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Mal'tsev and retral spaces

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Abstract

A space X is Mal'tsev if there exists a continuous map $M: X^3 \to X$ such that $M(x,y,y) = x = M(y,y,x)$. A space X is retral if it is a retract of a topological group. Every retral space is Mal'tsev. General methods for constructing Mal'tsev and retral spaces are given. An example of a Mal'tsev space which is not retral is presented. An example of a Lindelöf topological group with cellularity the continuum is presented. Constraints on the examples are examined. © 1997 Elsevier Science B.V.

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1. Introduction

The algebraic structure of topological groups has a profound effect on their topological properties: every T_0 topological group is Tychonoff [8], compact topological groups are dyadic [9] (indeed, Dugundji compact [14]), and every σ-compact topological group satisfies the countable chain condition [18]. This suggests two lines of investigation: first, isolating the algebraic heart of these results, and second, determining which 'nicely' embedded subspaces of topological groups inherit their topological properties. (Note that every Tychonoff space is embedded in a compact topological group.)

In [20], Uspenskij introduced the class of Mal'tsev spaces, and he has subsequently shown [21–23] that much of the behavior of topological groups transfers to Mal'tsev spaces.

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Definition. A space X is Mal'tsev if and only if there exists a continuous map $M : X^3 \to X$ such that $M(x, y, y) = x = M(y, y, x)$.

In the other direction, it cannot be denied that retracts are 'nicely embedded' in their ambient spaces.

Definition. A space X is retral if and only if it is a retract of a topological group.

The concepts of a Mal'tsev space and a retral space are closely related. Every retral space is Mal'tsev, for if $r : G \to X$ is a retraction of a topological group G onto the space X, then $M : X^3 \to X$ defined by $M(x, y, z) = r(x \cdot y^{-1} \cdot z)$ is a continuous Mal'tsev operation for X. The converse was proved for (countably) compact spaces by Sipacheva [17], and more recently this was extended by Reznichenko and Uspenskij [15]: they proved that every pseudocompact Mal'tsev space is retral. However, the basic question remained.

Basic Question. Is every Mal'tsev space retral?

The principal result of this paper is a negative answer to this question. Section 2 explains how to construct a wide variety of Mal'tsev spaces. These include the Michael line and Sorgenfrey line. At the start of our research, it was felt that neither of these 'should' be retral. Surprisingly, in Section 3 it is shown that they, and many other well-known spaces, *are* retral. This enables us to answer a longstanding open question about the cellularity of Lindelöf topological groups. The next section, Section 4, implements the basic idea underlying the construction of the example: if a space X has uncountable cellularity, but its free topological group (see below) $F(X)$ has countable cellularity, then X cannot be retral. The example is presented in Section 5. The final section examines how well-behaved we can make our example, and in doing so, we discover an unexpected link with Katetov's famous result that every (countably) compact space with hereditarily normal cube is metrizable.

Since T_0 topological groups are Tychonoff, we will assume all spaces to be Tychonoff. We note that many of our results will remain valid for Hausdorff spaces. It is vital for what follows that a space X is retral if and only if it is the retract of its (Markov) free topological group. See [11] for background information on the free topological group.

For a family γ of subsets of X and $F \subset X$, denote

$$\mathrm{St}(F, \gamma) = \{U \in \gamma\colon\ U \cap F \neq \emptyset\} \quad \text{and} \quad \mathrm{st}(F, \gamma) = \bigcup \mathrm{St}(F, \gamma).$$

For $F = \{x\}$, we will write $\mathrm{St}(x, \gamma)$ and $\mathrm{st}(x, \gamma)$ instead of $\mathrm{St}(\{x\}, \gamma)$ and $\mathrm{st}(\{x\}, \gamma)$.

For a space X, $F(X)$ ($A(X)$) is its free (free Abelian) topological group in Markov's sense. The set of all words in $F(X)$ ($A(X)$) of length not exceeding n is denoted as $F_n(X)$ ($A_n(X)$).

2. Constructing Mal'tsev spaces

The Mal'tsev operators we will construct are highly specific. Let us call a space X *2-Mal'tsev* if there exists a continuous Mal'tsev operator $M : X^3 \to X$ such that $M(x,y,z) \in \{x,z\}$. The class of all 2-Mal'tsev spaces is denoted as $\mathcal{M}_2$. For convenience, we define

$$\Pi_1 = \{(x,y,y)\colon\ x,y \in X\}, \qquad \Pi_2 = \{(y,x,y)\colon\ x,y \in X\},$$
$$\Pi_3 = \{(y,y,x)\colon\ x,y \in X\}, \qquad \Delta_3 = \{(x,x,x)\colon\ x \in X\}$$

for any space X.

Theorem 1. *Let X be a space. Then X is 2-Mal'tsev if and only if there are closed subspaces A and B of X^3 such that*
(1) $A \cup B = X^3$;
(2) $\Pi_1 \subseteq A$, $\Pi_3 \subseteq B$;
(3) $A \cap B \subseteq \Pi_2$.

Proof. Suppose first that $M : X^3 \to X$ witnesses that X is 2-Mal'tsev. Then the closed sets

$$A = \{(x,y,z)\colon\ M(x,y,z) = x\} \quad \text{and} \quad B = \{(x,y,z)\colon\ M(x,y,z) = z\}$$

are easily seen to satisfy conditions (1)–(3) above.

Conversely, given A and B satisfying (1)–(3), the map $M : X^3 \to X$ defined below is a continuous 2-Mal'tsev operator:

$$M(x,y,z) = \begin{cases} x, & \text{if } (x,y,z) \in A, \\ z, & \text{if } (x,y,z) \in B. \end{cases}$$

To see this, first note that M is well-defined by (1) and (3) and is a Mal'tsev operator by (2). To prove continuity, take any closed $C \subseteq X$. Observe that

$$M^{-1}C = \big((C \times X \times X) \cap A\big) \cup \big((X \times X \times C) \cap B\big).$$

This set is closed. □

The following lemma helps in identifying 2-Mal'tsev spaces. The simple proof is omitted.

Lemma 2. *The space (X,σ) is in $\mathcal{M}_2$ if there is a coarser topology τ on X such that one of (1)–(3) holds:*
(1) *(X,τ) is in $\mathcal{M}_2$;*
(2) *$(X,\tau)^3 \setminus \Delta_3$ is normal and strongly zero-dimensional;*
(3) *$((X,\tau)^2 \setminus \Delta) \times (X,\tau)$ is normal and strongly zero-dimensional.*

In particular, a space with a coarser strongly zero-dimensional metrizable topology is 2-Mal'tsev. We may extend this to many spaces constructed from strongly zero-dimensional metrizable spaces. A space X is said to be non-Archimedean if it can be obtained as follows: take any strongly zero-dimensional metrizable space, isolate all points of some subset, replace each point by a strongly zero-dimensional metrizable space, and repeat transfinitely, taking a subspace of the inverse limit at limit ordinals [13]. Alternatively, a space X is non-Archimedean if it has a base $\mathcal{B}$ which is a tree with respect to reverse inclusion and whenever $B \cap B' \neq \emptyset$ $(B, B' \in \mathcal{B})$, either $B \subseteq B'$ or $B' \subseteq B$. Evidently, every strongly zero-dimensional metrizable space is non-Archimedean, as is every linearly metrizable nonmetrizable space and the Michael line.

Theorem 3. *Let (X,σ) be a space with a coarser non-Archimedean topology τ. Then (X,σ) is 2-Mal'tsev.*

Proof. By Lemma 2, it is sufficient to show that the non-Archimedean space (X,τ) is 2-Mal'tsev. Let $\mathcal{B}$ be a base for (X,τ) as in the alternative characterization of non-Archimedean spaces given above. Set

$$C = \bigcup\{U \times V \times V\colon\ U, V \in \mathcal{B} \text{ and } U \cap V = \emptyset\}.$$

Note that C is open, $\Pi_1 \setminus \Delta_3 \subset C$, and $\Pi_3 \subset X^3 \setminus C$. Put $A = C \cup \Delta_3$ and $B = X^3 \setminus C$. Then conditions (1)–(3) from Theorem 1 hold, and we only need to check that A is closed in X^3. Let $(x,y,z) \in X^3 \setminus A$. For $u \in \{x,y,z\}$, select $U_u \in \mathcal{B}$ such that if $u, u' \in \{x,y,z\}$ and $u \neq u'$, then $U_u \cap U_{u'} = \emptyset$. Note that $U_x \times U_y \times U_z \cap \Delta_3 = \emptyset$.

Let us show that $U_x \times U_y \times U_z \cap C = \emptyset$. Assume the converse. Then there exist disjoint U and V in $\mathcal{B}$ such that

$$U_x \times U_y \times U_z \cap U \times V \times V \neq \emptyset.$$

Because $\mathcal{B}$ is a tree, $U_y \cap U_z = \emptyset, V \cap U_y \neq \emptyset$, and $V \cap U_z \neq \emptyset$, we have $U_y, U_z \subset V$. Therefore, $V \not\subset U_x$, and $U \cap U_x \neq \emptyset$ implies that $U_x \not\subset V$. Because that $\mathcal{B}$ is a tree, we obtain $U_x \cap V = \emptyset$. Thus, we have $(x,y,z) \in U_x \times V \times V$ and $U_x \cap V = \emptyset$. In other words, $(x,y,z) \in C$, which is a contradiction. □

3. Constructing retral spaces

There are two new situations in which we can assert that certain spaces are retral. One is a minor improvement of Sipacheva's positive solution to the basic problem for compact spaces mentioned in the introduction. The second reveals a subclass of the class of spaces with a coarser non-Archimedean topology which figured in Theorem 3 as retral. This subclass includes such spaces as the Sorgenfrey line, the Michael line, the natural Souslin line obtained from a Souslin tree, and a host of many other well known spaces.

An examination of Sipacheva's proof that compact Mal'tsev spaces are retral will show that to any Mal'tsev operator M on a space X, there corresponds a natural retraction, which we will denote r_M, of the free topological group of X onto X as sets. The question, of course, is whether this retraction is continuous. The line of reasoning in proving the Sipacheva theorem is as follows. Let X be an arbitrary space and $M = r_3 : X^3 \to X$ be a Mal'tsev operation. For all odd $n > 3$, we construct maps $r_n : X^n \to X$ such that if $(x_1, \ldots, x_n) \in X^n$ and $x_k = x_{k+1}$, then

$$r_n(x_1, \ldots, x_n) = r_{n-2}(x_1, \ldots, x_{k-1}, x_{k+3}, \ldots, x_n).$$

The maps r_n are constructed from r_3 using a certain formula. If r_3 is continuous, then r_n are also continuous. After that, we define r_M. First, we define it on the set

$$F_* = \left\{x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} \colon \varepsilon_i = \pm 1 \text{ and } \sum \varepsilon_i = 1\right\}$$

by $r_M(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}) = r_n(x_1, x_2, \ldots, x_n)$. F_* is a clopen subset of $F(X)$ containing X. The remaining part of $F(X)$ is mapped into an arbitrary fixed point of X. The retraction r_M is well defined. For a positive integer n, if the natural map $(X \oplus X^{-1})^n \to F_n(X)$ is quotient, then $r_M \upharpoonright F_n(X)$ is continuous. If, in addition, $F(X)$ has the inductive limit topology with respect to the decomposition $F(X) = \bigcup F_n(X)$, then r_M is continuous. Thus, the following assertion is valid.

Proposition 4. *Let M be a Mal'tsev operator for a space X. Suppose that $F(X)$ has the inductive limit topology and each natural map $(X \oplus X^{-1})^n \to F_n(X)$ is quotient. Then X is retral with the requisite continuous retraction r_M.*

Recall that a space X is said to be a k_ω space if $X = \bigcup_{n\in\omega} X_n$, where each X_n is compact, $X_n \subseteq X_{n+1}$, and $F \subseteq X$ is closed if $F \cap X_n$ is closed in X_n for all n in ω. It is easy to check that

(1) a finite sum of k_ω spaces is again k_ω,
(2) a closed subspace of a k_ω space is k_ω, and
(3) a Lindelöf locally compact space is k_ω.

It is known that if X is a k_ω, then $F(X)$ has the inductive limit topology [10], and the standard maps of $(X \oplus X^{-1})^n \to F_n(X)$ are quotient. Thus, Proposition 4 can be applied.

Proposition 5. *Let X be a k_ω space with a Mal'tsev operator M. Then r_M witnesses that X is retral.*

Since

(1) an arbitrary sum of retral (Mal'tsev) spaces is again retral (Mal'tsev),
(2) a retract of a retral (Mal'tsev) space is retral (Mal'tsev), and
(3) any locally compact paracompact space is the sum of Lindelöf locally compact subspaces,

we deduce:

Corollary 6. *Every locally compact paracompact Mal'tsev space is a retract of a topological group.*

We now turn to considering spaces with a coarser non-Archimedean topology. More specifically, we show that a space whose topology can be obtained from a non-

Archimedean topology by declaring (non-Archimedean) closed sets to be open is retral. We give two versions of this result. The first links in with the results above, but the method of proof only works in a restricted setting. The second requires directly constructing a retraction.

Theorem 7 (version 1). *Let (X,σ) be a space with a coarser zero-dimensional separable metrizable topology τ. Suppose, (X,σ) has a base consisting of τ-closed sets. Then (X,σ) is retral.*

Proof. Denote the Cantor space as $\mathbb{C}$, and for a closed subspace F of the Cantor space, define $\mathbb{C}_F$ to be the space with underlying set $\mathbb{C}$ and the topology obtained from $\mathbb{C}$ by adding F as an open (and closed) set. Observe that $\mathbb{C}_F$ is k_ω. Since every zero-dimensional separable metrizable space can be embedded in the Cantor space, we may assume that $X \subseteq \mathbb{C}$, and we may choose a collection $\mathcal{F}$ of closed subsets of $\mathbb{C}$ such that the collection of $F \cap X$, $F \in \mathcal{F}$, is a subbase for the topology on (X,σ).

Fix an $\mathcal{M}_2$ operator M for $\mathbb{C}$. Let r_M be the corresponding retraction of $F(\mathbb{C})$ onto $\mathbb{C}$ (as sets). Let r_X be r_M restricted to $F(X|\mathbb{C})$ (the subgroup of $F(\mathbb{C})$ generated by X). Note that r_X is a retraction onto X (as sets) by the hereditary property of $\mathcal{M}_2$.

We wish to show that r_X is continuous. To do this, it is sufficient to check that $r_X^{-1}(F \cap X)$ is closed in $F(X|\mathbb{C})$ for each $F \in \mathcal{F}$. Take any $F \in \mathcal{F}$ and consider $\mathbb{C}_F$. By Proposition 5, r_M is continuous for $\mathbb{C}_F$. Hence $r_X^{-1}(F \cap X) = (r_M^{-1}F) \cap F(X|\mathbb{C})$ is indeed closed in $F(X|\mathbb{C})$. $\square$

For a set X, we denote the free Boolean group of X as $B(X)$, and the set of finite subsets of X as $\exp_{<\omega} X$. There is a natural one-to-one map from $B(X)$ to $\exp_{<\omega} X$ defined by $g \mapsto \operatorname{supp} g$ for $g \in B(X)$, where

$$\operatorname{supp} g = \bigcap \{M \subset X\colon\ g \text{ in the algebraic envelope of } M\}.$$

Note that

$$g = \sum \{x\colon\ x \in \operatorname{supp} g\},$$
$$\operatorname{supp}(g+h) = \operatorname{supp} g \mathbin{\triangle} \operatorname{supp} h = (\operatorname{supp} g \setminus \operatorname{supp} h) \cup (\operatorname{supp} h \setminus \operatorname{supp} g)$$

for every $g, h \in B(X)$. For a partition γ of X, put

$$H(\gamma) = \{g \in B(X)\colon\ |\operatorname{supp} g \cap M| \text{ is even for } M \in \gamma\}.$$

Obviously, $H(\gamma)$ is a subgroup of $B(X)$.

For a zero-dimensional (i.e., with a clopen base) space X, let $B_z(X)$ denote the group $B(X)$ with the group topology for which $\{H(\gamma)\colon\ \gamma \text{ is an open partition of } X\}$ is a base at unity. It is easy to prove that X is embedded in $B_z(X)$ as a closed subset and $H(\gamma)$ is a clopen subgroup of $B_z(X)$ for an open partition γ.

Theorem 7 (version 2). *Let (X,σ) be a space with a coarser non-Archimedean topology τ. Suppose, (X,σ) has a base consisting of τ-closed sets. Then (X,σ) is retral. Moreover, X is a retract of $B_z(X)$ and therefore, of the free Abelian topological group $A(X)$ of X.*

Proof. Let $\mathcal{B}$ be a base of (X, τ) forming a tree with respect to reverse inclusion. Then X is naturally embedded into the set $T(\mathcal{B})$ of the branches of the tree $\mathcal{B}$ taken with the natural topology whose base consists of the sets $\{\xi \in T(\mathcal{B})\colon U \in \xi\}$, where $U \in \mathcal{B}$. Select one of the natural linear orders on $T(\mathcal{B})$ and denote its restriction on X as $<$. For $A, B \in \mathcal{B}$, we say that $A < B$ iff $a < b$ for any $a \in A$ and $b \in B$. Then for disjoint $A, B \in \mathcal{B}$, we have either $A < B$ or $B < A$.

For $\mathcal{P} \subset \mathcal{B}$, let $\mu(\mathcal{P})$ be the set of the elements of $\mathcal{P}$ that are maximum in $\mathcal{P}$ with respect to the inclusion order. Note that $\mathcal{P}$ is a refinement of $\mu(\mathcal{P})$, $\mu(\mathcal{P})$ is a disjoint family, and $\bigcup \mathcal{P} = \bigcup \mu(\mathcal{P})$.

Denote $B_* = \{g \in B_z(X)\colon |\operatorname{supp} g| \text{ is odd}\}$. We have $B_* = B_z(X) \setminus H(\{\{X\}\})$, therefore, B_* is clopen. For $g \in B_*$, put $\mathcal{O}(g) = \{U \in \mathcal{B}\colon |\operatorname{supp} g \cap U| \text{ is even}\}$ and $R(g) = \operatorname{supp} g \setminus \bigcup \mathcal{O}(g)$. Since $\bigcup \mu(\mathcal{O}(g)) = \bigcup \mathcal{O}(g)$ and $\mu(\mathcal{O}(g))$ is a disjoint family, $|\operatorname{supp} g \cap \bigcup \mathcal{O}(g)|$ is even. Therefore, $R(g)$ is nonempty, because $|\operatorname{supp} g|$ is odd. Put $r(g) = \min R(g)$.

Lemma. *Let $g, h \in B_*$, $\gamma \subset \mathcal{B}$ be a partition of X and $g + h \in H(\gamma)$. Then*
(1) $\operatorname{St}(R(g), \gamma) = \operatorname{St}(R(h), \gamma)$ *and*
(2) $\operatorname{st}(r(g), \gamma) = \operatorname{st}(r(h), \gamma)$.

Proof. Let $U \in \gamma$ and $U \cap R(g) = \emptyset$. To prove (1), it suffices to show that $U \cap R(h) = \emptyset$. Clearly, $X \setminus \operatorname{supp} g \subset \bigcup \mathcal{O}(g)$. Therefore, $R(g) = X \setminus \bigcup \mathcal{O}(g)$. Since $U \cap R(g) = \emptyset$, we have $U \subset \bigcup \mathcal{O}(g)$. This, the definition of $\mathcal{O}(g)$, and the fact that $\mathcal{B}$ is a tree with respect to reverse inclusion imply that $U \subset O$ for some $O \in \mathcal{O}(g)$. There exists $\gamma' \subset \gamma$ such that $O = \bigcup \gamma'$. Hence, $|\operatorname{supp}(g + h) \cap O|$ is even. Since $|\operatorname{supp} g \cap O|$ is even, $|\operatorname{supp} h \cap O|$ is also even, i.e., $O \in \mathcal{O}(h)$. Therefore, $R(h) \cap U \subset R(h) \cap O = \emptyset$.

Let us prove (2). The definition of $r(g)$ and the choice of the order on X imply that $\operatorname{st}(r(g), \gamma) < V$ for $V \in \operatorname{St}(R(g), \gamma) \setminus \{\operatorname{st}(r(g), \gamma)\}$. This and (1) imply $r(h) \in \operatorname{st}(r(g), \gamma)$. Hence, $\operatorname{st}(r(g), \gamma) = \operatorname{st}(r(h), \gamma)$. □

Clearly, $r\colon B_* \to X$ is a retraction. To prove the theorem, it suffices to show that r is continuous, because B_* is clopen. Let $g \in B_*$ and F' be a neighborhood of $r(g)$. Fix $W \in \mathcal{B}$ such that $\operatorname{supp} g \cap W = \{r(g)\}$. There exists a neighborhood F of $r(g)$ such that F is τ-closed and $F \subset W \cap F'$. Let $\gamma_* = \mu(\{U \in \mathcal{B}\colon U \cap F \neq \emptyset\})$ and $\gamma = \{F\} \cup \gamma_*$. We show that $r(g + H(\gamma)) \subset F$. Let $h \in g + H(\gamma)$. Put $\xi_* = \{U \in \gamma\colon U \cap W = \emptyset\}$. Note that $\xi = \xi_* \cup \{W\} \subset \mathcal{B}$ is a partition of X. Since γ refines ξ, we have $r(h) \in W$ by lemma. Suppose, $r(h) \notin F$. Then there exists $U \in \gamma_*$, $U \subset W$, such that $r(h) \in U$. We have $\operatorname{supp} g \cap U = \emptyset$. Because $g + h \in H(\gamma)$, $|\operatorname{supp}(g + h) \cap U|$ is even. We see that $|\operatorname{supp} h \cap U|$ is even, i.e., $U \in \mathcal{O}(h)$. This and the definition of $R(h)$ imply that $R(h) \cap U = \emptyset$, which contradicts the condition $r(h) \in R(h) \cap U$. □

One application of this result is the answer to a persistent open problem [20,2,3]. Uspenskij showed [20] that a Lindelöf Mal'tsev space has cellularity less than or equal to the continuum, but conjectured that the correct upper bound should be ω_1. However, there is a counter-example. Let X be a Michael–Bernstein line whose every finite power is Lindelöf and whose cellularity is $2^{\aleph_0}$ (see, e.g., Burke [1]). Consider the free topological group of X. By a standard argument, as X^n is Lindelöf for all $n \geqslant 1$, the free group is Lindelöf. By Theorem 7, there is a retraction of the free topological group onto X. But $c(X) = 2^{\aleph_0}$, hence, the cellularity of the free group must be the continuum.

Example 8. There is a Lindelöf topological group with cellularity $2^{\aleph_0}$.

In fact, one can go further in the analysis of the cellularity of free topological groups. For example, we have the following

Proposition 9. *Let X be a space and $\mathcal{F}$ be a collection of disjoint open subsets of X. Suppose, there exists a continuous metric d on X such that each $F \in \mathcal{F}$ is closed with respect to d. Then $c(F(X)) \geqslant c(A(X)) \geqslant |\mathcal{F}|$.*

Proof. Let $\{N_i\colon i \in \omega\}$ be a partition of ω, $|N_i| = \omega$ for $i \in \omega$. For $F \in \mathcal{F}$, $x \in X \setminus F$, and $m \in \omega$, denote

$$n(F, x, m) = \min\{i \in N_m\colon\ 8 \cdot 2^{-i} < d(x, F)\}.$$

Put

$$\gamma_m(F) = \{F\} \cup \{B(x, 2^{-n(F,x,m)})\colon\ x \in X \setminus F\},$$

where $B(x, \varepsilon) = \{y \in X\colon\ d(x, y) < \varepsilon\}$ for $\varepsilon > 0$. Then $\gamma_m(F)$ is an open cover of X. Let

$$U(F) = \bigcup_{k\in\omega} \left\{ \sum_{i=0}^{k} (x_i - y_i) \in A(X)\colon\ x_i, y_i \in U_i \text{ for some } U_i \in \gamma_i(F) \right\}.$$

Then (see [19]) $U(F)$ is an open neighborhood of unity in $A(X)$.

To prove the proposition, it suffices to show that $(F + U(F)) \cap (G + U(G)) = \emptyset$ for different $F, G \in \mathcal{F}$. Assume the opposite. Then there exist $n \in \omega$, $x_* \in F$, $x_i, y_i \in U_i \in \gamma_i(F)$, $u_* \in G$, and $u_i, v_i \in V_i \in \gamma_i(G)$, $i \leqslant n$, such that

$$x_* + \sum_{i=0}^{n} (x_i - y_i) = u_* + \sum_{j=0}^{n} (u_j - v_j).$$

Then there exist $s \leqslant n$, $\{i_0, \ldots, i_s\} \subset \{0, \ldots, n\}$ and $\{j_0, \ldots, j_s\} \subset \{0, \ldots, n\}$ such that

$$V_{j_0} \cap F \neq \emptyset, \qquad U_{i_s} \cap G \neq \emptyset, \qquad V_{j_t} \cap U_{i_t} \neq \emptyset$$

for any $t \leqslant s$, and for different $t_1, t_2 \leqslant s$, $i_{t_1} \neq i_{t_2}$ and $j_{t_1} \neq j_{t_2}$. For $t \leqslant s$, we fix $z_{2t} \in X \setminus G$, $z_{2t+1} \in X \setminus F$, and nonnegative integers $l(2t)$ and $l(2t+1)$ such that

$$l(2t) = n(G, z_{2t}, j_t), \qquad V_{j_t} = B(z_{2t}, 2^{-l(2t)}),$$
$$l(2t+1) = n(F, z_{2t+1}, i_t), \qquad V_{i_t} = B(z_{2t+1}, 2^{-l(2t+1)}).$$

Note that $|\{p \leqslant 2s+1\colon\ l(z_p) = n\}| \leqslant 2$ for any $n \in \omega$. We also have

$$F \cap B\big(z_0, 2^{-l(z_0)}\big) \neq \emptyset,$$
$$B\big(z_{2s+1}, 2^{-l(z_{2s+1})}\big) \cap G \neq \emptyset,$$
$$B\big(z_r, 2^{-l(z_r)}\big) \cap B\big(z_{r+1}, 2^{-l(z_{r+1})}\big) \neq \emptyset$$

for $r < 2s+1$. Choose $m \leqslant 2s+1$ such that z_m is a point of minimum for the function $p \mapsto l(p)$.

Suppose m is odd. Then $z_m \notin F$ and

$$d(z_m, F) \leqslant 2\sum_{p=0}^{2s} 2^{-l(z_p)} \leqslant 8 \cdot 2^{-l(z_m)} = 8 \cdot 2^{-n(F, z_m, i_*)},$$

where $i_* = i_{(m-1)/2}$. This contradicts the definition of $n(F, z_m, i_*)$. Even m are considered similarly. □

4. Relative cellularity and $M(X)$

To find a counter-example to the basic question, it would suffice to construct a 2-Mal'tsev space which has uncountable cellularity but whose free group does have the countable chain condition. In fact, rather less than countable cellularity of the free group is necessary.

Let X be a subspace of Y. Write $c(X, Y)$ for the supremum of all cardinals κ, where κ is the cardinality of a collection of pairwise disjoint open subsets of Y, each meeting X. Thus, $c(X, Y)$ measures the relative cellularity of X in Y.

Further, define $M(X) = \{xy^{-1}z \in F(X)\colon\ x, y, z \in X\}$. We write $M_G(X)$ for $M(X)$ considered as a subspace of $F(X)$. Let $q\colon X^3 \to M_G(X)$ be the natural map.

The topology of the free topological group is notoriously difficult to handle, but the topology of $M_G(X)$ may be identified in a straightforward fashion. Denote the universal uniformity on X as $\mathcal{U}_X$. Write $W(A, B) = q((A \times B) \cup (B \times A))$, where $A \subseteq X$ and $B \subseteq X^2$ is symmetric and contains the diagonal.

Claim 10. *Fix x in X. Then the collection of all $W(O, U)$ for open O containing x and U in $\mathcal{U}_X$ is a local base at x in $M_G(X)$.*

Proof. Let W^* be a neighborhood of the identity in $F(X)$ such that $x \cdot W^* \cap M_G(X) \subseteq W$. There exists a continuous seminorm $\|\cdot\|$ on $F(X)$ such that

$$x \cdot \big\{g \in F(X)\colon\ \|g\| \leqslant 2\big\} \subseteq W^*$$

(see [11]). Put

$$O = \big\{y \in X\colon\ \|xy^{-1}\| < 1,\ \|x^{-1}y\| < 1\big\} \quad \text{and}$$
$$U = \big\{(y, z) \in X \times X\colon\ \|zy^{-1}\| < 1,\ \|z^{-1}y\| < 1\big\}.$$

It is easy to see that for $g \in W(O, U)$, we have $\|x^{-1}g\| < 2$, therefore, $g \in W^*$. □

Now we may determine when a space X has countable relative cellularity in $M_G(X)$. Observe that the property (TG) mentioned in the statement of the claim is similar to Tkachenko's property (T) (see [18]).

Claim 11. *Let X be a space. Then the following conditions are equivalent:*

(i) (TG) *For any families $\{O_\alpha\}_{\alpha\in\omega_1}$ of nonempty open sets and $\{\gamma_\alpha\}_{\alpha\in\omega_1}$ of open normal covers, there exist distinct $\alpha, \beta \in \omega_1$ such that* $\mathrm{st}(O_\alpha, \gamma_\beta) \cap \mathrm{st}(O_\beta, \gamma_\alpha) \neq \emptyset$.

(ii) *For any family $\{O_\alpha\}_{\alpha\in\omega_1}$ of nonempty open subsets and $\{U_\alpha\}_{\alpha\in\omega_1} \subseteq \mathcal{U}_X$, there exist distinct $\alpha, \beta \in \omega_1$ such that $W(O_\alpha, U_\alpha) \cap W(O_\beta, U_\beta) \neq \emptyset$.*

(iii) $c(X, M_G(X)) \leqslant \aleph_0$.

Proof. Claim 10 implies that (ii) and (iii) are equivalent. Let us show that (i) and (ii) are equivalent. For a cover γ of X, put $U(\gamma) = \bigcup_{V\in\gamma} V \times V$. Note that the family $\{U(\gamma) : \gamma$ is a normal cover of $X\}$ forms a base of the universal uniformity $\mathcal{U}_X$ on X. The required equivalence now follows from the following simple assertion:

If $O_1, O_2 \subset X$ and γ_1, γ_2 are covers of X then $W(O_1, U(\gamma_1)) \cap W(O_2, U(\gamma_2)) \neq \emptyset$ if and only if $\mathrm{st}(O_1, \gamma_2) \cap \mathrm{st}(O_2, \gamma_1) \neq \emptyset$. □

We may summarize our approach to a counter-example in the following corollary. It follows from there observations: from above, a space X has the (TG) property if and only if $c(X, M_G(X)) \leqslant \aleph_0$; if X is a retract of Y, then $c(X) = c(X, Y)$; and every retral space X is a retract of $M_G(X)$.

Corollary 12. *A retral space satisfies the countable chain condition if and only if it has property* (TG).

5. Not every Mal'tsev space is retral

It will be convenient to introduce another property, stronger than (TG), which will be easier to check and which has the advantage of implying that the entire free topological group has the countable chain condition. We say that X has property (A) provided:

If $\mathcal{O}$ is an uncountable family of open subsets of X, then there exist an uncountable $\mathcal{O}_* \subseteq \mathcal{O}$ and $x_* \in X$ such that for any neighborhood V of x_*,

$$\left|\{O \in \mathcal{O}_*\colon\ O \cap V = \emptyset\}\right| < \omega.$$

Theorem 13. *Let X have property* (A). *Then*

(a) *any continuous image of X has property* (A);

(b) *if Y has* (A), *then $X \times Y$ also has* (A);

(c) *if $Y = \bigcup\{X_i\colon\ i < \omega\}$ and X_i has* (A) *for each $i < \omega$, then Y has* (A);

(d) *$F(X)$ has* (A);

(e) *X has* (TG);
(f) *if X is a retract of $M_G(X)$, then $c(X) \leqslant \omega$;*
(g) *if X is a topological group, then $c(X) \leqslant \omega$;*
(h) $c(F(X)) \leqslant \omega$.

Proof. Claims (a) and (c) are evident.

(b) Let $\mathcal{O} = \{U_\alpha \times V_\alpha\colon \alpha < \omega_1\}$ be a family of open subsets of $X \times Y$. As X has (A), there exist an uncountable $A \subseteq \omega_1$ and $x_* \in X$ such that for any neighborhood U_* of X_*, $|\{\alpha \in A\colon U_\alpha \cap U_* = \emptyset\}| < \omega$.

As Y has (A), there exist an uncountable $B \subseteq A$ and y_* such that for any neighborhood V_* of y_*, $|\{\alpha \in B\colon V_\alpha \cap V_* = \emptyset\}| < \omega$. The point (x_*, y_*) and the family $\{U_\alpha \times V_\alpha\colon \alpha \in B\}$ are what we need.

Claim (d) follows from (a), (b) and (c) by a standard argument.

(e) Let $\{O_\alpha\colon \alpha < \omega_1\}$ be a family of open subsets of X and $\{\gamma_\alpha\colon \alpha < \omega_1\}$ be a family of open normal covers of X. Since X has (A), there exist an uncountable $A \subset \omega_1$ and $x_* \in X$ such that for any neighborhood U of x_*, $|\{\alpha \in A\colon O_\alpha \cap U = \emptyset\}| < \omega$.

For $\alpha \in A$, choose $U_\alpha \in \gamma_\alpha$ with $x_* \in U_\alpha$ and put $K_\alpha = \{\beta \in A\colon O_\beta \cap U_\alpha = \emptyset\}$.

By construction, $|K_\alpha| < \omega$. It follows from the Δ-system lemma and a simple counting argument that there exist distinct $\alpha, \beta \in A$ such that $\alpha \notin K_\beta$ and $\beta \notin K_\alpha$, i.e., $O_\alpha \cap U_\beta \neq \emptyset$ and $O_\beta \cap U_\alpha \neq \emptyset$.

We have: $x_* \in \mathrm{st}(O_\alpha, \gamma_\beta) \cap \mathrm{st}(O_\beta, \gamma_\alpha) \neq \emptyset$.

Claim (f) follows from (d) and Claim 11. A topological group X is a retract of $F(X)$ and, therefore, of $M_G(X)$, hence, (g) is implied in (f). Finally, (h) follows from (g) and (d). □

Example 14. There is a space X such that X has a coarser separable zero-dimensional metrizable topology and $c(X) = 2^{\aleph_0}$, but $F(X)$ has countable cellularity.

Hence X is a 2-Mal'tsev space which is not retral.

Proof. Let $\mathbb{C}$ be the Cantor space. Let $\{M, M_\alpha\colon \alpha < 2^\omega\}$ be a partition of $\mathbb{C}$ such that, for each $\alpha < 2^\omega$, $M = M_\alpha = \mathbb{C}$. Strengthen the standard topology on $\mathbb{C}$ by declaring all sets $\{M_\alpha\colon \alpha < 2^\omega\}$ to be closed and open. The space X is $\mathbb{C}$ with this new topology.

By construction, X has a weaker separable zero-dimensional metrizable topology and its cellularity is the continuum. Once we have shown that X has (A), the remainder follows directly from Theorem 1, Corollary 12, and Theorem 13.

Thus, let $\{O_\beta\colon \beta < \omega_1$ be a family of open subsets of X. We have to find uncountable $B \subseteq \omega_1$ and $x_* \in X$ such that for any neighborhood U of x_*,

$$\bigl|\{\beta \in B\colon O_\beta \cap U = \emptyset\}\bigr| < \omega.$$

For $\beta < \omega_1$, find rational numbers $l_\beta < r_\beta \in \mathbb{C}$ and an ordinal $\alpha_\beta < 2^{\aleph_0}$ such that $(l_\beta, r_\beta) \cap M_{\alpha\beta} \subseteq O_\beta$. There exists an uncountable $A \subset \omega_1$ such that if $\beta_1, \beta_2 \in A$, then $l_{\beta_1} = l_{\beta_2} = l$ and $r_{\beta_1} = r_{\beta_2} = r$. If $|\{\beta \in A\colon \alpha_\beta = \alpha\}| = \omega_1$ for some $\alpha < 2^\omega$, then a point $x_* \in M_\alpha \cap (l, r)$ and the set $B = \{\beta \in A\colon \alpha_\beta = \alpha\}$ are the required ones.

Otherwise, there exists an uncountable $B \subseteq A$ such that if $\beta_1, \beta_2 \in B$ and $\beta_1 \neq \beta_2$, then $\alpha_{\beta_1} \neq \alpha_{\beta_2}$. Take any point $x_* \in M \cap (l, r)$. As the family

$$\left\{U \setminus \bigcup\{M_\alpha\colon\ \alpha \in K\}\colon\ x_* \in U,\ U \text{ is open in } \mathbb{C},\ K \text{ is a finite subset of } 2^\omega\right\}$$

is a base of the topology at x_*, x_* and B are as required. $\square$

6. The properties of (set) 2-Mal'tsev spaces and Katetov's theorem

The example in the preceding section answers the basic question: not every Mal'tsev space is retral. In the opposite direction, Reznichenko and Uspenskij have recently shown [15] that every pseudocompact Mal'tsev space is retral. However, there remain a number of natural questions.

Questions.

(1) Is every compact Mal'tsev space a retract of a compact topological group?

(2) Is every metrizable Mal'tsev space retral? Is every countable Mal'tsev space retral?

(3) Is every Mal'tsev Lindelöf Σ space retral?

Clearly, the technique elaborated above for finding Mal'tsev spaces which are not retral will not furnish us with negative answers to the second question. The third question also resists attack; indeed, every 2-Mal'tsev Lindelöf Σ space has a countable network. As will soon be seen, at this point, it is convenient to introduce a new class of spaces, which are closely related to 2-Mal'tsev spaces and are more natural from the *topological* viewpoint. Additionally, this leads us to a generalization of Katetov's theorem stating that a (countably) compact space with hereditarily normal cube is metrizable. Proofs are omitted and the results are not presented in their full generality. Interested readers are referred to [4].

A space X is said to be 2-set Mal'tsev if there is an upper semicontinuous set-valued map $M : X^3 \to X$ such that $M(x, y, y) = \{x\} = M(y, y, x)$ and $M(x, y, z) \subseteq \{x, z\}$. (Recall that a set-valued map $F : X \to Y$ is upper semicontinuous if

$$F^{-1}C = \{x \in X\colon\ F(x) \cap C \neq \emptyset\}$$

is closed for every closed $C \subseteq Y$.) A space X is said to be 2-weak Mal'tsev if there is a map $M : X^3 \to X$ such that $M(x, y, y) = x = M(y, y, x)$, M is continuous at all points (x, y, y) and (y, y, x), and $M(x, y, z) \in \{x, z\}$. Evidently, a 2-Mal'tsev space is 2-weak Mal'tsev. The following theorem is similar to Theorem 1, so only a sketch of the proof is given.

Theorem 15. *Let X be a space. Then the following conditions are equivalent:*

(i) *X is 2-set Mal'tsev;*

(ii) *X is 2-weak Mal'tsev;*

(iii) *there exist open subspaces U and V of $X^3 \setminus \Delta_3$ such that*

(1) $U \cap V = \emptyset$,

(2) $\Pi_3 \setminus \Pi_1 \subseteq U$, $\Pi_1 \setminus \Pi_3 \subseteq V$;

(iv) *there are closed subspaces A and B of X^3 such that*
  (1) $A \cup B = X^3$,
  (2) $\Pi_1 \subseteq A$ *and* $\Pi_3 \subseteq B$,
  (3) $(A \cap B) \cap (\Pi_1 \cup \Pi_3) \subseteq \Pi_3$.

Proof. First, suppose $M' : X^3 \to X$ witnesses that X is 2-set Mal'tsev. Define $M : X^3 \to X$ by: $M(x, y, z)$ is any point of $M'(x, y, z)$. This M demonstrates that X is 2-weak Mal'tsev.

Now suppose $M : X^3 \to X$ witnesses that X is 2-weak Mal'tsev. For all distinct x and y in X, pick open $U(x, y)$ containing x and open $V(x, y)$ containing y such that

$$M\big(U(x,y), V(x,y), V(x,y)\big) \cap \Pi_3 = \emptyset \quad \text{and}$$
$$M\big(V(x,y), V(x,y), U(x,y)\big) \cap \Pi_1 = \emptyset.$$

Then

$$U = \bigcup \big\{U(x,y) \times V(x,y) \times V(x,y)\colon\ x, y \in X, x \neq y\big\} \quad \text{and}$$
$$V = \bigcup \big\{V(x,y) \times V(x,y) \times U(x,y)\colon\ \text{distinct } x, y \in X\big\}$$

satisfy (1) and (2) of (iii).

Statements (iii) and (iv) are easily seen to be equivalent by taking complements.

Finally, suppose we have closed sets A and B as in (iv). Define a set-valued map $M : X^3 \to X$ by

$$M(x,y,z) = \begin{cases} \{x\}, & \text{if } (x,y,z) \in A \setminus B, \\ \{x, z\}, & \text{if } (x,y,z) \in A \cap B, \\ \{z\}, & \text{if } (x,y,z) \in B \setminus A. \end{cases}$$

Note that $M^{-1}C = ((C \times X \times X) \cap A) \cup ((X \times X \times C) \cap B)$ for any $C \subseteq X$. Thus, just as in the proof of Theorem 1, M is upper semicontinuous, and X is 2-set Mal'tsev. □

Lemma 16. *The space (X, σ) is 2-set Mal'tsev if there is a coarser topology τ on X such that one of* (1)–(3) *holds*:
  (1) (X, τ) *is 2-set Mal'tsev,*
  (2) $(X, \tau)^3 \setminus \Delta$ *is normal,*
  (3) $((X, \tau)^2 \setminus \Delta) \times (X, \tau)$ *is normal.*

A space X is a $\Sigma(\aleph_0)$ space [12] if there are a countable closed cover $\mathcal{K}$ and a cover $\mathcal{C}$ comprising countably compact sets such that for every $C \in \mathcal{C}$ and open $U \supseteq C$, there exists $K \in \mathcal{K}$ with $C \subseteq K \subseteq U$. Evidently, countably compact spaces are $\Sigma(\aleph_0)$. A 'Lindelöf Σ space' is a space which is both Lindelöf and $\Sigma(\aleph_0)$. Alternatively, the class of Lindelöf Σ spaces is the smallest class containing all compact and separable metrizable spaces that is closed under countable products, continuous image, and closed subspaces.

Theorem 17. *Let X be a 2-set Mal'tsev space. If X is a $\Sigma(\aleph_0)$ space, then X has a countable network.*

Corollary 18. *A countably compact space X such that $\Pi_1 \setminus \Pi_3$ and $\Pi_3 \setminus \Pi_1$ can be separated in X^3 by disjoint open sets is metrizable.*

In particular, if X is countably compact and either $X^3 \setminus \Delta_3$ or $X^3 \setminus \Pi_2$ is normal, then X is metrizable.

Remark. Let X be compact. Observe that $X^3 \setminus \Pi_2 = X \times (X^2 \setminus \Delta)$, and if $X^2 \setminus \Delta$ is paracompact, then $X \times (X^2 \setminus \Delta)$ is normal. Thus, Corollary 18 generalizes Gruenhage's theorem (in [6]):

If X is compact and $X^2 \setminus \Delta$ is paracompact, then X is metrizable.

We also note that there is an example of a nonmetrizable compact space X such that $X^2 \setminus \Delta$ is normal [7].

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