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Introducing Students to the Role of Assumptions in Mathematical Activity

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ABSTRACT

Assumptions play a fundamental role in disciplinary mathematical practice, especially concerning the relativity of truth. However, much is still unclear about ways to help students recognize key aspects of this role. In this paper, we propose a set of principles for task design to introduce students to the role of assumptions in mathematical activity, with particular attention to the following two learning goals: recognize that (1) a conclusion depends on the assumption(s) underlying the argument that led to it; and (2) making the underlying assumption(s) explicit is crucial to reaching consensus on the conclusion. In the context of a 3-year design research study, we first used existing literature to construct an initial version of task design principles which we then empirically tested and refined by designing and implementing two tasks in Japanese school classrooms. One of the tasks was in the area of functions at the secondary level and the other in the area of geometry at the elementary level. We analyze two classroom episodes to discuss the promise and evolution of our proposed task design principles. In addition, our analysis sheds light on the role of the teacher's instructional moves and the students' mathematical knowledge during the implementation of the designed tasks.

Introduction

Educational researchers have long discussed the importance of introducing disciplinary practices into school subjects (e.g., Bruner, 1977; Goldman, 2023; Stylianides et al., 2022), an element also emphasized in curriculum frameworks in several countries. Disciplinary practices in this context mean those that experts in the fields—such as scientists, historians, and mathematicians—employ for their investigations. For example, the Next Generation Science Standards in the US (National Research Council [NRC], 2013) include three dimensions, the first of which relates to science and engineering practices that aim to engage students with processes authentic to the activities of professional scientists and engineers.

In this paper, we address one important aspect of disciplinary practice in mathematics: the role of *assumptions* in mathematical activity. Assumptions refer to statements that people use or accept (often implicitly) and on which their assertions are based (Stylianides & Stylianides, 2023). Certain aspects of assumptions have been considered in mathematics education. For example, one of the Standards for Mathematical Practice in the US Common Core State Standards for Mathematics states that, “mathematically proficient students understand and use stated assumptions, definitions, and

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previously established results in constructing arguments” (Common Core State Standards Initiative, 2010, p. 6). The need, however, for students to understand how assumptions affect conclusions and how different assumptions can lead to different conclusions is less emphasized.

The relativity of truth, namely, whether a claim is true or false cannot be determined in an absolute manner but hinges on assumptions, plays a fundamental role in disciplinary mathematical practice (Davies et al., 2021; Kline, 1980), such as the development of non-Euclidean geometries. To promote authentic mathematical activity at the school level mirroring disciplinary activity (Lampert, 1992; Stylianides et al., 2022), several researchers have highlighted the importance of helping students become more aware of the role of assumptions in mathematical activity, especially in relation to the notion of proof. For example, Fawcett (1938), who treated proof as a key notion in school mathematics, listed four criteria for checking students’ understanding of the nature of proof, two of which closely relate to assumptions: Students understand “the necessity for assumptions or unproved propositions” and “that no demonstration proves anything that is not implied by the assumptions” (p. 10). More recently, Cabassut et al. (2012) discussed metaknowledge about proof and suggested that the connection between mathematical theorems and assumptions should be explicitly taught to promote students’ appropriate understanding of mathematical proof.

Despite the importance of assumptions throughout school mathematics, most of the related studies have focused on the context of proving at the secondary school or university levels including teacher education (e.g., Cabassut et al., 2012; Dawkins, 2014; Fawcett, 1938; Jahnke, 2007; Jahnke & Wambach, 2013; Komatsu, 2017; Parenti et al., 2007; Stylianides & Stylianides, 2023).¹ Considering assumptions only in the context of proving is limiting because proof is usually not taught at the elementary school level, and even when it is taught at the upper levels, it is challenging both for teachers to teach and for students to learn (Stylianides et al., 2017). Thus, extending the contexts where the notion of assumptions is encountered and studied, including different types of mathematical activities and areas, is vital for expanding students’ opportunities to learn about assumptions. As argued in other topics, such as early algebra (Kieran et al., 2016), introducing the nature of assumptions to younger students would lead to a better continuity between mathematics teaching and learning in early and later grades.

An exception among related studies is the work of Stylianides (2007b, 2016). Stylianides explained that, while assumptions are associated with advanced mathematics including axioms and proofs, there are other kinds of assumptions, some of which involve less advanced topics, that have to do with the relativity of conclusions. Consider, for example, the task of finding how many different addition sentences there are for 5 (Stylianides, 2007b). One answer to this task is four sentences: $1 + 4 = 5$; $2 + 3 = 5$; $3 + 2 = 5$; and $4 + 1 = 5$. However, this conclusion is valid only under certain assumptions, namely, that the task refers to the sum of two numbers within the set of natural numbers and that commutative expressions are regarded as different. This conclusion is invalid if different assumptions are made, such as that it is permissible to use more than two addends or non-integers. This illustration shows the possibility of expanding learning opportunities about assumptions for younger students. However, there is not enough research on how to introduce students to the role of assumptions in mathematical activity (Stylianides et al., 2017).

To further explore this issue, we focus on task design. Tasks have a significant impact on students’ mathematical understanding because students ordinarily learn mathematics through working on tasks (Watson & Ohtani, 2015). Our study is concerned with developing principles that can guide the design of specific tasks, rather than creating specific tasks (Kieran et al., 2015). This is because general principles can function as guidelines that teachers can use to adapt existing tasks according to their classroom situations while maintaining the essence of the original tasks (Komatsu & Jones, 2019; Stylianides & Stylianides, 2013). These principles can also make it easier for teachers and curriculum authors to create relevant tasks themselves. In addition to our focus on tasks, the instructional triangle metaphor proposed in the literature (Ball & Forzani, 2009; Cohen

et al., 2003; Straesser, 2007) suggests the importance of considering also other factors, including students' mathematical knowledge and the teacher's role. We specifically consider the teacher's role in this paper because whether a designed task leads to meaningful classroom activity depends on the teacher's instructional moves while implementing it in the classroom. In our discussion of the teacher's role, we will additionally consider the issue of students' knowledge as appropriate.

In this paper, we report on a 3-year design research study we conducted in Japan to address the following research questions:

RQ1. What principles might support the design of tasks aimed at introducing students to the role of assumptions in mathematical activity?

RQ2. What is the role of the teacher's instructional moves during the implementation of the designed tasks?

Design research has both pragmatic and theoretical orientations (Bakker & van Eerde, 2015; Cobb et al., 2017; Gravemeijer & Cobb, 2006). The two orientations can develop in dialectic, which was the case in this study. From a pragmatic standpoint, we aimed to promote among students of different ages two learning goals related to the role of assumptions in mathematical activity: to recognize that (1) a conclusion depends on the assumption(s) underlying the argument that led to it, and that (2) making the underlying assumption(s) explicit is crucial to reaching consensus on the conclusion. From a theoretical standpoint, we aimed to theorize the ways through which the learning goals were supported in our study primarily through the development and iterative refinement of task design principles for introducing students to the role of assumptions in mathematical activity. In the theoretical framework section, we present an early version of the task design principles we constructed based on existing studies, such as research on productive ambiguity (Foster, 2011; Grosholz, 2007; Marmur & Zazkis, 2022; Stylianides & Stylianides, 2023). We tested and refined these principles through three design research cycles, in each of which we designed a task, implemented it in one or more classrooms, and analyzed the classroom implementation (Gravemeijer & Cobb, 2006). In this paper, we describe and analyze two classroom episodes—one in an elementary school classroom and another in a secondary school classroom—to discuss the promise and evolution of the early version of task design principles. In the discussion section, we propose a refined set of task design principles. Our classroom-based explorations in this study differed in several aspects, such as the types of assumptions embedded in the designed tasks, their mathematical domains, and the school levels where the tasks were implemented, which may imply the potential of the refined task design principles for broader applicability.

Before explaining the theoretical framework, we clarify the scope of our research. While student learning about assumptions can be a long process, the learning goals addressed in our study were modest in that they were addressed in a single lesson in each classroom, aiming to introduce students to the role of assumptions in mathematical activity. We use *introduce* to emphasize that we were not expecting that following this lesson students would fully understand the importance of assumptions and get to habitually pay attention to assumptions in their subsequent mathematical work. Rather, we viewed each lesson where one of our designed tasks was implemented as a first step toward exposing students to the relativity of conclusions in mathematics and as setting up the stage for a productive engagement with the notion of assumptions in students' subsequent mathematical work.

Theoretical framework

Our study explores a way to raise students' awareness of the role of assumptions, consistent with the key role of assumptions in mathematics. Thus, we begin by discussing the role of assumptions in the discipline of mathematics and, building on this discussion, we explain the learning goals that our study sought to promote. We then propose a new and expanded classification of the

notion of assumptions we used in our study to help young students meaningfully engage in mathematical activity related to assumptions and achieve the intended learning goals. Following this, we construct a set of task design principles from relevant studies and conclude by considering the role of the teacher's instructional moves during the implementation of the designed tasks.

The role of assumptions in mathematical activity

One of the roles that assumptions play in mathematics is to introduce the idea of the relativity of truth (Fawcett, 1938). For example, Euclidean geometry was initially accepted as the standard form of geometry, considered to be absolutely true. However, mathematicians then discovered that the fifth postulate regarding parallel lines in Euclidean geometry is independent of its other postulates and created new geometries, the non-Euclidean geometries, by adopting axioms different from the fifth postulate. Thus, it was recognized that some propositions in Euclidean geometry are true only under the set of Euclidean axioms and that different conclusions may be derived under different sets of axioms. For instance, the sum of the interior angles of a triangle is 180 degrees in Euclidean geometry, but more or less than 180 degrees in non-Euclidean geometries. To date, new mathematical theories have been constructed by changing existing axioms or freely setting up axiom systems that meet certain criteria.

Another role of assumptions is to settle disputes about the truth of conjectures. Mathematicians have succeeded in settling such disputes and reaching consensus by addressing the assumptions underlying the conjectures. Take, for example, the conjecture that the limit of any convergent series of continuous functions is itself continuous (Lakatos, 1976). While Cauchy gave a proof of this conjecture in the nineteenth century, a counterexample to the conjecture was found in Fourier's work. Mathematicians tried various explanations for this puzzling situation in which both the proof for, and the counterexample against, the conjecture coexisted. Finally, Seidel analyzed Cauchy's proof and discovered a hidden lemma which, once included as a condition in the conjecture, confirmed its truth. In addition, the definition of convergence mentioned in the conjecture was studied during this period. As a result, the concept of uniform convergence was created, and mathematicians agreed that the conjecture is true if the convergence mentioned in the conjecture refers to uniform convergence.

While the above discussion is based on historical episodes in mathematics, assumptions play a similar role in contemporary mathematical practice. Indeed, in Davies et al. (2021) survey, where mathematicians were given several descriptions of the meaning of proof and were asked to evaluate those descriptions, all top three highest-evaluated descriptions related to assumptions. Two of these descriptions characterized proof as being "a comprehensive logical argument that a statement is true, based on clearly formulated assumptions" and "a checkable record of reasoning establishing a fact from agreed, more basic assumptions" (Davies et al., 2021, p. 7). Assumptions thus play an essential role in dealing with the relativity of truth and in mediating disagreement in order to reach consensus in mathematical practice.

Given these two roles of assumptions in the discipline of mathematics and the importance of promoting authentic mathematical activity in school mathematics (Lampert, 1992; Stylianides et al., 2022), we established two learning goals for our study (similar to the elements of knowledge about assumptions discussed in the context of prospective elementary teacher education in Stylianides & Stylianides, 2023). We stated these goals in the Introduction and repeat them here for ease of reference:

Learning Goal 1: To recognize that a conclusion depends on the assumption(s) underlying the argument that led to it.

Learning Goal 2: To recognize that making the underlying assumption(s) explicit is crucial to reaching consensus on the conclusion.

Classification of assumptions

Different scholars, including mathematics education researchers and mathematicians, consider the meaning of assumptions in different ways. We begin with Fawcett's (1938) seminal work to examine the notion of assumptions. Fawcett distinguished between assumptions and definitions; for him, assumptions meant statements accepted as true without proof. Typical examples of assumptions are axioms, such as the fifth postulate in Euclidean geometry regarding parallel lines. Yet, assumptions in school mathematics are taken more broadly than axioms in mathematical research. For instance, in Euclidean geometry, the statement that the sum of the interior angles of a triangle is 180 degrees is deduced from a set of axioms. However, in the Japanese national elementary school mathematics curriculum, this statement is studied without proof and is only empirically verified with protractors. This statement is treated as a "local axiom" (see Stylianides, 2007a, 2016) and is used as a basis to show that the sum of the interior angles of a quadrilateral is 360 degrees. Students later learn to prove this statement in secondary school.

While Fawcett (1938) distinguished definitions from assumptions, Stylianides (2007b) suggested that one possible form that assumptions can take in school mathematics relates to the choice of definition, such as in the case of a "trapezoid" for which there are two common, albeit nonequivalent, definitions: "a quadrilateral with at least one pair of parallel sides" versus "a quadrilateral with exactly one pair of parallel sides." In cases like this one, the choice of definition need not be connected to correctness but rather "could be related to personal preferences, beliefs, values or the theoretical framework of context to which one refers" (Zaslavsky, 2005, p. 301). In the previous example, opting for the first definition of a trapezoid would allow consideration of a rectangle as a special case of a trapezoid, which in turn could reflect a value of *hierarchical classifications* of concepts (de Villiers, 1994). We acknowledge that a definition, being a statement that establishes the meaning of a mathematical concept, is often differentiated from an axiom in that a definition is a classification—an object does or does not meet the definition and therefore is or is not a member of the class—while an axiom is not. Yet, in school mathematics, definitions of new concepts often begin being unclear, with more precision added to them as the objects they are meant to describe get developed and better understood by classroom participants (Ball & Bass, 2000; Stylianides, 2007b).

In this paper, we consider definitions as part of assumptions partly for two reasons, both of which relate to the similarity between definitions and axioms. First, the truth of statements is often relative to the choice of definitions. This can be observed in the previous example about a trapezoid, as the statement "every rectangle is a trapezoid" is true or false depending on the choice of the two nonequivalent definitions (Stylianides, 2007b). Second, definitions can be flexibly selected as long as they meet certain criteria (Ouvrier-Buffet, 2006; Usiskin et al., 2008; van Dormolen & Zaslavsky, 2003). This applies for equivalent definitions as much as it applies for nonequivalent definitions, which we discussed earlier. For example, a parallelogram is usually defined as a quadrilateral in which two pairs of opposite sides are parallel. However, there are other statements equivalent to this definition. It is logically acceptable to adopt one of these statements as an alternative definition of a parallelogram such as, for instance, defining a parallelogram as a quadrilateral in which two pairs of opposite sides are equal in length. Furthermore, some properties of a concept can be more useful than its standard definition when examining the relationship between the concept and other related concepts.

As we discussed in the introduction, previous research in mathematics education has mainly addressed assumptions in the context of proving (e.g., Fawcett, 1938; Jahnke & Wambach, 2013; Komatsu, 2017). By contrast, we include a broader context of *problem solving* to expand students' opportunities to experience the relativity of conclusions in mathematics. In Polya's (1957) words, our study considers not only "problems to prove" but also "problems to find" (p. 154). Furthermore, we extend the notion of assumptions to include two additional types related to mathematical tasks and their formulation.

various aspects, including its relationship with other concepts² and its utility for proving relevant properties (Dawkins, 2014; de Villiers, 1994; Rupnow & Randazzo, 2023). In contrast, we consider the conditions of tasks and the meanings of lay terms to be *local* as these assumptions are considered within specific tasks, as illustrated in the examples above.

Axioms and the conditions of tasks are included in the *propositional* category in Table 1 because both types of assumptions can be characterized in the form of propositions; an axiom can be represented as “if p , then q ,” and a condition means p in this propositional form.³ The definitions of mathematical concepts and the meanings of lay terms are included in the *terminological* category in Table 1 because both types of assumptions are related to specifying the meanings of words.

There are three additional comments we consider important to make for Table 1. First, Table 1 does not necessarily represent all types of assumptions in mathematics; not all researchers agree on the meaning of assumption, and we have inductively derived the local assumption category from examples observed in the literature. Second, global assumptions relate to both advanced mathematics, including disciplinary practice, and school mathematics, as illustrated by the fact that we referred to both levels of mathematics when discussing axioms and definitions. Likewise, although we have illustrated local assumptions with tasks from elementary school mathematics, this type of assumption is also relevant to advanced mathematics. For instance, concepts such as congruence, isomorphism, and homeomorphism originate from the need to make the lay notion of *the same* precise within different mathematical contexts. Another illustration is the aforementioned history of convergence where mathematicians discussed a condition of a conjecture and then developed the definition of uniform convergence. Third, as can be observed in these illustrations, the four elements in Table 1 are not independent, and there are interactions between global and local assumptions.

As may be inferred from the observation that global and local assumptions are involved in both advanced and school mathematics, there can be no clear developmental progression from one type of assumption to the other, and so attention to one type should not presuppose attention to the other. Our study focuses on the local assumption types (i.e., the conditions of tasks and the meanings of lay terms) in the context of problem solving at the school level. Our choice to focus on this has been motivated partly from the realization that local assumptions have received relatively less attention than global assumptions in school mathematics education research and practice. This limited attention to local assumptions is a major missed learning and teaching opportunity, especially in elementary school, as the combination of local assumptions and problem solving is expected to be accessible even to young students. Indeed, the previous examples show that elementary students, unfamiliar with proof, can experience the relativity of conclusions in mathematics. Also, the other previous examples about different concepts for sameness and convergence, which indicate local assumptions are relevant to the discipline of mathematics, suggest that the limited attention to local assumptions in school mathematics further constitutes a missed opportunity for an additional authentic relationship between school and disciplinary practice.

Task design principles for the role of assumptions in mathematical activity

As we mentioned earlier, in this study we aimed to develop certain principles of task design by using design research methodology that is both theoretical and pragmatic (Cobb et al., 2017). Regarding the theoretical aspects, Prediger (2019) considered that a theory consists of several types of elements, where design principles correspond to predictive theory elements. She then characterized a developmental process of design principles as progressing from humble predictive heuristics without any justification to refined predictive theory elements that are explained and justified from the results of existing studies. Similar to this type of refined predictive theory

element, below we construct and explain task design principles for the role of assumptions in mathematical activity based on the literature.

Previous research has suggested features of tasks that can be used for introducing students to the role of assumptions in mathematical activity. Stylianides (2007b) analyzed an instructional episode in which elementary students worked on a purposefully-designed task and noticed that their responses to the task depended on assumptions they implicitly made about the conditions of the task. Stylianides pointed out three features of the task used in the episode in the context of proving, two of which are relevant to a broader context including problem solving: “the conditions of the task were ambiguously stated and, therefore, were subject to different legitimate assumptions,” and “the conclusions of the arguments [...] that students constructed (or could construct) based on each assumption appeared to be contradictory, thus surfacing the role of assumptions” (Stylianides, 2007b, p. 379).

While these features relate to the specific task used in Stylianides’s (2007b) study, and in the teacher education intervention reported by Stylianides and Stylianides (2023) more recently, they can be seen as general principles for task design that allow students to experience the relativity of conclusions in mathematics. One of the two features of the task mentioned above relates to task ambiguity. Research has shown that *productive ambiguity* enhances students’ mathematical learning (Barwell, 2005; Foster, 2011; Grosholz, 2007; Marmur & Zazkis, 2022; Stylianides & Stylianides, 2023). Foster (2011), for example, criticized the general tendency to associate ambiguity with something negative that should be avoided, such as misunderstandings and misconceptions. He suggested several types of ambiguity, including paradigmatic and definitional ambiguities. He argued that “ambiguity permits [a] variety of thought and expression and allows (forces, even) alternatives to be considered, providing students with the opportunity to probe mathematical structure” (Foster, 2011, p. 6).

In the context of our study, and given our interest in the relativity of conclusions, we considered it strategic to leave some of the assumptions of tasks implicit or unspecified. The typical way of presenting a task to students and immediately clarifying its assumptions eliminates the possibility of multiple interpretations of the task and reinforces intuitive connections between the clarified assumptions and the expected answer. One way to raise students’ awareness of this connection between assumptions and answers is to break the typical way of task presentation by creating a situation where a task assumption is purposefully unstated by the task or the teacher so that the students can work on the task under their own (implicit and potentially varied) assumptions. As shown by Stylianides and Stylianides (2023), this situation can lead to inconsistencies in student responses during whole-class discussion, prompting students to explore reasons for the inconsistencies. In doing so, the teacher can help students recognize that their answers depend on their particular assumptions and that an explicitly stated assumption is required in the task for them to reach a consensus on the conclusion.

This hypothesis is also supported by several studies conducted by Komatsu and colleagues in various environments, including paper-and-pencil and digital environments (Komatsu et al., 2017; Komatsu & Jones, 2020, 2022). For example, in Komatsu’s (2017) study, a secondary school class tackled a task in which one of the conditions of a statement was implicit in the given diagram.⁴ Similar to Lakatos’s (1976) case study on polyhedra, students faced a contradictory situation in which they proved the statement but found a case in which the conclusion did not follow. They then noticed the existence of a hidden assumption in the task, and they made the assumption explicit to resolve the contradictory situation (for more details about Lakatos-style mathematical activity, see Komatsu, 2016).

However, research on ambiguity also suggests that, in some cases, leaving the assumptions of tasks unclarified is not sufficient to raise students’ awareness of task ambiguity. Some studies refer to Byers (2007) for his definition of ambiguity as “a single situation or idea that is *perceived* in two self-consistent but mutually incompatible frames of reference” (Byers, 2007, p. 28; emphasis

added). So it depends on the solvers whether a situation or idea can be perceived in multiple ways. Marmur and Zazkis (2022) investigated the kinds of ambiguities prospective elementary school teachers saw in a task asking whether $\frac{1}{6.5}$ was a fraction. Some teachers stated that this number was a fraction without seeing any ambiguity, whereas other teachers stated that this number was not a fraction. This result suggests that, if it is anticipated that students may not produce different legitimate answers on their own, it is important to suggest to them the possibility that these answers exist, provided of course that the pedagogical aim is to have such varied answers emerge and linked to different underpinning assumptions.

Based on the above discussion, we propose the following principles for task design to achieve the two learning goals targeted in our study (i.e., Learning Goals 1 and 2):

Task Design Principle 1: Create a task open to different legitimate assumptions by purposefully leaving some of the assumptions of the task implicit or unspecified.

Task Design Principle 2: Create a task that allows different answers that appear contradictory but are legitimate according to respective legitimate assumptions about the task.

Task Design Principle 3: Create a task in which the possibility of different legitimate answers is implied, if such answers are possible but may not be produced by students themselves.

Although each principle is formulated as “Create a task ...,” it can also be used for adapting existing tasks. In addition, the previous discussion suggests that, while Task Design Principles 1 and 2 are essential, Task Design Principle 3 is optional and can be used in case students may not suggest, on their own, the expected viable answers given the ambiguity embedded in the task. In this paper, we describe two lessons, one of which was conducted with a task based on the first two principles and another with a task based on all three principles. Although the three principles evolved through our design research, as we will explain later, we still consider these principles to be preliminary at this stage and examine the teaching episodes to develop a refined version of the principles later in this paper.

The teacher’s role in implementing tasks

While our study focuses primarily on task design, we recognize the importance of connections between task design and teachers’ perspectives (e.g., Sullivan et al., 2015) or actions (e.g., Stylianides, 2016). The relationship between tasks and teachers is particularly critical in implementing tasks with ambiguity because, unlike typical student work on tasks with clear assumptions, when ambiguity is present, students would likely provide diverse responses based on their respective assumptions. Thus, teachers must skillfully handle how to show respect toward student responses, on the one hand, and lead the classroom to important mathematical ideas, on the other (Stylianides & Stylianides, 2023). The role of teachers in fostering classroom learning has been examined in several studies (e.g., Forman & Ansell, 2002; Jackson et al., 2013; Komatsu, 2017; Komatsu & Jones, 2022; Leatham et al., 2015; Sherin, 2002; Stylianides & Stylianides, 2022, 2023), among which Stein et al. (2008) suggested five practices for orchestrating productive mathematical discussions:

- (1) anticipating likely student responses to cognitively demanding mathematical tasks, (2) monitoring students’ responses to the tasks during the explore phase, (3) selecting particular students to present their mathematical responses during the discuss-and-summarize phase, (4) purposefully sequencing the student responses that will be displayed, and (5) helping the class make mathematical connections between different students’ responses and between students’ responses and the key ideas. (Stein et al., 2008, p. 321)

We employ Stein et al. (2008) five practices for addressing the role of the teacher in implementing the designed tasks. Our purpose is not to examine whether the teachers in our study were aware of Stein et al.’s work and intentionally tried to use the five practices but rather to

draw on these practices as a framework for analyzing the teachers' instructional moves. Having said that, teachers in Japan, especially experienced teachers, often engage with these practices without being aware of Stein et al.'s work itself, such as circulating the classroom to monitor student work and purposefully selecting particular students to share their ideas (Fujii, 2014; Ohtani, 2014; Stigler & Hiebert, 1999); this suggests that the five practices fit our research context. We did not ask the teachers implementing our tasks to make particular instructional moves because we were interested in how the tasks would play out in their ordinary teaching situations. The only exception to this was that we shared with the teachers our expectations of student responses to the tasks we designed, especially students' varied interpretations of the tasks and their respective different answers arising from the ambiguity of the tasks.

Methods

Task design

In this paper, we describe and analyze the implementation of tasks in school classrooms as part of a 3-year design research, which included three research cycles. In each cycle, we designed a task based on certain principles, one or more teachers implemented the task, and we analyzed the task's implementation. To explicate our expectations and analysis of the task implementation, we employ the notion of hypothetical learning trajectory.

Hypothetical learning trajectory

In some research, a "hypothetical learning trajectory" is defined to consist of the student learning goal, the tasks to be used to promote student learning, and the hypotheses about the process of student activities (Simon, 1995; Simon & Tzur, 2004). In this study, in the context of classroom learning and for reasons explained elsewhere (Stylianides & Stylianides, 2023, pp. 8–9), we use a *hypothetical learning trajectory* to refer to a presumed route that a classroom community is likely to follow, in completing tasks, toward a specified learning goal (Stylianides & Stylianides, 2009, 2014, 2022, 2023). In design research that examines the means to support student learning, as in our study, a specific learning goal is established and tasks are designed and implemented to achieve this goal. Thus, hypotheses about student processes include the goal specified in the context of particular tasks. Whether the learning goal is achieved is evaluated by comparing the hypothetical learning trajectory to what really occurred in the classroom during the implementation of the tasks, called the *actual learning trajectory* (Leikin & Dinur, 2003; Marmur & Zazkis, 2021; Stylianides & Stylianides, 2009, 2014, 2022, 2023). When significant discrepancies are identified between hypothetical and actual learning trajectories, tasks are revised and re-implemented as part of an iterative research cycle of design, implementation, analysis, and improvement (Gravemeijer & Cobb, 2006).

In this paper, we are concerned with developing task design principles aimed at introducing students to aspects of assumptions in mathematical activity. To this end, we explain hypothetical learning trajectories in the context of specific tasks we designed. We compare these hypothetical learning trajectories to actual classroom activities in our description of two classroom episodes later in paper.

Design of the Function and Geometry Tasks

This paper focuses on the implementation of the two tasks presented in Figures 3 and 4. We refer to these tasks as the Function Task and the Geometry Task, respectively. Table 2 shows the type of assumption (see Table 1) and school level considered in each task, and Table 3 explains how each task was designed based on the three task design principles presented in the previous

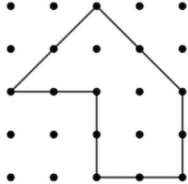
The table below represents y -values corresponding to the given x -values. Find y for $x = 6$.

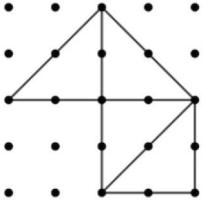
x	...	2	3	...	6	...
y	...	18	12

Figure 3. The Function Task (adapted from National Institute for Educational Policy Research [NIEPR], 2016).

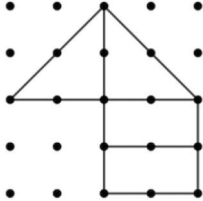
Daichi and Hinata solved the problem below in the following ways.

Problem. Divide the following figure into four same parts.





Daichi's solution



Hinata's solution

What do you think about these two solutions? Choose your thought from the options 1–4 below and explain the reason for choosing it.

- 1. While Daichi's solution is correct, Hinata's solution is not correct.
- 2. While Hinata's solution is correct, Daichi's solution is not correct.
- 3. Both Daichi's and Hinata's solutions are correct.
- 4. Both Daichi's and Hinata's solutions are not correct.

Figure 4. The Geometry Task (an adapted and expanded version of a task from Beckmann, 2005).

Table 2. Local type of assumption and school level considered in each task.

Task	Type of assumption (Table 1)	School level
The Function Task (Figure 3)	Condition of a task	Secondary school
The Geometry Task (Figure 4)	Meaning of a lay term mentioned in a task	Elementary school

section. We explain these two tasks below; in the next section, we will describe how these tasks and the design principles evolved throughout our research.

The Function Task was developed to address the condition of a task as a local type of assumption (see Table 1) and can be used with secondary students who are familiar with inverse proportions and linear functions. The crucial feature of this task regarding assumptions is that it does not specify the functional relationship between x and y . With this task design, we constructed a hypothetical learning trajectory in Table 4, showing our expectation that students' activity would

Table 3. Design of the tasks based on the task design principles.

Task	Relationship between the task and the task design principles
The Function Task (Figure 3)	<ul style="list-style-type: none"> • Principle 1: The functional relationship between x and y is not specified in the task. • Principle 2: The answer to the task is $y = 6$ if y is inversely proportional to x, and $y = -6$ if y is a linear function of x. (An infinite number of other legitimate answers are also possible.) • Principle 3: Not applicable.
The Geometry Task (Figure 4)	<ul style="list-style-type: none"> • Principle 1: The meaning of the term <i>same part</i> is not clarified in the task. • Principle 2: The answer to the task is Option 1 if the term “same” means that the divided parts are congruent, and Option 3 if this term means that the parts are equal in area. • Principle 3: Hinata’s solution, which can be legitimate but may not be produced by students themselves, is suggested.

Table 4. Hypothetical learning trajectories of the two tasks.

Learning goal	Task	Hypothesized student progression
Learning Goal 1: To recognize that a conclusion depends on the assumption(s) underlying the argument that led to it.	The Function Task (Figure 3)	<ul style="list-style-type: none"> • Students work on the task under their respective and implicit assumptions about the functional relationship between x and y. Some students consider the functional relationship as an inverse proportion and answer that $y = 6$, while others consider it as a linear function and answer that $y = -6$. • They notice this disagreement in the y-values in the subsequent whole-class discussion and realize that, for example, the answer $y = 6$ is legitimate <i>if</i> y is inversely proportional to x (Learning Goal 1). • They also recognize that, for example, if they specified that “y is inversely proportional to x” at the beginning of the task, then $y = 6$ would be the unique legitimate answer to the task (Learning Goal 2).
Learning Goal 2: To recognize that making the underlying assumption(s) explicit is crucial to reaching consensus on the conclusion.	The Geometry Task (Figure 4)	<ul style="list-style-type: none"> • Students work on the task under their respective and implicit assumptions about the meaning of the term “same” in the task. Some students consider the meaning of this term as “congruence” and choose Option 1, while others consider it as “equal areas” and choose Option 3. • They notice this disagreement in their choices in the subsequent whole-class discussion and realize that, for example, Option 1 is legitimate <i>if</i> the term “same” means congruence (Learning Goal 1). • They also recognize that, for example, if they stated “congruent parts” (instead of “same parts”) in the task, then Option 1 would be its unique legitimate answer (Learning Goal 2).

proceed in the following manner.⁵ According to this hypothetical learning trajectory, the students would first work on the task under different assumptions about the functional relationship and provide two contradictory answers, $y = 6$ and $y = -6$. Then, through discussing the validity of each answer with classmates, they would realize the relativity of their answers (Learning Goal 1)—for example, $y = 6$ is correct *if* y is inversely proportional to x . They would also recognize the importance of making explicit the functional relationship within the formulation of the task to determine the answer uniquely (Learning Goal 2)—for example, specifying “ y is inversely proportional to x ” at the beginning of the task would allow $y = 6$ to be its unique legitimate answer.

The Geometry Task was designed to address the meaning of a lay term as another type of local assumption (Table 1). It can be used with elementary students who are familiar with congruence and area of geometric figures. In this task, the meaning of the term *same part* is purposefully unspecified so that Hinata’s solution can be both valid or invalid under the respective assumptions, as discussed earlier. In addition, we presented as part of the task formulation Hinata’s solution, which students might not be able to devise by themselves, according to Task Design Principle 3. According to our hypothetical learning trajectory explained in Table 4, the students

would achieve the two learning goals targeted in this study in a similar way to what we have described earlier regarding the Function Task.

The two designed tasks are different in three aspects. The first difference is that the Function Task was designed according to Task Design Principles 1 and 2, which we consider essential as per our theoretical framework, while the Geometry Task was designed according to all three principles including Task Design Principle 3, which we consider optional (Table 3). The second difference relates to the two types of local assumptions in Table 1: The Function Task was designed to focus on a condition of a task, while the Geometry Task was designed to focus on the meaning of a lay term mentioned in the task (Table 2). Third, the two tasks relate to different mathematical domains—functions and geometry—and school levels—secondary and elementary (Table 2). Given these differences and previously described similarities between the two tasks, we considered them to be appropriate for an exploration of the flexibility and range of applicability of our task design principles.

Both the Function and Geometry Tasks are adaptations of existing tasks, and these adaptations are related to the task design principles of this study. The Function Task is adapted from a task used in the 2016 National Assessment of Academic Ability in Japan (NIEPR, 2016). The original version did not specify the functional relationship between x and y either, but rather it asked for the y -value when $x=4$, not when $x=6$. Based on a discussion with the teacher who implemented the task in his classrooms, we changed it to the case of $x=6$ because the difference in the y -values between the cases of inverse proportion and linear function in the case of $x=6$ is sharper, that is, $y=6$ or -6 which are opposite in signs. We took this reason to reflect the essence of Task Design Principle 2 (“different answers that appear contradictory”) in the task more clearly. Another reason was that we expected that this adaptation would lead students to spend more effort on tackling the task, and thus they would be more surprised when different answers were put forward. The Geometry Task was adapted from a task in Beckmann (2005). The difference from the original version is that we presented the two possible solutions to the task as multiple-choice options; our modification reflects Task Design Principle 3, as explained earlier.

Outline of the evolution of the task design principles

Next we explain how the design principles and the designed tasks evolved over the three research cycles of our study. Figure 5 depicts this developmental process. In cycle 1, we developed Task Design Principles 1 and 2 from the literature and used these principles in designing the Function Task. One teacher implemented this task in two lower secondary school classes, and the results were largely consistent with the hypothetical learning trajectory presented in Table 4 (Komatsu et al., 2019). In cycle 2, we further tested these two principles by implementing another task, presented in Figure 1, that incorporates the features of these principles in an elementary school class. However, the implementation did not work out as expected because the students provided only solution a shown in Figure 2; solution b was not proposed by any of them. This result led us to

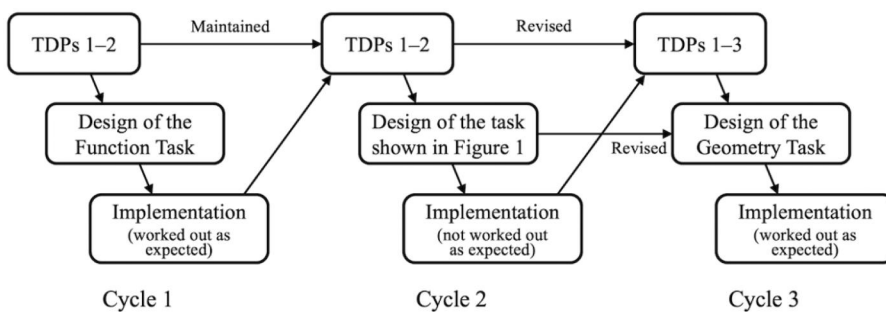


Figure 5. Developmental process of the Task Design Principles (TDPs).

realize that, even though various assumptions and conclusions are possible, students may not come up with them on their own, thus compromising the potential of the task to create a forum for raising and reflecting on the notion of assumptions. In cycle 3, we addressed this problem by adding Task Design Principle 3. We then used the three principles to design the Geometry Task (Figure 4). This task was implemented in three elementary school classrooms by two teachers, and the results nearly matched the hypothetical learning trajectory presented in Table 4. A difference between the hypothetical and actual learning trajectories observed in the class described in this paper was that students used a more specified expression than anticipated when revising the task to address the ambiguity. We discuss this issue later in the paper.

In this paper, we examine two classroom episodes, one from cycle 1 and another from cycle 3. As mentioned above, the three task design principles are still preliminary; at the end of the paper we will propose a more refined set of principles based on our analysis of these episodes.

The implementation of the tasks

The Function Task was implemented in a 50-minute lesson in two ninth-grade classes (aged 14–15 years) in a Japanese lower secondary school affiliated with a national university. It was implemented by the same teacher with 5 years of teaching experience. The mathematical attainment of students in the two classes was above average.⁶ In this paper, we focus on one class with 38 students because the two classes had similar characteristics, such as the extent of students' mathematical knowledge and class norms, and the implementation of the task played out almost identically in the two classes. Another reason for selecting this class is that the teacher was one of the two class teachers, responsible not only for teaching mathematics but also for the students' daily school life. We expected that the teacher would have a good relationship with the students and that the students would feel more comfortable to express their thoughts with him, which would be helpful for our data collection and analysis. The first author observed the students' extensive discussion during the lesson.

The Geometry Task was conducted in a 45-minute lesson in three fifth-grade classes (aged 10–11 years) across two Japanese public elementary schools. All three classes were composed of students with mixed mathematical attainments. The implementation was conducted by two teachers—one class by one teacher as a pilot and then two classes by another teacher. We focus on one of the two classes taught by the second teacher for the same reasons as for the Function Task. In particular, he was the regular teacher of the class, and so the study of the task implementation in this class would better reflect his everyday teaching situation since classroom teachers in Japanese elementary schools are responsible for teaching all school subjects, including mathematics.⁷ The class consisted of 22 students and the teacher had 20 years of teaching experience.

Regarding students' prior mathematical knowledge, the participating students had not been introduced to the notion of assumptions before, but they were familiar with the mathematical content appearing in the hypothetical learning trajectories presented in Table 4: inverse proportions and linear functions in the case of the Function Task, and the congruence and area of geometric figures in the case of the Geometry Task. In this paper, we also mention the students' prior knowledge about other mathematical topics. We are fairly confident in our description of their knowledge given the context of our research. Japan has a national curriculum developed by the Ministry of Education, Culture, Sports, Science and Technology, which highly prescribes what mathematical content should be taught in which grade. Textbooks are made by strictly complying with this curriculum,⁸ and teachers are required to follow textbooks in their daily classes. In this context, textbooks are essential resources for students' learning, and it can be assumed that students have studied a mathematical topic if it is included in the main part of the textbook they have used. Thus, we looked into the students' textbooks, including those from previous grades, to consider their expected prior mathematical knowledge.

We worked closely with the teachers in planning the implementation of the tasks in their classrooms. In each case, the first author shared the designed task with the teacher and discussed the intended learning goals and task design principles presented in the previous sections. In addition, we shared our anticipation of students' responses to the task, including some that are not mentioned in the hypothetical learning trajectory outlined in [Table 4](#). For example, in the case of the Function Task, we noted that students might suggest a step function because they had studied it before; in the case of the Geometry Task, students might provide another justification for the legitimacy of Option 3 by attending to the number of dots based on their previous related learning (we will revisit this point later in the paper). The teacher then created a lesson plan and discussed it with the first author. As mentioned earlier, the Function Task was modified from the original version at this stage. The teacher revised the lesson plan based on the discussion and again discussed the revised plan with the first author. The teacher then finalized the lesson plan and implemented the task in his classrooms, following the lesson plan closely as outlined below. The first author attended the implementations as a nonparticipating observer.

Data analysis

Each lesson was recorded with two or three video cameras. The data for our analysis included the video recordings, the transcripts made from these recordings, the students' worksheets, the teachers' lesson plans, the textbooks used by the students, and field notes taken by the first author during the observation of the lessons. We analyzed these data for each lesson in three phases. Phases 1–2 roughly correspond to RQ1, whereas phase 3 roughly corresponds to RQ2. In each phase, the first and second authors conducted the data analysis independently by working directly with the original data in the original language (Japanese). We then compared each analysis and discussed any discrepancies until we reached consensus.

In phase 1, we took a bottom-up approach, examining the video recordings and transcripts without any framework to divide the entire lesson into several parts. Here, a part means a period of time during which the class engaged with a particular type of activity. We then developed codes to represent these lesson parts, such as setting up the task, working on the task individually, sharing its answers, discussing why two different answers emerge, etc.

In phase 2, we adopted a top-down approach, focusing on the parts of the lesson related to the hypothetical learning trajectory presented in [Table 4](#), and compared the hypothetical and actual learning trajectories. In this way, we assessed whether Learning Goals 1 and 2 were met in the class. We paid attention to students' responses to the task, the teacher's supplementary questions, and students' responses to these questions. We looked at the class as a whole rather than at individual students' progression and, similar to Stylianides and Stylianides (2014, 2023), we judged that a student's thought expressed in the whole-class discussion was shared by the class if other students expressed similar thoughts or there was no objection from students during the class discussion. The latter was appropriate given that the students in the classes we worked with were expected, and felt comfortable, to express their ideas including objections during mathematics lessons. For example, we judged that Learning Goal 1 was met in a class—in other words, that the relevant part of the hypothetical learning trajectory in [Table 4](#) matched the actual learning trajectory—when a student shared the relative view of correctness stating that an answer to the task was correct under a particular assumption and other students agreed with this relative idea while none voiced any disagreements. To corroborate our analysis, we also looked at students' worksheets to examine how they engaged in the task and the teacher's questions and how they internalized the learning experience from the conducted lesson.

In phase 3, we turned to the teacher's role in implementing the task with a top-down approach in which we employed Stein et al. (2008) five practices for orchestrating productive whole-class discussions: anticipating, monitoring, selecting, sequencing, and connecting. We looked at the

teacher's lesson plan to see whether he had anticipated, or had considered our anticipation of, specific student responses. The video recordings were used for observing the teacher's monitoring activities. We regarded that the teacher selected a particular student's response if he called on the student to share his/her idea in a whole-class discussion and if he had anticipated this kind of response in his lesson plan and monitored the student's work while circulating the classroom. Regarding sequencing, we focused on the moment when different students' responses emerged and then the teacher chose one of them for discussion, and we checked his lesson plan to see whether this sequencing was intentional. We regarded a moment as connecting if the teacher asked students to examine different students' responses presented in a whole-class discussion or if he related a student's thought to the learning goals targeted by the implementation of the task related to assumptions. After this analysis, we additionally noted the teacher's instructional moves that did not correspond to Stein et al. (2008) five practices but were important for guiding the classroom to the intended learning goals. We then investigated whether these instructional moves were commonly observed in both the elementary and secondary school classes.

All student names in this paper are pseudonyms. The term "student" is used in the transcripts when we cannot identify the student who spoke; "students" is used when multiple students spoke. Student and teacher utterances and student comments from their worksheets were translated from Japanese into English by the first author. We have added words in square brackets to the transcripts for clarity; ellipses indicate that irrelevant parts and repetitions have been omitted for ease of reading. The number before the student's name (or "teacher") in each transcript indicates his/her turn's number in the classroom discussion. Some transcripts are not consecutive in number because we omit some conversations to focus on the entire flow of the classroom discussion.

Classroom implementation of the Function and Geometry Tasks

Episode 1: condition of task (the Function Task)

Relative legitimacy of answers

We begin with the description of implementing the Function Task shown in Figure 3 in a lower secondary classroom (Komatsu et al., 2019).⁹ The teacher started the lesson by introducing the task and asked students to work on it alone, without discussing it with their neighbors. The students spent five minutes on this work, during which time some students provided the answer $y = 6$, while others provided the answer $y = -6$. This disagreement surfaced during the subsequent whole-class discussion led by the teacher. Misaki first said, "I think y is 6," but a student objected to her answer, "That's wrong." Aoi then said, " $[y =] -6$," but Shun questioned her answer, saying, "What?" The teacher connected these students' responses by posing to the students the pre-planned question of which answer, $y = 6$ or $y = -6$, was correct:

- 31 Teacher: Well, now there are two answers. Which is correct?
- 32 Students: Both are correct answers.
- 33 Teacher: Really? Are both correct answers? Please think everyone. [The students start to discuss with their neighbors.] [...] [Speaking to the whole class] Please write your thoughts, write your thoughts [on the worksheets] about whether both answers are correct.
- 34 Nanami: What? Which is the conclusion?
- 35 Takumi: I don't know.
- 36 Teacher: Well, so, what do you think? Please write your thoughts.

As seen in this interaction, an anticipated sense of confusion emerged among students. While some of them considered both answers to be correct (line 32), others had no idea about which

the correct answer was (lines 34 and 35). The teacher took advantage of this confusion and asked students to think more carefully about whether both answers were correct and to describe their thoughts on their worksheets (lines 33 and 36). Afterward, the teacher asked students to share their thoughts in a discussion with the whole class:

- 38 Teacher: Well, I would like to listen. Okay, all of you have written quite well, so good. Ren, can you share? Yeah, Ren, can you share?
- 39 Ren: Um, I think both [answers] are correct. These two [$y = 6$ and $y = -6$]. Since we don't know what the function of this graph is, since we can interpret it as both an inverse proportion and a linear function, I think both are correct.
- 40 Teacher: You've described your opinion that both are correct. Well, let's listen to another student. Riko, can you share [your opinion]? What do you think?
- 41 Riko: Um, I also think both are correct. If we change how we read the table, it becomes an inverse proportion and also a linear function.

Although the teacher selected Ren and Riko based on his monitoring of their worksheets, their responses (lines 39 and 41) are representative of the students' responses and thus show that the classroom activity closely matched the hypothetical learning trajectory relevant to Learning Goal 1 (Table 4). In other words, the students understood that the legitimacy of their responses depended on their assumptions about the condition of the task, namely, the functional relationship between x and y : They indicated that y is 6 if y is inversely proportional to x , and that y is -6 if y is a linear function of x . All students in the class agreed that both answers could be legitimate, and most students shared or wrote down similar thoughts. For example, Mizuki wrote on her worksheet, "The answer is 6 if we interpret it as an inverse proportion, but -6 if it is a linear function." Similarly, Nanami wrote, "The graph can become a curve where the y -value is 6 and also become a line where the y -value is -6 , so I think both values are correct, depending on how we think about it."

Two ways to pin down the answer

While the students recognized the relativity of their answers, they started to feel dissatisfied with the task they tackled: Nanami said, "That's not good. [...] I don't know what I am solving." The teacher responded to the students' feelings of dissatisfaction by posing two pre-planned questions (written on the blackboard): "(1) Why are the answers divided [between $y = 6$ and -6]? (2) What should we do to determine a unique answer?" Students were given 10 min to consider these questions before sharing their thoughts in a new whole-class discussion. The students' responses were divided into two groups, represented by Kenta's and Shun's contributions, whom the teacher selected based on his anticipation and monitoring of student work:

- 75 Kenta: Um, um, [regarding question] number 1, [this] is because there is no explanation about whether y is proportional to x or inversely proportional. Um, if there are only points (2, 18) and (3, 12), equations are possible both for $y = 6$ and -6 . Um, if we interpret it as the graph of the inverse proportion of $y = 36/x$, um, we have 6. If we interpret it as the graph of the proportion of $-6x + 30$, we have -6 , I think. Well, [regarding question] number 2, I think, we should add an explanation, like when y is proportional to x or when y is inversely proportional to x . [Kenta mis-spoke here. He meant linear functions when talking about proportions.]
- 88 Shun: Um, because there are only two given points, the answer changes depending on whether we connect the two points with a line or connect [them] with a curve. [Regarding question] number 2, I think, if we determine the values of three or more points, the answer will be [uniquely] determined.

According to the first group of students (illustrated by Kenta's comment), the reason for the answers to the task to be divided, namely, $y=6$ or -6 , was the absence from the task description of a specification of the functional relationship between x and y ; the way this group suggested to address this problem was to clarify this relationship in the task. According to the second group of students (illustrated by Shun's comment), the cause of the two different answers was that only two pairs of values were given in the task. This group suggested that this problem could be solved by providing a third point with given values in the task.

Necessity of clarifying the conditions of the task

The teacher had anticipated these two kinds of student responses and addressed them in a pre-planned sequence, beginning with the second group's suggestion to give a third point with specific x - and y -values. He plotted the points (2, 18) and (3, 12) on the coordinate plane on the blackboard and said: "Suppose that, in the case of [...] 4, we have 6 [meaning $y=6$ for $x=4$]. Is this a linear function? We know three points." Here the teacher selected (4, 6) because the y -value in the case of $x=4$ is 6 if the function is assumed to be linear.

A student, Shota, responded to this teacher's question by suggesting an alternative possibility of the function, saying, "Maybe [it is] a step function." This suggestion of a step function, which the teacher had anticipated to come from students since the class had studied it before, greatly surprised the other students, who later agreed with Shota and acknowledged that the function in the task could not be uniquely determined even if they were given three pairs of x - and y -values. Below is the whole-class discussion after Shota drew the graph shown in Figure 6.

- 138 Teacher: [Speaking to the whole class] Well, in this case, what happens? This, this, well, what did you [Shota] draw?
- 139 Shota: A step function.
- 140 Teacher: Oh, yes, thank you. [Speaking to the whole class] Is this a linear function?
- 141 Students: No.
- 142 Teacher: Yeah. Let's review. In the case of 6, this 5, 6, the point in this case is ... [The teacher added the dotted vertical line in Figure 6 and suggested a question asking the y -value in the case of $x=6$.]
- 143 Nanami: Anything [would be] okay [meaning that the y -value can be any value].
- 144 Teacher: Can you predict [the y -value]?
- 145 Nanami: Anything [would be] okay.

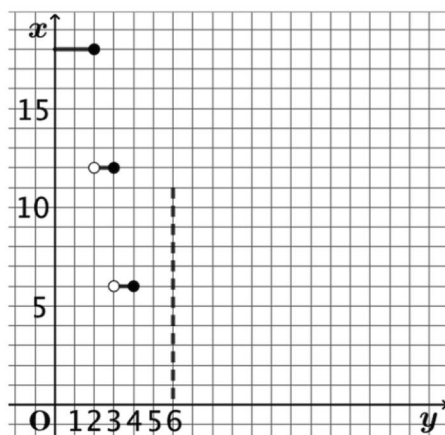


Figure 6. Step function drawn by Shota (the dotted vertical line was later added by the teacher).

- 146 Moe: [It can] quite suddenly change.
 147 Nanami: [It can] change suddenly, suddenly.

In the above interaction, the students stated that even if another point (4, 6) was added to the table given in the Function Task (Figure 3), the function could be a step function rather than linear, and the y -value in the case of $x = 6$ could not be uniquely determined (lines 143 and 145–147). Thus, the class recognized that specifying the third point was not a viable option for resolving the ambiguity of the task. Considering the step function motivated the class to look for another way to specify the answer. They returned to the way Kenta had proposed earlier (line 75), which was to specify the functional relationship between x and y in the task. In the end, the teacher summarized the way to address the task ambiguity as follows:

- 194 Teacher: Making this clear [meaning specifying the functional relationship] is good. I introduce, this [pointing “in the case of a linear function” and “in the case of an inverse proportion” written on the blackboard], this, the one shown at the beginning is, [...] this is called an *assumption*, *assumption*. So, I believe you see that the answer becomes unique by this assumption.

In this comment, the teacher connected and compared two different ideas for addressing the task’s ambiguity (lines 75 and 88). In addition, he connected the classroom activity to the notion of assumption by introducing this term and clarifying that the conditions, such as “ y is a linear function of x ,” were assumptions one could make based on the task’s phrasing. Furthermore, he mentioned that making assumptions explicit was crucial to reaching consensus. This comment by the teacher corresponds to the hypothetical learning trajectory related to Learning Goal 2 (Table 4), which the class achieved, as indicated by students’ comments at the end of the lesson when they summarized their learning. For example, Yuka said, “If there is no assumption, different functions can be considered. So, when we want to make the answer unique, we write the assumption.” Other students wrote similar comments on their worksheets. For instance, Kaito wrote, “I have learned that we cannot answer when we have this kind of problem without its assumption. The answer was not determined due to the step function, so [it is necessary to] show the assumption when we want to determine one answer.” Thus, the two questions (1) and (2) posed by the teacher and the subsequent classroom discussion led by him raised the students’ awareness of the importance of clarifying assumptions.

Episode 2: meaning of a lay term mentioned in the task (the Geometry Task)

Relative legitimacy of answers

Next, we describe and analyze the implementation of the Geometry Task presented in Figure 4 in an elementary school classroom. The teacher began the lesson by presenting the task and, similar to the case of the Function Task, gave the students 10 minutes to engage with it individually without discussing it with the students sitting nearby. The students then shared their thoughts in the whole-class discussion. Among 22 students, 17 selected Option 1—Daichi’s solution is correct, whereas Hinata’s solution is not correct—and justified their choice as follows:

- 16 Miyu: Well, Daichi divides it into four same triangles, but Hinata divides it into two triangles and two rectangles. Since it is written as “into four same parts” in the task, I think Hinata’s way of dividing two each, where the shapes are different, is wrong, and [...] Daichi is correct as he divides it into the same triangles.

- 20 Haruto: Well, because the problem is to divide it into four same parts, I think only Daichi's way of dividing it into all congruent triangles is correct.

The teacher then asked the class whether any students selected another option; four students raised their hands and argued in favor of Option 3—both Daichi's and Hinata's solutions are correct:

- 49 Yuto: [Option 3 is correct] because both consist of four parts.
 50 Teacher: [Speaking to the whole class] Um, what do you think? Both are four parts.
 51 Student: Right.
 52 Teacher: Right?
 53 Riku: Part is part. I want to add to it [what Yuto has said].
 54 Teacher: Yes, Riku.
 55 Riku: Well, it is because part is not shape. I think it is because the rectangles and triangles have the same, same size, and like the areas are the same.
 56 Teacher: [Speaking to the whole class] Do you understand what Riku has said? Is it true what Riku has said?
 57 Student: Ah, it is certainly the same.
 58 Eita: Certainly, the areas may be the same.

As seen in these interactions, the students had two contradictory thoughts regarding whether Hinata's solution was legitimate. Many students initially considered the meaning of *same* in the task as *congruence* and rejected Hinata's solution because the triangles and rectangles were not congruent (lines 16 and 20). However, Riku shared a different idea that considered the *same* as *equal area* and accepted Hinata's solution because the areas of the triangles and the rectangles were equal (line 55). Riku's argument prompted some students to reconsider their views, and they acknowledged the legitimacy of Hinata's solution (lines 57 and 58).

The teacher had anticipated that students would express these two conflicting ideas and connected these ideas by asking students to consider which Option, 1 or 3, was correct. The students discussed this with their neighbors, and the teacher then held a whole-class discussion for the students to share their thoughts. He initiated the class discussion by intentionally selecting Yui who he had heard talking while he was circulating around the classroom and monitoring student discussions. Yui shared another reason for the validity of Hinata's solution, saying, "I think it is Option 3 because [the problem] does not say the same shape, and the numbers of surrounding dots are also the same." Here, Yui pointed out that each of the two triangles and the two rectangles shown in Hinata's solution included an equal number of dots, namely, six dots, on its vertices and sides. The teacher anticipated this idea would come up because the students had previously studied a related topic of counting the numbers of people standing on various geometric shapes under a certain condition.¹⁰

While Yui's reasoning led more students to acknowledge the validity of Hinata's solution, some students began to pay attention to the relativity of their answers:

- 110 Teacher: You have begun to accept Option 3. Anyone who still chooses Option 1? [...]
 [Eita raises his hand] Okay, Eita.
 111 Eita: Well, what I said previously.
 112 Teacher: It's okay.
 113 Eita: I want to know the meaning of the *part*.
 114 Teacher: The meaning of the part. [Speaking to the whole class] Does it change depending on the meaning of the part?
 115 Students: [Many students express their opinions; inaudible]

- 116 Teacher: Can you specify what changes and how, depending on how we consider the part? Like, if we consider the part in this way, it will be this [meaning that the answer to the task will be a certain one]. [...] Is there anyone who can specify? [Miyu raises her hand] Okay, Miyu.
- 117 Miyu: If we consider the part as shape, the answer is Option 1, 1.
- 118 Teacher: If we consider it as shape, it is Option 1. But, can't we say this [triangle] and this [rectangle] are the same shapes?
- 119 Eita: No. [They are] quadrilateral and triangle.
- 120 Teacher: I see. If we mean shape, it is Option 1.
- 121 Riku: And, if we mean size, it is Option 3. Area, area.
- 122 Teacher: Ah, if we consider the part as area, [the answer is Option 3.] If we consider the term part as shape, both are correct. No, only Option 1 is correct, only Daichi's solution. If it means area, it is certainly the same, and it is Option 3.

This classroom discussion shows the close agreement between the hypothetical learning trajectory and the actual classroom activity regarding Learning Goal 1 (Table 4): The students understood that Option 1 is legitimate if the term *same part* in the task means that the divided parts are the same in shape (i.e., congruent), and that Option 3 is legitimate if this term means that the divided parts are the same in size or area (lines 117 and 121). These understandings are also clearly illustrated in the students' worksheets. For example, Kazuma wrote, "Different people considered *part* differently, and I thought it was absolutely Option 1, but I listened to others' opinions and realized that *there are other ways of thinking*" (emphasis in the original).

Clarification of the meaning of the lay term

Although the students admitted the relativity of their answers, they were not satisfied with this situation and were eager to determine an answer. To address the feelings of these students, the teacher asked the students to change the task formulation so that only Option 1 could be legitimate. More specifically, the teacher wrote on the blackboard, "Divide the following figure into _____," and asked the students to fill in the blank to construct a task where only Daichi's solution would be valid. The students worked on it for two minutes, and four students' ideas were shared in the subsequent whole-class discussion:

- 160 Kazuma: Divide the following figure into the same shapes.
- 162 Yui: Divide the following figure into triangles.
- 164 Miyu: Divide it into four congruent triangles.
- 166 Sakura: Divide it into four same shapes.

The teacher connected these four ideas by asking the class to check if these ideas allowed only Daichi's solution to be correct. Students rejected the idea suggested by Yui (line 162) because this sentence did not specify the number of triangles into which the figure would be divided. For example, the figure can be divided into 16 smaller congruent triangles. The teacher added that even if the number was specified as "... into four triangles," there would be other ways of dividing than Daichi's solution, such as dividing it into four non-congruent triangles.

The students considered that the most appropriate idea was the one proposed by Miyu (line 164). While this sentence allowed only Daichi's solution to be valid, it included excessive specification in that the term four congruent *figures*, not four congruent *triangles*, is sufficient. In other words, Miya's formulation was not minimal (Aktaş, 2016; van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005). The teacher raised this issue by asking, "Is the word 'triangles' necessary? Is [the word] 'congruent figures' wrong?" However, the students responded that the word

“triangles” was necessary, so they did not find the uneconomical aspect of Miya’s formulation problematic. We regard this as a minor point given that the participants were elementary school students unfamiliar with the mathematical custom of seeking minimal definitions.

The above analysis shows that the students wanted to unambiguously determine the answer to the task and used the notion of congruence for this purpose. This classroom activity corresponded to the hypothetical learning trajectory for Learning Goal 2 (Table 4), and the achievement of this goal was also evident from students’ worksheets. For instance, Yui wrote, “It was one word, but different people considered it differently, and the answers were also different, so I have realized that it is important to make a sentence that everyone considers in the same way by, for example, selecting a particular word that suits the intended answer.” Thus, this classroom lesson provided the students with an opportunity to recognize that specifying the meanings of lay terms in tasks is crucial for determining the conclusions uniquely.

Discussion

We begin the discussion by examining the classroom implementation of the two tasks to refine the task design principles proposed in the previous section; this refinement relates to RQ1 of this paper and is consistent with design-based research whereby each research cycle has the potential to generate new insights into the phenomenon of interest (Cobb et al., 2017). Following that, we address RQ2 by discussing the teacher’s role in implementing the designed tasks and extend this discussion to consider aspects of students’ mathematical knowledge. Finally, we suggest implications for teaching and future research.

The refinement of task design principles

In this study, we have addressed the role of assumptions in mathematical activity as an aspect of authentic disciplinary practice (Goldman, 2023; Lampert, 1992; Stylianides et al., 2022). We developed task design principles for this aspect and implemented tasks designed according to these principles in elementary and secondary classrooms. Our analysis of the two episodes illustrates the close correspondence between the hypothetical and actual learning trajectories, showing that Learning Goals 1 and 2 were achieved—recognition that a conclusion depends on the assumption(s) underlying the argument that led to it, and that making the underlying assumption(s) explicit is crucial to reaching consensus on the conclusion.

While the designed tasks were essential for achieving the intended learning goals, the analysis of the episodes indicates that extra sub-tasks the teachers gave to the students were also crucial for leading the classrooms to Learning Goal 2. These sub-tasks were the one asking, “What should we do to determine a unique answer?” in Episode 1, and the one asking students to fill in the blank in “Divide the following figure into _____” in Episode 2. Both tasks are similar in that each one invited the students to revise the task sentence to allow only one legitimate answer. In addition, the teachers encouraged the students to scrutinize whether their suggestions for revising the task sentences could eliminate the ambiguity of the tasks.

The relevance of this kind of opportunity to revise tasks has been discussed in several studies (Komatsu et al., 2014; Komatsu, 2017; Stylianides, 2007b). For example, Stylianides (2007b) suggested one possible instructional sequence regarding the role of assumptions in mathematical activity in the context of definitions. This sequence includes the question of which definitions could eliminate task ambiguity and allow everyone to work on the tasks under the same assumptions. Similarly, Komatsu (2017), who aimed to achieve a mathematical activity involving the interplay of proofs and refutations, showed the importance not only of tasks that lead to the emergence of counterexamples, but also of teacher questions that draw students’ attention to the modification of conjectures and proofs.

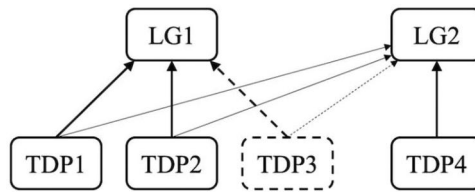


Figure 7. Relationship between the Learning Goals (LGs) and the Task Design Principles (TDPs).

Given the empirical findings of this study and the evidence from the literature, we propose an additional task design principle:

Task Design Principle 4: Create a task that invites students to revise the original task to eliminate its ambiguity and discuss whether the revision(s) can allow only one legitimate answer.

Figure 7 depicts the relationship between the two learning goals and the four task design principles. Task Design Principles 1–3 and 4 are mainly related to Learning Goals 1 and 2, respectively. The thinner arrows connecting Task Design Principles 1–3 and Learning Goal 2 mean that tasks designed according to these principles are prerequisites for achieving this learning goal. The dashed line box and arrows indicate that Task Design Principle 3 is optional.

This study used Task Design Principles 1–3 *a priori* to design the Function and Geometry Tasks, whereas we developed Task Design Principle 4 *a posteriori* from the additional tasks the teachers gave to students. Task Design Principle 4 justifies these teachers' task design approach. In this sense, the analysis of the classroom episodes shows that Task Design Principles 1–4 can form the basis for developing tasks aimed at introducing students to the role of assumptions in mathematical activity.

Although several existing studies have discussed the importance of assumptions in mathematics, the literature has primarily focused on global types of assumptions (Table 1) in the context of geometric proof where axioms and definitions are involved (e.g., Fawcett, 1938; Jahnke & Wambach, 2013; Komatsu, 2017). One weakness of such studies is that this context may become a barrier for many students because proofs are difficult for students to learn (Stylianides et al., 2017). When examining axioms and definitions, students must consider various aspects, including other definitions, axioms, propositions, and entire theories. Our study addresses this issue by considering the broader context of problem solving and focusing on local types of assumptions (Table 1). The two episodes described in this paper illustrate that students can experience the role of assumptions from the elementary school level and in diverse types of mathematical activities and areas.

Another issue that remained unclear in previous research is how to foster students' recognition of the role of assumptions (Stylianides et al., 2017). We have approached this issue by conducting a design study. In line with a key aim of design research, namely, to develop theoretical frameworks about supporting students' learning (Cobb et al., 2017; Prediger, 2019), we constructed task design principles based on the extant literature, including scholarship on the notion of productive ambiguity (Foster, 2011; Grosholz, 2007; Marmur & Zazkis, 2022; Stylianides & Stylianides, 2023). We also empirically tested and refined these principles through three research cycles, each of which consisted of task design, implementation, and analysis (Gravemeijer & Cobb, 2006). The tasks used in this study covered different types of assumptions (Table 1) and mathematical domains (functions and geometry) and were implemented in elementary and secondary schools. These results suggest a potentially broad range of applicability of the task design principles developed in the study.

We focused on developing general principles that underpin the design of specific tasks, rather than creating specific tasks per se, as the tasks used in this study are adaptations of existing tasks

(Beckmann, 2005; NIEPR, 2016). One advantage of developing such general principles is that they can provide teachers with a way to create tasks themselves and adapt tasks already designed by others or found in curricular resources (e.g., textbooks). In some cases, it may not be appropriate to implement a designed task as is because classrooms have their own contexts, including different teachers, students, and classroom norms. In such a case, it is necessary to modify the task, but it is often unclear which parts of the task can be changed and which others should be retained. Additional tasks may also be required. These can be based on principles of task design that allow teachers to adjust the task within reason (Komatsu & Jones, 2019; Stylianides & Stylianides, 2013). For example, in Episode 2, in the Geometry Task, although we presented students with Daichi's and Hinata's solutions from the outset according to Task Design Principle 3, some teachers may like these solutions to come from the students. In this case, Task Design Principle 3 suggests that the teachers can start with the task shown in Figure 1 (i.e., without presenting Daichi's and Hinata's solutions) and then suggest Hinata's solution if students do not come up with it on their own. Similarly, in the Function Task developed using Task Design Principles 1 and 2, students may give only one answer, for instance, $y = 6$, by considering the function as an inverse proportion. Task Design Principle 3 alerts the teacher to prepare for such a scenario, for example, by asking an additional question for the answer when the function is assumed to be linear. Thus, the task design principles are useful as flexible reference points that teachers can draw on when developing, revising, and adaptively implementing tasks.

As clarified in the Introduction, the goal of our study was to expose students to the importance of assumptions rather than to further examine whether this experience shaped the students' later learning. Nevertheless, we received a positive report from one of the elementary school teachers who participated in our study. After the study as part of a school event, he provided a task, "There is a bookshelf with a width of 56 cm, and it has just 7 books. How thick is one book?", to some groups of students belonging to other classes than his, and all the students answered, $56 \div 7 = 8$. He later gave the same task to his class, who participated in our study, and the students immediately responded that they could not answer the question in the task because the task did not specify the assumption that the books had the same width. The teacher shared other classroom episodes with us, showing that similar responses were observed in different tasks from different students. This teacher's report is an anecdote, but it does provide suggestive evidence about the possible sustained effect of the implementation of only one of our tasks on students' ways of thinking about mathematics in the area of assumptions. A future study is needed to explore this kind of longer-term issue.

To change students' habitual thinking, it is probably crucial to implement relevant tasks occasionally. Some aspects of the current study would be promising in this regard. The short duration of task implementation, one lesson of 45 or 50 minutes, would make it easier for teachers to incorporate it into their daily teaching practices. The principles developed in this study can be used for designing multiple tasks, essential for occasional task implementation. We used these principles for adapting tasks from the literature, rather than for designing new tasks. This can also be advantageous because it shows that teachers can adapt existing tasks in textbooks and elsewhere—across grade levels and mathematical topics—to offer their students multiple opportunities to experience the role of assumptions in mathematics. Such utilization of existing tasks additionally indicates that teachers do not need to feel burdened to design new tasks on their own.

The importance of the teacher's role and students' mathematical knowledge

The analysis of the classroom episodes presented in this paper indicates that factors other than task design have also contributed to the matching between the hypothetical and actual learning trajectories. The instructional triangle metaphor proposed in the literature (Ball & Forzani, 2009; Cohen et al., 2003; Straesser, 2007) delineates the relationships between the tasks and the teacher,

and between the tasks and the students. This paper has primarily focused on the former relationship.

Indeed, both episodes analyzed in this study illustrate the key role that the teachers played in the implementation of the tasks. While we analyzed the episodes using Stein et al. (2008) five practices for orchestrating classroom discussions, we also observed other notable teacher actions that do not correspond to these practices. One was that the teachers provided sub-tasks to complement the designed tasks, and we acknowledged this instructional move by adding Task Design Principle 4 to our list of principles.

Another key move was that when introducing the designed tasks to students at the beginning of the lessons, the task setup phase, the teachers in both episodes allowed several minutes for the students to tackle the tasks alone, without discussing their ideas with others. Some studies showed the importance of holding whole-class discussions in the task setup phase, where students can establish a *taken-as-shared* understanding of tasks (Cobb et al., 1992) and share their ideas about how to approach tasks (e.g., Komatsu & Jones, 2022). In contrast, both teachers in our study intentionally did not foster students' taken-as-shared understanding of the tasks in the setup phase. These teachers' strategic moves, similar to the move used by Stylianides and Stylianides (2023) in their intervention with prospective elementary teachers, were noteworthy: Establishing clear assumptions was one of the goals of the implemented lessons, and thus it was strategic to withhold prompting students to establish assumptions until they saw the need for it in the later phases of the lessons. One reason for which the implementation of the designed tasks worked out as expected in this study was the teachers' strategic setup.

Stein et al. (2008) five practices for orchestrating whole-class discussions were also observed in both episodes. This is clearly illustrated in Episode 1, where the teacher had anticipated that students would consider two ways to address the task's ambiguity and circulated the classroom to monitor their work and confirm that both ways were actually represented in it. He selected and invited Kenta and Shun to share their ideas (lines 75 and 88) and sequenced the ideas, beginning with Shun's. Building on Shota's suggestion of a step function, the teacher connected and compared these two ideas and connected the classroom activity to the notion of assumption.

While all five practices were crucial for the implementation of the tasks in this study, the most fundamental one was *anticipating* because, as Stein et al. (2008) stated, it formed the basis for the other practices in the later phases. This is particularly relevant to research involving ambiguity because, although ambiguity allows different responses from students, it does not mean that the teacher can proceed with the lesson arbitrarily, accepting whatever students come up with (Stylianides & Stylianides, 2023). Given that not all students' thoughts lead to meaningful ideas, the teacher needs to anticipate student work with attention to whether and how their work can be connected to the learning goal of the task. Indeed, in the case of the Function Task, we had expected the students' suggestion of a step function, which enabled the class to appreciate the significance of specifying the functional relationship in the task in order to uniquely determine the conclusion (Learning Goal 2). In this sense, anticipating has to be related not only to task implementation but also to task design and students' mathematical knowledge because, when designing a task, it is important to consider whether task ambiguity will evoke certain knowledge from the students that will lead to the different assumptions and respective conclusions expected in the task.

This consideration leads to the relevance of another relationship in the instructional triangle (e.g., Ball & Forzani, 2009), namely, the relationship between tasks and students. In particular, while the tasks used in this study presuppose certain mathematical knowledge owned by students, as seen in the mathematical concepts presented in Table 4, the classroom episodes we discussed show that additional mathematical knowledge can increase the extent to which students perceive or recognize task ambiguity and thus foster their understanding of the importance of assumptions. In Episode 1, the students were familiar with step functions, which led them to Learning

Goal 2, as mentioned above. Students in Episode 2 had previously studied the topic of counting the numbers of people on geometric shapes, and a student, Yui, proposed another justification related to this topic for selecting Option 3 in the Geometry Task. Yui's justification facilitated the students to move toward Learning Goal 1 where they acknowledged that Option 1 or 3 was correct depending on the meaning of the term *same part* in the task.

However, we do not mean to suggest that the more mathematical knowledge students have, the more successful the implementation of tasks will be. In Episode 1, for instance, if students had been familiar with additional kinds of functions, such as quadratic and cubic functions, the implementation of the task might have finished in a few minutes. In that case, students would have simply stated that they could not answer the question due to the lack of clarity about the task's conditions. One way in which ambiguity can be productive is if some students in a classroom interpret the task one way and others a different way. If they all interpret the task in the same way, or if they spot the ambiguity from various perspectives, the classroom discussion will likely not be particularly rich. In this sense, tasks need to be ambiguous to an appropriate extent, which depends on students' knowledge. Thus, it is essential to pay close attention to students' knowledge when designing tasks and/or deciding the appropriate student grade level for implementing designed tasks.

In sum, the above discussion shows that the relationships between the tasks and the teacher and between the tasks and the students are critical in task design research in the area of assumptions. In particular, our study, which employed the notion of productive ambiguity for task design (e.g., Foster, 2011; Stylianides & Stylianides, 2023), shows that the extent to which ambiguity intended in a task is productive for student learning depends not only on the task formulation but also on the role of the teacher implementing the task and the knowledge of students working on the task. Thus, while RQ2 of this paper focuses on the relationship between the tasks and the teacher, this paper also highlights a vital relationship between all three elements—tasks, teacher, and students—of the instructional triangle in the context of task design involving ambiguity and assumptions.

Implications for teaching and future research

An implication for teaching from the above discussion is that teachers can use the task design principles developed in this study to create relevant tasks and adapt existing ones. In doing so, they would need to consider several factors. For example, when designing a task, the teacher needs to pay close attention to students' knowledge, especially whether it would allow them to make the different assumptions and inferences that are expected in the task. In addition, the teacher would need to play a strategic and active role in implementing the task in class, for instance, by asking the students to engage in the task individually so that different conclusions based on different assumptions will naturally emerge in the subsequent classroom discussion. Orchestrating whole-class discussion is especially important for handling such students' varied ideas. It is critical for the teacher to connect the whole-class discussion to the intended learning goals (i.e., recognizing the role of assumptions in mathematical activity) while respecting students' diverse contributions (Stylianides & Stylianides, 2023).

An interesting issue arising from this study is the minimality of assumptions (Aktaş, 2016; van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005). In Episode 2, although using the term “congruent figures” can make only Daichi's solution legitimate, the students preferred the more specific expression of “congruent triangles.” In the context of our research, where multiple conclusions to a task emerge due to the lack of specificity of one or more of its assumptions, this kind of student preference would be understandable because students may prioritize their desire to uniquely determine the conclusion and think that less specific expressions may still leave the possibility of multiple legitimate conclusions. Although the

two classes described in this paper discussed whether the proposed possibilities (e.g., the four student ideas in lines 160–166 in Episode 2) could unambiguously determine the conclusions, and found that some of them could not, it may be worth asking an additional somewhat opposite question, namely, whether there are less specific expressions that can uniquely determine the conclusions. Future research can investigate when and how the mathematical habit of looking for minimal assumptions can be introduced to students.

Conclusions

Assumptions play a fundamental role in disciplinary mathematical practice with respect to the relativity of truth. However, most related studies in mathematics education have concentrated on the context of proving at the secondary school or university levels. Moreover, much of the research to date remains unclear about how to foster students' recognition of the importance of assumptions in mathematics. In this study, we took a step toward addressing these issues by conducting design research and considering a broader context, including the elementary school level (not just the secondary level) and problem-solving activities (not just proving activities). We provided a classification of assumptions and focused on local types of assumptions. We used the ambiguity of tasks to develop task design principles for introducing students to the role of assumptions in mathematical activity. We also discussed the significance of carefully considering the strategic and active role of the teacher in capitalizing on task ambiguity and examining the relationship between task ambiguity and students' available mathematical knowledge.

This study has some limitations. Although we have refined the principles of task design over three research cycles, the results of our study are based on the implementation of two tasks in a total of five classes. It is necessary to further test the task design principles by developing and implementing different tasks in different classrooms. Also, while the students participating in this study acknowledged the role of assumptions in mathematical activity, we did not track whether they continually paid attention to task assumptions during subsequent learning. Finally, our study focused on local assumptions, and it would be worthwhile to explore whether and how students' experiences with this type of assumption can be linked to their experiences with global assumptions in situations involving proofs, axioms, and definitions.

Notes

1. Assumptions have also been examined in research on mathematical modeling, where mathematics is employed to solve problems and explain phenomena in situations *outside* of mathematics (see Stylianides & Stylianides, 2023, for a related discussion). Our study differs from mathematical modeling studies as it is concerned with students' activity *within* mathematics.
2. For example, as we discussed earlier, special types of quadrilaterals, such as trapezoids and rectangles, can be defined in a hierarchical system whereby one type of quadrilateral is regarded as a particular case of another type.
3. For instance, an axiom related to congruent triangles in Hilbert theory can be described as “in $\triangle ABC$ and $\triangle A'B'C'$, if $AB \cong A'B'$, $AC \cong A'C'$, and $\angle BAC \cong \angle B'A'C'$, then $\angle ABC \cong \angle A'B'C'$.” Regarding the conditions of tasks, for example, one possible answer to the task asking the number of different addition sentences for 5 is “if the task refers to the sum of two numbers within the set of natural numbers and commutative expressions are regarded as different, then there are four sentences,” and the if-part is conditions of the task.
4. The task investigated by Komatsu (2017) was about proving a property of the perpendicular lines drawn from the vertices of the base of an isosceles triangle to the opposite sides of the triangle. The implicit condition in this task was the intersection of these perpendicular lines and sides.
5. The left column of Table 4 is identical across the two tasks as these tasks aim at common learning goals.
6. Our judgement of the secondary students' mathematical attainment as being above average is based on the fact that the school imposes an entrance examination, and thus has a selective admissions system. In the next paragraph, we mention that the elementary students in our study had mixed mathematical

attainments. Our judgement in this case is based on the fact that the participating elementary schools are public and do not impose entrance examinations to select students before they admit them. Also, the students studied mathematics with their regular classmates without any grouping by attainment.

7. There are some exceptions; for example, music is often taught by music teachers.
8. Textbook publishers need to get approval for publication from the Ministry. Publishers submit their textbooks to the Ministry, which closely reviews whether the textbooks are in accordance with the national curriculum.
9. The description in Komatsu et al. (2019) paper was briefer, and we provide a more detailed description and analysis in this paper.
10. One illustrative task in the student textbook is: “A person stands on each of the four vertices of a rectangle with 10m length and 6m width, and people line up on its four edges, 1m apart. How many people are there?”

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