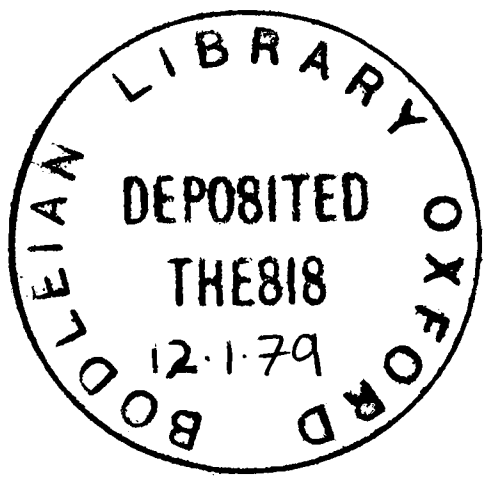
SOME PROBLEMS IN
COMBINATORIAL GEOMETRY

by

James G. Oxley



A thesis submitted to the

University of Oxford

for the degree of

Doctor of Philosophy

June, 1978

Acknowledgements

I thank my supervisor, Dr D.J.A. Welsh, who suggested the problems considered here, for much helpful advice, numerous valuable suggestions and many perceptive criticisms.

I also thank Dr P.D. Seymour and Dr Laurence R. Matthews for several useful discussions.

This work was financially supported by the Commonwealth Scientific and Industrial Research Organization (Australia) and their generosity is acknowledged with much gratitude.

Finally, I thank Mrs Sheila Robinson for her excellent typing.

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Thesis submitted for the degree of

Doctor of Philosophy, Trinity Term, 1978.

ABSTRACT

This thesis is in two parts. The first two chapters deal with infinite matroids and the remaining three chapters with finite matroids.

Chapter 1 has two main results. Firstly we prove that there is no duality function on the class of independence spaces which behaves like duality for finite matroids. Secondly we determine the unique class of preindependence spaces which contains the class of independence spaces and is closed under restriction, contraction and the natural duality function. Two other results of this chapter answer questions of Higgs and Welsh on infinite matroids.

The main result of Chapter 2 is a counter-example to a 1967 conjecture of Nash-Williams concerning packing disjoint spanning trees in countably infinite graphs.

In the third chapter we prove matroid analogues of various bounds on the chromatic number of a graph due to Matula, Szekeres and Wilf, and Brooks. These matroid results sharpen earlier bounds of Heron and Lindström. In the second half of the chapter we answer a number of questions of Mullin and Stanton related to the critical problem for binary matroids.

In Chapter 4 we prove the binary case of a matroid conjecture of Welsh which is a natural analogue of Gallai's theorem that the vertex-stability and vertex-covering numbers of a graph G sum to the number of vertices of G.

The Tutte polynomial of a graph is known to be related to the bond percolation model on the graph. In Chapter 5 we introduce a basic abstraction of classical percolation theory, namely percolation on clutters. We give new and simpler proofs of several results of Hammersley and bound the percolation probability above and below. One of the main results shows that the theory of the Tutte polynomial cannot be extended from matroids to arbitrary clutters.

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Preliminaries

1. Terminology and notation.

Throughout this thesis standard set-theoretic notation and terminology will be used. In particular if S and T are sets, then S\T denotes the set of elements of S that are not in T. The *symmetric difference* S Δ T of S and T is the set (S\T)∪(T\S) and the *cardinality* of S will be denoted by |S|.

The following are additional items of set-theoretic notation which will be used. A one-element set {x} will sometimes be denoted by x; for example, if X is a set, X ∪ x means X ∪ {x}. A subset U of a set T is called a *cofinite* subset of T if T\U is finite. A collection A of subsets of a set S will be called a *clutter* or *Sperner family* if no element of A properly contains another. The expression $U \subset\subset T$ will mean that U is a finite subset of T.

We shall assume familiarity with basic concepts from the theory of finite matroids and in general the matroid terminology we use will follow Welsh [86]. In particular if M is a matroid on a finite set S and $T \subseteq S$, then M^* denotes the matroid *dual* of M, while M|T and M.T denote respectively the *restriction* and *contraction* of M to T. If $T = \{x_1, x_2, \ldots, x_n\}$ we shall sometimes write M\T or $M\backslash x_1, x_2, \ldots, x_n$ for the restriction of M to S\T and M/T or $M/x_1, x_2, \ldots, x_n$ for the contraction of M to S\T. The *rank* and *corank* of T will be denoted by rk(T) and cork(T) respectively, and rk(S) will sometimes be written as rk(M). The simple matroid associated with the matroid M will be denoted by $\tilde{M}$ and the sets of circuits and cocircuits of M will be denoted by C(M) and $C^*(M)$ respectively. If M_1 and M_2 are matroids on disjoint sets, then

$M_1 \oplus M_2$ denotes their *direct sum*.

The first two chapters of this thesis deal with infinite matroids and graphs and a large number of definitions have been included in order to make the treatment as self-contained as possible. Many of these definitions extend well-established definitions for finite matroids to matroids on arbitrary sets. On finite matroids the newer definitions agree with the familiar definitions. Infinite graphs are considered briefly in Chapter 5, however, unless otherwise stated, all matroids, graphs and clutters in Chapters 3,4 and 5 are finite.

The graph-theoretic terminology used here will in general follow Bondy and Murty [4]. Except where otherwise stated, all graphs considered are undirected and may have loops and parallel edges. If G is a graph, then E(G) and V(G) denote respectively the set of edges and the set of vertices of G. All matroids associated with graphs will be defined on the edge-sets of the graphs.

If r is a non-negative real number, then [r] and {r} will denote respectively the greatest integer not exceeding r and the least integer not less than r. The sets of integers, positive integers and negative integers will be denoted by Z, Z^+ and Z^- respectively. The set of real numbers and the set of positive real numbers will be denoted by R and R^+ respectively.

2. Technical remarks.

The symbol // will denote the end or absence of a proof. References will be indicated by a number in square brackets followed by page or theorem numbers where appropriate.

Portions of the text which are to be highlighted will be

numbered to the left of the page. This numbering will be internal to sections and will be independent of the item being numbered. Item (3.2.6) is the sixth numbered item of the second section of Chapter 3. Note that §1.4 denotes the fourth section of Chapter 1.

At the beginning of the text there is a table of contents; at the end, an index of notation together with an index of definitions.

Chapter 1.

Infinite matroids

1. Introduction.

A finite matroid may be described in many equivalent ways: by its closure operator, its collection of independent sets, its bases or its circuits. Hence there are several potentially different approaches to the problem of extending the theory of finite matroids to matroids on infinite sets. The theory of preindependence and independence spaces (see Mirsky [52]) results from extending the independent set description of finite matroids. Infinite matroids have also been studied through their closure operators (see Klee [44] and Higgs [39]). The relationship between the independent set and operator approaches to infinite matroids is not as straightforward as in the finite case. In this chapter we prove a number of results linking these two approaches.

Suppose that S is a fixed infinite set and $\mathcal{S}$ is the collectic of independence spaces on S. Then a duality function δ on $\mathcal{S}$ is an involution of $\mathcal{S}$ such that for all I in $\mathcal{S}$ and all finite subsets T of S, the restriction of δI to T agrees with the finite dual of the contraction of I to T. In section 2 we prove that there is no such duality function on $\mathcal{S}$. This motivates a search for a distinguished class of preindependence spaces which includes $\mathcal{S}$, is closed under a suitable duality function and has well-defined operations of restriction and contraction under which it is closed. Some conditions which such a class must satisfy are determined in section 2 and in section 5 it is shown that the class of B-matroids which was introduced by Higgs [39] is the unique class of infinite matroids which fulfills all these conditions. This work was summarized in [59].

In the operator approach to infinite matroids as developed by Klee [44] and Higgs [39], duality is introduced at the beginning. Both authors define several dual pairs of conditions on operators and characterize certain types of operator in terms of their collections of circuits or their collections of bases. In the third section this work is reviewed and several classes of infinite matroids closed under restriction, contraction and duality are noted. In section 4 we study one such class, namely the class of *wIwE*-operators of Klee [44]. A *wIwE*-operator need not have circuits, bases or hyperplanes. We determine the families of sets which can occur as collections of independent sets of such operators. In addition we answer a question of Welsh [85] by proving that, whereas independence spaces fit naturally into the operator framework established by Klee, preindependence spaces do not.

In section 5 we settle a question of Higgs [40] concerned with *B*-matroids, and characterize *B*-matroids by their collections of independent sets and by their collections of bases. In [41], Higgs shows that an infinite generalization of the cycle matroid of a finite graph which admits two-way infinite paths as circuits need not be a *B*-matroid. In section 6 we prove that a similar generalization of the finite bicircular matroid is always a *B*-matroid. The final section, §7, shows that *B*-matroids are a subclass of the class of inductive exchange systems introduced by Brualdi and Scrimger [10]. The bulk of the material in this chapter is to be published in [50,60].

2. Preindependence spaces, independence spaces and duality.

A *preindependence structure* on a non-empty set S is a non-empty collection $\mathcal{I}$ of subsets of S (called *independent sets*) satisfying the following conditions (see Mirsky [52, p.90]).

(1.2.1) (Hereditary.) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$.

(1.2.2) (Finite exchange.) If X and Y are finite members of $\mathcal{I}$ and $|X| = |Y| + 1$, then there is an element x of $X \setminus Y$ such that $Y \cup x \in \mathcal{I}$.

Suppose that, in addition to (1.2.1) and (1.2.2), $\mathcal{I}$ satisfies the condition:

(1.2.3) (Finite character.) If $X \notin \mathcal{I}$, then some finite subset of X is not in $\mathcal{I}$.

Then $\mathcal{I}$ is an *independence structure* on S. Clearly when S is finite, $\mathcal{I}$ is a matroid on S. A *preindependence space* (*independence space*) is a pair $(S,\mathcal{I})$ where $\mathcal{I}$ is a preindependence structure (independence structure) on S. The terms "independenc space" and "preindependence space" will be used for both the collections $\mathcal{I}$ of independent sets and the pairs $(S,\mathcal{I})$.

A maximal independent subset of a preindependence space is called a *base*. The following condition on a preindependence space $(S,\mathcal{I})$ holds for all independence spaces (see, for example, [52, Theorem 6.1.2]).

(1.2.4) (Maximal condition.) If $X \in \mathcal{I}$, then there is a base of $(S,\mathcal{I})$ containing X.

Again suppose that $(S,\mathcal{I})$ is a preindependence space. If $T \subseteq S$ and

(1.2.5) $\mathcal{I}|T = \{X : X \subseteq T, X \in \mathcal{I}\}$,

then $(T,\mathcal{I}|T)$ is a preindependence space called the *restriction* of $\mathcal{I}$ to T. Clearly a restriction of an independence space is also an independence space.

Now if $(S,\mathcal{I})$ is an independence space and B is a base of $\mathcal{I}|(S\backslash T)$, then let

(1.2.6) $\mathcal{I}.T = \{X : X \subseteq T, X \cup B \in \mathcal{I}\}$.

$\mathcal{I}.T$ does not depend on the choice of the base B (see, for example, [8, Lemma 2.1]). Moreover, $(T,\mathcal{I}.T)$ is an independence space, the *contraction* of $\mathcal{I}$ to T.

In [10], Brualdi and Scrimger follow the independent set approach to infinite matroids and define an *exchange system* $(S,\mathcal{I})$ to be a preindependence space for which every restriction $(T,\mathcal{I}|T)$ satisfies the following additional condition.

(1.2.7) (Symmetric base exchange.) If B_1 and B_2 are bases of $\mathcal{I}|T$, then for each element x of $B_1\backslash B_2$ there is an element y of $B_2\backslash B_1$ such that $(B_1\backslash x)\cup y$ and $(B_2\backslash y)\cup x$ are bases of $\mathcal{I}|T$.

An *inductive exchange system* $(S,\mathcal{I})$ is an exchange system satisfying the maximal condition. Note that such systems satisfy the following strengthened symmetric exchange condition [10, p.246].

(1.2.8) For all $T \subseteq S$, if $A \in \mathcal{I}|T$ and B is a base of $\mathcal{I}|T$, then for each element x of $A\backslash B$ there is an element y of $B\backslash A$ such that $(B\backslash y)\cup x$ is a base of $\mathcal{I}|T$ and $(A\backslash x)\cup y \in \mathcal{I}|T$. If A is a base of $\mathcal{I}|T$, then $(A\backslash x)\cup y$ is a base of $\mathcal{I}|T$.

Inductive exchange systems will be looked at again in section 7 where they will be related to certain other classes of infinite matroids.

Consider now the extension of duality for finite matroids to duality for independence spaces. If $(S,\mathcal{I})$ is a

preindependence space, then let

(1.2.9) $\mathcal{I}^* = \{X : S\backslash X \text{ contains a base of } \mathcal{I}\}$.

If $(S,\mathcal{I})$ is an independence space, then $(S,\mathcal{I}^*)$ is a preindependence space with the maximal condition. But in general $(S,\mathcal{I}^*)$ need not satisfy the finite character condition (see, for example, [20, Example 20.4.1]). When S is finite, $(S,\mathcal{I}^*)$ is the *dual* matroid of $(S,\mathcal{I})$.

Let S be an arbitrary infinite set and $\mathcal{S}$ be the set of independence spaces on S. A *duality function* δ on $\mathcal{S}$ is a mapping from $\mathcal{S}$ into $\mathcal{S}$ such that for all $\mathcal{I}$ in $\mathcal{S}$:

(1.2.10) $\delta\delta\mathcal{I} = \mathcal{I}$; and

(1.2.11) $(\delta\mathcal{I})|T = (\mathcal{I}.T)^*$ for all $T \subset\subset S$.

The second of these conditions expresses agreement between δ and duality for finite matroids. Note that the function defined by (1.2.9) fails as a duality function only because it may map an element of $\mathcal{S}$ outside $\mathcal{S}$.

(1.2.12) Theorem. *There is no duality function on the collection* $\mathcal{S}$ *of independence spaces on an infinite set* S.

Proof Assume that there is a duality function δ on $\mathcal{S}$. Then for each non-negative integer i, let $\mathcal{I}_i = \{X : X \subseteq S, |X| \leq i\}$. Clearly $(S,\mathcal{I}_i)$ is an independence space. If $T \subset\subset S$, then $S\backslash T$ is infinite and so $\mathcal{I}_i|(S\backslash T)$ contains a base of $\mathcal{I}_i$. Thus $\mathcal{I}_i.T = \{\emptyset\}$, hence $(\mathcal{I}_i.T)^* = \{X : X \subseteq T\}$, and so, by (1.2.11), $T \in \delta\mathcal{I}_i$. It follows that $\delta\mathcal{I}_i$ contains all finite subsets of S and hence that $\delta\mathcal{I}_i = \{X : X \subseteq S\}$. Therefore, for i and j distinct non-negative integers, we have, by (1.2.10), that $\mathcal{I}_i = \delta(\delta\mathcal{I}_i) = \delta(\delta\mathcal{I}_j) = \mathcal{I}_j$; a contradiction.//

A natural response to the preceding result is to weaken (1.2.10) or (1.2.11) and seek a function satisfying the modified conditions. Las Vergnas [46, p.69] defines a function δ on S which maps S into S and satisfies (1.2.11) as well as the following weakened form of (1.2.10):

(1.2.13) $\delta\delta\delta I = \delta I$ for all I in S .

Alternatively one can look for a more general class of infinite matroids on which there is a permutation satisfying (1.2.10) and (1.2.11) or some pair of corresponding conditions. Several such classes emerge from the operator approach to infinite matroids. For the moment, however, we look for a suitable class within the independent set framework already established, attempting to modify this framework only minimally.

Formally we seek, for an arbitrary infinite set S, a collection of conditions on independent sets which define on every non-empty subset U of S, a distinguished class $\mathcal{D}_U$ of preindependence spaces so that:

(1.2.14) $\mathcal{D}_U$ includes all independence spaces on U.

(1.2.15) On $\mathcal{D}_U$, (1.2.5) and (1.2.6) give well-defined operations of restriction and contraction such that, if $V \subseteq U$, the restriction or contraction of a member of $\mathcal{D}_U$ is in $\mathcal{D}_V$.

(1.2.16) The function defined by (1.2.9) maps $\mathcal{D}_U$ into $\mathcal{D}_U$ and satisfies (1.2.10) and (1.2.11), where the latter is to hold for all finite subsets T of U.

Assume that for all non-empty subsets U of S such a class $\mathcal{D}_U$ of preindependence spaces on U exists. We now determine some conditions on the members of $\mathcal{D}_U$. Suppose that $I \in \mathcal{D}_U$

and $T \subseteq U$. If $I.T$ is to be well-defined, then $I|(U\backslash T)$ must have a base and moreover, $I.T$ must not depend on which base is chosen for $I|(U\backslash T)$. Thus the following condition holds.

(1.2.17) If B_1 and B_2 are bases of $I|(U\backslash T)$ and $X \subseteq T$, then $B_1 \cup X \in I$ if and only if $B_2 \cup X \in I$.

By (1.2.9), if $I \in \mathcal{D}_U$, then every element of I^* is contained in a base of I^*. That is (U,I^*) satisfies the maximal condition. But, by (1.2.16), $I^* \in \mathcal{D}_U$ and hence $(U,(I^*)^*)$ satisfies the maximal condition. We conclude, using (1.2.10), that every element of $\mathcal{D}_U$ satisfies the maximal condition and hence, by (1.2.15), every restriction of an element of $\mathcal{D}_U$ satisfies the maximal condition.

(1.2.18) Lemma. *Suppose that* $I \in \mathcal{D}_S$ *and* $U \subseteq S$. *Let* B_1 *and* B_2 *be bases of* $I|U$ *and* x *be in* $B_1\backslash B_2$. *Then there is an element* y *of* $B_2\backslash B_1$ *such that* $(B_1\backslash x)\cup y$ *is a base of* $I|U$.

Proof. Clearly B_1 and B_2 are bases of $I|(B_1 \cup B_2)$, and B_2 is a base of $I|((B_1 \cup B_2)\backslash x)$. If $B_1\backslash x$ is a base of $I|((B_1 \cup B_2)\backslash x)$, then by (1.2.17), since $(B_1\backslash x) \cup x \in I$, we have that $B_2 \cup x \in I$; a contradiction. Therefore $B_1\backslash x$ is not a base of $I|((B_1 \cup B_2)\backslash x)$. However, $I|((B_1 \cup B_2)\backslash x)$ satisfies the maximal condition and hence has a base B which properly contains $B_1\backslash x$. This implies that there is an element y of $B_2\backslash B_1$ such that $(B_1\backslash x) \cup y \in I|(B_1 \cup B_2)$. Thus $(B_1\backslash x) \cup y \in I|U$. Now consider the contraction $(I|U).(U\backslash(B_1\backslash x))$ of $I|U$. By (1.2.15), $\{x\}$ is a base of this preindependence space and $\{y\}$ is independent. Hence $\{y\}$ is a base of $(I|U).(U\backslash(B_1\backslash x))$ and so $(B_1\backslash x) \cup y$ is a base of $I|U$, as required. //

To summarize, the following are necessary conditions for $\mathcal{I}$ to belong to $\mathcal{D}_S$.

(1.2.19) $(S,\mathcal{I})$ is a preindependence space.

(1.2.20) If $T \subseteq S$, then $\mathcal{I}|T$ has the maximal condition.

(1.2.21) If $T \subseteq S$ and B_1 and B_2 are bases of $\mathcal{I}|T$, then for each element x of $B_1 \backslash B_2$ there is an element y of $B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y$ is a base of $\mathcal{I}|T$.

Section 5 shows that these conditions are sufficient to define a class of infinite matroids for which (1.2.14) - (1.2.16) are satisfied.

3. Closure operators and infinite matroids.

This section recalls some definitions and basic results from the papers of Klee [44] and Higgs [39,40,41] on infinite matroids. As far as possible the terminology will follow [44].

Let S be a non-empty set and f be a function from 2^S, the power set of S, into 2^S. Then f is an *operator* on S provided that the following conditions are satisfied for all $X \subseteq Y \subseteq S$.

(1.3.1) (Enlarging.) $X \subseteq f(X)$.

(1.3.2) (Isotonic.) $f(X) \subseteq f(Y)$.

Clearly the closure operator of a (finite) matroid is an operator in this sense; so is the closure operator of a topological space.

If f is an operator on S and f^* is defined by

$$f^*(X) = X \cup \{x : x \notin f(S\backslash(X \cup x))\} \quad \text{for all } X \subseteq S,$$

then f^* is an operator on S called the *dual* of f. Thus duality is built into this operator approach right from the beginning. Notice that

(1.3.3) $(f^*)^* = f$.

Moreover, if f is the closure operator of a finite matroid, then f^* is the closure operator of the dual matroid.

Again suppose that f is an operator on S. If $T \subseteq S$, then the *restriction*, f_T, of f to T is defined by

$f_T(X) = f(X) \cap T$ for all $X \subseteq T$.

The *contraction*, f^T, of f to T is defined, as for finite matroids, by

$$f^T(X) = f(X \cup (S\backslash T)) \cap T \text{ for all } X \subseteq T .$$

Clearly both f_T and f^T are operators on T. Moreover, the following familiar relationship holds (see [41, p.246]):

(1.3.4) $(f^*)_T = (f^T)^*$.

Note that (1.3.3) and (1.3.4) are the operator forms of (1.2.10) and (1.2.11) respectively.

An operator f on a set S is called *weakly idempotent (wI), weakly exchanging (wE), finitary (C_F) or cofinitary (H_F)* provided that the following conditions are satisfied for all elements p,x of S and all subsets Y of S.

(wI) If $x \in f(Y)$, then $f(x \cup Y) = f(Y)$.

(wE) If $p \in f(Y)$ and $p \notin f(Y \backslash x)$, then $x \in f(p \cup (Y \backslash x))$.

(C_F) If $p \in f(Y)$, then there is a finite subset U of Y such that $p \in f(U)$.

(H_F) If $p \notin f(Y)$, then there is a cofinite subset V of S such that $V \supseteq Y$ and $p \notin f(V)$.

Note that f satisfies *(wI)* if and only if f^* satisfies *(wE)*; f satisfies *(C_F)* if and only if f^* satisfies *(H_F)* . An operator satisfying *(wI)* and *(wE)* will be called simply a *wIwE*- operator. Similar abbreviations will be used for other types of operators.

Let f be an operator on S. A subset X of S is *independent (discrete* [39]) if $x \notin f(X \backslash x)$ for all x in X; otherwise X is *dependent*. The set X is *spanning (dense* [39]) or *non-spanning* according as $f(X) = S$ or $f(X) \subsetneq S$, and X is a *base* if X is both independent and spanning. A minimal dependent set is a *circuit* and a maximal non-spanning set is a *hyperplane*. To avoid ambiguity the above terms will sometimes be prefixed by the symbol for the relevant operator.

When the terminology of finite matroid theory is extended

as above, many familiar properties are preserved though some are not (see [44] and [86, Chapter 20]). The next result gives a well-known link between independence spaces and $wIwEC_F$- operators (see, for example, [86, 20.5.12]).

(1.3.5) Theorem. *Let* (S,I) *be an independence space. For all subsets* X *of* S, *let*

$$\sigma_I(X) = X \cup \{x : \textit{for some } I \textit{ in } I,\ I \subseteq X \textit{ and } I \cup x \notin I\}.$$

Then σ_I *is the unique* $wIwEC_F$*-operator on* S *having* I *as its collection of independent sets. Conversely, the collection of independent sets of a* $wIwEC_F$*-operator is an independence space.//*

Klee [44] has defined three other dual pairs of conditions on an operator f which involve modifying the conditions stated earlier.

(vwI) If X is finite, Y is independent and $X \subseteq f(Y)$, then $f(X \cup Y) = f(Y)$.

(vwE) If X is finite, $p \cup Y$ is spanning, $p \in f(Y)$ and $p \notin f(Y \setminus X)$, then $x \in f(p \cup (Y \setminus x))$ for some x in X.

(I) If $X \subseteq f(Y)$, then $f(X \cup Y) = f(Y)$.

(E) If $p \in f(Y)$ and $p \notin f(Y \setminus X)$, then $x \in f(p \cup (Y \setminus x))$ for some x in X.

(C) If $p \in f(Y)$, there is a minimal subset U of Y for which $p \in f(U)$ and U is independent.

(H) If $p \notin f(Y)$, there is a maximal subset V of S for which $V \supseteq Y$, $p \notin f(V)$ and $p \cup V$ is spanning.

Note that *(I)* is equivalent to the condition that $f(f(Y)) = f(Y)$.

The next result relates the ten conditions on operators defined above.

(1.3.6) Theorem (Klee [44, p.140]).

$I \Rightarrow wI \Rightarrow vwI$; $E \Rightarrow wE \Rightarrow vwE$;

$(wI \text{ and } H) \Rightarrow I$; $(wE \text{ and } C) \Rightarrow E$;

$(vwI \text{ and } C_F) \Rightarrow (I \text{ and } C)$; $(vwE \text{ and } H_F) \Rightarrow (E \text{ and } H)$;

$(wI \text{ and } H_F) \Rightarrow C$; $(wE \text{ and } C_F) \Rightarrow H$. //

An operator f on a finite set S satisfies *(vwI)* and *(vwE)* if and only if f is the closure operator of a (finite) matroid on S. Indeed Klee [44, p.141] uses the term "matroid" for a system (S,f) where f is a *vwIvwE*-operator on an arbitrary set S. We shall follow Klee in that the operators we consider will usually satisfy at least *(vwI)* and *(vwE)*.

The dual of a *vwIvwE*-operator is also a *vwIvwE*-operator and *(vwI)* is preserved under restriction. However, as the following example shows, *(vwE)* need not be preserved under restriction even in the presence of *(I)* .

(1.3.7) Example. Let $S = Z^+$ and define $f : 2^S \to 2^S$ by

$$f(X) = \begin{cases} X, & \text{for } X \text{ finite and } \{2,3\} \not\subseteq X ; \\ X \cup \{1\}, & \text{for } X \text{ finite and } \{2,3\} \subseteq X ; \\ S, & \text{for } X \text{ infinite.} \end{cases}$$

Clearly f is an operator and, since the f-spanning sets are precisely all infinite subsets of S, it follows that f satisfies *(vwE)* . Furthermore f satisfies *(I)* and hence *(vwI)* .

Now let $T = \{1,2,3\}$ and consider f_T. Put $Y = \{2,3\}$, $p = 1$ and $X = \{3\}$. Then $p \cup Y$ is f_T-spanning, $p \in f_T(Y)$ and $p \notin f_T(Y \backslash X)$. Yet $x \notin f(p \cup (Y \backslash x))$ for the only element x of X. That is, f_T does not satisfy *(vwE)* . //

It is not difficult to check that *(wI)*, *(wE)*, *(I)*, *(E)*, (C_F) and (H_F) are preserved under both restriction and contraction. (For the first four, see [41, p.247].) From this we get that the following classes of operators are closed under restriction, contraction and duality:

(i) *wIwE* -operators;

(ii) *IE* -operators;

(iii) IEC_FH_F -operators.

If we confine attention to operators satisfying some subset of Klee's ten properties and certainly at least *(vwI)* and *(vwE)*, then the above list includes all classes of such operators *known* to be closed under the three basic operations of restriction, contraction and duality. Two further classes, namely the classes of

(iv) *IECH* -operators; and

(v) *vwIvwECH* -operators

are closed under duality. It is an unsolved problem of Higgs [39, p.220] to determine whether a restriction (or contraction) of an *IECH* -operator is also an *IECH* -operator. We have also been unable to answer the same question for *vwIvwECH* -operators. A consequence of Theorem 1.3.6 is that (i)-(v) includes all classes of operators with the required properties.

The classes of *wIwE* -operators and *IE* -operators will be looked at in the next section. We note here that IEC_FH_F -operators can easily be characterized using Theorem 1.3.5 and the following result.

(1.3.8) Theorem (Mason [48, pp.258-259]). *If* (S,I) *is an independence space, then* (S,I^*) *is an independence space if and only if* (S,I) *is a direct sum of finite (connected) matroids.* //

A *B-matroid* (S,f) is an I-operator f on a non-empty set S such that for all $T \subseteq S$, if X is an independent subset of T, then there is a base of T containing X.

The following is a summary of some fundamental properties of B-matroids which will be used later.

(1.3.9) Theorem (Higgs [39, Propositions (12),(13),(16) and (9)])

(i) *The dual of a B-matroid is a B-matroid.*

(ii) *A restriction or contraction of a B-matroid is a B-matroid.*

(iii) *If* (S,f) *is a B-matroid, then* f *is an IECH-operator on* S.

(iv) *If* f *is a* $wIwEC_F$*-operator on* S, *then* (S,f) *is a B-matroid.* //

A problem of Higgs [39, P668] which still seems to be open is whether the converse of (1.3.9)(iii) is true.

4. Operators and preindependence spaces.

In this section those families of sets which can occur as the collection of independent sets of a *wIwE*-operator are characterized. It is also shown that, unlike independence spaces, preindependence spaces cannot be described in the operator framework established earlier.

Welsh [86, Example 20.5.4] notes that a *wIwE*-operator is not uniquely determined by its collection of independent sets. In fact, as the following example shows, an *IE*-operator is not even uniquely determined by the pair consisting of its collection of independent sets and its collection of spanning sets.

(1.4.1) Example. Let S = Z. Then S is the disjoint union of S_1, the set of non-negative integers, and S_2, the set of negative integers. Define $f_1 : 2^S \to 2^S$ and $f_2 : 2^S \to 2^S$ by

$$f_1(X) = \begin{cases} X, & \text{if } X \text{ is finite;} \\ (X \cap S_1) \cup S_2, & \text{if } X \cap S_1 \text{ is finite, } X \cap S_2 \text{ is infinite;} \\ S_1 \cup (X \cap S_2), & \text{if } X \cap S_1 \text{ is infinite, } X \cap S_2 \text{ is finite;} \\ S, & \text{if both } X \cap S_1 \text{ and } X \cap S_2 \text{ are infinite,} \end{cases}$$

and

$$f_2(X) = \begin{cases} (X \cap S_1) \cup \{0\} \cup S_2, & \text{if } X \cap S_1 \text{ is finite, } X \cap S_2 \text{ is infinite;} \\ f_1(X), & \text{otherwise.} \end{cases}$$

Both f_1 and f_2 are *IE*-operators on S. Moreover, each has the set of finite subsets of S as its collection of independent sets, and $\{X : X \cap S_1, X \cap S_2 \text{ are infinite}\}$ as its collection of spanning sets. //

Let *I* be the collection of independent sets of a

vwIwE-operator f on a set S. Then

(1.4.2) (S,$\mathcal{I}$) *is a preindependence space.*

This is an easy consequence of the fact that if T⊂⊂S, then f_T is a *vwIwE*-operator on T.

The next two lemmas generalize familiar finite results and their proofs are quite straightforward.

(1.4.3) Lemma. *Suppose that* f *is a wE-operator on a set* S. *If* I *is an* f*-independent subset of* S *and* $x \in S\backslash I$, *then* $x \in f(I)$ *if and only if* $I \cup x$ *is* f*-dependent.* //

(1.4.4) Lemma. *Let* f *be a vwIwE-operator on* S *and* B *be a maximal cofinite* f*-independent subset of a subset* Y *of* S. *Then* f(B) = f(Y). //

If $\mathcal{J}$ is a collection of subsets of a set T, a $\mathcal{J}$ subset of T is a subset of T which is in $\mathcal{J}$. By Lemmas 1.4.3 and 1.4.4, the collection $\mathcal{I}$ of independent sets of a *vwIwE*-operator on S satisfies the following condition.

(1.4.5) *Suppose that* $Y \subseteq S$ *and let* B *be a maximal cofinite* $\mathcal{I}$ *subset of* Y *and* I *be an* $\mathcal{I}$ *subset of* Y. *If* $x \in S\backslash Y$ *and* $B \cup x \in \mathcal{I}$, *then* $I \cup x \in \mathcal{I}$.

Moreover, we have the following result.

(1.4.6) Theorem. *Let* $\mathcal{I}$ *be a collection of subsets of a set* S. *Then* $\mathcal{I}$ *is the collection of independent sets of some vwIwE-operator on* S *if and only if* $\mathcal{I}$ *satisfies* (1.4.2) *and* (1.4.5).

Proof. From above we need only check the sufficiency of (1.4.2) and (1.4.5). Let (S,$\mathcal{I}$) be a preindependence space satisfying (1.4.5) and define $g : 2^S \to 2^S$ by

$$g(X) = \begin{cases} X \cup \{x : X \cup x \notin \mathcal{I}\}, & \text{if } X \in \mathcal{I}; \\ g(I_X), & \text{if X has a maximal cofinite } \mathcal{I} \text{ subset } I_X; \\ S, & \text{otherwise.} \end{cases}$$

To check that g is well-defined, suppose that $X \subseteq S$ and I_X^1 and I_X^2 are maximal cofinite $\mathcal{I}$ subsets of X. Then clearly $X \subseteq g(I_X^i)$ for i = 1,2. Now, if $x \in S\backslash X$, then by (1.4.5), $I_X^1 \cup x \notin \mathcal{I}$ if and only if $I_X^2 \cup x \notin \mathcal{I}$. That is, $x \in g(I_X^1)$ if and only if $x \in g(I_X^2)$.

Next we show that g is an operator. Clearly g is enlarging. Now suppose that $X \subseteq Y \subseteq S$, then if Y has no maximal cofinite $\mathcal{I}$ subset, $S = g(Y) \supseteq g(X)$. Alternatively, if $g(Y) = g(I_Y)$ for some maximal cofinite $\mathcal{I}$ subset I_Y of Y, then $g(X) = g(I_X)$ for some maximal cofinite $\mathcal{I}$ subset I_X of X. If $x \in g(X)\backslash Y$, then $I_X \cup x \notin \mathcal{I}$, so by (1.4.5), $I_Y \cup x \notin \mathcal{I}$ and hence $x \in g(I_Y)$. Thus $g(X) \subseteq g(Y)$.

We now prove that g satisfies $\cdot(vwI)$ by showing that g satisfies (wI). Suppose that $Y \in \mathcal{I}$ and $x \in g(Y)$. Then Y is a maximal cofinite $\mathcal{I}$ subset of $Y \cup x$ and so $g(Y \cup x) = g(Y)$. If $Y \notin \mathcal{I}$, $x \in g(Y)$ and Y has a maximal cofinite $\mathcal{I}$ subset, then a similar argument gives that $g(Y \cup x) = g(Y)$.

To check that f satisfies (wE), suppose $x \in g(Y \cup p)$ and $x \notin g(Y)$. Then $g(Y) = g(I_Y)$ where I_Y is a maximal cofinite $\mathcal{I}$ subset of Y. Since $g(Y \cup p) \neq g(Y)$, the set $I_Y \cup p$ is a maximal cofinite $\mathcal{I}$ subset of $Y \cup p$. Thus $g(Y \cup p) = g(I_Y \cup p)$. Now $x \notin g(I_Y)$, hence $I_Y \cup x \in \mathcal{I}$. Moreover, $(I_Y \cup p) \cup x \notin \mathcal{I}$, thus $p \in g(I_Y \cup x) = g(Y \cup x)$. That is, (wE) is satisfied.

It is straightforward to check that $\mathcal{I}$ is precisely the collection of g-independent sets and so this is omitted.//

Note that by Lemmas 1.4.3 and 1.4.4, any $vwIwE$-operator having $\mathcal{I}$ as its collection of independent sets must coincide

with g on sets containing maximal cofinite *I* subsets. It follows that g is maximal among such *vwIwE*-operators in the sense that if h is another such operator, then $h(X) \subseteq g(X)$ for all $X \subseteq S$.

In the proof of the preceding theorem, we showed that g satisfies *(wI)*. Thus:

(1.4.7) Corollary. *If I is a collection of subsets of a set* S, *then I is the collection of independent sets of a wIwE-operator on* S *if and only if I satisfies* (1.4.2) *and* (1.4.5).//

In Theorem 1.4.6, *(wE)* cannot be replaced by *(vwE)* for, as the following example shows, the collection of independent sets of a *vwIvwE*-operator need not be a preindependence space.

(1.4.8) Example. Let S be an infinite set and A be a finite subset of S having at least two elements. Define $f : 2^S \to 2^S$ by

$$f(X) = \begin{cases} X, & \text{if } X \subseteq A \text{ or } X \subset\subset S\backslash A; \\ A \cup (X \cap (S\backslash A)), & \text{if } X \cap A \neq \emptyset \text{ and } \emptyset \neq X \cap (S\backslash A) \subset\subset S\backslash A; \\ S, & \text{if } X \text{ is infinite.} \end{cases}$$

Clearly f is an operator on S.

The f-spanning sets are precisely all infinite subsets of S. Therefore f satisfies *(vwE)*. To see that f satisfies *(vwI)* observe that f satisfies the stronger condition *(I)*. Hence f is an *IvwE*-operator.

The collection of f-independent sets contains A as well as all finite subsets of S\A. Moreover, A is a maximal f-independent set. Thus the collection of f-independent sets is not a preindependence space. //

If *I* is the collection of independent sets of an *IE*-operator on a set S, then *I* satisfies (1.4.2) and the strengthening of (1.4.5) obtained by omitting the word "cofinite"

However I have been unable to prove an analogue of Theorem 1.4.6 for *IE*-operators.

Welsh [85, Problem 20.5.3] asks whether preindependence spaces can be described in operator terminology. Let P be the set of conditions on operators stated earlier. That is, P = {*(vwI), (wI), (I), (vwE), (wE), (E), (C), (C_F), (H), (H_F)*}. If Q $\subseteq$ P, an operator satisfying all of the conditions in Q will be called simply a Q-operator. Preindependence spaces cannot be described in terms of P. That is:

(1.4.9) Theorem. *There is no subset* K *of* P *for which both of the following statements are true.*

(i) The collection of independent sets of a K*-operator on an arbitrary set* S *is a preindependence space on* S.

(ii) If I *is a preindependence space on an arbitrary set* S*, then there is some* K*-operator on* S *having* I *as its collection of independent sets.*

Proof. Assume that there is a subset K of P for which both (i) and (ii) hold. We shall give examples of an $ECHC_FH_F$-operator and an $ICHC_FH_F$-operator whose collections of independent sets do not form preindependence spaces. From these examples, it follows that K $\not\subseteq$ {*(E),(C),(C_F),(H),(H_F)*} and K $\not\subseteq$ {*(I),(C),(C_F),(H),(H_F)*}. Therefore K $\cap$ {*(vwI),(wI),(I)*} $\neq \emptyset$ and K $\cap$ {*(vwE),(wE),(E)*} $\neq \emptyset$, and hence by Theorem 1.3.6, {*(vwI),(vwE)*} $\subseteq$ K.

(1.4.10) Example. Let S = {1,2,3} and define $f_1 : 2^S \to 2^S$ by $f_1(\emptyset) = \emptyset$, $f_1(1) = \{1,3\}$, $f_1(2) = \{2,3\}$, $f_1(3) = \{1,2,3\}$, and $f_1(X) = \{1,2,3\}$ for $|X| \geq 2$. Then it is easily checked that f_1 is an $ECHC_FH_F$-operator on S. Its collection I of independent sets is $\{\emptyset, \{1\},\{2\},\{3\},\{1,2\}\}$. Clearly (S,I) is not a preindependence space.//

(1.4.11) Example. Again let S = {1,2,3} and define $f_2 : 2^S \to 2^S$ by

$$f_2(X) = \begin{cases} X, \text{ if } |X| < 2 \text{ and } 3 \notin X ; \\ S, \text{ otherwise.} \end{cases}$$

Then it is easily checked that f_2 is an $ICHC_FH_F$-operator on S having the same collection of independent sets as f_1.//

The following example completes the proof by displaying a preindependence space I for which there is no $vwIvwE$-operator having the same collection of independent sets. Since $K \supseteq \{(vwI),(vwE)\}$, it follows that there is no K-operator having I as its collection of independent sets. This contradicts (1.4.9)(ii).

(1.4.12) Example. Let $S = Z^+$ and if $n_1,n_2,\ldots,n_j$ are distinct elements of S, then let $B_{n_1,n_2,\ldots,n_j}$ denote the set $S\setminus\{n_1,n_2,\ldots,n_j\}$. In addition, let

$$\mathcal{B} = \{B_{1,2,3},B_{1,2,4},B_{1,3,4},B_{2,3,4},B_{1,n},B_{2,n},B_{3,n},B_{4,n} \text{ (for all } n \geq 5)\}$$

and

$$I = \{X : X \subset\subset S \text{ or } X \subseteq B \text{ for some } B \text{ in } \mathcal{B}\}.$$

Clearly (S,I) is a preindependence space.

Now suppose that there is a $vwIvwE$-operator f on S having I as its collection of independent sets. Then f^* is a $vwIvwE$-operator on S and its set of spanning sets is

$\{X : S\setminus X \subset\subset S\}$ $\cup$

$\{X : X \supseteq \{1,2,3\},\{1,2,4\},\{1,3,4\} \text{ or } \{2,3,4\}\}$ $\cup$

$\{X : X \supseteq \{1,n\},\{2,n\},\{3,n\} \text{ or } \{4,n\} \text{ for some } n \geq 5\}$.

Let $\{a,b\}$ be a two-element subset of $\{1,2,3,4\}$. Then $\{a,b\}$ is not f^*-spanning. Thus, either there is an element z of $S\setminus\{1,2,3,4\}$ such that $z \notin f^*(\{a,b\})$, or not. In the first

case, since {b,z} is spanning, $a \in f^*(\{b,z\})$. Thus, $a \in f^*(b)$, as otherwise, by *(vwE)*, $z \in f^*(\{a,b\})$; a contradiction. Similarly since {a,z} is spanning, $b \in f^*(a)$. In the second case, there is an element z of $S \setminus \{1,2,3,4\}$ such that $z \in f^*(\{a,b\})$. It follows that {a,b} is not independent, since otherwise, by *(vwI)*, $f^*(\{a,b,z\}) = f^*(\{a,b\})$, which implies that {a,b} is spanning; a contradiction. Therefore $a \in f^*(b)$ or $b \in f^*(a)$. We conclude that for all two-element subsets {a,b} of {1,2,3,4}, either $a \in f^*(b)$ or $b \in f^*(a)$. Hence there are at least six ordered pairs (x,y) such that x and y are distinct elements of {1,2,3,4} and $x \in f^*(y)$.

Next let a,b and c be distinct elements of {1,2,3,4} and suppose that $a \in f^*(c)$ and $b \in f^*(c)$. If {c} is independent, then, by *(vwI)*, $f^*(\{a,b,c\}) = f^*(c)$ and so {c} is spanning; a contradiction. On the other hand, if {c} is dependent, then $c \in f^*(\emptyset)$ and so $f^*(c) = f^*(\emptyset)$. It follows that $\{a,b,c\} \subseteq f^*(\emptyset)$, and hence by *(vwI)*, $f^*(\{a,b,c\}) = f^*(\emptyset)$. Thus $\emptyset$ is spanning; a contradiction. Therefore, if $\{a,b,c,d\} = \{1,2,3,4\}$, at most one of the statements $a \in f^*(d)$, $b \in f^*(d)$ and $c \in f^*(d)$ is true. Hence there are at most four ordered pairs (x,y) such that x and y are distinct elements of {1,2,3,4} and $x \in f^*(y)$. However, it was shown earlier that there are at least six such pairs.

We conclude that there is no *vwIvwE*-operator on S having *I* as its collection of independent sets.//

This completes the proof of Theorem 1.4.9.//

Let $A = (A_i : i \in I)$ be a family of subsets of a non-empty

set S and let $\tau(A)$ be the collection of partial transversals of A. Then, as is well-known, $(S,\tau(A))$ is a preindependence space (see, for example, [52, Theorem 6.5.2]). Moreover:

(1.4.13) Lemma. (Brualdi and Scrimger [10, pp.247-248]). $(S,\tau(A))$ *is an exchange system satisfying* (1.2.8).//

We now show how $(S,\tau(A))$ may be related to the operator framework set up earlier.

(1.4.14) Theorem. *If* $A = (A_i : i \in I)$ *is a family of subsets of a set* S, *then there is some wIwE-operator on* S *having* $\tau(A)$ *as its collection of independent sets.*

Proof. It will be shown that $(S,\tau(A))$ satisfies (1.4.2) and (1.4.5). The required result then follows by Corollary 1.4.7.

As noted above, $(S,\tau(A))$ satisfies (1.4.2). Now suppose that $Y \subseteq S$ and let B be a cofinite base of Y and I be an independent subset of Y. Then since $I\backslash B$ is finite, we have on applying (1.2.8) to $\tau(A)|Y$ finitely many times, that there is a subset I' of $B\backslash I$ such that $|I'| = |I\backslash B|$ and $(B\backslash I')\cup(I\backslash B)$ is a base of Y. Let $B' = (B\backslash I')\cup(I\backslash B)$. Clearly $B' \supseteq I$.

For x in $S\backslash Y$, if $B \cup x \in \tau(A)$ and $I \cup x \notin \tau(A)$, then $B' \cup x \notin \tau(A)$. Hence $B \cup x$ and B' are bases of $Y \cup x$. Applying (1.2.8) again gives that there is an element y of $B'\backslash(B\cup x)$ such that $((B\cup x)\backslash x)\cup y$ is a base of $Y \cup x$. But $B \subsetneq B \cup y \subseteq Y$ and this contradiction establishes that $(S,\tau(A))$ satisfies (1.4.5).//

5. *B*-matroids.

In this section we characterize those families of sets which can occur as the collection of independent sets of a *B*-matroid and note that a *B*-matroid is uniquely determined by this collection. In addition we show that the class of *B*-matroids is the unique class of infinite matroids satisfying (1.2.14) - (1.2.16).

From [40, p.861], if (S,f) is a *B*-matroid, its collection $\mathcal{B}$ of bases satisfies the conditions:

(1.5.1) $\mathcal{B}$ is a clutter; and

(1.5.2) if $B_1, B_2 \in \mathcal{B}$ and $A \subseteq C \subseteq S$ where $A \subseteq B_1$ and $B_2 \subseteq C$, then there is an element B of $\mathcal{B}$ such that $A \subseteq B \subseteq C$.

Higgs [40, p.861] has asked the following question which we shall answer.

(1.5.3) If $\mathcal{B} \neq \emptyset$ and $\mathcal{B}$ satisfies (1.5.1) and (1.5.2), then does there exist a *B*-matroid having $\mathcal{B}$ as its collection of bases?

The next example shows that such a *B*-matroid need not exist.

(1.5.4) Example. Let $S = Z\backslash\{0\}$, the set of non-zero integers. Suppose that $\mathcal{B}$ is the collection of subsets of S consisting of Z^+ together with all sets of the form $(Z^+\backslash\{i_1,i_2,\ldots,i_n\})\cup\{-i_1,-i_2,\ldots,-i_n\}$ where $i_1,i_2,\ldots,i_n$ are n distinct elements of Z^+ and n takes all the values 1,2,3,... . Clearly $\mathcal{B}$ is a clutter.

Suppose that A is contained in an element of $\mathcal{B}$, C contains an element of $\mathcal{B}$ and $A \subseteq C$. Every element of $\mathcal{B}$ contains only finitely many elements of Z^- and all but finitely many elements of Z^+. Moreover, if $x \in Z^+ \cap (S\backslash C)$, then $-x \in C$, and if

$y \in Z^- \cap A$, then $-y \notin A$. Thus suppose that

$Z^- \cap A = \{-j_1,-j_2,\ldots,-j_n\}$ and that

$Z^+ \cap (S\backslash C) = \{j_{r_1},j_{r_2},\ldots,j_{r_s}, u_1,u_2,\ldots,u_t\}$ where

$\{r_1,r_2,\ldots,r_s\} \subseteq \{1,2,\ldots,n\}$ and $\{j_1,\ldots,j_n\} \cap \{u_1,\ldots,u_t\} = \emptyset$. Then

$$C \supseteq (Z^+\backslash\{j_1,j_2,\ldots,j_n, u_1,u_2,\ldots,u_t\})$$

$$\cup \{-j_1,-j_2,\ldots,-j_n, -u_1,-u_2,\ldots,-u_t\} \supseteq A$$

and hence $\mathcal{B}$ satisfies (1.5.2).

Now suppose that there is a *B*-matroid (S,f) having $\mathcal{B}$ as its set of bases. Then the collection $\mathcal{I}$ of independent sets of (S,f) is given by $\mathcal{I} = \{X : X \subseteq B \in \mathcal{B}\}$. Consider (Z^-,f_{Z^-}). By (1.3.9)(ii) this is a *B*-matroid. Its collection of independent sets is

$$\mathcal{I}|Z^- = \{X : X \in \mathcal{I}, X \subseteq Z^-\} = \{X : X \subset\subset Z^-\}.$$

Clearly $\mathcal{I}|Z^-$ does not satisfy the maximal condition and this contradicts the definition of a *B*-matroid.//

If f is an *I*-operator on a set S and B is a base of a subset Y of S, then f(B) = f(Y). Thus, in contrast to *wIwE*-operators, *B*-matroids are uniquely determined by their collections of independent sets. The next result should be compared with Corollary 1.4.7.

(1.5.5) Theorem. *The collection $\mathcal{I}$ of independent sets of a B-matroid on a set* S *satisfies the following conditions.*

(i) (S,$\mathcal{I}$) *is a preindependence space.*

(ii) *Every restriction of $\mathcal{I}$ satisfies the maximal condition.*

(iii) *Suppose that* $Y \subseteq S$ *and let* B *be a maximal $\mathcal{I}$ subset of* Y *and* I *be an $\mathcal{I}$ subset of* Y. *If* $x \in S\backslash Y$ *and* $B \cup x \in \mathcal{I}$, *then* $I \cup x \in \mathcal{I}$.

Conversely, if $\mathcal{I}$ is a collection of subsets of S *satisfying*

(i) - (iii), *then there is a unique B-matroid on* S *having* $\mathcal{I}$ *as its collection of independent sets.*

Proof. The first part is straightforward. For the second part the required I-operator is defined by

$$f(X) = \begin{cases} X \cup \{x : X \cup x \notin \mathcal{I}\}, & \text{if } X \in \mathcal{I}; \\ f(I_X), & \text{if } I_X \text{ is a maximal } \mathcal{I} \text{ subset of } X. \end{cases} //$$

The next result characterizes B-matroids in terms of their collections of bases. Notice the similarity to the corresponding result for finite matroids (see, for example, Welsh [86, p.14]). If A is a collection of subsets of a set S and $X \subseteq S$, then define A(X) to be the collection of maximal sets of the form $A \cap X$ where $A \in A$.

(1.5.6) Theorem. *A collection* B *of subsets of a set* S *is the set of bases of a B-matroid on* S *if and only if* B *satisfies the following conditions.*

(i) $\mathcal{B} \neq \emptyset$.

(ii) *If* $Y \subseteq X \subseteq S$ *and* $Y \subseteq B$ *for some* B *in* $\mathcal{B}$, *then* $Y \subseteq B_X$ *for some* B_X *in* $\mathcal{B}(X)$.

(iii) *Suppose that* $X \subseteq S$ *and* $B_1, B_2 \in \mathcal{B}(X)$. *If* $x \in B_1 \setminus B_2$, *then there is an element* y *of* $B_2 \setminus B_1$ *such that* $(B_1 \setminus x) \cup y \in \mathcal{B}(X)$.

Proof. The necessity of (i) and (ii) is clear. For (iii), use Theorem 1.5.5. For the converse, let $\mathcal{B}$ be a collection of subsets of S satisfying (i) - (iii) and let $\mathcal{I} = \{X : X \subseteq B \in \mathcal{B}\}$. Then, from (i) – (iii), $\mathcal{I}$ is a preindependence space. Moreover, by (ii), every restriction of $\mathcal{I}$ satisfies the maximal condition. To check that $\mathcal{I}$ satisfies (1.5.5)(iii), suppose that B is a maximal $\mathcal{I}$ subset of a subset Y of S. Let I be an $\mathcal{I}$ subset of Y and x be an element of $S \setminus Y$ such that $B \cup x \in \mathcal{I}$. If $I \cup x \notin \mathcal{I}$, then $I \subseteq B'$ where

B' is a maximal I subset of $Y \cup x$ not containing x. Clearly $B \cup x$ is a maximal I subset of $Y \cup x$, hence by (iii), since $x \in (B \cup x) \setminus B'$, there is an element y of $B' \setminus (B \cup x)$ such that $B \cup y$ is a maximal I subset of $Y \cup x$. But $B \cup y \subseteq Y$ and hence the choice of B is contradicted. We conclude that I satisfies (1.5.5)(i)-(iii) and the result follows by Theorem 1.5.5.//

Combining Theorems 1.5.5 and 1.5.6 we get the following:

(1.5.7) Corollary. *A collection I of subsets of a set* S *is the set of independent sets of a B-matroid on* S *if and only if I satisfies* (1.2.19) - (1.2.21).//

We now use this corollary to prove the main result of this section.

(1.5.8) Theorem. *If* S *is an infinite set, then the unique class of preindependence spaces defined on* S *and all its non-empty subsets such that* (1.2.14) - (1.2.16) *are satisfied is the class of B-matroids.*

Proof. Higgs [39, Proposition (11)] has proved that (1.2.6) gives a well-defined operation for *B*-matroids which agrees with the operator definition of contraction. Combining this with Theorem 1.3.9 implies that if, for all non-empty subsets U of S, we take $\mathcal{D}_U$ to be the class of *B*-matroids defined on U, then $\mathcal{D}_U$ satisfies (1.2.14) - (1.2.16). But in section 2 we showed that if for every non-empty subset U of S, $\mathcal{D}_U$ is an arbitrary class of preindependence spaces on U satisfying (1.2.14)-(1.2.16), then every member of $\mathcal{D}_U$ satisfies (1.2.19)-(1.2.21). The result now follows immediately by Corollary 1.5.7.//

Notice that it follows from this proof that there is some redundancy in the set of conditions (1.2.14) - (1.2.16). Indeed (1.2.14) and the fact that δ satisfies (1.2.11) are consequences of the other conditions on $\mathcal{D}_U$.

6. Infinite graphs and bicircular matroids.

In the preceding section it was shown that *B*-matroids are a class of preindependence spaces having many familiar properties of independence spaces. Higgs [41, p.252] has given an excluded subgraph characterization for when a certain infinite generalization of the cycle matroid of a finite graph is a *B*-matroid. In this section we shall prove that a similar generalization of the finite bicircular matroid is always a *B*-matroid.

The following result characterizes *wIEC*-operators in terms of their collections of circuits and should be compared with a similar, more familiar result for independence spaces.

(1.6.1) Theorem (Klee [44, p.143]). *The collection* Γ *of circuits of a wIEC-operator satisfies the following conditions.*

(i) Γ *is a clutter.*

(ii) *Suppose that* $C_1, C_2 \in \Gamma$. *If* $p \in C_1 \backslash C_2$ *and* $q \in C_1 \cap C_2$, *then there is an element* C_3 *of* Γ *such that* $p \in C_3 \subseteq (C_1 \cup C_2) \backslash q$.

Conversely, if Γ *is a collection of subsets of* S *satisfying (i) and (ii), then there is a unique wIEC-operator on* S *having* Γ *as its collection of circuits. This operator is given by*

$$g_\Gamma(X) = X \cup \{x : x \in C \subseteq X \cup x \text{ for some } C \text{ in } \Gamma\}$$

for all $X \subseteq S$. //

If, in the above, every element of Γ is finite, then the collection of g_Γ-independent subsets of S is an independence space on S. Moreover, if σ_I is as defined in Theorem 1.3.5, then $\sigma_I(X) = g_\Gamma(X)$ for all $X \subseteq S$.

Let G be an infinite graph. We may use the preceding theorem to define four *wIEC*-operators on the edge-set E(G) of G. If G_1 and G_2 are graphs, then G_2 is *homeomorphic from* G_1 if G_2

is obtained from G_1 by replacing each loop of G_1 by a (finite) cycle and each non-loop edge of G_1 by a path of finite, non-zero length.

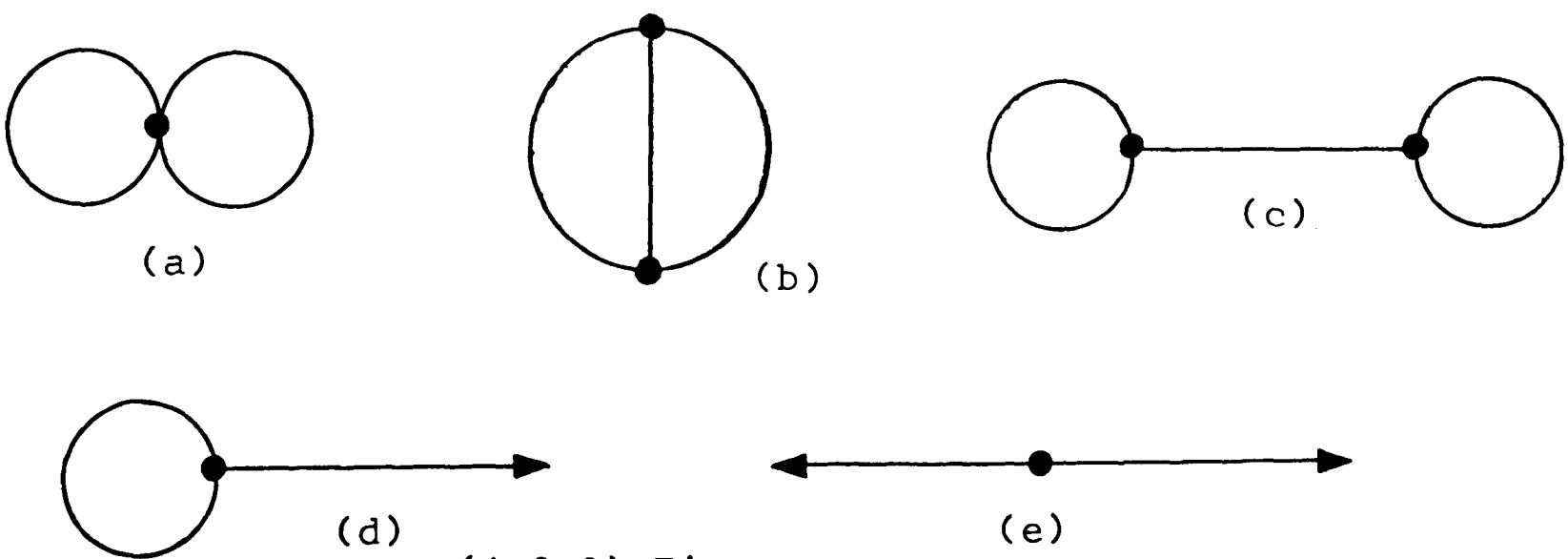
(1.6.2) Example. From the set $M(G)$ of finite cycles of G we obtain a *wIEC*-operator $g_{M(G)}$ on E(G) and, as above, an independence space - the familiar cycle matroid of an infinite graph.//

(1.6.3) Example. From the set of subgraphs of G homeomorphic from one of the graphs (a), (b) or (c) in Figure 1.6.6, we obtain an independence space on E(G). This is an obvious extension of the bicircular matroid of a finite graph (see [49] or [74]).//

The next two examples generalize the first two by admitting infinite circuits.

(1.6.4) Example (see [41] or [44]). If $\mathcal{G}(G)$ consists of all finite cycles of G together with all two-way infinite paths (see Figure 1.6.6(d)), then $g_{\mathcal{G}(G)}$ is a *wIEC*-operator on E(G). Higgs [41] calls this the *two-way path matroid* of G.//

(1.6.5) Example (see [44]). Let $K(G)$ be the collection of subgraphs of G homeomorphic from one of the five graphs shown in Figure 1.6.6 (where an arrow denotes a one-way infinite path). Then $g_{K(G)}$ is a *wIEC*-operator on E(G) and $(E(G), g_{K(G)})$ will be called the *infinite-bicircular matroid* of G.//



(1.6.6) Figure

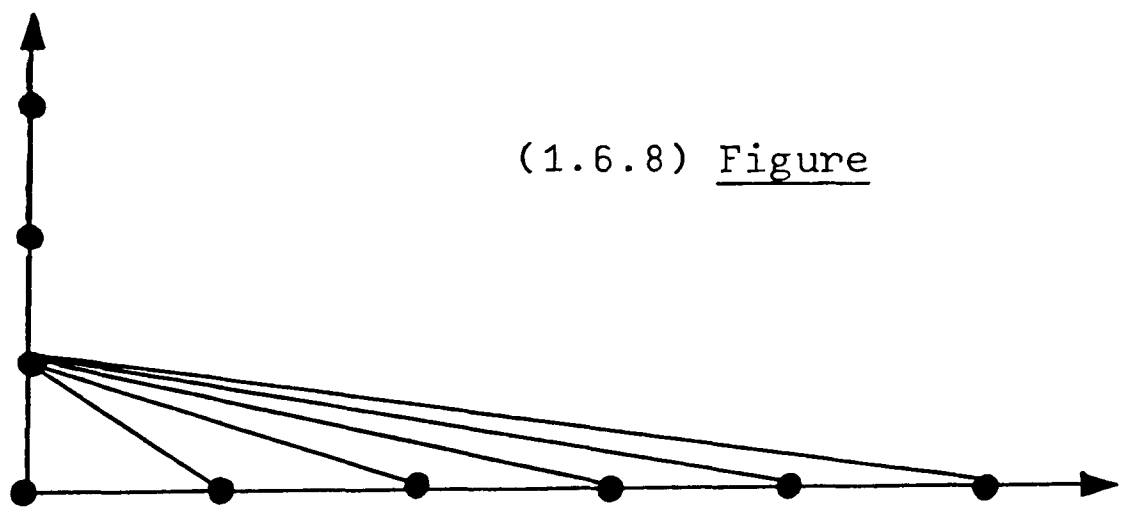
The graph shown in Figure 1.6.8 was introduced by Bean in [1, p.10]. In [41], Higgs calls it the *Bean graph* and his main result is the following:

(1.6.7) Theorem [41, p.252]. *For a graph* G *the following statements are equivalent.*

(i) $g_{G(G)}$ *is an IE-operator.*

(ii) $(E(G), g_{G(G)})$ *is a B-matroid.*

(iii) *No subgraph of* G *is homeomorphic from the Bean graph.//*



(1.6.8) Figure

This theorem prompts the questions as to when $g_{K(G)}$ is an *IE*-operator and when $(E(G), g_{K(G)})$ is a *B*-matroid. The main theorem of this section, the result of joint work with L.R. Matthews, answers these questions and has been published in [50].

(1.6.9) Theorem. *Let* G *be a graph. Then the infinite-bicircular matroid* $(E(G), g_{K(G)})$ *is a B-matroid.*

The proof of this theorem will use two lemmas.

(1.6.10) Lemma. *Let* G *be a graph. Then* $g_{K(G)}$ *is an IE-operator on* E(G).

Proof. We need only show that $g_{K(G)}$ satisfies (*I*). In this proof we shall write g for $g_{K(G)}$ and *K* for *K*(G).

Suppose that $X \subseteq E(G)$ and that $x \in g(g(X))\backslash g(X)$.

Then $x \in C \subseteq g(X) \cup x$ for some C in K. We distinguish two cases: when $C \cap (g(X)\backslash X)$ is finite, and when $C \cap (g(X)\backslash X)$ is infinite. In each case we get a contradiction, thereby showing that $g(g(X))\backslash g(X)$ is empty and hence that $g(g(X)) = g(X)$.

If $C \cap (g(X)\backslash X)$ is finite and $y \in C \cap (g(X)\backslash X)$, then since $y \in g(X)\backslash X$, there is an element C_y of K such that $y \in C_y \subseteq X \cup y$. Now $y \in C \cap C_y$ and $x \in C\backslash C_y$, hence by (1.6.1)(ii), there is an element C' of K such that $x \in C' \subseteq (C \cup C_y)\backslash y$. Since $|C' \cap (g(X)\backslash X)| < |C \cap (g(X)\backslash X)|$, after finitely many steps we get an element $C^{(n)}$ of K such that $x \in C^{(n)} \subseteq X \cup x$. Therefore $x \in g(X)$; a contradiction.

Next suppose that $C \cap (g(X)\backslash X)$ is infinite. Then C is homeomorphic from one of the graphs (d) or (e) in Figure 1.6.6. Let the edge x have endpoints u and v. Then either there is an edge y in $C \cap (g(X)\backslash X)$ and a finite path A_y in $X \cap C$ joining u to an endpoint of y, or there is no such path in $X \cap C$ from u to an edge of $C \cap (g(X)\backslash X)$. In the latter case, $X \cap C$ contains either a one-way infinite path from u, or a cycle joined to u by a path of finite (possibly zero) length. In the former case, since $y \in g(X)\backslash X$, there is an element C'_y of K such that $y \in C'_y \subseteq X \cup y$. But both A_y and $C'_y\backslash y$ are contained in X. A routine check of the five possibilities for C'_y shows that again X contains either a one-way infinite path from u, or a cycle joined to u by a finite path in X.

The argument for u may be repeated for v and then a check of the various possibilities (including the case $u = v$) gives that K contains an element C_x containing x such that $C_x \subseteq X \cup x$. Thus, when $C \cap (g(X)\backslash X)$ is infinite, $x \in g(X)$; the required

contradiction.//

(1.6.11) Lemma (Higgs [41,(5)(i)]). *Let* G *be a graph. Then the set of* $g_{G(G)}$*-independent sets satisfies the maximal condition.//*

We now complete the proof of the main theorem of this section.

Proof of Theorem 1.6.9.

By Lemma 1.6.10, $g_{K(G)}$ is an IE-operator. Since every restriction of $(E(G), g_{K(G)})$ is isomorphic to $(E(F), g_{K(F)})$ for some subgraph F of G, it suffices to show that every independent set of $(E(G), g_{K(G)})$ is contained in a base.

Suppose that A is a $g_{K(G)}$-independent subset of E(G) and let $\{A_j : j \in J\}$ be the set of connected components of the subgraph of G induced by A. Then for each j in J, A_j contains at most one cycle and we let $H = \{j \in J : A_j$ contains a cycle of G$\}$. Modify G to obtain a graph G' as follows. For each h in H, using the Axiom of Choice, select an edge e_h from the cycle contained in A_h; delete the edge e_h and insert a one-way infinite path P_h (of new vertices and edges) from an endpoint of e_h. Let

$$A'_h = (A_h \backslash e_h) \cup P_h \text{ for each h in H, and let}$$

$$A' = \left(\bigcup_{h \in H} A'_h \right) \cup \left(\bigcup_{j \in J \backslash H} A_j \right) .$$

Now apply Lemma 1.6.11 in G'. As A' is a $g_{G(G')}$-independent set, there is a $g_{G(G')}$-base T' containing A'. Let T be the subgraph of G corresponding to T'; that is, T is obtained from T' by deleting the path P_h and inserting the edge e_h for each h in H.

By construction, T is $g_{K(G)}$-independent and moreover, if e is an edge of G joining vertices in distinct components of the

subgraph T, then T ∪ e is $g_{K(G)}$-dependent. Let $\{T_i : i \in I\}$ be the set of components of T and let I' be the collection of members i of I for which T_i contains no finite cycle or one-way infinite path. Now recall that $g_{M(G)}(T_i)$ is the closure of T_i in the cycle matroid of G (see Example 1.6.2). If $i \in I'$ and $g_{M(G)}(T_i)\backslash T_i$ is non-empty, then a single element from $g_{M(G)}(T_i)\backslash T_i$ may be added to T_i without forming a $g_{K(G)}$-circuit. For each such i, using the Axiom of Choice, select such an element e'_i. Finally let

$$B = T \cup \{e'_i : i \in I',\ g_{M(G)}(T_i)\backslash T_i \neq \emptyset\}.$$

It is easy to check that B is $g_{K(G)}$-independent and $g_{K(G)}$-spanning. Hence, as required, B is a $g_{K(G)}$-base containing A.//

7. *B*-matroids and inductive exchange systems.

It is straightforward to show that, in Example 1.5.4, $(S,\mathcal{I})$ is an inductive exchange system (see section 2). However, $(Z^-,\mathcal{I}|Z^-)$ does not satisfy the maximal condition. Thus the restriction of an inductive exchange system need not be inductive. The next theorem shows that *B*-matroids are exactly those exchange systems for which every restriction is inductive. We shall use two lemmas.

(1.7.1) Lemma (Klee [44, p.142]). *Let* f *be a wE-operator. If* $p \in f(Y)$ *and* U *is a minimal subset of* Y *such that* $p \in f(U)$ *and* U *is independent, then* $p \cup U$ *is a circuit.*//

The next lemma extends a result of Brualdi [7, Lemma 1] for independence spaces.

(1.7.2) Lemma. *Let* B *be a base of a B-matroid* (S,f) *and* a *be an element of* $S\backslash B$. *Then there is a unique circuit* C_a *containing* a, *such that* $C_a \subseteq B \cup a$. *Moreover, if* $b \in B$, *then* $(B\backslash b) \cup a$ *is a base if and only if* $b \in C_a$.

Proof. As B is spanning, $a \in f(B)$. Thus since f satisfies (*C*) (see (1.3.9)(iii)), there is a minimal subset U of B such that $a \in f(U)$ and U is independent. By Lemma 1.7.1, $a \cup U$ is a circuit and clearly this circuit satisfies the requirements of the lemma. The fact that it is unique follows by (1.6.1)(ii), since f is an *IECH*-operator by (1.3.9)(iii). Thus let $a \cup U = C_a$.

If $b \in B$ and $(B\backslash b) \cup a$ is a base, then $(B\backslash b) \cup a$ does not contain C_a and hence $b \in C_a$. Conversely, if $b \in C_a \cap B$, then $b \in C_a \subseteq ((B\backslash b) \cup a) \cup b$ and hence, by Theorem 1.6.1, $b \in f((B\backslash b) \cup a)$. Thus $(B\backslash b) \cup a$ is spanning. Moreover, $(B\backslash b) \cup a$ is independent, as otherwise, by Lemma 1.4.3, $a \in f(B\backslash b)$ and so $B\backslash b$ is spanning; a contradiction. Therefore, if $b \in C_a \cap B$, then $(B\backslash b) \cup a$ is a base.//

We now prove the main result of this section which has as a consequence the fact that the bases of a *B*-matroid satisfy the symmetric exchange axiom (1.2.7).

(1.7.3) Theorem. *Let* (S,I) *be a preindependence space. Then* I *is the collection of independent sets of a B-matroid on* S *if and only if every restriction of* (S,I) *is an inductive exchange system.*

Proof. Suppose that B_1 and B_2 are bases of a *B*-matroid (S,f) having I as its collection of independent sets. Assume that $b_1 \in B_1\backslash B_2$ and let $S\backslash(B_1 \cap B_2) = T$. Then by (1.3.9)(ii) (T, f^T) is a *B*-matroid which by [39, Proposition (11)], has $B_1\backslash B_2$ and $B_2\backslash B_1$ as bases.

Now $b_1 \in T\backslash(B_2\backslash B_1)$ and therefore, by Lemma 1.7.2, there is a unique f^T-circuit C_{b_1} such that $b_1 \in C_{b_1} \subseteq (B_2\backslash B_1) \cup b_1$. Suppose that $((B_1\backslash B_2)\backslash b_1) \cup b$ is dependent for all elements b of $C_{b_1} \cap (B_2\backslash B_1)$. Then $C_{b_1}\backslash b_1 \subseteq f^T((B_1\backslash B_2)\backslash b_1)$ and so, by Theorem 1.6.1, $b_1 \in f^T(C_{b_1}\backslash b_1) \subseteq f^T((B_1\backslash B_2)\backslash b_1)$; that is, $B_1\backslash B_2$ is f^T-dependent; a contradiction. Therefore for some element b_2 of $C_{b_1} \cap (B_2\backslash B_1)$, the set $((B_1\backslash B_2)\backslash b_1) \cup b_2$ is independent. As $B_1\backslash B_2$ is a base, $(B_1\backslash B_2) \cup b_2$ is dependent and so $b_1 \in f^T(((B_1\backslash B_2)\backslash b_1) \cup b_2)$. Thus $((B_1\backslash B_2)\backslash b_1) \cup b_2$ is spanning and hence is an f^T-base. By Lemma 1.7.2, $((B_2\backslash B_1)\backslash b_2) \cup b_1$ is also an f^T-base and it follows that (S,I) and hence every *B*-matroid satisfies the symmetric exchange axiom. But every restriction of (S,I) is a *B*-matroid, hence every restriction of (S,I) satisfies the symmetric exchange axiom. Thus (S,I) is an inductive exchange system.

The converse is an immediate consequence of Corollary 1.5.7.//

Chapter 2.

A packing problem for infinite graphs and independence spaces.

1. Introduction.

In the preceding chapter we discussed the axiomatics of infinite matroids. In this much shorter chapter we discuss a particular problem for infinite matroids. The basic unsolved problem here is the following (see [86, Problem 20.3.3]). Given a family $(I_i : i \in I)$ of independence spaces on a set S, find necessary and sufficient conditions for the existence of a family $(B_i : i \in I)$ of pairwise disjoint subsets of S such that B_i is a base of I_i for all i in I.

Several special cases of this problem have been solved. Tutte [80] and Nash-Williams [55] independently found a necessary and sufficient condition for a finite graph to have t edge-disjoint spanning trees (where t is a positive integer). Later, Nash-Williams [56] conjectured that this result could be extended to countable graphs. Edmonds [19] generalized the former result by determining precisely when a (finite) matroid has t disjoint bases. Several extensions of this result to independence spaces were proved by Brualdi [9] though each of these retained some very strong finiteness restrictions on the independence spaces.

The two main results of this chapter are as follows. Firstly, in section 2, a counter-example is given to Nash-Williams's conjecture on edge-disjoint spanning trees for countable graphs. In fact, it is shown that a weakened form of the conjecture is also false. Secondly, in section 3, the following result is proved.

(2.1.1) Theorem. *Let* $(I_i : i \in I)$ *be a countable family of independence spaces on a countably infinite set* S. *Suppose that for all finite subsets* T *of* S, *there is a finite subset* U(T) *containing* T, *and a family* $(B_i' : i \in I)$ *of pairwise disjoint subsets of* U(T) *such that, for all* i *in* I, B_i' *is a base of* $I_i|U(T)$. *Then there is a family* $(B_i : i \in I)$ *of pairwise disjoint subsets of* S *such that* B_i *is a base of* I_i *for all* i *in* I.

The results from this chapter will appear in [61].

2. Nash-Williams's conjecture.

In this section we give a counter-example to a 1967 conjecture of Nash-Williams [56].

If G is a graph and $V(G) \cup E(G)$ is countable, then G will be called *countable*. G is *locally finite* if every vertex of G is incident with only finitely many edges. It is easy to show that a locally finite connected graph is countable (see, for example, [57, Theorem 2.4.1]).

If P is a finite partition of the set of vertices of a graph G, that is, P is a partition of $V(G)$ into finitely many subsets, then $E_P(G)$ denotes the set of edges of G which join vertices in different members of P. The next result solves the problem of packing spanning trees in *finite* graphs.

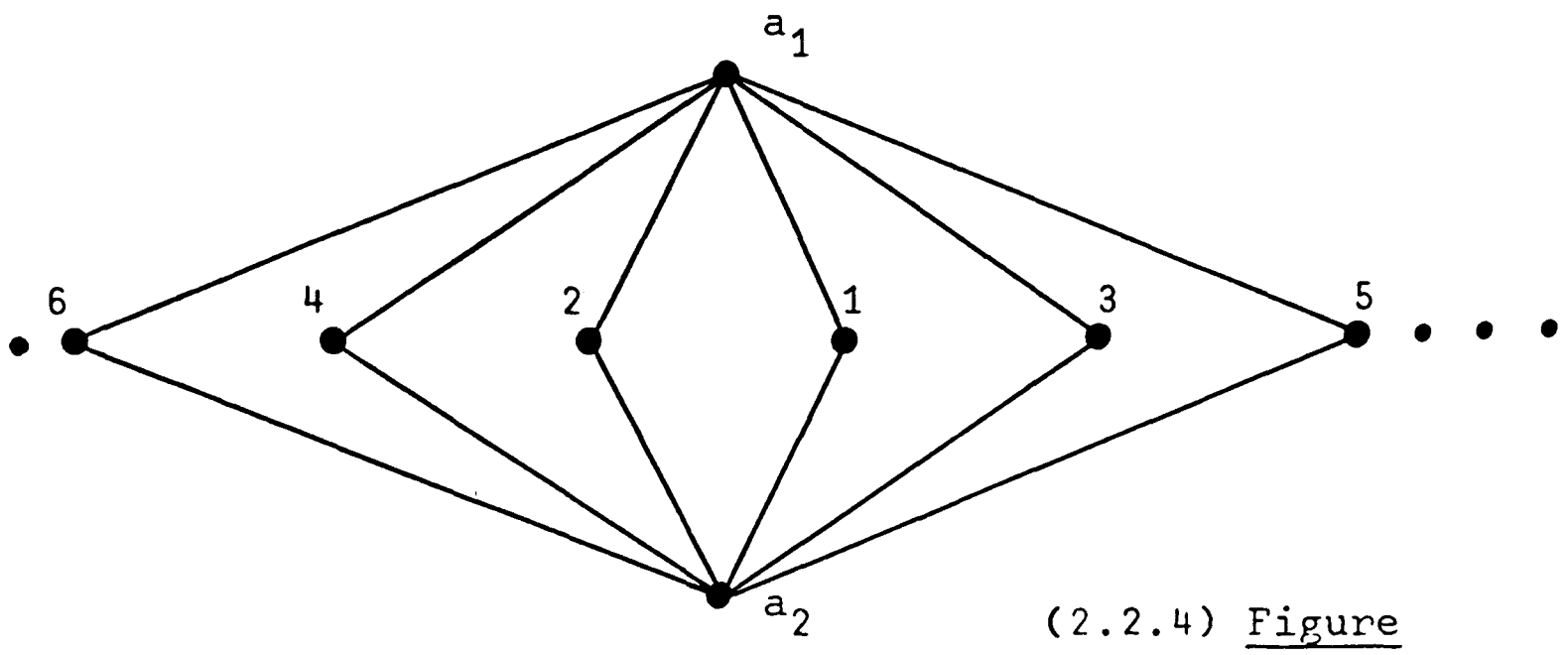
(2.2.1) Theorem [80, Theorem I; 55, Theorem 1]. *Let* G *be a finite graph and* t *be a positive integer. Then* G *has* t *edge-disjoint spanning trees if and only if*

(2.2.2) $\quad |E_P(G)| \geq t(|P| - 1)$

for every (finite) partition P *of* $V(G)$.//

Nash-Williams [56, Conjecture A] has conjectured that this theorem remains true for countable graphs. Clearly (2.2.2) is necessary for a countable graph to have t edge-disjoint spanning trees. The following example shows that it is not sufficient.

(2.2.3) Example. *The infinite graph* H *shown in Figure* 2.2.4 *satisfies* (2.2.2) *for* t = 2 *but does not have two edge-disjoint spanning trees.*



(2.2.4) Figure

Proof. If P is a partition of V(H) and P has a member which contains a_1, a_2 and all but a finite number, say j, of the vertices 1,2, ..., then $|E_P(H)| = 2j \geq 2(|P| - 1)$. If P has no such member, then $E_P(H)$ is infinite. Thus H satisfies (2.2.2) for t = 2. However, H does not have two edge-disjoint spanning trees. For, if T_1 and T_2 are such trees, then for all n in {1,2, ...}, one of the edges a_1n and na_2 is in T_1 and the other is in T_2. Hence there is no path of edges from a_1 to a_2 in T_1; a contradiction.//

Notice that if we add a new vertex a_3 to H, joining it to each of 1,2,3, ..., then we obtain a counter-example to the sufficiency of (2.2.2) when t = 3. Counter-examples for larger values of t are obtained by adding further such vertices.

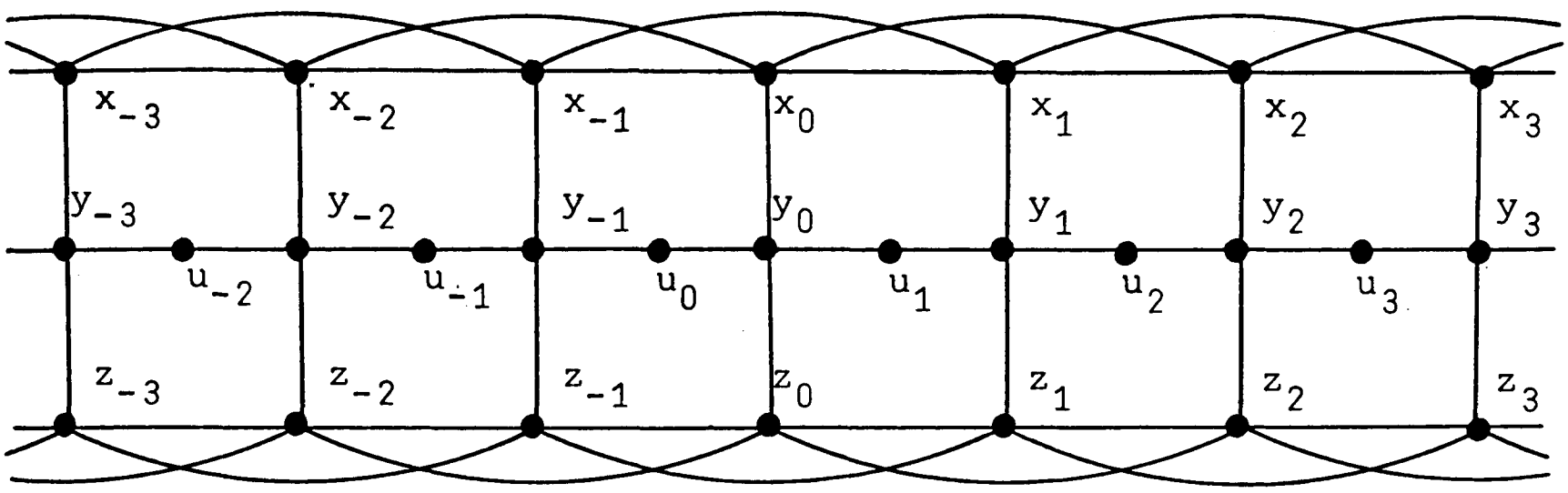
It is easy to see that condition (2.2.2) is a consequence of the requirement that every finite subset of V(G) be contained in t edge-disjoint trees of G. This suggests several stronger necessary conditions including the following.

(2.2.5) *Every finite set of vertices of* G *is contained in* t *edge-disjoint trees such that each of the* t *subgraphs which is obtained by deleting the edges of one of these trees is connected.*

However even for locally finite graphs this condition is not

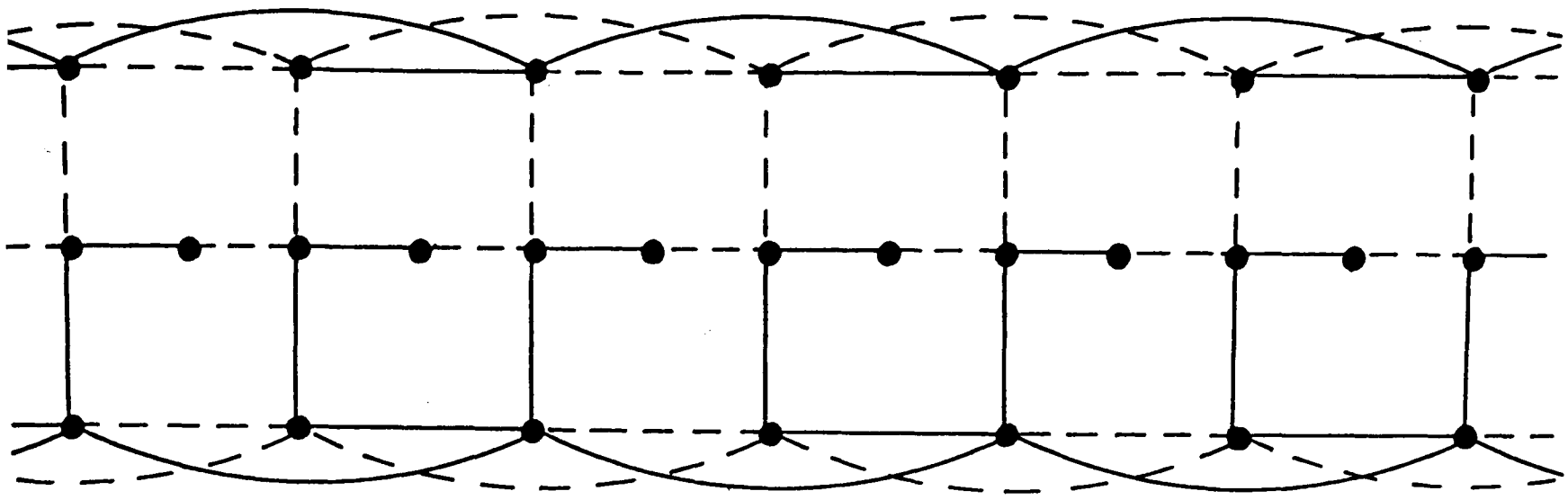
sufficient to ensure that G has t edge-disjoint spanning trees as the following example shows.

(2.2.6) Example. *The graph* K *shown in Figure* 2.2.7 *is locally-finite and satisfies* (2.2.5) *with* t = 2. *But* K *does not have two edge-disjoint spanning trees.*



(2.2.7) Figure

Proof. Suppose that T_1 and T_2 are edge-disjoint spanning trees of K. Then for each integer n, one of $y_{n-1}u_n$ and y_nu_n is in T_1 and the other is in T_2. Since T_1 is connected and spanning, for some integer j, both of the edges x_jy_j and y_jz_j are in T_1. Without loss of generality take j = 1. Then at least one of y_1u_2 and y_1u_1, say y_1u_2,is in T_2. It follows that y_2u_2 is in T_1. Moreover from above, one of y_0u_1 and y_1u_1 is in T_1. In either case the connectedness of T_2 is contradicted. Thus K does not have two edge-disjoint spanning trees. However K does have two edge-disjoint spanning forests, each of which has two components (see Figure 2.2.8). It is now clear that K satisfies (2.2.5) with t = 2.//



(2.2.8) Figure

The possibility of infinite cocycles in infinite graphs is clearly important in the preceding examples. Indeed Theorem 2.2.1 may be extended as follows.

(2.2.9) Corollary. *Let* G *be a graph having no infinite cocycles and* t *be a positive integer. Then* G *has* t *edge-disjoint spanning trees if and only if* (2.2.2) *holds for every finite partition* P *of* V(G).

Proof. The necessity of the condition is clear. For sufficiency note that we may assume that G has no loops. Then, since G has no infinite cocycles, we have by Theorem 1.3.8 (see also [46, Proposition 5.6]) that G has only finite blocks. The result now follows by Theorem 2.2.1.//

3. Packing bases in independence spaces on a countably infinite set.

In this section we shall prove Theorem 2.1.1. If $(I_i : i \in I)$ is a family of independence spaces on a set S, then $(I_i : i \in I)$ *admits a base packing* if and only if there is a family $(B_i : i \in I)$ of pairwise disjoint subsets of S such that B_i is a base of I_i for all i in I.

All bases of an independence space (S,I) have the same cardinality (see, for example, [86, Theorem 20.2.3]). Thus if $X \subseteq S$, the *rank* $rk(X)$ of X is the common cardinality of all bases of $I|X$. For $T \subseteq S$, the rank function of the contraction $I.T$ will be denoted by rk^T.

The proof of Theorem 2.1.1 will use the following infinite generalization of the matroid packing results of Edmonds [19] and Edmonds and Fulkerson [20]. An independence space (S,I) is called *finite* [9, p.266] if its collection I of independent sets is finite. Note that if $(n_i : i \in I)$ is a family of non-negative integers and $J = \{j : j \in I, n_j \neq 0\}$, then $\sum_{i \in I} n_i$ is to be interpreted as the finite sum $\sum_{i \in J} n_i$ or as ∞ according as J is finite or infinite.

(2.3.1) Theorem [9, Corollary 6.9]. *Let* $(I_i : i \in I)$ *be a family of finite independence spaces on a set* S *and suppose that* I_i *has rank function* rk_i. *Then* $(I_i : i \in I)$ *admits a base packing if and only if for all finite subsets* A *of* S,

$$|A| \geq \sum_{i \in I} rk_i^A(A) \quad .//$$

The next lemma contains the core of the proof of Theorem 2.1.1.

(2.3.2) Lemma. *Let* $(I_i : i \in I)$ *be a family of finite independence spaces on a set* S *and suppose that* $(I_i : i \in I)$ *admits a base packing. If* $T \subseteq S$ *and* $(I_i|(S\backslash T) : i \in I)$ *admits a base packing* $(B_i' : i \in I)$, *then* $(I_i : i \in I)$ *admits a base packing* $(B_i : i \in I)$ *such that* $B_i \supseteq B_i'$ *for all* i *in* I.

Proof. We show first that $(I_i.T : i \in I)$ admits a base packing. If rk_i is the rank function of I_i and $A \subset T$, then by Theorem 2.3.1, $|A| \geq \sum_{i \in I} rk_i^A(A)$. But, for all i in I, $(I_i.T).A = I_i.A$ and hence $rk_i^A(A) = (rk_i^T)^A(A)$. Applying Theorem 2.3.1 again establishes the existence of a base packing $(B_i'' : i \in I)$ for $(I_i.T : i \in I)$.

Now for each i in I let $B_i = B_i' \cup B_i''$. Then by (1.2.6), B_i is a base of I_i and clearly the sets $B_i (i \in I)$ are pairwise disjoint.//

Proof of Theorem 2.1.1. As S is countably infinite, take $S = \{1,2,3,\ldots\}$ and let $T_1 = \{1\}$. Then by assumption there is a finite subset $U(T_1)$ of S and a family $(B_i^1 : i \in I)$ of pairwise disjoint sets such that $T_1 \subseteq U(T_1)$ and B_i^1 is a base of $I_i|U(T_1)$ for all i in I.

Now suppose that $T_j, U(T_j)$ and $(B_i^j : i \in I)$ have been defined for all $j < n$. Then either $n \in U(T_{n-1})$ or $n \notin U(T_{n-1})$. In the former case, let $T_n = U(T_{n-1}) = U(T_n)$ and for all i in I, let $B_i^n = B_i^{n-1}$.

If $n \notin U(T_{n-1})$, then let $T_n = U(T_{n-1}) \cup \{n\}$. By assumption, there is a finite subset $U(T_n)$ of S such that $T_n \subseteq U(T_n)$ and $(I_i|U(T_n) : i \in I)$ admits a base packing. Since $(B_i^{n-1} : i \in I)$ is a base packing for $(I_i|U(T_{n-1}) : i \in I)$, it follows, by Lemma 2.3.2, that there is a base packing $(B_i^n : i \in I)$ for

$(I_i|U(T_n) : i \in I)$ such that B_i^n contains B_i^{n-1} for all i in I.

Finally, if $B_i = \bigcup_{n=1}^{\infty} B_i^n$ for all i in I, then it follows by the finite character condition that $(B_i : i \in I)$ is a base packing for $(I_i : i \in I)$.//

For graphic independence spaces Theorem 2.1.1 is a statement about edges. The next corollary restates this result in terms of vertices. If Y is a subset of the set of vertices of a graph G, then G[Y] will denote the subgraph of G induced by Y; that is the subgraph having vertex-set Y and edge-set those edges of G with both endpoints in Y.

(2.3.3)Corollary. *Let* G *be a countable connected graph and* t *be a positive integer. Suppose that for all finite subsets* X *of* V(G) *such that* G[X] *is connected, there is a finite subset* U(X)*of* V(G) *such that* U(X) *contains* X *and* G[U(X)] *has* t *edge-disjoint spanning trees. Then* G *has* t *edge-disjoint spanning trees.//*

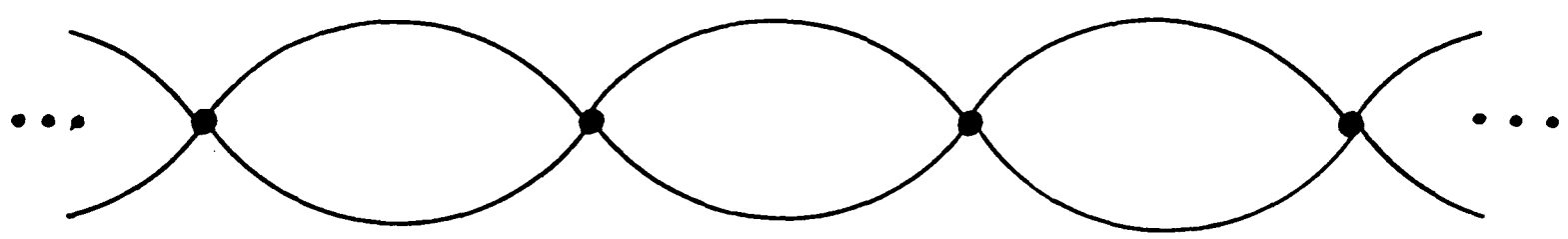
The infinite square lattice Q is an example which shows that the converse of Theorem 2.1.1 need not hold. By a result of Ringel [69, p.10], Q possesses two edge-disjoint (two-way) infinite Hamiltonian paths. Hence Q has two edge-disjoint spanning trees. However no non-trivial finite subgraph F of Q has a pair of edge-disjoint spanning trees. To see the latter consider the partition P_0 of V(F) in which each class contains exactly one vertex. No vertex of F has degree greater than 4 and F has sufficiently many vertices of degree less than 4 to ensure that (2.2.2) fails for P_0.

An easy modification of Theorem 2.1.1 shows that the following result is true.

(2.3.4) Theorem. *Let* $(I_i : i \in I)$ *be a countable family of independence spaces on a countably infinite set* S. *Suppose that for all finite subsets* T *of* S *there is a finite subset* U(T) *containing* T *such that* $(I_i|U(T) : i \in I)$ *admits a base packing which partitions* U(T). *Then* $(I_i : i \in I)$ *admits a base packing which partitions* S.//

The proof of Theorem 2.1.1 only uses the finite character condition at the end and it is not difficult to see that the theorem also holds when the family $(I_i : i \in I)$ is a family of *B*-matroids on S. However, the same is not true for Corollary 2.3.4. To see this, let I = {1,2} and let I_1 and I_2 be such that both I_1^* and I_2^* are isomorphic to the set of forests of the graph H shown in Figure 2.2.4; that is I_1 and I_2 are both isomorphic to the two-way path matroid (see Example 1.6.4) of the graph shown in Figure 2.3.5.

(2.3.5) Figure



Chapter 3.

Colouring and the critical problem.

1. Introduction.

The critical problem for combinatorial geometries was formulated by Crapo and Rota [14, Chapter 16] as an algebraic framework for several well-known colouring and flow problems for graphs and matroids representable over finite fields. The results of this chapter are all concerned with certain aspects of the critical problem.

In section 2 we show that for all matroids M without loops the maximum real root of the chromatic polynomial of M does not exceed the maximum size of a cocircuit of M. From this we deduce a bound on the chromatic number $\chi(M)$ of M where $\chi(M)$ is the least positive integer at which the chromatic polynomial of M is positive. This bound improves on a result of Heron [38]. For regular matroids sharper bounds on χ are proved which resemble the bounds of Brooks [6] and Szekeres/Wilf [76] for graphs. One such bound improves on a result of Lindström [47]. A key lemma of this section is used again in section 3 to generalize another result of Lindström [47] by giving an upper bound on the critical exponent of a matroid representable over GF(q).

Brylawski [12] and Heron [37] showed independently that a binary matroid is affine if and only if it is a disjoint union of cocircuits. In section 3 we show that if M is representable over GF(q) and M is a disjoint union of cocircuits, then M is affine. The converse is only true when $q = 2$. Most of the results from sections 2 and 3 have appeared in [62].

Mullin and Stanton [53] have defined an (m,k) cover of

V(k,2), the vector space of k-tuples over GF(2), to be a subset of the non-zero vectors of V(k,2) which has rank k and has non-empty intersection with every subspace of V(k,2) of rank k-m. Alternatively we note that an (m,k) cover is just the image in V(k,2) of a restriction M of PG(k-1,2) under some representation, where M has rank k and critical exponent greater than m. In sections 4,5 and 6, using the second approach, we solve several of the problems raised by Mullin and Stanton [53]. In section 4 this matroid approach gives an immediate proof of the result [53] that every (m,k) cover contains at least $2^{m+1} + k - m - 2$ elements. In addition we show that if M is a matroid representable over GF(q) and M is minimal having critical exponent n, then every cocircuit of M has at least q^{n-1} elements. Using this lemma we prove that for all $m \geq 2$, every (m,k) cover having exactly $2^{m+1} + k - m - 2$ elements is isomorphic to the direct sum of PG(m,2) and the free matroid on k - m - 1 elements. This answers a question from [53].

For each positive integer m, let $Q_m = \{|C| - k : C$ is a minimal (m,k) cover$\}$. The series connection of a minimal (m,h) cover having no coloops and a minimal (m,k) cover having no coloops is a minimal (m,h + k) cover. This is used in section 5 to answer another question from [53] by proving that for all $m \geq 2$, the set Q_m contains all but finitely many positive integers. In the case m = 2, series connections give that $Q_2 \supseteq Z^+ \setminus \{1,2,3,5,8\}$. In section 6 we prove a conjecture of Mullin and Stanton [53] that $5 \notin Q_2$. We also obtain a partial result on the problem of whether $8 \in Q_2$.

Note added in proof. The results from sections 4 and 5 can be generalized to matroids representable over GF(q) for q > 2 using arguments which are not significantly different from those given herein.

2. Colouring.

For arbitrary matroids one can define a polynomial which is closely related to the chromatic polynomial of a graph. The *chromatic* (or *characteristic*) *polynomial* $P(M;\lambda)$ of a matroid M on a set S is defined (see [86, p.262]) by

$$P(M;\lambda) = \sum_{A \subseteq S} (-1)^{|A|} \lambda^{rk\ S - rk\ A}$$

If M is the cycle matroid M(G) of a graph G which has $\kappa(G)$ connected components, then the chromatic polynomial $P_G(\lambda)$ of G satisfies

$$P_G(\lambda) = \lambda^{\kappa(G)} P(M(G);\lambda) .$$

Welsh [86, p.262] and Heron [37,38] have defined the *chromatic number* $\chi(M)$ of an arbitrary loopless matroid M by

$$\chi(M) = \min \{j \in Z^+ : P(M;j) > 0\} .$$

The binary projective geometry PG(n,2) has chromatic polynomial $P(PG(n,2);\lambda) = \prod_{i=0}^{n} (\lambda - 2^i)$ (see [37, p.99]). Thus,

$$\chi(PG(n,2)) = \begin{cases} 3, \text{ for n odd;} \\ 5, \text{ otherwise.} \end{cases}$$

Notice that if $n \geq 3$, then $P(PG(n,2);\lambda)$ vanishes and indeed takes negative values on integers greater than the chromatic number. Examples such as this prompt the introduction of another parameter π defined for an arbitrary loopless matroid M by

$$\pi(M) = \min \{j \in Z^+ : P(M;j + k) > 0 \text{ for all } k = 0,1,2,...\} .$$

Evidently for loopless graphic matroids χ and π are equal. However for any positive integer n, we have that

$$\pi(PG(n + 2,2)) - \chi(PG(n + 2,2)) > n .$$

In this section we derive an upper bound on $\pi(M)$ when M is an arbitrary loopless matroid, and a sharper bound when M is a loopless regular matroid. In a loopless 2-connected graph G, the set of edges incident with a particular vertex forms a cocircuit in the cycle matroid of G. As several well-known upper bounds on the chromatic number of a graph are stated in terms of the vertex degrees of the graph (see, for example, [4, Exercises 8.1.3, 8.1.5 and Theorem 8.4]), it seems natural to attempt to prove bounds on $\pi(M)$ which are obtained by substituting "cocircuit size" for "vertex degree" in the graph-theoretic results. Each of the bounds on $\pi(M)$ proved here is of this form. The first such result is deduced from a bound on $\xi(M)$, the maximum real root of $P(M;\lambda)$. The simple matroid associated with a matroid M will be denoted by $\tilde{M}$.

(3.2.1) Theorem. *If M is a loopless matroid, then*

$$\xi(M) \leq \max_{C^* \in C^*(M)} |C^*| \quad \textit{and so}$$

$$\pi(M) \leq 1 + \max_{C^* \in C^*(\tilde{M})} |C^*| \leq 1 + \max_{C^* \in C^*(M)} |C^*|.$$

The main part of the proof of this result is contained in Lemmas 3.2.6 and 3.2.7 which use the following four basic properties of the chromatic polynomial of a matroid M (see [86, p.263]).

(3.2.2) If M has a loop, then $P(M;\lambda) = 0$.

Let e be an element of M.

(3.2.3) If e is a coloop, then $P(M;\lambda) = (\lambda - 1)P(M\backslash e;\lambda)$.

(3.2.4) If e is neither a loop nor a coloop, then

$P(M;\lambda) = P(M\backslash e;\lambda) - P(M/e;\lambda)$.

Note that if e is a loop, then $M\backslash e \cong M/e$.

Hence combining (3.2.2) and (3.2.4) we get:

(3.2.5) if e is not a coloop, then

$P(M;\lambda) = P(M\backslash e;\lambda) - P(M/e;\lambda)$.

(3.2.6) Lemma. *Let* $\{x_1,x_2,\ldots,x_n\}$ *be a coindependent set in a matroid* M. *Then*

$$P(M;\lambda) = P(M\backslash x_1,x_2,\ldots,x_n;\lambda) + \sum_{j=2}^{n}\sum_{i=1}^{j-1} P(M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1}/x_i,x_j;\lambda) - \sum_{i=1}^{n} P(M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n/x_i;\lambda).$$

Proof. We proceed by induction on n. The result is immediate for n = 0. Now assume that the proposition is true for n - 1. Then, as x_n is not a coloop of $M\backslash x_1,\ldots,x_{n-1}$, by (3.2.5),

$$P(M\backslash x_1,\ldots,x_{n-1};\lambda) = P(M\backslash x_1,\ldots,x_{n-1},x_n;\lambda) - P(M\backslash x_1,\ldots,x_{n-1}/x_n;\lambda)$$

Moreover, if $1 \le i \le n - 1$, then x_n is not a coloop of $M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n-1}/x_i$ and therefore

$$\begin{aligned} &P(M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n-1}/x_i;\lambda)\\ &= P(M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n-1},x_n/x_i;\lambda)\\ &- P(M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n-1}/x_i,x_n;\lambda). \end{aligned}$$

It follows by induction that the result is true for all non-negative integers n.//

(3.2.7) Lemma. *Let* $\{x_1,x_2,\ldots,x_t\}$ *be a cocircuit of a matroid* M. *Then*

$$P(M;\lambda) = (\lambda - t)\,P(M\backslash x_1,\ldots,x_t;\lambda) + \sum_{j=2}^{t}\sum_{i=1}^{j-1} P(M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1}/x_i,x_j;\lambda).$$

Proof. Taking n = t - 1 in Lemma 3.2.6, we note that x_t is a coloop of $M\backslash x_1,\ldots,x_{t-1}$ and so

$P(M\backslash x_1,\ldots,x_{t-1};\lambda) = (\lambda - 1)P(M\backslash x_1,\ldots,x_{t-1},x_t;\lambda)$.

Furthermore x_t is not a coloop of $M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{t-1}/x_i$, hence

$$P(M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{t-1}/x_i;\lambda)$$

$$= P(M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_t/x_i;\lambda)$$

$$- P(M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{t-1}/x_i,x_t;\lambda) .$$

To complete the proof notice that x_i is a coloop of $N = M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_t$. Therefore $N/x_i \cong N\backslash x_i$ and so the chromatic polynomials of these two matroids are equal.//

Proof of Theorem 3.2.1. As $\xi(M) = \xi(\tilde{M})$, to prove this theorem it suffices to show that if M is a loopless matroid on a set S, then $\xi(M) \leq \max_{C^* \in C^*(M)} |C^*|$. We shall prove this proposition by induction on $|S|$.

If $|S| = 1$, then the result is immediate. Assume now that the proposition holds for all matroids on sets of fewer than n elements and let M be a loopless matroid on a set of size n. Let $\{x_1,x_2,\ldots,x_t\}$ be a cocircuit of M. Then we can suppose that $t < n$. By (3.2.2) and Lemma 3.2.7,

$$(3.2.8) \qquad \xi(M) \leq \max\{t,\ \xi(M\backslash x_1,x_2,\ldots,x_t),$$

$$\xi'(M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1}/x_i,x_j)(2 \leq j \leq t,$$

$$1 \leq i \leq j-1)\}$$

where $\xi'(N) = \begin{cases} \xi(N), & \text{if N has no loops;} \\ 0, & \text{otherwise.} \end{cases}$

Now $t \leq \max_{C^* \in C^*(M)} |C^*|$, and by the induction assumption, all the other terms on the right-hand side of (3.2.8) are bounded above by $\max_{C^* \in C^*(M)} |C^*|$. Therefore this number is an upper bound on $\xi(M)$. The required result follows by induction.//

As an immediate consequence of the preceding theorem we have the following:

(3.2.9) Corollary (Heron [38, p.193]). *If* M *is a loopless matroid, then*

$$\pi(M) \leq \text{cork}\,(\tilde{M}) + 2.//$$

For regular matroids Lemma 3.2.7 may be used again to sharpen the bound on π given in Theorem 3.2.1. Before stating this result we recall some definitions (see [86, p.170]). For a binary matroid M define the *circuit matrix* D(M) of M to be the incidence matrix of circuits against elements; that is, the entry in row i and column j of D(M) is 1 if the jth element is in the ith circuit, and is 0 otherwise. The *cocircuit matrix* $D^*(M)$ is the incidence matrix of cocircuits against elements. M is said to be *orientable* if there is an assignment of 1's and -1's to the non-zero entries of $D^*(M)$ and D(M) to give matrices $\vec{D}^*(M)$ and $\vec{D}(M)$ such that the rows of $\vec{D}^*(M)$ and $\vec{D}(M)$ are orthogonal over Z. The pair $(\vec{D}^*(M),\vec{D}(M))$ is called an *orientation* of M. It is well-known (see, for example, [86, p.175]) that M is orientable if and only if M is regular.

Now suppose that A is an additively-written abelian group and M is a regular matroid with orientation $(\vec{D}^*(M), \vec{D}(M))$. An A-*coboundary* of M is a function f from the ground set of M into A such that $f = \Sigma a_i r_i$ where the summation is over all rows r_i of $\vec{D}^*(M)$, and $a_i \in A$ for all i.
Thus if e is an element of M, then

$f(e) = \Sigma a_i r_i(e)$ where

$$a_i r_i(e) = \begin{cases} a_i, & \text{if } r_i(e) = 1; \\ -a_i, & \text{if } r_i(e) = -1; \\ 0, & \text{if } r_i(e) = 0. \end{cases}$$

The following theorem generalizes Matula's upper bound [51, Theorem 14] on the chromatic number of a graph, a bound which sharpens that given by Szekeres and Wilf [76]. The set of simple matroids which are restrictions of a matroid N will be denoted by $R(N)$.

(3.2.10) Theorem. *If* M *is a loopless regular matroid, then*

$$\chi(M) = \pi(M) \leq 1 + \max_{N \in R(M)} \left(\min_{C^* \in C^*(N)} |C^*| \right)$$

Proof. For all integers $n \geq 2$, let Z_n denote the ring of integers modulo n. By [13, Theorem III] (see also [12, Theorem 12.4]), if N is a regular matroid, then

(3.2.11) P(N;n) equals the number of nowhere-zero Z_n-coboundaries on N.

Therefore $P(N;n) \geq 0$. The upper bound on $\pi(M)$ now follows easily by induction using Lemma 3.2.7.

An easy consequence of a result of Tutte [81, 5.44] is that the following statements are equivalent for a regular matroid N (see [43, Proposition 1]):

(a) There is a nowhere-zero Z_n-coboundary on N.

(b) There is a nowhere-zero Z-coboundary on N with all values in $[1 - n, n - 1]$.

Using this equivalence and (3.2.11) the rest of the proof is straightforward. //

(3.2.12) Corollary (Lindström [47, Theorem 17]). *If* M *is a loopless regular matroid, then*

$$\pi(M) \leq 1 + \max_{p \in S} \left(\min_{p \in C^* \in C^*(M)} |C^*| \right)$$

Proof. It is clear that this result follows from Theorem 3.2.10 if we can show that

$$\max_{N \in R(M)} \left(\min_{C^* \in C^*(N)} |C^*| \right) \leq \max_{p \in S} \left(\min_{p \in C^* \in C^*(M)} |C^*| \right). \tag{3.2.13}$$

We shall prove this for an *arbitrary* loopless matroid M'.

Let $\max_{N\in R(M')} (\min_{C^*\in C^*(N)} |C^*|) = \min_{C^*\in C^*(M'|T)} |C^*| = |C^*_N|$

where $C^*_N \in C^*(M'|T)$. Choose x in C^*_N. If $C^*_0 \in C^*(M')$ and $x \in C^*_0$, then $C^*_0 \supseteq C^*_1$ for some C^*_1 in $C^*(M'|T)$. Thus $|C^*_0| \geq |C^*_1| \geq |C^*_N|$. It follows that

$$|C^*_N| \leq \min_{x\in C^*\in C^*(M')} |C^*| \leq \max_{p\in S} (\min_{p\in C^*\in C^*(M')} |C^*|).//$$

I have been unable to resolve whether the bounds on $\pi(M)$ given by Theorem 3.2.10 and Corollary 3.2.12 hold for arbitrary loopless matroids or even for loopless binary matroids.

Theorem 3.2.10 is used again to prove the following analogue of Brooks' Theorem [6, p.194].

(3.2.14) Theorem. *If* M *is a simple connected regular matroid, then* $\pi(M) \leq \max_{C^*\in C^*(M)} |C^*|$ *unless* M *is an odd circuit or a coloop.*

Proof. By Theorem 3.2.1, $\pi(M) \leq 1 + \max_{C^*\in C^*(M)} |C^*| = 1 + n$, say. If $\pi(M) = 1 + n$, then by Theorem 3.2.10, $\max_{N\in R(M)} (\min_{C^*\in C^*(N)} |C^*|) = n$. Thus for some $T \subseteq S$, all the cocircuits of $M|T$ have cardinality equal to $\max_{C^*\in C^*(M)} |C^*|$. Therefore no cocircuit of M intersects both T and S\T. Thus, as M is connected, S = T and so the cocircuits of M are equicardinal.

Murty [54] has shown that a simple connected binary matroid having equicardinal cocircuits is either a coloop, a circuit or a binary projective or affine geometry. From this it follows easily that M is either an odd circuit or a coloop.//

3. The critical problem.

If A is a subset of V(n,q), the vector space of n-tuples over GF(q), then a t-tuple $(f_1,f_2,\ldots,f_t)$ of linear functionals on V(n,q) is said to *distinguish* A if for all e in A, $f_i(e) \neq 0$ for some i in $\{1,2,\ldots,t\}$. Let M be a rank n matroid on a set S and suppose that M is representable over GF(q).

(3.3.1) Theorem (Crapo and Rota [14, p.16.4]). *If* $t \in Z^+$ *and* ϕ *is a representation of* M *in* V(n,q), *then the number of* t*-tuples of linear functionals on* V(n,q) *which distinguish* $\phi(S)$ *equals* $P(M;q^t)$.//

It follows that for a matroid M representable over GF(q)

(3.3.2) $P(M;q^t) \geq 0$ for all t in Z^+.

The *critical exponent* c(M;q) of M is defined by

$$(3.3.3)\quad c(M;q) = \begin{cases} \infty, \text{ if M has a loop;} \\ \min\{j \in Z^+ : P(M;q^j) > 0\}, \text{ otherwise.} \end{cases}$$

Thus if M has no loops, then

(3.3.4) $P(M;q^t) > 0$ for $t = c(M;q), c(M;q) + 1, \ldots$.

Projective and affine geometries will be referred to frequently in the subsequent discussion. We note here that among matroids representable over GF(q), PG(n,q) is the unique simple matroid having rank n + 1 and critical exponent n + 1 and AG(n,q) is the unique maximal simple matroid having rank n + 1 and critical exponent one.

The critical exponent has the following important alternative interpretation. If M is a rank n, loopless matroid representable over GF(q) and ϕ is a representation of M in V(n,q), then

$c(M;q)$ is the least number t of hyperplanes $H_1, H_2, \ldots, H_t$ of $V(n,q)$ such that $(\bigcap_{i=1}^{t} H_i) \cap \phi(S) = \emptyset$.

The next result should be compared with Theorem 3.2.10.

(3.3.5) Theorem. *If* M *is a loopless matroid on a set* S *and* M *is representable over* $GF(q)$, *then*

$$c(M;q) \leq \{\log_q (1 + \max_{N \in R(M)} (\min_{C^* \in C^*(N)} |C^*|))\}.$$

Proof. We proceed by induction on $|S|$. The result is true for $|S| = 1$. Assume it is true for all matroids on sets of fewer than n elements and let $|S| = n$. If M is not simple, then since $c(M;q) = c(\tilde{M};q)$ and $R(M) = R(\tilde{M})$, the result follows by the induction assumption. Thus suppose that M is simple and let $\{x_1, x_2, \ldots, x_t\}$ be a cocircuit of M of minimal size. Then by Lemma 3.2.7, (3.3.3) and (3.3.4),

$$c(M;q) \leq \max\{\{\log_q (t+1)\}, c(M\backslash x_1, x_2, \ldots, x_t; q)\}.$$

But $t = \min_{C^* \in C^*(M)} |C^*|$, thus $t \leq \max_{N \in R(M)} (\min_{C^* \in C^*(N)} |C^*|)$.

Moreover, by the induction assumption,

$$c(M\backslash x_1, x_2, \ldots, x_t; q) \leq \{\log_q (1 + \max_{N \in R(M\backslash x_1, \ldots, x_t)} (\min_{C^* \in C^*(N)} |C^*|))\}$$

$$\leq \{\log_q (1 + \max_{N \in R(M)} (\min_{C^* \in C^*(N)} |C^*|))\}.$$

The required result now follows by induction.//

The next result generalizes a theorem of Lindström [47, Theorem 18] who proved it for the case $q = 2$. A *cocircuit covering* of the ground set S of a matroid M is a collection $C_1^*, C_2^*, \ldots, C_j^*$ of cocircuits of M such that $S = \bigcup_{i=1}^{j} C_i^*$.

(3.3.6) Corollary. *Suppose that* M *is a matroid representable over* $GF(q)$. *If there is a covering of the ground set* S *of* M *with cocircuits each of size less than* q^t, *then* $c(M;q) \leq t$.

Proof. Since there is a covering of S with cocircuits each of size less than q^t, M has no loops and

$$\{\log_q(1 + \max_{p\in S}(\min_{p\in C^*\in C^*(M)} |C^*|))\} \leq \{\log_q(1 + (q^t - 1))\} = t.$$

The result now follows by (3.2.13).//

Now again let M be a rank n matroid on a set S and suppose that M is representable over GF(q). When M has no loops $c(M;q) = c(\tilde{M};q)$. Since by Theorem 3.3.1, c(M;q) does not depend on the representation chosen for M in V(n,q), it will be convenient in the proofs of Theorems 3.3.7 and 3.3.11 to identify $\tilde{M}$ with a restriction of V(n,q) to which it is isomorphic. The closure operator of V(n,q) will be denoted by σ. The following result gives an elementary, yet seemingly new property of the critical exponent.

(3.3.7) Theorem. *If* T *is a non-empty subset of* S, *then* $c(M|T;q) \leq c(M.T;q)$.

Proof. If M.T has a loop, then the result is immediate. Thus suppose that M.T has no loops. We may also assume that M is simple. Let $M' = V(n,q)/\sigma(S\backslash T)$. As M.T is loopless, $\sigma(S\backslash T) \cap T = \emptyset$. Now using [81, 3.331-4] we get that if $U = T \cup \sigma(S\backslash T)$, then $M'|T \cong V(n,q)|U.T$ and $M.T \cong V(n,q)|U.(U\backslash(S\backslash T))|T$. But every element of $U\backslash S$ is a loop of $V(n,q)|U.(U\backslash(S\backslash T))$, hence $M.T \cong M'|T$. Suppose that $\{H_1,H_2,\ldots,H_t\}$ is a minimal set of hyperplanes of M' such that $(\bigcap_{i=1}^{t} H_i) \cap T = \emptyset$. Then, as $\tilde{M}' \cong \widetilde{V(n',q)}$ where M.T has rank n', it follows that $t = c(M.T;q)$. But, for all $1 \leq i \leq t$, $H_i \cup \sigma(S\backslash T)$ is a hyperplane of V(n,q). Therefore, since $\sigma(S\backslash T) \cap T = \emptyset$, we have that $c(M|T;q) \leq t = c(M.T;q)$, as required.//

Two further elementary but useful properties of the critical exponent are as follows (see [14, Chapter 16]). For every non-empty subset T of S:

(3.3.8) $c(M|T;q) \leq c(M;q)$; and

(3.3.9) $c(M;q) \leq c(M|T;q) + c(M|(S\backslash T);q)$.

A loopless matroid representable over GF(q) is called *affine* [14, p.16.10] if it has critical exponent one. Evidently a loopless matroid M is affine if and only if $\tilde{M}$ is isomorphic to a restriction of an affine geometry.

The next theorem, which was proved independently by Brylawski [12, Theorem 10.3] and Heron [37, p.102], generalizes the well-known result that a graph is 2-colourable if and only if it is bipartite (see, for example, [57, p.229]).

(3.3.10) Theorem. *A binary matroid is affine if and only if it is a disjoint union of cocircuits.*//

This result motivated the following:

(3.3.11) Theorem. *Let* M *be a rank* n, *loopless matroid representable over* GF(q). *If the ground set* S *of* M *is a disjoint union of cocircuits, then* M *is affine.*

The proof of this theorem uses the following elementary result (see, for example, [86, Theorem 2.1.6]).

(3.3.12) Lemma. *A circuit and a cocircuit of a matroid cannot have exactly one common element.*//

Proof of Theorem 3.3.11. If M is a disjoint union of cocircuits, then by Lemma 3.3.12, so is $\tilde{M}$. Moreover $c(M;q) = c(\tilde{M};q)$ and hence we may assume that M is simple. Let S be a disjoint union of the cocircuits $C_1^*, C_2^*, \ldots, C_t^*$ of M. If $t = 1$, then $M \cong U_{1,1}$ and the result is immediate. Assume therefore that $t > 1$. For all $1 \leq i \leq t$, $S\backslash C_i^*$ is a hyperplane of M, hence

$S\backslash C_i^* = S \cap H_i$ where H_i is a hyperplane of $V(n,q)$. Let $\bigcap_{i=1}^{t} H_i = F$. Then F is a flat of $V(n,q)$ of rank $n - t$ and $F \cap S = \emptyset$. Moreover $\bigcup_{i=1}^{t} (\bigcap_{\substack{j\neq i \\ j=1}}^{t} H_j) \supseteq S$, hence

$$(\bigcup_{i=1}^{t} (\bigcap_{\substack{j\neq i \\ j=1}}^{t} H_j))\backslash F \supseteq S.$$

Now let $M' = V(n,q)/F$. Then M' has rank t and

$$\{(\bigcap_{\substack{j\neq i \\ j=1}}^{t} H_j)\backslash F\}_{1\leq i\leq t}$$

is a subset of the set of rank one flats of M'. It is a routine exercise in linear algebra to show that there is a hyperplane of $V(t,q)$ avoiding t linearly independent vectors. Therefore, as $\widetilde{M'} \cong \widetilde{V(t,q)}$, there is a hyperplane H' of M' such that

$$H' \cap ((\bigcup_{i=1}^{t} (\bigcap_{\substack{j\neq i \\ j=1}}^{t} H_j))\backslash F) = \emptyset.$$

Therefore

$$(H' \cup F) \cap ((\bigcup_{i=1}^{t} (\bigcap_{\substack{j\neq i \\ j=1}}^{t} H_j))\backslash F) = \emptyset \text{ and so } (H' \cup F) \cap S = \emptyset.$$

But $H' \cup F$ is a hyperplane of $V(n,q)$ and so $c(M;q) = 1$, as required.//

The converse of this theorem only holds if $q = 2$. To see this note that the affine plane $AG(2,q)$ has q^2 points, exactly q of which lie on every line. Hence every cocircuit of $AG(2,q)$ contains $q^2 - q$ points. But when $q > 2$, $q^2 - q$ does not divide q^2. Thus although $AG(2,q)$ is affine, it is only a disjoint union of cocircuits when $q = 2$.

The notion of an A-coboundary of a regular matroid was discussed in the preceding section. If the additively-written abelian group A is Z_2^k, the direct sum of k copies of Z_2, then we may extend this definition as follows. Let M be a binary

matroid and $A = Z_2^k$, then an A-*coboundary* of M is a function f from the ground set of M into A such that $f = \Sigma a_i r_i$ where the summation here is over the rows r_i of $D^*(M)$, and $a_i \in A$ for all i. It is easy to see that this agrees with the earlier definition when M is regular.

(3.3.13) Theorem (Lindström [47, Theorem 15]). *Let* M *be a binary matroid and* $A = Z_2^n$. *Then the number of nowhere-zero* A-*coboundaries of* M *is* $P(M;2^n)$.//

Two corollaries of this theorem will be used frequently in the next three sections. To prove the first of these we use a well-known fundamental property of binary matroids (see, for example, [86, Theorem 10.1.3]).

(3.3.14) Lemma. *If* $C_1, C_2, \ldots, C_j$ *are circuits of a binary matroid* M, *then* $C_1 \Delta C_2 \Delta \ldots \Delta C_j$ *is a disjoint union of circuits of* M.

(3.3.15) Corollary. *Let* M *be a loopless binary matroid on a set* S *and suppose that* t *is the least positive integer* p *such that* $S = \bigcup_{i=1}^{p} S_i$ *and* $M.S_i$ *is a disjoint union of cocircuits for all* $1 \le i \le p$. *Then* $t = c(M;2)$.

Proof. If $t = c(M;2)$, then $P(M;2^t) > 0$. Hence for $A = Z_2^t$, by Theorem 3.3.13, M has a nowhere-zero A-coboundary f. Now for each element y of A and each integer j in $\{1,2,\ldots,t\}$, let y_j be the jth co-ordinate of y and let $S_j = \{x \in S : f(x)_j = 1\}$. As f is nowhere-zero, $\bigcup_{i=1}^{t} S_i = S$.

Suppose that $D^*(M)$ has N rows, $r_1, r_2, \ldots, r_N$. Now $f = \sum_{i=1}^{N} a_i r_i$ where $a_i \in A$ for all $1 \le i \le N$, and for fixed j we may assume

that $\{(a_i)_j : 1 \le i \le N, (a_i)_j = 1\} = \{(a_1)_j, (a_2)_j, \ldots, (a_s)_j\}$. Hence for all x in S, $f(x)_j = \sum_{i=1}^{s} r_i(x)$. Thus S_j equals the symmetric difference of the cocircuits $C_1^*, C_2^*, \ldots, C_s^*$ of M which have $r_1, r_2, \ldots, r_s$ respectively as their incidence vectors. Hence by Lemma 3.3.14, S_j and hence $M.S_j$ is a disjoint union of cocircuits.

Now suppose that $S = \bigcup_{i=1}^{p} S_i$ where $M.S_i$ is a disjoint union of cocircuits for all $1 \le i \le p$. Then if $A = Z_2^p$, it is easy to define a nowhere-zero A-coboundary on S. Thus $c(M;2) \le p$ and the corollary is proved.//

The second corollary of Theorem 3.3.13 is a slight modification of the first and uses the following:

(3.3.16) Lemma (Welsh [84]). *A binary matroid is a disjoint union of cocircuits if and only if all its circuits have even cardinality.*//

(3.3.17) Corollary. *Let* M *be a loopless binary matroid on a set* S *and suppose that* t' *is the least positive integer* p' *such that* $S = \bigcup_{i=1}^{p'} S_i$ *and* $M|S_i$ *has no odd circuits for all* $1 \le i \le p'$. *Then* $t' = c(M;2)$.

Proof. If $t' = c(M;2)$, then by Corollary 3.3.15, $S = \bigcup_{i=1}^{t'} S_i$ where $M.S_i$ is a disjoint union of cocircuits for all $1 \le i \le t'$. But by Theorems 3.3.10 and 3.3.7, $1 = c(M.S_i;2) \ge c(M|S_i;2)$. Thus by Lemma 3.3.16 and Theorem 3.3.10, $M|S_i$ has no odd circuits. Now suppose that $S = \bigcup_{i=1}^{p'} S_i$ and $M|S_i$ has no odd circuits for all $1 \le i \le p'$. Then by (3.3.9), $c(M;2) \le \sum_{i=1}^{p'} c(M|S_i;2)$ and so by Lemma 3.3.16 and Theorem 3.3.10, $c(M;2) \le p'$.//

4. On a covering problem of Mullin and Stanton for binary matroids.

In this section and the two following we solve several problems of Mullin and Stanton [53] concerned with the critical problem for binary matroids.

If $k,m \in Z^+$ and $1 \leq m \leq k - 1$, then an (m,k)-*cover of* $V(k,2)$ is a subset of the non-zero vectors of $V(k,2)$ which has rank k and has non-empty intersection with every subspace of $V(k,2)$ of rank $k - m$. It follows that an (m,k) cover of $V(k,2)$ is the image in $V(k,2)$ of a restriction M of $PG(k-1,2)$ under a representation, where $rk\ M = k$ and $c(M;2) \geq m + 1$. We shall call a restriction of $PG(k-1,2)$ having rank k and critical exponent greater than m an (m,k)-*matroid*. A *minimal* (m,k)-matroid is an (m,k)-matroid for which no proper restriction is also an (m,k)-matroid. Tutte [82,p.25] has called a simple binary matroid having critical exponent greater than m an m-*block*. Thus the set of minimal m-blocks is precisely the union over all k in $\{1,2,\ldots,m-1\}$ of the sets of minimal (m,k)-matroids having no coloops. Tutte [82,83] and Datta [15,16] have studied in some detail a particular collection of minimal 2-blocks called *tangential 2-blocks*. These will only be discussed briefly. We note here that a tangential m-block is a simple binary matroid having critical exponent $m + 1$ such that every proper loopless contraction has critical exponent less than $m + 1$ (see [83, p.207]). For comparison we observe that a minimal m-block is just a simple binary matroid having critical exponent $m + 1$ such that every single-element contraction has critical exponent m.

Let $\eta(m,k)$ denote the least number of elements in any

(m,k)-matroid. Mullin and Stanton [53, Theorem 4.4] have determined $\eta(m,k)$. We give a proof of their result which is considerably shorter than the original. In addition we answer one of their questions by characterizing those (m,k)-matroids which have exactly $\eta(m,k)$ elements.

(3.4.1) Lemma (Mullin and Stanton [53, Theorem 2.1]). *For* $m \geq 1$, $\eta(m,k) \leq 2^{m+1} + k - m - 2$.

Proof. The matroid $PG(m,2) \oplus U_{k-m-1,k-m-1}$ is easily seen to be an (m,k)-matroid.//

From this lemma and the fact that $c(U_{k,k};2) = 1$, it follows that when $m = 1$, $\eta(m,k) = k + 1 = 2^{m+1} + k - m - 2$.

(3.4.2) Theorem. *For* $m > 1$,

(i) $\eta(m,k) = 2^{m+1} + k - m - 2$; *and*

(ii) *each* (m,k)-*matroid having* $\eta(m,k)$ *elements is isomorphic to* $PG(m,2) \oplus U_{k-m-1,k-m-1}$.

We shall use two lemmas in proving this result.

(3.4.3) Lemma. *If* M *is a matroid representable over* GF(q) *and* C^* *is a cocircuit of* M *such that* $c(M;q) - 1 = c(M\backslash C^*;q) = n$, *then* $|C^*| \geq q^n$. *Hence if* M *is minimal having critical exponent greater than* n, *then every cocircuit contains at least* q^n *elements.*

Proof. By Lemma 3.2.7, if $\{x_1,x_2,\ldots,x_t\}$ is a cocircuit of M, then

$$(3.4.4)\quad P(M;\lambda) = (\lambda - t)P(M\backslash x_1,\ldots,x_t;\lambda) + \sum_{j=2}^{t}\sum_{i=1}^{j-1} P(M\backslash x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1}/x_i,x_j;\lambda)\ .$$

If $c(M;q) - 1 = c(M\backslash x_1,\ldots,x_t;q) = n$, then put $\lambda = q^n$ in (3.4.4). Now $P(M;q^n) = 0$ and $P(M\backslash x_1,\ldots,x_t;q^n) > 0$. Moreover by (3.3.2), each term in the double summation is non-negative. We conclude

that $q^n - t \leq 0$.//

The next lemma proves the theorem for the case m = 2. The technique of the proof will be used several times throughout the remainder of this chapter.

(3.4.5) Lemma (Mullin and Stanton [53, Theorem 3.5]).

(i) $\eta(2,k) = k + 4$.

(ii) *If* M *is a* (2,k) *-matroid having* k + 4 *elements, then*

$M \cong PG(2,2) \oplus U_{k-3,k-3}$.

Proof. (i) By Lemma 3.4.1, $\eta(2,k) \leq k + 4$. Suppose that there is a minimal (2,k)-matroid N_o having fewer than k + 4 elements. Then N_o is a restriction of PG(k-1,2) and the latter has $2^k - 1$ elements. Since $k + 3 \leq 2^k - 1$ for $k \geq 3$, it follows that there is a (2,k)-matroid N on a set T having k + 3 elements. Suppose that T = {1,2,...,k + 3} and let B = {1,2,...,k} be a base of N. Then by Corollary 3.3.17, {k+1,k+2,k+3} is a circuit of N. Now choose an element i(k+1) from C(k+1,B)\{k+1} where C(k+1,B) is the fundamental circuit of k + 1 with respect to B. Then (B\{i(k+1)}) ∪ {k+1} is a base of N, hence by Corollary 3.3.17 again, {i(k+1), k+2,k+3} is a circuit of N. But by Lemma 3.3.14, {i(k+1), k+2,k+3}Δ{k+1, k+2, k+3} is a disjoint union of circuits of N. This contradicts the fact that N is simple. We conclude that $\eta(2,k) = k + 4$.

(ii) Let M be a (2,k)-matroid on a set S having k + 4 elements. Suppose that S = {1,2,...,k+4} and let B = {1,2,...,k} be a base of M. Now if D^* is a cobase of M, then $M|D^*$ is a simple binary matroid on a set of 4 elements and by Corollary 3.3.17, $M|D^*$ contains an odd circuit. Thus $M|D^*$, and hence M\B, is isomorphic to $U_{2,3} \oplus U_{1,1}$. Without loss of generality suppose

that $\{k+1, k+2, k+3\}$ is the circuit in $M\backslash B$. For $j = 1,2$, choose an element $i(k+j)$ from $C(k+j,B)\backslash\{k+j\}$ noting that these elements may be chosen to be distinct. Now $M|((S\backslash(B \cup \{k+j\})) \cup \{i(k+j)\}) \cong U_{2,3} \oplus U_{1,1}$, hence if F_3 is the rank 3 flat of M spanned by $S\backslash B$, then $\{i(k+1),i(k+2)\} \subseteq F_3$. Thus F_3 contains a 6 element subset T say. Since the only simple binary matroid of rank 3 on 6 elements is $M(K_4)$, we have that $M|T \cong M(K_4)$. Clearly $M(K_4)$ has a pair of disjoint bases B_1 and B_2, and B_1 is contained in a base B_1' of M. The set $S\backslash B_1'$ contains B_2 and an element x which is in neither B_1 nor B_2. As $M\backslash B_1' \cong U_{2,3} \oplus U_{1,1}$, it follows that $x \in F_3$. Therefore $|F_3| \geq 7$. Since M is simple and binary and rk M = 3, we conclude that $|F_3| = 7$ and $M|F_3 \cong PG(2,2)$. The result now follows since cork $(M|F_3)$ = cork M.//

Proof of Theorem 3.4.2. We shall prove both parts of this theorem together using induction on m. By Lemma 3.4.5, the proposition is true for m = 2 and all k. Now assume it is true for m - 1; that is assume that

(3.4.6) (i) $\eta(m-1,k) = 2^m + k - (m-1) - 2$; and

(3.4.6) (ii) every $(m-1,k)$-matroid having $\eta(m-1,k)$ elements is isomorphic to $PG(m-1,2) \oplus U_{k-m,k-m}$.

Let M be an (m,k)-matroid on a set S having $\eta(m,k)$ elements and let T be the set of coloops of M. If $N = M\backslash T$ and $k' = k - |T|$, then N is an (m,k')-matroid having $\eta(m,k')$ elements. Thus N is minimal having rank k' and critical exponent m + 1. But N has no coloops and hence N is minimal having critical exponent m + 1.

Let C^* be a cocircuit of N. Then by Lemma 3.4.3, $|C^*| \geq 2^m$ and by Lemma 3.4.1, N has at most $2^{m+1} + k' - m - 2$

elements. Hence $N\backslash C^*$ has at most $2^m + (k'-1) - (m-1) - 2$ elements. But $N\backslash C^*$ has rank $k' - 1$ and critical exponent m, hence $N\backslash C^*$ is an $(m-1,k'-1)$-matroid and so by (3.4.6)(i), $N\backslash C^*$ has at least $2^m + (k'-1) - (m-1) - 2$ elements. Thus $N\backslash C^*$ has exactly $2^m + (k'-1) - (m-1) - 2$ elements and $|C^*| = 2^m$. Hence by (3.4.6)(i)-(ii), $N\backslash C^* \cong PG(m-1,2) \oplus U_{k'-m-1,k'-m-1}$.

If y is a coloop of $N\backslash C^*$, then $y \cup C^*$ contains a cocircuit C_1^* of N containing y. But by Lemma 3.4.3, $|C_1^*| \geq 2^m$ and therefore $|C_1^* \Delta C^*| = 2$. However by Lemma 3.3.14, $C_1^* \Delta C^*$ is a disjoint union of cocircuits of N and hence we have a contradiction to Lemma 3.4.3. We conclude that $N\backslash C^*$ has no coloops; that is, $k' - m - 1 = 0$. Hence $N\backslash C^* \cong PG(m-1,2)$. But now N is a simple binary matroid of rank $m + 1$ having exactly $2^m - 1 + 2^m = 2^{m+1} - 1$ elements. Therefore $N \cong PG(m,2)$ and so $M \cong PG(m,2) \oplus U_{k-m-1,k-m-1}$ and $\eta(m,k) = 2^{m+1} + k - m - 2$.//

An n-*critical* graph [4, p.117] is a graph G having chromatic number n such that every proper subgraph of G has chromatic number less than n. The next result, a natural analogue of Lemma 3.4.3 for regular matroids, generalizes Dirac's well-known result [17, p.45] that an n-critical graph is $(n-1)$-edge-connected. The proof is very similar to the proof of Lemma 3.4.3 and hence is omitted.

(3.4.7) Theorem. *If M is a regular matroid and C^* is a cocircuit of M such that $\pi(M) - 1 = \pi(M\backslash C^*) = n$, then $|C^*| \geq n$. Hence if M is minimal having chromatic number $n + 1$, then every cocircuit of M contains at least n elements.*//

5. Minimal m-blocks.

In this section we shall show that if m is an integer greater than 1, then there exists a positive integer N(m) such that for all $n \geq N(m)$ there is a minimal m-block having corank n. This answers another of the questions of Mullin and Stanton [53].

If $m \in Z^+$, then let

$Q_m = \{\text{cork } M : M \text{ is a minimal } (m,k)\text{-matroid}\}$.

As every minimal (1,k)-matroid has k + 1 elements,

$Q_1 = \{1\}$ [53,§5]. However:

(3.5.1) Theorem. *If* $m \geq 2$, *then* Q_m *contains all but finitely many positive integers.*

This result will be proved by constructing suitable minimal (m,k)-matroids using the operation of series connection for matroids [11,§4]. Suppose that M and N are matroids with ground sets S and T respectively and assume that $S \cap T = \emptyset$. To form a series connection of M and N, choose an element x_1 of S and an element x_2 of T. These elements are called the *basepoints* of M and N respectively. Let x be a new element; that is $x \notin S \cup T$. Then the *series connection* S(M,N) of M and N with respect to the basepoints x_1 and x_2 is a matroid on $(S\backslash x_1) \cup (T\backslash x_2) \cup x$ whose circuits are the circuits of M not containing x_1, the circuits of N not containing x_2 and all sets of the form $(C_1\backslash x_1) \cup (C_2\backslash x_2) \cup x$ where $x_1 \in C_1$, $x_2 \in C_2$ and C_1 and C_2 are circuits of M and N respectively. For convenience we shall usually identify the elements x_1, x_2 and x so that S(M,N) is a matroid on $S \cup T$ where $S \cap T = \{x\}$. Note that S(M,N) is just the Nash-Williams union (see [86,p.121]) of matroids M' and N' on $S \cup T$ where M' is the direct sum of M and

the rank zero matroid on T\x and N' is the direct sum of N and the rank zero matroid on S\x.

The following basic properties of series connection will be used in proving Theorem 3.5.1.

(3.5.2) Lemma [11, Propositions 4.7 and 4.9]. *Suppose that* $e \in S$.

(i) *If* $e \neq x$, *then* $S(M,N)\backslash e = S(M\backslash e,N)$; *and*

(ii) *if* $e = x$, *then* $S(M,N)\backslash x = (M\backslash x) \oplus (N\backslash x)$.//

(3.5.3) Lemma [11, Theorem 6.16(v)]. *If the basepoint* x *of the series connection* S(M,N) *is neither a loop nor a coloop of* M *or* N, *then*

$$P(S(M,N);\lambda) = ((\lambda-2)/(\lambda-1))P(M;\lambda)P(N;\lambda) + P(M;\lambda)P(N/x;\lambda) + P(M/x;\lambda)P(N;\lambda).//$$

If M and N are graphic matroids, then S(M,N) takes a simple form. Suppose that G_1 and G_2 are graphs such that $M \cong M(G_1)$ and $N \cong M(G_2)$ and let e_1 and e_2 be edges of G_1 and G_2 respectively where $e_1 = a_1b_1$ and $e_2 = a_2b_2$. Let $G_{1,2}$ be the graph formed from G_1 and G_2 by identifying the vertices a_1 and a_2, deleting the edges e_1 and e_2 and adding a new edge $e_3 = b_1b_2$. The graph $G_{1,2}$ is called the *Hajós union* [24,p.40] or *conjunction* [58,p.180] of G_1 and G_2 with respect to e_1 and e_2. Now if S(M,N) is the series connection of M and N with respect to e_1 and e_2, then $S(M,N) \cong M(G_{1,2})$. The next result was motivated by a result of Hajós (see [4,Ex.8.1.11]) that if G_1 and G_2 are j-critical graphs, then their Hajós union is also j-critical.

(3.5.4) Lemma. *If* M *and* N *are matroids representable over* GF(q) *and each is minimal having critical exponent greater than* t

where $t \geq 1$, *then every series connection* $S(M,N)$ *of* M *and* N *is also representable over* $GF(q)$ *and minimal with critical exponent* $t + 1$.

Proof. Let M and N have ground sets S and T respectively where $S \cap T = \{x\}$ and suppose that $rk\ M = r_1$ and $rk\ N = r_2$. Let $\phi_1 : S \to V(r_1+r_2,q)$ and $\phi_2 : T \to V(r_1+r_2,q)$ be representations of M and N respectively where $\phi_1(S)$ is spanned by the first r_1 unit vectors of $V(r_1+r_2,q)$ and $\phi_2(T)$ is spanned by the last r_2 unit vectors of $V(r_1+r_2,q)$. Then define $\phi : S \cup T \to V(r_1+r_2,q)$ by

$$\phi(y) = \begin{cases} \phi_1(y), & \text{if } y \in S\backslash x; \\ \phi_2(y), & \text{if } y \in T\backslash x; \\ \phi_1(x) + \phi_2(x), & \text{if } y = x. \end{cases}$$

Clearly ϕ is a representation for $S(M,N)$, hence $S(M,N)$ is representable over $GF(q)$.

As neither M nor N has any loops or coloops, by Lemma 3.5.3, the chromatic polynomial of $S(M,N)$ is given by

$$P(S(M,N);\lambda) = ((\lambda-2)/(\lambda-1))P(M;\lambda)P(N;\lambda) + P(M;\lambda)P(N/x;\lambda) + P(M/x;\lambda)P(N;\lambda)$$

and, since M and N both have critical exponent greater than t, $P(M;q^t) = P(N;q^t) = 0$ and hence $P(S(M,N);q^t) = 0$; that is $c(S(M,N);q) > t$.

Now suppose that $e \in S \cup T$. If $e = x$, then by Lemma 3.5.2(ii), $S(M,N)\backslash e = (M\backslash x) \oplus (N\backslash x)$ and therefore $c(S(M,N)\backslash e;q) = \max\{c(M\backslash x;q), c(N\backslash x;q)\} = t$. If $e \in S\backslash x$, then by Lemma 3.5.2(i), $S(M,N)\backslash e = S(M\backslash e,N)$. If x is a coloop of $M\backslash e$, then $S(M\backslash e,N) \cong (M\backslash x,e) \oplus (N\backslash x) \oplus U_{1,1}$. Hence $P(S(M\backslash e,N);\lambda) = (\lambda-1)P(M\backslash x,e;\lambda)P(N\backslash x;\lambda)$ and as $P(M\backslash x,e;q^t) > 0$

and $P(N\backslash x;q^t) > 0$, we have $c(S(M,N)\backslash e;q) = t$. If x is not a coloop of $M\backslash e$, then by Lemma 3.5.3,

$$P(S(M,N)\backslash e;\lambda) = ((\lambda-2)/(\lambda-1))P(M\backslash e;\lambda)P(N;\lambda) + P(M\backslash e;\lambda)P(N/x;\lambda) + P(M\backslash e/x;\lambda)P(N;\lambda).$$

Therefore $P(S(M,N)\backslash e;q^t) = P(M\backslash e;q^t)P(N/x;q^t)$.

But $P(M\backslash e;q^t) > 0$ and

$P(N/x;q^t) = P(N\backslash x;q^t) - P(N;q^t) = P(N\backslash x;q^t) > 0$ and hence $c(S(M,N)\backslash e;q) = t$. A similar argument holds if $e \in T\backslash x$ and thus $S(M,N)$ is minimal having critical exponent greater than t; that is, $S(M,N)$ is minimal having critical exponent $t + 1$.//

It is clear that the cycle matroid of every (2^m+1)-critical graph is a minimal (m,k)-matroid for some k. Consider the following examples.

(3.5.5) Example. $M(K_{2^m+1})$ is a minimal (m,k)-matroid and

$$\text{cork}\,(M(K_{2^m+1})) = 2^{2m-1} - 2^{m-1}.$$

(3.5.6) Example. For every odd integer $n \geq 3$, a circuit C_n on n vertices is 3-critical. Now let the graph $G_{n,2^m-2} = C_n \vee K_{2^m-2}$ [4,p.58], that is, $G_{n,2^m-2}$ is obtained from C_n and K_{2^m-2} by joining every vertex of C_n to every vertex of K_{2^m-2}. It is clear (see [4,Ex.8.1.10]) that $G_{n,2^m-2}$ is (2^m+1)-critical and hence $M(G_{n,2^m-2})$ is a minimal (m,k)-matroid. Note that for $n = 3$, $G_{n,2^m-2} \cong K_{2^m+1}$. For $n = 5$,

$$\text{cork}\,(M(G_{n,2^m-2})) = 2^{2m-1} + 3(2^{m-1}) - 4.$$

(3.5.7) Example. The unique graph H which is the Hajós union of two copies of K_4 is 4-critical. Let $H_{2^m+1} = H \vee K_{2^m-3}$. Then again by [4,Ex.8.1.10], $M(H_{2^m+1})$ is a minimal (m,k)-matroid and

$$\text{cork}\,(M(H_{2^m+1})) = 2^{2m-1} + 5(2^{m-1}) - 7.$$

Proof of Theorem 3.5.1. If M and N are matroids on sets S and T respectively and neither M nor N has any loops or coloops, then S(M,N) is a matroid on a set of $|S| + |T| - 1$ elements and $rk(S(M,N)) = rk\ M + rk\ N$. Thus $cork\ (S(M,N)) = cork\ M + cork\ N - 1$

It follows that if M is a minimal (m,k)-matroid having no coloops, then for all n_1,n_2,n_3 in $Z^+ \cup \{0\}$,

$$cork\ M + n_1(cork(M(K_{2^m+1})) - 1) + n_2(cork\ (M(G_{5,2^m-2})) - 1) + n_3(cork\ (M(H_{2^m+1}) - 1)) \in Q_m.$$

Now it is easy to check that

$$g.c.d.\{cork\ (M(K_{2^m+1})) - 1,\ cork\ (M(G_{5,2^m-2})) - 1,\ cork\ (M(H_{2^m+1})) - 1\} = 1$$

and from this it follows that Q_m contains all but finitely many positive integers.

To verify the final step of the above we use the following elementary number theoretic argument. If $\{a_1,a_2,\ldots,a_n\} \subseteq Z^+$, then a *non-negative sum* of $a_1,a_2,\ldots,a_n$ is any integer of the form $y_1a_1 + y_2a_2 + \ldots + y_na_n$ where $\{y_1,y_2,\ldots,y_n\} \subseteq Z^+ \cup \{0\}$.
Now suppose that y_1 and y_2 are distinct relatively prime positive integers. Then by [36, Theorem 56,p.51], $\{0.y_2,1.y_2,\ldots,(y_1-1)y_2\}$ is a complete system of incongruent residues modulo y_1 [36,p.49].
Suppose that $a = y_1(y_2-1) + i$ for some $i \geq 0$.
Then, as $i \equiv j_iy_2 \pmod{y_1}$ for some j_i in $\{0,1,\ldots,y_1-1\}$, $j_iy_2 = i + h_iy_1$ for some $h_i \in Z^+ \cup \{0\}$. Since $j_i < y_1$, $h_i < y_2$ and so

$$a = y_1(y_2-1) + j_iy_2 - h_iy_1 = (y_2-1-h_i)y_1 + j_iy_2 .$$

Hence every integer greater than or equal to $y_1(y_2-1)$ is a non-negative sum of y_1 and y_2.

Now assume that y_1,y_2 and y_3 are positive integers such that g.c.d.$\{y_1,y_2,y_3\} = 1$. Then if g.c.d.$\{y_1,y_2\} = d$, g.c.d.$\{\frac{y_1}{d}, y_2\} = 1$ and hence every integer greater than or equal to $\frac{y_1}{d}(y_2-1)$ is a non-negative sum of $\frac{y_1}{d}$ and y_2.

Thus if $a \geq y_1(y_2-1)$ and d divides a, then a is a non-negative sum of y_1 and y_2. Since g.c.d.$\{y_1,y_2,y_3\} = 1$, g.c.d.$\{d,y_3\} = 1$ and so if $b \geq d(y_3-1)$, then b is a non-negative sum of d and y_3.

Now suppose that $c \in Z^+$ and $c \geq y_1(y_2-1) + d(y_3-1)$. Then $c = y_1(y_2-1) + d(y_3-1) + s$ where $s \geq 0$. Clearly $s = td + r$ where $t \geq 0$ and $0 \leq r < d$. Thus

$$c = (y_1(y_2-1) + td) + (d(y_3-1) + r).$$

But $d(y_3-1) + r = j_1y_3 + j_2d$ where $j_1,j_2 \in Z^+ \cup \{0\}$ and hence

$$c = (y_1(y_2-1) + (t + j_2)d) + j_1y_3 .$$

As d divides y_1, d divides $y_1(y_2-1) + (t+j_2)d$ and so $y_1(y_2-1) + (t+j_2)d = n_1y_1 + n_2y_2$ where $n_1,n_2 \in Z^+ \cup \{0\}$. We conclude that $c = n_1y_1 + n_2y_2 + j_1y_3$. That is, every integer greater than or equal to $y_1(y_2-1) + d(y_3-1)$ is a non-negative sum of y_1,y_2 and y_3. To get that Q_m is cofinite in Z^+, we apply this result taking

$$\{y_1,y_2,y_3\} = \{cork(M(K_{2^m+1}))-1, cork(M(G_{5,2^m-2}))-1, cork(M(H_{2^m+1}))-1\}. //$$

Note that if M and N are minimal m-blocks for $m \geq 1$, then although S(M,N) is always a minimal m-block, it is never tangential for by [11, Corollary 4.8], $S(M,N)/(T\backslash x) = M$. Indeed Tutte [83,p.207] has conjectured that the only tangential 2-blocks are PG(2,2), $M(K_5)$ and $M^*(G_P)$ where G_P is the Petersen graph.

6. The set Q_2.

In this section we determine the set Q_2 to within a single element. The Fano matroid, PG(2,2) is a minimal (2,3)-matroid and cork (PG(2,2)) = 4. Also $M(K_5)$ is a minimal (2,4)-matroid having corank 6. Taking series connections of copies of PG(2,2) and $M(K_5)$ shows that $4 + 3n_1 + 5n_2 \in Q_2$ and $6 + 3t_1 + 5t_2 \in Q_2$ for all n_1, n_2, t_1, t_2 in $Z^+ \cup \{0\}$. Thus $Q_2 \supseteq \{4,6,7,9,10,11,12,\ldots\}$. Moreover if $5 \in Q_2$, then $8 \in Q_2$. However in [53,§5] it is suggested that $5 \notin Q_2$. We shall prove this. The proof will use two lemmas. If M is a matroid on a set S, then N is a *parallel minor* of M if N can be written in the form $M/T_1\backslash T_2$ where $T_1 \subseteq S$, $T_2 \subseteq S\backslash T_1$ and, for each element x of T_2, there is an element y of $S\backslash(T_1 \cup T_2)$ such that x and y are parallel in M.

(3.6.1) Lemma. *Let* M *be a minimal* (2,k)*-matroid on a set* S *of* k + 5 *elements. Then* S *has no 7 element subset of rank 3 and no 9 element subset of rank 4.*

Proof. We shall prove this result when M has no coloops. The general result follows easily. Since M is simple and binary, $k \geq 4$. Moreover, if T is a 7 element subset of S of rank 3, then $M|T \cong PG(2,2)$. But M is minimal having critical exponent 3, hence if $e \in S\backslash T$, then by (3.3.8), $2 = c(M\backslash e;2) \geq c(M|T;2) = 3$; a contradiction. Therefore S has no 7 element subset of rank 3; that is the Fano matroid is not a restriction of M.

If U is a 9 element subset of S of rank 4, then cork (M|U) = cork M and since M has no coloops this implies that U = S. If M is graphic, then $M \cong M(G)$ where G is a connected graph having 5 vertices, 9 edges and chromatic number 5. As no

such graph exists, M is not graphic. Therefore if M is regular, then M has $M^*(K_5)$ or $M^*(K_{3,3})$ as a minor (see, for example, [86,p.176]). But $M^*(K_5)$ has rank 6 and $M^*(K_{3,3})$ has rank 4 and corank 5. Therefore $M \cong M^*(K_{3,3})$. Clearly $K_{3,3}$ can be covered by two subgraphs each of which is a disjoint union of circuits and so, by Corollary 3.3.15, $c(M;2) \leq 2$, contradicting the fact that $c(M;2) = 3$. Thus M is not regular and hence by a result of Bixby [3], M has a parallel minor isomorphic to PG(2,2) or $PG(2,2)^*$. If $M/T_1\backslash T_2 \cong PG(2,2)$ where $\widetilde{M/T_1} = M/T_1\backslash T_2$, then as M is simple, $|T_1| = |T_2| = 1$. But M is minimal having critical exponent 3 and therefore, as $P(M;\lambda) = P(M\backslash T_1;\lambda) - P(M/T_1;\lambda)$, we have that $c(M/T_1;2) = 2$. Thus $2 = c(M/T_1\backslash T_2;2) = c(PG(2,2);2) = 3$; a contradiction. Therefore M has a parallel minor isomorphic to $PG(2,2)^*$. Thus $PG(2,2)^*$ is isomorphic to $\widetilde{M}$. But $\widetilde{M} = M$ and M has 9 elements; a contradiction.//

(3.6.2) Lemma. *Let* M *be a minimal* (2,k)*-matroid on a set* S *of* k + 5 *elements. Then* S *has no 8 element subset of rank 4.*

Proof. Suppose that T is an 8 element subset of S of rank 4. We show firstly that M|T has two disjoint bases. By a result of Edmonds (see [19] or Theorem 2.3.1), this is so if and only if for all $A \subseteq T$, we have $|T\backslash A| \geq 2(\text{rk } T - \text{rk } A)$, that is, if and only if

(3.6.3) $|A| \leq 2 \text{ rk } A$.

Now, if $|A| \leq 3$, then as M is simple, (3.6.3) holds. If $4 \leq |A| \leq 6$, then since M is binary, M|T contains no 4-point line and hence $\text{rk } A \geq 3$ and (3.6.3) holds. If $|A| \geq 7$, then by Lemma 3.6.1, we have that $\text{rk } A \geq 4$ and again (3.6.3) holds. We

conclude that $M|T$ has a pair of disjoint bases, B_1 and B_2 say. Extend B_1 to a base B_1' of M. Then $S\backslash B_1' \supseteq B_2$, but $|S\backslash B_1'| = 5$, hence there is an element x of S such that $B_2 \cup x$ is a cobase of M. By Corollary 3.3.17, $B_2 \cup x$ is not independent in M. It follows that $B_1 \cup B_2 \cup x$ is a 9 element subset of S of rank 4. This contradicts Lemma 3.6.1.//

The technique of the next proof was used earlier in the proof of Lemma 3.4.5. A 3-*circuit* of a matroid N is a circuit of N of size three.

(3.6.4) Theorem. $5 \notin Q_2$.

Proof. Assume that $5 \in Q_2$, that is that there is a minimal (2,k)-matroid M on a set S of k + 5 elements. Let $S = \{1,2,\ldots,k+5\}$ and suppose that $B = \{1,2,\ldots,k\}$ is a base of M. Now by Corollary 3.3.17, if B^* is a cobase of M, then $M|B^*$ is a simple binary matroid containing an odd circuit. Thus $M|B^*$ and hence $M\backslash B$ is isomorphic to $U_{4,5}$ or $U_{2,3} \oplus U_{2,2}$ or L, where L is the matroid obtained from $M(K_4)$ by deleting an element (see [14, Figures 3.1-3.3]). In each case we show that S contains an 8 element subset of rank 4. The result then follows by Lemma 3.6.2.

Suppose that $M\backslash B \cong U_{4,5}$. For each j in $\{1,2,\ldots,5\}$, choose an element i(k+j) from $C(k+j,B)\backslash\{k+j\}$ where C(k+j,B) denotes the fundamental circuit of k+j with respect to B. If $B^*_j = ((S\backslash B)\cup\{i(k+j)\})\backslash\{k+j\}$, then B^*_j is a cobase of M. Now if $M|B^*_j \cong U_{4,5}$, then $(S\backslash B)\Delta B^*_j$ is a disjoint union of circuits. But $|(S\backslash B)\Delta B^*_j| = 2$ and M is simple. Thus $M|B^*_j \ncong U_{4,5}$. It follows that $M|B^*_j \cong U_{2,3} \oplus U_{2,2}$ or L. But $M|B^*_j$ has rank 4 so $M|B^*_j \cong U_{2,3} \oplus U_{2,2}$. Thus i(k+j) is contained in a 3-circuit C_j which is contained in B^*_j. Therefore

if F_4 is the rank 4 flat of M spanned by S\B, then $i(k+j) \in F_4$. If $i(k+j_1) = i(k+j_2)$ for $j_1 \neq j_2$, then $C_{j_1} \Delta C_{j_2} \subsetneq S\backslash B$ and hence $C_{j_1} \Delta C_{j_2}$ is independent. However $C_{j_1} \Delta C_{j_2}$ is a disjoint union of circuits, hence $C_{j_1} = C_{j_2}$. Without loss of generality, suppose that $C_1 = \{i(k+1), k+2, k+3\}$. Then $C_2 \neq C_1 \neq C_3$ and hence $i(k+2) \neq i(k+1) \neq i(k+3)$, although $i(k+2)$ and $i(k+3)$ may be equal. Now, as $i(k+3)$ was arbitrarily chosen in $C(k+3,B)\backslash\{k+3\}$ and $i(k+j) \in F_4$ for all j, $i(k+1) \notin C(k+3,B)$ and $C(k+3,B) \subseteq F_4$. But $|C(k+3,B)\backslash\{k+3\}| \geq 2$, therefore there is an element $i'(k+3)$ in $C(k+3,B)\backslash\{k+3,i(k+1),i(k+2)\}$, and $i'(k+3) \in F_4$. Hence F_4 contains a rank 4 subset of S of size 8. Thus for every cobase B^* of M we have $M|B^* \not\cong U_{4,5}$ and hence $M|B^* \cong U_{2,3} \oplus U_{2,2}$ or L.

Suppose now that $M\backslash B \cong L$ and let k+1 be the element of S\B contained in both 3-circuits of L. Choose an element i(k+1) from $C(k+1,B)\backslash\{k+1\}$. Then clearly $M|((S\backslash(B\cup\{k+1\}))\cup\{i(k+1)\}) \cong L$ and it follows, since M is simple and binary, that $M|((S\backslash B) \cup \{i(k+1)\}) \cong M(K_4)$. Now $M(K_4)$ has a pair of disjoint bases, B_1 and B_2 say, and B_1 is contained in a base B'_1 of M. The set $S\backslash B'_1$ contains B_2 and two other points x and y and by Corollary 3.3.17, $rk(B_2 \cup \{x,y\}) \leq 4$. Thus $B_1 \cup B_2 \cup \{x,y\}$ is an 8 element subset of S of rank not greater than 4. We conclude that for every cobase B^* of M we have $M|B^* \not\cong L$.

It remains to consider the case when $M\backslash B \cong U_{2,3} \oplus U_{2,2}$. Let $\{k+1, k+2, k+3\}$ be the 3-circuit in M\B and choose an element i(k+1) from $C(k+1,B)\backslash\{k+1\}$. Since $|C(k+2,B)\backslash\{k+2\}| \geq 2$, there is an element i(k+2) in $C(k+2,B)\backslash\{k+2,i(k+1)\}$. Let F_4 be the rank 4 flat of M spanned by S\B. Then clearly $\{i(k+1),i(k+2)\} \subseteq F_4$ Now either we can choose an element i(k+3) from $C(k+3,B)\backslash\{k+3,i(k+1),i(k+2)\}$ or not. In the first case,

$i(k+3) \in F_4$ and so S contains an 8 element subset of rank 4. In the second case, $C(k+3,B) = \{k+3,i(k+1),i(k+2)\}$ and hence $M|\{k+1,k+2,k+3,i(k+1),i(k+2)\} \cong L$. We may take $i(k+3)$ to be $i(k+1)$. Then as $M|((S\setminus(B \cup \{k+j\})) \cup \{i(k+j)\}) \cong U_{2,3} \oplus U_{2,2}$ for $j = 1,2,3$,it follows that either $k+4$ or $k+5$ is in the rank 3 flat F_3 of M spanned by $\{k+1,k+2,i(k+1)\}$. Without loss of generality suppose that $k+4 \in F_3$. Then $|F_3| \geq 6$, hence by Lemma 3.6.1, $|F_3| = 6$. Thus $M|F_3 \cong M(K_4)$ and the result follows as in the case when $M\setminus B \cong L$.//

By Theorem 3.4.2(i), the least member of Q_2 is 4. Thus to completely determine Q_2 we need only to decide whether $8 \in Q_2$. However, in this case the technique of the preceding proof seems unmanageable, since the number of binary matroids having 8 elements and containing an odd circuit is prohibitively large. A different approach gives the following partial result.

(3.6.5) Theorem. *If* M *is a minimal* (2,k)*-matroid on a set of* k+8 *elements, then* M *is not graphic.*

Proof. If M is graphic, then $M \cong M(G)$ where G is a simple connected graph having k+1 vertices and k+8 edges. In addition, G is minimal having chromatic number 5. . Let $(d_1,d_2,\ldots,d_{k+1})$ be the degree sequence of G where $d_1 \geq d_2 \geq \ldots \geq d_{k+1}$. Clearly G is neither a complete graph nor an odd circuit, hence by Brooks' Theorem (see [6, p.194]), $d_1 \geq 5$. Furthermore, as G is 5-critical, $d_{k+1} \geq 4$ [17, p.45]. Now $2|E(G)| = \sum_{i=1}^{k+1} d_i$. Thus $2k + 16 \geq 4k + 5$ and so $k \leq 5$. That is, G has at most 6 vertices. It follows since G is simple that G has exactly 6 vertices and 13 edges. Hence G is obtained from K_6 by deleting two edges. It is now easy to show that such a graph is not 5-critical.//

Chapter 4.

Circuit coverings and packings for matroids

1. Introduction.

If G is a finite graph having no loops, denote by $\alpha_v(G)$ the minimum cardinality of a set U of vertices such that every edge of G has at least one end-point in U and let $\beta_v(G)$ be the maximum cardinality of a set of mutually non-adjacent vertices of G. A well-known result of Gallai [28] is the following.

(4.1.1) If G has n vertices, then

$$\alpha_v(G) + \beta_v(G) = n.$$

Pursuing the analogy between vertices in graphs and cocircuits in matroids noted in the preceding chapter, we define the following parameters for an arbitrary loopless matroid M on a set S:

$\alpha(M)$ is the minimum cardinality of a set of cocircuits of M whose union is S and $\beta(M)$ is the maximum cardinality of a set of pairwise disjoint cocircuits of M. The following conjecture is based on (4.1.1).

(4.1.2) Conjecture (Welsh [87]). *If* M *is a loopless matroid having* $\gamma(M)$ *components, then*

(4.1.3) $$\alpha(M) + \beta(M) \leq \text{rk } M + \gamma(M) .$$

This conjecture was first published in [62] but little progress was made there towards resolving it. Subsequently I obtained a proof for the special case when M is binary and this is to be published in [64]. Very recently P.D. Seymour has proved the conjecture for arbitrary matroids. His proof is

quite different from mine and will appear in [70].

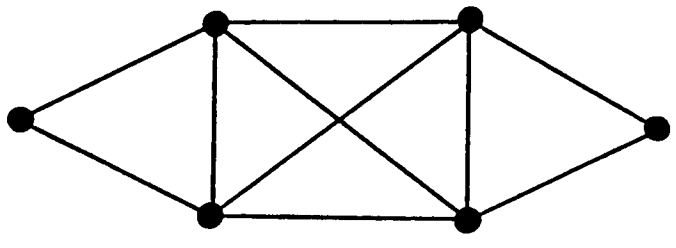
In section 2 of this chapter we prove several bounds on the parameters α and β. Most of these results have appeared in [62]. The proof of (4.1.2) for binary matroids is given in section 3.

2. Bounds on α and β.

It is clear that if G is a 2-connected loopless graph and M(G) is the cycle matroid of G, then $\alpha(M(G)) \leq \alpha_v(G)$ while $\beta(M(G)) \geq \beta_v(G)$. Moreover strict inequality may hold in both of these statements.

(4.2.1) Example. Let H be the graph shown in Figure 4.2.2.

(4.2.2) Figure



Then $\alpha_v(H) = 4$, $\alpha(M(H)) = 3$, $\beta_v(H) = 2$ and $\beta(M(H)) = 3$. //

Equality need not hold in (4.1.3). To see this note that the graph G which consists of two vertices joined by three edge-disjoint paths of length three satisfies

$rk(M(G)) + \gamma(M(G)) - \alpha(M(G)) - \beta(M(G)) = 1$.

An easy extension of this example gives examples of graphic matroids M for which $rk(M) + \gamma(M) - \alpha(M) - \beta(M)$ is arbitrarily large. Except for brief consideration in [70], the problem of determining those matroids for which equality holds in (4.1.3) seems almost untouched.

An immediate consequence of the definition of α is that for all loopless matroids M,

(4.2.3) $\alpha(M) = \min\{j \in Z^+ :$ M has hyperplanes $H_1, H_2, \ldots, H_j$ such that $\bigcap_{i=1}^{j} H_i = \emptyset\}$.

Hence

(4.2.4) $\alpha(M) \leq rk(M)$.

Furthermore, from [37,p.42], we have:

(4.2.5) If M is a simple connected matroid, then $\alpha(M) = rk(M)$

if and only if M has rank less than three or M is a projective geometry.

If $\{C_1^*, C_2^*, \ldots, C_j^*\}$ is a set of pairwise disjoint cocircuits of a loopless matroid M and $x_i \in C_i^*$ for all $1 \le i \le j$, then, by Lemma 3.3.12, $\{x_1, x_2, \ldots, x_j\}$ is independent in M. Thus

(4.2.6) $\beta(M) \le rk(M)$.

Similar arguments show that:

(4.2.7) If M is simple and connected, then

(i) $\beta(M) = rk(M)$ if and only if M is a coloop; and

(ii) if $\beta(M) = rk(M) - 1$, then $\alpha(M) = 2$.

Now recall from the preceding chapter that if M is a rank n loopless matroid representable over GF(q) and ϕ is a representation for M in V(n,q), then c(M;q) is the least number $H_1, H_2, \ldots, H_t$ of hyperplanes of <u>V(n,q)</u> such that $(\bigcap_{i=1}^{t} H_i) \cap \phi(S) = \emptyset$. Hence for such matroids M:

(4.2.8) $\alpha(M) \ge c(M;q)$.

The next result uses this fact together with Corollary 3.3.15 to link α, β, χ and π for loopless binary matroids. For an arbitrary loopless matroid M, let $\theta^*(M) = \max\{|C^*| : C^* \in C^*(M)\}$.

(4.2.9) <u>Theorem</u>. *Let* M *be a loopless binary matroid on a set* S. *Then*

(i) $\{\log_2\chi(M)\} \le \alpha(M) \le \beta(M)\{\log_2\pi(M)\}$; *and*

(ii) $\dfrac{|S|}{\theta^*(M)\{\log_2\pi(M)\}} \le \beta(M)$.

<u>Proof</u>. The left-hand inequality in (i) follows from (4.2.8) and the obvious inequality $\{\log_2\chi(M)\} \le c(M;2)$. Now suppose that $j = \{\log_2\pi(M)\}$. Then $c(M;2) \le j$ and so by Corollary 3.3.15,

$S = \bigcup_{i=1}^{j} S_i$ where $M.S_i$ is a disjoint union of cocircuits for all $1 \le i \le j$. If S_i is a union of t_i disjoint cocircuits, then $t_i \le \beta(M)$ and furthermore, $\sum_{i=1}^{j} t_i \ge \alpha(M)$ and $\theta^*(M) \sum_{i=1}^{j} t_i \ge |S|$. Thus $j\beta(M) \ge \alpha(M)$ and $j\beta(M)\theta^*(M) \ge |S|$. The right-hand inequality in (i) and inequality (ii) follow immediately.//

Clearly $\beta(M(K_n)) = 1$. Thus as $\chi(M(K_n)) = \pi(M(K_n)) = n$, it follows from (i) above that $\alpha(M(K_n)) = \{\log_2 n\}$.

The next two results give bounds on α and β for arbitrary loopless matroids.

Let

$$\theta(M) = \begin{cases} 1, & \text{if } M \text{ is free;} \\ \theta^*(M^*), & \text{otherwise.} \end{cases}$$

(4.2.10) Theorem. *Let M be a loopless matroid. Then*

$$\left\{\frac{rk(M)}{\theta^*(M)}\right\} \le \beta(M) \le rk\ M + 1 - \left\{\frac{\theta(M)}{2}\right\}.$$

Proof. Let $\{C_1^*, C_2^*, \ldots, C_\beta^*\}$ be a maximal set of pairwise disjoint cocircuits of M. Then $rk(\bigcup_{i=1}^{\beta} C_i^*) = rk(M)$ and so $\sum_{i=1}^{\beta} |C_i^*| \ge rk(M)$. But $|C_i^*| \le \theta^*(M)$ for all $1 \le i \le \beta$. Thus $\beta(M)\theta^*(M) \ge rk(M)$. This gives the lower bound on β.

By (4.2.6), the upper bound is certainly satisfied if $\theta(M) = 1$. If $\theta(M) > 1$, then let C be a circuit of M of maximum size. Add $rk(M) - \theta(M) + 1$ elements to C to get a subset B' of S which contains a base of M. Clearly B' intersects each of $C_1^*, C_2^*, \ldots, C_\beta^*$. Moreover, if $B' \cap C_i^*$ intersects C then by Lemma 3.3.12, since $B' \supseteq C$, $B' \cap C_i^*$ contains at least two elements of C. Therefore

$$\theta(M) = |C| \geq \sum_{i=1}^{\beta} |C_i^* \cap C|$$

$$\geq 2|\{i : 1 \leq i \leq \beta \text{ and } C_i^* \cap C \neq \emptyset\}| .$$

Thus

$$\theta(M) \geq 2(\beta(M) - (rk(M) - \theta(M) + 1))$$

and the upper bound on β follows.//

One lower bound on α comes from the obvious relation $\alpha(M)\theta^*(M) \geq |S|$. The next result gives another bound which is sometimes better and sometimes worse than this. If M is a rank r matroid on a set S and $\mathcal{I}$ is the collection of independent sets of M, then for $0 \leq j \leq r$, the j-*truncation* $M_{(j)}$ of M is the matroid on S which has $\{X \in \mathcal{I} : |X| \leq j\}$ as its collection of independent sets. If $X \subseteq S$, the rank $rk_{(j)}(X)$ of X in $M_{(j)}$ is $\min\{j, rk(X)\}$.

(4.2.11) Theorem. *Let* M *be a loopless matroid on a set* S. *Then*

$$\alpha(M) \geq \max_{\emptyset \neq A \subseteq S} \left\{\frac{|A|}{\text{cork}(A)+1}\right\} .$$

Proof. Let $\{C_1^*, C_2^*, \ldots, C_\alpha^*\}$ be a minimal set of cocircuits of M whose union is S. Then for each $1 \leq i \leq \alpha$, we may choose an element x_i from $C_i^* \setminus (\bigcup_{\substack{j=1 \\ j\neq i}}^{\alpha} C_j^*)$. Clearly $C_i^* \setminus x_i$ is M^*-independent. Furthermore by Lemma 3.3.12, $\{x_1, x_2, \ldots, x_\alpha\}$ is M-independent. Thus S is the union of one $M_{(\alpha)}$-independent set and α sets which are M^*-independent. Therefore for all $A \subseteq S$,

$\alpha\text{cork}(A) + rk_{(\alpha)}(A) \geq |A|$. But since $rk_{(\alpha)}(A) = \min\{\alpha, rk(A)\}$, we have that $\alpha\ \text{cork}(A) + \alpha \geq |A|$ for all $A \subseteq S$ and the result is immediate.//

The preceding theorem may also be deduced from a result for hypergraphs [2, Theorem 1, p.449].

3. Conjecture 4.1.2 for binary matroids.

The purpose of this section is to prove Conjecture 4.1.2 for binary matroids. In this section the ground set of a matroid N will be denoted by E(N). A flat of N of rank one will be called a 1-*flat* of N.

The proof of (4.1.2) for binary matroids will make frequent use of Lemmas 3.3.12 and 3.3.14 together with the following two well-known results (see, for example [86, Theorems 5.1.1 and 10.1.3]).

(4.3.1) Lemma. *Let* C^* *be a cocircuit of a matroid* M *and let* x *and* y *be distinct elements of* C^*. *Then there exists a circuit* C *of* M *such that* $C \cap C^* = \{x,y\}$. //

(4.3.2) Lemma. *If* M *is a binary matroid,* C *is a circuit of* M *and* C^* *is a cocircuit of* M, *then* $|C \cap C^*|$ *is even.* //

If M is a matroid having no coloops, then let $\alpha^*(M) = \alpha(M^*)$ and $\beta^*(M) = \beta(M^*)$. We now prove that the dual of Conjecture 4.1.2, and hence Conjecture 4.1.2, holds for binary matroids. This proof is based on a proof of the result for graphic matroids and is perhaps more easily understood in this light.

(4.3.3) Theorem. *Let* M *be a binary matroid having no coloops. Then*

$$\alpha^*(M) + \beta^*(M) \leq |E(M)| - rk(M) + \gamma(M). \tag{4.3.4}$$

Proof. We argue by induction on $|E(M)|$. Clearly (4.3.4) holds if $|E(M)| = 1$. Assume it hbolds for $|E(M)| < n$ and suppose that $|E(M)| = n$. We may also assume that M is connected for otherwise the result follows easily by the induction assumption.

Assume that M has a set of s pairwise parallel elements

($s \geq 2$). If $s = n$, then $M \cong U_{1,n}$ and (4.3.4) is easily verified.

(4.3.5) Lemma. *If* $\{x_1,x_2,\ldots,x_s\}$ *is a set of pairwise parallel elements of* M *and* $3 \leq s < n$, *then* (4.3.4) *holds by the induction assumption.*

Proof. Clearly $\gamma(M\backslash x_1,x_2) = \gamma(M)$ and $|E(M\backslash x_1,x_2)| = |E(M)| - 2$. By Lemma 3.3.12, $\{x_1,x_2\}$ does not contain a cocircuit of M and so $rk(M\backslash x_1,x_2) = rk(M)$. Moreover, since M is connected, $\alpha^*(M\backslash x_1,x_2) + 1 \geq \alpha^*(M)$. The lemma will thus be proved if we can show that

(4.3.6) $\beta^*(M\backslash x_1,x_2) \geq \beta^*(M) - 1$.

To verify this, suppose that $\{C_1,C_2,\ldots,C_{\beta^*}\}$ is a maximal set of pairwise disjoint circuits of M. If either $|(\bigcup_{i=1}^{\beta^*} C_i) \cap \{x_1,x_2\}| \leq 1$ or $\{x_1,x_2\} = C_i$ for some i in $\{1,2,\ldots,\beta^*\}$, then (4.3.6) is immediate. The only other possibility is that $x_1 \in C_i$ and $x_2 \in C_j$ for i and j distinct members of $\{1,2,\ldots,\beta^*\}$. But $\{x_1,x_2\} \in C(M)$ and hence $(C_j\backslash x_2) \cup x_1 \in C(M)$. It follows that there is a circuit of M, and hence of $M\backslash x_1,x_2$, contained in $(C_i \cup ((C_j\backslash x_2) \cup x_1))\backslash x_1$. Therefore (4.3.6) holds and so Lemma 4.3.5 is proved.//

By the above we may assume that:

(4.3.7) No 1-flat of M contains more than two elements.

We now distinguish three cases:

(I) M has a cocircuit of size two.

(II) Every cocircuit of M contains at least three elements but M has a cocircuit containing only two 1-flats.

(III) Every cocircuit of M contains at least three 1-flats.

Case I. If {c,d} is a cocircuit of M, then by Lemma 3.3.12, $\gamma(M/c) = \gamma(M)$ and cork (M/c) = cork (M). Moreover, by Lemma 3.3.12 again, $\alpha^*(M/c) = \alpha^*(M)$ and $\beta^*(M/c) = \beta^*(M)$. The result follows by applying the induction assumption to M/c.

Case II. Let C_1^* be a cocircuit of M containing at least three elements but only two 1-flats. Then by (4.3.7), either (i) $C_1^* = \{u,v,w\}$ where {u,v} and {w} are 1-flats of M; or (ii) $C_1^* = \{u,v,w,x\}$ where {u,v} and {w,x} are 1-flats of M.

Case II(i). By Lemma 3.3.12, since M is connected, M/w is connected. Moreover cork (M/w) = cork (M) and $\beta^*(M/w) \geq \beta^*(M)$. We shall show that $\alpha^*(M/w) \geq \alpha^*(M)$ from which (4.3.4) will follow by induction. Suppose that $\{C_1,C_2,\ldots,C_t\}$ is a minimal collection of circuits of M/w whose union is E(M/w). If for some i in $\{1,2,\ldots,t\}$, $C_i \cup w$ is a circuit of M, then (4.3.4) holds. Therefore suppose that for all $1 \leq i \leq t$, C_i is a circuit of M. The next lemma completes the proof for case II(i).

(4.3.8) Lemma. *Let* {c,d,e} *be a cocircuit of a binary connected matroid* N *where* {d,e} *and* {c} *are* 1*-flats of* N. *If* $\{C_1,C_2,\ldots,C_t\}$ *is a minimal collection of circuits of* N/c *covering* E(N/c) *and* $\{C_1,C_2,\ldots,C_t\} \subseteq C(N)$, *then there is a set of* t *circuits of* N *covering* E(N) *such that* c *is in exactly two of these circuits.*

Proof. Using Lemma 3.3.12 it is easy to show that N has a cocircuit D^* such that $D^* \cap \{c,d,e\} = \{d,e\}$. Clearly $D^* \in C^*(N/c)$.

Since $\{C_1,C_2,\ldots,C_t\} \subseteq C(N)$, we have by Lemma 3.3.12 that $\{d,e\} \in \{C_1,C_2,\ldots,C_t\}$, say $\{d,e\} = C_1$. Thus for $2 \leq i \leq t$, $C_i \cap \{d,e\} = \emptyset$. Now D^* properly contains

$\{d,e\}$ and therefore for some j in $\{2,3,\ldots,t\}$, say $j = 2$, $C_j \cap D^* \neq \emptyset$. Since N is connected there is a circuit of N containing c and intersecting C_2. Among such circuits choose one, say D_1, such that $|D_1 \backslash C_2|$ is minimal. Since $D_1 \cap \{c,d,e\} \neq \emptyset$, we have by Lemma 3.3.12, that D_1 contains exactly one of d and e, say d. Consider $D_1 \Delta C_2$. By Lemmas 3.3.14 and 3.3.12, this contains a circuit D_2 containing c and d. By the choice of D_1 it follows that $D_2 \backslash C_2 = D_1 \backslash C_2$. But $D_2 \backslash C_2 = D_1 \cap D_2$. Therefore by Lemma 3.3.14, $C_2 \backslash D_1 = D_2 \backslash D_1$ and so $D_2 = D_1 \Delta C_2$. But now $\{(D_1 \backslash d) \cup e, D_2, C_3, \ldots, C_t\}$ is a set of circuits of N covering E(N) and c is in both $(D_1 \backslash d) \cup e$ and D_2 but c is not in $C_3, C_4, \ldots, C_t$.//

Case II(ii). M/w has two components: a loop $\{x\}$ and $M/w \backslash x$. Thus $\gamma(M) = \gamma(M/w) - 1$. Moreover, $\beta^*(M/w) \geq \beta^*(M)$. We show that

(4.3.9) $\quad \alpha^*(M) \leq \alpha^*(M/w) - 1$

from which (4.3.4) follows using the induction assumption.

Let $\{C_1, C_2, \ldots, C_t\}$ be a minimal collection of circuits of M/w covering E(M/w). Then $\{x\} \in \{C_1, C_2, \ldots, C_t\}$ say $\{x\} = C_t$. If for distinct elements i and j of $\{1,2,\ldots,t-1\}$, $C_i \cup w$ and $C_j \cup w$ are circuits of M, then $C_i \cup w$ and $C_j \cup x$ are circuits of M and (4.3.9) holds. Next suppose that there is exactly one element i of $\{1,2,\ldots,t-1\}$ such that $C_i \cup w$ is a circuit of M. Then, as $x \notin \bigcup_{i=1}^{t-1} C_i$, Lemma 3.3.12 implies that $\{u,v\} \in \{C_1, C_2, \ldots, C_{t-1}\}$, say $\{u,v\} = C_1$. Now $|(C_i \cup w) \cap \{u,v,w,x\}|$ is non-zero and hence exceeds one, and $x \notin (C_i \cup w) \cap \{u,v,w,x\}$. Moreover $\{u,v\} \not\subseteq C_i \cup w$. Therefore C_i contains one of u and v, say u. But then $(C_i \backslash u) \cup v \cup w$ is a circuit of M. Thus

$(C_i \backslash u) \cup v \cup x$ is a circuit of M and

$\{(C_i \backslash u) \cup v \cup x, C_2, \ldots, C_{i-1}, C_i \cup w, C_{i+1}, \ldots, C_{t-1}\}$

is a collection of t - 1 circuits of M whose union is E(M). Hence (4.3.9) holds.

To complete case II(ii), suppose that for all i in $\{1,2,\ldots,t-1\}$, C_i is a circuit of M and hence of $M\backslash x$. In this case we have, on taking $N = M\backslash x$ in Lemma 4.3.8, that there is a covering X of $E(M\backslash x)$ with t - 1 circuits of $M\backslash x$ such that w is in exactly two of these circuits, say A_1 and A_2. Now $C(M\backslash x) \subseteq C(M)$ and $(A_2\backslash w) \cup x \in C(M)$. Thus replacing A_2 by $(A_2\backslash w) \cup x$ in X gives a covering of E(M) with t - 1 circuits of M; that is (4.3.9) holds.

Case III. If every cocircuit of M contains at least three 1-flats, then every cocircuit of $\tilde{M}$ contains at least three elements.

The next lemma is an analogue for binary matroids of a graph-theoretic result of Kaugars (see [35, p.31]).

(4.3.10) Lemma (Seymour [69]). *Let* N *be a simple connected binary matroid having no cocircuits of size less than three. Then* N *has a connected hyperplane.*

Proof. Suppose that every circuit of N has rk(N) + 1 elements. Then, since N has no cocircuits of size two, N has at least two circuits. Let y and z be distinct elements of a circuit C_1 of N and suppose that $x \in E(N)\backslash C_1$. Then $(C_1\backslash y) \cup x$ and $(C_1\backslash z) \cup x$ are circuits of N and hence by Lemma 3.3.14, $((C_1\backslash y) \cup x) \Delta ((C_1\backslash z) \cup x) = \{y,z\}$ is a disjoint union of circuits of N contradicting the simplicity of N.

We may therefore assume that N has a circuit of size less

than rk(N) + 1 and hence that E(N) has a non-empty subset A which is maximal with respect to being both connected and non-spanning. Clearly A is a flat of N.

As N is connected there is a circuit intersecting both A and its complement. Choose such a circuit C_1 so that $|C_1 \cap (E(N)\backslash A)| = j$ is minimal. We shall show that $j = 2$ from which it follows that A is a hyperplane and hence that A is the required connected hyperplane of N.

If $C_1 \supseteq E(N)\backslash A$ and c and d are distinct elements of $E(N)\backslash A$, then, by the choice of C_1, every circuit of N containing one of c and d also contains the other. Thus $E(N)\backslash\{c,d\}$ is non-spanning in N and, since N has no coloops, it follows that {c,d} is a cocircuit of N; a contradiction. Therefore $E(N)\backslash(A \cup C_1)$ is non-empty so let e be an element of this set. As $C_1 \cup A$ is connected, we have, by the choice of A, that $C_1 \cup A$ is spanning. Thus $C_1 \cup A$ contains a base B of N. Let C_2 be the fundamental circuit of e with respect to B. Then either $C_2 \cap A = \emptyset$ or not. In the first case, by Lemma 3.3.14, $C_1 \Delta C_2$ contains a circuit C_3 containing e. Then as $C_2\backslash e \subseteq C_1$, we have that $C_3 \cap A \neq \emptyset$. Hence $|C_3 \cap C_1 \cap (E(N)\backslash A)| \geq j - 1$ and so $|C_2| \leq 2$; a contradiction. Thus we may assume that $C_2 \cap A \neq \emptyset$. Then since $|C_2 \cap (E(N)\backslash A)| \geq j$, by Lemma 3.3.14, C_2 contains exactly $j - 1$ elements of $C_1 \cap (E(N)\backslash A)$. But $C_1 \Delta C_2$ contains a circuit C_4 containing e which, since M is simple, intersects A. Thus $j = 2$ as required.//

In case III, $\tilde{M}$ is connected and we have, by the preceding lemma, that $\tilde{M}$ has a connected hyperplane. Therefore M has a connected hyperplane, H say, and since rk $M \geq 2$, $H \neq \emptyset$.

Let $E(M)\backslash H = C^*$ and let $\{C_1, C_2, \ldots, C_{\beta^*}\}$ be a maximal set of pairwise disjoint circuits of M.

If there is an element x of $C^*\backslash(\bigcup_{i=1}^{\beta^*} C_i)$, then since C^* contains at least three 1-flats of M, if $u \in C^*\backslash\{x\}$, then there is an element v of $C^*\backslash\{x\}$ such that $\{u,v\}$ is not contained in a 1-flat and hence is not a circuit. By Lemma 4.3.1 it follows that there is a circuit of M containing $\{u,v\}$ and intersecting H. As $M|H$ is connected, we conclude that $M\backslash x$ is connected. The result follows by applying the induction assumption to $M\backslash x$.

Now suppose that $\bigcup_{i=1}^{\beta^*} C_i \supseteq C^*$. Then, since by Lemma 4.3.2, $|C_i \cap C^*|$ is even for all i, $|C^*|$ is even and so $|C^*| \geq 4$. We choose elements x and y from C^* as follows. If for some i in $\{1,2,\ldots,\beta^*\}$, $|C_i| = 2 = |C_i \cap C^*|$, then let $C_i = \{x,y\}$. Otherwise choose C_j such that $C_j \cap C^* \neq \emptyset$ and let x and y be any two elements of $C_j \cap C^*$. In either case a slight extension of the argument used in the preceding paragraph shows that $M\backslash x,y$ is connected. The result then follows by applying the induction assumption to $M\backslash x,y$.

This completes the proof of case III and thereby finishes the proof of Theorem 4.3.3.//

The above proof makes frequent use of the fact that M is binary and the method does not seem to generalize to arbitrary non-binary matroids. In particular Lemma 4.3.10 fails for $M = U_{3,5}$.

My original proof of Theorem 4.3.3 was only for graphic matroids and used, in place of Lemma 4.3.10, a result of Kaugars [35,p.31]. I am indebted to Dr P.D. Seymour for

suggesting that my proof could be generalized to binary matroids, and for communicating the required analogue of Kaugars' theorem.

Chapter 5.

Percolation and the Tutte polynomial

1. Introduction.

Classical percolation theory as introduced by Broadbent and Hammersley [5] is concerned with the spread of fluid through a randomly dammed medium. The media under consideration are usually locally finite, connected graphs in which each edge is, independently of all other edges, open or closed to fluid with probability p or 1 - p respectively. The theory of percolation and its applications in various physical fields has been reviewed by Frisch and Hammersley [27], Essam [23], Shante and Kirkpatrick [73] and Welsh [88]. The bibliographies of these review papers indicate that a vast amount of effort has been expended on the main problems of percolation theory. However it seems that since about 1965 the amount of theory proven as against numerical evidence gained is extremely small (see [88]). For example, the exact value of the critical probability for any regular crystal lattice seems to be unknown. In addition, little is known about the theoretical form of the percolation probability. In this chapter we study a very basic abstraction of classical percolation, namely percolation on clutters. This has an advantage over the classical theory in that for example it enables atom and bond percolation to be considered simultaneously. It also forces one to consider what special properties of graphs or regular crystal lattices are necessary for the proving of results in percolation theory. As will be seen, not very much more can be proved about the special case of percolation on regular crystal lattices than can be shown to

hold in the abstract case. In section 2 we discuss the percolation model on clutters. If A is a clutter on a finite set S, then we assign to each element of S, independently of all other elements, either the state open with probability p or the state closed with probability 1 - p. The *percolation probability* R(A;p) of A is then the probability that there is a member of A every element of which is open. Clearly R(A;p) is a polynomial in p. A fundamental property of this polynomial is that it satisfies a deletion-contraction formula of a slightly more general form than that satisfied by the chromatic polynomial of a matroid. This formula is used to get a new and shorter proof of the theorem of Hammersley [32] that fluid percolates more freely through a graph when the edges are randomly blocked with probability q than when the vertices are randomly blocked with probability q.

In section 3 we formalize an idea which Hammersley has used extensively in both classical percolation theory and first-passage percolation theory (see [29,33,34]). Essentially the idea is to bound the percolation probability above by the percolation probability of a structure which is obtained by pulling the original structure apart, thus reducing the statistical dependence between paths to a minimum. In section 4 we do the opposite and find a lower bound for the clutter percolation probability.

In 1947, Tutte [78] made a systematic study of the following natural set of functional equations defined on the class of all finite graphs:

(5.1.1) $f(G) = f(H)$, if $G \cong H$;

(5.1.2) $f(G) = f(G\backslash e) + f(G/e)$ provided e is neither a bridge nor a loop of G;

and

(5.1.3) $f(G_1 + G_2) = f(G_1)\, f(G_2)$,

where $G_1 + G_2$ denotes the disjoint union of the graphs G_1 and G_2. Tutte showed that subject to boundary conditions specifying f on bridges and loops, f is uniquely defined and is what is now called the Tutte polynomial of the cycle matroid of the graph. This result was extended to arbitrary matroids by Brylawski [12].

The connection between the Tutte polynomial of a graph and the percolation model on that graph has been noted by Essam [22], Fortuin and Kasteleyn [25] and Temperley and Lieb [77]. In the last three sections of this chapter, motivated by the percolation model and in particular by the fact that the clutter percolation probability satisfies a set of equations closely related to (5.1.1) - (5.1.3), we have attempted to extend the theory of the Tutte polynomial. In §5, we study a set of equations for matroids which generalize (5.1.1) - (5.1.3) and thereby obtain a slight extension of Brylawski's result on the Tutte polynomial for matroids. In §6 we show that the theory of the Tutte polynomial does not extend to arbitrary finite clutters and that matroids are the limiting structures for which a Tutte polynomial can be defined. In the final section, §7, we weaken the conditions that a Tutte polynomial must satisfy and show that the clutter percolation probability is the unique non-degenerate function which is defined for all finite non-empty clutters and which satisfies

these conditions.

This chapter is the result of joint work of the author and D.J.A. Welsh [65,66]. The percolation theory terminology used here is fairly standard. The term *regular crystal lattice* will mean either the infinite square lattice, the infinite triangular lattice or the infinite hexagonal lattice. If $p \in [0,1]$, then we shall usually write q for 1 - p. This differs from the notation of Chapter 3 where q was used to denote a prime power.

2. Percolation.

In this section the atom and bond percolation models for graphs are discussed and percolation on clutters is introduced. In addition a new proof is given of Hammersley's result that fluid percolates more readily in the bond percolation model on a graph than in the atom percolation model on the same graph.

Let G be a locally finite, directed or undirected graph with a countable vertex set. The *bond percolation model* on G supposes that each edge of G has, independently of all other edges, probability p of being *open* and probability q of being *closed*. If i,j are two vertices of G we let $P_{i,j}(p)$ denote the *interaction probability*, that is, the probability that there is a path of open edges in G joining i and j. When G is directed, this path must be directed from i to j.

Classical percolation theory (see [5]) is primarily concerned with the case when the graph G is some regular crystal lattice with a fixed origin 0 which is regarded as a source of fluid. In this case we denote by $P_n(p)$ the probability that there are open paths from 0 to at least n other vertices of the lattice. Clearly for a particular graph, $P_{i,j}(p)$ and $P_n(p)$ are closely related but very little seems to be known about their form. For example, it is easy to see that as $n \to \infty$, $P_n(p)$ converges to a function P(p) which represents the probability that fluid can spread from the origin to an infinite number of points. But although $P_n(p)$ is a polynomial in p for each n, a long-standing conjecture of Hammersley [31] that P(p) is a continuous function is still not settled even when the underlying graph is the infinite square lattice.

In the *atom percolation model* on G, each vertex of G has, independently of all other vertices, probability p of being *open* and probability q of being *closed*. In this model we shall call a path open if all its vertices (except possibly the first) are open. In the bond percolation model a path is called open if and only if all its edges are open. For the rest of the chapter, except where otherwise stated, all graphs considered will be undirected.

Let A be a clutter on a *finite* set S and suppose that we assign, to each element of S, independently of all other elements, either the state *open* with probability p or the state *closed* with probability q. If A is a subset of S we call A *open* if every element of A is open, and we denote by R(A;p) the probability that some member of the clutter A is open.

(5.2.1) Example. If A is the collection of edge-sets of paths joining two vertices i and j in a finite graph G, then R(A;p) is just the interaction probability $P_{i,j}(p)$.//

(5.2.2) Example. Let S be the edge-set of K_n, the complete graph on n vertices. Suppose that A is the collection of edge-sets of all maximal planar subgraphs of K_n, and let $\bar{A}$ be the complementary clutter of A, that is, $\bar{A} = \{S\backslash A : A \in A\}$. Then $R(\bar{A};q)$ is the probability that a random graph on n vertices is planar.//

An *hereditary property* H of graphs is a property of graphs such that if G has H and G' is a subgraph of G, then G' has H. It is clear that in Example 5.2.2 we are just treating a special hereditary property and that for any such property H, the probability that a random graph has H can be expressed as

the percolation probability of a suitable clutter.

If A is a clutter on a finite set S, then the *blocking clutter* or *blocker* b(A) of A is the clutter on S whose members are the minimal subsets of S which have non-empty intersection with every member of A (see Edmonds and Fulkerson [21]). It is easy to see that

$$b(b(A)) = A.$$

Moreover, in any assignment of the states open and closed to the elements of S, either some member of A is open or some member of b(A) is closed and these two events are mutually exclusive. Hence for any clutter A and all p in [0,1],

(5.2.3) $R(A;p) + R(b(A);q) = 1.$

Thus, in particular, if A and b(A) are isomorphic, then $R(A;\frac{1}{2}) = \frac{1}{2}$. An example of such a clutter is the set of paths from top to bottom in an n × (n-1) chessboard (see Seymour and Welsh [72]).

The following properties of clutters are well-known (see [68]). If A is a clutter on a finite set S and $T \subseteq S$, then the *restriction* $A|T$ and *contraction* $A.T$ of A to T are defined respectively by

$$A|T = \{A_i \in A : A_i \subseteq T\} \quad \text{and}$$

$$A.T = \{\text{minimal members of } \{A_i \cap T : A_i \in A\}\}.$$

Sometimes $A|T$ is called the *deletion* of $S\backslash T$ from A. It is easy to check that if $S \supseteq T_1 \supseteq T_2$, then

$$(A|T_1)|T_2 = A|T_2 \quad \text{and}$$

$$(A.T_1).T_2 = A.T_2 .$$

In addition, restriction and contraction are related through the following equation which resembles (1.2.11):

$$b(A)|T = b(A.T)$$

If $e \in S$, we shall write $A\backslash e$ for $A|(S\backslash e)$ and A/e for $A.(S\backslash e)$. An element e is *essential* for A if e belongs to every member of A. It is *redundant* for A if it belongs to no member of A.

If $A = (A_i : i \in I)$ and $A' = (A_j' : j \in J)$ are clutters on disjoint sets S and T, the *direct sum* $A \oplus A'$ of A and A' is the clutter on $S \cup T$ whose members are all sets of the form $A_i \cup A_j'$ $(i \in I, j \in J)$. The *union* $A \cup A'$ of A and A' is the clutter $\{X : X \in A \text{ or } X \in A'\}$ on $S \cup T$. Note that the operations of direct sum and union apply only to clutters on disjoint sets, hence whenever we write $A \oplus A'$ or $A \cup A'$ it will be implicitly assumed that A and A' have disjoint ground sets.

The present treatment of percolation on clutters is based on the following deletion-contraction and direct sum formulae which in principle allow $R(A;p)$ to be calculated for any non-empty clutter A on a finite set S. These formulae are easily proved - the first three by conditioning.

(5.2.4) If e is redundant, then $R(A;p) = R(A\backslash e;p)$.

(5.2.5) If e is essential, then $R(A;p) = pR(A/e;p)$.

(5.2.6) If e is neither redundant nor essential, then

$$R(A;p) = qR(A\backslash e;p) + pR(A/e;p).$$

(5.2.7) $R(A_1 \oplus A_2;p) = R(A_1;p)\, R(A_2;p)$.

(5.2.8) If $S = \{e\}$ and A is non-empty, then

$$R(A;p) = \begin{cases} p, & \text{if e is essential;} \\ 1, & \text{if e is redundant.} \end{cases}$$

Note that the constraint on e in (5.2.6) may be dropped and (5.2.4) - (5.2.6) may be stated as a single condition. The conditions have been written separately as later they will be compared with conditions on the Tutte polynomial for matroids.

Now if A is the collection of edge-sets of n-edge trees containing the origin in a finite section of a regular crystal lattice, then R(A;p) is the probability $P_n(p)$ discussed earlier; that is, the probability that at least n vertices of the regular crystal lattice other than 0 are wet by fluid in the bond percolation model. It appears from numerical evidence (see [75]) that the curves for $P_n(p)$ for the regular crystal lattices have a similar shape which can be characterized by the fact that they have exactly one point of inflection. Indeed some estimates [45] of the critical probability of a regular crystal lattice have been based on finding this point of inflection. One may ask whether this characteristic shape is a property of R(A;p) for all finite clutters A or whether it is an intrinsic property of clutters of paths in graphs (*graphic clutters*). The little progress made on this in [66] is contained in the next result.

(5.2.9) Theorem. *If* $p \in [0,1]$ *and* A *is a clutter on a non-empty finite set, then*

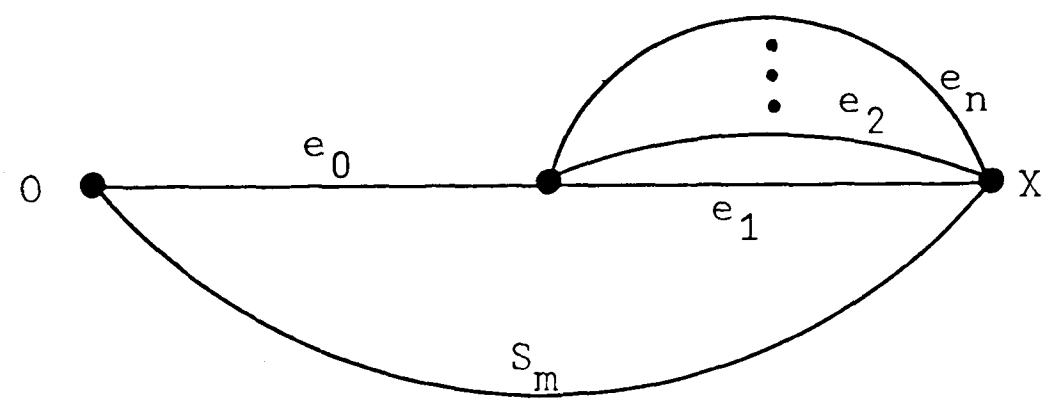
(i) R(A;p) *is a strictly monotonic increasing function of* p *unless every element of* A *is redundant in which case* $\frac{dR(A;p)}{dp}$ *is identically zero.*

(ii) *If* n_1 *and* n_1' *denote the numbers of members of size one of* A *and* b(A) *respectively, then*

$$\left.\frac{dR(A;p)}{dp}\right]_{p=0} = n_1 \quad \text{and} \quad \left.\frac{dR(A;p)}{dp}\right]_{p=1} = n_1' .$$

It follows easily from (ii) that when neither A nor b(A) has a member of size less than one, $\frac{d^2R(A;p)}{dp^2}$ has at least one zero in (0,1). If it could be shown that there is exactly one such zero, p_o say, then it would be natural to regard p_o as the *critical probability* of the clutter A. Unfortunately this is not true for arbitrary clutters.

(5.2.10) Example (A.W. Ingleton and P.D. Seymour [42]). Let A be the clutter of edge-sets of paths joining 0 to X in the graph $H_{m,n}$ shown in Figure 5.2.11 where $e_0, e_1, e_2, \ldots, e_n$ are edges and S_m is a path of length m.



(5.2.11) Figure

Then $R(A;p) = p^m + p(1-q^n) - p^{m+1}(1-q^n)$.

Taking m = n = 20, it is straightforward to check that R(A;p) has a point of inflection in each of the intervals (0.094,0.096), (0.500, 0.502) and (0.904, 0.906).//

It would be interesting to know for which clutters the percolation probability has a unique point of inflection and, in particular, whether finite sections of the regular crystal lattices have this property.

Proof of Theorem 5.2.9. We shall prove (i) by induction of the size of the ground set of A. Clearly (i) is true for all clutters on sets of size one. Suppose it is true for clutters

on sets of size less than n and let A be a clutter on a set S of n elements. Then from (5.2.6), if e is an element of S which is neither essential nor redundant for A,

$$\frac{dR(A;p)}{dp} = p\frac{dR(A/e;p)}{dp} + q\frac{dR(A\backslash e;p)}{dp} + R(A/e;p) - R(A\backslash e;p) . \tag{5.2.12}$$

Moreover, as e is neither essential nor redundant, $R(A\backslash e;p) < R(A;p) < R(A/e;p)$ for p in (0,1). Thus for p in (0,1), the right-hand side of (5.2.12) is positive and the monotonicity of R(A;p) follows. The rest of the proof of (i) is straightforward.

To prove (ii) it suffices to show that $\left.\frac{dR(A;p)}{dp}\right|_{p=0} = n_1$, for the rest of the proposition then follows from (5.2.3). But this is easy to check using induction on the size of the ground set of A and hence this is omitted.//

Even when A is a graphic clutter it seems difficult to find out much more about R(A;p).

As an example of the use of the deletion-contraction formula we give a short proof of a result of Hammersley [32]. Let G be a locally finite connected graph, O be a fixed vertex of G and X be a non-empty subset of V(G). Let $B_G(X;p)$ denote the probability that there is an open path from O to X in the bond percolation model on G and let $A_G(X;p)$ be the probability that there is an open path from O to X in the atom percolation model on G.

(5.2.13) Theorem. *Let G be a locally finite connected graph and X be a non-empty subset of* V(G). *Then the atom and bond percolation probabilities are related by*

$$A_G(X;p) \le B_G(X;p) . \tag{5.2.14}$$

Proof. The result will first be proved for finite connected graphs G using induction on $|E(G)|$. Clearly (5.2.14) holds when $|E(G)| = 1$. Assume (5.2.14) holds for all finite connected graphs with fewer than n edges and let G be a connected graph with n edges.

First note that we may assume that every edge of G lies on a path from 0 to some vertex in X. Now suppose that e is an edge of G. If we write $B_G(p)$ and $A_G(p)$ for $B_G(X;p)$ and $A_G(X;p)$ respectively, then

$$B_G(p) = \begin{cases} pB_{G/e}(p), & \text{if e is a bridge;} \\ pB_{G/e}(p) + qB_{G\setminus e}(p), & \text{otherwise.} \end{cases}$$

If e is the edge 0v, then we may assume that v is not equal to 0. It follows that

$$A_G(p) = \begin{cases} pA_{G/e}(p), & \text{if e is a bridge;} \\ pA_{G/e}(p) + qA_{G\setminus v}(p), & \text{otherwise.} \end{cases}$$

If e is a bridge, the required result follows immediately by the induction hypothesis. If e is not a bridge, then by the induction hypothesis, $B_G(p) \geq pA_{G/e}(p) + qA_{G\setminus e}(p)$. But clearly $A_{G\setminus e}(p) \geq A_{G\setminus v}(p)$. Thus (5.2.14) follows for finite connected graphs G.

To extend the result to a locally finite connected graph G and an arbitrary subset X of V(G) is straightforward and simply involves taking suitably chosen limits. First suppose that X is finite. For $w \in V(G)$ and $T \subseteq V(G)$, let d(w,T) be the length of the shortest path in G from w to some member of T. Clearly there is a finite connected subgraph G_0 of G

such that $V(G_0) \supseteq X \cup \{0\}$. For each positive integer j, let $V_j = \{v \in V(G) : d(v,V(G_0)) \leq j)\}$ and let $G_j = G[V_j]$, the subgraph of G induced by V_j. Since G is locally finite, G_j is finite for all j, hence $A_{G_j}(X;p) \leq B_{G_j}(X;p)$. Thus $\lim_{j\to\infty} A_{G_j}(X;p) \leq \lim_{j\to\infty} B_{G_j}(X;p)$. But $\lim_{j\to\infty} A_{G_j}(X;p) = A_G(X;p)$ and $\lim_{j\to\infty} B_{G_j}(X;p) = B_G(X;p)$, hence for G locally finite and X finite, $A_G(X;p) \leq B_G(X;p)$. To get the result when X is infinite, note first that since G is locally finite and connected, X is countably infinite. Hence if we take an increasing sequence $X_1, X_2, \ldots$ of non-empty finite subsets of X whose union is X, then

$$B_G(X;p) = \lim_{j\to\infty} B_G(X_j;p) \geq \lim_{j\to\infty} A_G(X_j;p) = A_G(X;p) \quad . \; //$$

As a corollary of Theorem 5.2.13 we get the following theorem of Hammersley [32]. In a locally finite graph G, if A(p) and B(p) represent the probabilities that from a source vertex 0, fluid percolates to an infinite number of vertices of G in the atom and bond percolation models respectively, then

(5.2.15) $A(p) \leq B(p) \qquad (0 \leq p \leq 1)$.

In fact Hammersley has also proved this result when G is a directed graph. It is not difficult to see that only slight modifications need to be made to the above to give a proof for this case.

Notice that in this section there has been no attempt to

study percolation on infinite clutters and the passage to infinity in the particular case of percolation on locally finite graphs has been achieved by taking limits through a sequence of finite graphs.

3. An upper bound for the percolation probability.

The best upper bound for the percolation probability P(p) seems to be derived from a recursion formula of Hammersley and Walters [34,p.831] which though effective numerically does not seem to give an explicit closed form. Here we give an upper bound for the percolation probability for arbitrary finite clutters and we prove a general result formalizing the ideas of Hammersley [29,33] that fluid percolates more freely in a graph as the amount of statistical dependence between paths is reduced. In particular we obtain an upper bound for the percolation probability for a graph G in terms of the Bethe approximation to G. If $A = \{A_1, A_2, \ldots, A_n\}$ is a clutter on a finite set S, then suppose that $|A_1| \leq |A_2| \leq \ldots \leq |A_n|$. If A has r redundant elements, the *profile* of A is

$(|A_1|, |A_2|, \ldots, |A_n| ; r)$.

(5.3.1) Theorem. *Let* $(a_1, a_2, \ldots, a_n)$ *be a non-empty family of positive integers which are ordered so that* $a_1 \leq a_2 \leq \ldots \leq a_n$. *Then among those clutters* A *with profile* $(a_1, a_2, \ldots, a_n; 0)$, *the unique clutter which maximizes* R(A;p) *is the clutter whose members are pairwise disjoint.*

(5.3.2) Corollary. *If* $A = \{A_1, A_2, \ldots, A_n\}$ *is a clutter, then for all* p *in* [0,1],

$$R(A;p) \leq 1 - \prod_{i=1}^{n} (1 - p^{|A_i|}). //$$

Before proving Theorem 5.3.1 we give a few applications.

(5.3.3) Example. In a regular crystal lattice, let s(n) denote the number of paths which start at the origin 0 and which end on reaching a point a distance n from 0. If D(n;p) is the probability that fluid from 0 percolates a distance n from 0, then

(5.3.4) $D(n;p) \leq 1 - (1-p^n)^{s(n)}$.

To verify this take A_i ($1 \leq i \leq s(n)$) to be the edge-set of the ith path from 0 which ends on reaching a point a distance n from 0. Then (5.3.4) follows by Corollary 5.3.2.//

(5.3.5) Example. Let t(n) be the number of trees in a regular crystal lattice which have n + 1 vertices including the origin. Then the probability $P_n(p)$ discussed earlier satisfies

(5.3.6) $P_n(p) \leq 1 - (1-p^n)^{t(n)}$ .//

Not much seems to be known about the form of the functions s(n) and t(n).

The next result which is known as the FKG-inequality has numerous applications in percolation theory (see [71]) and in fact contains the core of the proof of Theorem 5.3.1. Let D be a finite distributive lattice and μ be a function from D into the non-negative real numbers such that μ is not identically zero. Then for any real-valued function f on D, the μ-*average* of f is

$$\langle f \rangle = \Big(\sum_{x \in D} \mu(x) f(x) \Big) \Big/ \Big(\sum_{x \in D} \mu(x) \Big) .$$

(5.3.7) Theorem [26]. *If* μ *satisfies* $\mu(x \vee y)\mu(x \wedge y) \geq \mu(x)\mu(y)$ *for all* x,y *in* D *and* f *and* g *are both non-decreasing (or both non-increasing) real-valued functions on* D, *then*

$$\langle fg \rangle \geq \langle f \rangle \langle g \rangle .//$$

Theorem 5.3.1 follows easily from the next result.

(5.3.8) Theorem. *Suppose that* $A = \{A_1, A_2, \ldots, A_n\}$ *is a clutter on a finite set* S. *Let* $A_1 = \{A_1, A_2, \ldots, A_j\}$ *and* $A_2 = \{A_{j+1}, A_{j+2}, \ldots, A_n\}$ *where* $1 \leq j \leq n-1$. *Then if* A_1' *and*

A_2' *are clutters on disjoint sets such that* A_1' *and* A_2' *are isomorphic to* A_1 *and* A_2 *respectively,*

$$R(A;p) \leq R(A_1' \cup A_2';p) \text{ for all } 0 \leq p \leq 1.$$

Proof. Let C_i $(1 \leq i \leq n)$ be the event that A_i is not open. We can regard the sample space Ω as the cartesian product of $|S|$ copies of $\{0,1\}$ with the understanding that a member ω of Ω is an $|S|$-tuple $(\omega_1,\omega_2,\ldots,\omega_{|S|})$ in which $\omega_i = 1$ if and only if the ith element of S is open. Then in an obvious sense Ω forms a distributive lattice and the probability of the event C_i is

$$P[C_i] = \sum_{X \not\supseteq A_i} p^{|X|} q^{|S\setminus X|} .$$

Hence if ψ_i is the function defined on the lattice Ω by

$$\psi_i(X) = \begin{cases} 0, & \text{if } X \supseteq A_i; \\ 1, & \text{otherwise,} \end{cases}$$

then

$$P[C_1 \cap \ldots \cap C_n] = \sum_{X \subseteq S} \psi_1(X)\psi_2(X)\ldots\psi_n(X) p^{|X|} q^{|S\setminus X|} .$$

Now $\psi_1,\psi_2,\ldots,\psi_n$ are non-increasing set functions and if μ is the function from Ω into R defined by $\mu(X) = p^{|X|} q^{|S\setminus X|}$ for all $X \in \Omega$, then μ satisfies the equation $\mu(X)\mu(Y) = \mu(X\cup Y)\mu(X\cap Y)$. Hence we may apply the FKG-inequality to get

$$P[C_1 \cap C_2 \cap \ldots \cap C_n] \geq P[C_1 \cap \ldots \cap C_j] P[C_{j+1} \cap \ldots \cap C_n].$$

Thus

$$R(A;p) \leq 1 - P[C_1 \cap \ldots \cap C_j] P[C_{j+1} \cap \ldots \cap C_n].$$

But in $A_1' \cup A_2'$, the events corresponding to $C_1 \cap \ldots \cap C_j$ and $C_{j+1} \cap \ldots \cap C_n$ are independent and hence

$$R(A_1' \cup A_2';p) = 1 - P[C_1 \cap \ldots \cap C_j] P[C_{j+1} \cap \ldots \cap C_n].$$

Theorem 5.3.8 now follows immediately.//

Proof of Theorem 5.3.1. If A_1 and A_2 are clutters on disjoint sets, then it is easy to check that $b(A_1 \cup A_2) = b(A_1) \oplus b(A_2)$. It follows by (5.2.3) that

$$R(A_1 \cup A_2; p) = 1 - (1-R(A_1;p))(1-R(A_2;p)) .$$

Using this with Theorem 5.3.8 it is not difficult to complete the proof.//

In the case when A is the collection of edge-sets of paths joining vertices 0 and X in a finite graph G, Theorem 5.3.1 says that the interaction probability $P_{0,X}(p)$ is maximized when all the paths between 0 and X are edge-disjoint.

We now show how the ideas above can be used to formalize work of Hammersley who has several times [29; 33,p.667; 34,p.837] used the principle that fluid percolates more freely through the Bethe tree of a crystal structure than through the crystal itself. We begin by defining the *Bethe tree* T(G) of a locally finite connected graph G with respect to some fixed vertex 0 of G. The tree T(G) will be specified completely by (i) describing the sets of vertices $\{R_j : 0 \le j < \infty\}$ where $R_0 = \{0\}$ and R_j is the set of vertices a distance j from 0 in T(G); and (ii) listing the edges joining members of R_j to members of R_{j-1} for $1 \le j < \infty$.

If $v \in V(G)$, let $\{\zeta_1, \zeta_2, \ldots, \zeta_t\}$ be the set of paths of G of length j which join 0 to v. There will be a distinct vertex of R_j for each pair (v, ζ_i) where $1 \le i \le t$. Suppose that ζ is the path with vertex set $\{0, v_1, v_2, \ldots, v_{j-1}, v\}$ and edge-set $\{e_1, e_2, \ldots, e_j\}$ where e_j joins v_{j-1} and v. Then if ζ' is the path obtained by deleting e_j from ζ, it is clear that ζ' has length j-1 and hence (v_{j-1}, ζ') is a

member of R_{j-1}. We join (v_{j-1},ζ') and (v,ζ) by an edge. Thus each member of R_j is joined to a single member of R_{j-1} and it is easy to see that as j runs through the positive integers, this construction gives a tree T(G) rooted at 0.

If X is a subset of V(G), we let X_T denote the subset of V(T(G)) defined by $(v,\zeta) \in X_T$ if and only if $v \in X$.

If $p \in [0,1]$, then as in section 2, $B_G(X;p)$ denotes the probability that there is a path from 0 to X in the bond percolation model on G.

(5.3.9) Theorem. *For a locally finite connected graph* G *and a non-empty subset* X *of* V(G), *if* 0 *is a fixed vertex of* G, *then* $B_G(X;p) \leq B_{T(G)}(X_T;p)$ *where* T(G) *is the Bethe tree of* G *with respect to* 0.

Proof. Suppose first that G is finite. Then we may assume that each edge of G is on some path from 0 to X. Let A and A_T be the clutters of edge-sets of paths from 0 to X in G and from 0 to X_T in T(G) respectively. Then $B_G(X;p) = R(A;p)$ and $B_{T(G)}(X_T;p) = R(A_T;p)$. We argue by induction on $|A|$. If there is only a single path of G joining 0 to X, then the result is immediate. Assume the result for all finite graphs in which 0 and X are joined by fewer than n paths and let G be a graph in which 0 and X are joined by exactly n paths where $n \geq 2$. Suppose that in G, the vertex 0 is incident with the edges $e_1,e_2,\ldots,e_j$. Then if $j = 1$, since $n \geq 2$, we can continue contracting edges incident with 0 one at a time until a graph is obtained in which 0 is incident with at least two distinct edges. As $T(G)/e_1 \cong T(G/e_1)$ when $j = 1$, the following argument for the case when $j \geq 2$ will also deal with the case $j = 1$.

The clutter A may be partitioned into two sets A' and A'' where a path ψ from 0 to X belongs to A'' if and only if the first edge of ψ is e_j. Then exactly as in the proof of Theorem 5.3.8, the FKG-inequality (Theorem 5.3.7) implies that

$$(5.3.10)\quad R(A;p) = 1 - P[\text{no member of } A \text{ is open}] \leq 1 - P[\text{no member of } A' \text{ is open}]P[\text{no member of } A'' \text{ is open}].$$

But in the Bethe tree T(G) the sets of paths corresponding to A' and A'', say A'_T and A''_T, have disjoint ground sets. Thus, by independence,

$$(5.3.11)\quad P[\text{no member of } A_T \text{ is open}] = P[\text{no member of } A'_T \text{ is open}]\, P[\text{no member of } A''_T \text{ is open}].$$

Now if $G[A']$ denotes the subgraph of G whose edges are the edges of paths in A', then

$B_{G[A']}(X;p) = R(A';p)$ and $B_{T(G[A'])}(X_T;p) = R(A'_T;p)$.

Thus, as $|A'| < |A|$, $R(A';p) \leq R(A'_T;p)$.

Likewise $R(A'';p) \leq R(A''_T;p)$ and using (5.3.10) and (5.3.11), the required result follows for finite graphs G.

If G is infinite but locally finite, then the result follows by taking appropriate limits, as in the proof of Theorem 5.2.13.//

Let $A_G(p)$ and $B_G(p)$ denote the probabilities that from a source vertex 0, fluid percolates to an infinite number of vertices of a locally finite graph G in the atom and bond percolation models respectively. Then a straightforward consequence of the preceding result is the following:

(5.3.12) Corollary. *If* G *is a locally finite connected graph and* O *is a fixed vertex of* G, *then for all* p *in* [0,1], $B_G(p) \leq B_{T(G)}(p)$ *and* $A_G(p) \leq A_{T(G)}(p)$, *where* T(G) *is the Bethe tree of* G *with respect to* O.

Proof. The first inequality follows easily from (5.3.9). To get the second, we use (5.2.15) together with the fact that for a locally finite tree T, $A_T(p) = B_T(p)$ [32].//

4. A minimal result.

In the preceding section we derived an upper bound on the percolation probability for finite clutters. In this section we obtain a lower bound for the percolation probability.

(5.4.1) Theorem. *Let* $(a_1,a_2,\ldots,a_n)$ *be a family of positive integers. If* $A = \{A_1,A_2,\ldots,A_n\}$ *is a clutter such that* $|A_i| = a_i$ *for all* $1 \le i \le n$, *then*

(i) $R(A;p) \ge p^{a_1} + p^{a_2}q + p^{a_3}q^2 + \ldots + p^{a_n}q^{n-1}$.

(ii) This lower bound is maximized when $a_1,a_2,\ldots,a_n$ *are ordered so that* $a_1 \le a_2 \le \ldots \le a_n$.

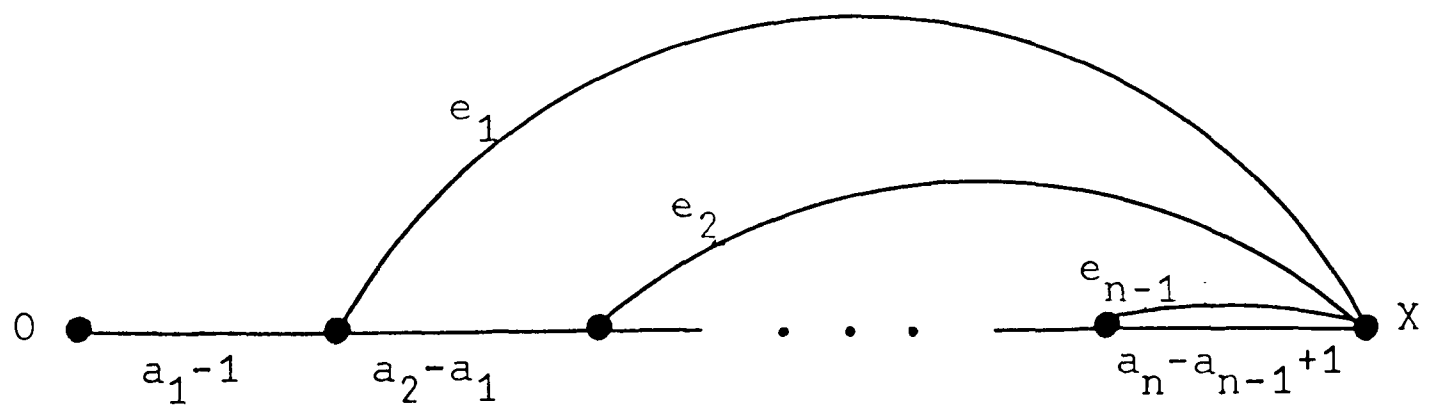
Moreover,

(iii) if $a_1,a_2,\ldots,a_n$ *are so ordered, the unique clutter which has profile* $(a_1,a_2,\ldots,a_n;0)$ *and which attains the lower bound is the clutter of edge-sets of paths joining* 0 *and* X *in the graph* H *shown in Figure* 5.4.2.

(iv) Among all clutters with profile $(a_1,a_2,\ldots,a_n;0)$, *this path clutter has the minimum number of blockers.*

(5.4.2) Figure

($e_1,e_2,\ldots,e_n$ are edges of H. The other segments are paths, the number of edges in each path being marked.)



Note that although the path clutter of H minimizes the number of blockers, it need not be the only clutter with profile

$(a_1,a_2,\ldots,a_n;0)$ which has the minimum number of blockers. This contrasts with the case of maximizing the number of blockers when there is a unique clutter having profile $(a_1,a_2,\ldots,a_n;0)$, namely the clutter which maximizes $R(A;p)$ (see Theorem 5.3.1).

Before proving Theorem 5.4.1 we note a corollary which gives a necessary condition for the existence of a Sperner family. If A is a clutter on a finite set S, the *upper ideal* $U(A)$ of A is defined by $U(A) = \{X \subseteq S : X \supseteq A \in A\}$.

(5.4.3) Corollary. *Let* $A = \{A_1,A_2,\ldots,A_n\}$ *be a clutter on a* t*-set. Then*

$$|U(A)| \geq \sum_{i=1}^{n} 2^{t-i+1-|A_i|}$$

Proof. Take $p = q = \frac{1}{2}$ in (5.4.1)(i) and note that $R(A;p) = \dfrac{|U(A)|}{2^t}$ .//

Proof of Theorem 5.4.1. Assume that $(a_1,a_2,\ldots,a_n)$ is an n-tuple of positive integers which are ordered so that $a_1 \leq a_2 \leq \ldots \leq a_n$. Associate with every such n-tuple the pair (n,a_1). Totally order this countable collection of pairs lexicographically. We use induction on this ordered set of pairs to prove (i) when $a_1,a_2,\ldots,a_n$ are ordered as above. The first member of the set is $(1,1)$ and (i) is true for a clutter consisting of one set of size one. Assume that (i) is true for all clutters whose associated pairs are less than (n,a_1) and let $A = \{A_1,A_2,\ldots,A_n\}$ be a clutter such that $|A_i| = a_i$ for all $1 \leq i \leq n$. If A has an essential element e, then we may suppose that $A \neq \{\{e\}\}$. Now writing $R(A)$ for

$R(A;p)$ we have $R(A) = pR(A/e)$. But A/e has as its associated pair $(n, a_1 - 1)$ and $a_1 - 1 \geq 1$. Therefore, by the induction assumption,

$$R(A/e) \geq p^{a_1 - 1} + p^{a_2 - 1} q + \ldots + p^{a_n - 1} q^{n-1}$$

and so

$R(A) \geq p^{a_1} + p^{a_2} q + \ldots + p^{a_n} q^{n-1}$ as required.

We may now suppose that A has no essential elements. Fick an element e from A_1 and let

$\{j_1 = 1, j_2, \ldots, j_t\} = \{j : e \in A_j\}$ and

$\{i_1, i_2, \ldots, i_s\} = \{i : e \notin A_i\}$ where $j_1 < j_2 < \ldots < j_t$ and $i_1 < i_2 < \ldots < i_s$. Then $R(A) = pR(A/e) + qR(A \backslash e)$ and clearly the pairs associated with A/e and $A \backslash e$ are less than the pair associated with A.

If $A_1 = \{e\}$, then $A/e = \{\emptyset\}$. Using this and the induction assumption gives

$$R(A) \geq p + q(p^{a_2} + p^{a_3} q + \ldots + p^{a_n} q^{n-1}).$$ That is,

$R(A) \geq p^{a_1} + p^{a_2} q + \ldots + p^{a_n} q^{n-1}$ as required.

Now suppose $a_1 \geq 2$. Then A/e is the collection of minimal members of $\{A_i \backslash e : A_i \in A\}$. In particular,

$$A_e = \{A_{j_1} \backslash e, A_{j_2} \backslash e, \ldots, A_{j_t} \backslash e\} \subseteq A/e,$$

but A/e may also contain some sets A_i for i in $\{i_1, i_2, \ldots, i_s\}$.

Thus $R(A_e) \leq R(A/e)$ and so $R(A) \geq pR(A_e) + qR(A \backslash e)$.

Hence, by the induction assumption,

$$R(A) \geq p(p^{a_{j_1} - 1} + p^{a_{j_2} - 1} q + \ldots + p^{a_{j_t} - 1} q^{t-1}) + q(p^{a_{i_1}} + p^{a_{i_2}} q + \ldots + p^{a_{i_s}} q^{s-1})$$

$$= p^{a_{j_1}} + p^{a_{j_2}} q + \ldots + p^{a_{j_t}} q^{t-1} + p^{a_{i_1}} q + p^{a_{i_2}} q^2 + \ldots + p^{a_{i_s}} q^s .$$

Notice that $j_1 = 1$ and $j_z \geq z$ for all $2 \leq z \leq t$.

Similarly $i_1 \geq 2$ and in general, $i_w \geq w + 1$ for all $1 \leq w \leq s$. Therefore as $0 \leq q \leq 1$,

$$R(A) \geq p^{a_1} + p^{a_{j_2} j_2 - 1} q^{j_2 - 1} + \ldots + p^{a_{j_t} j_t - 1} q^{j_t - 1}$$
$$+ p^{a_{i_1} i_1 - 1} q^{i_1 - 1} + p^{a_{i_2} i_2 - 1} q^{i_2 - 1} + \ldots + p^{a_{i_s} i_s - 1} q^{i_s - 1}$$
$$= p^{a_1} + p^{a_2} q + \ldots p^{a_n} q^{n-1} .$$

This completes the proof of (i) when $a_1, a_2, \ldots, a_n$ are ordered so that $a_1 \leq a_2 \leq \ldots \leq a_n$. Let $a_1, a_2, \ldots, a_n$ be so ordered and let $(b_1, b_2, \ldots, b_n)$ be a permutation of the symbols $(a_1, a_2, \ldots, a_n)$. We show that if $c \leq d$ and $i \geq 1$, then

(5.4.4) $p^c q^{i-1} + p^d q^i \geq p^d q^{i-1} + p^c q^i$ (where $q^0 = 1$ for all q).

From this it follows easily that

$$p^{a_1} + p^{a_2} q + \ldots + p^{a_n} q^{n-1} \geq p^{b_1} + p^{b_2} q + \ldots p^{b_n} q^{n-1}$$

and hence (i) and (ii) hold. If $c = d$ or $q = 0$, then (5.4.4) is immediate. Thus suppose that $c < d$ and $q > 0$. Now $q \leq 1$. Therefore

$$(1 - p^{d-c}) q \leq 1 - p^{d-c} .$$

That is, $q + p^{d-c} \leq 1 + p^{d-c} q$. Hence $p^c q^i + p^d q^{i-1} \leq p^c q^{i-1} + p^d q^i$ and (5.4.4) is proved.

It is easy to show by induction on n that when A is the collection $\mathcal{T}_H$ of edge-sets of paths joining 0 and X in the graph H of Figure 5.4.2, then equality holds in (i). The proof of the uniqueness in (iii) is again by induction on the size of the ground set of A. The one additional fact required is that if $R(A;p) = p^{a_1} + p^{a_2} q + \ldots + p^{a_n} q^{n-1}$ and $n>1$, then A has $a_1 - 1$ essential elements. This follows from (5.2.9)(ii).

To complete the proof of Theorem 5.4.1 it remains only to verify (iv). From Figure 5.4.2, we note that $b(\mathcal{T}_H)$ consists

of $a_1 - 1$ sets of size 1, $a_k - a_{k-1}$ sets of size k for $1<k<n$, and $a_n - a_{n-1} + 1$ sets of size n. That is, $b(T_H)$ has exactly a_n sets. But if A' is an arbitrary clutter with profile $(a_1,a_2,\ldots,a_n;0)$, then

(5.4.5) $b(A')$ has at least a_n members.

To see this, suppose that $A_i' \in A'$ and $|A_i'| = a_i$. For any element z of A_i', there is a member Z of $b(A')$ such that $A_i' \cap Z = \{z\}$, otherwise A_i' is not a minimal set intersecting every member of $b(A')$, contrary to the fact that $b(b(A')) = A'$. Thus, in particular, taking $i = n$, we have that $b(A')$ has at least a_n members. This proves (5.4.5) and completes the proof of Theorem 5.4.1.//

Finally note that the lower bound in Theorem 5.4.1 applies to all clutters on finite sets. However, when it is applied to say, percolation on the regular crystal lattices, the lower bound it gives for $P_n(p)$ is rather weak and hard to calculate. In this case there seems to be no good lower bound known.

5. A matroid equation.

The deletion-contraction formula for the clutter percolation probability $R(A;p)$ was noted in section 2 and has been a fundamental tool in the last three sections. We recall from §3.2 that the chromatic polynomial of a matroid satisfies a similar deletion-contraction formula which was equally important in the treatment of colouring for matroids. The chromatic polynomial of a matroid is closely linked to a more general two-variable function, the Tutte polynomial. In this section and the two following we look more closely at polynomials for matroids and clutters and ask whether there is a clutter analogue of the Tutte polynomial.

If M is a matroid on a set S, then the *Whitney rank generating function* $R_W(M;x,y)$ of M is defined by

$$R_W(M;x,y) = \sum_{X \subseteq S} x^{rkS-rkX}\, y^{|X|-rkX} .$$

The *Tutte polynomial* of M is defined by $T(M;x,y) = R_W(M;x-1,y-1)$ and is related to the chromatic polynomial studied in §3.2 through the equation

$$P(M;\lambda) = (-1)^{rkS}\, T(M;1-\lambda,0).$$

In this section we consider the following problem. Does there exist a unique real-valued function f defined on the class of all matroids and satisfying the following conditions?

(5.5.1) If matroids M and N are isomorphic, then $f(M) = f(N)$.

(5.5.2) For fixed non-zero real numbers a and b and any matroid M,

$$f(M) = af(M\backslash e) + bf(M/e)$$

provided e is neither a loop nor a coloop of M.

(5.5.3) If M_1 and M_2 are matroids on disjoint sets, then

$$f(M_1 \oplus M_2) = f(M_1)f(M_2) .$$

The following result is a slight extension of a theorem

of Brylawski [12] (see [86, Theorem 15.4.7]).

(5.5.4) Theorem. *There is a unique real-valued function* f *satisfying* (5.5.1) - (5.5.3) *together with the boundary conditions*

(5.5.5) $f(U_{1,1}) = x$, *and*

(5.5.6) $f(U_{0,1}) = y$.

This function is given for any matroid M *on a set* S *by*

(5.5.7) $f(M) = a^{|S|-rkS}\, b^{rkS}\, T(M;b^{-1}x,\ a^{-1}y)$

where T *is the Tutte polynomial of* M.

Proof. It is easy to check that f as defined in (5.5.7) satisfies (5.5.1) - (5.5.3). The uniqueness follows from the uniqueness of the Tutte polynomial (see, for example, [86, Theorem 15.4.7]).//

(5.5.8) Example. Let M be a matroid on a finite set S and suppose that each element of S has, independently of all other elements, a probability q of being deleted from M. The resulting restriction minor $\omega(M)$ of M will be called a *random submatroid* of M, corresponding in the obvious way to a random graph when M is the cycle matroid of the complete graph. Suppose $Q(p;M)$ is the probability that $\omega(M)$ has the same rank as M. Then $Q(p;M) = R(\mathcal{B}(M);p)$ where $\mathcal{B}(M)$ is the clutter of bases of M. Thus provided e is neither a loop nor a coloop of M,

$$Q(p;M) = qQ(p;M\backslash e) + pQ(p;M/e)$$

and

$$Q(p;M_1 \oplus M_2) = Q(p;M_1)Q(p;M_2).$$

Also

$$Q(p;M) = \begin{cases} p, \text{ if } M \text{ is a coloop;} \\ 1, \text{ if } M \text{ is a loop,} \end{cases}$$

and hence by Theorem 5.5.4,

$$Q(p;M) = q^{|S|-rkS}\, p^{rkS}\, T(M;1,q^{-1}).$$

By a similar argument, if $r(M;\theta) = E(\theta^{rk(\omega(M))})$ denotes the probability generating function of the rank of $\omega(M)$ we have that when e is neither a loop nor a coloop of M,

$$r(M;\theta) = qr(M\backslash e;\theta) + p\theta r(M/e;\theta) .$$

Moreover, $r(U_{1,1};\theta) = q + p\theta$, and $r(U_{0,1};\theta) = 1$ and $r(M_1 \oplus M_2;\theta) = r(M_1;\theta)r(M_2;\theta)$.

Hence by Theorem 5.5.4,

$$r(M;\theta) = q^{|S|-rkS}(p\theta)^{rkS}T(M;\tfrac{q}{p\theta} + 1, \tfrac{1}{q}).$$

6. On (not) extending the Tutte polynomial.

In view of (5.2.4) - (5.2.8) and Theorem 5.5.4, it is natural to ask whether the theory of the Tutte polynomial can be extended to arbitrary finite clutters. First recall that on the singleton set {e} there are three clutters. Writing (S,A) for a clutter A on a set S, these are ({e},{{e}}), ({e},{∅}) and the empty clutter ({e},∅). We call ({e},{{e}}) the *essential* clutter and ({e},{∅}) the *redundant* clutter. Then if a and b are non-zero real numbers we ask if there exists a function f(A;x,y) of two real variables x and y defined on the class of all finite non-empty clutters such that the following rules are satisfied.

(5.6.1) If A and A' are isomorphic clutters, then

$$f(A;x,y) = f(A';x,y).$$

(5.6.2) If e is neither essential nor redundant for A, then

$$f(A;x,y) = af(A\backslash e;x,y) + bf(A/e;x,y).$$

(5.6.3) $f(A_1 \oplus A_2;x,y) = f(A_1;x,y)f(A_2;x,y)$.

(5.6.4) $f(\{\{e\},\{\{e\}\});x,y) = x$ and

$f(\{\{e\},\{\emptyset\}\});x,y) = y$.

The main result of this section shows that Theorem 5.5.4 is best possible in the sense that matroids are the limiting structures for which a Tutte polynomial can be defined.

(5.6.5) Theorem. *If* a *and* b *are fixed non-zero real numbers, then a function* f(A;x,y) *satisfying* (5.6.1) - (5.6.4) *is uniquely defined for a non-empty clutter* A *if and only if* A *is the collection of bases of a matroid.*

Before proving this result we consider the following:

(5.6.6) Example. Let $S = \{1,2,3\}$ and $A = \{\{1\},\{2,3\}\}$ and suppose that $f(A;x,y)$ obeys the rules (5.6.1) - (5.6.4). Then by (5.6.2),

$$f(A;x,y) = af(A\backslash 1;x,y) + bf(A/1;x,y).$$

That is,

$$f(A;x,y) = af((\{2,3\},\{\{2,3\}\});x,y) + bf((\{2,3\},\{\emptyset\});x,y)$$

where, for example, $(\{2,3\},\{\emptyset\})$ denotes the clutter $\{\emptyset\}$ on the set $\{2,3\}$. It now follows by (5.6.3) and (5.6.4) that

(5.6.7) $f(A;x,y) = ax^2 + by^2$.

However, we also have by (5.6.2) that

$$\begin{aligned} f(A;x,y) &= af(A\backslash 2;x,y) + bf(A/2;x,y) \\ &= af((\{1,3\},\{\{1\}\});x,y) + bf((\{1,3\},\{\{1\},\{3\}\});x,y) \\ &= axy + bf((\{1,3\},\{\{1\},\{3\}\});x,y), \text{ by (5.6.3) and (5.6.4).} \end{aligned}$$

Hence by (5.6.2) and (5.6.4) we have

(5.6.8) $f(A;x,y) = axy + abx + b^2y$.

Since a and b are both non-zero, it follows on comparing (5.6.7) and (5.6.8) that in this case $f(A;x,y)$ is not uniquely defined.//

Proof of Theorem 5.6.5. In view of Theorem 5.5.4, we have only to prove that if $f(A;x,y)$ satisfies (5.6.1) - (5.6.4) and is uniquely defined, then A is the set of bases of a matroid on S. From (5.6.3) and (5.6.4) it is clear that

(5.6.9) $$f(A;x,y) = \begin{cases} xf(A/e;x,y), & \text{if } e \text{ is essential;} \\ yf(A\backslash e;x,y), & \text{if } e \text{ is redundant.} \end{cases}$$

We shall use induction on the size n of the ground set S. Let $\mathcal{Q}_n$ be the collection of clutters on sets of size n. The theorem is true when $n = 1$; suppose it is true for all positive integers $j < n$. Let A be in $\mathcal{Q}_n$ and suppose that f is uniquely defined for A. If A has an essential element e, then by (5.6.9),

$f(A;x,y) = xf(A/e;x,y)$ and as f is uniquely defined on A, it is uniquely defined on A/e. Hence, by the induction hypothesis, A/e is the set of bases of a matroid on S\e. Thus, as $A = \{A \cup e : A \in A/e\}$, it follows that A is the set of bases of a matroid on S. A similar argument gives the result when A has a redundant element. Hence we may suppose that each element of S is neither essential nor redundant for A.

Let A_1 and A_2 be distinct members of A and suppose that $e \in A_1 \backslash A_2$. We shall show that there is an element g of $A_2 \backslash A_1$ such that $(A_2 \backslash g) \cup e \in A$. It then follows that A is the set of bases of a matroid on S. The proof of this will distinguish three cases:

(I) $A_1 \cup A_2 \neq S$;

(II) $A_1 \cup A_2 = S$ and $A_1 \cap A_2 \neq \emptyset$; and

(III) $A_1 \cup A_2 = S$ and $A_1 \cap A_2 = \emptyset$.

Consider the first of these cases.

(I) If $A_1 \cup A_2 \neq S$, then let h be in $S \backslash (A_1 \cup A_2)$. By (5.6.2), $f(A;x,y) = af(A \backslash h;x,y) + bf(A/h;x,y)$. As f(A;x,y) is uniquely defined, so is f(A\h;x,y). Therefore, since $A \backslash h \in \mathcal{Q}_{n-1}$, it follows that A\h is the set of bases of a matroid on S\h. But $A_1, A_2 \in A \backslash h$, hence there is an element g of $A_2 \backslash A_1$ such that $(A_2 \backslash g) \cup e \in A \backslash h$ and thus $(A_2 \backslash g) \cup e \in A$.

(II) If $A_1 \cup A_2 = S$ and $A_1 \cap A_2 \neq \emptyset$, we choose h in $A_1 \cap A_2$ and then, since f(A;x,y) is uniquely defined, f(A/h;x,y) is also uniquely defined. But $A/h \in \mathcal{Q}_{n-1}$ and so, by the induction hypothesis, A/h is the set of bases of a matroid on S\h. Since $A_1 \backslash h, A_2 \backslash h \in A/h$, there is an element g of

$(A_2\backslash h)\backslash(A_1\backslash h)$ such that $(A_2\backslash\{h,g\}) \cup e \in A/h$. Hence, either $(A_2\backslash g) \cup e \in A$ or $(A_2\backslash\{h,g\}) \cup e \in A$. Suppose the latter, then $e \in ((A_2\backslash\{h,g\}) \cup e) \cap A_1$, so that $A_1\backslash e$ and $A_2\backslash\{h,g\}$ are in A/e. Hence, by the induction hypothesis, since A/e is the set of bases of a matroid, $|A_1\backslash e| = |A_2\backslash\{h,g\}|$, that is, $|A_1| = |A_2| - 1$. But since $A_1\backslash h, A_2\backslash h \in A/h$, $|A_1| = |A_2|$ and we have a contradiction. Hence $(A_2\backslash g) \cup e \in A$.

(III) Suppose that $A_1 \cup A_2 = S$ and $A_1 \cap A_2 = \emptyset$. If $u \in A_1$, then $A_1\backslash u \in A/u$ and there is a subset A_2' of A_2 such that $A_2' \in A/u$. Thus, by the induction assumption, $|A_2'| = |A_1\backslash u|$. Hence either

(i) $A_2' = A_2$ and $|A_1| = |A_2| + 1$; or

(ii) $A_2' \neq A_2$ and $A_2' \cup u \in A$ and $|A_2| > |A_2'| = |A_1| - 1$.

Choose t from A_2. By the same argument as above, there is a subset A_1' of A_1 such that $A_1' \in A/t$ and $|A_1'| = |A_2| - 1$. Moreover either

(iii) $A_1' = A_1$ and $|A_1| = |A_2| - 1$; or

(iv) $A_1' \neq A_1$ and $A_1' \cup t \in A$ and $|A_1| > |A_1'| = |A_2| - 1$.

First note that (i) and (iii) cannot both hold. Suppose that (i) and (iv) hold. Then either $A_1' = \emptyset$ or not. In the first case, $A_2 = \{t\}$ and by (i), $|A_1| = 2$. Since $A_1 \cup A_2 = S$, this forces $S = \{t,u,w\}$ and $A = \{\{t\},\{u,w\}\}$. But by Example 5.6.6, f is not uniquely defined for this clutter. Thus we may assume that $A_1' \neq \emptyset$. Choose z in A_1'. Then A/z is the set of bases of a matroid on $S\backslash z$. Now since $(A_1' \cup t)\backslash z$ and $A_1\backslash z$ are in A/z, we have $|(A_1' \cup t)\backslash z| = |A_1\backslash z|$, hence $|A_1'| = |A_1| - 1$. But then by (iv), $|A_1| = |A_2|$, contrary to (i).

If (ii) and (iii) hold, then interchanging the roles of A_1

and A_2, A_1' and A_2' and u and t in the preceding argument again gives a contradiction. Hence (ii) and (iv) hold and so $|A_1| = |A_2|$. It follows that $A_2' = A_2\backslash c$ for some c in A_2 and hence $(A_2\backslash c) \cup u \in A$. As $A_1 \cap A_2 = \emptyset$, $c \in A_2\backslash A_1$. Hence if we let u = e and g = c, then we obtain the required result. This completes the proof of Theorem 5.6.5.//

Now suppose that each clutter in conditions (5.6.1) - (5.6.4) is replaced by its blocker. Then (5.6.2) and (5.6.3) become

(5.6.2)' If e is not redundant for A and $\{e\} \notin A$, then $f(A;x,y) = af(A/e;x,y) + bf(A\backslash e;x,y)$.

(5.6.3)' $f(A_1 \cup A_2;x,y) = f(A_1;x,y)f(A_2;x,y)$.

Therefore since the blocker of the essential clutter $(\{e\},\{\{e\}\})$ is itself, but the blocker of the redundant clutter $(\{e\},\{\emptyset\})$ is the empty clutter $(\{e\},\emptyset)$, the analogue of Theorem 5.6.5 is as follows.

(5.6.10) Theorem. *If* a *and* b *are fixed non-zero real numbers, then a function* f(A;x,y) *satisfying* (5.6.1), (5.6.2)', (5.6.3)' *and*

(5.6.4)' $f((\{e\},\{\{e\}\});x,y) = x$ *and* $f((\{e\},\emptyset);x,y) = y$

is uniquely defined for a finite clutter $A \neq \{\emptyset\}$ *if and only if* A *is the collection of circuits of a matroid.*

Proof. This follows easily from Theorem 5.6.5 using the fact that the blocker of the set of bases of a matroid is the set of circuits of the dual matroid.//

7. Polynomials for clutters.

By Theorem 5.6.5, if a and b are fixed non-zero real numbers and x and y are independent real variables, then there is no function on the class of all finite non-empty clutters which satisfies (5.6.1) - (5.6.4). However by (5.2.4) - (5.2.8), when $x = p$, $y = 1$, $a = 1 - p$ and $b = p$, the clutter percolation probability $R(A;p)$ satisfies (5.6.1) - (5.6.4) for all p in $[0,1]$. This prompts the questions: for which points (a,b,x,y) of R^4 are there uniquely defined functions satisfying (5.6.1) - (5.6.4) for all finite non-empty clutters and what are these functions? The next theorem shows that although there are seven planes in R^4 for which there exist uniquely defined functions satisfying the required conditions, there is only one non-degenerate such function, this being an extension of the percolation probability. We now list the six degenerate functions and the planes in which they are defined. Notice that each of these functions says very little about the structure of the clutter A.

(5.7.1) In the plane $\{(a,b,0,0)\}$, $f(A) = 0$.

(5.7.2) In the plane $\{(a,0,0,y)\}$, $f(A) = \begin{cases} y^{|S|}, & \text{if } A = \{\emptyset\}; \\ 0, & \text{otherwise.} \end{cases}$

(5.7.3) In the plane $\{(0,b,x,0)\}$, $f(A) = \begin{cases} x^{|S|}, & \text{if } A = \{S\}; \\ 0, & \text{otherwise.} \end{cases}$

(5.7.4) In the plane $\{(0,0,x,y)\}$, $f(A) = \begin{cases} x^{|T|}y^{|S\setminus T|}, & \text{if } A = \{T\} \text{ for some } T \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$

(5.7.5) In the plane $\{(0,b,x,b)\}$, $f(A) = x^{e(A)}\, b^{|S|-e(A)}$ where $e(A)$ is the number of essential elements of A.

(5.7.6) In the plane $\{(a,b,a+b,a+b)\}$, $f(A) = (a+b)^{|S|}$.

The *upper polynomial* $U(A;z)$ of a clutter A is defined by $U(A;z) = \sum_{j=0}^{|S|} u_j z^j$ where S is the ground set of A and u_j is the number of members of $U(A)$ of size j.

(5.7.7) Theorem. *The sets of points* (a,b,x,y) *of* R^4 *for which there is a uniquely defined function* f *satisfying* (5.6.1) - (5.6.4) *for all finite non-empty clutters are given by the six planes* (5.7.1) - (5.7.6) *each of which corresponds to a degenerate function, together with the plane* $\{(a,b,b,a+b)\}$ *in which the following function satisfies the required conditions:*

$$f(A) = a^{|S|}\, U(A;ba^{-1}) .$$

Proof. We shall assume that there is a uniquely defined function f satisfying (5.6.1) - (5.6.4) and determine how a,b,x and y must be related.

Let $S_1 = \{1,2,3,4\}$ and $A_1 = \{\{1,2\},\{2,3\},\{3,4\}\}$.

Then by (5.6.2),

$f(A_1) = af(A_1\backslash 1) + bf(A_1/1)$

$= af(\{2,3,4\},\{\{2,3\},\{3,4\}\}) + bf(\{2,3,4\},\{\{2\},\{3,4\}\})$

$= ax(ax+by) + b(ax^2+by^2)$, by (5.6.9), (5.6.2), (5.6.4) and Example 5.6.6.

That is, $f(A_1) = a^2x^2 + abxy + abx^2 + b^2y^2$.

Alternatively, by (5.6.2),

$f(A_1) = af(A_1\backslash 2) + bf(A_1/2)$

$= af(\{1,3,4\},\{\{3,4\}\}) + bf(\{1,3,4\},\{\{1\},\{3\}\})$

$= ax^2y + abxy + b^2y^2$, by (5.6.9), (5.6.2) and (5.6.4).

Thus if f is uniquely defined, then

$a^2x^2 + abxy + abx^2 + b^2y^2 = ax^2y + abxy + b^2y^2$.

Hence $ax^2(a+b-y) = 0$ and so

(5.7.8) $a = 0$ or $x = 0$ or $y = a + b$.

From Example 5.6.6, we have that if f is uniquely defined, then

(5.7.9) $ax^2 + by^2 = axy + abx + b^2y$.

If $y = a + b$, this gives

$ax^2 + a^2b + 2ab^2 + b^3 = a^2x + 2abx + ab^2 + b^3$, and so

$a(x-b)(x-(a+b)) = 0$. Hence

(5.7.10) $a = 0$ or $x = b$ or $x = a + b$.

On combining (5.7.8) and (5.7.10) we get four possibilities:

(i) $x = y = a + b$;

(ii) $x = b$, $y = a + b$;

(iii) $x = 0$; and

(iv) $a = 0$.

In case (i) it is a straightforward exercise to verify by induction on $|S|$ that $f(A) = (a+b)^{|S|}$ satisfies (5.6.1) - (5.6.4) for all finite non-empty clutters A. The most interesting case is (ii). Here we know that when $a + b = 1$ and a and b are both non-negative, if $a = q$ and $b = p$, then the percolation probability $R(A;p) = \sum_{X \in U(A)} p^{|X|} q^{|S\setminus X|}$ satisfies (5.6.1) - (5.6.4). In fact it is not difficult to show that

(5.7.11) $$f(A) = \sum_{X \in U(A)} b^{|X|} a^{|S\setminus X|}$$

satisfies (5.6.1) - (5.6.4) for arbitrary a and b. But this function is precisely the function $a^{|S|} U(A;ba^{-1})$.

To deal with cases (iii) and (iv) we shall use another constraint on a,b,x and y, this being derived from the following example. Let $S_2 = \{1,2,3,4\}$ and $A_2 = \{\{1,2\},\{3\},\{4\}\}$. Then

$f(A_2) = af(A_2\backslash 4) + bf(A_2/4)$ and $f(A_2) = af(A_2\backslash 1) + bf(A_2/1)$.

Completing these two expansions we find that

$$a^2x^2 + aby^2 + by^3 = a^2xy + aby^2 + a^2bx + ab^2y + b^2y^2$$

and hence

$$a^2x^2 + by^3 = a^2xy + a^2bx + ab^2y + b^2y^2 \quad . \tag{5.7.12}$$

Consider case (iii). From (5.7.9), we have that $by^2 = b^2y$ and so $y = 0$, $b = 0$ or $y = b$. If $y = 0$, then the unique well-defined function satisfying (5.6.1) - (5.6.4) in the plane $\{(a,b,0,0)\}$ is $f(A) = 0$.

If $x = 0$ and $b = 0$, then it is easy to check that

$$f(A) = \begin{cases} y^{|S|}, & \text{if } A = \{\emptyset\}, \\ 0, & \text{otherwise,} \end{cases}$$

satisfies (5.6.1) - (5.6.4).

Finally, if $x = 0$ and $y = b$, then from (5.7.12) it follows that $ab^3 = 0$ and so $a = 0$ or $b = 0$. The second case has already been considered. The first case is included in case (iv).

There remains only case (iv) to consider to complete the proof of Theorem 5.7.7 and this is straightforward.//

The upper polynomial $U(A;z) = \sum_{j=0}^{|S|} u_j z^j$ is closely related to a polynomial defined for an arbitrary simplicial complex by Wilf [89]. However, although such simplex polynomials have in one sense been characterized [18], the characterization is rather unmanageable and does not seem to throw any light on the sort of problem that has been considered here.

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Index of Notation

The following is a list of the most frequently used notation together with the number of the page on which it was introduced.

$A_G(X;p) = A_G(p)$ . . . 102

$B_G(X;p) = B_G(p)$ . . . 102

$c(M;q)$ . . . critical exponent of matroid M, 55

cork (T) . . . corank of set T, v

$D(n;p)$ . . . 106

$E(G)$ . . . set of edges of graph G, vi

$E(M)$ . . . set of elements of matroid M, 84

f_T . . . restriction of operator f to T, 9

f^T . . . contraction of operator f to T, 9

f^* . . . dual of operator f, 9

g.c.d. . . . greatest common divisor, 71

g_Γ . . . 28

GF(q) . . . Galois field of q elements, 46

K_n . . . complete graph on n vertices, 70

$P[C_i]$ . . . probability of event C_i, 108

$P_G(\lambda)$ . . . chromatic polynomial of graph G, 48

$P_{i,j}(p)$ . . . interaction probability, 96

$P(M;\lambda)$ . . . chromatic polynomial of matroid M, 48

$P_n(p)$ . . . 96

q . . . In Chapter 5, denotes 1-p, 95

Q_m . . . 47, 67

[r] . . . greatest integer not exceeding r, vi

{r} . . . least integer not less than r, vi

R(A;p) . . . percolation probability of clutter A, 93,97

rk(T) . . . rank of set T, v

$rk_{(j)}(T)$ . . . rank of T in j-truncation, 83

$R_W(M;x,y)$ . . . Whitney rank generating function of M, 118

$T(M;x,y)$ . . . Tutte polynomial of M, 118

$U(A;z)$ . . . 127

$V(G)$ . . . set of vertices of a graph G, vi

$V(n,q)$ . . . vector space of n-tuples over GF(q), 47

$A(X)$ . . . 25

$C(M)$ . . . set of circuits of M, v

$C^*(M)$ . . . set of cocircuits of M, v

$\mathcal{D}_U$ . . . 6

$G(G)$ . . . 29

$K(G)$ . . . 29

$M(G)$ . . . 29

$R(M)$ . . . set of simple matroids which are restrictions of M, 53

$U(A)$ . . . upper ideal of clutter A, 114

R . . . set of real numbers, vi

R^+ . . . set of positive real numbers, vi

Z . . . set of integers, vi

Z^+ . . . set of positive integers, vi

Z^- . . . set of negative integers, vi

Z_n . . . ring of integers modulo n, 53

$\alpha(M)$ . . . 78

$\alpha^*(M)$ . . . $\alpha(M^*)$, 84

$\alpha_v(G)$ . . . 78

$\beta(M)$ . . . 78

$\beta^*(M)$ . . . $\beta(M^*)$, 84

$\beta_v(G)$ . . . 78

$\gamma(M)$ . . . number of components of matroid M, 78

δ . . . 5

$\eta(m,k)$ . . . 62

$\theta(M)$ . . . 80

$\theta^*(M)$ . . . 81

$\xi(M)$ . . . maximum real root of $P(M;\lambda)$, 49

$\pi(M)$ . . . 48

$\tau(A)$ . . . set of partial transversals of family A, 22

$\chi(M)$ . . . chromatic number of matroid M, 48

Set-theoretic notation.

$S \setminus T$ . . . difference of sets S and T, v

$S \Delta T$ . . . symmetric difference of sets S and T, v

$U \subset\subset T$ . . . U is a finite subset of T, v

2^S . . . set of subsets of set S, 9

$\emptyset$ . . . empty set, 5

Special matroids

$U_{j,n}$ . . . j-uniform matroid on n elements, 58, 63

PG(n,q) . . . projective geometry of rank n+1 over GF(q), 47, 55

AG(n,q) . . . affine geometry of rank n+1 over GF(q), 55

M(G) . . . cycle matroid of graph G, 68

$M^*(G)$ . . . dual of M(G), 72

Operations on matroids: M is a matroid.

M^* . . . dual matroid of M, v

$\tilde{M}$ . . . simple matroid associated with M, v

M|T . . . restriction of M to T, v

M.T . . . contraction of M to T, v

M\T . . . deletion of T from M, v

M/T . . . contraction of T from M, v

S(M,N) . . . series connection of matroids M and N, 67

$M \oplus N$ . . . direct sum of matroids M and N, vi

$M_{(j)}$ . . . j-truncation of matroid M, 83

Operations on graphs: G is a graph.

G\e . . . deletion of edge e from G, 103

G\v . . . deletion of vertex v from G, 103

G/e . . . contraction of edge e from G, 103

G[Y] . . . subgraph of G induced by $Y \subseteq V(G)$, 44

GVH . . . 70

Operations on clutters: A is a clutter on a finite set.

b(A) . . . blocker of A, 98

$\overline{A}$ . . . complementary clutter of A, 97

A|T . . . restriction of A to T, 98

A.T . . . contraction of A to T, 98

A\e . . . deletion of element e from A, 99

A/e . . . contraction of element e from A, 99

$A \oplus A'$ . . . direct sum of clutters A and A' on disjoint sets, 99

$A \cup A'$ . . . union of clutters A and A' on disjoint sets, 99

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