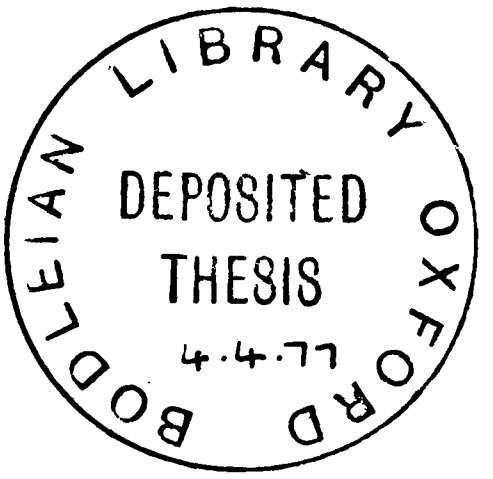
PROBLEMS IN

FUNCTIONAL ANALYSIS



Abstract of
Thesis submitted for the Degree of
Doctor of Philosophy in the University of Oxford
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by

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A B S T R A C T

This thesis is concerned with two, rather different, areas of analysis. A major part is devoted to problems in distributive lattice theory. Motivated by the Priestley representation theorem for a distributive lattice L we obtain information about the lattices of ideals, filters and congruence relations of L. Using purely lattice-theoretic means we give extension theorems for homomorphisms in the class of pseudocomplemented distributive lattices. These are analogous to the classical Hahn-Banach theorem. Finally, L^p-spaces are constructed from matrix measures and their properties investigated.

Chapter 1 marshals material used in subsequent chapters. In §1.2 the Priestley representation theorem is stated which allows a distributive lattice with zero and identity to be identified with the clopen decreasing subsets of a certain ordered topological space; the relevant theory of ordered spaces is summarized. The class of distributive lattices with pseudocomplementation is defined in §1.3 together with the equational subclasses. The Priestley duality for a pseudocomplemented lattice L is given as well as for the centre and dense set of L.

If L is a distributive lattice with zero denoted 0, the lattice of ideals, I(L), is a pseudocomplemented distributive lattice. The relationship between L and I(L) is investigated in Chapter 2. In §2.2 it is shown that if L is a pseudo-complemented lattice, the centre of I(L) is the minimal completion of the centre of L. This generalizes a result of Stone's for Boolean algebras. The problem, posed by Grätzer, concerning the classification of those L for which I(L) belongs to a particular equational subclass is considered in §2.3 to §2.6. In §2.3 it is shown, using the above result, that if L has an identity, denoted 1, I(L) is a Stone algebra if and only if L is a Stone algebra and the centre of L is complete. Similar results are obtained for lattices without 1 in §2.4 and §2.5. In §2.6 I(L) belongs to other equational subclasses and a classification in terms of the dual space is given. Katriňák has given equivalent conditions on L in order for I(L) to be a Stone algebra. His proofs are lattice-theoretic and do not explain the close relationship between the centres.

The congruence relations K(L) of a distributive lattice L as well as the filters F(L), if L contains 1, form pseudo-complemented lattices. Chapter 3 concerns these lattices as Stone algebras. Using the principle of duality §3.3 briefly describes those L for which F(L) is a Stone algebra. The congruence relations on a lattice are required to preserve all operations defined. §3.4, §3.5 and §3.8 classify the distributive lattices, pseudocomplemented distributive lattices and

distributive double p-algebras, respectively, which have congruence relations which are Stone algebras. If L is a distributive lattice with 0, 1, K(L) is a Stone algebra if and only if the Boolean completion of L is complete. This follows easily from §2.3. For L a pseudocomplemented distributive lattice, the subsets of the dual space corresponding to elements in the centre of K(L) are explicitly described and it is shown that K(L) is a Stone algebra if and only if L is a dual Stone algebra, the dual centre of L is complete, and the dense set is relatively complemented. Similar results are proved for distributive double p-algebra congruences. In §3.6, Stone algebras of interated congruence relations are discussed.

Chapter 4 investigates, in the class of pseudocomplemented distributive lattices L, Hahn-Banach theorems concerning the extension of a map f defined on a subset A of L, bounded below by a meet-homomorphism m and above by a join-homomorphism j, to a homomorphism h defined on the whole of L and likewise bounded. In §4.2 a summary of similar results for Boolean algebras due to Monteiro and Bacsich and for distributive lattices due to Cignoli is given. By placing conditions in §4.3 on the semi-morphisms, such as m mapping the dense set to the dense set, h preserves pseudocomplementation. The range of the afore-mentioned maps is a complete Boolean algebra which is the injective object in these categories. However in the class of Stone algebras there is a wider set of injective objects. In §4.4 an extension theorem in the class of Stone algebras is given in which conditions relating

m to j seem necessary if the maps are no longer to have values in a Boolean algebra. An example is given ensuring that these conditions do not reduce the problem to the Boolean algebra case.

In Chapter 5 an analogue to the classical L^p-space theory is developed for an L^p-space constructed from a matrix measure. An n x n positive matrix measure (μ_{ij}) on a general measure space (X,Σ,ν) is defined, in §5.2, in terms of n x n ν-integrable complex-valued functions (m_{ij}). In §5.3 $L^p(\mu_{ij})$-spaces, $1 \leq p \leq \infty$, of n-tuples of ν-measurable functions satisfying a norm condition involving the (m_{ij}) are shown to be Banach spaces. This is achieved by pointwise "diagonalizing" the matrix (μ_{ij}) and showing that the L^p-space constructed from the diagonal matrix is isomorphic to $L^p(\mu_{ij})$. The "diagonal" space is easily dealt with as problems are solved using classical results concerning $L^p(\nu)$. Kaz showed that a Hilbert space $L^2(\mu_{ij})$ can be constructed where (μ_{ij}) is defined on the bounded Borel sets of the real line R. A diagonal procedure is used relying on properties of R. Binding and Browne studied corresponding Banach spaces using non-constructive proofs. §5.2 alters the definitions of a matrix measure in order to work in general measure spaces. In §5.4 the converse of Hölder's inequality is discussed in $L^p(\mu_{ij})$. This is used in §5.5 to establish that the continuous dual of $L^p(\mu_{ij})$ is $L^q(\mu_{ij})$ for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ and the dual of $L^1(\mu_{ij})$ is $L^\infty(\mu_{ij})$.

P R O B L E M S I N

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Chapter 1

PRELIMINARIES

§1 Introduction

The basic definitions and results concerning lattices may be found in (3), (8) and (18). The theory of Boolean algebras is expounded in (37) and (21). In this chapter we shall outline a few concepts used in Chapters 2, 3 and 4 for the purpose of establishing notation. The topological approach to Chapters 2 and 3 relies on the Priestley representation theorem for distributive lattices ((33) and (34)) which we shall briefly describe together with the relevant theory of ordered spaces. This theorem is an extension of pioneer work done by Stone in (41) and when the distributive lattice is a Boolean algebra gives the well-known characterization of the dual of a Boolean algebra as a compact and totally disconnected space ((21) §18).

§2 The Priestley representation theorem

Given a subset E of a partially ordered set X, define:

$$i(E) = \{x \in X : x \geqslant y \text{ for some } y \in E\}$$

$$d(E) = \{x \in X : x \leqslant y \text{ for some } y \in E\}.$$

E is <u>increasing</u> (<u>decreasing</u>) if $i(E) = E$ ($d(E) = E$). If $(X, \mathcal{T}, \leqslant)$ is an <u>ordered space</u> (by which we shall mean a set X, with a topology $\mathcal{T}$, endowed with a partial order relation $\leqslant$), then the set $\mathcal{U}$ consisting of the increasing sets in $\mathcal{T}$ and the set $\mathcal{L}$ consisting of the decreasing sets in $\mathcal{T}$ each define a topology on X. $(X, \mathcal{T}, \leqslant)$ is said to be <u>monotonically separated</u> if, given $x, y \in X$, $x \not\leqslant y$, there exist disjoint sets $U \in \mathcal{U}$, $V \in \mathcal{L}$ such that $x \in U$, $y \in V$. $(X, \mathcal{T}, \leqslant)$ is said to be <u>totally order disconnected</u> (9) if, given $x, y \in X$ with $x \not\leqslant y$, there exist disjoint $\mathcal{T}$-clopen sets $U \in \mathcal{U}$, $V \in \mathcal{L}$ such that $x \in U$, $y \in V$.

The main theorems of (33) allow a distributive lattice L with zero and identity elements (denoted by 0,1) to be identified with the lattice of clopen decreasing subsets of a compact totally order disconnected space $(X, \mathcal{T}, \leqslant)$. If L has 0, but no 1, L may be identified with the lattice of compact open decreasing subsets of a monotonically separated partially ordered space $(X, \mathcal{T}, \leqslant)$, see (34). X, the <u>dual space</u> of L, is the set of prime ideals of L, partially ordered by set inclusion and appropriately topologized. This partial order is the reverse of that used in (33), (34) and (35) but is described in (36).

If L is a distributive lattice with 0 (1) and has dual space $(X, \mathcal{T}, \leqslant)$ then L is relatively complemented if and only

$\leqslant$ is the discrete order

If L and L_1 are distributive lattices with 0 and X, X_1 their dual spaces, there is a one-to-one correspondence between lattice homomorphisms from L_1 into L and continuous increasing maps from X into X_1.

For $a \in L$, let a' denote the corresponding subset of the dual space X. We shall use set-theoretic symbols to denote lattice operations in the dual space.

The concept of a C-space and the superficially similar I-space is crucial to many of the proofs in Chapters 2 and 3. We call $(X, \mathcal{T}, \leqslant)$ a C_i-space (C_d-space) if a subset F of X is closed implies i(F) (d(F)) is closed. A space which is simultaneously a C_i-space and a C_d-space will be referred to as a C-space. Analogously, $(X, \mathcal{T}, \leqslant)$ is an I_i-space if G open in X implies i(G) open. An I_d-space and an I-space are defined in the obvious way.

LEMMA 1.1 ((34) Lemma 2)

i) $(X, \mathcal{T}, \leqslant)$ is a C_i-space if and only if

$i(\mathcal{T}\text{-cl}E) = \mathcal{L}\text{-cl}E$ for any subset E of X.

ii) $(X, \mathcal{T}, \leqslant)$ is an I_d-space if and only if

$\mathcal{T}\text{-cl}(i(E)) = \mathcal{L}\text{-cl}\ E$ for any subset E of X.

Similarly for C_d-spaces and I_i-spaces.

Remark. We note that every compact ordered space is a C-space. This is a consequence of Nachbin's result that i(F), d(F) are closed if F is a compact subset of a monotonically separated space ((32) p.44).

Throughout, where topological properties of the ordered space $(X, \mathcal{T}, \leqslant)$ are mentioned without explicit reference to which topology is intended, that topology is $\mathcal{T}$.

§3 Pseudocomplemented distributive lattices

Let L be a distributive lattice with 0. L is pseudocomplemented if for any $a \in L$, there is $a^* \in L$ such that $a \wedge a^* = 0$ and $a \wedge b = 0$, for $b \in L$, implies $b \leqslant a^*$. Two important subsets associated with a pseudocomplemented distributive lattice are:

i) The centre of L, S(L), which is $\{a^* : a \in L\}$ and is a Boolean algebra if $\wedge_{S(L)}$ is defined as in L and $\vee_{S(L)}$ is defined by $a \vee_{S(L)} b = (a^* \wedge b^*)^*$;

ii) The dense set of L, D(L), defined as $\{a : a \in L, a^* = 0\}$.

Pseudocomplementation is an operation on the distributive lattice L.

The following theorem summarizes some of the basic properties used in studying pseudocomplemented distributive lattices. i), ii) and v) show that the operation $a \to a^{**}$ is a closure operation on L, the set of closed elements being S(L).

THEOREM 1.2. (cf.((18), Theorem 15.1)). Let L be a pseudocomplemented distributive lattice. Then for $a, b \in L$:

i) $a^* = a^{***}$.

ii) $a \leqslant a^{**}$.

iii) $(a \vee b)^* = a^* \wedge b^*$.

iv) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

v) $a \leqslant b$ implies $a^* \geqslant b^*$.

Let B_{-1} be the class of one-element lattices with pseudocomplementation; let B_0 denote the class of Boolean algebras; for $n \geq 1$, let B_n be the class of pseudocomplemented distributive lattices satisfying the identity:

(L_n) $(x_1 \wedge \ldots \wedge x_n)^* \vee (x_1^* \wedge \ldots \wedge x_n)^* \vee \ldots \vee (x_1 \wedge \ldots \wedge x_n^*)^* = 1$;

let B_ω be the class of all pseudocomplemented distributive lattices. Lee (28) has shown that the $B_n (-1 \leq n \leq \omega)$ are the only equational classes of distributive lattices with pseudocomplementation, and $B_{-1} \subset B_0 \subset \ldots \subset B_n \subset \ldots \subset B_\omega$. For $n \geq 0$, $L \in B_n$ if and only if every prime ideal of L contains at most n distinct minimal prime ideals. Let B_1 be the class of Stone algebras ((10),(11)) i.e. L is a Stone algebra if L is a pseudocomplemented distributive lattice such that for all $x \in L$, $x^* \vee x^{**} = 1$.

THEOREM 1.3 ((18) Lemma 15.2) Let L be a pseudocomplemented distributive lattice. The following conditions are equivalent:

i) L satisfies the identity (L_n);

ii) L satisfies the implication:

$x_i \wedge x_j = 0$, $i \neq j$, $i,j = 0,\ldots,n$ imply that

$x_0^* \vee \ldots \vee x_n^* = 1$;

iii) L satisfies the implication:

$x_i \wedge x_j = 0$, $i \neq j$, $i, j = 0,\ldots,n, x_0,\ldots,x_n \in S(L)$

imply that $x_0^* \vee \ldots \vee x_n^* = 1$.

Priestley in (36) and (35) describes the dual spaces of pseudocomplemented distributive lattices and Stone algebras. We note a few results. In any compact totally order disconnected space $(X, \mathcal{T}, \leqslant)$ by Zorn's lemma there exists for $x \in X$ at least one point $y \leqslant x$ such that y is minimal with respect to the partial order $\leqslant$. Let X_m denote the set of minimal points.

PROPOSITION 1.4.((36) Proposition 4) If L is a pseudocomplemented distributive lattice with dual space X the dual of D(L) is $X-X_m$.

PROPOSITION 1.5.((36) Proposition 5) If L is a pseudocomplemented distributive lattice with dual space X, then the dual of S(L) is $(X_m, \mathcal{T}|X_m)$.

Let L be a pseudocomplemented distributive lattice with dual space X. Then $L \in B_n$, $1 \leqslant n < \omega$ if and only for each $x \in X$, $m(x)=d(x) \cap X_m$ is a subset of X_m of cardinality not greater than n.

Chapter 2

LATTICES OF IDEALS

§ 1 Introduction

The lattice of ideals of a distributive lattice with zero is a pseudocomplemented distributive lattice. In this chapter an attempt is made to classify distributive lattices according to the particular equational class of distributive lattices with pseudocomplementation that the ideal lattice belongs. This problem was posed by Grätzer ((18) Problem 66, p.192). The results are summarized in a table at the end of §2.7.

In addition, in §2.2 we show that if L is a pseudocomplemented distributive lattice, the centre of the ideal lattice, I(L), is the minimal completion of the centre of L. In §2.4 we obtain corresponding results where L is a generalized Stone algebra.

This important link between L and I(L) is used to classify those distributive lattices L for which I(L) is a Stone algebra. At the time we did not know of Katriňák's classifications ((24) Theorems 6,7 and (26) Correction p.143). However he does not explain the close relationship between L and I(L). Hashimoto ((22) Theorem 4.3) established when I(L) is a Boolean algebra. In §2.6 we consider the remaining equational classes to which I(L) may belong.

§2 The minimal completion of the centre of L

In this section, L is a distributive lattice with 0 and 1. We show that I(L), the lattice of ideals of L, is a complete pseudocomplemented distributive lattice. If L is a pseudocomplemented distributive lattice we show S(I(L)) is the minimal completion of S(L). This is consistent with the result of Stone for Boolean rings ((40) Theorem 2.4,p.236). Some of these results are to appear in (44).

A one-to-one correspondence between ideals, U, of a distributive lattice L with 0 and 1 and open decreasing subsets, U', in the dual space of L was noted by Priestley ((33) p.189). The open subsets of a topological space form a complete distributive lattice, so this is true of I(L). Lemma 2.2 shows that I(L) is pseudocomplemented.

First we need:

LEMMA 2.1 *If* $(X,\mathcal{T},\leqslant)$ *is a compact ordered space then* $i(\mathcal{T}\text{-cl}E) = \mathcal{L}\text{-cl}E$ *for any subset* E *of* X.

Proof. This follows from Lemma 1.1 i) and the subsequent remark.

LEMMA 2.2 Let $U \in I(L)$ and U' be the corresponding open decreasing set in X. $I(L)$ is pseudocomplemented and the pseudocomplement of U, U^*, corresponds to the open decreasing set $U'^* = X - i(\mathcal{T}\text{-cl } U')$.

Proof. By Lemma 2.1, $i(\mathcal{T}\text{-cl}U')$ is a closed increasing set and so $X - i(\mathcal{T}\text{-cl } U')$ corresponds to an element of $I(L)$. $U' \subseteq i(\mathcal{T}\text{-cl } U')$ implies $U' \cap X - i(\mathcal{T}\text{-cl } U') = \emptyset$. Suppose V' is an open decreasing subset of X with $U' \cap V' = \emptyset$. V' open implies $V' \cap \mathcal{T}\text{-cl}U' = \emptyset$. Now V' decreasing gives $V' \cap i(\mathcal{T}\text{-cl } U') = \emptyset$ and $V' \subseteq X - i(\mathcal{T}\text{-cl}U')$. Hence U' has pseudocomplement $U'^* = X - i(\mathcal{T}\text{-cl}U')$. This means $I(L)$ is pseudocomplemented, completing the proof.

Note that since $S(I(L))$ is the set of closed elements of the complete lattice $I(L)$, $S(I(L))$ will be complete.

If L is a pseudocomplemented distributive lattice with dual space X, then the dual of $S(L)$ is $(X_m, \mathcal{T}|X_m)$ (Proposition 1.5). In fact $(X_m, \mathcal{T}|X_m)$ equals $(X_m, \mathcal{L}|X_m)$ by:

LEMMA 2.3 Let X be the dual space of a pseudocomplemented distributive lattice. Then in X_m, the relative topologies $\mathcal{T}|X_m$ and $\mathcal{L}|X_m$, induced by $\mathcal{T}$ and $\mathcal{L}$ respectively, coincide. Hence $(X_m, \mathcal{L}|X_m)$, is a compact Hausdorff space with clopen base.

Proof. Obviously $\mathcal{L}|X_m \subseteq \mathcal{T}|X_m$. Now let P be a basic element of $\mathcal{T}|X_m$. P is clopen. Take $x \in P$, $y \in X_m - P$. There exists $A(x,y)$, $\mathcal{T}$-clopen decreasing subset of X such that $x \in A(x,y)$, $y \notin A(x,y)$ by the total order disconnectedness of X. Fix x. $X_m - P$ is $\mathcal{T}|X_m$-clopen and since X_m is $\mathcal{T}$-compact, $X_m - P$ is covered by finitely many of the sets $X_m - A(x,y)$.

The intersection A(x) of the associated sets A(x,y) is a clopen decreasing neighbourhood of x with $A(x) \cap X_m$ contained in P. By compactness, $P = A \cap X_m$ where A is a finite union of sets A(x), and so is clopen decreasing in X.

The following three lemmas although rather technical are indispensable:

LEMMA 2.4 ((35) Propositions 1,2, p.128) A distributive lattice L with 0, 1 is pseudocomplemented if and only if, for each clopen decreasing subset a' is the dual space X of L, i(a') is open in X. $a'^* = X - i(a')$ and L is a Stone algebra if and only if $U' \in \mathcal{L}$ implies $i(U') \in \mathcal{L}$.

Proof. Let $a \in L$ and suppose i(a') is open. Then X–i(a') is disjoint from a' and is clopen decreasing. Also, if b' is decreasing, $a' \cap b' = \emptyset$ implies that $i(a') \cap b' = \emptyset$. It follows that a' has pseudocomplement X–i(a') and so L is pseudocomplemented.

Conversely suppose that each a in L has pseudocomplement a* in L. To show that i(a') is open we show that $i(a') = X - a'^*$. $a' \cap a'^* = \emptyset$ and a'^* is decreasing so $a'^* \subseteq X - i(a')$. If x does not belong to the closed decreasing set i(a'), then total order disconnectedness of X provides a clopen decreasing set b' such that $x \in b' \subseteq X - i(a')$. $a' \cap b' = \emptyset$, so $b' \subseteq a'^*$. Hence $X - i(a') \subseteq a'^*$.

For the last statement suppose L is a Stone algebra. Then, for all $a \in L$, $a'^* \cup a'^{**} = X$. The above implies $i(a') = a'^{**}$ which is open decreasing. Any open decreasing set is the union of its clopen decreasing subsets and the operator i preserves unions. Thus $i(U') \in \mathcal{L}$ whenever $U' \in \mathcal{L}$. For the converse, note that, if $a \in L$ implies that $i(a') \in \mathcal{L}$, then $a'^{**} = X - i(X - i(a')) = i(a')$. This completes the proof.

LEMMA 2.5 Let X be the dual space of a pseudocomplemented distributive lattice and $U' \in \mathcal{L}$. Then $i(\mathcal{T}\text{-cl } U' \cap X_m) = i(\mathcal{T}\text{-cl } U')$.

Proof. As in ((36), p.219-220), let 2^{X_m} be the collection of all non-empty closed subsets of X_m. Define for $P \subseteq X_m$.

$$B_P = \{F \in 2^{X_m} : F \subseteq P\}$$

$$B^P = \{F \in 2^{X_m} : F \cap P \neq \emptyset\}$$

and give 2^{X_m} the topology generated by the sub-base consisting of all sets B_P, B^P for P clopen in X_m. By Lemma 2.4, $m : X \to 2^{X_m}$ is continuous since $m^{-1}(B^P) = i(P)$. U' decreasing implies $m(y) \subseteq U' \cap X_m$ for all $y \in U'$. Now let $\{y_\alpha\}$ be a net in U' with $y_\alpha \to y$ in $\mathcal{T}$. Then $m(y_\alpha) \to m(y)$. We show $m(y) \cap (\mathcal{T}\text{-cl}U' \cap X_m) \neq \emptyset$. If not, by total disconnectedness of X_m, there exists a $\mathcal{T} \mid X_m$-clopen set P such that $m(y) \subseteq P$ and

$$\mathcal{T}\text{-cl}U' \cap X_m \subseteq X_m - P .$$

Thus $m(y) \in B_P$ and for $\alpha \geq \alpha_0$, say, $m(y_\alpha) \in B_P$. Hence

$$m(y_\alpha) \cap (X_m - P) = \emptyset$$

and

$$m(y_\alpha) \cap (\mathcal{T}\text{-cl}U' \cap X_m) = \emptyset$$

for $\alpha \geq \alpha_0$ which gives the required contradiction.

Now take $x \in i(\mathcal{T}\text{-cl}U')$, and suppose $x \geq y$, where $y \in \mathcal{T}\text{-cl}U'$. By the above there is some

$$z \in m(y) \cap (\mathcal{T}\text{-cl}U' \cap X_m).$$

Transitivity gives $x \geq y \geq z$. Thus $x \in i(\mathcal{T}\text{-cl}U' \cap X_m)$. Obviously

$$i(\mathcal{T}\text{-cl}U' \cap X_m) \subseteq i(\mathcal{T}\text{-cl}U')$$

which completes the proof.

If L is a Stone algebra we have a corollary which is essential to some of the proofs of Chapter 3, §5 and §8.

COROLLARY 2.6. Let X be the dual space of a Stone algebra. For $U' \in \mathcal{L}$, $m(\mathcal{T}\text{-cl}U') \subseteq \mathcal{T}\text{-cl}U'$.

Proof: By the proof of Lemma 2.5 we have that for $U' \in \mathcal{L}$ and $y \in \mathcal{T}\text{-cl}U'$, $m(y) \cap (\mathcal{T}\text{-cl}U' \cap X_m) \neq \emptyset$. L is a Stone algebra so $m(y)$ is a singleton and $m(y) \in \mathcal{T}\text{-cl}U'$.

Even with the conditions of Corollary2.6, we do not have $\mathcal{T}\text{-cl}U'$ decreasing although $\mathcal{T}\text{-cl}U' \supseteq d(\mathcal{T}\text{-cl}U') \cap X_m$ for $U' \in \mathcal{L}$. However, we prove the following and refer to it in Chapter 3,Theorem 3.12.

THEOREM 2.7 Let L be a distributive lattice with 0, 1. L is relatively pseudocomplemented if and only if the dual space X has the property: $U' \in \mathcal{L}$ implies $\mathcal{T}$-cl U' is decreasing.

Proof. It is easily shown that if $U' \in \mathcal{L}$, $\mathcal{T}\text{-cl}U'$ is decreasing if and only if $\mathcal{T}\text{-cl}U' = \mathcal{U}\text{-cl}U'$. We show that for $U' \in \mathcal{L}$ this is

equivalent to X being an I_i-space. If X is an I_i-space then $\mathcal{T}$-clU' = $\mathcal{U}$-clU' for U' ε $\mathcal{L}$ by Lemma 1.1. To prove the converse, let P be $\mathcal{T}$-clopen in X, then i(P) is $\mathcal{T}$-closed as X is a C_i-space. Thus X–i(P)ε $\mathcal{L}$ which implies, by hypothesis, $\mathcal{T}$-cl(X–i(P)) is decreasing and

$$i(P) \cap \mathcal{T}\text{-cl}(X\text{–}i(P)) = \emptyset.$$

Now if x ε i(P), not every $\mathcal{T}$-neighbourhood of x meets X–i(P) and i(P) must be open. Let G be any $\mathcal{T}$-open set. Then G is the union of clopen sets, and the operator i preserves unions so i(G) will be open.

Let G be an open subset of X and suppose i(G) is open. If a',b' are any clopen decreasing subsets of X we show that

$$c' = X\text{–}i(a' \cap (X\text{–}b'))$$

is the subset corresponding to a$\rightarrow$b (see(3)p.173). c' is clopen decreasing by hypothesis. Also c' $\cap$ a' $\subseteq$ b' since if x ε c' $\cap$ a', then x$\notin$i(a' $\cap$ (X–b')) but x ε a'. So x $\not\geqslant$ y for any y ε a$\dot{\cap}$(X–b') but x ε a' which implies x ε b'. Now suppose P' $\cap$ a' $\subseteq$ b' for some clopen decreasing set P'. We show P' $\subseteq$ c'. If there exists y ε P'–c', then y ε P' $\cap$ i(a' $\cap$ (X–b')). Thus y ε P' and y $\geqslant$ z for some z ε a' $\cap$ (X–b'). Now P' decreasing implies z ε P' and so z ε P' $\cap$ a'. But z ε X–b' which contradicts our hypothesis that P' $\cap$ a' $\subseteq$ b'. This implies L is relatively pseudocomplemented.

To prove the converse, we suppose L is relatively pseudo-complemented. Then for any a', b' clopen decreasing in the dual space X, X–i(a' ∩ (X–b') is clopen decreasing. It is easily proved from properties of ordered spaces that every $\mathcal{T}$-clopen subset of X is of the form $\bigcup_{1\leq i\leq n} (P_i \cap Q_i)$ where P_i, $X-Q_i$ are clopen decreasing subsets of X. If $A = P \cap Q$, for P, X–Q clopen decreasing, then X–i(P ∩ X–(X–Q)) is clopen implies i(A) is clopen. If G is an arbitrary clopen subset, $G = \bigcup_{1\leq i\leq n} A_i, A_i = P_i \cap Q_i$ as above and i preserves unions so i(G) is clopen. For arbitrary open G, $G = \bigcup_{\alpha} G_\alpha$, G_α clopen, and again i preserves unions so i(G) is open. This completes the proof.

LEMMA 2.8. Let X be the dual space of a pseudocomplemented distributive lattice. For $U' \in \mathcal{L}$,

$$\mathcal{T}\text{-cl}U' \cap X_m = \mathcal{T}\text{-cl}(U' \cap X_m) \cap X_m$$

and

$$\mathcal{L}\text{-cl}U' \cap X_m = \mathcal{L}\text{-cl}(U' \cap X_m) \cap X_m .$$

Proof. Certainly $\mathcal{T}\text{-cl}(U' \cap X_m) \cap X_m \subseteq \mathcal{T}\text{-cl}U' \cap X_m$. Let $z \in \mathcal{T}\text{-cl}U' \cap X_m$ and $\{y_\alpha\}$ be a net in U' with limit z. U' decreasing implies $m(y_\alpha) \subseteq U'$, for all α, and $y_\alpha \to z$ implies $m(y_\alpha) \to m(z) = \{z\}$ by the continuity of m. Suppose P is $\mathcal{T}|X_m$-clopen with $z \in P$. Then $z \in B^P$. Hence $m(y_\alpha) \in B^P$ and $m(y_\alpha) \cap P \neq \emptyset$ for $\alpha \geq \alpha_0$, say. Let

$$q_\alpha \in m(y_\alpha) \cap P \subseteq U' \cap X_m \cap P .$$

P an arbitrary clopen set containing z implies $q_\alpha \to z$ in $\mathcal{T}|X_m$ and $z \in \mathcal{T}\text{-cl}(U' \cap X_m) \cap X_m$.

The first statement is proved.

By Lemma 2.3,

$$\mathcal{L}|X_m\text{-cl}(U' \cap X_m) = \mathcal{T}|X_m\text{-cl}(U' \cap X_m).$$

But

$$\mathcal{T}\text{-cl}(U' \cap X_m) \cap X_m = \mathcal{T}\text{-cl}U' \cap X_m$$

by the above, and Lemma 2.1 gives $\mathcal{T}\text{-cl}U' \cap X_m = \mathcal{L}\text{-cl}U' \cap X_m$. The second statement follows.

The Stone representation theorem implies easily that every Boolean algebra can be embedded in a complete Boolean algebra. However a more refined completion can be defined as follows. A <u>completion</u> of a Boolean algebra A is a complete Boolean algebra B together with a monomorphism h from A into B, such that:

i) if $\vee_i a_i = a$ in A, then $\vee_i h(a_i) = h(a)$ in B,

ii) the complete algebra generated by h(A) in B is B itself.

A completion (B,h) is <u>minimal</u> if corresponding to every completion (C,k), there exists a complete monomorphism f from B into C such that $fh = k$ ((37) p.152, (21) p.92).

THEOREM 2.9 ((21) §21, Theorem 11, p.93). Let A be the dual algebra of a Boolean space X, B the algebra of regular open sets in X, and h the identity mapping from A into B, then (B,h) is a minimal completion of A.

Remark. Suppose (B,h) is a minimal completion of A. Let C be a complete Boolean algebra and k a monomorphism from A into C. Then (C, k)is also a minimal completion of A if (B,h) is isomorphic to (C,k) i.e. if there exists an isomorphism g from C onto B such that h = gk. This is because infinite joins and meets are invariant under isomorphisms ((37), p.56).

For a ϵ L, let (a] be the principal ideal generated by a.

THEOREM 2.10 Let L be a pseudocomplemented distributive lattice. (S(I(L)),k) is the minimal completion of S(L), where k is the map $a^* \to (a]^*$.

Proof. We show that there is an isomorphism g from S(I(L)) onto the complete Boolean algebra B of regular open subsets of $(X_m, \mathcal{T}|X_m)$. $(X_m, \mathcal{T}|X_m)$ is the dual of the Boolean algebra S(L). By Theorem 2.9 the regular open subsets of $(X_m, \mathcal{T}|X_m)$ with the identity mapping h is the minimal completion of S(L). We also show h = gk so the above remark completes the proof.

Let U' be an open decreasing subset of X and define g by

$$g(U'^*) = (X - \mathcal{T}\text{-cl}U') \cap X_m.$$

Now

$$(X - \mathcal{T}\text{-cl}U') \cap X_m = (X - (\mathcal{T}\text{-cl}U' \cap X_m)) \cap X_m$$

by Lemma 2.8, which equals $X_m - \mathcal{T}|X_m\text{-cl}(U' \cap X_m)$, a regular open subset of $(X_m, \mathcal{T}|X_m)$. So g takes the sets corresponding to S(I(L)) into regular open subsets of $(X_m, \mathcal{T}|X_m)$. Let U',V' be open decreasing subsets of X.

If

$$\mathcal{T}\text{-cl}U' \cap X_m = \mathcal{T}\text{-cl}V' \cap X_m,$$

then

$$i(\mathcal{T}\text{-cl}U' \cap X_m) = i(\mathcal{T}\text{-cl}V' \cap X_m)$$

and by Lemma 2.5 $i(\mathcal{T}\text{-cl}U') = i(\mathcal{T}\text{-cl}V')$. This implies $U'^* = V'^*$ and g is one-to-one.

A trivial computation shows that g is a meet-homomorphism.

To show g preserves complements we must show that for $U'^* \in S(I(L))$, $g(U'^{**}) = (g(U'^*))^*$. Now $g(U'^{**}) = (X - \mathcal{T}\text{-cl}U'^*) \cap X_m$ and

$$(g(U'^*))^* = (X - (\mathcal{T}\text{-cl}(U'^* \cap X_m)) \cap X_m.$$

So g preserves complements if and only if

$$(X - \mathcal{T}\text{-cl}U'^*) \cap X_m = (X - (\mathcal{T}\text{-cl}(U'^* \cap X_m)) \cap X_m.$$

The equation is true by Lemma 2.8 as U'^* is an open decreasing subset of X.

To show g is onto, let U be regular open in $\mathcal{T}|X_m$. By Lemma 2.3, $U = U' \cap X_m$ for some $U' \in \mathcal{L}$. For $P \subseteq X_m$, let

$$P^{\perp} = X_m - (\mathcal{L}|X_m\text{-cl}P).$$

U regular implies

$$U = (U' \cap X_m)^{\perp\perp}$$

$$= (X_m - (\mathcal{L}|X_m\text{-cl}(U' \cap X_m)))^{\perp}$$

$$= (X_m - (\mathcal{L}\text{-cl}(U' \cap X_m) \cap X_m))^{\perp}$$

By Lemma 2.8, this gives $U = (X_m - (\mathcal{L}\text{-cl}U' \cap X_m))^{\perp}$ which is $((X - \mathcal{L}\text{-cl}U') \cap X_m)^{\perp}$. Let $V' = X - \mathcal{L}\text{-cl}U'$. Then $V' \in \mathcal{L}$ and

$$U = (V' \cap X_m)^{\perp}$$

$$= X_m - (\mathcal{L}|X_m\text{-cl}(V' \cap X_m))$$

$$= X_m - (\mathcal{L}\text{-cl}(V' \cap X_m) \cap X_m).$$

Again Lemma 2.8 gives $U = X_m - (\mathcal{L}\text{-cl}V' \cap X_m)$. So $U = (X - \mathcal{L}\text{-cl}V') \cap X_m = g(V'^*)$, by definition of g. This completes the proof that g is an isomorphism.

The identity map h of Theorem 2.9 is that which takes the element $a^* \in S(L)$ to its corresponding subset of X_m, $(X - a') \cap X_m$ (see ((36), Proposition 5)). Now $g\,k(a^*) =$

$= g((a]^*) = g(X - i(a')) = (X - a') \cap X_m = (X - \mathcal{T}\text{-cl}a') \cap X_m$ since a' is compact. Thus $g\,k = h$ and it follows that S(I(L)) is the minimal completion of S(L).

A Boolean algebra is a pseudocomplemented distributive lattice with centre the whole Boolean algebra. Thus as an immediate corollary to Theorem 2.10 we have that for a Boolean algebra L, S(I(L)) is the minimal completion of L (cf. Stone (40) Theorem 2.4, p.236).

§3 The ideal lattice is a Stone algebra; L has 0,1

Let us assume L is a distributive lattice with 0 and 1. In §2.2 we showed that I(L) is a pseudocomplemented distributive lattice. In this section we classify those L for which I(L) is a Stone algebra and summarize results concerning I(L) as a Boolean algebra.

The following is a consequence of a result of Hashimoto ((22) Theorem 4.3 p.163).

THEOREM 2.11 *Let L be a distributive lattice with 0 and 1. Then I(L) is a Boolean algebra (i.e. $I(L) \in B_0$) if and only if L is a finite Boolean algebra.*

Proof. Let I(L) be a Boolean algebra. For $U \in I(L)$ let U' be the corresponding subset of X. $U' \cup U'^* = X$ implies $U' = i(\mathcal{T}\text{-cl}(U'))$. Hence every open decreasing set is closed and increasing. Let $x, y \in X$ with $x < y$. By total order disconnectedness of X there exists an open decreasing subset U', such that $x \in U'$, $y \notin U'$. But U' is increasing by the previous remark so $x < y$ implies $y \in U'$ giving a contradiction. Thus the order on X is discrete. We also have every open set is closed. Thus $\{x\}$ is open for each $x \in X$ and by compactness X must be finite. Hence L is a finite Boolean algebra.

Conversely, if L is a finite Boolean algebra then the ideals of L correspond to the set of all subsets of X which is a Boolean algebra. This completes the proof.

G. Grätzer((18) Problem 66, p.192) suggests classifying those lattices L for which $I(L) \in B_n$, $n \geq 1$. In this section we give a solution for $n = 1$. At the time that we solved the problem we did not know of Katriňák's classification ((24) Theorem 6, p.138, (26) Correction, p.143). However his method does not give an explanation of the close relationship between L and I(L).

Now let X be a topological space. X is said to be extremally disconnected if X is a Hausdorff space and the closure of each open set is open.

It can easily be shown that a Hausdorff space X is extremally disconnected if and only if every two disjoint open sets have disjoint closures.

An important theorem in the theory of Boolean algebras is:

THEOREM 2.12 The dual algebra A of a Boolean space X is complete if and only if X is extremally disconnected.

Proof. See ((21) §21 Theorem 10 p.92).

THEOREM 2.13 Let L be a distributive lattice with 0 and 1. I(L) is a Stone algebra if and only if L is a Stone algebra and S(L) is a complete Boolean algebra.

Proof. Let I(L) be a Stone algebra. Then $(a]^* \vee (a]^{**} = L$ for any $a \in L$. $1 \in L$ implies $(a]^* \vee (a]^{**} = (1]$ and $(a]^* \wedge (a]^{**} = (0]$. By considering these as subsets of the dual space, it is obvious that $(a]^*$ and $(a]^{**}$ are principal ideals and $(a]^* = (a^*]$ and $(a]^{**} = (a^{**}]$. Hence L is a Stone algebra.

Let P and Q be open in $(X_m, \mathcal{L}|X_m)$ with $P = P' \cap X_m$ and $Q = Q' \cap X_m$ where $P', Q' \in \mathcal{L}$. If $P \cap Q = \emptyset$, $P' \cap Q' = \emptyset$ as they are decreasing sets. I(L) a Stone algebra implies $P' \cup Q'^* = X$, by Theorem 1.3. Hence

$$i(\mathcal{T}\text{-cl}P') \cap i(\mathcal{T}\text{-cl}Q') = \emptyset$$

and

$$\mathcal{L}\text{-cl}P' \cap \mathcal{L}\text{-cl}Q' = \emptyset$$

by Lemma 2.1. Thus .

$$\mathcal{L}|X_m\text{-cl}P \cap \mathcal{L}|X_m\text{-cl}Q = \emptyset$$

and $(X_m, \mathcal{L}|X_m)$ is extremally disconnected. By Lemma 2.3 and the fact that the dual of S(L) is $(X_m, \mathcal{T}|X_m)$, we may apply Theorem 2.12 and have S(L) a complete Boolean algebra.

To prove the converse suppose S(L) is a complete Boolean algebra. Then $(X_m, \mathcal{T}|X_m)$ is extremally disconnected. Let g be the * - isomorphism from S(I(L)) onto the regular open subsets of $(X_m, \mathcal{T}|X_m)$ defined in Theorem 2.10. If $U', V' \in \mathcal{L}$ with $U'^* \cap V'^* = \emptyset$, then, $g(U'^*) \cap g(V'^*) = \emptyset$ and by extremal disconnectedness

$$\mathcal{T}|X_m - cl(g(U'^*)) \cap \mathcal{T}|X_m - cl(g(V'^*)) = \emptyset .$$

Thus

$$(X_m - \mathcal{T}|X_m - cl(g(U'^*))) \cup (X_m - \mathcal{T}|X_m - cl(g(V'^*))) = X_m$$

or

$$(g(U'^*))^* \cup (g(V'^*))^* = X_m .$$

Now $g(U'^{**}) \cup g(V'^{**}) = X_m$ implies $U'^{**} \cup V'^{**} = X$. By Theorem 1.3, I(L) is a Stone algebra.

We now give an example of a Stone algebra. Let L be a distributive lattice with 0, 1. L is a Post algebra if the following conditions are satisfied:

i) L has a subset

$$C = \{0 = c_0 \leqslant c_1 \leqslant \ldots \leqslant c_{n-1} = 1\}$$

for $n \geqslant 2$ and such that each $x \in L$ can be written as

$$x = \sum_{i=1}^{n} a_i \wedge c_i$$

where, for $1 \leqslant i \leqslant n-1$, $a_i \in C(L)$ the set of complemented elements of L;

ii) if $a \in C(L)$ and $a \wedge c_i \leqslant c_{i-1}$ for some i, $1 \leqslant i \leqslant n-1$, then $a = 0$.

THEOREM 2.14 A Post algebra L is a Stone algebra where S(L) = C(L).

Proof. A Post algebra is a coproduct of C(L) and C. Now C(L) is a Boolean algebra, C is a chain and as these are both Stone algebras and coproducts are preserved in the equational class of Stone algebras, L is a Stone algebra. In any Stone algebra, the centre equals the set of complemented elements.

THEOREM 2.15 ((17) Theorem 23) If L is a Post algebra, L is complete if and only if C(L) is complete.

COROLLARY 2.16 If L is a Post algebra, I(L) is a Stone algebra, if and only if L is complete.

§4 The ideal lattice is a Stone algebra; L has 0

In this section we consider distributive lattices with 0, but no 1. Such a lattice, L, is locally pseudocomplemented if, for any $a \in L$, $[0,a]$ is a pseudocomplemented lattice. It is a generalized Stone algebra if, for any $a \in L$, $[0,a^*]$ is a Stone algebra. In Lemmas 2.18, 2.21, 2.23 we give characterizations of the dual spaces of these lattices.

If L is a generalized Stone algebra we show that S(I(L)) is the completion of a lattice μ_L. If L has an 1, μ_L is, in fact, just S(L) so this extends the result in §2.2.

We then give necessary and sufficient conditions on a distributive lattice L with 0 in order that I(L) is a Stone algebra.

We note that as a consequence of Hashimoto's result ((22) Theorem 4.3 p.163) we have the following:

THEOREM 2.17 The ideals of a lattice L form a Boolean algebra if and only if L is a relatively complemented lattice with 0 (a generalized Boolean algebra) and L satisfies the descending chain condition.

Proof. Similar to that of Theorem 2.11. Note that for each $a \in L$, a' is compact in the dual space of L.

LEMMA 2.18 Let L be a distributive lattice with 0. L is locally pseudocomplemented if and only if for each $\mathcal{T}$-compact open decreasing subset a' in the dual space X of L, i(a') is $\mathcal{T}$-open.

Proof. Let $a \in L$ and suppose that i(a') is open, i(a') is closed by the remark following Lemma 1.1. Let $d \in L$ and $0 \leqslant a \leqslant d$, then $i_{d'}(a') = i(a') \cap d'$ will be $\mathcal{T}|d'$-clopen. d' is $\mathcal{T}$-compact open so $d' - i_{d'}(a')$ will be $\mathcal{T}$-compact open. Now if $0 \leqslant b \leqslant d$ and $a \wedge b = 0$ then $a' \cap b' = \emptyset$ implies $i_{d'}(a') \cap b' = \emptyset$ and $b' \subseteq d' - i_{d'}(a')$. Thus L is locally pseudocomplemented.

Conversely suppose $[0,d]$ is pseudocomplemented for each $d \in L$. If $0 \leqslant a \leqslant d$ we can show a*d, the pseudocomplement of a in $[0,d]$, corresponds to the element $c' = d' - i_{d'}(a')$. Certainly $(a*d)' \subseteq c'$. If $x \in d' - i_{d'}(a')$ using $\mathcal{T}$-compactness of d' and since $i_{d'}(a')$ is closed we obtain a $\mathcal{T}|d$-clopen decreasing set p' such that $x \in p' \subseteq c'$. Hence $a' \cap p' = \emptyset$ and $p' \subseteq (a*d)'$. Hence $(a*d)' = c'$ and $i_{d'}(a')$ is $\mathcal{T}$-open. Now $i(a') = \bigcup_{d \in L} i_{d'}(a')$, which completes the proof.

LEMMA 2.19 If E is a $\mathcal{T}$-compact subset of a monotonically separated space, $i(\mathcal{T}\text{-cl}E) = \mathcal{L}\text{-cl}E$.

Proof. We always have $i(\mathcal{T}\text{-cl}E) \subseteq i(\mathcal{L}\text{-cl}E) = \mathcal{L}\text{-cl}E$. Equality occurs if and only if $i(\mathcal{T}\text{-cl}E)$ is closed, since $\mathcal{L}\text{-cl}E$ is the smallest closed increasing set containing E. By the remark following Lemma 1.1, $i(\mathcal{T}\text{-cl}E)$ is closed. This completes the proof.

Using the same method of proof as Lemma 2.2 we have:

LEMMA 2.20 Let L be a distributive lattice with 0. Then I(L) is a pseudocomplemented distributive lattice. If $U \in I(L)$, U corresponds to an open decreasing set U', and U* corresponds to the open decreasing set $U'^* = X - \mathcal{L}\text{-cl}U'$.

Katriňák ((24), Lemma 8 p.134) has shown that a distributive lattice L with 0 is a generalized Stone algebra if and only if $(a]^* \vee (a]^{**} = L$ for any $a \in L$.

LEMMA 2.21 X is the dual space of a generalized Stone algebra L if and only if $U' \in \mathcal{L}$ implies $i(U') \in \mathcal{L}$.

Proof. Necessity: for $a \in L$, obviously $a' = (a]'$. So $a'^* \cup a'^{**} = X$ implies $a'^* \cup (X - \mathcal{L}\text{-cl}a'^*) = X$ by Lemma 2.20 and $a'^* = \mathcal{L}\text{-cl}a'^*$. Thus $X - \mathcal{L}\text{-cl}\ a'$ is $\mathcal{L}$-closed and by Lemma 2.19 $i(a')$ is $\mathcal{L}$-open. If $U' \in \mathcal{L}$, $U' = \bigcup_\alpha a'_\alpha$, a'_α compact open decreasing. Now $i(U') = \bigcup_\alpha i(a'_\alpha)$, since i preserves unions, which is $\mathcal{L}$-open.

Sufficiency: by Lemmas 2.19 and 2.20

$$a'^* \cup a'^{**} = (X - i(a')) \cup (X - \mathcal{L}\text{-cl}(X - i(a')))$$

Since $i(a') \in \mathcal{L}$, this gives $a'^* \cup a'^{**} = X$.

LEMMA 2.22 Let X be the dual space of a distributive lattice L with 0. Then $(X_m, \mathcal{L}|X_m)$ is a Hausdorff space with clopen base.

Proof. Let $x, y \in X_m$ with $x \neq y$. X is monotonically separated so $y \not\leq x$ implies there exists $U' \in \mathcal{L}$ such that $x \in U'$ and $y \notin U'$. We can take U' compact open decreasing i.e. a basic element. For $z \in U'$, $y \not\leq z$ so again X monotonically separated implies there exists V'_z, compact open increasing, with $z \in V'_z$ and $y \notin V'_z$.

Now $\{V'_z\}_{z \in U'}$ is a cover for U'. Hence $U' \subseteq \bigcup_{i=1}^{n} V'_i$, say. Now

$$y \in X - (\bigcup_{i=1}^{n} V'_i) = \bigcap_{i=1}^{n} (X - V'_i) = W'$$

say, where W' is open decreasing. Hence $x \in U' \cap X_m$ and $y \in W' \cap X_m$ where $U' \cap W' \cap X_m = \emptyset$. Now $\{a' \cap X_m : a \in L\}$ is an open basis for $\mathcal{L}|X_m$. But $(a]'^* \in \mathcal{L}$ and

$$(a]'^* \cap X_m = (X - i(a')) \cap X_m$$

by Lemmas 2.19 and 2.20. Thus $(X - a') \cap X_m$ is $\mathcal{L}|X_m$ open and $a' \cap X_m$ is $\mathcal{L}|X_m$-closed. This completes the proof.

LEMMA 2.23 Let L be a locally pseudocomplemented distributive lattice and X be its dual space. The following are equivalent:

i) X is the dual space of a generalized Stone algebra;

ii) for each $x \in X$ there exists a unique $m(x) \in X_m$ with $m(x) \leqslant x$;

iii) for any decreasing sets $P', Q' \subseteq X$, $i(P' \cap Q') = i(P') \cap i(Q')$.

Proof. i) implies ii): a Zorn's Lemma argument shows that each $x \in X$ is minimized by at least one element in X_m. Suppose $x \geqslant y, z$ where $y, z \in X_m$, $y \neq z$. Then there exist open decreasing sets P', Q' such that $y \in P' \cap (X-Q')$, $z \in Q' \cap (X-P')$ using the Hausdorff property of $(X_m, \mathcal{L}|X_m)$. Thus $x \in i(P') \cap i(Q')$ which is decreasing by Lemma 2.21. Hence $y \in i(P') \cap i(Q')$ and so by the minimality of y, $y \in P' \cap Q'$ contradicting the choice of P',Q'.

ii) implies iii): let P', Q' be decreasing. If $i(P')$ and $i(Q')$ are disjoint, iii) holds trivially. If $x \in i(P') \cap i(Q')$ then $x \geqslant z$ and $x \geqslant y$ for some $z \in P'$, $y \in Q'$. Thus $x \geqslant m(z)$ and $x \geqslant m(y)$. By ii) $m(x) = m(z) = m(y)$. Hence $x \in i(P' \cap Q')$. Trivially $i(P' \cap Q') \subseteq i(P') \cap i(Q')$.

iii) implies i): let $d \in L$. Then by assumption $[0,d]$ is a pseudocomplemented lattice. If d' is the subset of X corresponding to d and P', Q' are any decreasing sets contained in d' then $i_{d'}(P' \cap Q') = i_{d'}(P') \cap i_{d'}(Q')$. By Lemma 2.4, for pseudocomplemented lattices, this implies $[0,d]$ is a Stone algebra as we show $i_{d'}(P')$ is decreasing. Let $x \in i_{d'}(P')$, $y \leqslant x$. Then $x \geqslant y \geqslant z \in X_m$. Also $x \geqslant z_1 \in P' \cap X_m$. $x \in i_{d'}(\{z\}) \cap i_{d'}(\{z_1\}) = i_{d'}(\{z\} \cap \{z_1\})$. Hence $z = z_1$, so $y \in i_{d'}(P')$.

Let L be a distributive lattice with 0. For any $a \in L$, let $\mathcal{M}(a)$ be the set of minimal prime ideals not containing a Let

$$\mu_L = \{\mathcal{M}(a) : a \in L\} .$$

It can easily be shown that if X is the dual space of L and a' the subset of X corresponding to a, the subset of X corresponding to $\mathcal{M}(a)$ is $a' \cap X_m$.

LEMMA 2.24 Let L be a generalized Stone algebra. The dual of the lattice μ_L is $(X_m, \mathcal{L}|X_m)$ and μ_L is a generalized Boolean algebra.

Proof. By Lemma 2.22 we know that $(X_m, \mathcal{L}|X_m)$ is a Hausdorff space with $\{a' \cap X_m : a \in L\}$ as clopen base. We show that for each a, $a' \cap X_m$ is compact. Suppose $\{U_\alpha\}$ is a $\mathcal{L}|X_m$ open cover for $a' \cap X_m$. Then for each α, $U_\alpha = U'_\alpha \cap X_m$ for some $U'_\alpha \in \mathcal{L}$ and $a' \cap X_m \subseteq \bigcup_\alpha (U'_\alpha \cap X_m)$. Obviously $a' \subseteq \bigcup_\alpha i(U'_\alpha)$. By Lemma 2.21, $i(U'_\alpha) \in \mathcal{L}$ for each α, and

$$a' \subseteq \bigcup_{j=1}^{n} i(U'_j)$$

by compactness. Therefore

$$a' \cap X_m \subseteq \bigcup_{j=1}^{n} i(U'_j) \cap X_m$$

which implies $a' \cap X_m \subseteq \bigcup_{j=1}^{n} U_j$. By the Priestley duality theorem, $(X_m, \mathcal{L}|X_m)$ is the dual of μ_L. Since $(X_m, \mathcal{L}|X_m)$ has discrete order, μ_L is a generalized Boolean algebra.

LEMMA 2.25 Let X be the dual space of a generalized Stone algebra L. Then m is continuous from $(X, \mathcal{L})$ onto $(X_m, \mathcal{L}|X_m)$.

Proof. Let $P \in \mathcal{L}|X_m$. Then $P = P' \cap X_m$ for some $P' \in \mathcal{L}$. Now

$$m^{-1}(P) = i(P' \cap X_m) = i(P') \cap i(X_m)$$

by Lemma 2.23. Hence $m^{-1}(P) = i(P')$ which is $\mathcal{L}$-open by Lemma 2.21.

LEMMA 2.26 Let X be the dual space of a generalized Stone algebra. Then:

i) if $U' \in \mathcal{L}$, $\mathcal{L}\text{-cl}U' \cap X_m = \mathcal{L}\text{-cl}(U' \cap X_m) \cap X_m$,

ii) if $U', V' \in \mathcal{L}$, $\mathcal{L}\text{-cl}U' \cap X_m = \mathcal{L}\text{-cl}\,V' \cap X_m$ implies $\mathcal{L}\text{-cl}U' = \mathcal{L}\text{-cl}V'$.

Proof. i) Obviously

$$\mathcal{L}\text{-cl}(U' \cap X_m) \cap X_m \subseteq \mathcal{L}\text{-cl}U' \cap X_m .$$

Suppose $z \in \mathcal{L}\text{-cl}U' \cap X_m$. Then there is a net y_α, $y_\alpha \to z$ and $y_\alpha \in U'$ for all α. U' decreasing implies $m(y_\alpha) \in U' \cap X_m$ for all α. By Lemma 2.25 $m(y_\alpha) \to m(z) = z$ in $\mathcal{L}|X_m$. Hence $z \in \mathcal{L}|X_m\text{-cl}(U' \cap X_m) \cap X_m$.

ii) Suppose $\mathcal{L}\text{-cl}U' \cap X_m = \mathcal{L}\text{-cl}V' \cap X_m$, $U', V' \in \mathcal{L}$. If $\mathcal{L}\text{-cl}U' \neq \mathcal{L}\text{cl}V'$, we can suppose, without loss of generality, that these exists $x \in \mathcal{L}\text{-cl}U' - \mathcal{L}\text{-cl}V'$. Let y_α be a net in U' with $y_\alpha \to x$. By Lemma 2.25, $m(y_\alpha) \to m(x)$ in $\mathcal{L}|X_m$. U' decreasing implies $m(y_\alpha) \in U'$. So

$$m(x) \in \mathcal{L}|X_m\text{-cl}(U' \cap X_m) = \mathcal{L}\text{-cl}(U' \cap X_m) \cap X_m .$$

Thus $m(x) \in \mathcal{L}\text{-cl}V' \cap X_m$ by hypothesis. Hence $x \in \mathcal{L}\text{-cl}V'$ contradicting the choice of x.

THEOREM 2.27 Let L be a generalized Stone algebra. Then S(I(L)) is isomorphic to the Boolean algebra of regular open subsets of $(X_m, \mathcal{L}|X_m)$. Thus S(I(L)) is the minimal completion of μ_L.

Proof. Similar to that of Theorem 2.10, using Lemma 2.26, the isomorphism being g where

$$g(U'^*) = X-(\mathcal{L}\text{-cl}U') \cap X_m = X_m-(\mathcal{L}|X_m\text{-cl}(U' \cap X_m)).$$

THEOREM 2.28 Let L be a distributive lattice with 0. I(L) is a Stone algebra if and only if L is a generalized Stone algebra and μ_L is a conditionally complete generalized Boolean algebra.

Proof. Let I(L) be a Stone algebra. Then for any $a \in L$, $(a]^* \vee (a]^{**} = L$ implies L is a generalized Stone algebra. As in the proof of Theorem 2.13 we can show $(X_m, \mathcal{L}|X_m)$ is extremally disconnected. Suppose $\{a'_\alpha \cap X_m : a_\alpha \in L\}$ is bounded above by $b' \cap X_m$, which is a compact subspace of $\mathcal{L}|X_m$ by Lemma 2.24. Thus $b' \cap X_m$ will be a Boolean space and as it is extremally disconnected its dual algebra will be complete by Theorem 2.12. Therefore $\{a'_\alpha \cap X_m\}$ will have a supremum in $b' \cap X_m$, say $(c' \cap b') \cap X_m$ which equals $d' \cap X_m$ for some $d \in L$.

Conversely, suppose L is a generalized Stone algebra and μ_L is conditionally complete. Let $U \in (X_m, \mathcal{L}|X_m)$. We show that for all $x \in \mathcal{L}|X_m\text{-cl}U$, x is a $\mathcal{L}|X_m$-interior point of U. Now $x \in a' \cap X_m$ for some $a \in L$. $a' \cap X_m$ open implies $x \in \mathcal{L}|X_m\text{-cl}\,(a' \cap U \cap X_m)$. As above, we can show that since μ_L is conditionally complete, each $a' \cap X_m$ will be extremally disconnected in the relative topology induced by $\mathcal{L}|X_m$. Thus the closure in this relative topology of $a' \cap U \cap X_m$ will be open. But $a' \cap X_m$ is $\mathcal{L}|X_m$ clopen so closures and interiors in the relative topology coincide with closures and interiors in $\mathcal{L}|X_m$. Thus $\mathcal{L}|X_m\text{-cl}\,(a' \cap U \cap X_m)$ is $\mathcal{L}|X_m$ open and is contained in $\mathcal{L}|X_m\text{-cl}U$. Hence $(X_m, \mathcal{L}|X_m)$ is extremally disconnected. Again, as in Theorem 2.13, using Theorem 2.27 this implies I(L) is a Stone algebra.

By definition L is a generalized Stone algebra if and only if for each $a \in L$, $[0,a]$ is a Stone algebra. It can be shown that the dual of the centre with respect to a of $[0,a]$, $S_a([0,a])$, is $a' \cap X_m$ with the relative topology induced by $\mathcal{L}|X_m$. Theorem 2.28 implies that if L is a generalized Stone algebra, I(L) is a Stone algebra if and only if $(X_m, \mathcal{L}|X_m)$ is extremally disconnected. By Theorem 2.12 we have:

COROLLARY 2.29 ((24) Theorem 7,p.141) Let L be a distributive lattice with 0. I(L) is a Stone algebra if and only if L is a generalized Stone algebra and $S_a([0,a])$ is complete.

§5 The ideal lattice is a Stone algebra; L is quasicomplemented, normal

In this section let L be a distributive lattice with 0, no 1, but suppose L is normal and quasicomplemented as described by Cornish (13) and Speed (38), (39). The problem of when I(L) is a Stone algebra is a special case of that in §2.4 but these lattices provide rather nicer conditions. They also provide an introduction to §2.6.

A distributive lattice L with 0 is called normal if each prime ideal contains a unique minimal prime ideal. This is equivalent to the condition: for any $a,b \in L$, $a \wedge b = 0$ implies $(a]^* \vee (b]^* = L$ ((13) p.201). L is quasicomplemented ((13) p.207 (38)p.290-292) if for each a in L there exists $\bar{a}$ in L such that $(a]^{**} = (\bar{a}]^*$; sectionally quasicomplemented ((13)Definition 5.2p.208) if each interval $[0,a]$ is quasicomplemented.

Cornish ((13) Theorem 5.7,p.209) has shown that a distributive lattice L with 0 is a generalized Stone algebra if and only if it is both normal and sectionally quasicomplemented. As an immediate consequence of Theorem 2.28 we have:

COROLLARY 2.30 *Let L be a distributive lattice with 0. I(L) is a Stone algebra if and only if L is normal, sectionally quasicomplemented and μ_L is conditionally complete.*

It is obvious from the definition in §2.4 that a Stone algebra can be considered as a generalized Stone algebra with 1. Similarly, Cornish ((13) Proposition 5.5, p.208) has shown that a distributive lattice L with 0 is quasicomplemented if and only if it is sectionally quasicomplemented and possesses an element d such that $(d]^* = 0$.

Speed ((38) Proposition 3.3, p.292) has shown that μ_L, as defined in §2.4, is a Boolean algebra if and only if L is quasicomplemented.

LEMMA 2.31 *Let L be a quasicomplemented distributive lattice and X be the dual space of L. Then the dual space of μ_L is $(X_m, \mathcal{L}|X_m)$ and μ_L is a Boolean algebra.*

Proof. Since $\{a' : a \in L\}$ is a compact open base for $\mathcal{L}$, as in Lemma 2.22 we can show that $\{a' \cap X_m : a \in L\}$ is a clopen base for the Hausdorff space $(X_m, \mathcal{L}|X_m)$. Now $X_m - (a' \cap X_m) = (a]'^* \cap X_m = (\bar{a}]'^{**} \cap X_m$ for some $\bar{a} \in L$ by hypothesis. It is easily seen that $(\bar{a}]'^{**} \cap X_m = \bar{a} \cap X_m$. Hence $\{a' \cap X_m : a \in L\}$ is a basis for the closed sets. We now show X_m is $\mathcal{L}|X_m$-compact. By the above we need only consider a family of closed sets of the form $\{a'_\alpha \cap X_m : a_\alpha \in L\}$.

Suppose $\{a'_\alpha \cap X_m : a_\alpha \in L\}$ has the finite intersection property. Then for any finite n, $\bigcap_{\alpha=1}^{n} a'_\alpha \cap X_m \neq \emptyset$ implies $a_1 \wedge \ldots \wedge a_n \neq 0$. Using Stone's Lemma ((18) p.74) we may enclose the filter generated by $\{a_\alpha\}$ in a prime filter whose complement is a minimal prime ideal not meeting $\{a_\alpha\}$. Let $X-z$, $z \in X_m$, be the subset corresponding to the minimal prime ideal. Then $z \in a'_\alpha \cap X_m$, for all α.

Thus $(X_m, \mathcal{L}|X_m)$ is a Boolean space and has dual μ_L by the definition of μ_L in §2.4.

THEOREM 2.32 ((39) Corollary 6.6,p.67) *Let L be a distributive lattice with 0. The following conditions are equivalent:*

i) *μ_L is a complete Boolean algebra;*

ii) *for any $A \subseteq L$, there is $a \in L$: $A^{**} = (a]^*$.*

The next theorem is a corollary of Theorem 2.28, but it can be proved in a somewhat different manner using results described above.

THEOREM 2.33 *Let L be a distributive lattice with 0 and an element d such that $(d]^* = 0$. I(L) is a Stone algebra if and only if L is a normal, quasicomplemented lattice and μ_L is complete.*

Proof. Let I(L) be a Stone algebra. Then $(a]^* \vee (a]^{**} = L$ for any $a \in L$ implies L is a generalized Stone algebra and so L is normal and quasicomplemented. Let $U, V \in \mathcal{L}|X_m$ be

disjoint sets with $U = U' \cap X_m$, $V = V' \cap X_m$ for U', $V' \in \mathcal{L}$. Now $U' \cap V' = \emptyset$ implies

$$(X - \mathcal{L}\text{-cl}U') \cup (X - \mathcal{L}\text{-cl}V') = X$$

and

$$\mathcal{L}\text{-cl}U' \cap \mathcal{L}\text{-cl}V' = \emptyset .$$

Hence

$$\mathcal{L}|X_m\text{-cl}U \cap \mathcal{L}|X_m\text{-cl}V = \emptyset$$

which implies $(X_m, \mathcal{L}|X_m)$ is extremally disconnected. Hence μ_L is complete by Lemma 2.31 and Theorem 2.12.

For the converse suppose L is normal, quasicomplemented and μ_L is complete. Let $U,V \in I(L)$ be such that $U^* \wedge V^* = 0$. Then Theorem 2.32 implies there exists $a, b \in L$ such that $U^{**} = (a]^*$ and $V^{**} = (b]^*$. Now $U^* \wedge V^* = 0$ implies $(a]^{**} \wedge (b]^{**} = 0$. Since $(a] \leq (a]^{**}$ and $(b] \leq (b]^{**}$ this implies $a \wedge b = 0$. By normality, $(a]^* \vee (b]^* = L$ i.e. $U^{**} \vee V^{**} = L$ and I(L) is a Stone algebra.

§6 The ideal lattice is a B_n-algebra

In this section we consider the problem of characterizing a distributive lattice L with 0 such that $I(L) \in B_n$, $1 < n < \omega$. This is considerably more difficult than considering when $I(L) \in B_1$ as is shown by Theorem 2.36.

Let L be a distributive lattice with 0. L is called n-normal for $n \geq 1$ if each prime ideal in L contains at most n minimal prime ideals.

L is n-normal is equivalent ((14) Theorem 3.4p.50) to the condition: for any $x_0, x_1, \ldots, x_n \in L$ such that $x_i \wedge x_j = 0$, $i \neq j$, $i, j = 0, \ldots, n$, $(x_0]^* \vee \ldots \vee (x_n]^* = L$.

THEOREM 2.34 Let L be a distributive lattice with 0. For $n \geq 1$, $I(L) \in B_n$ implies L is an n-normal lattice. If L is a pseudocomplemented lattice, $I(L) \in B_n$ implies $L \in B_n$.

Proof. Let $x_0, \ldots, x_n \in L$ be such that $x_i \wedge x_j = 0$, $i,j = 0, \ldots, n$. Then $I(L) \in B_n$ implies $(x_0]^* \vee \ldots \vee (x_n]^* = L$. Now the algebras in B_n are exactly the pseudocomplemented members of the class of n-normal lattices so the proof is complete.

THEOREM 2.35 Suppose L is a distributive lattice with 0. If L is an n-normal, quasicomplemented lattice and μ_L is a complete Boolean algebra then $I(L) \in B_n$, $n \geq 1$.

Proof. Let $U_i \in I(L)$, $0 \leq i \leq n$, be such that $\{U_i^*\}$ is pairwise disjoint. Now μ_L a complete Boolean algebra implies, by Theorem 2.32, that for each i there is $a_i \in L$ such that $U_i^{**} = (a_i]^*$. By hypothesis, $(a_i]^{**} \wedge (a_j]^{**} = 0$, $i \neq j$, so $a_i \wedge a_j = 0$, $i \neq j$. L is n-normal so $(a_0]^* \vee \ldots \vee (a_n]^* = L$ i.e. $U_0^{**} \vee \ldots \vee U_n^{**} = L$. By Theorem 1.3, $I(L) \in B_n$.

A necessary and sufficient condition on L for I(L) to belong to B_n seems difficult to obtain. In terms of the dual space X of L we have:

THEOREM 2.36 Let L be an n-normal lattice with dual space X. $I(L) \in B_n$, $n \geq 1$ if and only if U_i' $\mathcal{L}$-open in X, $0 \leq i \leq n$, and pairwise disjoint implies $\mathcal{L}\text{-cl}U_0' \cap \ldots \cap \mathcal{L}\text{-cl}U_n' = \emptyset$.

Proof. Necessity: let $I(L) \in B_n$.

Let U_i', $0 \leq i \leq n$ be pairwise disjoint subsets in $\mathcal{L}$. Then as in Lemma 2.20 each U_i' corresponds to a member of I(L) and by hypothesis

$$(X - \mathcal{L}\text{-cl}U_0') \cup \ldots \cup (X - \mathcal{L}\text{-cl}U_n') = X.$$

Thus

$$\mathcal{L}\text{-cl}U_0' \cap \ldots \cap \mathcal{L}\text{-cl}U_n' = \emptyset.$$

Sufficiency: let $U_0, \ldots, U_n \in I(L)$ be such that $U_i \wedge U_j = 0$ for $i \neq j$, $i,j, = 0, \ldots, n$. Let U_i' be the $\mathcal{L}$-open subset of X corresponding to U_i. Then $U_i' \cap U_j' = \emptyset$ for $i \neq j$, $i,j = 0, \ldots, n$ implies

$$\mathcal{L}\text{-cl}U_0' \cap \ \ldots \ \cap \mathcal{L}\text{-cl}U_n' = \emptyset.$$

Thus

$$(X - \mathcal{L}\text{-cl}U_0') \cup \ \ldots \ \cup (X - \mathcal{L}\text{-cl}U_n') = X$$

and by Lemma 2.20

$$U_0^* \cup \ \ldots \ \cup U_n^* = L.$$

Thus

$$I(L) \in B_n.$$

When we considered $I(L) \in B_1$ the condition on two disjoint open sets of the dual space X having disjoint closures implied extremal disconnectedness and hence completeness (Theorem 2.28). With three or more sets a different topological property arises for which there seems to be no obvious lattice-theoretic translation.

§7 Iterated ideal lattices

Let L be a distributive lattice with 0. Consider the sequence of iterated ideal lattices : $I(L)$, $I^2(L)$,..., $I^n(L)$,... where, for $n > 1$, $I^n(L)$ is the lattice of ideals of $I^{n-1}(L)$. Since $I(L)$ is a pseudocomplemented distributive lattice, $I^2(L)$ is a pseudocomplemented distributive lattice and similarly for $I^n(L)$. We consider the implication, on L, of $I^n(L)$ being a Stone algebra and vice versa.

THEOREM 2.37 *Let L be a distributive lattice with 0. Then the following conditions are equivalent:*

i) $I^n(L)$ *is a Boolean algebra*, $n > 1$;

ii) $I(L)$ *is a finite Boolean algebra*;

iii) L *is a finite Boolean algebra*.

Proof. ii) and iii) are equivalent by Theorem 2.17 and the finiteness of $I(L)$ implying L is finite. We show i) is equivalent to ii). Let $I^n(L)$, $n > 1$ be a Boolean algebra. Then $I^{n-1}(L)$ is a pseudocomplemented lattice so by Theorem 2.11 must be a finite Boolean algebra. By iteration, $I(L)$ is a finite Boolean algebra. Now suppose $I(L)$ is a finite Boolean algebra. Then $I^2(L)$ is a Boolean algebra and is finite. By induction on n, $I^n(L)$ is a Boolean algebra $n > 1$. This completes the proof.

However in the "Stone algebra case" we have:

THEOREM 2.38 Let L be a distributive lattice with 0. For $n \geq 1$ the following conditions are equivalent:

i) $I^n(L)$ is a Stone algebra;

ii) $I(L)$ is a Stone algebra;

iii) L is a generalized Stone algebra and μ_L is conditionally complete.

Proof. ii) and iii) are equivalent by Theorem 2.28. We show i) is equivalent to ii). This is true if $n = 1$. Suppose $I^n(L)$ is a Stone algebra for $n > 1$. Now $I^{n-1}(L)$ is a pseudocomplemented lattice so is a Stone algebra by Theorem 2.13. By iteration $I(L)$ is a Stone algebra. Conversely, suppose $I(L)$ is a Stone algebra. We have $S(I(L))$ is complete so $I^2(L)$ is a Stone algebra by Theorem 2.13. By induction, $I^n(L)$ is a Stone algebra for $n \geq 1$.

Some of the results of Chapter 2 are summarized in the table. The conditions given are necessary and sufficient.

I D E A L S

	I(L) is a Boolean algebra	I(L) is a Stone algebra	I(L) is a B_n-algebra $1 \leq n < \omega$
L is a distributive lattice with 0	L is a generalized Boolean algebra satisfying the descending chain condition	L is a generalized Stone algebra and μ_1 is condition-ally complete generalized Boolean algebra	L is n-normal and the dual space X of L satisfies (*)
L is a distributive lattice with 0,1	L is a finite Boolean algebra	L is a Stone algebra and S(L) is complete Boolean algebra	L is n-normal and the dual space X of L satisfies (*)
L is a pseudocomplemented distributive lattice	L is a finite Boolean algebra	L is a Stone algebra and S(L) is complete Boolean algebra	$L \in B_n$ and the dual space X of L satisfies (*)
L is a Stone algebra	L is a finite Boolean algebra	S(L) is complete Boolean algebra	The dual space X of L satisfies (*)
L is a Boolean algebra	L is finite	L is complete	The dual space X of L satisfies (*) with $\mathcal{T} = \mathcal{L}$

(*) U_i $\mathcal{L}$-open in X, $0 \leq i \leq n$, and pairwise disjoint implies

$\mathcal{L}\text{-cl}U_0 \cap \ldots \cap \mathcal{L}\text{-cl}U_n = \emptyset$

Chapter 3

LATTICES OF FILTERS AND CONGRUENCE RELATIONS

§1 Introduction

The principle of duality is described in §3.2 and this notion is used throughout the chapter.

The dual concept of an ideal is a filter. As the filters of a distributive lattice with 1 form a pseudocomplemented lattice it is natural to consider the problem dealt with in Chapter 2 in the context of filters. In §3.3 we classify those distributive lattices with 1 for which the lattice of filters is a Stone algebra.

If L is a distributive lattice with 0 and 1 a congruence relation on L is required to preserve $\wedge$ and $\vee$ operations as well as any other operations defined on L. The sets of congruences on a distributive lattice, a pseudocomplemented distributive lattice and a distributive double p-algebra are pseudocomplemented distributive lattices. There is a good deal of literature concerned with the problem of when a congruence lattice of a lattice is Boolean. For the three different types of lattices mentioned above, the answers are given in ((22) Theorem 8.4), ((6) Theorem 3.1), and ((5) Theorem 4). In each of the three classes of lattices we classify the lattices which have congruence relations forming a Stone algebra. This is done in §3.4, §3.5 and §3.8.

The problem of when iterated congruence lattices and iterated lattices of filters form Stone algebras is much more complicated than the same problem for ideal lattices, discussed in §2.7. We briefly mention this in §3.3 and deal with it in §3.6.

§2 The principle of duality

In this section we describe the principle of duality which yields nothing profound but it can save a lot of work. We give definitions of a few dual concepts.

For a lattice L with partial order $\leqslant$, consider the relation defined on L by $a \leqslant' b$ if and only if $b \leqslant a$. It is evident that $\leqslant'$ partially orders L and in fact $(L,\leqslant')$ is a lattice in which $a \wedge' b = a \vee b$ and $a \vee' b = a \wedge b$. $(L, \leqslant')$ is the dual of $(L,\leqslant)$. Now, if Φ is a lattice theoretic proposition, and if in Φ we replace all occurences of $\vee$ and $\wedge$ by $\vee'$ and $\wedge'$ respectively, then we have the dual of Φ. For example, the dual of a distributive lattice L is distributive since for $a,b, c \in L$, $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$ if and only if $(a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$.

The dual concept of an ideal is a filter F which is a sublattice of a lattice L such that if $f \in F$ and $a \in L$ then $a \vee f \in F$.

The dual of a pseudocomplemented distributive lattice is a dually pseudocomplemented distributive lattice which is a distributive lattice L with 0 and 1 such that for each $a \in L$ there exists a dual pseudocomplement $a^{+} \in L$ such that $a \vee b = 1$ if and only if $b \geq a^{+}$. We define:

$$\overline{S}(L) = \{a^{+} \in L : a \in L\}$$

$$\overline{D}(L) = \{a \in L : a^{+} = 1\}.$$

A dual Stone algebra is a dually pseudocomplemented distributive lattice satisfying $a^{+} \wedge a^{++} = 0$ for each $a \in L$.

§3 Lattices of filters

We classify those distributive lattices L with 1 for which $F(L) \in B_0, B_1$. We could give lengthy proofs of theorems dual to those of Chapter 2 but using the principle of duality we are able to by-pass these.

Let $\overline{L}$ be the dual of a lattice L. Then F(L) is distributive and isomorphic to $I(\overline{L})$. $I(\overline{L})$ is a complete pseudocomplemented lattice if $\overline{L}$ is a distributive lattice with 0. Hence F(L) is a complete pseudocomplemented lattice if L is a distributive lattice with 1. We first consider those L for which $F(L) \in B_0$.

THEOREM 3.1. *Let L be a distributive lattice with 1. F(L) is a Boolean algebra if and only if L is a relatively complemented lattice satisfying the ascending chain condition.*

Proof. By duality and Theorem 2.17.

THEOREM 3.2. *Let L be a distributive lattice with 0 and 1. The following conditions are equivalent:*

i) F(L) *is a Boolean algebra;*

ii) I(L) *is a Boolean algebra;*

iii) L *is a finite Boolean algebra.*

Proof. By duality and Theorem 2.11.

THEOREM 3.3 Let L be a distributive lattice with 0 and 1. F(L) is a Stone algebra if and only if L is a dual Stone algebra and $\overline{S}(L)$ is complete.

Proof. F(L) is a Stone algebra if and only if $I(\overline{L})$ is a Stone algebra. By Theorem 2.13, this is true if and only if $\overline{L}$ is a Stone algebra and $S(\overline{L})$ is complete. By duality, this is equivalent to L being a dual Stone algebra and $\overline{S}(L)$ being complete.

Let L be a distributive lattice with 1. We define L to be a generalized dual Stone algebra if for every $a \in L$, $[a,1]$ is a dual Stone algebra. If L is a distributive lattice with 1, for each $a \in L$, let:

$$\mathcal{M}(a) = \{\text{maximal prime ideals not containing } a\}$$

and let

$$\nu_L = \{\mathcal{M}(a) : a \in L\}.$$

THEOREM 3.4 Let L be a distributive lattice with 1. F(L) is a Stone algebra if and only if L is a generalized dual Stone algebra and ν_L is a conditionally complete, relatively complemented distributive lattice with 1.

Proof. F(L) is a Stone algebra if and only if $I(\overline{L})$ is a Stone algebra. By Theorem 2.28, this is equivalent to $\overline{L}$ being a generalized Stone algebra and $\mu_{\overline{L}}$ being a conditionally complete generalized Boolean algebra. Now $\overline{L}$ has a 0 if and only if L has an 1. Each interval in $\overline{L}$ of the form $[0, a]$

is a Stone algebra if and only if $[a,1]$ is a dual Stone algebra in L. By considering the dual spaces of L and $\overline{L}$ it can be shown $\mu_{\overline{L}}$ is the dual of ν_L. The result follows.

Considering iterated filter lattices by Theorem 3.1 we have:

THEOREM 3.5 Let L be a distributive lattice with 1. The following conditions are equivalent:

i) L is a finite Boolean algebra;

ii) F(L) is a finite Boolean algebra;

iii) $F^n(L)$ is a Boolean algebra, $n > 1$.

To classify those L for which $F^n(L)$ is a Stone algebra is more complicated. If $F^n(L)$ is a Stone algebra, $n > 1$, then $F^{n-1}(L)$ is a dual Stone algebra. Thus we no longer have a sequence of Stone algebras. The same problem arises in the question of iterated congruence lattices which is dealt with, in some depth, in §3.6 so we do not go into any detail here.

§4 Lattice congruence relations

An equivalence relation θ on a lattice L is called a lattice congruence relation of L if $a_0 \equiv b_0(\theta)$ and $a_1 \equiv b_1\ (\theta)$ imply that

$$a_0 \wedge a_1 \equiv b_0 \wedge b_1\ (\theta) \text{ and } a_0 \vee a_1 \equiv b_0 \vee b_1\ (\theta).$$

Let K (L) denote the set of lattice congruences on L.

In this section we classify those distributive lattices L with 0 for which the congruence lattice is a Stone algebra. At the time, we did not know of Katriňák's result ((25)Theorem 7).

Let $(X, \mathcal{T}, \leqslant)$ be the dual space of a distributive lattice L with 0. Then, for each closed subset θ' of X, the relation $a_1 \equiv a_2\ (\theta)$ if and only if $a_1' \cap \theta' = a_2' \cap \theta'$ defines a congruence relation θ on L. Conversely, given $\theta \in K(L)$, the dual space of $L \mid \theta$ may be identified with a closed subspace X_θ of X via the map dual to the canonical epimorphism $\pi_\theta : L \to L \mid \theta$, so that $\pi_\theta(a_1) = \pi_\theta(a_2)$ if and only if $a_1' \cap X_\theta = a_2' \cap X_\theta$ ((34) p.515-517). The closed sets form a dually pseudocomplemented lattice. Thus the dual correspondence above implies K(L) is a pseudocomplemented lattice.

We summarize some results on those L for which K(L) is a Boolean algebra.

Hashimoto ((22) Theorem 8.4, p.179) has given a more general result than Theorem 3.7 for distributive lattices without 0, 1. We give proofs for the lattices we are considering for completeness.

THEOREM 3.6 Let L be a distributive lattice with 0 and 1. Then K(L) is a Boolean algebra if and only if L is finite.

Proof. K(L) is a Boolean algebra if and only if the open subsets of X, the dual space of L, form a complemented lattice. This is true if and only if every closed set is open. If $\{x\}$ is clopen for all $x \in X$, the compactness implies X, and hence L, is finite. Conversely if L is finite then the topology $\mathcal{T}$ is discrete. This completes the proof.

THEOREM 3.7 Let L be a distributive lattice with 0. K(L) is a Boolean algebra if and only if every interval $[0,a]$, $a \in L$, is of finite length.

Proof. Similar to that of Theorem 3.6 using the fact that for each $a \in L$, a' is compact in the dual space of L.

Let L be a distributive lattice with 0 (and 1) and let $(X,\mathcal{T},\leqslant)$ be the dual space of L. If the order relation in the dual space is replaced by the discrete order we obtain a space $(X,\mathcal{T})$ which has a base of compact open sets. These sets form a Boolean ring (algebra) L_B which is, up to isomorphism, the minimal Boolean ring (Boolean) extension of the distributive lattice L (cf (29), (34), Proposition 26 p.528).

We now classify those L for which K(L) is a Stone algebra.

THEOREM 3.8 Let L be a distributive lattice with 0, 1. K(L) is a Stone algebra if and only if L_B is a complete Boolean algebra.

Proof. K(L) is isomorphic with the lattice of open subsets of $(X,\mathcal{T})$. These correspond, as in §2.2, to the ideals of L_B. By Theorem 2.13, $I(L_B)$ is a Stone algebra if and only if L_B is complete.

COROLLARY 3.9 Let L be a Boolean algebra. K(L) is a Stone algebra if and only if L is complete.

THEOREM 3.10 Let L be a distributive lattice with 0. K(L) is a Stone algebra if and only if for each $a \in L_B$, $[o,a]$ is a complete Boolean algebra.

Proof. As in Theorem 3.8 using Corollary 2.29.

COROLLARY 3.11 Let L be a generalized Boolean algebra. K(L) is a Stone algebra if and only if for each a ε L, [o,a] is a complete Boolean algebra.

Iqbalunissa ((23) Theorem 1, p.19) has given necessary and sufficient conditions on a complete lattice L (not necessarily distributive) in order that K(L) is a Stone algebra. This raises the problem of whether there is a distributive lattice L with 0 and 1 which is not complete although L_B is complete.

Using Priestley's description ((34) Proposition 16,p.521) of the dual of a complete distributive lattice L we have L_B is complete if and only if in the dual space of L $\mathcal{T}$-cl G ε $\mathcal{T}$, for all G ε $\mathcal{T}$, whereas L is complete if and only if $\mathcal{U}$-clVε $\mathcal{L}$, for all V ε $\mathcal{L}$. Thus the existence of a lattice which is not complete although L_B is complete seems likely although it may be difficult to obtain. We can show:

THEOREM 3.12 Let L be a relatively pseudocomplemented distributive lattice with 0 and 1. Then L is complete if the minimal Boolean completion of L is complete.

Proof. If L is a relatively pseudocomplemented lattice the dual space $(X, \mathcal{T}, \leqslant)$ of L is an I_i space, as shown in the proof of Theorem 2.7, thus by Lemma 1.1 we have $\mathcal{T}$-cl V =$\mathcal{U}$-clV, for all V ε $\mathcal{L}$. Now L_B is complete so $\mathcal{T}$-cl V ε $\mathcal{T}$. Thus $\mathcal{U}$-clV ε $\mathcal{L}$ and L is complete.

§5 Congruence relations of pseudocomplemented distributive lattices

This section is concerned with the congruence relations of pseudocomplemented distributive lattices where, as mentioned in Chapter 1, pseudocomplementation is an operation on L. Let L be a pseudocomplemented distributive lattice. K(L) is the set of lattice congruence relations on L which also satisfy for a, b ε L:

$$\theta \varepsilon K(L) \text{ and } a \equiv b\ (\theta) \text{ implies } a^* \equiv b^*\ (\theta)\ .$$

So θ is required to preserve *. In this section we describe those pseudocomplemented distributive lattices for which K(L) is a Stone algebra.

If $\theta \varepsilon K(L)$, we describe the subset θ' of the dual space X of L which corresponds to θ.

LEMMA 3.13 ((36) Proposition 3,p.217) Let X, X_1 be dual spaces of L,L_1, pseudocomplemented distributive lattices, and let $f: X \to X_1$ be continuous and increasing. Then the dual map $\Phi: L_1 \to L$ preserves pseudocomplementation if and only if

$$f(m(x)) = m(f(x)) \text{ for all } x \varepsilon X\ .$$

LEMMA 3.14 (cf. (36) Proposition 13, Adams (1)). Let L be a pseudocomplemented distributive lattice. Then θ is a congruence of L if and only if θ' is a closed subset of X and is such that $m(\theta') \subseteq \theta'$.

Proof. Preservation of pseudocomplements by θ is equivalent to preservation of pseudocomplements by the canonical epimorphism $\pi_\theta : L \to L|\theta$. Regarding the dual of π_θ as a map f_θ from X into X, it is sufficient to show that $f_\theta(m(x)) = m(f_\theta(x))$ for all $x \in X$ if and only if $m(\theta') \subseteq \theta'$ where $\theta' = f_\theta(X)$ (Lemma 3.13). It is easily seen that each of these conditions is equivalent to

$$y \in X_m,\ y \leqslant z \in \theta' \text{ implies } y \in \theta'.$$

Note. If L is a pseudocomplemented distributive lattice, and F is a filter in L, F generates a pseudocomplementation-preserving congruence, θ_F, in the following way:

$a_1 \equiv a_2 (\theta_F)$ if and only if $a_1 \wedge a = a_2 \wedge a$ for some $a \in F$. That F is a *-congruence is easily seen by the fact that F', the subset of the dual space X of L corresponding to F, is the closed decreasing subset, $\bigcap_{a \in F} a'$.

Using Lemma 3.14 we now show that K(L) is a pseudocomplemented distributive lattice for L a pseudocomplemented distributive lattice. We call a subset E of a partially ordered space X a c-subset if E is $\mathcal{T}$-closed and $m(E) \subseteq E$. We show that the c-subsets of X form a dually pseudocomplemented lattice. By Lemma 3.14, K(L) will then be pseudocomplemented.

LEMMA 3.15 If X is the dual space of a pseudocomplemented distributive lattice, and E a c-subset of X, then E has dual pseudocomplement

$$E^{+} = \mathcal{T}\text{-cl}\ (X-E) \cup m\ (\mathcal{T}\text{-cl}(X-E)).$$

Proof. $m(\mathcal{T}\text{-cl}(X-E))$ is $\mathcal{T}$-closed since X is a C_d-space and X_m is $\mathcal{T}$-closed. Hence E^{+} is $\mathcal{T}$-closed and it is a c-subset. Certainly $E \cup E^{+} = X$. Now if $E \cup G = X$ for some c-subset G of X then $G \supseteq X-E$. G a c-subset implies

$$G \supseteq \mathcal{T}\text{-cl}(X-E) \cup m\ (\mathcal{T}\text{-cl}X-E))$$

so E^{+} is the dual pseudocomplement of E.

The question arises as to when K(L), L a pseudocomplemented distributive lattice, is a Boolean algebra or a Stone algebra. The first problem was solved by Berman ((6), Theorem 3.1,p.29) using lattice-theoretic methods.

THEOREM 3.16 Let L be a pseudocomplemented distributive lattice. K(L) is a Boolean algebra if and only if L is a finite Boolean algebra.

Proof. Suppose K(L) is a Boolean algebra. Let a' be clopen decreasing in X, the dual space of L. Then a' is a c-subset so it is complemented. By Lemma 3.15,

$$X-a' = a'^{+} = (X-a') \cup m(X-a').$$

Hence X–a' is decreasing. Thus every clopen increasing set is decreasing and the order on X is discrete. Now the *-preserving congruences of a Boolean algebra are the same as the lattice congruences, using De Morgan's laws ((21) p.5).. By Theorem 3.6 L is a finite Boolean algebra.

The converse also follows by Theorem 3.6.

We now consider the question of classifying those pseudocomplemented distributive lattices L for which K(L) is a Stone algebra.

In any compact totally order disconnected space $(X, \mathcal{T}, \leqslant)$, by Zorn's lemma there exists for each $x \in X$, at least one point $y \geqslant x$ such that y is maximal with respect to the partial order $\leqslant$. The set of all points maximal with respect to $\leqslant$ will be denoted by X_M.

THEOREM 3.17 Let L be a pseudocomplemented distributive lattice and K(L) be the lattice of congruences on L. Then K(L) is a Stone algebra if and only if L is a dual Stone algebra, $\overline{S}(L)$ is complete and D(L) is a relatively complemented distributive lattice with 1.

Proof. Suppose K(L) is a Stone algebra. This implies, by Lemmas 3.14, 3.15, that the closed c-subsets of X form a dual Stone algebra. We show that if U is a c-subset of X, U^+ is clopen atonic.

By hypothesis $U^+ \cap U^{++} = \emptyset$. By definition, $U^+ \cup U^{++} = X$, so $U^+ = X - U^{++}$ which implies U^+ is clopen. Now suppose $y \in U^+$ and $x \leqslant y$. Then $x \notin U^+$ implies $x \in U^{++}$. U^{++} is a c-subset so $m(x) \subseteq U^{++}$. But $x \leqslant y$ implies $m(x) \subseteq m(y)$. Now $m(y) \subseteq U^+$ since U^+ is a c-subset, so $m(x) \subseteq U^+$. This contradicts $U^+ \cap U^{++} = \emptyset$. Similarly we can show U^+ is increasing.

Conversely, suppose L is a dual Stone algebra, $\overline{S}(L)$ is complete and that D(L) is relatively complemented. This implies that the dual of D(L), $X \to X_m$ will have the discrete order so we can write $X = X_m \cup X_M$. As in the preceding paragraph, we can show that if U is a c-subset, U must be decreasing. So to prove that K(L) is a Stone algebra we must show that the closed decreasing subsets of X form a dual Stone algebra. This is equivalent to showing that F(L) is a Stone algebra which is true by Theorem 3.3.

We note that if L is a Boolean algebra, the dual space of L has discrete order which shows that lattice congruences and complement - preserving lattice congruences are the same. This also follows by De Morgan's Laws. So, as in Corollary 3.9, K(L) is a Stone algebra if and only if L is complete.

COROLLARY 3.18 Let L be a finite pseudocomplemented distributive lattice. K(L) is a Stone algebra if and only if L is a dual Stone algebra and D(L) is a Boolean algebra.

Proof. Follows directly from Theorem 3.17 since $\overline{S}(L)$ finite implies $\overline{S}(L)$ complete and D(L) finite and relatively complemented implies D(L) is a Boolean algebra.

An example to show that D(L) being relatively complemented is necessary is furnished by the following.

Let C be the four-element chain {o,a,b,1}. C is a Stone algebra and a dual Stone algebra with $\overline{S}$ (C) finite, if we define:

$$0^* = 1,$$

$$a^* = 0,$$

$$b^* = 0,$$

$$1^* = 0,$$

and

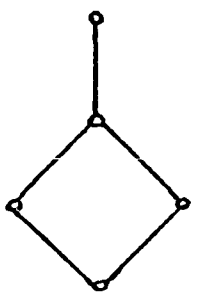
$$0^+ = 1,$$

$$a^+ = 1,$$

$$b^+ = 1,$$

$$1^+ = 0.$$

Now D(C) is the chain {a,b,1} which is not a Boolean algebra and K(C), which has diagram:



is not a Stone algebra. It is a B_2-algebra.

However, if A is the three-element chain {0,a,1}. Then A is a Stone algebra and a dual Stone algebra if we define * and $^+$ as above. $\overline{S}$(A) is finite and D(A) = {a,1} is a Boolean algebra. K(A) is a three-element chain which is a Stone algebra.

THEOREM 3.19 ((6) Theorems 3.2,3.3) Let L be a pseudocomplemented distributive lattice. L is finite and D(L) is a Boolean algebra if and only if L is dually isomorphic with K(L).

Proof. If L is finite and D(L) is a Boolean algebra, as in the proof of Theorem 3.17, the dual space of L, X, must be $X_M \cup X_m$ and the topology on X is discrete. Certainly every decreasing set is a c-subset. On the other hand if U is a c-subset of X and $y < x$ for some $x \in U$, then D(L) a Boolean algebra implies $y \in m(x)$. So $y \in U$. Thus decreasing subsets are in one-to-one correspondence with c-subsets and L is dually isomorphic with K(L).

Conversely, if L is dually isomorphic with K(L), every c-subset of X is clopen decreasing. In particular, for each $x \in X$, d(x) is a c-subset of X. So d(x) is open. Now $X = \bigcup_{x \in X} d(x)$ so $X = \bigcup_{i=1}^{n} d(x_i)$ for some x_i. Hence X and L must be finite. If $x,y \in X - X_m$, $x \lessdot y$, then $\{y\} \cup m(\{y\})$ is a c-subset of X so it must be decreasing. This implies $x \in \{y\} \cup m(\{y\})$ so $x=y$. Hence the order on $X-X_m$ is discrete and D(L) is a Boolean algebra.

Theorem 3.19 and Corollary 3.18 yield:

COROLLARY 3.20 Let L be a finite pseudocomplemented distributive lattice. If K(L) is a Stone algebra, then L is dually isomorphic with K(L).

§6 Iterated lattices of congruence relations

In §2.7 we considered those lattices L for which the iterated ideal lattices $I^n(L)$ form Boolean algebras or Stone algebras. In this section iterated lattices of congruence relations $K^n(L)$ are discussed.

Let L be a distributive lattice with 0 and 1. Then if K(L) is the set of lattice congruences on L, we can consider $K^n(L)$, $n > 1$, to be the set of lattice congruences on $K^{n-1}(L)$.

THEOREM 3.21 Let L be a distributive lattice with 0, 1. The following conditions are equivalent:

i) $K^n(L)$ is a Boolean algebra, $n \geq 1$;

ii) K(L) is a Boolean algebra;

iii) L is finite.

Proof. For $n > 2$, $K^n(L)$ a Boolean algebra implies $K^{n-1}(L)$ is finite by Theorem 3.6. This implies $K^{n-2}(L)$ is finite, since if $K^{n-2}(L)$ has dual space X with only finitely many open sets, then there are only finitely many sets of the form $X-\{x\}$ which gives finitely many points $x \in X$. Again, by Theorem 3.6, $K^{n-1}(L)$ is a Boolean algebra. The result follows by iteration.

We can now take L and K(L) as above but let $K^n(L)$, $n > 1$, be the *-preserving lattice congruences on $K^{n-1}(L)$. By Theorem 3.16 we can show Theorem 3.21, with the new meaning for $K^n(L)$, $n > 1$, remains true. This is because lattice congruences of a Boolean algebra are the same as *-preserving lattice congruences - easily shown by the fact that there is no order structure on the dual space of a Boolean algebra.

Finally by Theorem 3.16, considering *-preserving lattice congruences one can easily prove:

THEOREM 3.22 Let L be a pseudocomplemented distributive lattice. The following conditions are equivalent:

i) $K^n(L)$ is a Boolean algebra, $n \geq 1$;

ii) K(L) is a Boolean algebra;

iii) L is a finite Boolean algebra.

Suppose L is a distributive lattice with 0 and K(L) the lattice congruences on L. Let $K^2(L)$ be the lattice congruences on K(L). Then by Theorem 3.8, $K^2(L)$ is a Stone algebra if and only if $(K(L))_B$ is a complete Boolean ring. Now K(L) is a complete lattice so the problem arises as to whether L a complete distributive lattice with 0 implies L_B complete. Again the answer is not known, but as L complete implies $\mathcal{L}\text{-cl}(U) \in \mathcal{U}$, for all $U \in \mathcal{U}$, whereas L_B is complete only if $\mathcal{T}\text{-cl}(G) \in \mathcal{T}$, for all $G \in \mathcal{T}$, it seems unlikely. However, in general we have:

THEOREM 3.23. *Let L be a distributive lattice with 0, and, $n > 1$, $K^n(L)$ the lattice congruences on $K^{n-1}(L)$. Then $K^n(L)$ is a Stone algebra if and only if $(K^{n-1}(L))_B$ is complete.*

Suppose L is a distributive lattice with 0, 1. Then K(L) is a pseudocomplemented distributive lattice so we may consider the lattice of *-preserving lattice congruences, $K^2(L)$, on K(L). By Theorem 3.17, if $K^2(L)$ is a Stone algebra, K(L) must be a dual Stone algebra.

LEMMA 3.24. *Let L be a distributive lattice with 0, 1. Then K(L) is dually pseudocomplemented if and only if L is finite.*

Proof. K(L) is dually isomorphic with the lattice of $\mathcal{T}$-closed subsets of X, the dual space of L. Now the lattice of $\mathcal{T}$-closed subsets is pseudocomplemented if and only if L is finite. Suppose $\{x\}$, $x \in X$, has pseudocomplement $\{x\}^*$. Suppose$\{x\}$ is not open. Then $\{x\}^* \neq X-\{x\}$. Let $y \in X-(\{x\} \cup \{x\}^*)$. $(X, \mathcal{T})$ is compact and Hausdorff so there exist disjoint open sets W, V such that $x \in W$, $\{x\}^* \cup \{y\} \subseteq V$. X–W is a closed set containing $\{x\}^* \cup \{y\}$ and $\{x\} \cap (X-W) = \emptyset$. This contradicts the definition of $\{x\}^*$. Thus $\{x : x \in X\}$ is an open cover for X. By compactness, X is finite.

Conversely, if L is finite obviously K(L) is finite and so is dually pseudocomplemented.

Now if L is a finite distributive lattice, by Theorem 3.6, $K(L)$ is a finite Boolean algebra and so are $K^2(L)$, $K^3(L)$.... Thus we have proved:

THEOREM 3.25. *Let L be a distributive lattice with 0 and 1. The following conditions are equivalent:*

i) $K^2(L)$ *is a Stone algebra;*

ii) $K(L)$ *is a Boolean algebra;*

iii) L *is finite;*

iv) $K^2(L)$ *is a finite Boolean algebra.*

Now let L be a distributive lattice with 0, 1. Let $K^3(L)$ be the *-preserving lattice congruences on $K^2(L)$. Then if $K^3(L)$ is a Stone algebra, $K^2(L)$ is a dual Stone algebra. This implies that the c-subsets of the dual space of $K(L)$ form a Stone algebra.

LEMMA 3.26. *Let L be a pseudocomplemented distributive lattice. $K(L)$ is a dual Stone algebra if and only if L is a Stone algebra and $S(L)$ is finite.*

Proof. Let X be the dual space of L. For $x \in X$, $d(x)$ is a c-subset of X. By hypothesis, the c-subsets of X form a Stone algebra. We show $(d(x))^* = X - i(d(x))$. Since $(d(x))^*$ is a c-subset, $i(d(x)) \cap (d(x))^* = \emptyset$, so $(d(x))^* \subseteq X - i(d(x))$. If $y \in X - (i(d(x)) \cup (d(x))^*)$, then $d(y) \cup (d(x))^*$ is a c-subset such that its intersection with $d(x)$ is empty. Hence $(d(x))^* = X - i(d(x))$.

This implies $i(d(x))$ is open for each $x \in X$. In particular for $x \in X_m$, $i(x)$ is open. Now $X = \bigcup_{x \in X_m} i(x)$. By compactness $X = \bigcup_{j=1}^{n} i(x_j)$, $x_j \in X_m$, so X_m is finite. Hence S(L) is finite.

If U is a c-subset of X, $U^{**} \cup U^* = X$. Also $U^* \cap U^{**} = \emptyset$, so $U^{**} = X-(U^*)$. We can show $U^* = X-i(d(U))$. Certainly $(X-i(d(U)) \cap U = \emptyset$ so $U^* \supseteq X-i(d(U))$. For any $x \in U^*$, $d(x) \cap X_m \subseteq U^*$ so $m(x) \subseteq X-U$. Suppose there exists $x \in U^*$ such that $x \notin X-i(d(U))$. Then $x \in i(d(U))$ so $x \geqslant y$ for $y \in d(U)$. Now $m(y) \subseteq U$, and $m(x) \supseteq m(y)$ so $m(x) \cap U \supseteq m(y) \cap U$ and so $m(x) \cap U \neq \emptyset$. This contradicts $m(x) \subseteq X-U$. Now $U^* = X-i(d(U)$ implies $i(d(U))$ is open and we know it is closed and increasing. We show it is decreasing. Let $x \in i(d(U))$ and suppose $y < x$. Then $y \notin i(d(U))$ implies $y \in U^{**}$. Hence $m(y) \subseteq U^{**}$. But $m(y) \subseteq m(x) \subseteq i(d(U))$, since it is a c-subset. Thiscontradicts $U^* \cap U^{**} = \emptyset$.

If a' is clopen decreasing, we have shown $i(a') \in \mathcal{L}$. Any open decreasing set is the union of its clopen decreasing subsets and the operator i preserves unions. Thus $i(U) \in \mathcal{L}$ whenever $U \in \mathcal{L}$. By Lemma 2.4, L is a Stone algebra.

Conversely, let U be a c-subset of X. If we show U^* is clopen atonic, then U^{**} must be $X-U^*$ which implies the c-subsets form a Stone algebra, and K(L) is a dual Stone algebra. We show that$U^* = X-i(d(U))$. Certainly $U^* \supseteq X-i(d(U))$. If $x \in U^* \cap i(d(U))$, then $m(x) \in U^*$ and $x \geqslant y \in d(U)$. But $y \in d(U)$ implies $y \leqslant z$ for $z \in U$. Since U is a c-subset, $m(y) = m(z) \in U$. But $x \geqslant y$ implies $m(y) = m(x) \in U^*$. This contradicts $U \cap U^* = \emptyset$. Hence $X-i(d(U)) = U^*$.

We have $S(L)$ and hence X_m is finite. Thus $X = \bigcup_{i=1}^{n} i(x_j)$, for distinct $x_j \in X_m$. Let $y \in i(x_j)$ and suppose $y \in i(x_k)$. Then $y \geq x_j$ and $y \geq x_k$ implies $x_j = x_k$ if L is a Stone algebra. Hence the $i(x_j)$ are disjoint and $i(x_j)$ is clopen for all j. Also if $z \in i(x_j)$ and $y \leq z$ then $m(y) = m(z) = x_j$ so $y \in i(x_j)$. Hence $i(x_j)$ is decreasing for all j.

Now $i(d(U)) = i(d(U) \cap X_m) = i(\bigcup_{i=1}^{n} (x_j)) = \bigcup_{i=1}^{n} i(x_j)$, for some n and $x_j \in X_m$, if L is a Stone algebra and $S(L)$ is finite. We have $i(x_j)$ clopen atonic for each j so $i(d(U))$ is clopen atonic. This completes the proof.

By Theorem 3.17 and Lemma 3.26 we have:

COROLLARY 3.27 *Let L be a distributive lattice with 0 and 1. If $K^3(L)$ is a Stone algebra, then $K^2(L)$ is a dual Stone algebra and $K(L)$ is a Stone algebra.*

The following two corollaries are proved by induction on n using Theorems 3.17, 3.25 and Lemma 3.26. For $n > 1$, $K^n(L)$ is the lattice of *-preserving congruences on $K^{n-1}(L)$.

COROLLARY 3.28. *Let L be a distributive lattice with 0 and 1. For $n \geq 1$, $K^{2n+1}(L)$ a Stone algebra implies $K(L)$, $K^3(L)$, ..., $K^{2n+1}(L)$ are Stone algebras and $K^2(L)$,...$K^{2n}(L)$ are dual Stone algebras. If $n \geq 1$, $K^{2n}(L)$ is a Stone algebra if and only if $K^r(L)$ is a finite Boolean algebra $1 \leq r \leq 2n$.*

COROLLARY 3.29. Let L be a pseudocomplemented distributive lattice. For $n \geq 0$, $K^{2n+1}(L)$ a Stone algebra implies $K(L)$, $K^3(L),\ldots,K^{2n+1}(L)$ are Stone algebras and L, $K^2(L),\ldots,K^{2n}(L)$ are dual Stone algebras. If $n \geq 1$, $K^{2n}(L)$ a Stone algebra implies L, $K^2(L),\ldots,K^{2n}(L)$ are Stone algebras and $K(L),\ldots,K^{2n-1}(L)$ are dual Stone algebras.

§7 Dual spaces of dually pseudocomplemented distributive lattices

In the following section we discuss congruence relations of double p-algebras. In order to do this we need to know more about the dual spaces of dually pseudocomplemented distributive lattices. These results could be obtained using the principle of duality and some of Priestley's results in (36) but we give constructive proofs to gain a clear understanding of the spaces and to establish notation.

LEMMA 3.30. A distributive lattice L with 0 and 1 is dually pseudocomplemented if and only if for each clopen increasing subset a' in the dual space X of L, d(a') is open in X. For $a \in L$, $a'^{+} = d(X-a')$.

Proof. Let $b \in L$ and b' be the corresponding clopen decreasing subset of X and suppose d(X–b') is open. By Lemma 1.1 d(X–b') is clopen. Obviously $d(X-b') \cup b' = X$. Now if $c' \cup b' = X$ for $c' \subseteq X$ and clopen decreasing, then $c' \supseteq X-b'$. Also c' decreasing implies $c' \supseteq d(X-b')$. Hence d(X–b') corresponds to b^{+}.

Conversely, suppose that each b in L has a dual pseudocomplement, b^+. If a' is a clopen increasing subset of the dual space X, a' = X–b' for some b' clopen decreasing. To show d(a') is open, we show d(a') corresponds to b^+. Certainly $b'^+ \supseteq d(a')$. Now if $x \in b'^+ - d(a')$, d(a') closed decreasing implies there exists a clopen increasing subset c' containing x but with $d(a') \subseteq X-c'$. This implies $(X-c') \cup b' = X$. X–c' is clopen decreasing, hence $X-c' \supseteq b'^+$. But $x \in b'^+$ and $x \notin X-c'$ giving a contradiction. The proof is completed by noting that we have shown for $a \in L$, $a'^+ = d(X-a')$.

As in the proof of Theorem 3.17, we let X_M be the set of maximal points of a partially ordered space X.

LEMMA 3.31. If $(X, \mathcal{T}, \leqslant)$ is the dual space of a dually pseudocomplemented distributive lattice, then X_M is $\mathcal{T}$-closed in X.

Proof. Let $x \in X-X_M$. Take $y \in X_M$, $y > x$. Then $y \not\leq x$ so there exists a clopen increasing set P with $y \in P$, $x \notin P$. By Lemma 3.30 d(P)–P is a $\mathcal{T}$-open neighbourhood of x disjoint from X_M.

COROLLARY 3.32. For E a $\mathcal{T}$-closed subset of X, $M(E) = i(E) \cap X_M$ is $\mathcal{T}$-closed in X.

LEMMA 3.33. If L is a dually pseudocomplemented distributive lattice with dual space X, then the dual of $\overline{D}(L)$ is $X-X_M$.

Proof. By definition and Lemma 3.30, $a \in \overline{D}(L)$ if and only if $a' \subseteq X-X_M$. Let $Z = \overline{D}(L) \cup \{1\}$. Define an equivalence relation R on X by letting x R y if and only if x = y or x,y both belong to X_M. X|R is a compact totally order disconnected space under the quotient topology and quotient order (cf.(34) Proposition 14(ii),p.519). The lattice of clopen decreasing subsets of X|R is isomorphic to Z. X|R is homeomorphic and order isomorphic to the one-point order compactification of $X-X_M$. By the representation theorem ((34) Proposition 25,p.527) for a distributive lattice without 1 the dual space of $\overline{D}(L)$ is $X-X_M$.

LEMMA 3.34. If L is a dually pseudocomplemented distributive lattice with dual space X, then the dual of $\overline{S}(L)$ is $(X_M, \mathcal{U}|X_M)$.

Proof. Let $a \in L$. Then $a'^{+} = d(X-a')$. Let Ψ be a map defined by $\Psi(d(X-a')) = (X-a') \cap X_M$.

For a', b' clopen decreasing subsets of X, $(X-a') \cap X_M = (X-b') \cap X_M$ implies $d(X-a') \cap X_M = d(X-b') \cap X_M$. If $d(X-a') \neq d(X-b')$ then suppose $y \in d(X-a') - d(X-b')$. Then $y \leqslant x$ for some $x \in X-a'$. Now X–a' increasing implies $x \leqslant z \in (X-a') \cap X_M$ for some z. But $z \in (X-b') \cap X_M$ implies $y \in d(X-b')$ giving a contradiction. So Ψ is one-to-one.

Similarly one can show Ψ is well-defined. A trivial computation shows Ψ is a join-homomorphism. One can show Ψ is a meet-homomorphism using the fact that $a^+ \underset{\overline{S}(L)}{\wedge} b^+ = (a \wedge b)^{++}$.

To show Ψ is onto, let c be clopen in $(X_M, \mathcal{U}|X_M)$. Then $c = U \cap X_M$ for some U, $\mathcal{U}$-clopen. Let $a' = X-U$. a' is clopen decreasing and $\Psi(a'^+) = (X-a') \cap X_M = c$.

§8 Congruence relations of distributive double p-algebras

We define a distributive double p-algebra to be a distributive lattice which is both pseudocomplemented and dually pseudocomplemented. A double Stone algebra is a Stone algebra and a dual Stone algebra. In any double Stone algebra the sets $S(L)$ and $\overline{S}(L)$ coincide. If L is a distributive double p-algebra, $K(L)$ is the set of lattice congruences on L which preserve pseudocomplementation and dual pseudocomplementation. We describe these congruences in terms of subsets of the dual space and consider conditions on L in order that $K(L)$ is a Stone algebra.

Let $\theta \in K(L)$ then as in §3.4, since θ is a distributive lattice congruence, θ corresponds to some closed subset θ' of the dual space X. In addition, by Lemma 3.14, as θ preserves pseudocomplementation θ' must contain $m(\theta')$. Dually, if $M(\theta') = i(\theta') \cap X_M$, in order that θ preserve dual pseudocomplementation, θ' must contain $M(\theta')$.

If E is a subset of a partially ordered space X, we call E an e-subset if E is $\mathcal{T}$-closed and E contains m(E) and M(E).

LEMMA 3.35. Let L be a distributive double p-algebra. K(L) is a complete pseudocomplemented distributive lattice.

Proof. Since the correspondence $\theta \to \theta'$ is a dual isomorphism, we need only show that the e-subsets of the dual space X are dually pseudocomplemented. Let U be an e-subset of X and consider $\{U_\alpha : U_\alpha$ is an e-subset and $U_\alpha \cup U = X\}$. Obviously $X \in \{U_\alpha\}$. Now $\bigcap_\alpha U_\alpha$ is a closed subset of X. Also $\bigcap_\alpha U_\alpha \cup U = \bigcap_\alpha (U_\alpha \cup U) = \bigcap_\alpha X = X$. If $y \in \bigcap_\alpha U_\alpha$, then $y \in U_\alpha$ for all α so $m(y) \subseteq U_\alpha$ for all α and $m(y) \subseteq \bigcap_\alpha U_\alpha$. Similarly $M(y) \subseteq \bigcap_\alpha U_\alpha$. Thus $\bigcap_\alpha U_\alpha$ is the dual pseudocomplement of U.

In the case where L is a double Stone algebra and U is an e-subset of the dual space X of L, we can obtain an explicit expression for the dual pseudocomplement of U:

LEMMA 3.36. Let L be a double Stone algebra. If U is an e-subset of the dual space X of L, the dual pseudocomplement of U is

$$U^+ = \mathcal{T}\text{-cl}(X-U) \cup m(\mathcal{T}\text{-cl}(X-U)) \cup M(\mathcal{T}\text{-cl}(X-U)).$$

Proof. U^+ is $\mathcal{T}$-closed by Corollary 3.32 and its dual. L is a double Stone algebra, so for each $x \in X$, $m(x)$ and $M(x)$ are singletons. If $z \in U^+$ we show $m(z) \in U^+$. Certainly if $z \in \mathcal{T}\text{-cl}(X-U)$ then

$$m(z) \in m(\mathcal{T}\text{-cl}(X-U)) \subseteq U^+ .$$

If $z \in m(\mathcal{T}\text{-cl}(X-U))$, then $m(z) = z$ so

$$m(z) \in m(\mathcal{T}\text{-cl}(X-U)) \subseteq U^+.$$

If $z \in M(\mathcal{T}\text{-cl}(X-U))$, then $z = M(y)$ for some $y \in \mathcal{T}\text{-cl}(X-U)$. Now

$$m(y) \in m(\mathcal{T}\text{-cl}(X-U)) \subseteq U^+.$$

$z \geqslant y$ implies $m(z) = m(y)$ as L is a Stone algebra, so $m(z) \in U^+$. Dually we can show that for each $z \in U^+$, $M(z) \in U^+$. Thus U^+ is an e-subset of X. If V is an e-subset of X such that $V \cup U = X$ it is easy to show $V \supseteq U^+$, so U^+ is the dual pseudocomplement of U.

For a general distributive double p-algebra it is very difficult to find an explicit expression for the dual pseudocomplement U^+ of an e-subset U of the dual space. Since although U^+ must contain $\mathcal{T}\text{-cl}(X-U)$, $m(\mathcal{T}\text{-cl}(X-U))$ would not in general contain $m(M(\mathcal{T}\text{-cl}(X-U)))$ and dually.

We now give an example of a distributive double p-algebra congruence. A filter F of a distributive double p-algebra L is said to be <u>normal</u> if $f \in F$ implies $f^{+*} \in F$. We note that for $a \in L$, $a^{+*} \leqslant a$. The lattice of normal filters will be denoted by $F^n(L)$. Beazer ((5) Theorem 2) has described a congruence θ_F on a distributive double p-algebra which is generated by a normal filter as the relation defined by:

$a \equiv b\ (\theta_F)$ if and only if $avf^+ = bvf^+$, for some $f \in F$.

That a normal filter F does generate a congruence follows from:

LEMMA 3.37. Let L be a distributive double p-algebra with dual space X. F is a normal filter if and only if F', the subset of X corresponding to F, is a closed atonic set.

Proof. Suppose F is a normal filter. As F is a filter, F' is a closed decreasing subset of X, in fact $F' = \bigcap_{a\in F} a'$. We show F' is increasing. Let $y \in F'$ and $z > y$. Then $z \notin F'$ implies $z \notin a'$, some $a \in F$. Thus $z \in (X-a')$ and $y \in d(X-a')$. Hence $y \in i(d(X-a'))$ which implies

$$y \notin X-i(\, d(X-a')) = a'^{+*}$$

by Lemmas 2.4 and 3.30. But if $y \in F'$, $y \in a'$, for all $a \in F$, so $y \in a'^{+*}$ for all $a \in F$ since $a \in F$ implies $a^{+*} \in F$. This gives a contradiction.

Conversely, let F' be a closed atonic set. Then F' corresponds to a filter F, since F' is closed decreasing, where $F' = \bigcap_{a\in F} a'$. We must show that for $a \in F$, $a^{+*} \in F$. This is equivalent to showing that if $a' \supseteq F'$, $(a^{+*})' \supseteq F'$. Now $F' \subseteq a'$ implies $X-F' \supseteq X-a'$. Also $X-F'$ is atonic, so $X-F' \supseteq i(d(X-a'))$ and $F' \subseteq X-i(d(X-a')) = a'^{+*}$.

Let L be a distributive double p-algebra. A further example of a congruence on L was described by Varlet ((45)p.220) by:

$$x \equiv y\ (\Phi) \text{ if and only } x^* = y^* \text{ and } x^+ = y^+.$$

This congruence plays a fundamental role in the study of double p-algebras, see (5). It is easy to see that if X is the dual space of L, the e-subset of X corresponding to Φ is $X_M \cup X_m$. Let ω be the zero congruence, then it is easily seen that $\Phi = \omega$ if and only if $X = X_M \cup X_m$.

Let L be a distributive lattice with 0 and 1. The set of complemented elements of L will be denoted by C(L).

$$C(L) = \{a \in L : a^{+*} = a\}.$$

It is easy to see that $a \in C(L)$ if and only if a', the subset of the dual space X of L corresponding to a, is clopen atonic.

If L is a distributive double p-algebra, given any $a \in L$, we can define $a^{n(+*)}$ inductively as follows:

$$a^{1(+*)} = a^{+*},\ a^{(k+1)+*} = a^{k(+*)+*} \text{ for every } k \geq 1.$$

The chain $a \geq a^{1(+*)} \geq a^{2(+*)} \geq \ldots \geq a^{n(+*)} \geq \ldots$ is called the normal chain determined by a. Beazer ((5) Theorem 4) has shown:

THEOREM 3.38. If L is a distributive double p-algebra then K(L) is a Boolean algebra if and only if the following conditions hold:

i) $\Phi = \omega$,

ii) every normal chain in L is finite,

iii) C(L) is finite.

We now classify those L for which K(L) is a Stone algebra.

THEOREM 3.39. Let L be a distributive double p-algebra. K(L) is a Stone algebra if and only if C(L) is dually isomorphic with S(K(L)).

Proof. Suppose K(L) is a Stone algebra. We show that if U is an e-subset of the dual space X of L, U^{+} is clopen atonic. Now K(L) is a Stone algebra if and only if E(X), the lattice of e-subsets of X, is a dual Stone algebra. Thus $U^{+} \cap U^{++} = \emptyset$.

Now we always have $U^+ \cup U^{++} = X$, so $U^+ = X-U^{++}$. Hence U^+ is clopen. We show U^+ is decreasing. Let $x \in U^+$, $y < x$, and suppose $y \notin U^+$. Then $y \in U^{++}$. U^{++} is an e-subset so $m(y) \subseteq U^{++}$. But $y < x$ implies $m(y) \subseteq m(x)$. U^+ is an e-subset so $m(x) \subseteq U^+$. Hence $m(y) \subseteq U^+$. This contradicts $U^+ \cap U^{++} = \emptyset$. Dually we can show that U^+ is increasing.

Thus $\overline{S}(E(X)) \subseteq \{$clopen atonic subsets of $X\}$. But if U is clopen atonic in X, then obviously $U^+ = X-U$. So $U = U^{++}$ which implies $U \in \overline{S}(E(X))$. Hence $\overline{S}(E(X)) = \{$clopen atonic subsets of $X\}$. Since the clopen atonic subsets of X correspond to C(L) we have S(K(L)) dually isomorphic with C(L).

Conversely, suppose S(K(L)) is dually isomorphic with C(L). For U an e-subset in X, U^+ is clopen atonic, so $U^{++} = X-U^+$. Hence $U^+ \cap U^{++} = \emptyset$ which implies E(X) is a dual Stone algebra and K(L) is a Stone algebra.

THEOREM 3.40. Let L be a distributive double p-algebra. If K(L) is a Stone algebra, $F^n(L)$ is a sublattice of K(L) and $S(F^n(L)) = S(K(L))$.

Proof. Since $F^n(L)$ is dually isomorphic with the lattice of closed atonic subsets of X, it is easy to see that $F^n(L)$ is a $(\wedge, \vee)$ sublattice of K(L). If K(L) is a Stone algebra, then by Theorem 3.39, U a closed atonic subset of X implies U^+ is clopen and atonic. So $F^n(L)$ is a pseudocomplemented sublattice of K(L). In particular $S(F^n(L)) \subseteq S(K(L))$.

We show the reverse inclusion holds. Let $V \in S(K(L))$, and V' be the corresponding e-subset of X. Then $V' \in \overline{S}(E(X))$, so V' is clopen atonic. Hence V' does correspond to a normal filter. Since $V' = V'^{++}$, $V \in S(F^n(L))$. This completes the proof.

If L is a double Stone algebra, $S(L) = \overline{S}(L) = C(L)$. By Theorem 3.39 we have:

COROLLARY 3.41. Let L be a double Stone algebra. K(L) is a Stone algebra if and only if S(L), or $\overline{S}(L)$, is dually isomorphic with S(K(L)).

THEOREM 3.42. Let L be a double Stone algebra. If K(L) is a Stone algebra, $D(L) \cap \overline{D}(L)$ is relatively complemented.

Proof. Suppose K(L) is a Stone algebra. As the dual of $D(L) \cap \overline{D}(L)$ is $X-(X_M \cup X_m)$ we must show $X-(X_M \cup X_m)$ has the discrete order in order for $D(L) \cap \overline{D}(L)$ to be relatively complemented. Let $y_1, y_2 \in X-(X_M \cup X_m)$ with $y_1 < y_2$, say. By total order disconnectedness of X, there exists a clopen decreasing subset U such that $y_1 \in U$ and $y_2 \in X-U$. Now $U \cup M(U)$ is an e-subset, since U is closed and decreasing. The e-subsets form a dual Stone algebra so $(U \cup M(U))^+$ is clopen atonic. Now

$$y_2 \in X-(U \cup M(U)) \subseteq (U \cup M(U))^+.$$

Also $(U \cup M(U))^+$ atonic and $y_1 < y_2$ implies $y_1 \in (U \cup M(U))^+$. But

$$(U \cup M(U))^+ =$$

$$= \mathcal{T}\text{-cl}(X-(U \cup M(U))) \cup m(\mathcal{T}\text{-cl}(X-(U \cup M(U)))) \cup M(\mathcal{T}\text{-cl}(X-(U \cup M(U))))$$

and $y_1 \in X-(X_M \cup X_m)$ implies $y_1 \in \mathcal{T}\text{-cl}(X-(U \cup M(U)))$. But $\mathcal{T}\text{-cl}(X-(U \cup M(U))) \subseteq \mathcal{T}\text{-cl}(X-U) = X-U$. This contradicts the choice of $y_1 \in U$.

It is easy to show that L is a finite distributive lattice if and only if the dual space $(X, \mathcal{T}, \leq)$ has discrete topology $\mathcal{T}$. We use this to show:

THEOREM 3.43. Let L be a finite double Stone algebra. K(L) is a Stone algebra if and only if the following conditions hold:

i) $D(L) \cap \overline{D}(L)$ is a Boolean algebra;

ii) if $z \in X_M$ (X_m) and $y_1, y_2 \in X-(X_M \cup X_m)$ then $z > y_1$ and $z > y_2$ $(z < y_1,\ z < y_2)$ implies $y_1 = y_2$.

Proof. Suppose K(L) is a Stone algebra. Condition i) is true by Theorem 3.42. Suppose condition ii) is not true and that if X is the dual space of L for some $z \in X_M$ and $y_1, y_2 \in X-(X_M \cup X_m)$, $y_1 \neq y_2$ but $y_1 < z$ and $y_2 < z$. Now

$$U = \{y_1\} \cup m(y_1) \cup M(y_1)$$

is a closed e-subset of X and

$$U^+ = (X-U) \cup m(X-U) \cup M(X-U)$$

since every subset of X is closed. $y_2 \notin X_M \cup X_m$ so $y_2 \notin U$ which implies $y_2 \in U^+$. This is atonic. Hence $y_1 \in U^+$. Now $y_1 \notin X_M \cup X_m$ so $y_1 \in X-U$. This contradicts the definition of U. Dually we can show that for $z \in X_m$, $y_1, y_2 \in X-(X_M \cup X_m)$ and $z < y_1$, $z < y_2$ implies $y_1 = y_2$.

On the other hand if U is an e-subset of X we show

$$U^+ = (X-U) \cup m(X-U) \cup M(X-U)$$

is atonic which implies E(X) is a dual Stone algebra.

Suppose $y \in U^+$ and $z < y$.

Case a) If $y \in M(X-U)$, and $z \in X_m$ then $z \in U^+$ since U^+ is an e-subset. Suppose $z \notin X_m$, and $y = M(v)$, $v \in X-U$. Suppose $v \in X-(X_M \cup X_m)$. Then $y = M(m(v))$ which implies $m(z) = m(y) = m(v)$. If $z \in U$, $z > m(z)$ and $v > m(z)$ so, by condition ii), $z = v$. If $v \in X_M$, then $y \in (X-U) \cap X_M$. If $z \in U$, then $m(y) = m(z) \in U$ and so $M(m(y)) = y \in U$ giving a contradiction. If $v \in X_m$, then $z \geq v$, $v \in (X-U) \cap X_m$. If $z \in U$ then $m(z) = v \in U$ giving a contradiction.

Case b) If $y \in m(x-U)$, obviously $z < y$ implies $z = y \in m(X-U) \subseteq U^+$.

Case c) Let $y \in X-U$. If $y \in (X-U) \cap X_M$ and $z < y$, then part of the proof of case a) shows $z \in U^+$. If $y \in (X-U) \cap X_m$ and $z < y$, then case b) shows $z \in U^+$. If $y \in (X-U) \cap (X-(X_M \cup X_m))$, then $z < y$ implies $z \in m(X-U)$ which implies $z \in U^+$.

Thus U^+ is decreasing. Dually we can show U^+ is increasing. This completes the proof.

We note that in order for K(L) to be a Stone algebra in Theorem 3.39, the conditions on L involve K(L). This is not so if L is a finite double Stone algebra as shown by Theorem 3.43. If we use the following theorem, we can remove K(L) from the conditions for certain infinite L.

THEOREM 3.44. If L is a distributive double p-algebra the following statements hold:

i) $\Phi = \omega$ if and only if K(L) is isomorphic with $F^n(L)$ ((5) Corollary 1),

ii) every normal chain in L is finite if and only if $F^n(L)$ is isomorphic with F(C(L)) ((5) Theorem 5).

COROLLARY 3.45. Let L be a double p-algebra and suppose

i) $\Phi = \omega$,

ii) every normal chain in L is finite.

Then K(L) is a Stone algebra if and only if C(L) is complete.

Proof. Suppose K(L) is a Stone algebra. Then Theorem 3.39, implies S(K(L)) is dually isomorphic with C(L). Now K(L) is complete so S(K(L)) is complete. (cf. remark following Lemma 2.2). Hence C(L) is complete since infinite joins and meets are preserved by Boolean isomorphisms ((37) p.65).

Conversely i) and ii), by Theorem 3.44, imply $K(L) \cong F(C(L))$. By Theorem 3.3, F(C(L)) is a Stone algebra if C(L) is complete, so K(L) is a Stone algebra.

COROLLARY 3.46. Let L be a double Stone algebra and suppose $\Phi = \omega$. Then K(L) is a Stone algebra if and only if S(L) is complete.

Proof. As L is a Stone algebra, S(L) coincides with C(L). Thus for any $a \in L$, $a^{+*} = a^{n(+*)}$ so every normal chain in L is finite. Corollary 3.45 completes the proof.

Some of the results of Chapter 3 are summarized in the tables. The conditions given are necessary and sufficient. The deleted squares in the second table are of no interest as differing algebraic operations are involved.

F I L T E R S

	F(L) is a Boolean algebra	F(L) is a Stone algebra
L is a distributive lattice with 1	L is a relatively complemented lattice satisfying the ascending chain condition	For all $x \in L$ $[x,1]$ is a dual Stone algebra and ν_L is conditionally complete
L is a distributive lattice with 0 and 1	L is a finite Boolean algebra	L is a dual Stone algebra and $\overline{S}(L)$ is complete
L is a dually pseudocomplemented distributive lattice	L is a finite Boolean algebra	L is a dual Stone algebra and $\overline{S}(L)$ is complete
L is a dual Stone algebra	L is a finite Boolean algebra	$\overline{S}(L)$ is complete
L is a Boolean algebra	L is finite	L is complete

CONGRUENCES

	Lattice congruences form a Boolean algebra	Lattice congruences form a Stone algebra	Lattice congruences preserving * form a Boolean algebra	Lattice congruences preserving * form a Stone algebra	Lattice congruences preserving $*,^+$ form a Boolean algebra	Lattice congruences preserving $*,^+$ form a Stone algebra
L is a distributive lattice with 0	Every closed interval of L has finite length	For $x \in L_B$, $[0, x]$ is a complete Boolean algebra				
L is a distributive lattice with 0 and 1	L is finite	L_B is a complete Boolean algebra				
L is a pseudocomplemented distributive lattice	L is finite	L_B is a complete Boolean algebra	L is a finite Boolean algebra	L is a dual Stone algebra; D(L) is relatively complemented; S(L) is complete		
L is a Stone algebra	L is finite	L_B is a complete Boolean algebra	L is a finite Boolean algebra	L is double Stone algebra; D(L) is relatively complemented; S(L) is complete		
L is a distributive double p-algebra	L is finite	L_B is a complete Boolean algebra	L is a finite Boolean algebra	L is a dual Stone algebra; D(L) is relatively complemented; S(L) is complete	$\Phi = \omega$; every normal chain in L is finite; C(L) is finite	C(L) is dually isomorphic with S(K(L))
L is a double Stone algebra	L is finite	L_B is a complete Boolean algebra	L is a finite Boolean algebra	D(L) is relatively complemented; S(L) is complete	$\Phi = \omega$; S(L) is finite	S(L) is dually isomorphic with S(K(L))
L is a Boolean algebra	L is finite	L is complete	L is finite	L is complete	L is finite	L is complete

Chapter 4

HAHN-BANACH THEOREMS FOR PSEUDOCOMPLEMENTED DISTRIBUTIVE LATTICES

§1 Introduction

Monteiro has proved a theorem ((31) Theorem 3) which extends a homomorphism with range in a complete Boolean algebra, defined on a subalgebra of a Boolean algebra and dominated by a join-homomorphism, to a homomorphism defined on the whole Boolean algebra. This result is the analogue of the Hahn-Banach Theorem ((15) p.105) for Boolean algebras. Cignoli (12) generalized this result to distributive lattices. These results are summarized in §4.2.

In §4.3 we prove similar extension theorems for pseudo-complemented distributive lattices where the homorphisms are required to preserve the *-operation. In all the classes of lattices mentioned, the injective objects are the complete Boolean algebras. However in the class of Stone algebras the injective objects are certain complete Stone algebras which are not necessarily Boolean algebras. In §4.4 we obtain extension theorems for morphisms on Stone algebras with range in an injective object of this class.

§2 Hahn-Banach theorems for Boolean algebras and distributive lattices

In this section we summarize some known results for use in §4.3 and §4.4. We describe a Hahn-Banach theorem for distributive lattices given by Cignoli (12). When the distributive lattice is a Boolean algebra this gives a theorem of Bacsich (2) which generalizes Monteiro's theorem mentioned in §4.1.

Let L_1, L_2 be distributive lattices with 0,1. A homomorphism $h: L_1 \to L_2$ is a map satisfying the conditions:

i) $h(0) = 0$,

ii) $h(1) = 1$,

iii) $h(a \wedge b) = h(a) \wedge h(b)$,

iv) $h(a \vee b) = h(a) \vee h(b)$.

The set of all homomorphisms from L_1 into L_2 will be denoted by $H(L_1,L_2)$. A map $m : L_1 \to L_2$ satisfying conditions i),ii) and iii) is called a meet-homomorphism and a map $j : L_1 \to L_2$ satisfying conditons i), ii) and iv) is called a join-homomorphism. $M(L_1,L_2)$ ($J(L_1,L_2)$) will denote the set of all meet-(join-) homomorphisms from L_1 into L_2.

$h \in H(L_1,L_2)$ is said to be between $m \in M(L_1,L_2)$ and $j \in J(L_1,L_2)$ if $m(a) \leqslant h(a) \leqslant j(a)$ for any a in L_1.

LEMMA 4.1. ((12) p.336). Let L be a distributive lattice with 0, 1 and B a Boolean algebra. Let $m \in M(L,B)$, $j \in J(L,B)$. In order that a map $h : L \to B$ be a homomorphism which lies between m and j it is necessary and sufficient that the following conditions hold:

i) $h(0) = 0$ and $h(1) = 1$,

ii) if $a_1,\ldots,a_n$ and $b_1,\ldots,b_m,c,d$ are elements in L, the relation

$$a_1 \wedge \ldots \wedge a_n \wedge c \leq b_1 \vee \ldots \vee b_m \vee d$$

implies that

$$h(a_1) \wedge \ldots \wedge h(a_n) \wedge m(c) \leq h(b_1) \vee \ldots \vee h(b_m) \vee j(d).$$

The above result suggests the following definition. Let A be a subset of L containing the elements 0, 1 and let $m \in M(L,B)$ and $j \in J(L,B)$. A map $f : A \to B$ is said to be between m and j in A if $f(0) = 0$, $f(1) = 1$ and if for any elements $a_1,\ldots,a_n,b_1,\ldots,b_m$ in A and any elements c, d in L, the relation

$$a_1 \wedge \ldots \wedge a_n \wedge c \leq b_1 \vee \ldots \vee b_m \vee d$$

implies that

$$f(a_1) \wedge \ldots \wedge f(a_n) \wedge m(c) \leq f(b_1) \vee \ldots \vee f(b_m) \vee j(d).$$

THEOREM 4.2 ((12) Theorem, p.337). Let L be a distributive lattice with 0, 1 and B a complete Boolean algebra. Let $m \in M(L,B)$, $j \in J(L,B)$, A a subset of L containing the elements 0 and 1 and f a map from A into B. Then there exists an $h \in H(L,B)$ which lies between m and j and such that $h(a) = f(a)$ for any a in A if and only if f lies between m and j in A.

If L is a Boolean algebra and h a homomorphism from L into B then, by De Morgan's Laws ((21) p.5) h also preserves complements and we have the following theorem which is slightly more general than that of Bacsich ((2) Theorem 3.2).

THEOREM 4.3. Let L be a Boolean algebra and B a complete Boolean algebra. Let $m \in M(L,B)$, $j \in J(L,B)$, A a subset of L containing 0, 1 and f a map from A into B. Then there exists a Boolean homomorphism from L into B which lies between m and j and such that $h(a) = f(a)$ for any a in A if and only if f lies between m and j in A.

§3 Hahn-Banach theorems for pseudocomplemented distributive lattices

In this section we prove extension theorems for pseudocomplemented distributive lattices where homomorphisms are required to preserve the *-operation.

Let L_1 , L_2 be pseudocomplemented distributive lattices. A *-homomorphism from L_1 into L_2 is a map $h : L_1 \to L_2$, satisfying the conditions:

$$\text{i)} \quad h(0) = 0,$$

$$\text{ii)} \quad h(1) = 1,$$

$$\text{iii)} \quad h(a \wedge b) = h(a) \wedge h(b),$$

$$\text{iv)} \quad h(a \vee b) = h(a) \vee h(b),$$

$$\text{v)} \quad h(a^*) = (h(a))^*.$$

The set of all *-homomorphisms from L_1 into L_2 will be denoted by $H^*(L_1, L_2)$. A map $m : L_1 \to L_2$ satisfying conditions i), ii), iii) and v) is called a *-meet-homomorphism and a map $j : L_1 \to L_2$ satisfying conditions i), ii), iv) and v) is called a *-join-homomorphism. $M^*(L_1,L_2)$ ($J^*(L_1,L_2)$) will denote the set of all *-meet-(join-)homomorphisms from L_1 into L_2.

If we restrict Cignoli's Theorem to pseudocomplemented distributive lattices we do not get an extension theorem for *-morphisms as we did when we restricted the theorem to Boolean algebras. However we show we can obtain extension theorems by placing restrictions on the semi-homomorphisms of §4.2 .

LEMMA 4.4. Let L be a pseudocomplemented distributive lattice and B a complete Boolean algebra. Let $m \in M^*(L,B)$, $j \in J(L,B)$. Then $h : L \to B$ is a *-homomorphism lying between m and j if and only if:

i) $h(0) = 0$ and $h(1) = 1$,

ii) if $a_1,\ldots,a_n$, $b_1,\ldots,b_m$, c, d are elements in L, the relation

$$a_1 \wedge \ldots \wedge a_n \wedge c \leqslant b_1 \vee \ldots \vee b_m \vee d$$

implies that

$$h(a_1) \wedge \ldots \wedge h(a_n) \wedge m(c) \leqslant h(b_1) \vee \ldots \vee h(b_m) \vee j(d).$$

Proof. It is plain that a *-homomorphism that lies between m and j must satisfy i) and ii).

To prove the sufficiency observe first that since $m(1) = 1$ and $j(0) = 0$, the conditions:

$$a \wedge b \wedge 1 \leqslant (a \wedge b) \vee 0,$$

$$(a \wedge b) \wedge 1 \leqslant a \vee 0,$$

$$(a \wedge b) \wedge 1 \leqslant b \vee 0,$$

imply that $h(a \wedge b) = h(a) \wedge h(b)$. Dually it follows that $h(a \vee b) = h(a) \vee h(b)$. Moreover it follows from $a \wedge 1 \leqslant 0 \vee a$ that:

$$h(a) = h(a) \wedge m(1) \leqslant h(0) \vee j(a) = j(a).$$

Similarly $m(a) \leqslant h(a)$. Now for any a in L, $a^* \wedge a = 0$ implies:

$$h(a^*) \wedge h(a) = h(a^* \wedge a) = h(0) = 0.$$

Thus $h(a^*) \leqslant (h(a))^*$. But $m(a) \leqslant h(a)$ implies

$$(h(a))^* \leqslant (m(a))^* = m(a^*) \leqslant h(a^*),$$

since m preserves *. Hence $h \in H^*(L,B)$.

The above result suggests the following definition . Let L be a pseudocomplemented distributive lattice and B a complete Boolean algebra. Let A be a subset of L containing 0, 1 and $m \in M^*(L,B)$, $j \in J(L,B)$. A map $f : A \to B$ is said to be between m and j in A if $f(0) = 0$, $f(1) = 1$ and if for any elements $a_1,\ldots,a_n,b_1,\ldots,b_m$ in A and any elements c, d in L, the relation

$$a_1 \wedge \ldots \wedge a_n \wedge c \leq b_1 \vee \ldots \vee b_m \vee d$$

implies

$$f(a_1) \wedge \ldots \wedge f(a_n) \wedge m(c) \leq f(b_1) \vee \ldots \vee f(b_m) \vee j(d).$$

THEOREM 4.5. Let L be a pseudocomplemented distributive lattice and B a complete Boolean algebra. Let $m \in M^*(L,B)$ and $j \in J(L,B)$. Let A be a subset of L containing the elements 0, 1 and f a map from A into B. Then there exists $h \in H^*(L,B)$ which lies between m and j and $h(a) = f(a)$ for all a in A if and only if f lies between m and j in A.

Proof. The necessity of the condition follows at once by Lemma 4.4.

To prove the sufficiency, let f lie between m and j in A. By Theorem 4.2 there exists $h \in H(L,B)$ which lies between m and j and such that $h(a) = f(a)$ in A. Lemma 4.1 implies h satisfies conditions i) and ii) of Lemma 4.4. Since $m \in M^*(L,B)$ and $j \in J(L,B)$, Lemma 4.4 gives $h \in H^*(L,B)$.

LEMMA 4.6. ((18) Ex.22,p.175) Let L_1, L_2 be pseudocomplemented distributive lattices and $h \in H(L_1, L_2)$. Then $h \in H^*(L_1,L_2)$ if and only if $h : D(L_1) \to D(L_2)$ and $h : S(L_1) \to S(L_2)$.

Proof. Let $h \in H(L_1,L_2)$, $h : D(L_1) \to D(L_2)$, and $h : S(L_1) \to S(L_2)$. For any $a \in L_1$, let $h(a) = b \in L_2$. We must show $h(a^*) = b^*$. Certainly $h(a^*) = c^*$ for some $c^* \in S(L_2)$. Now $h(a^*) \wedge h(a) = h(a^* \wedge a) = h(0) = 0$. Hence $h(a^*) \leqslant (h(a))^*$ and so $c^* \leqslant b^*$. But $h(a^*) \vee h(a) = h(a^* \vee a) \in D(L_2)$. Thus $(c^* \vee b)^* = 0$ which implies $c^{**} \wedge b^* = 0$. Thus $b^* \leqslant c^{***} = c^*$.

Conversely, if $h \in H^*(L_1,L_2)$ it is easy to show that $h : S(L_1) \to S(L_2)$ and $h : D(L_1) \to D(L_2)$.

LEMMA 4.7. Let L be a pseudocomplemented distributive lattice and B a complete Boolean algebra. Let $m \in M(L,B)$, $m : D(L) \to D(B) = 1$, $j \in J(L,B)$. Then $h : L \to B$ is a *-homomorphism lying between m and j if and only if:

i) $h(0) = 0$ and $h(1) = 1$,

ii) if $a_1,\ldots,a_n,b_1,\ldots,b_m,c,d$ are elements in L, the relation

$$a_1 \wedge \ldots \wedge a_n \wedge c \leqslant b_1 \vee \ldots \vee b_m \vee d$$

implies that

$$h(a_1) \wedge \ldots \wedge h(a_n) \wedge m(c) \leqslant h(b_1) \vee \ldots \vee h(b_m) \vee j(d).$$

Proof. To prove the sufficiency, as in the proof of Lemma 4.4, we can show $h \in H(L,B)$. Now $h : S(L) \to B(=S(B))$. Also, for $a \in D(L)$, $m(a) \leqslant g(a)$ implies $(g(a))^* \leqslant (m(a))^* = 0$, by hypothesis. Hence $g : D(L) \to D(B)(= 1)$. By Lemma 4.6, $h \in H^*(L,B)$.

It is plain that any homomorphism which lies between m and j must satisfy conditions i) and ii). This completes the proof.

This result suggests the following definition.

Let A be a subset of L containing 0, 1 and $m \in M(L,B)$, $m : D(L) \to 1$, $j \in J(L,B)$. A map $f: A \to B$ is between m and j in A if $f(0) = 0$, $f(1) = 1$ and if for any elements $a_1,\ldots,a_n,b_1,\ldots,b_m$ in A and c,d in L:

$$a_1 \wedge \ldots \wedge a_n \wedge c \leq b_1 \vee \ldots \vee b_m \vee d$$

implies

$$f(a_1) \wedge \ldots \wedge f(a_n) \wedge m(c) \leq f(b_1) \vee \ldots \vee f(b_m) \vee j(d).$$

THEOREM 4.8. Let L be a pseudocomplemented distributive lattice and B a complete Boolean algebra. Let $m \in M(L,B)$, $m : D(L) \to 1$, $j \in J(L,B)$. Let A be a subset of L containing the elements 0, 1 and f a map from A into B. Then there exists $h \in H^*(L,B)$ which lies between m and j and $h(a) = f(a)$ for all a in A if and only if f lies between m and j in A.

Proof. Same as that for Theorem 4.5 using Lemma 4.7 instead of Lemma 4.4.

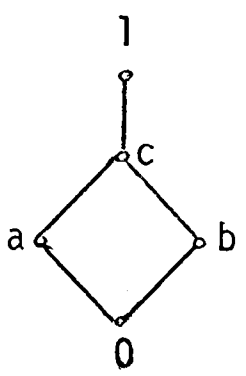
Remark. If $m = j$ in Theorem 4.8 then take $g = m = j$. We certainly have $g \in H(L,B)$ and $m \leq g$ implies $g : D(L) \to 1$. Also $g : S(L) \to B(= S(B))$ and Lemma 4.6 gives the result.

If $m \in M^*(L,B)$, then $m : D(L) \to 1$ since for $a \in D(L)$,

$$(m(a))^* = m(a^*) = m(0) = 0.$$

However, the following simple example shows that $m \in M(L,B)$, $m : D(L) \to 1$ does not imply $m \in M^*(L,B)$.

EXAMPLE. Let B be the two-element Boolean algebra and L be the lattice with diagram:



L is a pseudocomplemented distributive lattice if we define:

$$a^* = b, \quad b^* = a, \quad 1^* = c^* = 0, \quad 0^* = 1.$$

Let us define $m : L \to B$ by:

$$m(1) = m(c) = 1,$$

$$m(a) = m(b) = m\,(0) = 0.$$

. It is easy to show that m is a meet-homomorphism, and $m : \{1,c\} = D(L) \to 1$. But $m \notin M^*(L,B)$, since $m(a^*) = m(b) = 0$ whereas $(m(a))^* = 0^* = 1$.

§4 Hahn-Banach theorems for Stone algebras

We recall that the injective objects in the class of pseudocomplemented distributive lattices, the class of distributive lattices with 0, 1 and the class of Boolean algebras are the complete Boolean algebras. However in the class of Stone algebras there is a wider set of injective objects. Influenced by Bacsich's proof of Monteiro's theorem (2) in this section we prove extension theorems in the class of Stone algebras for maps with range in an injective Stone algebra.

LEMMA 4.9. *Let L_1, L_2 be pseudocomplemented distributive lattices. Suppose $m \in M(L_1, L_2)$, $m : D(L_1) \to D(L_2)$ and $j \in J(L_1, S(L_2))$. Then $H : L_1 \to L_2$ is a *-homomorphism lying between m and j if and only if*

i) $h(0) = 0$ *and* $h(1) = 1$,

ii) *if* $a_1, \ldots, a_n, b_1, \ldots, b_m, c, d$ *are elements in* L, *the relation* $a_1 \wedge \ldots \wedge a_n \wedge c \leq b_1 \vee \ldots \vee b_m \vee d$ *implies that*

$$h(a_1) \wedge \ldots \wedge h(a_n) \wedge m(c) \leq h(b_1) \vee \ldots \vee h(b_m) \vee j(d).$$

Proof. Same as that for Lemma 4.7 as we did not rely on the fact that the range was a Boolean algebra.

This result gives rise to the usual definition.

Let L_1, L_2 be pseudocomplemented distributive lattices and A a subset of L containing 0,1. Suppose $m \in M(L_1, L_2)$, $m : D(L_1) \to D(L_2)$ and $j \in J(L_1, S(L_2))$. A map $f : A \to L_2$ is between m and j in A if $f(0) = 0$, $f(1) = 1$ and for any elements $a_1,\ldots,a_n,b_1,\ldots,b_m$ in A,c,d in L_1 the relation:

$$a_1 \wedge \ldots \wedge a_n \wedge c \leq b_1 \vee \ldots \vee b_m \vee d$$

implies that

$$f(a_1) \wedge \ldots \wedge f(a_n) \wedge m(c) \leq f(b_1) \vee \ldots \vee f(b_m) \vee j(d).$$

LEMMA 4.10. ((18) Ex.21, p.175) Let L_1, L_2 be Stone algebras and suppose $h \in H(L_1,L_2)$. Then $h \in H^*(L_1,L_2)$ if and only if $h : D(L_1) \to D(L_2)$.

Proof. Suppose $h \in H(L_1,L_2)$ and $h : D(L_1) \to D(L_2)$. Let $a \in L_1$, then

$$h(a^*) \vee h(a^{**}) = h(a^* \vee a^{**}) = h(1) = 1.$$

Also

$$h(a^*) \wedge h(a^{**}) = h(a^* \wedge a^{**}) = h(0) = 0.$$

So $h(a^*)$ has a complement, $h(a^{**})$. Now as L_2 is a Stone algebra, the set of complemented elements in L_2 is the same as $S(L_2)$. Hence $h : S(L_1) \to S(L_2)$. By Lemma 4.6, $h \in H^*(L_1,L_2)$. The converse is obvious.

Let B be a Boolean algebra. We define ((19) p.478):

$$B^{<n+1>} = \{(a_0,a_1,\ldots,a_n) \in B^{n+1} : a_0 \leqslant a_1 \wedge \ldots \wedge a_n\}.$$

Then $B^{<n+1>}$ is a pseudocomplemented distributive lattice with $0 = (0,\ldots,0)$, $1 = (1,\ldots,1)$, co-ordinatewise meet and join operations and

$$(a_0,a_1,\ldots,a_n)^* = (a_1' \wedge \ldots \wedge a_n', a_1', \ldots, a_n'),$$

where a' on the right-hand side of the equation denotes the complement of a in the Boolean algebra. For any Boolean algebra B, $B^{<2>}$ is a Stone algebra.

LEMMA 4.11. *Let L be a Stone algebra and 2 be the two-element Boolean algebra. Let* $m \in M(L,2^{<2>})$, $m : D(L) \to D(2^{<2>})$, $j \in J(L,S(2^{<2>}))$ *and suppose* $m^{-1}(0) = j^{-1}(0)$. *Let A be a subset of L containing 0,1 and* $f : A \to 2^{<2>}$. *Then there exists* $h \in H^*(L,2^{<2>})$ *which lies between m and j and h = f in A if and only if f lies between m and j in A.*

Proof. The necessity follows at once from Lemma 4.9.

To prove the sufficiency, consider the projection maps from $2^{<2>}$ onto 2 defined by $p_i(a_1,a_2) = a_i$, $i = 1,2$ and $a_1 \leqslant a_2$. It is easily shown that $p_i \in H(2^{<2>},2)$ for each i. Now

$$p_2(a_1,a_2)^* = p_2(a_2',a_2') = a_2' = (p_2(a_1,a_2))'.$$

Hence $p_2 \in H^*(2^{<2>},2)$. But

$$p_1(a_1,a_2)^* = p_1(a_2',a_2') = a_2'$$

whereas

$$(p_1(a_1,a_2))' = a_1'.$$

So $p_i \notin H^*(2,^{<2>},2)$. For i = 1, 2 define:

$$f_i : A \to 2 \quad \text{by } p_i f = f_i;$$

$$m_i : L \to 2 \quad \text{by } p_i m = m_i;$$

$$j_i : L \to 2 \quad \text{by } p_i j = j_i.$$

It can be shown $m_i \in M(L,2)$ i = 1,2 and $m_2 : D(L) \to D(2)$. Also $j_i \in J(L,2)$, i = 1,2. As each p_i preserves order, f_i lies between m_i and j_i, i = 1,2.

By Theorem 4.8 there exists $h_2 \in H^*(L,2)$ which lies between m_2 and j_2 and $h_2(a) = f_2(a)$ for all a in A.

We now show that the map f_1 lies between m_1 and h_2 in the set A i.e. for $a_1,\ldots,a_n,b_1,\ldots,b_m$ in A, c,d in L:

$$a_1 \wedge \ldots \wedge a_n \wedge c \leqslant b_1 \vee \ldots \vee b_m \vee d$$

implies

i) $f_1(a_1) \wedge \ldots \wedge f_1(a_n) \wedge m_1(c) \leqslant f_1(b_1) \vee \ldots \vee f_1(b_m) \vee h_2(d)$.

Since f_1 lies between m_1 and j_1 we certainly have:

ii) $f_1(a_1) \wedge \ldots \wedge f_1(a_n) \wedge m_1(c) \leqslant f_1(b_1) \vee \ldots \vee f_1(b_m) \vee j_1(d)$.

Case a): If the left-hand side of i) is 0 the inequality always holds.

Case b): If the left-hand side of i) is 1 and $f_1(b_1) \vee \ldots \vee f_1(b_m) = 1$ again the inequality holds.

Case c): If the left-hand side of i) is 1 and $f_1(b_1) \vee \ldots \vee f_1(b_m) = 0$ then ii) implies $j_2(d) = 1$. As $j_1 = j_2$, by hypothesis, this implies $j(d) = 1$. Hence $m(d) \neq 0$ since $m^{-1}(0) = j^{-1}(0)$ by hypothesis. Thus m(d) = (0,1) or (1,1). In either case, $m_2(d) = 1$. Since $m_2 \leqslant h_2 \leqslant j_2$, $h_2(d) = 1$ and i) holds.

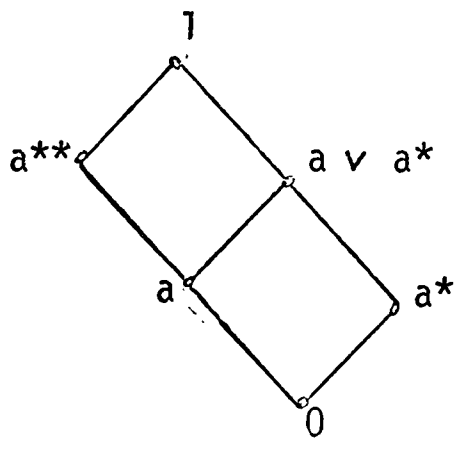
By Theorem 4.2 there exists $h_1 \in H(L,2)$ which lies between m_1 and h_2 and such that $h_1(a) = f_i(a)$ for a in A.

For $a \in L$ define $h(a) = (h_1(a),h_2(a))$. $h_1 \preccurlyeq h_2$ by construction so $(h_1(a),h_2(a)) \in 2^{<2>}$. $h_1,h_2 \in H(L,2)$ implies $h \in H(L, 2^{<2>})$. Also h lies between m and j by construction. Now $m \preccurlyeq h$, so for each $a \in D(L)$, $m(a) \preccurlyeq h(a)$ implies $(h(a))^* \preccurlyeq (m(a))^* = 0$ by hypothesis. Thus $h : D(L) \to D(2^{<2>})$ and $h \in H^*(L,2^{<2>})$ by Lemma 4.10.

We note that $2^{<2>}$ is the three-element chain but this does not simplify the above proof.

Let us consider the statement of Lemma 4.11. The condition $m^{-1}(0) = j^{-1}(0)$, implies j is a homomorphism. As $m : D(L) \to D(2^{<2>})$ and $m \preccurlyeq j$, we have $j:D(L) \to D(2^{<2>})$ which implies, by Lemma 4.10, $j \in H^*(L,2^{<2>})$. Note that this does not imply m is a join-homomorphism nor does it imply that we can take $h = j$ as j and f may differ on A and h must extend f. The following example demonstrates this.

EXAMPLE. Let L be the Stone algebra with diagram:



Let us consider $2^{<2>}$ as the three-element chain $\{0,b,1\}$.

Define $j \in J(L,S(2^{<2>})$ such that:

$$j(0) = j(a^*) = 0,$$

$$j(1) = j(a^{**}) = j(a \vee a^*) = j(a) = 1.$$

Define $m \in M(L,2^{<2>})$, $m : D(L) \to D(2^{<2>})$ by:

$$m(0) = m(a^*) = 0,$$

$$m(1) = 1,$$

$$m(a^{**}) = m(a) = m(a \vee a^*) = b.$$

Let $A = \{0, a \vee a^*, 1\}$ and define:

$$f(0) = 0,$$

$$f(a \vee a^*) = b,$$

$$f(1) = 1.$$

Then $m^{-1}(0) = j^{-1}(0)$ but $m \notin J(L,2^{<2>})$ since $m(a^{**} \vee a^*) = m(1) = 1$ whereas $m(a^{**}) \vee m(a^*) = b$.

We can easily show that f lies between m and j and if we define h such that:

$$h(0) = h(a^*) = 0,$$

$$h(a \vee a^*) = h(a) = b,$$

$$h(a^{**}) = h(1) = 1,$$

then $h \in H^*(L,2^{<2>})$, h extends f and $h \neq j$.

In fact a condition such as $m^{-1}(0) = j^{-1}(0)$ would seem necessary if we wish to have f, m and j mapping into a Stone algebra.

Let L be a Stone algebra. Suppose $m \in M(L,2^{<2>})$, $m : D(L) \to D(2^{<2>})$ and $j \in J(L,S(2^{<2>}))$. Let A be a subset of L containing 0,1 and $f : A \to 2^{<2>}$. Suppose there exists $h \in H^*(L,2^{<2>})$ such that h lies between m and j and $h(a) = f(a)$ for a in A.

Now consider the projection maps $p_i : 2^{<2>} \to 2$, $i = 1,2$, defined by $p_i(a_1,a_2) = a_i$. For $i = 1, 2$ define:

$$f_i : A \to 2 \text{ by } p_i f = f_i ,$$

$$h_i : L \to 2 \text{ by } p_i h = h_i ,$$

$$m_i : L \to 2 \text{ by } p_i m = m_i ,$$

$$j_i : L \to 2 \text{ by } p_i j = j_i .$$

It can be shown $m_i \in M(L,2)$, $j_i \in J(L,2)$ and $h_i \in H(L,2)$. h_i lies between m_i and j_i and h_i extends f_i which also lies between m_i and j_i.

As the range of h is in $2^{<2>}$, $h_1(a) \leqslant h_2(a)$ for $a \in L$. Hence $m_1 \leqslant h_1 \leqslant h_2$ and as h_1 extends f_1, by Theorem 4.2, f_1 must lie between m_1 and h_2. Thus for $a_1,\ldots,a_n,b_1,\ldots,b_m$ in A c, d in L,

$$a_1 \wedge \ldots \wedge a_n \wedge c \leqslant b_1 \vee \ldots \vee b_m \vee d$$

implies

$$f_1(a_1) \wedge \ldots \wedge f_1(a_n) \wedge m_1(c) \leqslant f_1(b_1) \vee \ldots \vee f_1(b_m) \vee h_2(d).$$

This shows that for a given m, j, and f in order for f lying between m and j to imply there exists an h extending f and lying between m and j, we must place restrictions on m,j and f such that f lying between m and j must at least imply f_1 lies between m_1 and h_2.

If we wish to exclude the condition $m^{-1}(0) = j^{-1}(0)$ from the statement of Lemma 4.11, we could take $f : A \to S(2^{<2>})$. Then take $g_1 = g_2$ but this is just Theorem 4.8. Otherwise some condition on m and j such that for $a_1,\ldots,a_n,b_1,\ldots,b_m$ in A,c d in L satisfying

$$a_1 \wedge \ldots \wedge a_n \wedge c \leqslant b_1 \vee \ldots \vee b_m \vee d$$

we have

$$f_1(a_1) \wedge \ldots \wedge f_1(a_n) \wedge m_1(c) \leqslant f_1(b_1) \vee \ldots \vee f_1(b_m) \vee j_1(d)$$

implies

$$f_1(a_1) \wedge \ldots \wedge f_1(a_n) \wedge m_1(c) \leqslant f_1(b_1) \vee \ldots \vee f_1(b_m) \vee h_2(d),$$

will be sufficient.

Using Lemma 4.11 we are able to extend maps lying between morphisms which are defined on a Stone algebra and which have range in $B^{<2>}$ for some complete atomic Boolean algebra B.

As B is complete and atomic, $B = 2^I$ for some set I ((21) Theorem 5 p.70). An element in $B^{<2>}$ is of the form $((a_\alpha), (b_\alpha))$ with $a_\alpha \leqslant b_\alpha$ for each $\alpha \in I$, and $(a_\alpha), (b_\alpha) \in 2^I$. Let $p_i : B^{<2>} \to 2^{<2>}$ for $i \in I$ be the map defined by:

$$p_i((a_\alpha),(b_\alpha)) = (a_i,b_i).$$

For each i define:

$$f_i : A \to 2^{<2>} \text{ by } p_i f = f_i;$$

$$m_i : L \to 2^{<2>} \text{ by } p_i m = m_i;$$

$$j_i : L \to 2^{<2>} \text{ by } p_i j = j_i.$$

LEMMA 4.12. Let L be a Stone algebra and B a complete atomic Boolean algebra. Let $m \in M(L,B^{<2>})$, $m : D(L) \to D(B^{<2>})$ and $j \in J(L, S(B^{<2>}))$. Let A be a subset of L and f a map from A into $B^{<2>}$. Suppose, as defined above, $m_i^{-1}(0) = j_i^{-1}(0)$, $i \in I$. Then there exists $h \in H^*(L,B^{<2>})$ such that $h(a) = f(a)$ for all a in A and h lies between m and j if and only if f lies between m and j.

Proof. The necessity follows at once from Lemma 4.9. We prove the sufficiency. It is easy to show that for $i \in I$, $p_i \in H^*(B^{<2>},2^{<2>})$. Thus each $m_i \in M(L,2^{<2>})$, $m_i : D(L) \to D(2^{<2>})$, $j_i \in J(L,S(2^{<2>}))$ and by hypothesis $j_i^{-1}(0) = m_i^{-1}(0)$.

Let $H_i = \{g \in H^*(L,2^{<2>}) : g(a) = f_i(a), a \in A, m_i \leqslant g \leqslant j_i\}$. Then H_i is non-void by Lemma 4.11. For each i choose $h_i \in H_i$ and define $h : L \to B^{<2>}$ such that $p_i h = h_i$. It is easily shown that $h \in H^*(L,B^{<2>})$. By construction $h(a) = f(a)$ for $a \in A$ and h lies between m and j.

Let I be an algebra in an equational category K . I is injective if and only if every morphism $\gamma : B \to A$ from a subalgebra B of an algebra A in K can be extended to a morphism $\alpha : A \to I$.

THEOREM 4.13 ((4) Theorem 2,p.345). A Stone algebra is injective if and only if it has the form $B_1 \times B_2^{<2>}$ where B_1 and B_2 are complete Boolean algebras.

Let B be a complete Boolean algebra then there exists an embedding $e : B \to E$ where E is a complete atomic Boolean algebra. Define $d : B^{<2>} \to E^{<2>}$ by $d(a_1,a_2) = (e(a_1),e(a_2))$. It is easy to show d is a *-preserving monomorphism. As in Lemma 4.12 we can define projection maps $p_i : E^{<2>} \to 2^{<2>}$ in order to obtain $p_i dm = (dm)_i$, $p_i dj = (dj)_i$ which map L into $2^{<2>}$.

LEMMA 4.14. Let L be a Stone algebra and B a complete Boolean algebra. Let $m \in M(L,B^{<2>})$, $m : D(L) \to D(B^{<2>})$ and $j \in J(L, S(B^{<2>}))$. Let A be a subset of L containing 0,1 and f a map from A into $B^{<2>}$. Suppose, as defined above, $(dj)_i^{-1}(0) = (dm)_i^{-1}(0)$, $i \in I$. Then there exists $h \in H^*(L,B^{<2>})$ such that $h(a) = f(a)$ for all a in A and h lies between m and j if and only if f lies between m and j.

Proof. The necessity follows at once from Lemma 4.9. We prove the sufficiency. Let d be as above. Then $dm \in M(L,E^{<2>})$, $dm : D(L) \to D(E^{<2>})$ and $dj \in J(L,S(E^{<2>}))$. $df : A \to E^{<2>}$ lies between dm and dj and $(dj)_i^{-1}(0) = (dm)_i^{-1}(0)$. So by Lemma 4.12 there exists $g \in H^*(L,E^{<2>})$ such that $g = df$ on A and g lies between dm and dj.

Let $r \in H^*(E^{<2>},B^{<2>})$ be such that $rd = 1_{B^{<2>}}$. This is possible as $B^{<2>}$ is an injective object in the category of Stone algebras. Let $h = rg$. Then $h \in H^*(L,B^{<2>})$ and by construction $h|A = f$. It can easily be shown h lies between m and j.

By Theorem 4.13 every injective Stone algebra I is of the form $B_1 \times B_2^{<2>}$ where B_1, B_2 are complete Boolean algebras. Thus define $p_1 : I \to B_1$, $p_2 : I \to B_2^{<2>}$ where $p_i(a_1, a_2) = a_i$. Let $p_i m = m_i$, $p_i j = j_i$. Using Theorem 4.8 and Lemma 4.14 we can easily prove:

THEOREM 4.15. Let L be a Stone algebra and I an injective Stone algebra. Let $m \in M(L,I)$, $\mathfrak{m} : D(L) \to D(I)$ and $j \in J(L,S(I))$. Let A be a subset of L containing 0,1, and f a map from A into I. Suppose $p_2 m$ and $p_2 j$ satisfy the condition relating them in Lemma 4.14. Then there exists $h \in H^*(L,I)$ such that $h(a) = f(a)$ for all a in A and h lies between m and j if and only if f lies between m and j.

It might be thought that as the injective objects in the equational class of pseudocomplemented distributive lattices B_2 are of the form $M \times N^{<3>}$ ((19) Theorem 2,p.483) where M and N are complete Boolean algebras one might be able to prove a theorem similar to Theorem 4.15 for B_2. The method used in Lemma 4.11 will not generalize to $2^{<3>}$ however. If we defined projection maps $p_1, p_2, p_3 : 2^{<3>} \to 2$ we could define homomorphisms h_2 and h_3 in the same way as we defined h_2 in Lemma 4.11. We now need an h_1 such that $h_1(a) \leqslant h_2(a) \wedge h_3(a)$, $a \in L$. However $h_2 \wedge h_3$ is not necessarily a join homomorphism so we cannot consider it as we considered h_2 in Lemma 4.11.

Chapter 5

L^p-SPACES FROM MATRIX MEASURES

§1 Introduction

Kaz, in (27), seems to have been the first to have shown that a Hilbert space, $L^2(\mu)$, can be constructed from an n x n positive matrix measure (μ_{ij}) defined on the bounded Borel sets of the real line. A diagonal procedure is used relying on properties of the real line (see (16) p.1337-1346). Binding and Browne, in (7), describe corresponding Banach spaces and their continuous duals using non-constructive proofs.

In this chapter we show that by altering the definition of a matrix measure (§5.2) we may work in general measure spaces. We then develop an analogue to the classical L^p-space theory using constructive methods in §5.3. In order to obtain the integral representation of the continuous linear functionals on $L^p(\mu)$, $1 \leqslant p < \infty$ in §5.5 we discuss, in §5.4, the converse of Hölder's inequality in $L^p(\mu)$, $1 \leqslant p \leqslant \infty$, under weaker conditions than those in ((42) p. 380-381). A summary of the results in this chapter is given in (43).

The integration theory needed is expounded in (16); linear algebra theory in (30); functional analysis in (16).

§2 Matrix measures

We shall use the following notation. If $M = (m_{ij})$ is an $n \times n$ matrix and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are n-tuples of complex numbers we shall write the summation

$$\Sigma m_{ij}\, x_i\, \overline{y}_j \text{ as } y^*Mx.$$

More generally we shall denote $\overline{M}^{\,T}$ by M^*.

Let (X, Σ, ν) be a measure space by which we mean a set X, a σ-field Σ, and a countably additive set function ν, which may be complex-valued, defined on Σ.

We shall call a complex-valued function f ν-integrable if $f = u + iv$ where u and v are real ν-measurable functions on X and if $f \in L'(\nu)$.

Let $\{m_{ij} : 1 \leqslant i,j \leqslant n\}$ be a family of complex-valued ν-integrable functions on X. We define a family (μ_{ij}) of set functions, such that for each $e \in \Sigma$

$$\mu_{ij}(e) = \int_e m_{ij}(t)\, d\nu(t), \quad 1 \leqslant i,j \leqslant n.$$

Suppose the matrix $M(t) = (m_{ij}(t))$ is a positive semi-definite Hermitian matrix ν-almost everywhere. In fact we can assume it is Hermitian for all $t \in X$ as we can change the functions m_{ij} on a ν-null set without altering the μ_{ij}. Then (μ_{ij}) is a positive matrix measure and (m_{ij}) is called the matrix of densities of (μ_{ij}) with respect to ν. This alters the definition in ((16) p.1337-1338) where (m_{ij}) is defined in terms of (μ_{ij}).

§3 The L^p-space and the diagonal space

In this section we define the space $L^p(\mu_{ij})$ and show it is a Banach space by constructing a related "diagonal" space $L^p(\eta)$.

Let $\mathcal{L}^p(\mu_{ij})$ denote the space of n-tuples $F = (F_1, \ldots, F_n)$ of ν-measurable functions on X for which

$$\|F\| = \left[\int_X [F^*(t)\, M(t)\, F(t)]^{p/2}\, d\nu(t)\right]^{1/p} < \infty, \qquad 1 \leqslant p < \infty,$$

and

$$\|F\| = \nu\text{-ess sup } [F^*(t)\, M(t)\, F(t)]^{1/2} < \infty, \quad p = \infty.$$

Note that $F^*(t)\, M(t)\, F(t) \geqslant 0$ for ν-almost all t, so that the above integral is positive. By changing M on a null set, $F^*(t)\, M(t)\, F(t) \geqslant 0$ for all t, and $\|F\|$ remains unchanged.

An element $F \in \mathcal{L}^p(\mu_{ij})$ will be said to be a μ_{ij}-null function if $\|F\| = 0$. We define an equivalence relation on $\mathcal{L}^p(\mu_{ij})$: F is equivalent to G if F-G is a μ_{ij}-null function. The set of these equivalence classes will be denoted by $L^p(\mu_{ij})$. If $[F] \in L^p(\mu_{ij})$ then we define $[F] = F$ and instead of writing $[F]$ we write F. The spaces $L^p(\mu_{ij})$, which are clearly linear, will be shown to be normed spaces later. For convenience we write $L^p(\mu)$ for $L^p(\mu_{ij})$.

We now construct a diagonal space, $L^p(n)$.

LEMMA 5.1. *If* $\Sigma m_{ij}(t)\, x_i \bar{x}_j \geq 0$ *for all* $t \in X$, $i,j = 1,\ldots,n$, *then for each* t *there exists a complex non-singular linear transformation specified by the equations*

$$\begin{aligned} y_1 &= x_1 + R_{12}\, x_2 + R_{13}\, x_3 + \ldots + R_{1n}\, x_n \\ y_2 &= \qquad x_2 + R_{23}\, x_3 + \ldots + R_{2n}\, x_n \\ &\cdots \\ y_n &= \qquad\qquad\qquad\qquad x_n \end{aligned}$$

which changes $\sum_{i,j=1}^{n} m_{ij}(t) x_i \bar{x}_j$ *into the diagonal form* $\sum_{i=1} \lambda_i(t) |y_i|^2$ *where* R_{ij} *and* λ_i *are measurable functions with* $\lambda_i \geq 0$ $i,j = 1,\ldots,n$.

Proof. Since $M(t)$ is positive semi-definite, for each i

$$m_{ii}(t) = 0 \text{ implies } m_{ij}(t) = 0 \text{ for } j = 1, \ldots, n.$$

Thus $M(t)$ is zero or $m_{ii}(t) \neq 0$ for some i. We may assume without loss of generality $m_{11}(t) \neq 0$. Then, as in ((30) p.398-399)

$$\begin{aligned} \phi_1 &= \Sigma m_{ij}(t)\, x_i \bar{x}_j \\ &= m_{11}(t) \left| x_1 + \sum_{j=2}^{n} R_{1j}(t)\, x_j \right|^2 + \phi_2(x_2,\ldots,x_n) \end{aligned}$$

where the function R_{1j} is chosen so that $m_{11}(t)\, R_{1j}(t) = m_{1j}(t)$ $j=2,\ldots,n$ hence is measurable and $\Phi_2(x_2,\ldots,x_n)$ is a positive semi-definite quadratic form in $(n-1)$ variables.

Let:

$$\lambda_1(t) = \begin{cases} m_{11}(t) & \text{if } m_{11}(t) > 0 \\ 0 & \text{otherwise} \end{cases}.$$

If, for some i, $m_{ii}(t) = 0$, take $R_{ij}(t) = 0$ $j = 1,\ldots,n$. Complete the proof by induction noting that each R_{ij} and λ_i is a measurable function as it is found in terms of other measurable functions.

This method of diagonalization is non-trivial and ensures that the R_{ij} and λ_i are measurable whereas there may be other diagonal functions which are not measurable as the following example shows:

Consider the matrix of measurable functions

$$M(t) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where a, b are complex numbers $a \neq b$.

Let N be a non-measurable set and define:

$$f(t) = \begin{cases} a & t \in N \\ b & t \in R - N \end{cases}$$

$$g(t) = \begin{cases} b & t \in N \\ a & t \in R - N \end{cases} .$$

f, g are non-measurable functions.

Let

$$R(t) = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 & t \in N \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & t \in R\text{-}N. \end{cases}$$

Then for each t, $R(t)$ is unitary but $R(t)$ is also non-measurable.

$$R^{-1}(t)M(t)R^{-1}(t) = \begin{bmatrix} f & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is non-measurable. So there does exist a (non-measurable) unitary matrix taking a measurable matrix into a non-measurable diagonal matrix. Nevertheless there is a matrix $R(t) = 1$ which gives measurable eigenvalues.

The method used by Kaz (see also (16) p.1342) relies heavily on the topology on the real line.

If $\Lambda = (\lambda_i)$, $R = (R_{ij})$, then

$$M = R^* \Lambda R \text{ and } \Lambda = R^{*-1} M R^{-1}.$$

Now consider the set functions η_k, $1 \leqslant k \leqslant n$, defined by

$$\eta_k(e) = \int_e \lambda_k(t)\, d\nu(t),$$

for each $e \in \Sigma$. The properties of $M(t)$ ensure that $\Lambda(t)$ will be Hermitian and positive semi-definite for all $t \in X$. Thus (η_k) is a positive matrix measure defined on (X,Σ,ν) - the <u>diagonal matrix measure</u> corresponding to (μ_{ij}).

The space $L^p(\eta)$ is defined in the same manner as $L^p(\mu)$. Note that if $F = (F_1,\ldots,F_n) \in L^p(\mu)$ it does not follow that the n-tuples $(0,\ldots,F_i,\ldots 0) \in L^p(\mu)$, $i = 1,\ldots,n$, as the following example shows (cf. (16) p.1345):

Let ν be a finite positive regular measure defined on the bounded Borel subsets of the real line, R. Suppose $p = 2$, and

$$M(t) = \begin{bmatrix} +1 & -1 \\ & \\ -1 & +1 \end{bmatrix} \qquad \text{for all } t \in R.$$

Then

$$(\mu_{ij}(e)) = \begin{bmatrix} \nu(e) & -\nu(e) \\ & \\ -\nu(e) & \nu(e) \end{bmatrix} \qquad \text{for all bounded Borel subsets } e,$$

so that

$$\int_{-\infty}^{\infty} \Big| \sum_{i,j=1}^{2} m_{ij}(t)\, F_i(t)\, \overline{F}_j(t) \Big|\, d\nu(t) = \int_{-\infty}^{\infty} |F_1(t)-F_2(t)|^2 d\nu(t).$$

Thus if (F_1, F_2) is any pair of functions whose difference belongs to $L^2(\nu)$, $(F_1,F_2) \in L^2(\mu)$ even though neither $(F_1,0)$ nor $(0,F_2)$ need belong to $L^2(\mu)$.

However, if $F = (F_1,\ldots,F_n) \in L^p(\eta)$, $1 \leqslant p < \infty$, we have:

$$\left(\int_X \left(\sum_{i=1}^{n} \lambda_i(t) \, |F_i(t)\,|^2 \right)^{p/2} d\nu(t) \right)^{1/p} < \infty$$

$$\Longrightarrow \left(\int_X (\lambda_i(t)|F_i(t)|^2)^{p/2} \, d\nu(t) \right)^{1/p} < \infty$$

$$\Longrightarrow (0,\ldots F_i,\ldots,0) \in L^p(\eta) \quad i = 1,\ldots,n,$$

and $F = \sum_{i=1}^{n} (0,\ldots,F_i,\ldots,0)$.

Similarly for $p = \infty$.

LEMMA 5.2. $L^p(\eta)$ is a normed linear space, $1 \leqslant p \leqslant \infty$.

Proof. ii) By definition, for $F \in L^p(\eta)$, $\|F\| = 0$ implies $F = 0$, $1 \leqslant p \leqslant \infty$.

ii) If α is any complex number, it is easily seen $\|\alpha F\| = |\alpha| \, \|F\|$, for $F \in L^p(\eta)$, $1 \leqslant p \leqslant \infty$.

iii) We must show $\|F + G\| \leqslant \|F\| + \|G\|$, $F, G \in L^p(\eta)$, $1 \leqslant p \leqslant \infty$. Let

$$f_i(t) = \begin{cases} \lambda_i^{1/2}(t) F_i(t) & \text{if } \lambda_i(t) > 0 \\ 0 & \text{if } \lambda_i(t) = 0 . \end{cases}$$

Let $f = (f_1 \lambda_1^{-1/2},..,f_n\lambda_n^{-1/2})$ where $f_i(t)\lambda_i^{-1/2}(t)$ is to denote zero when $\lambda_i(t) = 0$. If $1 \leqslant p < \infty$, $f \in L^p(\eta)$.

$$\|F - f\|^p = \int_X \left[\sum_{i=1}^{n} \lambda_i(t) \left| F_i(t) - f_i(t) \, \lambda_i^{-1\,2}(t) \right|^2 \right]^{p/2} d\nu(t) = 0,$$

because the integrand is always zero. Similarly for $p = \infty$.

As we are dealing with equational classes, we need only show $\|f + g\| \leqslant \|f\| + \|g\|$ where g is defined in terms of G. This is easily proved using Minkowski's inequality for the space $L^p(\nu)$.

Let $F_i \varepsilon\ L^p(\lambda_i^{p/2}\nu)$ $i = 1,\ldots,n$ have the Banach space norm, $1 \leqslant p < \infty$,

$$\|F_i\| = \left(\int_X \lambda_i(t)^{p/2}|F_i(t)|^p d\nu(t)\right)^{1/p} .$$

Let $\oplus L^p(\lambda_i^{p/2}\nu)$ be the direct sum ((16) p.89) with $F \varepsilon \oplus L^p(\lambda_i^{p/2}\nu)$ having norm

$$\|F\| = \max_{1 \leqslant i \leqslant n} \|F_i\| .$$

Similarly define $\oplus L^\infty(\lambda_i^{1/2}\nu)$.

LEMMA 5.3 $L^p(\eta)$ and $\oplus L^p(\lambda_i^{p/2}\nu)$ are isomorphic spaces with equivalent norms $1 \leqslant p < \infty$. Similarly for $\oplus L^\infty(\lambda_i^{1/2}\nu)$ and $L^\infty(\eta)$.

Proof. Let $F = (F_1,\ldots,F_n) \varepsilon L^p(\eta)$ $1 \leqslant p < \infty$. Then:

$$\left(\int_X \left[\sum_{i=1}^n \lambda_i(t)|F_i(t)|^2\right]^{p/2} d\nu(t)\right)^{1/p} < \infty$$

$$\Rightarrow \quad \left(\int_X \lambda_i(t)^{p/2}|F_i(t)|^p\, d\nu(t)\right)^{1/p} < \infty \qquad i = 1,\ldots,n$$

$$\Rightarrow \quad F_i \varepsilon L^p(\lambda_i^{p/2}\nu) \qquad i = 1,\ldots,n .$$

If $F_i \varepsilon L^p(\lambda_i^{p/2}\nu)$ $i = 1,\ldots,n$ then $\lambda_i^{1/2}F_i \varepsilon L^p(\nu)$

$$\Rightarrow \sum_{i=1}^n \lambda_i^{1/2}|F_i| \varepsilon L^p(\nu)$$

$$\Rightarrow \left(\int_X \left[\sum_{i=1}^n \lambda_i(t)^{1/2}|F_i(t)|\right]^p d\nu(t)\right)^{1/p} < \infty$$

$$\Rightarrow \left(\int_X \left[\sum_{i=1}^n \lambda_i(t)|F_i(t)|^2\right]^{p/2} d\nu(t)\right)^{1/p} < \infty, \text{ since}$$

$$\sum_{i=1}^n \lambda_i|F_i|^2 \leqslant \left(\sum_{i=1}^n \lambda_i^{1/2}|F_i|\right)^2 .$$

Thus $F = (F_1,\ldots,F_n) \varepsilon L^p(\eta)$.

Now we show norm equivalence. Let $F \in L^p(\eta)$.

$$\|F\| = \left(\int_X \left[\sum_{i=1}^n \lambda_i(t) |F_i(t)|^2 \right]^{p/2} d\nu(t) \right)^{1/p}$$

$$\leq \left(\int_X \left[\sum_{i=1}^n \lambda_i(t)^{1/2} |F_i(t)| \right]^p d\nu(t) \right)^{1/p}$$

$$\leq \sum_{i=1}^n \left(\int_X [\lambda_i(t)^{1/2} |F_i(t)|]^p d\nu(t) \right)^{1/p} \text{ by Minkowski's inequality}$$

$$\leq n \max_{1 \leq i \leq n} \left(\int_X \lambda_i(t)^{p/2} |F_i(t)|^p d\nu(t) \right)^{1/p}$$

$= n \max_{1 \leq i \leq n} \|F_i\|$ where $F_i \in L^p(\lambda_i{}^{p/2}\nu)$. Also

$$\|F\| \geq \max_{1 \leq i \leq n} \left(\int_X [\lambda_i(t)^{1/2} |F_i(t)|]^p d\nu(t) \right)^{1/p}$$

$$= \max_{1 \leq i \leq n} \|F_i\| \text{ where } F_i \in L^p(\lambda_i{}^{p/2}\nu).$$

The case $p = \infty$ is similar.

DEFINITION 5.4 Let $G = (G_1, \ldots, G_n) \in L^p(\eta)$. Define G to be a simple (measurable, bounded, characteristic) function if each $G_i \lambda_i{}^{1/2}$ is a simple (measurable, bounded characteristic) function in $L^p(\nu)$.

Remark. By Lemma 5.3 we note that $\|F_i\| = \|(0, \ldots, F_i, \ldots 0)\|$ where the norm on the left is that in $L^p(\lambda_i{}^{p/2}\nu)$ and on the right is in $L^p(\eta)$.

THEOREM 5.5 $L^p(\eta)$ is a Banach space.

Proof. This follows immediately from Lemma 5.3 using the fact that the norm on $L^p(\eta)$ is equivalent to that on $\oplus L^p(\lambda_i^{p/2}\nu)$ which is a Banach space, $1 \leq p < \infty$. Similarly for $L^\infty(\eta)$.

COROLLARY 5.6. $L^p(\mu)$ is a Banach space.

Proof. Consider the map between the spaces given by $R : L^p(\mu) \to L^p(\eta)$, where $R = (R_{ij})$ as in Lemma 5.1. Then

$$\|RF\|^p = \int_X [RF^*(t)\,\Lambda(t)\,RF(t)]^{p/2} d\nu(t)$$

$$= \int_X [F^*(t)\,M(t)\,F(t)]^{p/2} d\nu(t)$$

$$= \|F\|^p.$$

R is clearly an isometry with inverse, R^{-1}. As $L^p(\eta)$ is a Banach space, R^{-1} induces a Banach space structure on $L^p(\mu)$. Note that by the above, we have $L^p(\mu)$ isometrically isomorphic to $L^p(\eta)$.

§4 The converse of Hölder's inequality

In this section the converse of Hölder's inequality is discussed in $L^p(\mu)$, $1 \leqslant p \leqslant \infty$, under weaker conditions than those in ((42), 380-1).

LEMMA 5.7. *Let* y_i, $1 \leqslant i \leqslant n$, *be complex numbers*, $\frac{1}{p} + \frac{1}{q} = 1$ *where* $1 < p < \infty$, *and* $\lambda_i \geqslant 0$. *If*

$$x_i = y_i \, \phi^{\frac{1}{2}(q-1)} \text{ and } \phi = \sum_{i=1}^{n} \lambda_i |y_i|^2 \geqslant 0,$$

then

$$\Big[\sum_{i=1}^{n} \lambda_i |x_i|^2\Big]^{p/2} = \Big[\sum_{i=1}^{n} \lambda_i |y_i|^2\Big]^{q/2} = \sum_{i=1}^{n} \lambda_i x_i \bar{y}_i .$$

Proof.
$$\Big(\sum_{i=1}^{n} \lambda_i |x_i|^2\Big)^{p/2} = \phi^{\frac{1}{2}(q-2)p} \Big[\sum_{i=1}^{n} \lambda_i |y_i|^2\Big]^{p/2}$$

$$= \phi^{\frac{1}{2}p(q-2)+p/2}$$

$$= \phi^{\frac{1}{2}p(q-1)}$$

$$= \phi^{q/2}.$$

Also
$$\sum_{i=1}^{n} \lambda_i x_i \bar{y}_i = \sum_{i=1}^{n} \lambda_i |y_i|^2 \phi^{\frac{1}{2}(q-1)} = \phi^{q/2}.$$

LEMMA 5.8. Let $1 < p < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$, and $G = (G_1,\ldots,G_n)$ be defined ν-almost everywhere over X, where we take (X,Σ,ν) σ - finite or G measurable. If

$$\left|\int_X \sum_{i=1}^{n} \lambda_i(t)F_i(t)\overline{G}_i(t)d\nu(t)\right| \leqslant M \|F\| \quad .$$

holds for every bounded measurable $F = (F_1,\ldots,F_n)$ in $L^p(\eta)$ and some constant M, then $G \in L^q(\eta)$ and $\|G\| \leqslant M$.

Proof. We first show that if (X,Σ,ν) is σ-finite then G is measurable. Let $X = \cup X_j$ where, for each j, $\nu(X_j) < \infty$. Fix X_j. Consider the element $(0,\ldots,\chi\lambda_i^{-1/2},\ldots,0) \in L^p(\eta)$ where χ is the characteristic function of X_j in $L^p(\nu)$, see Definition 5.4, and $\chi(t)\lambda_i^{-1/2}(t) = 0$ when $\lambda_i(t) = 0$.

By hypothesis

$$\left|\int_{X_j} \lambda_i(t)\chi(t)\lambda_i^{-1/2}(t)\overline{G}_i(t)d\nu(t)\right| \leqslant M,$$

so

$$\left|\int_{X_j} \lambda_i^{1/2}(t)\overline{G}_i(t)d\nu(t)\right| \leqslant M$$

and $\lambda_i^{1/2}\, G_i$ is integrable for each i. By Definition 5.4, G is integrable and hence measurable on each X_j and so measurable.

Now let $F_r = (F_{1r},\ldots,F_{nr})$ be defined on X, with

$$F_{ir}(t) = \begin{cases} \overline{G}_i(t)\ \phi\ (t)^{\frac{1}{2}(q-1)} & (|\overline{G}_i(t)||\phi(t)|^{\frac{1}{2}(q-1)} < r) \quad (*) \\ 0 & (\text{otherwise}) \end{cases}$$

where

$$\phi\ (t) = \sum_{i=1}^{n} \lambda_i(t)|G_i(t)|^2.$$

Then for each $t \in X$,

$$\Big[\sum_{i=1}^{n} \lambda_i(t)|F_{ir}(t)|^2\Big]^{p/2} = \sum_{i=1}^{n} \lambda_i(t)\, F_{ir}(t)\overline{G}_i(t),$$

for equality holds if (*) applies for every i, by Lemma 5.7. If(*) holds only for some i the remaining terms are zero and the argument still applies. Thus from the given integral we have

$$\Big| \int_X \Big[\sum_{i=1}^{n} \lambda_i(t)|F_{ir}(t)|^2\Big]^{p/2} d\nu(t)\Big| \leq M^q.$$

By (*)

$$\lim_{r \to \infty} F_{ir}(t) = \overline{G}_i(t)\, \phi\,(t)^{\frac{1}{2}(q-1)}$$

ν-almost everywhere, so that

$$\int_X \Big[\sum_{i=1}^{n} \lambda_i(t)|F_{ir}(t)|^2\Big]^{p/2} d\nu(t) \to \int_X \Big[\sum_{i=1}^{n} \lambda_i(t)|G_i(t)|^2\Big]^{q/2} d\nu(t),$$

by Lebesgue's dominated convergence theorem ((16),p.151). Thus $G \in L^q(\eta)$ and $\|G\| \leq M$.

LEMMA 5.9. *Let* (X,Σ,ν) *be* σ*-finite. Let* $G = (G_1,\ldots,G_n)$ *be defined* ν*-almost everywhere on* X *and let*

$$\left|\int_X \sum_{i=1}^{n} \lambda_i(t)F_i(t)\overline{G}_i(t)d\nu(t)\right| \leq M\|F\|$$

hold for all $F = (F_1,\ldots,F_n)$ *bounded in* $L^1(\eta)$. *Then* $G \in L^\infty(\eta)$ *and*

$$\nu\text{-ess sup}\left[\sum_{i=1}^{n} \lambda_i(t)|G_i(t)|^2\right]^{1/2} \leq M.$$

Proof. Let $X = \cup X_j$ where, for each j, $\nu(X_j) < \infty$. We can show G is measurable using the same method as in the proof of Lemma 5.8. Now, as the inequality

$$\left|\int_X \lambda_i(t)F_i(t)\overline{G}_i(t)d\nu(t)\right| \leq M\int_X \lambda_i^{1/2}(t)|F_i(t)|d\nu(t)$$

holds for each i and $F_i \in L^1(\lambda_i^{1/2}\nu)$, it follows that

$$|\lambda_i^{1/2}(t)\,\overline{G}_i(t)| \leq M$$

ν-almost everywhere, $1 \leq i \leq n$. By Definition 5.4, $G \in L^\infty(\eta)$. Suppose that $[\Sigma\lambda_i(t)|G_i(t)|^2]^{1/2} \geq N > 0$ on some set $E \subseteq X_j$. Assuming $\|G\| \neq 0$ define

$$F_i(t) = \begin{cases} \overline{G}_i(t)\left[\sum_{j=1}^{n} \lambda_j(t)|G_j(t)|^2\right]^{-1/2} & (t \in E) \\ 0 & (\text{otherwise}). \end{cases}$$

Then F is bounded in $L^1(\eta)$, by Definition 5.4, and on E

$$\sum_{i=1}^{n} \lambda_i F_i \overline{G}_i = \Big[\sum_{i=1}^{n} \lambda_i |G_i|^2\Big]^{1/2},$$

and

$$\sum_{i=1}^{n} \lambda_i(t)|F_i(t)|^2 = 1.$$

Then for the above F,

$$N\ \nu(E) \leqslant \Big| \int_X \sum_{i=1}^{n} \lambda_i(t)F_i(t)\overline{G}_i(t)d\nu(t)\Big| \leqslant M\ \nu(E).$$

So $N \leqslant M$ unless E is a null set.

We have shown that for $t \in X_j$,

$$\nu\text{-ess sup}\Big| \sum_{i=1}^{n} \lambda_i(t)|G_i(t)|^2\Big|^{1/2} \leqslant M.$$

This is true for each j, so $\|G\| \leqslant M$.

LEMMA 5.10. Let $p = \infty$, and (X,Σ,ν) be a measure space. Let $G = (G_1,\ldots,G_n)$ be defined ν-almost everywhere on X and suppose

$$\Big| \int_X \sum_{i=1}^{n} \lambda_i(t)F_i(t)\overline{G}_i(t)d\nu(t)\Big| \leqslant M\|F\|$$

holds for each $F \in L^\infty(\eta)$ and some constant M. Then $G \in L'(\eta)$ and $\|G\| \leqslant M$.

Proof. Choose $(0,\ldots,\lambda_i^{-1/2},\ldots,0) \in L^\infty(\eta)$ where $\lambda_i^{-1/2}(t) = 0$ if $\lambda_i(t) = 0$.

Using the given inequality, we have

$$\left| \int_X \lambda_i^{1/2}(t)\overline{G}_i(t)d\nu(t) \right| \leqslant M$$

for each i, and hence G is measurable.

Suppose $\|G\| \neq 0$ and define

$$F_i(t) = \overline{G}_i(t)\Big[\sum_{i=1}^{n} \lambda_i(t)|G_i(t)|^2\Big]^{-1/2}.$$

Then $\sum_{i=1}^{n}\lambda_i(t)|F_i(t)|^2 = 1$, so $F \in L^\infty(\eta)$.

By hypothesis

$$\left| \int_X \sum_{i=1}^{n} \lambda_i(t)|G_i(t)|^2\Big[\sum_{i=1}^{n} \lambda_i(t)|G_i(t)|^2\Big]^{-1/2} d\nu(t) \right| \leqslant M.$$

Thus $G \in L^1(\eta)$ and $\|G\| \leqslant M$.

COROLLARY 5.11. *Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* (X,Σ,ν) *be a measure space. Suppose* $G = (G_1,\ldots,G_n)$ *is defined* ν*-almost everywhere on* X *and is measurable. If*

$$\left| \int_X G^*(t)\, M(t)\, F(t)\, d\nu(t) \right| \leqslant N\, \|F\|$$

holds for every bounded measurable $F = (F_1,\ldots,F_n)$ *in* $L^p(\mu)$ *and some constant* N, *then* $G \in L^q(\mu)$ *and* $\|G\| \leqslant N$.

Proof.

$$\left| \int_X G^*(t)M(t)F(t)d\nu(t) \right| = \left| \int_X G^*(t)R^*(t)\Lambda(t)R(t)F(t)d\nu(t) \right| \leqslant N\|RF\|$$

as R is an isometry. By Lemma 5.8, $RG \in L^q(\eta)$ and $\|RG\| \leqslant N$.

Hence $G \in L^q(\mu)$ and $\|G\| \leqslant N$.

COROLLARY 5.12. *Let* $p = 1$, $q = \infty$. *If* (X,Σ,ν) *is a* σ*-finite measure space, suppose* $G = (G_1,\ldots,G_n)$ *is defined* ν*-almost everywhere on* X. *If*

$$\left| \int_X G^*(t)M(t)F(t)d\nu(t) \right| \leq N\|F\|$$

holds for every bounded measurable $F = (F_1,\ldots,F_n)$ *in* $L^1(\mu)$ *and some constant* N, *then* $G \in L^\infty(\mu)$ *and* $\|G\| \leq N$.

Proof. Similar to that of Corollary 5.11, using Lemma 5.9.

COROLLARY 5.13. *Let* $p = \infty$, $q = 1$. *Suppose* $G = (G_1,\ldots,G_n)$ *is defined* ν*-almost everywhere on* X. *If*

$$\left| \int_X G^*(t)M(t)F(t)d\nu(t) \right| \leq N\|F\|$$

holds for each $F \in L^\infty(\mu)$ *and some constant* N, *then* $G \in L^1(\mu)$ *and* $\|G\| \leq N$.

Proof. Similar to that of Corollary 5.11, using Lemma 5.10.

§5 The conjugate of L^p

We now show that the conjugate of an $L^p(\mu)$-space, $1 \leqslant p < \infty$ is an $L^q(\mu)$-space where for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$; $p = 1$, $q = \infty$.

LEMMA 5.14. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. If $F \in L^p(\eta)$, $G \in L^q(\eta)$, we have by the Cauchy and Hölder inequalities,*

$$\left|\int_X [G^*(t)\Lambda(t)F(t)]d\nu(t)\right| \leqslant \int_X [F^*(t)\Lambda(t)F(t)]^{1/2}[G^*(t)\Lambda(t)G(t)]^{1/2}d\nu(t$$

$$\leqslant \|F\|_p\|G\|_q .$$

Similarly for the case $p = 1$, $q = \infty$.

THEOREM 5.15. *Let $1 \leqslant p < \infty$ and (X,Σ,ν) be a measure space. For $p = 1$, we take (X,Σ,ν) as a σ-finite measure space. Then there is a one-to-one, linear, norm-preserving correspondence between the normed conjugate space of $L^p(\mu)$ and $L^q(\mu)$, such that if $\theta \in L^p(\mu)^*$ and $g = (g_1,\ldots,g_n) \in \mathcal{L}^q(\mu)$ is a function which represents the element of $L^q(\mu)$ corresponding to θ, we have*

$$\theta(f) = \int_X g^*(t)M(t)f(t)d\nu(t)$$

for each $f = (f_1,\ldots,f_n) \in \mathcal{L}^p(\mu)$.

The norm-preserving feature of the correspondence means that

$$\|\theta\| = \left[\int_X [g^*(t)M(t)g(t)]^{q/2} d\nu(t)\right]^{1/q}$$

if $1 < p < \infty$, and

$$\|\theta\| = \nu\text{-ess sup } [g^*(t)M(t)g(t)]^{1/2}$$

in the case $p = 1$ in which case $q = \infty$.

Proof. We show it is sufficient to prove the theorem for the space $L^p(\eta)$. Let $f \in L^p(\mu)$, and $Rf = F \in L^p(\eta)$. Given $\theta \in L^p(\mu)^*$ we can find $\Psi \in L^p(\eta)^*$ with $\|\theta\| = \|\Psi\|$ and $\theta(f) = \Psi(F)$ for every $f \in L^p(\mu)$. Suppose we have $G \in L^q(\eta)$ such that for every $F \in L^p(\eta)$,

$$\Psi(F) = \int_X G^*(t)\Lambda(t)F(t)d\nu(t),$$

and where $\|G\| = \|\Psi\|$. Then

$$\Psi(F) = \int_X G^*(t)R^{*-1}(t)M(t)R^{-1}(t)F(t)d\nu(t)$$

$$= \int_X (R^{-1}G)^*(t)M(t)(R^{-1}F)(t)d\nu(t).$$

Let $R^{-1}G = g$. Then $g \in L^q(\mu)$ and R^{-1} being an isometry implies $\|g\| = \|\theta\|$.

Thus we need only consider the space $L^p(\eta)$.

Let $G = (G_1,\ldots,G_n) \in \mathcal{L}^q(\eta)$ and define θ by

$$\theta(F) = \int_X \sum_{i=1}^{n} \lambda_i(t)F_i(t)\overline{G}_i(t)d\nu(t)$$

for each $F = (F_1,\ldots,F_n) \in \mathcal{L}^p(\eta)$. Then θ is obviously linear and is continuous by Lemma 5.14 with $\|\theta\| \leq \|G\|$, and the mapping $G \to \theta$ defines a continuous linear mapping of $L^q(\eta)$ into $L^p(\eta)^*$. By Lemmas 5.8 and 5.9 if $\theta(F) = 0$ for every F, $\|G\| = 0$. Thus the mapping is one-to-one.

Now let $\theta \in L^p(\eta)^*$. By Lemma 5.3 if $F = (F_1,\ldots,F_n) \in L^p(\eta)$ we have $F_i \in L^p(\lambda_i^{p/2}\nu)$ $i = 1,\ldots,n$.

Define θ_i on $L^p(\lambda_i^{p/2}\nu)$ by

$$\theta_i(F_i) = \theta(0,\ldots,F_i,\ldots,0).$$

θ_i is obviously linear, and is continuous because

$$|\theta_i(F_i)| \leq \|\theta\| \, \|(0,\ldots,F_i,\ldots 0)\|$$
$$= \|\theta\| \, \|F_i\|.$$

Thus $\theta_i \in L^p(\lambda_i^{p/2}\nu)^*$ for each i. By a well-known result, see ((16) p.286) there exists $\overline{G}_i \in L^q(\lambda_i^{q/2}\nu)$ such that

$$\theta_i(F_i) = \int_X \lambda_i(t)F_i(t)\overline{G}_i(t)d\nu(t).$$

Note that if $\overline{G}_i \in L^q(\lambda_i^{q/2}\nu)$, by Definition 5.4, $G = (G_1,\ldots,G_n) \in L^q(\eta)$.

Now

$$\theta(F) = \sum_{i=1}^{n} \theta(0,\ldots,F_i,\ldots,0)$$

$$= \sum_{i=1}^{n} \theta_i(F_i)$$

$$= \sum_{i=1}^{n} \int_X \lambda_i(t)F_i(t)\overline{G}_i(t)d\nu(t)$$

$$= \int_X \sum_{i=1}^{n} \lambda_i(t)F_i(t)\overline{G}_i(t)d\nu(t) .$$

This is true for all $F \in L^p(\eta)$. Hence

$$\left| \int_X \sum_{i=1}^{n} \lambda_i(t)F_i(t)\overline{G}_i(t)d\nu(t) \right| = |\theta(F)| \leq \|\theta\|\|F\| .$$

By Lemmas 5.8 and 5.9 $\|G\| \leq \|\theta\|$.

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