alpha_2(T_2, d, 0, e_1)$ on grid points, since it is needed for the terminal condition of α_1 .

A.1.1.3 Option Pricing Problem

We use the obvious modification of the earlier scheme described in Appendix A.1.1.1 to solve the option pricing partial differential equation (3.17).

A.1.2 Monte Carlo Simulation of Cumulative Emissions

For an index $k \in \mathbb{N} \cup \{0\}$, let (D_k, E_k, A_k) denote the discrete time approximation to the processes (D, E, A) on the time grid $0 < \Delta t < 2\Delta t < \dots < n_k \Delta t = T$. At each time step we calculate A_k by interpolating the discrete approximation $\alpha_{i,l}^k$ at D_k, E_k , beginning with the initial values $D_0 = d_0$ and $E_0 = e_0$. The approximations (D_k, E_k) are obtained using a simple Euler scheme (cf. [Glasserman, 2004]). The discretized version of D is forced to be instantaneously reflecting at the boundaries $D_k = 0$ and $D_k = \bar{\xi}$.

Using this discretization we simulate n_c paths and, as usual, calculate the mean cumulative emissions \hat{E}_T , given by

$$\hat{E}_T := \frac{1}{n_c} \sum_{i=1}^{n_c} E_{n_k}^i,$$

where $E_{n_k}^i$ denotes the outcome of the simulation of the i th path. The corresponding standard error $\hat{\sigma}_{\hat{E}}$ is obtained by

$$\hat{\sigma}_{\hat{E}} := \sqrt{\frac{1}{n_c(n_c - 1)} \sum_{i=1}^{n_c} (E_{n_k}^i - \hat{E}_T)^2}.$$

A.2 Numerical Schemes for the (4+1)-Dimensional Model

This appendix explains the numerical schemes that were used to produce the results in §5.4. Since the schemes are natural extensions of the ones explained in Appendix A.1 we shall be brief in our presentation.

A.2.1 Numerical Solution of the Allowance Pricing Problem

We comment on the numerical scheme employed to solve the nonlinear partial differential equation satisfied by the allowance price in the case of the model including stochastic fuel prices (§5.2).

A.2.1.1 Candidate Pricing PDE

Similar to our discussion in the main text of Chapter 3, in §3.4.2, we briefly address the question of the need to prescribe boundary conditions for the problem (5.15) given

the original stochastic problem (5.14) (cf. the discussion in §3.4.2). In the present case, the Fichera function f reads

$$\begin{aligned} f(t, d, s_c, s_g, e) := & \left(\mu_d - \frac{1}{2} \frac{\partial}{\partial d} \sigma_d^2 \right) n_d + \left(\mu_c - \frac{1}{2} \frac{\partial}{\partial s_c} \sigma_c^2 - \frac{\partial}{\partial s_c} \rho \sigma_c \sigma_g \right) n_c \\ & + \left(\mu_g - \frac{1}{2} \frac{\partial}{\partial s_g} \sigma_g^2 - \frac{\partial}{\partial s_g} \rho \sigma_c \sigma_g \right) n_g + \mu_e n_e, \quad \text{on } \partial U_T, \end{aligned} \quad (\text{A.1})$$

where $n := (n_d, n_c, n_g, n_e)$ is the inward normal vector to the boundary.

Evaluating (A.1) for the choice of coefficients presented in §5.4.1.4 and §5.4.1.5, reveals that no boundary conditions are necessary except when $e = \bar{e}$, where we prescribe

$$\alpha = \exp(-r(T-t))\pi, \quad \text{on } U_T|_{e=\bar{e}}. \quad (\text{A.2})$$

In addition we need to specify an asymptotic condition for large values of s_c and s_g . We choose to consider solutions that, for $j \in J$, satisfy

$$\frac{\partial \alpha}{\partial s_j} \sim 0, \quad \text{on } U_T|_{s_j \rightarrow \infty}. \quad (\text{A.3})$$

A.2.1.2 An Implicit-Explicit Finite Difference Scheme

We approximate the domain \bar{U}_T by a finite grid spanning $[0, T] \times [0, \bar{\xi}] \times [0, \bar{s}_c] \times [0, \bar{s}_g] \times [0, \bar{e}]$. For the discretization we choose mesh widths Δd , Δs_c , Δs_g , Δe and a time step Δt . The discrete mesh points $(t_k, d_m, s_{c_i}, s_{g_l}, e_n)$ are then defined by

$$\begin{aligned} t_k &:= k\Delta t, & d_m &:= m\Delta d, \\ s_{c_i} &:= i\Delta s_c, & s_{g_l} &:= l\Delta s_g, & e_n &:= n\Delta e. \end{aligned}$$

The finite difference scheme we employ produces approximations $\alpha_{m,i,l,n}^k$, which are assumed to converge to the true solution α as the mesh width tends to zero.

The partial differential equation (5.15) is posed backwards in time with a terminal condition and we choose a backward finite difference for the time derivative. In order to achieve better stability properties we make the part of the scheme relating to the linear terms of the operator \mathcal{N} implicit; the part relating to the first order derivative in the e -direction is made explicit in order to handle the nonlinearity.

We now use the same scheme as described in Appendix A.1.1.1 for the case of the $(2+1)$ dimensional model, the only addition being the s_c and s_g -directions, which are treated similarly to the d -direction. At the boundary corresponding to $s_c = \bar{s}_c$ and $s_g = \bar{s}_g$ we apply the asymptotic condition (A.3) as a boundary condition.

A.2.2 Numerical Calculation of Spread Prices

For the approximation of the values of clean spread options in §5.4.2 we used the following Monte Carlo method.

A.2.2.1 Time Discretisation of SDEs

Let $(D_k, S_k^c, S_k^g, E_k, A_k)$ denote the discrete time approximation to the FBSDE solution (D, S^c, S^g, E, A) on the time grid $0 < \Delta t < 2\Delta t < \dots < n_k \Delta t = \tau$. At each time step we calculate A_k by interpolating the discrete approximation $\alpha_{m,i,l,n}^k$ at (D_k, S_k^c, S_k^g, E_k) , beginning with the initial values $D_0 = d_0, S_0^c = s_0^c, S_0^g = s_0^g, E_0 = e_0$. The approximations (D_k, S_k^c, S_k^g, E_k) are obtained using a simple Euler scheme (cf. [Glasserman, 2004]). The discretized version of D is forced to be instantaneously reflecting at the boundaries $D_k = 0$ and $D_k = \bar{\xi}$; similarly, the discretized versions of S^c and S^g are made instantaneously reflecting at $S_k^c = 0$ and $S_k^g = 0$.

A.2.2.2 Monte Carlo Calculation of Option Prices

Using this discretization we simulate n_{mc} paths and, as usual, for $t \in [0, \tau)$, calculate the mean spark spread price \hat{V}_t , given by

$$\hat{V}_t := \exp(-r(\tau - t)) \frac{1}{n_{mc}} \sum_{i=1}^{n_{mc}} (b(D_{n_k}^i, S_{n_k}^{c,i}, S_{n_k}^{g,i}, A_{n_k}^i) - h_v S_{n_k}^{g,i} - e_v A_{n_k}^i)^+,$$

where the index i refers to the simulation scenario. The corresponding standard error $\hat{\sigma}_v$ is given by

$$\hat{\sigma}_v := \sqrt{\frac{1}{n_{mc}(n_{mc} - 1)} \sum_{i=1}^{n_{mc}} (V_{n_k}^i - \hat{V}_\tau)^2}.$$

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