

ISSN 1471-0498



**DEPARTMENT OF ECONOMICS**

**DISCUSSION PAPER SERIES**

**DEPARTURES FROM SLUTSKY SYMMETRY IN NONCOOPERATIVE  
HOUSEHOLD DEMAND MODELS**

**Valérie Lechene and Ian Preston**

Number 52

December 2000

Manor Road Building, Oxford OX1 3UQ

# Departures from Slutsky Symmetry in Noncooperative Household Demand Models\*

Valérie Lechene  
Wadham College, Oxford

Ian Preston<sup>†</sup>  
University College London and Institute for Fiscal Studies

September 2000

## Abstract

Maximisation of utility by a single consumer subject to a linear budget constraint is well known to imply strong restrictions on the properties of demand functions. Empirical applications to data on households however frequently reject these restrictions. In particular such data frequently show a failure of Slutsky symmetry - the restriction of symmetry on the matrix of compensated price responses. Browning and Chiappori (1998) show that under assumptions of efficient within-household decision making, the counterpart to the Slutsky matrix for demands from a  $k$  member household will be the sum of a symmetric matrix and a matrix of rank  $k - 1$ . We establish the rank of the departure from Slutsky symmetry for couples under the assumption of Nash equilibrium in individual demands with both partners contributing to all public goods. We show that the Slutsky matrix is the sum of a symmetric matrix and another of rank at most 2. This result implies not only that the Browning-Chiappori assumption of efficiency can be tested against other models within the class of those based on individual optimisation, but also that the hypothesis of Nash equilibrium in demands has testable content against a general alternative.

JEL: D11, C72

Keywords: Nash equilibrium, Intra-household allocation, Slutsky symmetry.

---

\***Preliminary.** We are grateful for comments from Martin Browning, Pierre-André Chiappori and participants at the Econometric Society World Congress 2000. This research has been partly funded by the ESRC Centre for the Microeconomic Analysis of Fiscal Policy at the Institute for Fiscal Studies and by the Leverhulme Trust. The views expressed are nonetheless those of the authors alone.

<sup>†</sup>Address for Correspondence: Ian Preston, Department of Economics, University College London, Gower Street, London WC1E 6BT. Email: i.preston@ucl.ac.uk

# 1 Introduction

Maximisation of utility by a single consumer subject to a linear budget constraint is well known to imply strong restrictions on the properties of demand functions. Empirical applications to data on households however frequently reject these restrictions (see for example, Browning and Meghir (1991), Banks, Blundell and Lewbel (1997) or Deaton (1990)). In particular such data frequently show a failure of Slutsky symmetry - the restriction of symmetry on the matrix of compensated price responses.

At the same time, the inadequacy of the single consumer model as a description of decision making for households with more than one member is increasingly being recognised. The problem in assuming families to behave as if they had the preferences of an individual was well recognised by Samuelson (1956), who established strong conditions under which such a simplified representation would hold. A large body of recent research has investigated alternative models accommodating more realistic descriptions of within-household decision-making processes.

An important advance is made by Browning and Chiappori (1998), who show that under assumptions of efficient within-household decision making, the counterpart to the Slutsky matrix for demands from a  $k$  member household will be the sum of a symmetric matrix and a matrix of rank  $k - 1$ . Tests on Canadian data are found to reject symmetry for couples but not for single individual households. However it is impossible to reject the hypothesis that the departure from symmetry for the sample of couples has rank 1 as required. This work is important not only in filling a gap in our theoretical understanding of demand behaviour but also in the prospect which it presents of reconciling demand theory and data on consumer behaviour.

However the assumption of efficiency is not satisfied in all models of household behaviour which have been suggested. It clearly holds, for instance, in the Nash bargaining models of Manser and Brown (1980), McElroy and Horney (1981) or McElroy (1990), and has been the central assumption of papers such as Browning, Bourguignon, Chiappori and Lechene (1994) or Bourguignon and Chippori (1994). However it is not a property of noncooperative models such as those of Ulph (1988) or Woolley (1988). While the inability of Browning and Chiappori to reject the symmetry and rank condition for couples is intriguing, it is not clear what if any power it has as a test of efficiency of intrahousehold decisions unless one understands the nature of the departure from symmetry under the principal alternative models of household decision making. In particular if noncooperative models were to give rise to a departure of similar rank then this would obviously not be a feature of demand behaviour which would be of use in discriminating between these alternative assumptions about within household behaviour. On the other hand, if the departure from symmetry under noncooperative behaviour were to be of greater rank then the Browning-Chiappori result would not only promise to reconcile assumptions of optimising behaviour with demand data but also provide evidence in favour of the collectively rational model against other descriptions of within-household decision making.

In this paper we establish the rank of the departure from Slutsky symmetry for couples under the assumption of Nash equilibrium in individual demands. We show that, for the case of equilibria with both partners contributing to all goods, the Slutsky matrix is the sum of a symmetric matrix and another of rank at most 2. We also establish conditions on partners' preferences under which such equilibria can hold showing that such cases are unlikely unless preferences over public goods are similar. For cases in which one or other partner fails to contribute to some public goods the rank of the deviation can generally be expected to be higher. This result implies not only that the Browning-Chiappori assumption of efficiency can be tested against other models within the class of those based on individual optimisation, but also that the hypothesis of Nash equilibrium in demands has testable content against a general alternative.

## 2 A non-cooperative Nash model of household demands

### 2.1 General framework

Suppose a household consists of two individuals labelled  $A$  and  $B$ . The household spends on a set of  $m$  private goods  $q$  and  $n$  public goods,  $Q$ . The quantities purchased by the individuals are  $q^A$ ,  $q^B$ ,  $Q^A$  and  $Q^B$  with total household quantities being  $q = q^A + q^B$  and  $Q = Q^A + Q^B$ . Individual utility functions are  $u^A(q^A, Q)$  and  $u^B(q^B, Q)$ . The two partners bring in separate incomes of  $y^A$  and  $y^B$ . Household income is denoted  $y$ . Prices of the two sorts of goods are vectors  $p$  and  $P$ .

Each person decides on the purchases made from their own income so as to maximise their own utility subject to the spending decisions of the partner. Hence,  $A$  chooses  $q^A$  and  $Q^A$  to solve

$$\max_{q^A, Q^A} u^A(q^A, Q^A + Q^B) \text{ s. t. } p'q^A + P'Q^A \leq y^A, Q^A \geq 0$$

and  $B$  chooses  $q^B$  and  $Q^B$  to solve

$$\max_{q^B, Q^B} u^B(q^B, Q^A + Q^B) \text{ s. t. } p'q^B + P'Q^B \leq y^B, Q^B \geq 0.$$

A noncooperative within household equilibrium consists of a set of quantities simultaneously solving these two problems. We assume continuity of the equilibrium in prices and incomes  $(p, P, y^A, y^B)$ .

We need to distinguish between two types of public good in equilibrium - those which are purchased in positive quantities by both partners (*jointly contributed public goods*) and those which are purchased in positive quantities by only one partner (*individually contributed public goods*). There are therefore three types of equilibrium to consider - those where all public goods are jointly contributed (*fully interior equilibria*), those where some but not all public goods

are jointly contributed (*partly interior equilibria*) and those where all public goods are individually contributed (*fully boundary equilibria*).

## 2.2 Fully interior solution

We consider firstly the case of fully interior solutions, where both partners contribute to all public goods. In such equilibria the nonnegativity constraint on public goods contributions bites on neither partner. We can write the agents' problems as

$$\max_{q^A, Q} u^A(q^A, Q) \text{ s. t. } p'q^A + P'Q \leq y - p'q^B, \quad Q \geq Q^B$$

and

$$\max_{q^B, Q} u^B(q^B, Q) \text{ s. t. } p'q^B + P'Q \leq y - p'q^A, \quad Q \geq Q^A.$$

and the quantities purchased will satisfy

$$\begin{aligned} q^A &= f^A(y - p'q^B, p, P) \\ q^B &= f^B(y - p'q^A, p, P) \\ Q &= F^A(y - p'q^B, p, P) \\ &= F^B(y - p'q^A, p, P). \end{aligned}$$

where  $f^A(\cdot)$  and  $F^A(\cdot)$  are (unrationed Marshallian) demand functions corresponding to  $A$ 's preferences and together satisfying the usual demand properties including Slutsky symmetry and  $f^B(\cdot)$  and  $F^B(\cdot)$  are demand functions corresponding to  $B$ 's preferences, of which the same is true.

By substitution between the first two equations

$$q^A = f^A(y - p'f^B(y - p'q^A, p, P), p, P)$$

which makes plain that the equilibrium value of  $q^A$  is determined by total household income  $y$  independently of the distribution between the partners. Further substitution establishes the same to be true for  $q^B$  and  $Q$ . This is the income pooling result of Bergstrom, Blume and Varian (1986) and other authors. Provided that a redistribution of income between the partners does not take the household out of fully interior equilibrium, quantities consumed in equilibrium are invariant

## 2.3 Separability restrictions

The conditions on preferences under which such equilibria can exist are certainly strong. Since both partners face the same relative prices, the same total quantities of the jointly contributed public goods can constitute solutions to the individual optimisation problems only if marginal rates of substitution between these public goods are the same for both partners. Given continuity in prices

this needs to hold throughout a neighbourhood of the equilibrium public good quantities - in other words, indifference curves in  $Q$ -space must coincide locally for the two partners.

One assumption that would allow this is that preferences over the public goods are separable within the preferences of both partners and that the individual sub-utility functions over these goods are the same ie

$$\begin{aligned} u^A(q^A, Q) &= v^A(q^A, \nu(Q)) \\ u^B(q^B, Q) &= v^B(q^B, \nu(Q)) \\ &\text{for some } \nu(Q) \end{aligned}$$

If private goods quantities  $q^A$  and  $q^B$  are variation-free across such equilibria<sup>1</sup> then this appears to be a necessary condition for existence of fully interior equilibria. For the time being we confine our attention to such a case. This clearly covers the special case in which there is only one public good.

The implications of this are extremely useful. In particular we know that two stage budgetting will hold (Blackorby, Primont and Russell 1978) with a common lower stage and therefore

$$\begin{aligned} F^A(y - p'q^B, p, P) &= g(P'F^A(y - p'q^B, p, P), P) \\ F^B(y - p'q^A, p, P) &= g(P'F^B(y - p'q^A, p, P), P) \\ &\text{for some } g(.,.). \end{aligned}$$

Hence

$$\begin{cases} F_y^i = g_x P' F_y^i \\ F_p^i = g_x P' F_p^i \\ F_P^i = g_x (P' F_P^i + Q') + g_P \\ \text{for } i = A, B \end{cases} \quad (1)$$

where  $g_x$  is the partial derivative of  $g$  with respect to its first argument.

Thus, the derivatives of the partners' demand functions are closely linked - for example, Engel curves for public goods are proportional

$$F_y^A / P' F_y^A = F_y^B / P' F_y^B.$$

Also, if we define  $R^i = [I_m - F_y^i P' / P' F_y^i]$  we have the following useful equations

$$\begin{aligned} R F_y^i &= 0 \\ R F_p^i &= 0 \\ R F_P^i &= g_P + g_x Q' \equiv \psi \\ \text{for } i &= A, B \end{aligned} \quad (2)$$

$$\begin{aligned} R F_y^i &= 0 \\ R F_p^i &= 0 \\ R F_P^i &= g_P + g_x Q' \equiv \psi \\ \text{for } i &= A, B \end{aligned} \quad (3)$$

where we may note that  $\psi$ , being a Slutsky matrix corresponding to the preferences  $\nu(Q)$ , is symmetric (and that  $\psi = 0$  in the case of a single public good).

---

<sup>1</sup>That is to say the value taken by  $q^A$  does not restrict the possible value taken by  $q^B$  at equilibrium.

## 2.4 Demand responses

Demand responses follow from

$$M \begin{pmatrix} dq^A \\ dq^B \\ dQ \end{pmatrix} = N_1 dy + N_2 \begin{pmatrix} dp \\ dP \end{pmatrix}$$

where

$$M = \begin{pmatrix} I_m & f_y^A p' & 0 \\ f_y^B p' & I_m & 0 \\ 0 & F_y^A p' & I_n \end{pmatrix} = \begin{pmatrix} I_m & ap' & 0 \\ bp' & I_m & 0 \\ 0 & F_y^A p' & I_n \end{pmatrix}$$

$$N_1 = \begin{pmatrix} f_y^A \\ f_y^B \\ F_y^A \end{pmatrix} = \begin{pmatrix} a \\ b \\ F_y^A \end{pmatrix} \quad \text{and} \quad N_2 = \begin{pmatrix} f_p^A - aq^{B'} & f_p^A \\ f_p^B - bq^{A'} & f_p^B \\ F_p^A - F_y^A q^{B'} & F_p^A \end{pmatrix}$$

Hence

$$\begin{pmatrix} dq^A \\ dq^B \\ dQ \end{pmatrix} = M^{-1} N_1 dy + M^{-1} N_2 \begin{pmatrix} dp \\ dP \end{pmatrix}$$

In typical budget surveys, we observe total household purchases rather than spending by individual members. In terms of household purchases  $q$  and  $Q$  we have

$$\begin{pmatrix} dq \\ dQ \end{pmatrix} = EM^{-1} \left( N_1 dy + N_2 \begin{pmatrix} dp \\ dP \end{pmatrix} \right)$$

where

$$E = \begin{pmatrix} I_m & I_m & 0 \\ 0 & 0 & I_n \end{pmatrix}$$

is an appropriate aggregating matrix.

The matrix  $M$  has a block upper triangular structure which makes it readily invertible. In a convenient representation

$$M^{-1} = \begin{pmatrix} I_m + \frac{B}{(1-AB)} ap' & -\frac{1}{(1-AB)} ap' & 0 \\ -\frac{1}{(1-AB)} bp' & I_m + \frac{A}{(1-AB)} bp' & 0 \\ \frac{B}{(1-AB)} F_y^A p' & -\frac{1}{(1-AB)} F_y^A p' & I_n \end{pmatrix}$$

$$EM^{-1} = \begin{pmatrix} I_m + \frac{1}{(1-AB)} (Ba + b)p' & I_m - \frac{1}{(1-AB)} (a + Ab)p' & 0 \\ \frac{B}{(1-AB)} F_y^A p' & -\frac{1}{(1-AB)} F_y^A p' & I_n \end{pmatrix}$$

where adding-up implies that  $A = 1 - P'F_y^A = p'a$  and  $B = 1 - P'F_y^B = p'b$ . Note that  $A$ ,  $B$  and therefore expressions of  $A$  and  $B$  alone are scalar.

## 2.5 Pseudo-Slutsky matrix

Browning and Chiappori (1998) define the pseudo-Slutsky matrix. In the present context we can see this as what would be calculated in place of the Slutsky matrix if the household were treated as behaving according to the unitary model. Thus the pseudo-Slutsky matrix has the form:

$$\Psi = EM^{-1} \left( N_2 + N_1 \begin{pmatrix} q \\ Q \end{pmatrix} \right)' = EM^{-1}\Phi$$

where

$$\begin{aligned} \Phi &= \begin{pmatrix} f_p^A + aq^{A'} & f_P^A + aQ' \\ f_p^B + bq^{B'} & f_P^B + bQ' \\ F_p^A + F_y^A q^{A'} & F_P^A + F_y^A Q' \end{pmatrix} \\ &= \begin{pmatrix} \Psi_{11}^A & \Psi_{12}^A \\ \Psi_{11}^B & \Psi_{12}^B \\ \Psi_{21}^A & \Psi_{22}^A \end{pmatrix}. \end{aligned}$$

Note that the terms in  $\Phi$  are all elements of the underlying true individual Slutsky matrices ie those corresponding to the individual decision problems.

By adding-up, we have the useful restrictions

$$\begin{aligned} p'f_y^i + P'F_y^i &= 1 \\ q^{i'} + p'f_p^i + P'F_p^i &= 0 \\ p'f_P^i + Q' + P'F_P^i &= 0 \\ \text{for } i &= A, B \end{aligned}$$

from which it follows that

$$\begin{aligned} p'\Psi_{11}^i + P'\Psi_{21}^i &= 0 \\ p'\Psi_{12}^i + P'\Psi_{22}^i &= 0 \\ \text{for } i &= A, B. \end{aligned}$$

Also from (2),

$$\begin{aligned} R\Psi_{21}^i &= 0 \\ R\Psi_{22}^i &= g_P + g_x Q' \equiv \psi \\ \text{for } i &= A, B \end{aligned} \tag{4}$$

By substitution, we can derive expressions for each of the terms in  $\Psi$

$$\begin{aligned}
\Psi_{11} &= \left(I + \frac{1}{(1-AB)}(Ba-b)p'\right)\Psi_{11}^A + \left(I - \frac{1}{(1-AB)}(a-Ab)p'\right)\Psi_{11}^B \\
&= \Psi_{11}^A + \Psi_{11}^B + \frac{1}{(1-AB)}[(b-Ba)P'\Psi_{21}^A + (a-Ab)P'\Psi_{21}^B] \\
\Psi_{12} &= \left(I + \frac{1}{(1-AB)}(Ba-b)p'\right)\Psi_{12}^A + \left(I - \frac{1}{(1-AB)}(a-Ab)p'\right)\Psi_{12}^B \\
&= \Psi_{12}^A + \Psi_{12}^B + \frac{1}{(1-AB)}[(b-Ba)P'\Psi_{22}^A + (a-Ab)P'\Psi_{22}^B] \\
\Psi_{21} &= \frac{1}{(1-AB)}BF_y^A p'\Psi_{11}^A - \frac{1}{(1-AB)}F_y^A p'\Psi_{11}^B + \Psi_{21}^A \\
&= \Psi_{21}^A + \Psi_{21}^B - \frac{1}{(1-AB)}BF_y^A P'\Psi_{21}^A - \left(I - \frac{1}{(1-AB)}F_y^A P'\right)\Psi_{21}^B \\
&= \Psi_{21}^A + \Psi_{21}^B - \frac{1}{(1-AB)}[BF_y^A P'\Psi_{21}^A + AF_y^B P'\Psi_{21}^B] + R^B \Psi_{21}^B \\
&= \Psi_{21}^A + \Psi_{21}^B - \frac{1}{(1-AB)}[BF_y^A P'\Psi_{21}^A + AF_y^B P'\Psi_{21}^B] \\
\Psi_{22} &= (1-AB)^{-1}BF_y^A p'\Psi_{12}^A - (1-AB)^{-1}F_y^A p'\Psi_{12}^B + \Psi_{22}^A \\
&= \Psi_{22}^A + \Psi_{22}^B + \frac{1}{(1-AB)}BF_y^A P'\Psi_{22}^A - \left(I - \frac{1}{(1-AB)}F_y^A P'\right)\Psi_{22}^B \\
&= \Psi_{22}^A + \Psi_{22}^B + \frac{1}{(1-AB)}[BF_y^A P'\Psi_{22}^A - AF_y^B P'\Psi_{22}^B] + R^B \Psi_{22}^B \\
&= \Psi_{22}^A + \Psi_{22}^B + \frac{1}{(1-AB)}[BF_y^A P'\Psi_{22}^A - AF_y^B P'\Psi_{22}^B] + \psi
\end{aligned}$$

Combining these expressions, it can be shown that the pseudo-Slutsky matrix  $\Psi$  admits the decomposition

$$\Psi = \Psi^A + \Psi^B + \Lambda + \Delta$$

where

$$\begin{aligned}
\Psi^A &= \begin{pmatrix} \Psi_{11}^A & \Psi_{12}^A \\ \Psi_{21}^A & \Psi_{22}^A \end{pmatrix}' \\
\Psi^B &= \begin{pmatrix} \Psi_{11}^B & \Psi_{12}^B \\ \Psi_{21}^B & \Psi_{22}^B \end{pmatrix} \\
\Lambda &= \begin{pmatrix} 0 & 0 \\ 0 & \psi \end{pmatrix} \\
\Delta &= \frac{1}{1-AB} \begin{pmatrix} b-Ba \\ -BF_y^A \end{pmatrix} P' \begin{pmatrix} \Psi_{21}^A & \Psi_{22}^A \end{pmatrix} \\
&\quad + \frac{1}{1-AB} \begin{pmatrix} a-Ab \\ -AF_y^B \end{pmatrix} P' \begin{pmatrix} \Psi_{21}^B & \Psi_{22}^B \end{pmatrix}.
\end{aligned}$$

Both  $\Psi^A$  and  $\Psi^B$  are individual Slutsky matrices and therefore symmetric as is  $\Lambda$ . The departure from symmetry therefore depends on the properties of the matrix  $\Delta$ . This, being the sum of two outer products of vectors, is plainly of rank no greater than 2 in general.

**Theorem 1** *At fully interior equilibria with public goods separable from private goods in the preferences of the two partners, the rank of the deviation from Slutsky symmetry is no greater than 2.*

It is interesting to ask under what conditions the rank can be less than 2. The rank of the deviation reduces to 1 if either  $\begin{pmatrix} b - Ba \\ -BF_y^A \end{pmatrix}$  is proportional to  $\begin{pmatrix} a - Ab \\ -AF_y^B \end{pmatrix}$  or  $P' \begin{pmatrix} \Psi_{21}^A & \Psi_{22}^A \end{pmatrix}$  is proportional to  $P' \begin{pmatrix} \Psi_{21}^B & \Psi_{22}^B \end{pmatrix}$ . Remembering that  $F_y^A/(1-A) = F_y^B/(1-B)$ , it can be shown that the former is true only if  $a/A = b/B$ , ie Engel curves for private goods are proportional. One special case of interest in which both are true is that in which the partners have identical preferences.

**Corollary 2** *At fully interior equilibria with identical preferences and public goods separable from private goods, the rank of the deviation from Slutsky symmetry is no greater than 1.*

A further special case is that in which  $\Delta$  is itself symmetric and the deviation from symmetry therefore disappears. This can be shown to hold for Cobb Douglas preferences.

## References

- [1] Banks, J., R. Blundell and A. Lewbel, 1997, Quadratic Engel curves and consumer demand, *Review of Economics and Statistics* 79, 527-539.
- [2] Becker, G. S., 1991, *A Treatise on the Family*, Cambridge: Harvard University Press.
- [3] Bergstrom, T.C., L. Blume and H. Varian, 1986, On the private provision of public goods, *Journal of Public Economics* 29, 25-59.
- [4] Blackorby, C., D. Primont and R. R. Russell, 1978, *Duality, Separability and Functional Structure: Theory and Economic Applications*, North-Holland, Amsterdam.
- [5] Browning, M., F. Bourguignon, P.-A. Chiappori and V. Lechene, 1994, Incomes and outcomes: a structural model of intrahousehold allocations, *Journal of Political Economy* 102, 1067-1096.

- [6] Bourguignon, F. and P.-A. Chiappori, 1994, The collective approach to household behaviour, in: *The Measurement of Household Welfare*, ed. R. Blundell, I. Preston and I. Walker, Cambridge: Cambridge University Press, 70-85.
- [7] Browning, M. and C. Meghir, 1991, The effects of male and female labor supply on commodity demands, *Econometrica* 59, 925-951.
- [8] Browning, M. and P.-A. Chiappori, 1998, Efficient intra-household allocations: a general characterisation and empirical tests, *Econometrica* 66, 1241-1278.
- [9] Deaton, A. S., 1990, Price elasticities from survey data: extensions and Indonesian results, *Journal of Econometrics* 44, 281-309.
- [10] Diewert, W. E., 1977, Generalized Slutsky conditions for aggregate consumer demand functions, *Journal of Economic Theory* 15, 353-362.
- [11] Kemp, M., 1984, A note on the theory of international transfers, *Economics Letters* 14, 259-262.
- [12] Manser, M. and M. Brown, 1980, Marriage and household decision making: a bargaining analysis, *International Economic Review* 21, 31-44.
- [13] McElroy, M. B., 1990, The empirical content of Nash-bargained household behaviour, *Journal of Human Resources* 25, 559-583.
- [14] McElroy, M. B. and M. J. Horney, 1981, Nash-bargained decisions: toward a generalisation of the theory of demand, *International Economic Review* 22, 333-349.
- [15] Samuelson, P. A., 1956, Social indifference curves, *Quarterly Journal of Economics* 70, 1-22.
- [16] Ulph, D. T., 1988, A general non-cooperative Nash model of household consumption behaviour, *University of Bristol Working Paper* 88/205.
- [17] Warr, P., 1983, The private provision of a public good is independent of the distribution of income, *Economics Letters* 13, 207-211.
- [18] Woolley, F., 1988, A non-cooperative model of family decision making, *STICERD Discussion Paper No TIDI/125*, London School of Economics.