

Line defects in the small elastic constant limit of a three-dimensional Landau-de Gennes model

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Abstract We consider the Landau-de Gennes variational model for nematic liquid crystals, in three-dimensional domains. More precisely, we study the asymptotic behaviour of minimizers as the elastic constant tends to zero, under the assumption that minimizers are uniformly bounded and their energy blows up as the logarithm of the elastic constant. We show that there exists a closed set $\mathcal{S}_{\text{line}}$ of finite length, such that minimizers converge to a locally harmonic map away from $\mathcal{S}_{\text{line}}$. Moreover, $\mathcal{S}_{\text{line}}$ restricted to the interior of the domain is a locally finite union of straight line segments. We provide sufficient conditions, depending on the domain and the boundary data, under which our main results apply. We also discuss some examples.

Keywords Landau-de Gennes model · Q -tensors · asymptotic analysis · topological singularities · line defects · rectifiable sets · stationary varifolds

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1 Introduction

1.1 Variational theories for nematic liquid crystals

A nematic liquid crystal is matter in an intermediate state between liquid and crystalline solid. Molecules can flow and their centers of mass are randomly distributed, but the molecular axes tend to self-align locally. As a result, the material is anisotropic with respect to optic and electromagnetic properties. In the so-called uniaxial nematics, the molecules are often rod-shaped and, although they may carry a permanent dipole, there are as many dipoles ‘up’ as

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there are ‘down’. Therefore, the material symmetry group contains the rotations around the molecular axis and the reflection symmetry which exchanges the two orientations of the axis. The long-range orientational order due to the self-alignment of the molecules is broken in some places, called *defects*. The word nematic itself refers to the line defects (see Friedel, [32]):

I am going to use the term nematic ($\nu\eta\mu\alpha$, thread) to describe the forms, bodies, phases, etc. of the second type... because of the linear discontinuities, which are twisted like threads, and which are one of their most prominent characteristics.

In addition to line defects, also called *disclinations*, nematic media exhibit “hedgehog-like” point singularities. According to the topological theory of ordered media (see e.g. [53, 64, 66]), both kinds of defects are described by the homotopy groups of a manifold, which parametrizes the possible local configurations of the material.

Several models for uniaxial nematic liquid crystals have drawn the attention of the mathematical community. The most popular continuum theories which are based on a finite-dimensional order parameter space are probably the Oseen-Frank, the Ericksen and the Landau-de Gennes theories. In the Oseen-Frank theory [31], the material is modeled by a unit vector field $\mathbf{n} = \mathbf{n}(x) \in \mathbb{S}^2$, which represents the preferred direction of molecular alignment. The elastic energy, in the simplest setting, reduces to the Dirichlet functional

$$E(\mathbf{n}) := \frac{1}{2} \int_{\Omega} |\nabla \mathbf{n}|^2, \quad (1)$$

where $\Omega \subseteq \mathbb{R}^3$ is the physical domain. In this case, least-energy configurations are but harmonic maps $\mathbf{n}: \Omega \rightarrow \mathbb{S}^2$. As such, minimizers have been widely studied in the literature (the reader is referred to e.g. [39] for a general review of this subject). Schoen and Uhlenbeck [61] proved that minimizers are smooth away from a discrete set of points singularities. Brezis, Coron and Lieb [17] investigated the precise shape of minimizers around a point defect x_0 , and proved that

$$\mathbf{n}(x) \simeq \pm R \frac{x - x_0}{|x - x_0|} \quad \text{for } |x - x_0| \ll 1, \quad (2)$$

where R is a rotation. These “hedgehog-like” point defects are associated with a non-trivial homotopy class of maps $\mathbf{n}: \partial B_r(x_0) \rightarrow \mathbb{S}^2$, i.e. a non-trivial element of $\pi_2(\mathbb{S}^2)$. Interesting results are also available for the full Oseen-Frank energy, which consists of various terms accounting for splay, twist and bend deformations. Hardt, Kinderlehrer and Lin [37] proved the existence of minimizers and partial regularity, i.e. regularity out of an exceptional set whose Hausdorff dimension is strictly less than 1. As for the local behaviour of minimizers around the defects, the picture is not as clear as for the Dirichlet energy (1), but at least the stability of “hedgehog-like” singularities such as (2) as been completely analyzed (see [45] and the references therein). However, the partial regularity result of [37] implies that the Oseen-Frank theory cannot account for line defects.

Ericksen theory is less restrictive, because it allows variable orientational order. Indeed, the configurations are described by a pair $(s, \mathbf{n}) \in \mathbb{R} \times \mathbb{S}^2$, where \mathbf{n} is the preferred direction of molecular alignment and the scalar s measures the degree of ordering. In this theory, defects are identified by the condition $s = 0$, which correspond to complete disordered states. Under suitable assumptions, minimizers can exhibit line singularities and even planar discontinuities (see [47, Theorem 7.2]). Explicit examples were studied by Ambrosio and Virga [6] and Mizel, Roccato and Virga [54]. However, the Ericksen theory — as the Oseen-Frank theory — excludes configurations which might have physical reality. Ericksen himself was aware of this, since he presented his theory as a “kind of compromise” [29, p. 98] between physical intuition and mathematical simplicity. Indeed, both the Oseen-Frank and the Ericksen theory do not take into account the material symmetry, that is, the configurations represented by \mathbf{n} and $-\mathbf{n}$ are physically indistinguishable. Moreover, these theories postulate that, at each point of the medium, there is at most one preferred direction of molecular orientation. Configurations for which such a preferred direction exists are called *uniaxial*, because they have one axis of rotational symmetry. If no preferred direction exists, the configuration is called *isotropic* (in the Ericksen theory, this corresponds to $s = 0$).

The Landau-de Gennes theory [27] allows for a rather complete description of the local behaviour of the medium, because it accounts for *biaxial*¹ configurations as well. A state is called biaxial when it has no axis of rotational symmetry, but three orthogonal axes of reflection symmetry instead (see [56] for more details). What makes the Landau-de Gennes theory so rich is the order parameter space. Configurations are described by matrices (the so-called Q -tensors), which can be interpreted as renormalized second-order moments of the microscopic distribution of molecules with respect to the orientation.

In this paper, we aim at describing line defects in a special asymptotic limit of the Landau-de Gennes theory, namely, when the elastic constant of the nematic material is very small, compared to the typical length-scale of the domain. Two simplifying assumptions are postulated here. First, we neglect the effect of external electromagnetic fields. To induce non-trivial behaviour in minimizers, we couple the problem with non-homogeneous Dirichlet boundary conditions (strong anchoring). Second, we adopt the one-constant approximation, that is we drop out several terms in the expression of the elastic energy, and we are left with the gradient-squared term only. These assumptions, which drastically reduce the technicality of the problem, are common in the mathematical literature on this subject (see e.g. [28, 33, 40, 41, 46, 52]). For the two-dimensional case, the analysis of the analogous problem is presented in [21, 35].

¹ Here “uniaxial” and “biaxial” refer to *arrangements* of molecules, not to the molecules themselves which are always assumed to be uniaxial.

1.2 The Landau-de Gennes functional

As we mentioned before, the local configurations of the medium are described by Q -tensors, i.e. elements of

$$\mathbf{S}_0 := \{Q \in M_3(\mathbb{R}) : Q^\top = Q, \operatorname{tr} Q = 0\}.$$

This is a real linear space, of dimension five, which we endow with the scalar product $Q \cdot P := Q_{ij}P_{ij}$ (Einstein's convention is assumed). This choice of the configurations space can be justified as follows. At a microscopic scale, the distribution of molecules around a given point $x \in \Omega$, as a function of orientation, can be represented by a probability measure μ_x on the unit sphere \mathbb{S}^2 . The measure μ_x satisfies to the condition $\mu_x(B) = \mu_x(-B)$ for all $B \in \mathcal{B}(\mathbb{S}^2)$, which accounts for the head-to-tail symmetry of the molecules. Then, the simplest meaningful way to condense the information conveyed by μ_x is to consider the second-order moment

$$Q = \int_{\mathbb{S}^2} \left(\mathbf{n}^{\otimes 2} - \frac{1}{3} \operatorname{Id} \right) d\mu_x(\mathbf{n}).$$

We denote by $\mathbf{n}^{\otimes 2}$ the matrix defined by $(\mathbf{n}^{\otimes 2})_{i,j} := \mathbf{n}_i \mathbf{n}_j$, for each $i, j \in \{1, 2, 3\}$. The quantity Q is renormalized, so that the isotropic state $\mu_x = \mathcal{H}^2 \llcorner \mathbb{S}^2$ corresponds to $Q = 0$. As a result, Q is a symmetric traceless matrix. (The interested reader is referred e.g. to [56] for further details.)

The (simplified) Landau-de Gennes functional reads

$$E_\varepsilon(Q) := \int_\Omega \left\{ \frac{1}{2} |\nabla Q|^2 + \frac{1}{\varepsilon^2} f(Q) \right\}, \quad (\text{LG}_\varepsilon)$$

where $Q: \Omega \rightarrow \mathbf{S}_0$ is the configuration of the medium, located in a bounded container $\Omega \subseteq \mathbb{R}^3$. The function f is the quartic Landau-de Gennes potential, defined by

$$f(Q) = k - \frac{a}{2} \operatorname{tr} Q^2 - \frac{b}{3} \operatorname{tr} Q^3 + \frac{c}{4} (\operatorname{tr} Q^2)^2 \quad \text{for } Q \in \mathbf{S}_0. \quad (3)$$

This expression for f has been derived by a formal expansion in powers of Q . All the terms are invariant by rotations so that f is independent of the coordinate frame. This potential allows for multiple local minima, with a first-order isotropic-nematic phase transition (see [27, 65]). The positive parameters a , b and c depend on the material and the temperature (which is assumed to be uniform), while k is just an additive constant, which plays no role in the minimization problem. The potential f is bounded from below, so we determine uniquely the value of k by requiring $\inf f = 0$. The (small) parameter ε^2 is proportional to the material-dependent elastic constant. For each $0 < \varepsilon < 1$, we assign a boundary datum $g_\varepsilon \in H^1(\partial\Omega, \mathbf{S}_0)$ and we restrict our attention to minimizers Q_ε of (LG_ε) in the class

$$H_{g_\varepsilon}^1(\Omega, \mathbf{S}_0) := \{Q \in H^1(\Omega, \mathbf{S}_0) : Q = g_\varepsilon \text{ on } \partial\Omega \text{ in the sense of traces}\}.$$

When ε is small, the term $\varepsilon^{-2}f(Q)$ in (LG_ε) forces minimizers to take their values close to $\mathcal{N} := f^{-1}(0)$. This set can be characterized as follows (see [52, Proposition 9]):

$$\mathcal{N} = \left\{ s_* \left(\mathbf{n}^{\otimes 2} - \frac{1}{3} \text{Id} \right) : \mathbf{n} \in \mathbb{S}^2 \right\}, \quad (4)$$

where the constant s_* is defined by

$$s_* = s_*(a, b, c) := \frac{1}{4c} \left(b + \sqrt{b^2 + 24ac} \right).$$

Thus, \mathcal{N} is a smooth submanifold of \mathbf{S}_0 , diffeomorphic to the real projective plane \mathbb{RP}^2 , called *vacuum manifold*. The topology of \mathcal{N} plays an important role, for a map $\Omega \rightarrow \mathcal{N}$ may encounter topological obstructions to regularity. Sources of obstruction are the homotopy groups $\pi_1(\mathcal{N}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $\pi_2(\mathcal{N}) \simeq \mathbb{Z}$, which are associated with line and point singularities, respectively. There is a remarkable difference with the Oseen-Frank model at this level, for \mathbb{S}^2 is a simply connected manifold, so topological obstructions result from $\pi_2(\mathbb{S}^2)$ only. Despite this fact, a strong connection between the Oseen-Frank and Landau-de Gennes theories was established by Majumdar and Zarnescu [52], then by Nguyen and Zarnescu [57], who addressed the asymptotic analysis of minimizers of (LG_ε) , in three-dimensional domains. Their results imply that, when $\Omega, \partial\Omega$ are simply connected and $g_\varepsilon = g \in C^1(\partial\Omega, \mathcal{N})$, minimizers Q_ε of (LG_ε) converge in $H^1(\Omega, \mathbf{S}_0)$ to a map of the form

$$Q_0(x) = s_* \left(\mathbf{n}_0^{\otimes 2}(x) - \frac{1}{3} \text{Id} \right)$$

where $\mathbf{n}_0 \in H^1(\Omega, \mathbb{S}^2)$ is a minimizer of (1). The convergence is locally uniform, away from singularities of Q_0 . Also in this case, line defects do not appear in the limiting map, although point defects analogous to (2) might occur. Indeed, their assumptions on the domain and boundary datum are strong enough to guarantee the uniform energy bound

$$E_\varepsilon(Q_\varepsilon) \leq C \quad (5)$$

for an ε -independent constant C , and obtain H^1 -compactness. In this paper, we work in the logarithmic energy regime

$$E_\varepsilon(Q_\varepsilon) \leq C (|\log \varepsilon| + 1), \quad (6)$$

which is compatible with singularities of codimension two, in the small ε -limit.

There are analogies between the functional (LG_ε) and the Ginzburg-Landau energy for superconductivity, which reduces to

$$E_\varepsilon^{\text{GL}}(u) := \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 \right\} \quad (7)$$

when no external field is applied. Here the unknown is a complex-valued function u . There is a rich literature about the asymptotic behaviour, as $\varepsilon \rightarrow 0$,

of critical points satisfying a logarithmic energy bound such as (6). It is well-known that, under appropriate assumptions, critical points converge to maps with topology-driven singularities of codimension two. In two-dimensional domains, the theory has been developed after Bethuel, Brezis and Hélein's work [11]. In the three-dimensional case, the asymptotic analysis of minimizers was performed by Lin and Riviere [48], and extended to non-minimizing critical points by Bethuel, Brezis and Orlandi [12]. Later, Jerrard and Sonar [44] and Alberti, Baldo, Orlandi [1] proved independently that $|\log \varepsilon|^{-1} E_\varepsilon^{\text{GL}} \Gamma$ -converges, when $\varepsilon \rightarrow 0$, to a functional on integral currents of codimension two. This functional essentially measures the length of defect lines, weighted by some quantity that accounts for the topology of the defect.

1.3 Main results

For each fixed $\varepsilon > 0$, a classical argument of Calculus of Variations shows that minimizers of (LG_ε) exist as soon as $g_\varepsilon \in H^{1/2}(\partial\Omega, \mathbf{S}_0)$ and are regular in the interior of the domain. Our main result deals with their asymptotic behaviour as $\varepsilon \rightarrow 0$.

Theorem 1 *Let Ω be a bounded, Lipschitz domain. Assume that there exists a positive constant M such that, for any $0 < \varepsilon < 1$, there hold*

$$E_\varepsilon(Q_\varepsilon) \leq M(|\log \varepsilon| + 1) \quad \text{and} \quad \|Q_\varepsilon\|_{L^\infty(\Omega)} \leq M. \quad (\text{H})$$

Then, there exist a subsequence $\varepsilon_n \searrow 0$, a closed set $\mathcal{S}_{\text{line}} \subseteq \overline{\Omega}$ and a map $Q_0 \in H_{\text{loc}}^1(\Omega \setminus \mathcal{S}_{\text{line}}, \mathcal{N})$ such that the following holds.

- (i) $\mathcal{S}_{\text{line}} \cap \Omega$ is a countably \mathcal{H}^1 -rectifiable set, and $\mathcal{H}^1(\mathcal{S}_{\text{line}} \cap \Omega) < +\infty$.
- (ii) $Q_{\varepsilon_n} \rightarrow Q_0$ strongly in $H_{\text{loc}}^1(\Omega \setminus \mathcal{S}_{\text{line}}, \mathcal{N})$.
- (iii) Q_0 is locally minimizing harmonic in $\Omega \setminus \mathcal{S}_{\text{line}}$, that is for every ball $B \subset\subset \Omega \setminus \mathcal{S}_{\text{line}}$ and any $P \in H^1(B, \mathcal{N})$ which satisfies $P = Q_0$ on ∂B we have

$$\frac{1}{2} \int_B |\nabla Q_0|^2 \leq \frac{1}{2} \int_B |\nabla P|^2.$$

- (iv) *There exists a locally finite set $\mathcal{S}_{\text{pts}} \subseteq \Omega \setminus \mathcal{S}_{\text{line}}$ such that Q_0 is smooth in $\Omega \setminus (\mathcal{S}_{\text{line}} \cup \mathcal{S}_{\text{pts}})$ and $Q_\varepsilon \rightarrow Q_0$ locally uniformly in $\Omega \setminus (\mathcal{S}_{\text{line}} \cup \mathcal{S}_{\text{pts}})$.*

By saying that $\mathcal{S}_{\text{line}}$ is countably \mathcal{H}^1 -rectifiable we mean that there exists a decomposition

$$\mathcal{S}_{\text{line}} = \bigcup_{j \in \mathbb{N}} \mathcal{S}_j,$$

where $\mathcal{H}^1(\mathcal{S}_0) = 0$ and, for each $j \geq 1$, the set \mathcal{S}_j is the image of a Lipschitz map $K_j \subset \mathbb{R} \rightarrow \mathbb{R}^3$. In addition to the singular set $\mathcal{S}_{\text{line}}$ of dimension one, the limiting map Q_0 may have a set of point singularities \mathcal{S}_{pts} . This is consistent with the regularity results for minimizing harmonic maps [34, 61]. Later on, we will discuss examples where $\mathcal{S}_{\text{line}}$ and \mathcal{S}_{pts} are non-empty.

Theorem 1 is local in nature. In particular, boundary conditions play no particular role in the proof of this result, although they need to be imposed to induce non-trivial behaviour of minimizers. Theorem 1 can be adapted to the analysis near the boundary of the domain, under additional assumptions on the boundary datum. The necessary modifications are sketched in Section 4.4.

The singular set $\mathcal{S}_{\text{line}}$ is defined as the concentration set for the energy densities of minimizers. In other words, thanks to (H) we find a subsequence $\varepsilon_n \searrow 0$ and a measure $\mu_0 \in \mathcal{M}_b(\overline{\Omega}) := C(\overline{\Omega})'$ such that

$$\left\{ \frac{1}{2} |\nabla Q_{\varepsilon_n}|^2 + \frac{1}{\varepsilon_n^2} f(Q_{\varepsilon_n}) \right\} \frac{dx}{|\log \varepsilon_n|} \rightharpoonup^* \mu_0 \quad \text{in } \mathcal{M}_b(\overline{\Omega}),$$

then we define $\mathcal{S}_{\text{line}}$ as the support of μ_0 . A more precise description of the limit measure μ_0 is given by the following result. We set

$$\kappa_* := \frac{\pi}{2} s_*^2.$$

As we will see in Section 2.2, this number quantifies the energy cost associated with a topological defect of codimension two.

Proposition 2 *The measure $\mu_0 \llcorner \Omega$ is naturally associated with a stationary varifold, and*

$$\lim_{r \rightarrow 0} \frac{\mu_0(\overline{B}_r(x))}{2r} = \kappa_* \quad \text{for } \mu_0\text{-a.e. } x \in \Omega.$$

For any open set $K \subset\subset \Omega$, $\mathcal{S}_{\text{line}} \cap \overline{K}$ is the union of a finite number of closed straight line segments L_1, \dots, L_p . After possible subdivision, assume that for each $i \neq j$, either L_i and L_j are disjoint or they intersect at a common endpoint. Then, the following properties hold.

- (i) If $D \subset\subset K$ is a closed disk which intersects $\mathcal{S}_{\text{line}}$ at a single point x_0 and x_0 is not an endpoint for any L_i , then the homotopy class of Q_0 restricted to ∂D is non-trivial.
- (ii) Suppose that $x_0 \in K$ is an endpoint of exactly q segments L_{i_1}, \dots, L_{i_q} . Then q is even.

The definition of stationary varifold is given in [63, Chapter 4]. Varifolds are a generalization of differentiable manifolds, introduced by Almgren [4] in the context of Calculus of Variations, and stationary varifolds can be thought as a weak notion of minimal manifolds. Proposition 2 relies heavily on the structure theorem for stationary varifolds of dimension one [3, Theorem p. 89].

Inside the domain, the singular set is a locally finite union of line segments. Branching points are not excluded by this result, but only an even number of branches can originate from each point. (However, we expect that branching points should not arise — see the concluding remarks, Section 1.4.) Moreover, the set $\mathcal{S}_{\text{line}}$ is a topological singularity, i.e. it is associated with a non-trivial homotopy class of the map Q_0 . Therefore, from the physical point of view $\mathcal{S}_{\text{line}}$ corresponds to the “thin disclination lines” of index $\pm 1/2$ (see, e.g. [22]).

The “thick disclination lines” of index ± 1 are not included in $\mathcal{S}_{\text{line}}$ because the order parameter Q_0 can be defined continuously throughout their cores, thanks to the “escape in the third dimension” proposed by Cladis and Kléman [24]. Proposition 2 also excludes disclination loops in the interior of the domain, although loops of radius larger than some critical value R_c are expected to occur (see [22, p. 519]). However, in the limit as $\varepsilon \rightarrow 0$ we have that $R_c \rightarrow +\infty$, therefore any defect loop which is contractible in Ω should become unstable, shrink and eventually disappear. On the other hand, disclination loops may occur at the boundary of the domain. In Section 5.4, we show by an example that the singular set $\mathcal{S}_{\text{line}}$ may touch $\partial\Omega$, even if the boundary datum is smooth. In this case, the conclusion of Proposition 2 does not hold any more.

We provide sufficient conditions for the estimate (H) to hold, in terms of the domain and the boundary data. Here is our first condition.

(H₁) Ω is a bounded, smooth domain and $\{g_\varepsilon\}_{0 < \varepsilon < 1}$ is a bounded family in $H^{1/2}(\partial\Omega, \mathcal{N})$.

The uniform $H^{1/2}$ -bound is satisfied if, for instance, $g_\varepsilon = g: \partial\Omega \rightarrow \mathcal{N}$ has a finite number of disclinations. This means, there exists a finite set $\Sigma \subseteq \partial\Omega$ such that g is smooth on $\partial\Omega \setminus \Sigma$ and, for each $x_0 \in \Sigma$, we can write

$$g(x) = s_* \left\{ \left(\tau_1 \cos(k\theta(x)) + \tau_2 \sin(k\theta(x)) \right)^{\otimes 2} - \frac{1}{3} \text{Id} \right\} + \text{smooth terms of order } \rho(x) \quad (8)$$

as $x \rightarrow x_0$. Here $k \in \frac{1}{2}\mathbb{Z}$, $(\rho(x), \theta(x))$ are geodesic polar coordinates centered at x_0 and (τ_1, τ_2) is an orthonormal pair in \mathbb{R}^3 .

Proposition 3 *Condition (H₁) implies (H).*

Alternatively, one can assume

- (H₂) $\Omega \subseteq \mathbb{R}^3$ is a bounded Lipschitz domain, and it is bilipschitz equivalent to a handlebody (i.e. a 3-ball with a finite number of handles attached).
 (H₃) There exists $M_0 > 0$ such that, for any $0 < \varepsilon < 1$, we have $g_\varepsilon \in (H^1 \cap L^\infty)(\partial\Omega, \mathbf{S}_0)$ and

$$E_\varepsilon(g_\varepsilon, \partial\Omega) \leq M_0 (|\log \varepsilon| + 1), \quad \|g_\varepsilon\|_{L^\infty(\partial\Omega)} \leq M_0.$$

As an example of sequence satisfying (H₃), one can take smooth approximations of a map $g: \partial\Omega \rightarrow \mathcal{N}$ of the form (8). For instance, we can take

$$g_\varepsilon(x) := \eta_\varepsilon(\rho(x))g(x) \quad (9)$$

where $\eta_\varepsilon \in C^\infty[0, +\infty)$ is such that

$$\eta_\varepsilon(0) = \eta'_\varepsilon(0) = 0, \quad \eta_\varepsilon(\rho) = 1 \text{ if } \rho \geq \varepsilon, \quad 0 \leq \eta_\varepsilon \leq 1, \quad |\eta'_\varepsilon| \leq C\varepsilon^{-1}.$$

Proposition 4 *If (H₂) and (H₃) are satisfied, then (H) holds.*

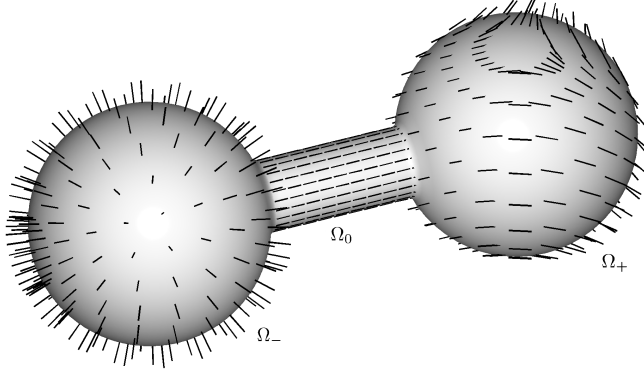


Fig. 1 The domain considered in Section 7: two unit balls (Ω_- on the left, Ω_+ on the right) joined by a cylinder Ω_0 of length $2L$ and radius r . The (unoriented) director field associated to the boundary datum is also represented. The boundary datum restricted to the boundary of Ω_- , Ω_+ defines non-trivial homotopy classes in $\pi_2(\mathcal{N})$, $\pi_1(\mathcal{N})$ respectively.

Remark 5 Hypothesis (H_2) is *not* the same as asking Ω to be a bounded Lipschitz domain with connected boundary. Let $K \subseteq \mathbb{S}^3$ be a (open) tubular neighborhood of a trefoil knot. Then K is a solid torus, i.e. K is diffeomorphic to $\mathbb{S}^1 \times B_1^2$, but $\mathbb{S}^3 \setminus K$ is *not* a solid torus. In fact, $\mathbb{S}^3 \setminus K$ is not even a handlebody, because

$$\pi_1(\mathbb{S}^3 \setminus K) = \text{the knot group of the trefoil knot} = \langle x, y \mid x^2 = y^3 \rangle$$

while the fundamental group of any handlebody is free. By composing with a stereographic projection, one constructs a smooth domain $\overline{\Omega} \subseteq \mathbb{R}^3$ diffeomorphic to $\mathbb{S}^3 \setminus K$. In particular, $\partial\Omega$ is a torus but Ω does not satisfy (H_2) .

Given an arbitrary domain, one can construct examples where line defects occur.

Proposition 6 *For each bounded domain $\Omega \subseteq \mathbb{R}^3$ of class C^1 , there exists a family of boundary data $\{g_\varepsilon\}_{0 < \varepsilon < 1}$ satisfying (H_3) and a number $\alpha > 0$ such that*

$$E_\varepsilon(Q) \geq \alpha (|\log \varepsilon| - 1)$$

for any $Q \in H_{g_\varepsilon}^1(\Omega, \mathbf{S}_0)$ and any $0 < \varepsilon < 1$. Moreover, $\mathcal{S}_{\text{line}}$ is non-empty.

The functions g_ε are smooth approximations of a map $\partial\Omega \rightarrow \mathcal{N}$, which has point singularities of the form (9).

Finally, we consider an example where both $\mathcal{S}_{\text{line}}$ and \mathcal{S}_{pts} are non-empty. The domain consists of two balls of radius 1, joined by a cylinder of radius $r \in (0, 1/2)$ and length $2L$. The boundary datum, which is defined in Section 7, is uniaxial and has two point defects. In Figure 1, we represent the behaviour of the boundary datum or, more precisely, the direction of the eigenspace corresponding to the leading eigenvalue (that is, the average orientation of the molecules at each point). This map defines non-trivial homotopy classes both in $\pi_1(\mathcal{N})$ and in $\pi_2(\mathcal{N})$.

Proposition 7 *There exists a positive number L^* such that, if $L \geq L^*$, then both $\mathcal{S}_{\text{line}}$ and \mathcal{S}_{pts} are non-empty, and there exists $x_0 \in \mathcal{S}_{\text{pts}}$ such that $\text{dist}(x_0, \mathcal{S}_{\text{line}}) \geq L/2$.*

In other words, if the cylinder is long enough then the limit configuration has line defects and at least one point defect, which is far from the line defects. Although the boundary datum defines a non-trivial class in $\pi_2(\mathcal{N})$, topological arguments alone are not enough to conclude that $\mathcal{S}_{\text{pts}} \neq \emptyset$, for there exist maps $\Omega \rightarrow \mathcal{N}$ which satisfy the boundary condition and $\mathcal{S}_{\text{line}} \neq \emptyset$, $\mathcal{S}_{\text{pts}} = \emptyset$ (see Remark 72). Proposition 7 is inspired by Hardt and Lin's paper [38], where the existence of non-smooth minimizing harmonic maps, satisfying a boundary condition of degree zero, is proved. However, there is an additional difficulty here, namely minimizers are not uniformly bounded in H^1 as $\varepsilon \rightarrow 0$. We take care of this issue by adapting some ideas of the proof of Theorem 1.

Let us spend a few words on the proof of the main result, Theorem 1. The core of the argument is a concentration property for the energy, which can be stated as follows.

Proposition 8 *Assume that the condition (H) holds. For any $0 < \theta < 1$ there exist positive numbers η , ϵ_0 and C such that, for any $x_0 \in \Omega$, $R > 0$ satisfying $\overline{B}_R(x_0) \subseteq \Omega$ and any $0 < \varepsilon \leq \epsilon_0 R$, if*

$$E_\varepsilon(Q_\varepsilon, B_R(x_0)) \leq \eta R \log \frac{R}{\varepsilon} \quad (10)$$

then

$$E_\varepsilon(Q_\varepsilon, B_{\theta R}(x_0)) \leq CR.$$

Proposition 8 implies that either the energy on a ball blows up at least logarithmically, or it is bounded on a smaller ball. Combining this fact with covering arguments, one proves that the energy concentrates on a set $\mathcal{S}_{\text{line}}$ of finite length. Then, the asymptotic behaviour of minimizers away from $\mathcal{S}_{\text{line}}$ can be studied using well-established techniques, e.g. arguing as in [52].

Roughly speaking, the proof of Proposition 8 goes as follows. Condition (10) implies that the problem can be reformulated in terms of \mathcal{N} -valued maps. Indeed, by an average argument we find $r \in (\theta R, R)$ such that the energy of Q_ε on the sphere of radius r is controlled by $C\eta|\log \varepsilon|$. On the other hand, the energy per unit length associated with a topological defect line is of the order of $\kappa_* |\log \varepsilon|$ (see the estimates by Jerrard and Sandier [43, 60]). Therefore, if η is small compared to κ_* , the sphere of radius r intersects no topological defect line of Q_ε . This makes it possible to approximate Q_ε with an \mathcal{N} -valued map P_ε , defined on a sphere of radius r' close to r . Now, since the sphere is simply connected, P_ε can be lifted to a \mathbb{S}^2 -valued map, i.e. one can write

$$P_\varepsilon(x) = s_* \left(\mathbf{n}_\varepsilon^{\otimes 2}(x) - \frac{1}{3} \text{Id} \right) \quad \text{for } x \in \partial B_{r'}(x_0)$$

for a smooth vector field $\mathbf{n}_\varepsilon: \partial B_{r'}(x_0) \rightarrow \mathbb{S}^2$. Thus, we have reduced to a problem which is formulated in terms of vector fields, and we can apply the methods by Hardt, Kinderlehrer and Lin [37, Lemma 2.3] to obtain boundedness of the energy. In other words, Condition (10) enable us to reduce the asymptotic analysis of the Landau-de Gennes problem to the analysis of the Oseen-Frank problem. Extension results are needed in several steps of this proof, for instance to interpolate between Q_ε and P_ε in order to construct an admissible comparison map. Various results in this direction are discussed in detail in Section 3. In particular, we prove variants of Luckhaus' lemma [49, Lemma 1] which are fit for our purposes.

Remark 9 Condition (H) is *not* sufficient to obtain compactness for minimizers of the Ginzburg-Landau energy. Indeed, Brezis and Mironescu [18] constructed a sequence² of minimizers $u_\varepsilon \in H^1(B_1^2, \mathbb{C})$ such that

$$E_\varepsilon^{\text{GL}}(u_\varepsilon, B_1^2) \ll |\log \varepsilon| \quad \text{and} \quad |u_\varepsilon| \leq 1,$$

yet $\{u_\varepsilon\}_{0 < \varepsilon < 1}$ does not have subsequences converging a.e. on sets of positive measure. Brezis and Mironescu's example relies on *oscillations of the phase*. Indeed, the u_ε 's can be lifted to \mathbb{R} -valued functions φ_ε 's (that is $u_\varepsilon \approx \exp(i\varphi_\varepsilon)$), but the latter are not uniformly bounded in L^∞ . This phenomenon does not occur in our case, because the \mathcal{N} -valued map P_ε is lifted to a unit vector field. The finiteness of the fundamental group $\pi_1(\mathcal{N})$ yields the compactness of the universal covering of \mathcal{N} , hence better compactness properties for the minimizers.

1.4 Concluding remarks and open questions

Several questions about minimizers of the Landau-de Gennes functional on three-dimensional domains remain open. A first question concerns the behaviour of the singular set $\mathcal{S}_{\text{line}}$. Since $\mathcal{S}_{\text{line}}$ is obtained as a limit of minimizers, one would expect that it inherits from Q_ε minimizing properties. It is natural to conjecture that $\mathcal{S}_{\text{line}}$ is a relative cycle, whose homology class is determined by the domain and the boundary datum, and that $\mathcal{S}_{\text{line}}$ has minimal length in its homology class. For instance, if the domain is convex and the boundary data has a finite number of point singularities x_1, \dots, x_p of the form (8), then $\mathcal{S}_{\text{line}}$ should be a union of non-intersecting straight lines connecting the x_i 's in pairs, in such a way that the total length of $\mathcal{S}_{\text{line}}$ is minimal. (Notice that, by topological arguments, the number p must be even.) However, if the domain is non-convex, then a part of $\mathcal{S}_{\text{line}}$ may lie on the boundary.

It would be interesting to study the structure of minimizers Q_ε in the core of line defects. For instance, *does the core of line defects contain biaxial phases?* Contreras and Lamy [25] and Majumdar, Pisante and Henao [51]

² Throughout the paper, the word “sequence” will be used to denote family of functions indexed by a continuous parameter as well.

proved that the core of point singularities, in dimension three, contains biaxial phases when the temperature is low enough. Their proofs use a uniform energy bound such as (5), so they do not apply directly to singularities of codimension two. However, the analysis of point defects on two-dimensional domains (see e.g. [21, 28, 42]) suggests that line defects may also contain biaxial phases, when the temperature is low. A related issue is the analysis of *singularity profiles*. Let $x_0 \in \mathcal{S}_{\text{line}}$ and let Π be an orthogonal plane to $\mathcal{S}_{\text{line}}$, passing through the point x_0 . Set

$$P_{\varepsilon, x_0}(y) := Q_\varepsilon(x_0 + \varepsilon y) \quad \text{for } y \in \Pi.$$

This defines a bounded sequence in $L^\infty(\Pi, \mathbf{S}_0)$, such that

$$\|\nabla P_{\varepsilon, x_0}\|_{L^2(K)} = \|\nabla Q_\varepsilon\|_{L^2(x_0 + \varepsilon K)} \leq C(K) \quad \text{for every } K \subset \subset \Pi.$$

Therefore, up to a subsequence we have $P_{\varepsilon, x_0} \rightharpoonup P_{x_0}$ in $H_{\text{loc}}^1(\Pi, \mathbf{S}_0)$. The map P_{x_0} contains the information on the fine structure of the defect core. What can be said about P_{x_0} ?

In another direction, investigating the asymptotic behaviour of a more general class of functionals in the logarithmic energy regime is a challenging issue. For instance, one may consider functionals with more elastic energy terms and/or choose different potentials, such as the sextic potential

$$\begin{aligned} f(Q) := & -\frac{a_1}{2} \operatorname{tr} Q^2 - \frac{a_2}{3} \operatorname{tr} Q^3 + \frac{a_3}{4} (\operatorname{tr} Q^2)^2 + \frac{a_4}{5} (\operatorname{tr} Q^2) (\operatorname{tr} Q^3) \\ & + \frac{a_5}{6} (\operatorname{tr} Q^2)^3 + \frac{a'_5}{6} (\operatorname{tr} Q^3)^2 \end{aligned}$$

(see [26, 36]) or the singular potential proposed by Ball and Majumdar [8]. From this point of view, it is interesting to remark that the proof of Proposition 8 is quite robust, as it is based on variational arguments alone and does not use the structure of the Euler-Lagrange system. Therefore, the proof of Proposition 8 could be adapted to other choices of the elastic energy density, provided that they are quadratic in the gradient, and other potentials f , possibly infinite-valued outside some convex set, provided that they satisfy some non-degeneracy conditions around their set of minimizers (see Lemma 14). This would yield local H^1 -compactness, away from a singular set of codimension two, for minimizers of more general Landau-de Gennes functionals. However, proving stronger compactness for minimizers (e.g., with respect to the uniform norm) and understanding the structure of the singular set $\mathcal{S}_{\text{line}}$ for the general Landau-de Gennes energy are completely open questions.

The paper is organized as follows. Section 2 deals with general facts about the space of Q -tensors and Landau-de Gennes minimizers. In particular, lower estimates for the energy of Q -tensor valued maps are established in Section 2.2, by adapting Jerrard's and Sandier's arguments. Section 3 contains several extension results, which are a fundamental tool for the proof of our main theorem. Section 4 aims at proving Theorem 1, and in particular it contains the proof of Proposition 8 (Section 4.1). The asymptotic analysis away from the singular lines is carried out in Section 4.2, while the singular set $\mathcal{S}_{\text{line}}$ is

defined and studied in Section 4.3. In Section 4.4, we work out the analysis of minimizers near the boundary of the domain. Section 5 deals with the proof of Proposition 2. We first show the stationarity of μ_0 (Section 5.1); then, with the help of an auxiliary problem (Section 5.2), we compute the density of μ_0 and conclude the proof (Section 5.3). In Section 5.4, we construct an example where μ_0 concentrates at the boundary of the domain. Section 6 deals with the proofs of Propositions 3 (Section 6.1), 4 and 6 (Section 6.2). Finally, in Section 7 we prove Proposition 7 by constructing an example where the limit configuration Q_0 has both lines and point singularities.

2 Preliminary results

Throughout the paper, we use the following notation. We denote by $B_r^k(x)$ (or, occasionally, $B^k(x, r)$) the k -dimensional open ball of radius r and center x , and by $\overline{B}_r^k(x)$ the corresponding closed ball. When $k = 3$, we omit the superscript. When $x = 0$, we write B_r^k or B_r . Balls in the matrix space \mathbf{S}_0 will be denoted $B_r^{\mathbf{S}_0}(Q)$ or $B_r^{\mathbf{S}_0}$. For any $Q \in H^1(\Omega, \mathbf{S}_0)$ and any k -submanifold $U \subseteq \Omega$, we set

$$e_\varepsilon(Q) := \frac{1}{2} |\nabla Q|^2 + \frac{1}{\varepsilon^2} f(Q), \quad E_\varepsilon(Q, U) := \int_U e_\varepsilon(Q) \, d\mathcal{H}^k.$$

The function $e_\varepsilon(Q)$ will be called the energy density of Q . For any map Q , we set $E_\varepsilon(Q, \emptyset) := 0$. Additional notation will be set later on.

2.1 Properties of \mathbf{S}_0 and f

We discuss general facts about Q -tensors, which are useful in order to have an insight into the structure of the target space \mathbf{S}_0 . The starting point of our analysis is the following representation formula.

Lemma 10 *For all fixed $Q \in \mathbf{S}_0 \setminus \{0\}$, there exist two numbers $s \in (0, +\infty)$, $r \in [0, 1]$ and an orthonormal pair of vectors (\mathbf{n}, \mathbf{m}) in \mathbb{R}^3 such that*

$$Q = s \left\{ \mathbf{n}^{\otimes 2} - \frac{1}{3} \text{Id} + r \left(\mathbf{m}^{\otimes 2} - \frac{1}{3} \text{Id} \right) \right\}.$$

Given Q , the parameters $s = s(Q)$, $r = r(Q)$ are uniquely determined. The functions $Q \mapsto s(Q)$ and $Q \mapsto r(Q)$ are continuous on $\mathbf{S}_0 \setminus \{0\}$, and are positively homogeneous of degree 1 and 0, respectively.

Slightly different forms of this formula are often found in the literature (e.g. [52, Proposition 1]). The proof is a straightforward computation sketched in [21, Lemma 3.2], so we omit it here.

Remark 11 The parameters $s(Q)$, $r(Q)$ are given by

$$s(Q) = 2\lambda_1(Q) + \lambda_2(Q), \quad r(Q) = \frac{\lambda_1(Q) + 2\lambda_2(Q)}{2\lambda_1(Q) + \lambda_2(Q)},$$

where $\lambda_1(Q) \geq \lambda_2(Q) \geq \lambda_3(Q)$ are the eigenvalues of Q .

Following [52, Proposition 15], the vacuum manifold $\mathcal{N} := f^{-1}(0)$ can be characterized as follows:

$$Q \in \mathcal{N} \quad \text{if and only if } s(Q) = s_* \text{ and } r(Q) = 0, \quad (11)$$

where

$$s_* := \frac{1}{4c} \left(b + \sqrt{b^2 + 24ac} \right).$$

There is another set which is important for our analysis, namely

$$\mathcal{C} := \left\{ Q \in \mathbf{S}_0 : \lambda_1(Q) = \lambda_2(Q) \right\},$$

i.e. \mathcal{C} is the set of matrices whose leading eigenvalue has multiplicity > 1 . This is a closed subset of \mathcal{C} , and it is a cone (i.e., $\mu Q \in \mathcal{C}$ for any $Q \in \mathcal{C}$, $\mu \in \mathbb{R}^+$). By Remark 11, we see that

$$Q \in \mathcal{C} \quad \text{if and only if } s(Q) = 0 \text{ or } r(Q) = 1.$$

Then, applying Lemma 10 and the identity $\text{Id} = \mathbf{n}^{\otimes 2} + \mathbf{m}^{\otimes 2} + \mathbf{p}^{\otimes 2}$ (where $(\mathbf{n}, \mathbf{m}, \mathbf{p})$ is any orthonormal, positive basis of \mathbb{R}^3), we see that $Q \in \mathcal{C}$ if and only if there exist $s \geq 0$ and $\mathbf{p} \in \mathbb{S}^2$ such that

$$Q = -s \left(\mathbf{p}^{\otimes 2} - \frac{1}{3} \text{Id} \right). \quad (12)$$

Therefore, \mathcal{C} is the cone over the projective plane \mathbb{RP}^2 . Away from \mathcal{C} , it is possible to define locally a continuous map $Q \mapsto \mathbf{n}(Q)$, which selects a unit eigenvector $\mathbf{n}(Q)$ associated with $\lambda_1(Q)$ (see e.g. [7, Section 9.1, Equation (9.1.41)]). In particular, the map $\varrho: \mathbf{S}_0 \setminus \mathcal{C} \rightarrow \mathcal{N}$ defined by

$$\varrho(Q) := s_* \left(\mathbf{n}(Q)^{\otimes 2} - \frac{1}{3} \text{Id} \right) \quad \text{for } Q \in \mathbf{S}_0 \setminus \mathcal{C} \quad (13)$$

is continuous. It was proven in [21, Lemma 3.10] that ϱ gives a retraction by deformation of $\mathbf{S}_0 \setminus \mathcal{C}$ onto the vacuum manifold \mathcal{N} .

Lemma 12 *The retraction ϱ is of class C^1 on $\mathbf{S}_0 \setminus \mathcal{C}$. Moreover, ϱ coincides with the nearest-point projection onto \mathcal{N} , that is*

$$|Q - \varrho(Q)| \leq |Q - P| \quad (14)$$

holds for any $Q \in \mathbf{S}_0 \setminus \mathcal{C}$ and any $P \in \mathcal{N}$, with strict inequality if $P \neq \varrho(Q)$.

Proof. By definition of \mathcal{C} , the leading eigenvalue $\lambda_1(Q)$ is simple for any $Q \in \mathbf{S}_0 \setminus \mathcal{C}$. Then, classical differentiability results for the eigenvectors (see e.g. [7, Section 9.1]) imply that the map $Q \mapsto \mathbf{n}(Q)$ in (13) is of class $C^1(\mathbf{S}_0 \setminus \mathcal{C})$, whence $\varrho \in C^1(\mathbf{S}_0 \setminus \mathcal{C})$.

To show that ϱ is the nearest point projection onto \mathcal{N} , we pick an arbitrary $Q \in \mathbf{S}_0 \setminus \mathcal{C}$ and $P \in \mathcal{N}$. By applying Lemma 10 and (11), we write

$$Q = s \left(\mathbf{n}^{\otimes 2} - \frac{1}{3} \text{Id} \right) + sr \left(\mathbf{m}^{\otimes 2} - \frac{1}{3} \text{Id} \right) \quad \text{and} \quad P = s_* \left(\mathbf{p}^{\otimes 2} - \frac{1}{3} \text{Id} \right)$$

for some numbers $s > 0$ and $0 \leq r < 1$, some orthonormal pair (\mathbf{n}, \mathbf{m}) and some unit vector \mathbf{p} . We compute that

$$\begin{aligned} |Q - P|^2 &= \left| s\mathbf{n}^{\otimes 2} + sr\mathbf{m}^{\otimes 2} - s_*\mathbf{p}^{\otimes 2} - \frac{1}{3}(s + sr - s_*)\text{Id} \right|^2 \\ &= C(s, r, s_*) - 2s_*s \left\{ (\mathbf{n} \cdot \mathbf{p})^2 + r(\mathbf{m} \cdot \mathbf{p})^2 \right\} \end{aligned}$$

where $C(s, r, s_*)$ is a constant which only depends on s, r and s_* . For the last equality, we have used the identities $\mathbf{u}^{\otimes 2} \cdot \mathbf{v}^{\otimes 2} = (\mathbf{u} \cdot \mathbf{v})^2$ and $\mathbf{u}^{\otimes 2} \cdot \text{Id} = |\mathbf{u}|^2$, which hold for any vectors \mathbf{u}, \mathbf{v} . Given s, r, \mathbf{n} and \mathbf{m} , we minimize the right-hand side with respect to \mathbf{p} , subject to the constraint

$$(\mathbf{n} \cdot \mathbf{p})^2 + (\mathbf{m} \cdot \mathbf{p})^2 \leq 1.$$

Since $r < 1$, one easily sees that the minimum is achieved if and only if $\mathbf{p} = \pm \mathbf{n}$, that is $P = \varrho(Q)$. \square

We also consider the function $\phi: \mathbf{S}_0 \rightarrow \mathbb{R}$, given by

$$\phi(Q) = s_*^{-1} (\lambda_1(Q) - \lambda_2(Q)). \quad (15)$$

It is clear that $\phi \geq 0$, and $\phi(Q) = 0$ if and only if $Q \in \mathcal{C}$. Moreover, by Remark 11 we have

$$\phi(Q) := s_*^{-1} s(Q)(1 - r(Q)),$$

therefore $\phi(Q) = 1$ if $Q \in \mathcal{N}$, thanks to (11).

Lemma 13 *The function ϕ is Lipschitz continuous on \mathbf{S}_0 , of class C^1 on $\mathbf{S}_0 \setminus \mathcal{C}$ and satisfies*

$$\sqrt{2}s_*^{-1} \leq |D\phi(Q)| \leq 2s_*^{-1} \quad \text{for any } Q \in \mathbf{S}_0 \setminus \mathcal{C}.$$

Proof. Thanks to standard regularity results for the eigenvalues (see e.g. [7, Equation (9.1.32)]), we immediately deduce that ϕ is locally Lipschitz continuous on \mathbf{S}_0 and $\phi \in C^1(\mathbf{S}_0 \setminus \mathcal{C})$. Let $(\mathbf{n}, \mathbf{m}, \mathbf{p})$ be an orthonormal set of eigenvectors relative to $(\lambda_1, \lambda_2, \lambda_3)$ respectively. Then, for any $Q \in \mathbf{S}_0 \setminus \mathcal{C}$ there holds

$$s_* |D\phi(Q)| = \max_{B \in \mathbf{S}_0, |B|=1} |\partial_B \phi(Q)| = \max_{B \in \mathbf{S}_0, |B|=1} |\mathbf{n} \cdot B\mathbf{n} - \mathbf{m} \cdot B\mathbf{m}|$$

(the last identity follows by differentiating (15), with the help of [7, Equation (9.1.32)] again). This implies $s_* |\mathrm{D}\phi(Q)| \leq 2$. Now, set

$$B_0 := \frac{1}{\sqrt{2}} (\mathbf{n}^{\otimes 2} - \mathbf{m}^{\otimes 2}) \in \mathbf{S}_0.$$

Since $|\mathbf{n}^{\otimes 2}| = |\mathbf{m}^{\otimes 2}| = 1$ and $\mathbf{n}^{\otimes 2} \cdot \mathbf{m}^{\otimes 2} = 0$, it is straightforward to check that $|B_0| = 1$, so

$$s_* |\mathrm{D}\phi(Q)| \geq |\mathbf{n} \cdot B_0 \mathbf{n} - \mathbf{m} \cdot B_0 \mathbf{m}| = \frac{1}{\sqrt{2}} (|\mathbf{n}|^2 + |\mathbf{m}|^2) = \sqrt{2}. \quad \square$$

We conclude our discussion on the structure of the target space \mathbf{S}_0 by proving a couple of properties of the Landau-de Gennes potential f , defined by (3).

Lemma 14 *There exists a constant $\gamma = \gamma(a, b, c) > 0$ with the following properties. For any $Q \in \mathbf{S}_0$, there holds*

$$f(Q) \geq \gamma (1 - \phi(Q))^2. \quad (\mathrm{F}_0)$$

For any $Q_0 \in \mathcal{N}$ and any matrix $P \in \mathbf{S}_0$ which is orthogonal to $T_{Q_0} \mathcal{N}$, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \Big|_{t=0} f(Q_0 + tP) \geq \gamma |P|^2. \quad (\mathrm{F}_1)$$

As a consequence of (F₁), there exist γ' and $\delta_0 > 0$ such that, if $Q \in \mathbf{S}_0$ satisfies $\mathrm{dist}(Q, \mathcal{N}) \leq \delta_0$, then

$$f(Q) \geq \gamma' \mathrm{dist}^2(Q, \mathcal{N}) \quad (\mathrm{F}_2)$$

and

$$f(tQ + (1-t)\varrho(Q)) \leq \gamma' t^2 f(Q) \quad (\mathrm{F}_3)$$

for any $0 \leq t \leq 1$.

Proof of (F₀). Using the representation formula of Lemma 10, we can compute $\mathrm{tr} Q^2$ and $\mathrm{tr} Q^3$ as functions of $s := s(Q)$, $t := s(Q)r(Q)$. This yields

$$\begin{aligned} f(Q) &= k - \frac{a}{3} (s^2 - st + t^2) - \frac{b}{27} (2s^3 - 3s^2t + 3st^2 - 2t^3) \\ &\quad + \frac{c}{9} (s^2 - st + t^2)^2 =: \tilde{f}(s, t). \end{aligned}$$

We know that $(s_*, 0)$ is the unique minimizer of \tilde{f} (see e.g. [52, Proposition 15]), so $\mathrm{D}^2 \tilde{f}(s_*, 0) \geq 0$. Moreover, it is straightforward to compute that

$$\det \mathrm{D}^2 \tilde{f}(s_*, 0) > 0$$

thus $\mathrm{D}^2 \tilde{f}(s_*, 0) > 0$. As a consequence, there exist two numbers $\delta > 0$ and $C > 0$ such that

$$\tilde{f}(s, sr) \geq C(s_* - s)^2 + Cs^2r^2 \quad \text{if } (s - s_*)^2 + s^2r^2 \leq \delta. \quad (16)$$

The left-hand side in this inequality is a polynomial of order four with leading term $\frac{c}{9}(s^2 - st + t^2)^2 \geq \frac{c}{36}(s^2 + t^2)^2$, while the right-hand side is a polynomial of order two. Therefore, there exists a positive number M such that

$$\tilde{f}(s, sr) \geq C(s_* - s)^2 + Cs^2r^2 \quad \text{if } (s - s_*)^2 + s^2r^2 \geq M. \quad (17)$$

Finally, we have $\tilde{f}(s, t) > 0$ for any $(s, t) \neq (s_*, 0)$, so there exists a positive constant C' such that

$$\tilde{f}(s, sr) \geq C' \quad \text{if } \delta < (s - s_*)^2 + s^2r^2 \leq M. \quad (18)$$

Combining (16), (17) and (18), and modifying the value of C if necessary, for any $Q \in \mathbf{S}_0$, $s = s(Q)$, $r = r(Q)$ we obtain

$$\tilde{f}(s, sr) \geq C(s_* - s)^2 + Cs^2r^2 \geq \frac{Cs_*^2}{2} \left(1 - \frac{s}{s_*} + \frac{sr}{s_*}\right)^2 = \frac{Cs_*^2}{2} (1 - \phi(Q))^2.$$

□

Proof of (F₁). Since the group $\text{SO}(3)$ acts transitively on the manifold \mathcal{N} and the potential f is preserved by the action, it suffices to check (F₁) in case

$$Q_0 = s_* \left(\mathbf{e}_3^{\otimes 2} - \frac{1}{3} \text{Id} \right). \quad (19)$$

Indeed, for any $Q \in \mathbf{S}_0$ there exists $\mathbf{n} \in \mathbb{S}^2$ such that

$$\varrho(Q) = s_* \left(\mathbf{n}^{\otimes 2} - \frac{1}{3} \text{Id} \right),$$

and there exists a matrix $R \in \text{SO}(3)$ such that $R\mathbf{n} = \mathbf{e}_3$. As is easily checked, the function $\xi_R: Q \mapsto RQR^\top$ maps isometrically \mathbf{S}_0 onto itself. Then, (14) implies that ξ_R commutes with ϱ , so

$$\varrho(\xi_R(Q)) = \xi_R(\varrho(Q)) = s_* \left(R\mathbf{n}(R\mathbf{n})^\top - \frac{1}{3} \text{Id} \right) = s_* \left(\mathbf{e}_3\mathbf{e}_3^\top - \frac{1}{3} \text{Id} \right).$$

On the other hand, f is invariant by composition with ξ_R (i.e. $f \circ \xi_R = f$) because it is a function of the scalar invariants of Q . Therefore, we assume WLOG that Q_0 is given by (19).

In [21, Lemma 3.5], it is shown that a matrix $P \in \mathbf{S}_0$ is orthogonal to $T_{Q_0}\mathcal{N}$ if and only if it can be written in the form

$$P = \begin{pmatrix} -\frac{1}{3}(s_* + x_0) + x_2 & x_1 & 0 \\ x_1 & -\frac{1}{3}(s_* + x_0) - x_2 & 0 \\ 0 & 0 & \frac{2}{3}(s_* + x_0) \end{pmatrix}$$

for some $x = (x_0, x_1, x_2) \in \mathbb{R}^3$. Then, one can write $f(Q_0 + tP)$ as a function of t, x and compute the second derivatives. The proof of (F₁) is reduced to a straightforward computation, which we omit here. □

Proof of (F₂)-(F₃). These properties follow from (F₁), by a Taylor expansion of f around Q_0 . □

2.2 Energy estimates in 2-dimensional domains

In the analysis of the Ginzburg-Landau functional, a very useful tool are the estimates proved by Jerrard [43] and Sandier [60]. These estimates provide a lower bound for the energy of complex-valued maps defined on a two-dimensional disk, depending on the topological properties of the boundary datum. More precisely, if $u \in H^1(B_1^2, \mathbb{C})$ satisfies $|u(x)| = 1$ for a.e. $x \in \partial B_1^2$ (plus some technical assumptions) then

$$E_\varepsilon^{\text{GL}}(u, B_1^2) \geq \pi |d| |\log \varepsilon| - C, \quad (20)$$

where $E_\varepsilon^{\text{GL}}$ is the Ginzburg-Landau energy, defined by (7), and d denotes the topological degree of $u/|u|$ on ∂B_1^2 , i.e. its winding number. The aim of this subsection is to generalize this result to tensor-valued maps and the Landau-de Gennes energy.

Since we work in the H^1 -setting, we have to take care of a technical detail. Set $A := B_1^2 \setminus B_{1/2}^2$. Let $Q \in H^1(B_1^2, \mathbf{S}_0)$ be a given map, which satisfies

$$\phi_0(Q, A) := \operatorname{ess\,inf}_A \phi \circ Q > 0. \quad (21)$$

In case Q is continuous, Condition (21) is equivalent to

$$Q(x) \notin \mathcal{C} \quad \text{for every } x \in \bar{A}.$$

For a.e. $r \in (1/2, 1)$ the restriction of Q to ∂B_r^2 is an H^1 -map (due to Fubini theorem) and hence, by Sobolev injection, it is a continuous map which satisfies $Q(x) \notin \mathcal{C}$ for every $x \in \partial B_r^2$. Therefore, $\varrho \circ Q$ is well defined and continuous on ∂B_r^2 . Moreover, its homotopy class is independent of r . If $\varrho \circ Q$ is continuous, then $\varrho \circ Q$ itself provides a homotopy between $\varrho \circ Q|_{\partial B_{r_1}^2}$ and $\varrho \circ Q|_{\partial B_{r_2}^2}$, for any r_1 and r_2 . Otherwise, by convolution (as in [62, Proposition p. 267]) one constructs a smooth approximation $(\varrho \circ Q)_\delta : A \rightarrow \mathcal{N}$ such that $(\varrho \circ Q)_\delta \rightarrow \varrho \circ Q$ in $H^1(A, \mathbf{S}_0)$ when $\delta \rightarrow 0$. By Fubini theorem and Sobolev injection, we have $(\varrho \circ Q)_\delta \rightarrow \varrho \circ Q$ uniformly on ∂B_r^2 for a.e. $r \in (1/2, 1)$. Therefore, the maps $\varrho \circ Q|_{\partial B_r^2}$ belong to the same homotopy class, for a.e. r . By abuse of notation, this homotopy class will be called “homotopy class of $\varrho \circ Q$ restricted to the boundary” or also “homotopy class of the boundary datum”.

Proposition 15 *There exist positive constants M and κ_* , depending only on f , with the following property. Let $0 < \varepsilon < 1$ and $Q \in H^1(B_1^2, \mathbf{S}_0)$ be given. Assume that Q satisfies (21) and the homotopy class of $\varrho \circ Q|_{\partial B_1^2}$ is non-trivial. Then*

$$E_\varepsilon(Q, B_1^2) \geq \kappa_* \phi_0^2(Q, A) |\log \varepsilon| - M.$$

The energetic cost associated with topological defects is quantified by a number κ_* , defined by (26) and explicitly computed in Lemma 19:

$$\kappa_* = \frac{\pi}{2} s_*^2.$$

This number plays the same role as the quantity $\pi|d|$ in (20). The quantity $\phi_0^2(Q, A)$ at the right-hand side has been introduced for technical reasons. Notice that $\phi = 1$ on \mathcal{N} , so $\phi_0(Q, A) = 1$ if $Q|_A$ takes values in \mathcal{N} .

Before dealing with the proof of Proposition 15, we state an immediate consequence.

Corollary 16 *Let ε, R be two numbers such that $0 < \varepsilon < R/2$. Given a map $Q \in H^1(B_R^2, \mathbf{S}_0)$, suppose that the restriction to the boundary belongs to $H^1(\partial B_R^2, \mathbf{S}_0)$ and that*

$$\phi_0(Q, \partial B_R^2) := \operatorname{ess\,inf}_{\partial B_R^2} \phi \circ Q > 0.$$

If the homotopy class of $\varrho \circ Q|_{\partial B_R^2}$ is non-trivial, then

$$E_\varepsilon(Q, B_R^2) + (\log 2)R E_\varepsilon(Q, \partial B_R^2) \geq \kappa_* \phi_0^2(Q, \partial B_R^2) \log \frac{R}{\varepsilon} - M.$$

In particular, if Q satisfies $Q = g$ on ∂B_R^2 for some non-trivial $g \in H^1(\partial B_R^2, \mathcal{N})$, then Corollary 16 implies

$$E_\varepsilon(Q, B_R^2) \geq \kappa_* \log \frac{R}{\varepsilon} - M$$

for a constant $M = M(R, g)$. (Compare with [43, Theorem 3.1], [60, Theorem 1], and [23, Proposition 6.1].)

Proof of Corollary 16. We apply Proposition 15 to $\epsilon := 2\varepsilon/R$ and the map $\tilde{Q} \in H^1(B_1^2, \mathbf{S}_0)$ defined by

$$\tilde{Q}(x) := \begin{cases} Q\left(\frac{Rx}{|x|}\right) & \text{if } x \in A := B_1^2 \setminus B_{1/2}^2 \\ Q(2Rx) & \text{if } x \in B_{1/2}^2. \end{cases}$$

Notice that $\phi_0(\tilde{Q}, A) = \phi_0(Q, \partial B_R^2)$. Then, by a change of variable, we deduce

$$\begin{aligned} \kappa_* \phi_0^2(Q, \partial B_R^2) \log \frac{R}{\varepsilon} - C &\leq E_\epsilon(\tilde{Q}, B_1^2) \\ &\leq E_\epsilon(Q, B_{1/2}^2) + \int_{1/2}^1 E_\epsilon(\tilde{Q}, \partial B_r^2) \, dr \\ &= E_\varepsilon(Q, B_R^2) + \int_{1/2}^1 \frac{R}{r} E_{2\varepsilon/r}(Q, \partial B_R^2) \, dr \\ &\leq E_\varepsilon(Q, B_R^2) + (\log 2)R E_\varepsilon(Q, \partial B_R^2). \quad \square \end{aligned}$$

A generalization of the Jerrard-Sandier estimate (20) has already been proved by Chiron, in his PhD thesis [23]. Given a smooth, compact manifold without boundary, Chiron considered maps into the *cone over \mathcal{N}* , that is

$$X_{\mathcal{N}} := ((0, +\infty) \times \mathcal{N}) \cup \{0\} \ni u = (|u|, u/|u|)$$

(with a metric defined accordingly). He obtained an estimate which is analogous to (20). In case $\mathcal{N} = \mathbb{S}^1$, one has $X_{\mathbb{S}^1} \simeq \mathbb{C}$ and the standard estimate (20) is recovered. Given a map $u: U \subseteq \mathbb{R}^k \rightarrow X_{\mathcal{N}}$, a key step in Chiron's arguments is to decompose the gradient of u in terms of modulus and phase, that is

$$|\nabla u|^2 = |\nabla |u||^2 + |u|^2 |\nabla (u/|u|)|^2 \quad \text{a.e. on } U. \quad (22)$$

Chiron's result does not apply to tensor-valued maps, because the space \mathbf{S}_0 do not coincide with the cone over \mathcal{N} (the latter only contains uniaxial matrices, while \mathbf{S}_0 also contains biaxial matrices). However, one can prove an estimate in the same spirit as (22), namely, the gradient of a map $\Omega \rightarrow \mathbf{S}_0$ is controlled from below by the gradients of $\phi \circ Q$ and $\varrho \circ Q$.

Lemma 17 *Let $U \subseteq \mathbb{R}^k$ be a domain and let $Q \in C^1(U, \mathbf{S}_0)$. There holds*

$$|\nabla Q|^2 \geq \frac{s_*^2}{3} |\nabla (\phi \circ Q)|^2 + (\phi \circ Q)^2 |\nabla (\varrho \circ Q)|^2 \quad \mathcal{H}^k\text{-a.e. on } U,$$

where we have set $(\phi \circ Q)|\nabla(\varrho \circ Q)|(x) := 0$ if $Q(x) \in \mathcal{C}$.

Proof. First of all, notice that $\varrho \circ Q$ is well-defined on the set where $Q \notin \mathcal{C}$, or equivalently, the set where $\phi \circ Q > 0$. Therefore, the right-hand side always makes sense. Because of our choice of the norm, we have

$$|\nabla \psi|^2 = \sum_{i=1}^k |\partial_i \psi|^2$$

for any scalar or tensor-valued map ψ . Thus, it suffices to prove the lemma when ∇ is replaced by the partial derivative operator ∂_i , then sum over $i \in \{1, \dots, k\}$. In view of this remark, we assume WLOG that $k = 1$.

Since ϕ is Lipschitz continuous (Lemma 13), we have $\phi \circ Q \in W_{\text{loc}}^{1,\infty}(U)$. Moreover, $\phi \circ Q = 0$ on $Q^{-1}(\mathcal{C})$. Therefore, we have $(\phi \circ Q)' = 0$ a.e. on $Q^{-1}(\mathcal{C})$ and the lemma is trivially satisfied a.e. on $Q^{-1}(\mathcal{C})$.

For the rest of the proof, we fix a point $x \in U \setminus Q^{-1}(\mathcal{C})$ so that $\phi \circ Q, \varrho \circ Q$ are of class C^1 in a neighborhood of x , and the leading eigenvalue of $Q(x)$ has multiplicity one. We are going to distinguish a few cases, depending on whether the others eigenvalues of $Q(x)$ have multiplicity one as well. Suppose first that $r(Q(x)) > 0$: in this case, all the eigenvalues of $Q(x)$ are simple. Using Lemma 10 and the results in [7], the map Q can be locally written as

$$Q = s \left(\mathbf{n}^{\otimes 2} - \frac{1}{3} \text{Id} \right) + sr \left(\mathbf{m}^{\otimes 2} - \frac{1}{3} \text{Id} \right), \quad (23)$$

where $s, r, \mathbf{n}, \mathbf{m}$ are C^1 functions defined in a neighborhood of x , satisfying the constraints

$$s > 0, \quad 0 < r < 1, \quad |\mathbf{n}| = |\mathbf{m}| = 1, \quad \mathbf{n} \cdot \mathbf{m} = 0.$$

Then, $\varrho \circ Q$ is of class C^1 in a neighborhood of x , and we can compute $|Q'|$, $|(\varrho \circ Q)'|$ in terms of s , r , \mathbf{n} , \mathbf{m} and their derivatives. Setting $t := sr$, a straightforward computation gives

$$s_*^2(\phi \circ Q)'^2 = s'^2 - 2s't' + t'^2, \quad |(\varrho \circ Q)'|^2 = 2s_*^2 |\mathbf{n}'|^2$$

and

$$\begin{aligned} |Q'|^2 &= \frac{2}{3} \left(s'^2 - s't' + t'^2 \right) + 2s^2 |\mathbf{n}'|^2 + 2t^2 |\mathbf{m}'|^2 + 4st(\mathbf{n}' \cdot \mathbf{m})(\mathbf{n} \cdot \mathbf{m}') \\ &\geq \frac{s_*^2}{3} (\phi \circ Q)'^2 + 2s^2 \left(|\mathbf{n}'|^2 + r^2 |\mathbf{m}'|^2 + 2r(\mathbf{n}' \cdot \mathbf{m})(\mathbf{n} \cdot \mathbf{m}') \right) \end{aligned} \quad (24)$$

Let $\mathbf{p} := \mathbf{n} \times \mathbf{m}$, so that $(\mathbf{n}, \mathbf{m}, \mathbf{p})$ is an orthonormal, positive frame in \mathbb{R}^3 . By differentiating the orthogonality conditions for $(\mathbf{n}, \mathbf{m}, \mathbf{p})$, we obtain

$$\mathbf{n}' = \alpha \mathbf{m} + \beta \mathbf{p}, \quad \mathbf{m}' = -\alpha \mathbf{n} + \gamma \mathbf{p}$$

for some smooth, real-valued functions α , β , γ . Then, from (24) and (2.2) we deduce

$$\begin{aligned} |Q'|^2 - \frac{s_*^2}{3} (\phi \circ Q)'^2 &\geq 2s^2 (\alpha^2 + \beta^2 + r^2(\alpha^2 + \gamma^2) - 2r\alpha^2) \\ &\geq 2s^2(1-r)^2(\alpha^2 + \beta^2) \\ &= s_*^{-2}s^2(1-r)^2 |(\varrho \circ Q)'|^2 = (\phi \circ Q)^2 |(\varrho \circ Q)'|^2, \end{aligned}$$

so the lemma holds at the point x .

If $r(Q) = 0$ in a neighborhood of x then the function \mathbf{m} might not be well-defined. However, the previous computation still make sense because $t = sr$ vanishes in a neighborhood of x , and from (2.2), (24) we deduce that the lemma holds at x . We still have to consider one case, namely, $r(Q(x)) = 0$ but $r(Q)$ does not vanish identically in a neighborhood of x . In this case, there exists a sequence $x_k \rightarrow x$ such that $r(Q(x_k)) > 0$ for each $k \in \mathbb{N}$. By the previous discussion the lemma holds at each x_k , and the functions $\phi \circ Q$, $(\varrho \circ Q)'$ are continuous (by Lemmas 13 and 12). Passing to the limit as $k \rightarrow +\infty$, we conclude that the lemma is satisfied at x as well. \square

The regularity of Q in Lemma 17 can be relaxed.

Corollary 18 *The map $\tau: \mathbf{S}_0 \rightarrow \mathbf{S}_0$ given by*

$$\tau: Q \mapsto \begin{cases} s_* \phi(Q) \varrho(Q) & \text{if } Q \in \mathbf{S}_0 \setminus \mathcal{C} \\ 0 & \text{if } Q \in \mathcal{C} \end{cases}$$

is Lipschitz-continuous. Moreover, for any $Q \in H^1(U, \mathbf{S}_0)$ there holds $\tau \circ Q \in H^1(U, \mathbf{S}_0)$ and

$$\frac{1}{4} |\nabla(\tau \circ Q)|^2 \leq \frac{s_*^2}{3} |\nabla(\phi \circ Q)|^2 + (\phi \circ Q)^2 |\nabla(\varrho \circ Q)|^2 \leq |\nabla Q|^2 \quad (25)$$

holds \mathcal{H}^k -a.e. on U .

Proof. By differentiating τ and applying Lemma (17) to the map $Q = \text{Id}_{\mathbf{S}_0}$, we obtain

$$\frac{1}{4} |\text{D}\tau|^2 \leq \frac{s_*^2}{3} |\text{D}\phi|^2 + \phi^2 |\text{D}\varrho|^2 \leq C \quad \text{on } \mathbf{S}_0 \setminus \mathcal{C}.$$

Using this uniform bound, together with $\tau \in C(\mathbf{S}_0, \mathbf{S}_0)$ and $\tau|_{\mathcal{C}} = 0$, it is not hard to conclude that τ has bounded derivative in the sense of distributions, therefore τ is a Lipschitz function and the lower bound in (25) holds. The upper bound follows easily from Lemma (17), by a density argument. \square

Following an idea of Chiron [23], we can associate with each homotopy class of maps $\mathbb{S}^1 \rightarrow \mathcal{N}$ a positive number which measures the energy cost of that class. Since \mathcal{N} is a real projective plane, quantifying the energy cost of non-trivial maps is simple, because there is a unique homotopy class of such maps. Define

$$\kappa_* := \inf \left\{ \frac{1}{2} \int_{\mathbb{S}^1} |P'(\theta)|^2 d\theta : P \in H^1(\mathbb{S}^1, \mathcal{N}) \text{ is non-trivial} \right\}. \quad (26)$$

Thanks to the compact embedding $H^1(\mathbb{S}^1, \mathcal{N}) \hookrightarrow C^0(\mathbb{S}^1, \mathcal{N})$, a standard argument shows that the infimum is achieved. The Euler-Lagrange condition for (26) implies that minimizers are geodesics in \mathcal{N} . Moreover, we have the following property.

Lemma 19 *A minimizer for (26) is given by*

$$P_0(\theta) := s_* \left(\mathbf{n}_*(\theta)^{\otimes 2} - \frac{1}{3} \text{Id} \right) \quad \text{for } 0 \leq \theta \leq 2\pi,$$

where $\mathbf{n}_*(\theta) := (\cos(\theta/2), \sin(\theta/2), 0)^\top$. In particular, there holds

$$\kappa_* = \frac{\pi}{2} s_*^2.$$

Sketch of the proof. The lemma has been proved, e.g., in [21, Lemma 3.6, Corollary 3.7], but we sketch the proof for the convenience for the reader. Let $\psi: \mathbb{S}^2 \rightarrow \mathcal{N}$ be the universal covering of \mathcal{N} , that is

$$\psi(\mathbf{n}) := s_* \left(\mathbf{n}^{\otimes 2} - \frac{1}{3} \text{Id} \right) \quad \text{for } \mathbf{n} \in \mathbb{S}^2. \quad (27)$$

For any $\mathbf{n} \in \mathbb{S}^2$ and any tangent vector $\mathbf{v} \in T_{\mathbf{n}}\mathbb{S}^2$, one computes that

$$|\text{d}\psi(\mathbf{n})\mathbf{v}|^2 = 2s_*^2 |\mathbf{v}|^2,$$

that is, the pull-back metric induced by ψ coincides with the first fundamental form of \mathcal{N} , up to a constant factor. Therefore, the Levi-Civita connections associated with the two metrics are the same, because the Christoffel symbols are homogeneous functions, of degree zero, of the coefficients of the metric. As a consequence, a loop P is a geodesic in \mathcal{N} if and only if it can be written as $P =$

$\psi \circ \mathbf{n}$, where $\mathbf{n}: [0, 2\pi] \rightarrow \mathbb{S}^2$ is a geodesic path in \mathbb{S}^2 such that $\mathbf{n}(2\pi) = \pm \mathbf{n}(0)$. Moreover, P has a non-trivial homotopy class if and only if $\mathbf{n}(2\pi) = -\mathbf{n}(0)$. Therefore, $P := \psi \circ \mathbf{n}$ is a minimizing geodesic in the non-trivial class if and only if \mathbf{n} is half of a great circle in \mathbb{S}^2 parametrized by arc-length, and the lemma follows. \square

By adapting Sandier's arguments in [60], we can bound from below the energy of \mathcal{N} -valued maps, in terms of the quantity (26). We use the following notation: for any $V \subset \subset \mathbb{R}^2$, we define the radius of V as

$$\text{rad}(V) := \inf \left\{ \sum_{i=1}^n r_i : \bar{V} \subseteq \bigcup_{i=1}^n B(a_i, r_i) \right\}.$$

We clearly have $\text{rad}(V) \leq \text{diam}(\bar{V})$ and, since for bounded sets there holds $\text{diam}(\bar{V}) = \text{diam}(\partial V)$, we obtain that

$$\text{rad}(V) \leq \text{diam}(\partial V). \quad (28)$$

Lemma 20 *Let V be a subdomain of B_1^2 , and take a number $\rho > 0$ such that $\text{dist}(V, \partial B_1^2) \geq 2\rho$. For any $P \in H^1(B_1^2 \setminus V, \mathcal{N})$ such that $P|_{\partial B_1^2}$ is homotopically non-trivial, there holds*

$$\frac{1}{2} \int_{B_1^2 \setminus V} |\nabla P|^2 \, d\mathcal{H}^2 \geq \kappa_* \log \frac{\rho}{\text{rad}(V)}.$$

Sketch of the proof. Suppose, at first, that $V = B_r^2$ with $0 < r < 1$ and P is smooth. Then, computing in polar coordinates, we obtain

$$\begin{aligned} \frac{1}{2} \int_{B_1^2 \setminus B_r^2} |\nabla P|^2 \, d\mathcal{H}^2 &= \frac{1}{2} \int_r^1 \int_{\mathbb{S}^1} \left(\rho |\partial_\rho P|^2 + \frac{1}{\rho} |\partial_\theta P|^2 \right) \, d\theta \, d\rho \\ &\stackrel{(26)}{\geq} \kappa_* \int_r^1 \frac{d\rho}{\rho} = \kappa_* \log \frac{1}{r} \end{aligned}$$

so the lemma is satisfied for any $0 < \rho \leq 1$. By a density argument, the same estimate holds for any $P \in H^1(B_1^2 \setminus B_r^2, \mathcal{N})$. For a general V , the lemma can be proved arguing exactly as in [60, Proposition p. 385]. (Assuming additional $W^{1,\infty}$ -bounds on P , the lemma could also be deduced by the arguments of [43, Theorems 3.1 and 4.1].) \square

Finally, we can prove the main result of this subsection.

Proof of Proposition 15. We argue as in [23, Theorem 6.1] and [21, Proposition 3.11]. As a first step, we suppose that Q is smooth. Reminding that $A := B_1^2 \setminus B_{1/2}^2$, we have

$$\phi_0 := \phi_0(Q, A) = \min_A \phi \circ Q \stackrel{(21)}{>} 0.$$

Moreover, there must be

$$\min_{\overline{B}_1^2} \phi \circ Q = 0, \quad (29)$$

otherwise $\varrho \circ Q$ would be a well-defined, continuous map $\overline{B}_1^2 \rightarrow \mathcal{N}$ and the boundary datum would be topologically trivial. For each $\lambda > 0$, we set

$$\begin{aligned} \Omega_\lambda &:= \{x \in B_1^2 : \phi \circ Q(x) > \lambda\}, \quad \omega_\lambda := \{x \in B_1^2 : \phi \circ Q(x) < \lambda\} \\ \Gamma_\lambda &:= \partial\Omega_\lambda \setminus \partial\Omega = \partial\omega_\lambda. \end{aligned}$$

Notice that Ω_λ , ω_λ , and Γ_λ are non empty for every $\lambda \in (0, \phi_0)$, due to (29). We also set

$$\Theta(\lambda) := \int_{\Omega_\lambda} |\nabla(\varrho \circ Q)|^2 \, d\mathcal{H}^2, \quad \nu(\lambda) := \int_{\Gamma_\lambda} |\nabla(\phi \circ Q)| \, d\mathcal{H}^1.$$

Lemma 17 entails

$$\int_{B_1^2} |\nabla Q|^2 \geq \int_{B_1^2} \left\{ \frac{1}{2} |\nabla(\phi \circ Q)|^2 + (\phi \circ Q)^2 |\nabla(\varrho \circ Q)|^2 \right\} \, d\mathcal{H}^2$$

and, applying the coarea formula, we deduce

$$\begin{aligned} E_\varepsilon(Q) &\geq \frac{1}{2} \int_0^{\phi_0} \int_{\Gamma_\lambda} \left(\frac{1}{2} |\nabla(\phi \circ Q)| + \frac{2f(Q)}{\varepsilon^2 |\nabla(\phi \circ Q)|} \right) \, d\mathcal{H}^1 \, d\lambda \\ &\quad - 2 \int_0^{+\infty} \lambda^2 \Theta'(\lambda) \, d\lambda. \end{aligned} \quad (30)$$

Thanks to Sard lemma, a.e. $\lambda \in (0, \phi_0)$ is a proper regular value of $\phi \circ Q$, so dividing by $|\nabla(\phi \circ Q)|$ makes sense. Let us estimate the terms in the right-hand side of (30), starting from the second one. Lemma 14, (F₀) implies that

$$f(Q) \geq C(1 - \lambda)^2 \quad \text{on } \Gamma_\lambda.$$

Therefore, with the help of the Hölder inequality we deduce

$$\begin{aligned} \int_{\Gamma_\lambda} \frac{2f(Q)}{\varepsilon^2 |\nabla(\phi \circ Q)|} \, d\mathcal{H}^1 &\geq \frac{C(1 - \lambda)^2}{\varepsilon^2} \int_{\Gamma_\lambda} \frac{1}{|\nabla(\phi \circ Q)|} \, d\mathcal{H}^1 \\ &\geq \frac{C(1 - \lambda)^2 \mathcal{H}^1(\Gamma_\lambda)^2}{\varepsilon^2 \nu(\lambda)}. \end{aligned} \quad (31)$$

Moreover, we have

$$\mathcal{H}^1(\Gamma_\lambda) \geq 2 \operatorname{diam}(\Gamma_\lambda) \stackrel{(28)}{\geq} 2 \operatorname{rad}(\omega_\lambda).$$

Combining this with (30) and (31), we find

$$\begin{aligned} E_\varepsilon(Q) &\geq \frac{1}{2} \int_0^{\phi_0} \left\{ \frac{1}{2} \nu(\lambda) + \frac{C(1 - \lambda)^2 \operatorname{rad}^2(\omega_\lambda)}{\varepsilon^2 \nu(\lambda)} \right\} \, d\lambda - \int_0^{+\infty} \lambda^2 \Theta'(\lambda) \, d\lambda \\ &\geq \int_0^{\phi_0} \frac{C}{\varepsilon} |1 - \lambda| \operatorname{rad}(\omega_\lambda) \, d\lambda - \int_0^{+\infty} \lambda^2 \Theta'(\lambda) \, d\lambda. \end{aligned} \quad (32)$$

The second line follows by the elementary inequality $a + b \geq 2\sqrt{ab}$. As for the last term, we integrate by parts. Since Θ is compactly supported and non-negative, for all $\lambda_0 > 0$ we have

$$-\int_{\lambda_0}^{+\infty} \lambda^2 \Theta'(\lambda) d\lambda = 2 \int_{\lambda_0}^{+\infty} \lambda \Theta(\lambda) d\lambda + \lambda_0^2 \Theta(\lambda_0) \geq 2 \int_{\lambda_0}^{\phi_0} \lambda \Theta(\lambda) d\lambda$$

and, letting $\lambda_0 \rightarrow 0$, by monotone convergence ($\Theta \geq 0$, $-\Theta'$ is a non-negative Borel measure) we conclude that

$$-\int_0^{+\infty} \lambda^2 \Theta'(\lambda) d\lambda \geq 2 \int_0^{\phi_0} \lambda \Theta(\lambda) d\lambda.$$

Now, for any $\lambda \in (0, \phi_0)$ we have $\omega_\lambda \subseteq B_{1/2}^2$, so $\text{dist}(\omega_\lambda, \partial B_1^2) \geq 1/2$. Therefore, by applying Lemma 20 we obtain

$$\Theta(\lambda) \geq -\kappa_* \log(\text{rad}(\omega_\lambda)) - \kappa_* \log 4.$$

Thus, (32) implies

$$F_t(Q) \geq \int_0^{\phi_0} \left\{ \frac{C}{\varepsilon} |1 - \lambda| \text{rad}(\omega_\lambda) - 2\kappa_* \lambda \log(\text{rad}(\omega_\lambda)) \right\} d\lambda - C.$$

An easy analysis shows that the function $r \in (0, +\infty) \mapsto C\varepsilon^{-1}|1 - \lambda|r - 2\kappa_* \lambda \log r$ has a unique minimizer r_* , which is readily computed. As a consequence, we obtain the lower bound

$$\begin{aligned} F_t(Q) &\geq \int_0^{\phi_0} \left\{ 2\kappa_* \lambda - 2\kappa_* \lambda \log \frac{C\varepsilon\kappa_*\lambda}{|1 - \lambda|} \right\} d\lambda - C \\ &= -2\kappa_* \int_0^{\phi_0} \left\{ \lambda \log \varepsilon - \lambda + \lambda \log \frac{C\kappa_*\lambda}{|1 - \lambda|} \right\} d\lambda - C \end{aligned}$$

All the terms are locally integrable functions of λ , so the proposition is proved in case Q is smooth.

Given any Q in H^1 , we can reduce to previous case by means of a density argument, inspired by [62, Proposition p. 267]. For $\delta > 0$, let χ^δ be a standard mollification kernel and set $Q^\delta := Q * \chi^\delta$. (In order to define the convolution at the boundary of Ω , we extend Q by standard reflection on a neighborhood of the domain.) Then, $\{Q^\delta\}_{\delta>0}$ is a sequence of smooth maps, which converge to Q strongly in H^1 and, by Sobolev embedding, in L^4 . This implies $E_\varepsilon(Q^\delta) \rightarrow E_\varepsilon(Q)$ as $\delta \rightarrow 0$. Moreover, for any $x \in A$ we have

$$\begin{aligned} \text{dist}(\phi \circ Q^\delta(x), [\phi_0, +\infty)) &\leq \int_{B_\delta^2(x)} |\phi \circ Q^\delta(x) - \phi \circ Q(y)| d\mathcal{H}^2(y) \\ &\leq C \int_{B_\delta^2(x)} |Q^\delta(x) - Q(y)| d\mathcal{H}^2(y), \end{aligned} \tag{33}$$

where the last inequality follows by the Lipschitz continuity of ϕ (Lemma 13). Now, we can adapt the Poincaré inequality and combine it with the Hölder inequality to obtain

$$\begin{aligned} \int_{B_\delta^2(x)} |Q^\delta(x) - Q(y)| \, d\mathcal{H}^2(y) &\leq C\delta \int_{B_\delta^2(x)} |\nabla Q| \, d\mathcal{H}^2 \\ &\leq C\delta^2 \left(\int_{B_\delta^2(x)} |\nabla Q|^2 \, d\mathcal{H}^2 \right)^{1/2}. \end{aligned}$$

This fact, combined with (33), implies

$$\text{dist}(\phi \circ Q^\delta(x), [\phi_0, +\infty)) \leq C \left(\int_{B_\delta^2(x)} |\nabla Q|^2 \, d\mathcal{H}^2 \right)^{1/2} \rightarrow 0$$

uniformly in $x \in A$ as $\delta \rightarrow 0$ so, in particular, $\phi_0(Q^\delta, A) \rightarrow \phi_0(Q, A)$ as $\delta \rightarrow 0$. Then, since the proposition holds for each Q^δ , by passing to the limit as $\delta \rightarrow 0$ we see that it also holds for Q . \square

2.3 Basic properties of minimizers

We conclude the preliminary section by recalling some basic facts about minimizers of (LG_ε) .

Lemma 21 *Minimizers Q_ε of (LG_ε) exist and are of class C^∞ in the interior of Ω . Moreover, for any $U \subset\subset \Omega$ they satisfy*

$$\varepsilon \|\nabla Q_\varepsilon\|_{L^\infty(U)} \leq C(U).$$

Sketch of the proof. The existence of minimizers follows by standard method in Calculus of Variations. Minimizers solve the Euler-Lagrange system

$$-\varepsilon^2 \Delta Q_\varepsilon - aQ_\varepsilon - bQ_\varepsilon^2 + \frac{b}{3} \text{Id} |Q_\varepsilon|^2 + c |Q_\varepsilon|^2 Q_\varepsilon = 0 \quad (34)$$

on Ω , in the sense of distributions. The term $\text{Id} |Q_\varepsilon|^2$ is a Lagrange multiplier, associated with the tracelessness constraint. The elliptic regularity theory, combined with the uniform L^∞ -bound of Assumption (H), implies that each component $Q_{\varepsilon,ij}$ is of class C^∞ in the interior of the domain. The $W^{1,\infty}(U)$ -bound follows by interpolation results, see [10, Lemma A.1, A.2]. \square

Lemma 22 (Stress-energy identity) *For any $i \in \{1, 2, 3\}$, the minimizers satisfy*

$$\partial_j (e_\varepsilon(Q_\varepsilon) \delta_{ij} - \partial_i Q_\varepsilon \cdot \partial_j Q_\varepsilon) = 0 \quad \text{in } \Omega$$

in the sense of distributions.

Proof. Since Q_ε is of class C^∞ in the interior of the domain by Lemma 21, we can differentiate the products and use the chain rule. For each i we have

$$\begin{aligned}
& \partial_j (e_\varepsilon(Q_\varepsilon) \delta_{ij} - \partial_i Q_\varepsilon \cdot \partial_j Q_\varepsilon) \\
&= \partial_i \partial_k Q_\varepsilon \cdot \partial_k Q_\varepsilon + \frac{1}{\varepsilon^2} \frac{\partial f(Q_\varepsilon)}{\partial Q_{pq}} \partial_i Q_{\varepsilon, pq} \\
&\quad - \partial_i \partial_j Q_\varepsilon \cdot \partial_j Q_\varepsilon - \partial_i Q_\varepsilon \cdot \partial_j \partial_j Q_\varepsilon \\
&\stackrel{(34)}{=} \partial_k \partial_k Q_\varepsilon \cdot \partial_i Q_\varepsilon - \frac{b}{3} |Q_\varepsilon|^2 \text{Id} \cdot \partial_i Q_\varepsilon - \partial_i Q_\varepsilon \cdot \partial_j \partial_j Q_\varepsilon = 0
\end{aligned}$$

where we have used that $\text{Id} \cdot \partial_i Q_\varepsilon = 0$, because Q_ε is traceless. \square

Lemma 23 (Pohozaev identity) *Let $G \subset \Omega$ be a Lipschitz subdomain and x_0 a point in G . Then, there holds*

$$\begin{aligned}
E_\varepsilon(Q_\varepsilon, G) &+ \frac{1}{2} \int_{\partial G} \boldsymbol{\nu}(x) \cdot (x - x_0) |\partial_{\boldsymbol{\nu}} Q_\varepsilon|^2 \, d\mathcal{H}^2 \\
&= \int_{\partial G} \boldsymbol{\nu}(x) \cdot (x - x_0) e_\varepsilon(Q_\varepsilon) \, d\mathcal{H}^2 \\
&\quad - \int_{\partial G} (\nabla Q_\varepsilon) \boldsymbol{\nu}(x) \cdot (\nabla Q_\varepsilon) P_{\partial G}(x - x_0) \, d\mathcal{H}^2,
\end{aligned}$$

where $\boldsymbol{\nu}(x)$ is the outward normal to ∂G at x and $P_{\partial G}(x - x_0)$ is the component of $x - x_0$ that is tangent to ∂G .

This identity can be proved arguing exactly as in [11, Theorem III.2] (the reader is also referred to [52, Lemma 2] for more details). The Pohozaev identity has a very important consequence, which is obtained by taking $G = B_r(x_0)$ (see e.g. [52, Lemma 2] for a proof).

Lemma 24 (Monotonicity formula) *Let $x_0 \in \Omega$, and let $0 < r_1 < r_2 < \text{dist}(x_0, \partial\Omega)$. Then*

$$r_1^{-1} E_\varepsilon(Q_\varepsilon, B_{r_1}(x_0)) \leq r_2^{-1} E_\varepsilon(Q_\varepsilon, B_{r_2}(x_0)).$$

Here is another useful consequence of the Pohozaev identity, whose proof is straightforward.

Lemma 25 *Assume that $G \subseteq \Omega$ is star-shaped, i.e. there exists $x_0 \in G$ such that $\boldsymbol{\nu}(x) \cdot (x - x_0) \geq 0$ for any $x \in \partial G$. Then*

$$E_\varepsilon(Q_\varepsilon, G) \leq 3 \, \text{diam}(G) \, E_\varepsilon(Q_\varepsilon, \partial G).$$

3 Extension properties

3.1 Extension of \mathbb{S}^2 - and \mathcal{N} -valued maps

In some of our arguments, we will encounter extension problems for \mathcal{N} -valued maps. This means, given $g: \partial B_r^k \rightarrow \mathcal{N}$ (for $k \in \mathbb{N}$, $k \geq 2$ and $r > 0$) we look for a map $Q: B_r^k \rightarrow \mathcal{N}$ satisfying $Q = g$ on ∂B_r^k , with a control on the energy of Q . When the datum g is smooth enough and no topological obstruction occur, this problem can be reformulated in terms of \mathbb{S}^2 -valued maps. Indeed, if the homotopy class of g is trivial then g can be *lifted*, i.e. there exists a map $\mathbf{n}: \partial B_r^k \rightarrow \mathbb{S}^2$, as regular as g , such that the diagram

$$\begin{array}{ccc} & \mathbb{S}^2 & \\ \mathbf{n} \nearrow & \downarrow \psi & \\ \partial B_r^k & \xrightarrow{g} & \mathcal{N} \end{array}$$

commutes. Here ψ is the universal covering map of \mathcal{N} , given by (27). In other words, the function \mathbf{n} satisfies

$$g(x) = (\psi \circ \mathbf{n})(x) \quad \text{for (almost) every } x \in \partial B_r^k. \quad (35)$$

Physically speaking, the vector field \mathbf{n} determines an orientation for the boundary datum g , therefore a map g which admits a lifting is said to be *orientable*. Since \mathbb{S}^2 is a simply connected manifold, \mathbb{S}^2 -valued maps are easier to deal with than \mathcal{N} -valued map, and extension results are known. Therefore, one can find an \mathbb{S}^2 -valued extension \mathbf{w} of \mathbf{n} , then apply ψ to define an extension $P := \psi \circ \mathbf{w}$ of g . Thus, one proves extension results for \mathcal{N} -valued maps, which will be crucial in the proof of Proposition 8. We will denote by ∇_{\top} the tangential gradient at the boundary, i.e. $\nabla_{\top} g(x)$ is the restriction of $\nabla g(x)$ to the tangent plane to ∂B_r^k at a point x .

Lemma 26 *There exists a constant $C > 0$ such that, for any $r > 0$, $k \geq 3$ and any $g \in H^1(\partial B_r^k, \mathcal{N})$, there exists $P \in H^1(B_r^k, \mathcal{N})$ which satisfies $P = g$ on ∂B_r^k and*

$$\|\nabla P\|_{L^2(B_r^k)}^2 \leq C r^{k/2-1/2} \|\nabla_{\top} g\|_{L^2(\partial B_r^k)}.$$

In Lemma 26, the two sides of the inequality have different homogeneities in P , g . This fact is of main importance, for the arguments of Section 4 rely crucially on it.

In case $k = 2$, it makes sense to consider the homotopy class of a H^1 -boundary datum g , due to Sobolev embedding. If the homotopy class is trivial, we have the following

Lemma 27 *There exists a constant $C > 0$ such that, for any $r > 0$ and any $g \in H^1(\partial B_r^2, \mathcal{N})$ with trivial homotopy class, there exists $P \in H^1(B_r^2, \mathcal{N})$ satisfying $P = g$ on ∂B_r^2 and*

$$\|\nabla P\|_{L^2(B_r^2)}^2 \leq C r \|\nabla_{\top} g\|_{L^2(\partial B_r^2)}^2.$$

If the homotopy class of the boundary datum g is non-trivial, then there is no extension $P \in H_g^1(B_r^2, \mathcal{N})$. However, we can still find an extension $P \in H^1(B_r^2, \mathbf{S}_0)$ whose energy satisfies a logarithmic upper bound.

Lemma 28 *There exists a constant $C > 0$ such that, for any $0 < \varepsilon < 1$, $r > 0$ and any $g \in H^1(\partial B_r^2, \mathcal{N})$ with non-trivial homotopy class, there exists $P_\varepsilon \in H^1(B_r^2, \mathbf{S}_0)$ such that $P_\varepsilon = g$ on ∂B_r^2 and*

$$E_\varepsilon(P_\varepsilon, B_r^2) \leq \kappa_* \log \frac{r}{\varepsilon} + C \left(r \|\nabla_\top g\|_{L^2(\partial B_r^2)}^2 + 1 \right).$$

We also prove an extension results on a cylinder, in dimension three. Given positive numbers L and r , set $\Lambda := B_r^2 \times [-L, L]$ and $\Gamma := \partial B_r^2 \times [-L, L]$. Let $g \in H^1(\Gamma, \mathcal{N})$ be a boundary datum, which is only defined on the lateral surface of the cylinder. By Fubini theorem and Sobolev embedding, the restriction of g to $\partial B_r^2 \times \{z\}$ has a well-defined homotopy class, for a.e. $z \in [-L, L]$. Moreover, arguing by density (as in Section 2.2) we see that this class is independent of z . We call it the homotopy class of g .

Lemma 29 *For any $0 < \varepsilon < 1$ and any $g \in H^1(\Gamma, \mathcal{N})$ with non-trivial homotopy class, there exists $P_\varepsilon \in H^1(\Lambda, \mathbf{S}_0)$ such that $P_\varepsilon = g$ on Γ ,*

$$E_\varepsilon(P_\varepsilon, \Lambda) \leq \kappa_* L \log \frac{r}{\varepsilon} + CL \left(\frac{L}{r} + \frac{r}{L} \right) \|\nabla_\top g\|_{L^2(\Gamma)}^2 + CL$$

and

$$E_\varepsilon(P_\varepsilon, B_r^2 \times \{z\}) \leq \kappa_* \log \frac{r}{\varepsilon} + C \left(\frac{L}{r} + \frac{r}{L} \right) \|\nabla_\top g\|_{L^2(\Gamma)}^2 + C$$

for $z \in \{-L, L\}$.

In both the inequalities, the prefactors of the H^1 -seminorm of g are probably not optimal, but the leading order terms are sharp (see Corollary 16).

A useful technique to construct extensions of \mathbb{S}^2 -valued maps has been proposed by Hardt, Kinderlehrer and Lin [37].

Lemma 30 (Hardt, Kinderlehrer and Lin, [37]) *Given $k \geq 2$ and a map $\mathbf{n} \in H^1(\partial B_r^k, \mathbb{S}^2)$, there exists a map $\mathbf{w} \in H^1(B_r^k, \mathbb{S}^2)$ which satisfy $\mathbf{w} = \mathbf{n}$ on ∂B_r^k ,*

$$\|\nabla \mathbf{w}\|_{L^2(B_r^k)}^2 \leq C_k r^{k/2-1/2} \|\nabla_\top \mathbf{n}\|_{L^2(\partial B_r^k)} \quad (36)$$

and

$$\|\nabla \mathbf{w}\|_{L^2(B_r^k)}^2 \leq C_k r \|\nabla_\top \mathbf{n}\|_{L^2(\partial B_r^k)}^2. \quad (37)$$

Sketch of the proof. The existence of such an extension has been proved by Hardt, Kinderlehrer and Lin (see [37, proof of Lemma 2.3, Equation (2.3)]). Although the proof has been given in the case $k = 3$, the same argument applies to any $k \geq 2$. The proof relies on the following identity: if $\mathbf{v}: B_r^k \rightarrow \mathbb{R}^3$ satisfies $-\Delta \mathbf{v} = 0$ on B_r^k and $\mathbf{v} = \mathbf{n}$ on ∂B_r^k , then

$$r \|\nabla_\top \mathbf{n}\|_{L^2(\partial B_r^k)}^2 = (k-2) \|\nabla \mathbf{v}\|_{L^2(B_r^k)}^2 + r \|\partial_\nu \mathbf{v}\|_{L^2(\partial B_r^k)}^2, \quad (38)$$

where $\boldsymbol{\nu}$ is a unit normal to ∂B_r^k . This identity is obtained by multiplying both sides of the equation $-\Delta \mathbf{v} = 0$ with $x \cdot \nabla \mathbf{v}$ and integrating by parts. Then, by integrating the equality $|\nabla \mathbf{v}|^2 = \nabla \cdot ((\mathbf{v} - \mathbf{n}_*) \nabla \mathbf{v})$, where $\mathbf{n}_* := \int_{\partial B_r^k} \mathbf{n}$, we deduce

$$\|\nabla \mathbf{v}\|_{L^2(B_r^k)}^2 = \int_{\partial B_r^k} (\mathbf{n} - \mathbf{n}_*) \partial_{\boldsymbol{\nu}} \mathbf{v} \, d\mathcal{H}^{k-1} \stackrel{(38)}{\leq} \|\mathbf{n} - \mathbf{n}_*\|_{L^2(\partial B_r^k)} \|\nabla \mathbf{v}\|_{L^2(\partial B_r^k)},$$

so $\|\nabla \mathbf{v}\|_{L^2(B_r^k)}^2$ is bounded by the right-hand side of (36) and, thanks to the Poincaré inequality, also by (37). Finally, one defines \mathbf{w} by composing \mathbf{v} with a suitable projection onto the unit sphere \mathbb{S}^2 . \square

We state now a lifting property for Sobolev maps. This subject has been studied extensively, among others, by Bethuel and Zheng [15], Bourgain, Brezis and Mironescu [16], Bethuel and Chiron [13], Ball and Zarnescu [9] (in particular, in the latter a problem closely related to the Q -tensor theory is considered).

Lemma 31 *Let \mathcal{M} be a smooth, compact, simply connected surface, possibly with boundary. Then, any map $g \in H^1(\mathcal{M}, \mathcal{N})$ has a lifting, i.e. there exists $\mathbf{n} \in H^1(\mathcal{M}, \mathbb{S}^2)$ which satisfies (35). Moreover, there holds*

$$|\nabla g|^2 = 2s_*^2 |\nabla \mathbf{n}|^2 \quad \mathcal{H}^2\text{-a.e. on } \mathcal{M}. \quad (39)$$

If \mathcal{M} has a boundary then $\mathbf{n}|_{\partial \mathcal{M}}$ is a lifting of $g|_{\partial \mathcal{M}}$. Moreover, if $g|_{\partial \mathcal{M}} \in H^1(\partial \mathcal{M}, \mathcal{N})$ then $\mathbf{n}|_{\partial \mathcal{M}} \in H^1(\partial \mathcal{M}, \mathbb{S}^2)$.

Sketch of the proof. The identity (39) follows directly by (35), by a straightforward computation. The existence of a lifting is a well-known topological fact, when g is of class C^1 . In case $g \in H^1$ and \mathcal{M} is a bounded, smooth domain in \mathbb{R}^2 , the existence of a lifting has been proved by Ball and Zarnescu [9, Theorem 2]. The proof, which is based on the density of smooth maps in $H^1(\mathcal{M}, \mathcal{N})$ (see [62]), carries over to more general manifolds \mathcal{M} . In case \mathcal{M} has a boundary and $g|_{\partial \mathcal{M}} \in H^1$, one can adapt the density argument and find a lifting \mathbf{n} such that $\mathbf{n}|_{\partial \mathcal{M}} \in H^1$. If $\tilde{\mathbf{n}}$ is any other lifting of g , then $\mathbf{n} \cdot \tilde{\mathbf{n}}$ is an H^1 -map $\mathcal{M} \rightarrow \{1, -1\}$ and so, by a slicing argument, either $\tilde{\mathbf{n}} = \mathbf{n}$ a.e. or $\tilde{\mathbf{n}} = -\mathbf{n}$ a.e (see [9, Proposition 2]). In particular, any lifting $\tilde{\mathbf{n}}$ of g satisfies $\tilde{\mathbf{n}}|_{\partial \mathcal{M}} \in H^1$. \square

Combining Lemmas 30 and 31, we obtain easily Lemmas 26 and 27.

Proof of Lemmas 26 and 27. Consider Lemma 26 first. Let $\mathbf{n} \in H^1(\partial B_r^k, \mathbb{S}^2)$ be a lifting of g , whose existence is guaranteed by Lemma 31, and let $\mathbf{w} \in H^1(B_r^k, \mathbb{S}^2)$ be the extension given by Lemma 30. Then, the map defined by

$$P(x) := s_* \left(\mathbf{w}^{\otimes 2}(x) - \frac{1}{3} \text{Id} \right) \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in B_r^k$$

has the desired properties. The proof of Lemma 27 is analogous. \square

Proof of Lemma 28. By a scaling argument, we can assume WLOG that $r = 1$. Let $h(x) := P_0(2x)$ for $x \in \partial B_{1/2}^2$, where P_0 is given by Lemma 19, and let $w_\varepsilon: B_{1/2}^2 \rightarrow \mathbf{S}_0$ be given by

$$w_\varepsilon(x) := \eta_\varepsilon(|x|)h\left(\frac{x}{|x|}\right) \quad \text{for } x \in B_{1/2}^2,$$

where

$$\eta_\varepsilon(\rho) := \begin{cases} 1 & \text{if } \rho \geq \varepsilon \\ \varepsilon^{-1}\rho & \text{if } 0 \leq \rho < \varepsilon. \end{cases} \quad (40)$$

Then, w_ε belongs to $H^1(B_{1/2}^2, \mathbf{S}_0)$ and

$$E_\varepsilon(w_\varepsilon, B_{1/2}^2) \leq \kappa_* |\log \varepsilon| + C. \quad (41)$$

Indeed,

$$|\nabla w_\varepsilon|^2 = |\partial_\rho w_\varepsilon|^2 + \frac{1}{\rho^2} |\nabla_\top w_\varepsilon|^2 \begin{cases} \leq C\varepsilon^{-1} & \text{where } \rho \leq \varepsilon \\ = \rho^{-2} |\nabla_\top h|^2 & \text{where } \rho \geq \varepsilon, \end{cases}$$

and $w_\varepsilon(x) \in \mathcal{N}$ if $|x| \geq \varepsilon$. Therefore, we have

$$\begin{aligned} E_\varepsilon(w_\varepsilon, B_{1/2}^2) &\leq \frac{1}{2} \int_\varepsilon^{1/2} \frac{d\rho}{\rho} \int_{\mathbb{S}^1} |\nabla_\top h|^2 d\mathcal{H}^1 + E_\varepsilon(w_\varepsilon, B_\varepsilon^2) \\ &\leq \frac{1}{2} (|\log \varepsilon| - \log 2) \int_{\mathbb{S}^1} |\nabla_\top h|^2 d\mathcal{H}^1 + C, \end{aligned}$$

whence (41) follows.

To complete the proof of the lemma, we only need to interpolate between g and h by a function defined on the annulus $D := B_1^2 \setminus B_{1/2}^2$. Up to a bilipschitz equivalence, D can be thought as the unit square $(0, 1)^2$ with an equivalence relation identifying two opposite sides of the boundary, as shown in Figure 2. We assign the boundary datum g on the bottom side, and h on the top side. Since \mathcal{N} is path-connected, we find a smooth path $c: [0, 1] \rightarrow \mathcal{N}$ connecting $g(0, 0)$ to $h(0, 1)$. By assigning c as a boundary datum on the lateral sides of the square, we have defined an H^1 -map $\partial[0, 1]^2 \rightarrow \mathcal{N}$, homotopic to $g * c * h * \tilde{c}$. (Here, the symbol $*$ stands for composition of paths, and \tilde{c} is the reverse path of c .) Since the square is bilipschitz equivalent to a disk, it is possible to apply Lemma 27 and find $\tilde{v} \in H^1([0, 1]^2, \mathcal{N})$ such that

$$\int_{[0, 1]^2} |\nabla \tilde{v}|^2 d\mathcal{H}^2 \leq C \left(\|\nabla g\|_{L^2(\partial B_1^2)}^2 + \|\nabla h\|_{L^2(\partial B_{1/2}^2)}^2 + \|c'\|_{L^2(0, 1)}^2 \right). \quad (42)$$

Passing to the quotient $[0, 1]^2 \rightarrow D$, we obtain a map $v \in H^1(D, \mathbf{S}_0)$. Now, the function $P: B_1^2 \rightarrow \mathbf{S}_0$ defined by $P := v$ on D and $P := w_\varepsilon$ on $B_{1/2}^2$ satisfies the lemma. Indeed, the energy of P is bounded by (41) and (42), and the H^1 -norms of h and c are controlled by a constant depending only on \mathcal{N} . \square

Proof of Lemma 29. By an average argument, we find $z_0 \in [-L/4, L/4]$ such that

$$\|\nabla_{\top} g\|_{L^2(\partial B_r^2 \times \{z_0\})}^2 \leq \frac{8}{L} \|\nabla_{\top} g\|_{L^2(I)}^2. \quad (43)$$

To avoid notation, we assume WLOG that $z_0 = 0$. We construct an \mathcal{N} -valued map \tilde{P} , defined over the “cylindrical annulus” $(B_r^2 \setminus B_{r/2}^2) \times [-L, L]$, such that

$$\tilde{P}(\rho e^{i\theta}, z) = \begin{cases} g(\rho e^{i\theta}, z) & \text{for } \rho = r \\ g(\rho e^{i\theta}, 0) & \text{for } \rho = r/2, \end{cases}$$

where $(\rho, \theta, z) \in [0, r] \times [0, 2\pi] \times [-L, L]$ are the cylindrical coordinates. Then, since \tilde{P} restricted to $\partial B_{r/2}^2 \times [-L, L]$ is independent of the z -variable, we can apply Lemma 28 to construct an extension in the inner cylinder $B_{r/2}^2 \times [-L, L]$.

The map \tilde{P} is defined as follows:

$$\tilde{P}(\rho e^{i\theta}, z) := \begin{cases} g(\rho e^{i\theta}, z'(\rho, z)) & \text{if } \rho_0(\rho, z) \leq \rho \leq r \text{ and } |z| \leq L \\ g(\rho e^{i\theta}, 0) & \text{if } r/2 \leq \rho < \rho_0(\rho, z) \text{ and } |z| \leq L, \end{cases}$$

where

$$z'(\rho, z) := \frac{2L}{r} \operatorname{sign}(z)(\rho - r) + z, \quad \rho_0(\rho, z) := r - \frac{r}{2L}|z|.$$

Note that $|z'(\rho, z)| \leq L$ if $\rho_0(\rho, z) \leq \rho \leq r$ and $|z| \leq L$. We compute the H^1 -seminorm of \tilde{P} . For simplicity, we restrict our attention to the upper half-cylinder. We have

$$\begin{aligned} & \|\nabla \tilde{P}\|_{L^2((B_r^2 \setminus B_{r/2}^2) \times [0, L])}^2 \\ &= \int_0^{2\pi} \int_0^L \int_{\rho_0(\rho, z)}^r \left\{ \left(\frac{4L^2}{r^2} + 1 \right) \rho |\partial_z g|^2(\rho e^{i\theta}, z'(\rho, z)) \right. \\ & \quad \left. + \rho^{-1} |\partial_{\theta} g|^2(\rho e^{i\theta}, z'(\rho, z)) \right\} d\rho dz d\theta \\ & \quad + \int_0^{2\pi} \int_0^L \int_{r/2}^{\rho_0(\rho, z)} \rho^{-1} |\partial_{\theta} g|^2(\rho e^{i\theta}, 0) d\rho dz d\theta \\ &\leq \frac{r}{2L} \int_0^{2\pi} \int_0^L \int_0^z \left\{ \left(\frac{4L^2}{r^2} + 1 \right) r |\partial_z g|^2 + 2r^{-1} |\partial_{\theta} g|^2 \right\}(\rho e^{i\theta}, \xi) d\xi dz d\theta \\ & \quad + (\log 2)L \int_0^{2\pi} |\partial_{\theta} g|^2(\rho e^{i\theta}, 0) d\theta \\ &\leq r \left(\frac{4L^2}{r^2} + 1 \right) \int_0^{2\pi} \int_0^L \left\{ r |\partial_z g|^2 + r^{-1} |\partial_{\theta} g|^2 \right\}(\rho e^{i\theta}, \xi) d\xi d\theta \\ & \quad + (\log 2)L \int_0^{2\pi} |\partial_{\theta} g|^2(\rho e^{i\theta}, 0) d\theta. \end{aligned}$$

An analogous estimates holds on the lower half-cylinder. Therefore,

$$\begin{aligned} \|\nabla \tilde{P}\|_{L^2((B_r^2 \setminus B_{r/2}^2) \times [-L, L])}^2 &\leq \left(\frac{4L^2}{r} + r \right) \|\nabla_{\top} g\|_{L^2(\Gamma)}^2 \\ &\quad + (\log 2) r L \|\nabla_{\top} g\|_{L^2(\partial B_r^2 \times \{0\})}^2 \end{aligned}$$

and so, due to (43), we have

$$\|\nabla \tilde{P}\|_{L^2((B_r^2 \setminus B_{r/2}^2) \times [-L, L])}^2 \leq C \left(\frac{L^2}{r} + r \right) \|\nabla_{\top} g\|_{L^2(\Gamma)}^2. \quad (44)$$

By applying Lemma 28 (and (43)) to $g(\cdot, 0)$, we find an extension $\tilde{P}_\varepsilon \in H^1(B_{r/2}^2, \mathbf{S}_0)$ which satisfies

$$E_\varepsilon(\tilde{P}_\varepsilon, B_{r/2}^2) \leq \kappa_* \log \frac{r}{\varepsilon} + \frac{Cr}{L} \|\nabla_{\top} g\|_{L^2(\Gamma)}^2 + C. \quad (45)$$

Define the map P by letting $P(\rho e^{i\theta}, z) := \tilde{P}(\rho e^{i\theta}, z)$ if $r/2 < \rho \leq r$ and $P_\varepsilon(\rho e^{i\theta}, z) := \tilde{P}_\varepsilon(\rho e^{i\theta})$ if $\rho \leq r/2$. By integrating (44) with respect to $z \in [-L, L]$, and combining the resulting inequality with (44), we give an upper bound for the energy of P_ε on Λ . Moreover, there holds

$$\begin{aligned} &\|\nabla \tilde{P}\|_{L^2((B_r^2 \setminus B_{r/2}^2) \times \{L\})}^2 \\ &= \int_0^{2\pi} \int_{r/2}^r \left\{ \left(\frac{4L^2}{r^2} + 1 \right) \rho |\partial_z g|^2 (r e^{i\theta}, z'(\rho, L)) \right. \\ &\quad \left. + \rho^{-1} |\partial_\theta g|^2 (r e^{i\theta}, z'(\rho, L)) \right\} d\rho d\theta \\ &\leq \frac{r}{2L} \int_0^{2\pi} \int_0^L \left\{ \left(\frac{4L^2}{r^2} + 1 \right) r |\partial_z g|^2 + 2r^{-1} |\partial_\theta g|^2 \right\} (r e^{i\theta}, \xi) d\xi d\theta \\ &\leq \frac{r}{L} \left(\frac{4L^2}{r^2} + 1 \right) \|\nabla_{\top} g\|_{L^2(\Gamma)}^2. \end{aligned}$$

This inequality combined with (45), gives an upper bound for the energy of P_ε on $B_r^2 \times \{L\}$; a similar inequality holds on $B_r^2 \times \{-L\}$. This concludes the proof. \square

3.2 Luckhaus' lemma and its variants

In many of our arguments, it will be useful to compare the energy of a minimizer Q_ε of (LG_ε) on, say, a ball B_r with the energy of another map $P_\varepsilon: B_r \rightarrow \mathbf{S}_0$. However, it may happen that $P_\varepsilon \neq Q_\varepsilon$ on ∂B_r , so P_ε is not an admissible comparison map. To correct this, we need to construct a function which interpolates between P_ε and Q_ε over a thin spherical shell.

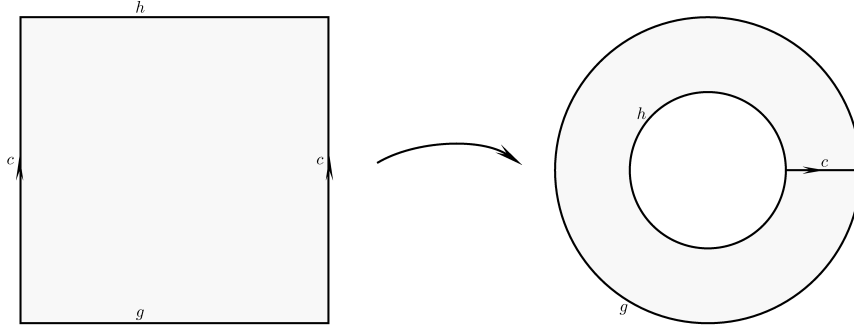


Fig. 2 A square can be mapped into an annulus, by identifying a pair of opposite sides.

The problem may be stated as follows. Set $\epsilon := \varepsilon/r$, and suppose that $0 < \epsilon < 1$. Let $u_\epsilon: \partial B_1 \rightarrow \mathbf{S}_0$, $v_\epsilon: \partial B_1 \rightarrow \mathcal{N}$ be two H^1 -maps. We look for a (small) number $h(\epsilon) > 0$ and a map $\varphi_\epsilon: A_\epsilon := B_1 \setminus B_{1-h(\epsilon)} \rightarrow \mathbf{S}_0$, such that

$$\varphi_\epsilon(x) = u_\epsilon(x) \quad \text{and} \quad \varphi_\epsilon(x - h(\epsilon)x) = v_\epsilon(x) \quad \text{for } \mathcal{H}^2\text{-a.e. } x \in \partial B_1 \quad (46)$$

and the energy $E_\epsilon(\varphi_\epsilon, A_\epsilon)$ satisfies a suitable bound. In some cases, only the function u_ϵ is prescribed, and we need to find *both* v_ϵ and φ_ϵ .

Luckhaus proved an interesting interpolation lemma (see [49, Lemma 1]), which turned out to be useful in several applications. When both u_ϵ and v_ϵ take values in the manifold \mathcal{N} , Luckhaus' lemma gives an extension φ_ϵ satisfying (46) and

$$\sup_{x \in A_\epsilon} \text{dist}(\varphi_\epsilon(x), \mathcal{N}) + \int_{A_\epsilon} |\nabla \varphi_\epsilon|^2 \leq C(u_\epsilon, v_\epsilon, h(\epsilon)).$$

For the convenience of the reader, and for future reference, we recall Luckhaus' lemma. Since the term $\epsilon^{-2}f(\varphi_\epsilon)$ is not taken into account here, we drop the subscript ϵ in the notation.

Lemma 32 (Luckhaus, [49]) *For any $\beta \in (1/2, 1)$, there exists a constant $C > 0$ with this property. For any fixed numbers $0 < \lambda \leq 1/2$, $0 < \sigma < 1$ and any $u, v \in H^1(\partial B_1, \mathcal{N})$, set*

$$K := \int_{\partial B_1} \left\{ |\nabla u|^2 + |\nabla v|^2 + \frac{|u - v|^2}{\sigma^2} \right\} d\mathcal{H}^2.$$

Then, there exists a function $\varphi \in H^1(B_1 \setminus B_{1-\lambda}, \mathbf{S}_0)$ satisfying (46),

$$\text{dist}(\varphi(x), \mathcal{N}) \leq C\sigma^{1-\beta}\lambda^{-1/2}K^{1/2}$$

for a.e. $x \in B_1 \setminus B_{1-\lambda}$ and

$$\int_{B_1 \setminus B_{1-\lambda}} |\nabla \varphi|^2 \leq C\lambda(1 + \sigma^2\lambda^{-2})K.$$

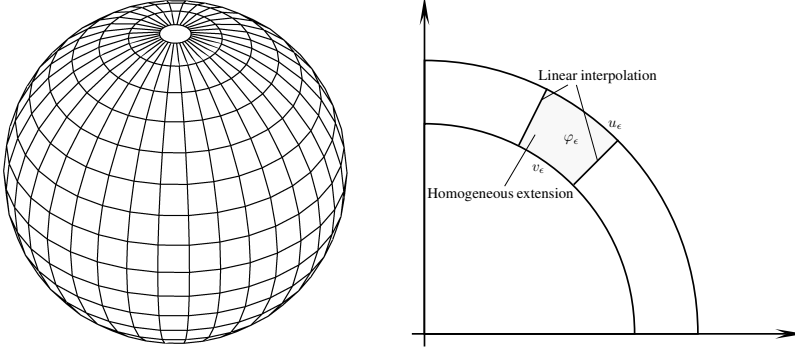


Fig. 3 Left: a grid on a sphere. Right: the Luckhaus' construction. Given two maps u_ϵ, v_ϵ (respectively defined on the outer and inner boundary of a thin spherical shell), we construct a map φ_ϵ by using linear interpolation on the boundary of the cells, and homogeneous extension inside each cell.

The idea of the proof is illustrated in Figure 3. One constructs a grid on the sphere ∂B_1 with suitable properties. The map φ is defined by linear interpolation between u and v on the boundary of the cells. Inside each cell, φ is defined by a homogeneous extension. By choosing carefully the grid on ∂B_1 , and using Sobolev embeddings, one can bound the L^∞ -distance between u and v on the boundary of the cells, in terms of K . This yields bounds both on $\text{dist}(\varphi(x), \mathcal{N})$ and on the gradient of φ .

We will discuss here a couple of variants of this lemma. In our first result, we suppose that only the map $u_\epsilon: \partial B_1 \rightarrow \mathbf{S}_0$ is prescribed, so we need to find both $v_\epsilon: \partial B_1 \rightarrow \mathcal{N}$ and φ_ϵ . Approximating u_ϵ with a \mathcal{N} -valued map v_ϵ may be impossible, due to topological obstructions. However, this is possible if the energy of u_ϵ is small, compared to $|\log \epsilon|$. More precisely, we assume that

$$E_\epsilon(u_\epsilon, \partial B_1) \leq \eta_0 |\log \epsilon| \quad (47)$$

for some small constant $\eta_0 > 0$. For technical reasons, we also require a L^∞ -bound on u_ϵ , namely

$$\|u_\epsilon\|_{L^\infty(\partial B_1)} \leq \kappa \quad (48)$$

where κ is an ϵ -independent constant. In the applications, u_ϵ will be a Landau-de Gennes minimizer and (48) will be satisfied, because of (H).

Proposition 33 *For any $\kappa > 0$, there exist positive numbers η_0, ϵ_1, C with the following property. For any $0 < \eta \leq \eta_0$, any $0 < \epsilon \leq \epsilon_1$ and any $u_\epsilon \in (H^1 \cap L^\infty)(\partial B_1, \mathbf{S}_0)$ satisfying (47)–(48), there exist maps $v_\epsilon \in H^1(\partial B_1, \mathcal{N})$ and $\varphi_\epsilon \in H^1(B_1 \setminus B_{1-h(\epsilon)}, \mathbf{S}_0)$ which satisfy (46),*

$$\frac{1}{2} \int_{\partial B_1} |\nabla v_\epsilon|^2 \, d\mathcal{H}^2 \leq C E_\epsilon(u_\epsilon, \partial B_1), \quad (49)$$

$$E_\epsilon(\varphi_\epsilon, B_1 \setminus B_{1-h(\epsilon)}) \leq C h(\epsilon) E_\epsilon(u_\epsilon, \partial B_1) \quad (50)$$

for $h(\epsilon) := \epsilon^{1/2} |\log \epsilon|$.

We will discuss the proof of this proposition later on. Before that, we remark that v_ϵ effectively approximates u_ϵ , i.e. their distance — measured in a suitable norm — tends to 0 as $\epsilon \rightarrow 0$.

Corollary 34 *Under the same assumptions of Proposition 33, there holds*

$$\|u_\epsilon - v_\epsilon\|_{L^2(\partial B_1)} \leq Ch^{1/2}(\epsilon) E_\epsilon^{1/2}(u_\epsilon, \partial B_1).$$

Notice that the right-hand side tends to 0 as $\epsilon \rightarrow 0$, due to (47) and the choice of $h(\epsilon)$.

Proof. We can estimate the L^2 -distance between u_n and v_n thanks to (46), the Hölder inequality and (50):

$$\begin{aligned} \|u_\epsilon - v_\epsilon\|_{L^2(\partial B_1)}^2 &\stackrel{(46)}{=} \int_{\partial B_1} |\varphi_\epsilon(x) - \varphi_\epsilon(x - h(\epsilon)x)|^2 d\mathcal{H}^2(x) \\ &\leq h(\epsilon) \int_{\partial B_1} \int_{1-h(\epsilon)}^1 |\nabla \varphi_\epsilon(tx)|^2 dt d\mathcal{H}^2(x) \\ &\leq \frac{h(\epsilon)}{(1-h(\epsilon))^2} E_\epsilon(\varphi_\epsilon, B_1 \setminus B_{1-h(\epsilon)}) \stackrel{(50)}{\leq} Ch(\epsilon) E_\epsilon(u_\epsilon, \partial B_1). \end{aligned}$$

□

Combining Lemma 32 and Proposition 33, we obtain a third extension result. In this case, both the boundary values u, v are prescribed and, unlike Luckhaus' lemma, we provide a control over the potential energy of the extension $\epsilon^{-2}f(\varphi_\epsilon)$.

Proposition 35 *Let $\{\sigma_\epsilon\}_{\epsilon>0}$ be a positive sequence such that $\sigma_\epsilon \rightarrow 0$, and let u_ϵ, v_ϵ be given functions in $H^1(\partial B_1, \mathbf{S}_0)$. For all $\epsilon > 0$, assume that u_ϵ satisfies (48), that $v_\epsilon(x) \in \mathcal{N}$ for \mathcal{H}^2 -a.e. $x \in \partial B_1$ and that*

$$\int_{\partial B_1} \left\{ |\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2} f(u_\epsilon) + |\nabla v_\epsilon|^2 + \frac{|u_\epsilon - v_\epsilon|^2}{\sigma_\epsilon^2} \right\} d\mathcal{H}^2 \leq C \quad (51)$$

for an ϵ -independent constant C . Set

$$\nu_\epsilon := h(\epsilon) + \left(h^{1/2}(\epsilon) + \sigma_\epsilon \right)^{1/4} (1 - h(\epsilon)).$$

Then, there exist a number $\epsilon_1 > 0$ and, for $0 < \epsilon \leq \epsilon_1$, a function $\varphi_\epsilon \in H^1(B_1 \setminus B_{1-\nu_\epsilon}, \mathbf{S}_0)$ which satisfies (46) and

$$E_\epsilon(\varphi_\epsilon, B_1 \setminus B_{1-\nu_\epsilon}) \leq C\nu_\epsilon.$$

The assumption (51) could be replaced by a logarithmic bound, of the order of $\eta_0 |\log \epsilon|$ for small $\eta_0 > 0$, with additional assumptions on σ_ϵ . However, the result as it is presented here suffices for our purposes.

Proof of Proposition 35. Thanks to (51) and (48), we can apply Proposition 33 to the function u_ϵ . We obtain two maps $w_\epsilon \in H^1(\partial B_1, \mathcal{N})$ and $\varphi_\epsilon^1 \in H^1(B_1 \setminus B_{1-h(\epsilon)}, \mathbf{S}_0)$, which satisfy

$$\begin{aligned} \varphi_\epsilon^1(x) &= u_\epsilon(x) \quad \text{and} \quad \varphi_\epsilon^1(x - h(\epsilon)x) = w_\epsilon(x) \quad \text{for } \mathcal{H}^2\text{-a.e. } x \in \partial B_1, \\ \int_{\partial B_1} |\nabla w_\epsilon|^2 \, d\mathcal{H}^2 &\leq C, \\ E_\epsilon(\varphi_\epsilon^1, B_1 \setminus B_{1-h(\epsilon)}) &\leq Ch(\epsilon). \end{aligned} \quad (52)$$

Corollary 34, combined with (51), entails

$$\|w_\epsilon - v_\epsilon\|_{L^2(\partial B_1)} \leq \|w_\epsilon - u_\epsilon\|_{L^2(\partial B_1)} + \|u_\epsilon - v_\epsilon\|_{L^2(\partial B_1)} \leq C \left(h^{1/2}(\epsilon) + \sigma_\epsilon \right).$$

Therefore, setting $\tilde{\sigma}_\epsilon := h^{1/2}(\epsilon) + \sigma_\epsilon$, we have

$$\int_{\partial B_1} \left\{ |\nabla w_\epsilon|^2 + |\nabla v_\epsilon|^2 + \frac{|w_\epsilon - v_\epsilon|^2}{\tilde{\sigma}_\epsilon^2} \right\} d\mathcal{H}^2 \leq C$$

Then, we can apply Lemma 32 to v_ϵ and w_ϵ , choosing $\sigma = \tilde{\sigma}_\epsilon$, $\beta = 3/4$ and $\lambda := \tilde{\sigma}_\epsilon^{1/4}$. By rescaling, we find a map $\varphi_\epsilon^2 \in H^1(B_{1-h(\epsilon)} \setminus B_{1-\nu_\epsilon}, \mathbf{S}_0)$ which satisfies

$$\begin{aligned} \int_{B_{1-h(\epsilon)} \setminus B_{1-\nu_\epsilon}} |\nabla \varphi_\epsilon^2|^2 &\leq C \tilde{\sigma}_\epsilon^{1/4} (1 - h(\epsilon)) \\ \text{dist}(\varphi_\epsilon^2(x), \mathcal{N}) &\leq C \tilde{\sigma}_\epsilon^{1/8} \quad \text{for all } x \in B_{1-h(\epsilon)} \setminus B_{1-\nu_\epsilon}. \end{aligned} \quad (53)$$

Since $\tilde{\sigma}_\epsilon \rightarrow 0$, there exists $\epsilon_1 > 0$ such that $\varphi_\epsilon^2(x) \notin \mathcal{C}$ for any $0 < \epsilon \leq \epsilon_1$ and x . Therefore, the function

$$\varphi_\epsilon(x) := \begin{cases} \varphi_\epsilon^1(x) & \text{if } x \in B_1 \setminus B_{1-h(\epsilon)} \\ \varrho \circ \varphi_\epsilon^2(x) & \text{if } x \in B_{1-h(\epsilon)} \setminus B_{1-\nu_\epsilon} \end{cases}$$

is well-defined, belongs to $H^1(B_1 \setminus B_{1-\nu_\epsilon}, \mathbf{S}_0)$, satisfies (46) and

$$E_\epsilon(\varphi_\epsilon, B_1 \setminus B_{1-\nu_\epsilon}) = E_\epsilon(\varphi_\epsilon^1, B_1 \setminus B_{1-h(\epsilon)}) + \int_{B_{1-h(\epsilon)} \setminus B_{1-\nu_\epsilon}} |\nabla \varphi_\epsilon^2|^2 \stackrel{(52)-(53)}{\leq} C \nu_\epsilon. \quad \square$$

Sections 3.3–3.5 are devoted to the proof of Proposition 33, which we sketch here. From now on, we assume that there exists a positive constant M such that

$$E_\epsilon(u_\epsilon, \partial B_1) \leq M |\log \epsilon| \quad \text{for all } 0 < \epsilon < 1. \quad (\text{M}_\epsilon)$$

The assumption (47) clearly implies (M_ϵ) . As in Luckhaus' arguments, the key ingredient of the construction is the choice of a grid on the unit sphere ∂B_1 , with special properties. In Section 3.3 we construct a family of grids $\{\mathcal{G}^\epsilon\}$,

whose cells have size controlled by $h(\epsilon)$, and we prove that there exists $\epsilon_1 > 0$ such that

$$\text{dist}(u_\epsilon(x), \mathcal{N}) \leq \delta_0 \quad \text{for any } \epsilon \in (0, \epsilon_1) \text{ and any } x \in R_1^\epsilon.$$

Here R_1^ϵ denotes the 1-skeleton of \mathcal{G}^ϵ , i.e. the union of the boundaries of all the cells, and δ_0 is given by Lemma 14. In particular, the composition $\varrho \circ u_\epsilon$ is well-defined on R_1^ϵ when $\epsilon < \epsilon_1$. We wish to extend $\varrho \circ u_\epsilon$ to a map $v_\epsilon: \partial B_1 \rightarrow \mathcal{N}$. This may be impossible, depending on the homotopy properties of u_ϵ . A sufficient condition for the existence of v_ϵ is the following:

- (C $_\epsilon$) For any 2-cell K of \mathcal{G}^ϵ , the loop $\varrho \circ u_\epsilon|_{\partial K}: \partial K \rightarrow \mathcal{N}$ is homotopically trivial.

This condition makes sense for any $u_\epsilon \in H^1(\partial B_1, \mathbf{S}_0)$, for we construct \mathcal{G}^ϵ in such a way that u_ϵ restricted to R_1^ϵ belongs to $H^1 \hookrightarrow C^0$.

In Section 3.4, we assume that (M $_\epsilon$) and (C $_\epsilon$) hold and we construct a function $v_\epsilon \in H^1(\partial B_1, \mathcal{N})$, whose energy is controlled by the energy of u_ϵ . Once v_ϵ is known, we construct φ_ϵ by Luckhaus' method. Particular care must be taken here, as we need to bound the potential energy of φ_ϵ as well. Finally, in Section 3.5 we show that the logarithmic bound (47), for a small enough constant η_0 , implies that Condition (C $_\epsilon$) is satisfied. The argument relies on the results of Section 2.2, which imply a logarithmic lower bound for the energy if $\varrho \circ u_\epsilon|_{\partial K}$ is non-trivial for at least one 2-cell K .

3.3 Good grids on the sphere

Consider a decomposition of ∂B_1 of the form

$$\partial B_1 = \bigcup_{j=0}^2 \bigcup_{i=1}^{k_j} K_{i,j},$$

where the sets $K_{i,j}$ are mutually disjoint, and each $K_{i,j}$ is bilipschitz equivalent to a j -dimensional ball. The collection of all the $K_{i,j}$'s will be called a *grid* on ∂B_1 . Each $K_{i,j}$ will be called a j -cell of the grid. We define the j -skeleton of the grid as

$$R_j := \bigcup_{i=1}^{k_j} K_{i,j} \quad \text{for } j \in \{0, 1, 2\}.$$

For our purposes, we need to consider grids with some special properties.

Definition 36 Let $h: (0, \epsilon_1] \rightarrow (0, +\infty)$ be a fixed function. A family of grids $\mathcal{G} := \{\mathcal{G}^\epsilon\}_{0 < \epsilon \leq \epsilon_1}$ will be called a *good family of grids of size h* if there exists a constant $C_{\mathcal{G}} > 0$ which satisfies the following properties.

- (G $_1$) For each ϵ, i, j , there exists a bilipschitz homeomorphism $\varphi_{i,j}^\epsilon: K_{i,j}^\epsilon \rightarrow B_{h(\epsilon)}^j$ such that

$$\|D\varphi_{i,j}^\epsilon\|_{L^\infty} + \|D(\varphi_{i,j}^\epsilon)^{-1}\|_{L^\infty} \leq C_{\mathcal{G}}.$$

(G₂) For all $p \in \{1, 2, \dots, k_1\}$ we have

$$\#\{q \in \{1, 2, \dots, k_2\} : K_{p,1}^\epsilon \subseteq \partial K_{q,2}^\epsilon\} \leq C_{\mathcal{G}},$$

i.e., each 1-cell is contained in the boundary of at most $C_{\mathcal{G}}$ 2-cells.

(G₃) We have

$$E_\epsilon(u_\epsilon, R_1^\epsilon) \leq C_{\mathcal{G}} h^{-1}(\epsilon) E_\epsilon(u_\epsilon, \partial B_1),$$

where R_1^ϵ denotes the 1-skeleton of \mathcal{G}^ϵ .

(G₄) There holds

$$\int_{R_1^\epsilon} f(u_\epsilon) d\mathcal{H}^1 \leq C_{\mathcal{G}} h^{-1}(\epsilon) \int_{\partial B_1} f(u_\epsilon) d\mathcal{H}^2.$$

Of course, this definition depends on the family $\{u_\epsilon\}$, which we assume to be fixed throughout the arguments of this section.

Lemma 37 *For any strictly positive function h , a good family of grids of size h exists.*

Proof. On the unit cube $\partial[0, 1]^3$, consider the uniform grid of size $\lceil h^{-1}(\epsilon) \rceil^{-1}$, i.e. the grid spanned by the points

$$(\lceil h^{-1}(\epsilon) \rceil^{-1} \mathbb{Z}^3) \cap \partial[0, 1]^3$$

(where $\lceil x \rceil$ is, by the definition, the smallest integer k such that $k \geq x$). By applying a bilipschitz homeomorphism $[0, 1]^3 \rightarrow \bar{B}_1$, one obtains a grid \mathcal{F}^ϵ on ∂B_1 which satisfy (G₁)–(G₂). Denote by T_1^ϵ the 1-skeleton of \mathcal{F}^ϵ . By an average argument, as in [49, Lemma 1], we find a rotation $\omega \in \text{SO}(3)$ such that

$$E_\epsilon(u_\epsilon, \omega(T_1^\epsilon)) \leq C h^{-1}(\epsilon) E_\epsilon(u_\epsilon, \partial B_1)$$

and

$$\int_{\omega(T_1^\epsilon)} f(u_\epsilon) d\mathcal{H}^1 \leq C h^{-1}(\epsilon) \int_{\partial B_1} f(u_\epsilon) d\mathcal{H}^2.$$

Thus,

$$\mathcal{G}^\epsilon := \{\omega(K) : K \in \mathcal{F}^\epsilon\}$$

is a good family of grids of size h . □

Good families of grids enjoy the following property.

Lemma 38 *Let \mathcal{G} be a good family of grids on ∂B_1 , of size h . Assume that (M_ε) holds, and that there exists $\alpha \in (0, 1)$ such that*

$$\lim_{\epsilon \rightarrow 0} \epsilon^\alpha h^{-1}(\epsilon) = 0. \tag{54}$$

Then, there holds

$$\lim_{\epsilon \rightarrow 0} \sup_{x \in R_1^\epsilon} \text{dist}(u_\epsilon(x), \mathcal{N}) = 0.$$

Proof. The arguments below are adapted from [1, Lemmas 3.4 and 3.10] (the reader is also referred to [14, Lemmas 2.2, 2.3 and 2.4]). Since the Landau-de Gennes potential satisfies (F₂) by Lemma 14, there exist positive numbers β , C and a continuous function $\psi: [0, +\infty) \rightarrow \mathbb{R}$ such that

$$\begin{cases} \psi(s) = \beta s^2 & \text{for } 0 \leq s < \delta_0 \\ 0 < \psi(s) \leq C & \text{for } s \geq \delta_0 \\ \psi(\text{dist}(v, \mathcal{N})) \leq f(v) & \text{for any } v \in \mathbf{S}_0. \end{cases}$$

Denote by G a primitive of $\psi^{1/6}$, and set $d_\epsilon := \text{dist}(u_\epsilon, \mathcal{N})$. Since the function $\text{dist}(\cdot, \mathcal{N})$ is 1-Lipschitz continuous, we have $d_\epsilon \in H^1(\Omega, \mathbb{R})$ and $|\nabla d_\epsilon| \leq |\nabla u_\epsilon|$. Moreover, $\psi(d_\epsilon) \leq f(u_\epsilon)$ by construction of ψ . Thus, (M_ε) and (G₃) entail

$$h(\epsilon) \int_{R_1^\epsilon} \left\{ \frac{1}{2} |\nabla d_\epsilon|^2 + \epsilon^{-2} \psi(d_\epsilon) \right\} d\mathcal{H}^1 \leq MC_{\mathcal{G}} |\log \epsilon|.$$

By applying Young's inequality $a + b \geq Ca^{3/4}b^{1/4}$, we obtain

$$\begin{aligned} MC_{\mathcal{G}} |\log \epsilon| &\geq C\epsilon^{-1/2} h(\epsilon) \int_{R_1^\epsilon} |\nabla d_\epsilon|^{3/2} \psi^{1/4}(d_\epsilon) d\mathcal{H}^1 \\ &= C\epsilon^{-1/2} h(\epsilon) \int_{R_1^\epsilon} |\nabla G(d_\epsilon)|^{3/2} d\mathcal{H}^1. \end{aligned} \quad (55)$$

Fix a 1-cell K of \mathcal{G}^ϵ . We control the oscillations of $G(d_\epsilon)$ over K thanks to the Sobolev embedding $W^{1,3/2}(K) \hookrightarrow C^0(K)$ and (55):

$$\begin{aligned} \left(\text{osc}_K G(d_\epsilon) \right)^{3/2} &\leq Ch^{1/2}(\epsilon) \int_K |\nabla G(d_\epsilon)|^{3/2} d\mathcal{H}^1 \\ &= C\epsilon^{1/2} h^{-1/2}(\epsilon) |\log \epsilon|. \end{aligned}$$

In view of (54), we obtain

$$\text{osc}_{R_1^\epsilon} G(d_\epsilon) \rightarrow 0$$

as $\epsilon \rightarrow 0$. The function G is a continuous and strictly increasing, so G has a continuous inverse. This implies

$$\text{osc}_{R_1^\epsilon} d_\epsilon \rightarrow 0 \quad (56)$$

as $\epsilon \rightarrow 0$. On the other hand, (M_ε), (G₃) and (54) yield

$$\int_K \psi(d_\epsilon) d\mathcal{H}^1 \leq \frac{1}{h(\epsilon)} \int_{R_1^\epsilon} f(u_\epsilon) d\mathcal{H}^1 \rightarrow 0 \quad (57)$$

as $\epsilon \rightarrow 0$, for any 1-cell K of \mathcal{G}_ϵ . As we will see in a moment, this implies

$$\sup_K \int_K d_\epsilon d\mathcal{H}^1 \rightarrow 0. \quad (58)$$

Combining (58) with (56), we conclude that d_ϵ converges uniformly to 0 as $\epsilon \rightarrow 0$.

Now, we check that (58) holds. There exists a constant $\lambda > 0$ such that

$$\|d_\epsilon\|_{L^\infty(\Omega)} \leq \lambda$$

(this follows from the uniform L^∞ -estimate for u_ϵ , (48)). For any $\delta \in (0, \lambda)$, set

$$\psi_*(\delta) := \inf_{\delta \leq s \leq \lambda} \psi(s) > 0.$$

Then,

$$\begin{aligned} \frac{\mathcal{H}^1(\{d_\epsilon \geq \delta\} \cap K)}{\mathcal{H}^1(K)} \psi_*(\delta) &\leq \frac{1}{\mathcal{H}^1(K)} \int_{\{d_\epsilon \geq \delta\} \cap K} \psi(d_\epsilon) d\mathcal{H}^1 \\ &\leq \int_K \psi(d_\epsilon) d\mathcal{H}^1. \end{aligned} \quad (59)$$

Thus, for any 1-cell K , we have

$$\begin{aligned} 0 \leq \int_K d_\epsilon d\mathcal{H}^1 &= \frac{1}{\mathcal{H}^1(K)} \int_{\{d_\epsilon \leq \delta\} \cap K} d_\epsilon d\mathcal{H}^1 + \frac{1}{\mathcal{H}^1(K)} \int_{\{d_\epsilon \geq \delta\} \cap K} d_\epsilon d\mathcal{H}^1 \\ &\leq \frac{\mathcal{H}^1(\{d_\epsilon \leq \delta\} \cap K)}{\mathcal{H}^1(K)} \delta + \frac{\mathcal{H}^1(\{d_\epsilon \geq \delta\} \cap K)}{\mathcal{H}^1(K)} \lambda \\ &\stackrel{(59)}{\leq} \delta + \frac{\lambda}{\psi_*(\delta)} \int_K \psi(d_\epsilon) d\mathcal{H}^1 \\ &\stackrel{(57)}{\leq} \delta + \frac{C\lambda}{\psi_*(\delta)} \epsilon^2 h^{-1}(\epsilon) |\log \epsilon|. \end{aligned}$$

We pass to the limit first as $\epsilon \rightarrow 0$, then as $\delta \rightarrow 0$. Using (54), we deduce (58). \square

Remark 39 As a byproduct of the proof, under the assumptions of Lemma 38 the following property holds. For any $\delta > 0$, there exists a positive number ϵ_1 which only depend on M , κ , $C_{\mathcal{G}}$ and the potential f , such that

$$\text{dist}(u_\epsilon(x), \mathcal{N}) \leq \delta$$

for any $0 < \epsilon \leq \epsilon_1$ and any $x \in R_1^\epsilon$. Here M , κ and $C_{\mathcal{G}}$ are given respectively by (M_ϵ) , (48), and Definition 36.

3.4 Construction of v_ϵ and φ_ϵ

First, we construct the approximating map $v_\epsilon: \partial B_1 \rightarrow \mathcal{N}$.

Lemma 40 *Assume that (M_ϵ) , (C_ϵ) hold. There exists $\epsilon_1 > 0$ such that, for any $0 < \epsilon \leq \epsilon_1$, there exists a map $v_\epsilon \in H^1(\partial B_1, \mathcal{N})$ which satisfy (49),*

$$v_\epsilon(x) = \varrho(u_\epsilon(x)) \quad \text{and} \quad |u_\epsilon(x) - v_\epsilon(x)| \leq \delta_0 \quad (60)$$

for every $x \in R_1^\epsilon$.

Proof. To construct v_ϵ , we take a family $\mathcal{G} = \{\mathcal{G}^\epsilon\}_{\epsilon>0}$ of grids of size

$$h(\epsilon) := \epsilon^{1/2} |\log \epsilon| \quad (61)$$

(such a family exists by Lemma 37). Condition (54) is satisfied for $\alpha = 1/2$, so by Lemma 38 there exists $\epsilon_1 > 0$ such that

$$\text{dist}(u_\epsilon(x), \mathcal{N}) \leq \delta_0 \quad \text{for any } \epsilon \in (0, \epsilon_1) \text{ and any } x \in R_1^\epsilon. \quad (62)$$

The constant δ_0 is given by Lemma 14. In particular, the formula

$$v_\epsilon(x) := \varrho(u_\epsilon(x)) \quad \text{for all } x \in R_1^\epsilon$$

defines a function $v_\epsilon \in H^1(R_1^\epsilon, \mathbf{S}_0)$ which satisfies (60).

To extend v_ϵ inside each 2-cell, we use Lemma 27. Fix a 2-cell K of \mathcal{G}_ϵ . Since we have assumed that (C_ϵ) holds, $v_{\epsilon|_{\partial K}}$ is homotopically trivial. Therefore, Lemma 27 and (G_1) imply that there exists $v_{\epsilon,K} \in H^1(K, \mathcal{N})$ such that $v_{\epsilon,K} = v_\epsilon$ on ∂K and

$$\int_K |\nabla v_{\epsilon,K}|^2 d\mathcal{H}^2 \leq Ch(\epsilon) \int_{\partial K} |\nabla v_\epsilon|^2 d\mathcal{H}^1.$$

Define $v_\epsilon: \partial B_1 \rightarrow \mathcal{N}$ by setting $v_\epsilon := v_{\epsilon,K}$ on each 2-cell K . This function agrees with $v_{\epsilon|_{R_1^\epsilon}}$ previously defined by (60), hence the notation is not ambiguous. Moreover, $v_\epsilon \in H^1(\partial B_1, \mathcal{N})$ and

$$\begin{aligned} \int_{\partial B_1} |\nabla v_\epsilon|^2 d\mathcal{H}^2 &\leq \sum_K \int_K |\nabla v_\epsilon|^2 d\mathcal{H}^2 \leq Ch(\epsilon) \sum_K \int_{\partial K} |\nabla v_\epsilon|^2 d\mathcal{H}^1 \\ &\stackrel{(G_2)}{\leq} Ch(\epsilon) \int_{R_1^\epsilon} |\nabla v_\epsilon|^2 d\mathcal{H}^1 \stackrel{(60)}{\leq} Ch(\epsilon) \int_{R_1^\epsilon} |\nabla u_\epsilon|^2 d\mathcal{H}^1 \\ &\stackrel{(G_3)}{\leq} CE_\epsilon(u_\epsilon, \partial B_1), \end{aligned}$$

where the sum runs over all the 2-cells K of \mathcal{G}_ϵ . Thus v_ϵ satisfies (49), so the lemma is proved. \square

Now, we construct the interpolation map $\varphi_\epsilon: \partial B_1 \rightarrow \mathbf{S}_0$.

Lemma 41 *Assume that the conditions (M_ϵ) , (C_ϵ) are satisfied. Then, for any $0 < \epsilon \leq \epsilon_1$ there exists a map $\varphi_\epsilon \in H^1(B_1 \setminus B_{1-h(\epsilon)}, \mathbf{S}_0)$ which satisfies (46) and (50).*

Proof. Set $A_\epsilon := B_1 \setminus B_{1-h(\epsilon)}$. The grid \mathcal{G}^ϵ on ∂B_1 induces a grid $\hat{\mathcal{G}}^\epsilon$ on A_ϵ , whose cells are

$$\hat{K} := \left\{ x \in \mathbb{R}^3 : 1 - h(\epsilon) \leq |x| \leq 1, \frac{x}{|x|} \in K \right\} \quad \text{for each } K \in \mathcal{G}^\epsilon.$$

If K is a cell of dimension j , then \hat{K} has dimension $j + 1$. For $j \in \{0, 1, 2\}$, we call \hat{R}_j^ϵ the union of all the $(j + 1)$ -cells of $\hat{\mathcal{G}}^\epsilon$.

The function φ_ϵ is constructed as follows. If $x \in \partial B_1 \cup \partial B_{1-h(\epsilon)}$, then $\varphi_\epsilon(x)$ is determined by (46). If $x \in \hat{R}_1^\epsilon \cup \hat{R}_1^\epsilon$, we define $\varphi_\epsilon(x)$ by linear interpolation:

$$\varphi_\epsilon(x) := \frac{1-|x|}{h(\epsilon)} u_\epsilon \left(\frac{x}{|x|} \right) + \frac{h(\epsilon)-1+|x|}{h(\epsilon)} v_\epsilon \left(\frac{x}{|x|} \right). \quad (63)$$

For any 3-cell \hat{K} of \mathcal{G}_ϵ , we extend homogeneously (of degree 0) the function $\varphi_{\epsilon|_{\partial \hat{K}}}$ on \hat{K} . This gives a map $\varphi_\epsilon \in H^1(\hat{K})$, because \hat{K} is a cell of dimension 3. As a result, we obtain a map $\varphi_\epsilon \in H^1(A_\epsilon, \mathbf{S}_0)$ which satisfies (46).

To complete the proof of the lemma, we only need to bound the energy of φ_ϵ . Since φ_ϵ has been obtained by homogeneous extension on cells of size $h(\epsilon)$, we have

$$\begin{aligned} E_\epsilon(\varphi_\epsilon, A_\epsilon) &\stackrel{(G_1)}{\leq} Ch(\epsilon) \sum_{\hat{K}} E_\epsilon(\varphi_\epsilon, \partial \hat{K}) \\ &\stackrel{(G_2)}{\leq} Ch(\epsilon) \left\{ E_\epsilon(u_\epsilon, \partial B_1) + E_\epsilon(v_\epsilon, \partial B_{1-h(\epsilon)}) + E_\epsilon(\varphi_\epsilon, \hat{R}_1^\epsilon) \right\}, \end{aligned} \quad (64)$$

where the sum runs over all the 3-cells \hat{K} of \mathcal{G}^ϵ . To conclude the proof, we invoke the following fact.

Lemma 42 *We have*

$$E_\epsilon(\varphi_\epsilon, \hat{R}_1^\epsilon) \leq C (\epsilon^2 h^{-2}(\epsilon) + 1) E_\epsilon(u_\epsilon, \partial B_1).$$

From (64) and Lemma 42 we get

$$\begin{aligned} E_\epsilon(\varphi_\epsilon, A_\epsilon) &\leq Ch(\epsilon) \left\{ (\epsilon^2 h^{-2}(\epsilon) + 1) E_\epsilon(u_\epsilon, \partial B_1) + E_\epsilon(v_\epsilon, \partial B_{1-h(\epsilon)}) \right\} \\ &\stackrel{(49)}{\leq} Ch(\epsilon) (\epsilon^2 h^{-2}(\epsilon) + 1) E_\epsilon(u_\epsilon, \partial B_1) \end{aligned}$$

and, thanks to our choice (61) of $h(\epsilon)$, we conclude that (50) holds, so Lemma 41 is proved. \square

Remark 43 We can keep track of the constants in the proof of Lemmas 40 and 41. By doing so, one sees that the constant C given by Proposition 33 (Equations (49) and (50)) only depends on $C_{\mathcal{G}}$ and the potential f .

Proof of Lemma 42. We consider first the contribution of the potential energy. Thanks to (F₃), (63) and (60), we deduce that

$$f(\varphi_\epsilon(x)) \leq C \left(\frac{1-|x|}{h(\epsilon)} \right)^2 (f \circ u_\epsilon) \left(\frac{x}{|x|} \right) \quad \text{for } x \in \hat{R}_1^\epsilon.$$

By integration, this gives

$$\int_{\hat{R}_1^\epsilon} f(\varphi_\epsilon) d\mathcal{H}^2 \leq Ch(\epsilon) \int_{R_1^\epsilon} f(u_\epsilon) d\mathcal{H}^2. \quad (65)$$

Now, we consider the elastic part of the energy. Using again (63), we have

$$\int_{\hat{R}_1^\epsilon} |\nabla \varphi_\epsilon|^2 \, d\mathcal{H}^2 \leq Ch^{-1}(\epsilon) \int_{R_1^\epsilon} |u_\epsilon - v_\epsilon|^2 \, d\mathcal{H}^1. \quad (66)$$

The condition (F₂) on the Landau-de Gennes potential, together with (60), implies

$$\int_{R_1^\epsilon} |u_\epsilon - v_\epsilon|^2 \, d\mathcal{H}^1 \leq C \int_{R_1^\epsilon} f(u_\epsilon) \, d\mathcal{H}^1. \quad (67)$$

Using (65), (66) and (67), we deduce that

$$E_\epsilon(\varphi_\epsilon, \hat{R}_1^\epsilon) \leq C (h^{-1}(\epsilon) + \epsilon^{-2}h(\epsilon)) \int_{R_1^\epsilon} f(u_\epsilon) \, d\mathcal{H}^1.$$

Because of Condition (G₄) in Definition 36, we obtain

$$E_\epsilon(\varphi_\epsilon, \hat{R}_1^\epsilon) \leq C (h^{-2}(\epsilon) + \epsilon^{-2}) \int_{\partial B_1} f(u_\epsilon) \, d\mathcal{H}^2$$

so the lemma follows easily. \square

3.5 Logarithmic bounds for the energy imply (C_ε)

The aim of this subsection is to establish the following lemma, and conclude the proof of Proposition 33.

Lemma 44 *There exists $\eta_1 = \eta_1(\mathcal{N}, C_{\mathcal{G}}, M, \epsilon_1)$ such that, if $0 < \epsilon < \epsilon_1$ and u_ϵ satisfies (M_ε), (48) but not (C_ε), then*

$$E_\epsilon(u_\epsilon, \partial B_1) \geq \eta_1 |\log \epsilon|.$$

Once Lemma 44 is proved, Proposition 33 follows in an elementary way.

Proof of Proposition 33. Choose $\eta_0 := \eta_1/2$. If u_ϵ satisfies (47) with this choice of η_0 and (48), then it must satisfy Condition (C_ε), otherwise Lemma 44 would yield a contradiction. Then, the proposition follows by Lemmas 40 and 41. \square

Proof of Lemma 44. By assumption, Condition (C_ε) is not satisfied, so there exists a 2-cell $K^* \in \mathcal{G}^\epsilon$ such that $\varrho \circ u_{\epsilon|_{\partial K^*}}$ is non-trivial. By Definition 36, there exists a bilipschitz homeomorphism $\varphi: K_* \rightarrow B_{h(\epsilon)}$ which satisfies (G₁). Therefore, up to composition with φ we can assume that K_* is a 2-dimensional disk, $K_* = B_{h(\epsilon)}^2$. Lemma 38 implies that $u_\epsilon(x) \notin \mathcal{C}_0$ for every $x \in \partial K_*$, for $0 < \epsilon \leq \epsilon_1$. Then, by applying Corollary 16 we deduce

$$E_\epsilon(u_\epsilon, K_*) + Ch(\epsilon)E_\epsilon(u_\epsilon, \partial K_*) \geq \kappa_* \phi_0^2(u_\epsilon, \partial K_*) \log \frac{h(\epsilon)}{\epsilon} - C$$

Notice that $\phi_0(u_\epsilon, \partial K_*) \geq 1/2$ if δ_0 is small enough, because of (62). On the other hand, condition (G₃) yields

$$E_\epsilon(u_\epsilon, K_*) + Ch(\epsilon)E_\epsilon(u_\epsilon, \partial K_*) \leq CE_\epsilon(u_\epsilon, \partial B_1).$$

Due to the previous inequalities and (61), we infer

$$E_\epsilon(u_\epsilon, \partial B_1) \geq C \left\{ \log \left(\epsilon^{-1/2} |\log \epsilon| \right) - 1 \right\} \geq C \left(\frac{1}{2} |\log \epsilon| - 1 \right)$$

for all $0 < \epsilon \leq \epsilon_1 < 1$, so the lemma follows. \square

4 Compactness of Landau-de Gennes minimizers: Proof of Theorem 1

4.1 Concentration of the energy: Proof of Proposition 8

The whole section aims at proving Theorem 1. In this subsection, we prove Proposition 8 by applying the results of Section 3.

Let η_0, ϵ_1 be given by Proposition 33, and set $\epsilon_0 := \epsilon_1 \theta$. Throughout the section, the same symbol C will be used to denote several different constants, possibly depending on θ and ϵ_1 , but not on ε, R . To simplify the notation, from now on we assume that $x_0 = 0$. For a fixed $0 < \varepsilon \leq \epsilon_0 R$, define the set

$$D^\varepsilon := \left\{ r \in (\theta R, R) : E_\varepsilon(Q_\varepsilon, \partial B_r) \leq \frac{2\eta}{1-\theta} \log \frac{R}{\varepsilon} \right\}.$$

The elements of D^ε are the “good radii”, i.e. $r \in D^\varepsilon$ means that we have a control on the energy on the sphere of radius r . Assume that the condition (10) is satisfied. Then, by an average argument we deduce that

$$\mathcal{H}^1(D^\varepsilon) \geq \frac{(1-\theta)R}{2}. \quad (68)$$

For any $r \in D^\varepsilon$ we have

$$E_\varepsilon(Q_\varepsilon, \partial B_r) \leq \frac{2\eta}{1-\theta} \left(\log \frac{r}{\varepsilon} - \log \theta \right),$$

since $R \leq \theta^{-1}r$. By choosing η small enough, we can assume that

$$E_\varepsilon(Q_\varepsilon, \partial B_r) \leq \eta_0 \log \frac{r}{\varepsilon} \quad \text{for any } r \in D^\varepsilon \text{ and } 0 < \varepsilon \leq \epsilon_0 R. \quad (69)$$

In particular, our choice of η depends on $\epsilon_1, \eta_0, \theta$.

Lemma 45 *For any $0 < \varepsilon \leq \epsilon_0 R$ and any $r \in D^\varepsilon$, there holds*

$$E_\varepsilon(Q_\varepsilon, B_r) \leq CR \left(E_\varepsilon^{1/2}(Q_\varepsilon, \partial B_r) + 1 \right).$$

A similar inequality was obtained by Hardt, Kinderlehrer and Lin in [37, Lemma 2.3, Equation (2.3)], and it played a crucial role in the proof of their energy improvement result.

Proof of Lemma 45. To simplify the notations, we get rid of r by a scaling argument. Set $\epsilon := \varepsilon/r$, and define the function $u_\epsilon: B_1 \rightarrow \mathbf{S}_0$ by

$$u_\epsilon(x) := Q_\varepsilon(rx) \quad \text{for all } x \in B_1.$$

Notice that $\epsilon \leq \epsilon_1$, since $\varepsilon \leq \epsilon_0 R$ and $\theta R < r$. The lemma will be proved once we show that

$$E_\epsilon(u_\epsilon, B_1) \leq CE_\epsilon^{1/2}(u_\epsilon, \partial B_1) + 1 \quad (70)$$

(multiplying both sides of (70) by $r \leq R$ yields the lemma). Since we have assumed that $r \in D^\varepsilon$ we have, by (69),

$$E_\epsilon(u_\epsilon, \partial B_1) \leq \eta_0 |\log \epsilon|.$$

Moreover, u_ϵ satisfies the L^∞ -bound (48), due to (H). Therefore, we can apply Proposition 33 and find $v_\epsilon \in H^1(\partial B_1, \mathcal{N})$, $\varphi_\epsilon \in H^1(A_\epsilon, \mathbf{S}_0)$ which satisfy

$$\begin{aligned} \varphi_\epsilon(x) = u_\epsilon(x) \quad \text{and} \quad \varphi_\epsilon(x - h(\epsilon)x) = v_\epsilon(x) \quad \text{for } \mathcal{H}^2\text{-a.e. } x \in \partial B_1 \\ \int_{\partial B_1} |\nabla v_\epsilon|^2 d\mathcal{H}^2 \leq CE_\epsilon(u_\epsilon, \partial B_1), \end{aligned} \quad (71)$$

$$E_\epsilon(\varphi_\epsilon, A_\epsilon) \leq Ch(\epsilon)E_\epsilon(u_\epsilon, \partial B_1). \quad (72)$$

Here $h(\epsilon) := \epsilon^{1/2} |\log \epsilon|$ and $A_\epsilon := B_1 \setminus B_{1-h(\epsilon)}$. By applying Lemma 26 to v_ϵ , we find a map $w_\epsilon \in H^1(B_1, \mathcal{N})$ such that $w_\epsilon|_{\partial B_1}$ and

$$\int_{B_1} |\nabla w_\epsilon|^2 \leq C \left\{ \int_{\partial B_1} |\nabla v_\epsilon|^2 d\mathcal{H}^2 \right\}^{1/2} \stackrel{(71)}{\leq} CE_\epsilon^{1/2}(u_\epsilon, \partial B_1). \quad (73)$$

Now, define the function $\tilde{w}_\epsilon: B_1 \rightarrow \mathbf{S}_0$ by

$$\tilde{w}_\epsilon(x) := \begin{cases} \varphi_\epsilon(x) & \text{for } x \in A_\epsilon \\ w_\epsilon\left(\frac{x}{1-h(\epsilon)}\right) & \text{for } x \in B_{1-h(\epsilon)}. \end{cases}$$

The energy of \tilde{w}_ϵ in the spherical shell A_ϵ is controlled by (72). Due to our choice of the parameter $h(\epsilon)$, we deduce that

$$E_\epsilon(\tilde{w}_\epsilon, A_\epsilon) \leq 1$$

provided that ϵ_1 is small enough. Combining this with (73), we obtain

$$E_\epsilon(\tilde{w}_\epsilon, B_1) \leq CE_\epsilon^{1/2}(u_\epsilon, \partial B_1) + 1.$$

But \tilde{w}_ϵ is an admissible comparison function for u_ϵ on B_1 , because $\tilde{w}_\epsilon = u_\epsilon$ on ∂B_1 . Thus, the minimality of u_ϵ implies (70). \square

Lemma 45 can be seen as a non-linear differential inequality for the function $y: r \in (\theta R, R) \mapsto E_\varepsilon(Q, B_r)$. The conclusion of the proof of Proposition 8 follows now by a simple ODE argument.

Lemma 46 *Let α, β be two positive numbers. Let $y \in W^{1,1}([r_0, r_1], \mathbb{R})$ be a function such that $y' \geq 0$ a.e., and let $D \subseteq (r_0, r_1)$ be a measurable set such that $\mathcal{H}^1(D) \geq (r_1 - r_0)/2$. If the function y satisfies*

$$y(r) \leq \alpha y'(r)^{1/2} + \beta \quad \text{for } \mathcal{H}^1\text{-a.e. } r \in D, \quad (74)$$

then there holds

$$y(r_0) \leq \beta + \frac{2\alpha^2}{r_1 - r_0}.$$

Proof. If there exists a point $r_* \in (r_0, r_1)$ such that $y(r_*) \leq \beta$, then $y(r_0) \leq \beta$ (because y is an increasing function) and the lemma is proved. Therefore, we can assume WLOG that $y - \beta > 0$ on (r_0, r_1) . Then, Equation (74) and the monotonicity of y imply

$$\frac{y'(r)}{(y(r) - \beta)^2} \geq \alpha^{-2} \mathbb{1}_D(r)$$

for a.e. $r \in (r_0, r_1)$, where $\mathbb{1}_D$ is the characteristic function of D (that is, $\mathbb{1}_D(r) = 1$ if $r \in D$ and $\mathbb{1}_D(r) = 0$ otherwise). By integrating this inequality on $(0, r)$, we deduce

$$\frac{1}{y(r_0) - \beta} - \frac{1}{y(r) - \beta} \geq \alpha^{-2} \mathcal{H}^1((r_0, r) \cap D)$$

for any $r \in (r_0, r_1)$. Since we have assumed that $\mathcal{H}^1(D) \geq (r_1 - r_0)/2$, we obtain

$$\mathcal{H}^1((r_0, r) \cap D) \geq \left(r - \frac{r_0 + r_1}{2}\right)^+ := \max\left\{r - \frac{r_0 + r_1}{2}, 0\right\}$$

so, via an algebraic manipulation, we get

$$y(r) \geq \beta + \frac{y(r_0) - \beta}{1 - \alpha^{-2}(r - (r_0 + r_1)/2)^+(y(r_0) - \beta)}$$

for any $r \in (r_0, r_1)$. Since y is well-defined (and finite) up to $r = r_1$, there must be

$$1 - \frac{r_1 - r_0}{2\alpha^2} (y(r_0) - \beta) > 0,$$

whence the lemma follows. \square

Conclusion of the proof of Proposition 8. Thanks to Lemma 45 and (68), we can apply Lemma 46 to the function $y(r) := E_\varepsilon(Q_\varepsilon, B_r)$, for $r \in (\theta R, R)$, and the set $D := D^\varepsilon$. This yields

$$E_\varepsilon(Q_\varepsilon, B_{\theta R}) \leq CR,$$

so the proposition is proved. \square

4.2 Uniform energy bounds imply convergence to a harmonic map

In this subsection, we suppose that minimizers satisfy

$$E_\varepsilon(Q_\varepsilon, B_R(x_0)) \leq CR \quad (75)$$

on a ball $B_r(x_0) \subset\subset \Omega$. In interesting situations, where line defects appear, such an estimate is not valid over the whole of Ω but it is satisfied locally, away from a singular set. The main result of this subsection is the following:

Proposition 47 *Assume that $\overline{B}_R(x_0) \subseteq \Omega$ and that (75) is satisfied for some $R, C > 0$. Fix $0 < \theta < 1$. Then, there exist a subsequence $\varepsilon_n \searrow 0$ and a map $Q_0 \in H^1(B_{\theta R}(x_0), \mathcal{N})$ such that*

$$Q_{\varepsilon_n} \rightarrow Q_0 \quad \text{strongly in } H^1(B_{\theta R}(x_0), \mathbf{S}_0).$$

The map Q_0 is minimizing harmonic in $B_{\theta R}(x_0)$, that is, for any comparison map $Q \in H^1(B_{\theta R}(x_0), \mathcal{N})$ such that $Q = Q_0$ on $\partial B_{\theta R}(x_0)$ there holds

$$\frac{1}{2} \int_{B_{\theta R}(x_0)} |\nabla Q_0|^2 \leq \frac{1}{2} \int_{B_{\theta R}(x_0)} |\nabla Q|^2.$$

In general, we cannot expect the map Q_0 to be smooth (see the example of Section 7). In contrast, by Schoen and Uhlenbeck's partial regularity result [61, Theorem II] we know that there exists a finite set $\mathcal{S}_{\text{pts}} \subseteq B_{\theta R}(x_0)$ such that Q_0 is smooth on $B_{\theta R}(x_0) \setminus \mathcal{S}_{\text{pts}}$. Accordingly, the sequence $\{Q_{\varepsilon_n}\}$ will not converge uniformly to Q_0 on the whole of $B_{\theta R}(x_0)$, in general, but we can prove the uniform convergence away from the singularities of Q_0 .

Proposition 48 *Let $K \subseteq B_{\theta R}(x_0)$ be such that Q_0 is smooth on the closure of K . Then $Q_{\varepsilon_n} \rightarrow Q_0$ uniformly on K .*

The asymptotic behaviour of minimizers of the Landau-de Gennes functional, in the bounded-energy regime (75), was already studied by Majumdar and Zarnescu in [52]. In that paper, H^1 -convergence to a harmonic map and local uniform convergence away from the singularities of Q_0 were already proven. However, in our case some extra care must be taken, because of the local nature of our assumption (75).

Proof of Proposition 47. Up to a translation, we assume that $x_0 = 0$. In view of (75), there exists a subsequence $\varepsilon_n \searrow 0$ and a map $Q_0 \in H^1(B_R, \mathbf{S}_0)$ such that

$$Q_{\varepsilon_n} \rightarrow Q_0 \quad \text{weakly in } H^1(B_R, \mathbf{S}_0), \text{ strongly in } L^2(B_R, \mathbf{S}_0) \text{ and a.e.}$$

Using Fatou's lemma and (75) again, we also see that

$$\int_{B_R} f(Q_0) \leq \liminf_{n \rightarrow +\infty} \varepsilon_n^2 E_{\varepsilon_n}(Q_{\varepsilon_n}, B_R) \leq \liminf_{n \rightarrow +\infty} \varepsilon_n^2 CR = 0,$$

hence $f(Q_0) = 0$ a.e. or, equivalently,

$$Q_0(x) \in \mathcal{N} \quad \text{for a.e. } x \in B_1.$$

By means of a comparison argument, we will prove that Q_{ε_n} 's actually converge *strongly* in H^1 . Fatou's lemma combined with (75) gives

$$\int_{\theta R}^R \liminf_{n \rightarrow +\infty} E_{\varepsilon_n}(Q_{\varepsilon_n}, \partial B_r) dr \leq \liminf_{n \rightarrow +\infty} E_{\varepsilon_n}(Q_{\varepsilon_n}, B_R \setminus B_{\theta R}) \leq CR. \quad (76)$$

Therefore, the set

$$\left\{ r \in (0, R] : \liminf_{n \rightarrow +\infty} E_{\varepsilon_n}(Q_{\varepsilon_n}, \partial B_r) > \frac{2C}{1-\theta} \right\}$$

must have length $\leq (1-\theta)R/2$, otherwise (76) would be violated. In particular, there exist a radius $r \in (\theta R, R]$ and a relabeled subsequence such that

$$E_{\varepsilon_n}(Q_{\varepsilon_n}, \partial B_r) \leq \frac{2C}{1-\theta}.$$

For ease of notation we scale the variables, setting $\epsilon_n := \varepsilon_n/r$,

$$u_n(x) := Q_{\varepsilon_n}(rx) \quad \text{and} \quad u_*(x) := Q_0(rx) \quad \text{for } x \in B_1.$$

The scaled maps satisfy

$$u_n \rightarrow u_* \quad \text{weakly in } H^1(B_1, \mathbf{S}_0), \text{ strongly in } L^2(B_1, \mathbf{S}_0) \text{ and a.e.,} \quad (77)$$

$$u_*(x) \in \mathcal{N} \quad \text{for a.e. } x \in B_1, \quad (78)$$

$$E_{\epsilon_n}(u_n, \partial B_1) \leq C. \quad (79)$$

By (77) and the trace theorem, $u_n \rightharpoonup u_*$ weakly in $H^{1/2}(\partial B_1, \mathbf{S}_0)$ and hence, by compact embedding, strongly in $L^2(\partial B_1, \mathbf{S}_0)$. Moreover, by (79) $u_n \rightharpoonup u_*$ weakly in $H^1(\partial B_1, \mathbf{S}_0)$, so

$$\frac{1}{2} \int_{\partial B_1} |\nabla u_*|^2 d\mathcal{H}^2 \leq \limsup_{n \rightarrow +\infty} E_{\epsilon_n}(u_n, \partial B_r) \leq C. \quad (80)$$

We are going to apply Proposition 35 to interpolate between u_n and u_* . Set $\sigma_n := \|u_n - u_*\|_{L^2(\partial B_1)}$. Then $\sigma_n \rightarrow 0$ and

$$\int_{\partial B_1} \left\{ |\nabla u_n|^2 + \frac{1}{\epsilon_n} f(u_n) + |\nabla u_*|^2 + \frac{|u_n - u_*|^2}{\sigma_n} \right\} d\mathcal{H}^2 \leq C,$$

because of (79), (80). Moreover, the L^∞ -estimate (48) is satisfied by Lemma 21. Thus, Proposition 35 applies. We find a positive sequence $\nu_n \rightarrow 0$ and functions $\varphi_n \in H^1(B_1 \setminus B_{1-\nu_n}, \mathbf{S}_0)$ which satisfy

$$\varphi_n(x) = u_n(x), \quad \varphi_n(x - \nu_n x) = u_*(x)$$

for \mathcal{H}^2 -a.e. $x \in \partial B_1$ and

$$E_{\epsilon_n}(\varphi_n, B_1 \setminus B_{1-\nu_n}) \leq C\nu_n. \quad (81)$$

Now, let $w_* \in H^1(B_1, \mathcal{N})$ be a minimizing harmonic extension of $u_*|_{\partial B_1}$, i.e.

$$\frac{1}{2} \int_{B_1} |\nabla w_*|^2 \leq \frac{1}{2} \int_{B_1} |\nabla w|^2 \quad (82)$$

for any $w \in H^1(B_1, \mathcal{N})$ such that $w|_{\partial B_1} = u_*|_{\partial B_1}$. Such a function exists by classical results (see e.g. [62, Proposition 3.1]). Define $w_n: B_1 \rightarrow \mathbf{S}_0$ by

$$w_n(x) := \begin{cases} \varphi_n(x) & \text{if } x \in B_1 \setminus B_{1-\nu_n} \\ w_*\left(\frac{x}{1-\nu_n}\right) & \text{if } x \in B_{1-\nu_n}. \end{cases}$$

The map w_n is an admissible comparison map for u_n , i.e. $w_n \in H^1(B_1, \mathbf{S}_0)$ and $w_n|_{\partial B_1} = u_n|_{\partial B_1}$. Hence,

$$E_{\epsilon_n}(u_n, B_1) \leq E_{\epsilon_n}(w_n, B_1) = \frac{1-\nu_n}{2} \int_{B_1} |\nabla w_*|^2 + E_{\epsilon_n}(w_n, B_1 \setminus B_{1-\nu_n}).$$

When we take the limit as $n \rightarrow +\infty$, $\nu_n \rightarrow 0$ and the energy in the shell $B_1 \setminus B_{1-\nu_n}$ converges to 0, due to (81). Keeping (77) in mind, we obtain

$$\begin{aligned} \frac{1}{2} \int_{B_1} |\nabla u_*|^2 &\leq \liminf_{n \rightarrow +\infty} \frac{1}{2} \int_{B_1} |\nabla u_n|^2 \leq \limsup_{n \rightarrow +\infty} \frac{1}{2} \int_{B_1} |\nabla u_n|^2 \\ &\leq \limsup_{n \rightarrow +\infty} E_{\epsilon_n}(u_n, B_1) \leq \frac{1}{2} \int_{B_1} |\nabla w_*|^2 \leq \frac{1}{2} \int_{B_1} |\nabla u_*|^2, \end{aligned}$$

where the last inequality follows by the minimality of w_* , (82). But this implies

$$\lim_{n \rightarrow +\infty} \frac{1}{2} \int_{B_1} |\nabla u_n|^2 = \frac{1}{2} \int_{B_1} |\nabla u_*|^2,$$

which yields the strong H^1 convergence $u_n \rightarrow u_*$, as well as

$$\lim_{n \rightarrow +\infty} \frac{1}{\epsilon_n} \int_{B_1} f(u_n) = 0. \quad (83)$$

Moreover, u_* must be a minimizing harmonic map.

Scaling back to Q_{ϵ_n} , Q_0 , we have shown that $Q_{\epsilon_n} \rightarrow Q_0$ strongly in $H^1(B_r, \mathbf{S}_0)$ and that Q_0 is minimizing harmonic in B_r , where $r \geq \theta R$. In particular, the proposition holds true. \square

Once Proposition 47 is established, Proposition 48 can be proved arguing as in Majumdar and Zarnescu's paper [52]. The arguments in [52, Proposition 4] rely on the condition

$$\lim_{n \rightarrow +\infty} \frac{1}{\epsilon_n^2} \int_{B_{\theta R}(x_0)} f(Q_{\epsilon_n}) = 0,$$

which follows as a byproduct of Equation (83) in the previous proof.

4.3 The singular set

In this subsection, we complete the proof of Theorem 1 by defining the singular set $\mathcal{S}_{\text{line}}$ and showing that it is a rectifiable set of finite length. For each $0 < \varepsilon < 1$, define the measure μ_ε by

$$\mu_\varepsilon(B) := \frac{E_\varepsilon(Q_\varepsilon, B)}{|\log \varepsilon|} \quad \text{for } B \in \mathcal{B}(\overline{\Omega}). \quad (84)$$

In view of our main assumption (H), the measures $\{\mu_\varepsilon\}_{0 < \varepsilon < 1}$ have uniformly bounded mass. Therefore, we may extract a subsequence $\varepsilon_n \searrow 0$ such that

$$\mu_{\varepsilon_n} \rightharpoonup^* \mu_0 \quad \text{weakly}^* \text{ in } \mathcal{M}(\overline{\Omega}) := C(\overline{\Omega})'. \quad (85)$$

Let $\mathcal{S}_{\text{line}}$ be the support of μ_0 . By definition, $\mathcal{S}_{\text{line}}$ is a closed subset of $\overline{\Omega}$. Let η be given by Proposition 8, corresponding to the choice $\theta = 1/2$.

Lemma 49 *Let $x_0 \in \Omega$ and $R > 0$ be such that $\overline{B}_R(x_0) \subset \Omega$. If*

$$\mu_0(\overline{B}_R(x_0)) < \eta R \quad (86)$$

then

$$\mu_0(B_{R/2}(x_0)) = 0,$$

that is $B_{R/2}(x_0) \subseteq \Omega \setminus \mathcal{S}_{\text{line}}$.

Proof. In force of (85) and (86), we know that

$$\limsup_{n \rightarrow +\infty} \frac{E_{\varepsilon_n}(Q_{\varepsilon_n}, B_R(x_0))}{R \log(\varepsilon_n/R)} < \eta.$$

In particular, the assumption (10) is satisfied along the subsequence $\{\varepsilon_n\}$. Then, we can apply Proposition 8 with $\theta = 1/2$ and we obtain

$$E_{\varepsilon_n}(B_{R/2}(x_0)) \leq CR$$

for n large enough. Due to (85), we deduce

$$\mu_0(B_{R/2}(x_0)) \leq \liminf_{n \rightarrow +\infty} \mu_{\varepsilon_n}(B_{R/2}(x_0)) = 0. \quad \square$$

By the monotonicity formula (Lemma 24), for any $x \in \Omega$ the function

$$r \in (0, \text{dist}(x, \partial\Omega)) \mapsto \frac{\mu_0(\overline{B}_r(x))}{2r}$$

is non-decreasing, so the limit

$$\Theta(x) := \lim_{r \rightarrow 0^+} \frac{\mu_0(\overline{B}_r(x))}{2r} \quad (87)$$

exists. The function Θ is usually called the (1-dimensional) density of μ_0 (see [63, p. 10]).

Lemma 50 *For all $x \in \mathcal{S}_{\text{line}} \cap \Omega$, we have $\Theta(x) \geq \eta/2$.*

Proof. This follows immediately by Lemma 49. Indeed, if $x \in \mathcal{S}_{\text{line}} \cap \Omega$ then for any $r > 0$ we have $\mu_0(B_r(x)) > 0$, so Lemma 49 implies

$$\frac{\mu_0(\overline{B}_{2r}(x))}{4r} \geq \frac{\eta}{2}.$$

Passing to the limit as $r \rightarrow 0$, we conclude. \square

The strict positivity of Θ has remarkable consequences.

Proposition 51 *The set $\mathcal{S}_{\text{line}} \cap \Omega$ is countably \mathcal{H}^1 -rectifiable, and*

$$\mathcal{H}^1(\mathcal{S}_{\text{line}} \cap \Omega) < +\infty.$$

Moreover, there holds

$$(\mu_0 \llcorner \Omega)(B) = \int_{B \cap \mathcal{S}_{\text{line}} \cap \Omega} \Theta(x) \, d\mathcal{H}^1(x) \quad \text{for all } B \in \mathcal{B}(\overline{\Omega}).$$

Proof. Lemma 50, together with [63, Theorem 3.2.(i), Chapter 1] and (H), implies

$$\mathcal{H}^1(\mathcal{S}_{\text{line}} \cap \Omega) \leq 2\eta^{-1}\mu_0(\Omega) \leq 2\eta^{-1}M < +\infty.$$

Moreover, since the 1-dimensional density of $\mu_0 \llcorner \Omega$ exists and is essentially bounded away from zero, the support is a \mathcal{H}^1 -rectifiable set and $\mu_0 \llcorner \Omega$ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner (\mathcal{S}_{\text{line}} \cap \Omega)$. This fact was proved by Moore [55] and is a special case of Preiss' theorem [58, Theorem 5.3], which holds true for measures in \mathbb{R}^n having positive k -dimensional density, for any $k \leq n$. Thus, there exists a positive, \mathcal{H}^1 -integrable function $g: \Omega \rightarrow \mathbb{R}$ such that

$$(\mu_0 \llcorner \Omega)(B) = \int_{B \cap \mathcal{S}_{\text{line}} \cap \Omega} g(x) \, d\mathcal{H}^1(x)$$

for any $B \in \mathcal{B}(\overline{\Omega})$. By Besicovitch differentiation theorem, there holds

$$\lim_{r \rightarrow 0^+} \frac{\mu_0(\overline{B}_r(x))}{\mathcal{H}^1(B_r(x) \cap \mathcal{S}_{\text{line}})} = g(x)$$

for \mathcal{H}^1 -a.e. $x \in \mathcal{S}_{\text{line}} \cap \Omega$. On the other hand, because $\mathcal{S}_{\text{line}} \cap \Omega$ is rectifiable and has finite length, [30, Theorem 3.2.19] implies that

$$\lim_{r \rightarrow 0^+} \frac{\mathcal{H}^1(B_r(x) \cap \mathcal{S}_{\text{line}})}{2r} = 1$$

for \mathcal{H}^1 -a.e. $x \in \mathcal{S}_{\text{line}} \cap \Omega$. Combining these facts with (87), we obtain that $\Theta = g$ \mathcal{H}^1 -a.e. on $\mathcal{S}_{\text{line}} \cap \Omega$, so the proposition follows. \square

To complete the proof of Theorem 1, we check that Q_{ε_n} locally converge to a harmonic map, away from $\mathcal{S}_{\text{line}}$.

Proposition 52 *There exists a map $Q_0 \in H_{\text{loc}}^1(\Omega \setminus \mathcal{S}_{\text{line}}, \mathcal{N})$ such that, up to a relabeled subsequence,*

$$Q_{\varepsilon_n} \rightarrow Q_0 \quad \text{strongly in } H_{\text{loc}}^1(\Omega \setminus \mathcal{S}_{\text{line}}, \mathbf{S}_0).$$

The map Q_0 is minimizing harmonic on every ball $B \subset\subset \Omega \setminus \mathcal{S}_{\text{line}}$. Moreover, there exists a locally finite set $\mathcal{S}_{\text{pts}} \subseteq \Omega \setminus \mathcal{S}_{\text{line}}$ such that Q_0 is of class C^∞ on $\Omega \setminus (\mathcal{S}_{\text{line}} \cup \mathcal{S}_{\text{pts}})$, and

$$Q_{\varepsilon_n} \rightarrow Q_0 \quad \text{locally uniformly in } \Omega \setminus (\mathcal{S}_{\text{line}} \cup \mathcal{S}_{\text{pts}}).$$

Proof. Fix an open subset $K \subset\subset \Omega \setminus \mathcal{S}_{\text{line}}$. Combining Proposition 8 with a standard covering argument, we deduce that minimizers Q_ε satisfy

$$E_\varepsilon(Q_\varepsilon, K) \leq C = C(K),$$

therefore they are weakly compact in $H^1(K, \mathbf{S}_0)$. It follows from Proposition 47 that the convergence is strong, and any limit map Q_0 is locally minimizing harmonic. Then, on each ball $B \subset\subset \Omega \setminus \mathcal{S}_{\text{line}}$ there exists a finite set $X_B \subseteq B$ such that $Q_0 \in C^\infty(B \setminus X_B, \mathbf{S}_0)$, because of [61, Theorem II]. Therefore $Q_0 \in C^\infty(\Omega \setminus \mathcal{S}_{\text{line}} \cup \mathcal{S}_{\text{pts}})$, where $\mathcal{S}_{\text{pts}} := \cup_B X_B$ is locally finite in $\Omega \setminus \mathcal{S}_{\text{line}}$. The locally uniform convergence $Q_{\varepsilon_n} \rightarrow Q_0$ on $\Omega \setminus (\mathcal{S}_{\text{line}} \cup \mathcal{S}_{\text{pts}})$ follows from Proposition 48 and a covering argument. \square

4.4 The analysis near the boundary

Proposition 8, which is the key step in the proof of our main theorem, has been proven on balls contained in the domain. In this subsection, we aim at proving a similar result in case the ball intersects the boundary of Ω . For this purpose, we need an additional assumption on the behaviour of the boundary datum. Let Γ be a relatively open subset of $\partial\Omega$. We assume that

(H _{Γ}) For any $0 < \varepsilon < 1$, there holds $g_\varepsilon \in (H_{\text{loc}}^1 \cap L_{\text{loc}}^\infty)(\Gamma, \mathbf{S}_0)$. Moreover, for any $K \subset\subset \Gamma$ there exists a constant C_K such that

$$E_\varepsilon(g_\varepsilon, K) \leq C_K \quad \text{and} \quad \|g_\varepsilon\|_{L^\infty(K)} \leq C_K$$

for any $0 < \varepsilon < 1$.

For instance, the families of boundary data given by (8) and (9) satisfies Condition (H _{Γ}) on $\Gamma := \partial\Omega \setminus \Sigma$.

Proposition 53 *Assume that the conditions (H) and (H _{Γ}) hold. For any $0 < \theta < 1$ there exist positive numbers η , ϵ_0 and C such that, for any $x_0 \in \overline{\Omega}$, $R > 0$ satisfying $\overline{B}_R(x_0) \cap \partial\Omega \subseteq \Gamma$ and any $0 < \varepsilon \leq \epsilon_0 R$, if*

$$E_\varepsilon(Q_\varepsilon, B_R(x_0) \cap \Omega) \leq \eta R \log \frac{R}{\varepsilon} \tag{88}$$

then

$$E_\varepsilon(Q_\varepsilon, B_{\theta R}(x_0) \cap \Omega) \leq CR.$$

By a standard covering argument, we see that Proposition 53 implies the weak compactness of minimizers up to the boundary. More precisely, we have

Corollary 54 *Let Γ be a relatively open subset of $\partial\Omega$. Assume that the conditions (H) and (H_Γ) are satisfied. Then, there exist a subsequence $\varepsilon_n \searrow 0$, a closed set $\mathcal{S}_{\text{line}} \subseteq \overline{\Omega}$ and a map $Q_0 \in H_{\text{loc}}^1((\Omega \cup \Gamma) \setminus \mathcal{S}_{\text{line}}, \mathcal{N})$ which satisfy (i)–(iv) in Theorem 1 and*

$$Q_{\varepsilon_n} \rightharpoonup Q_0 \quad \text{weakly in } H_{\text{loc}}^1((\Omega \cup \Gamma) \setminus \mathcal{S}_{\text{line}}, \mathbf{S}_0).$$

The set $\mathcal{S}_{\text{line}}$ is again defined as the support of the measure μ_0 , where μ_0 is a weak * limit of $\{\mu_\varepsilon\}_{0 < \varepsilon < 1}$ in $C(\overline{\Omega})'$ and the μ_ε 's are given by (84). The proofs in Section 4.3 remain unchanged. We cannot expect strong H^1 convergence of minimizers up to the boundary, unless some additional assumption on the boundary datum is made. Moreover, the intersection $\mathcal{S}_{\text{line}} \cap \Gamma$ may be non-empty. An example is given in Section 5.4, Proposition 62.

Proof of Proposition 53. For the sake of simplicity, we assume that $x_0 = 0$ and set $F_\varepsilon(r) := E_\varepsilon(Q_\varepsilon, B_r \cap \Omega)$ for $0 < r < R$. The coarea formula implies

$$F_\varepsilon(r) = \int_0^r E_\varepsilon(Q_\varepsilon, \partial B_s \cap \Omega) ds$$

for $0 < r < R$, so $F'_\varepsilon(r) = E_\varepsilon(Q_\varepsilon, \partial B_r \cap \Omega)$ for a.e. $0 < r < R$. Define the set

$$\tilde{D}^\varepsilon := \left\{ r \in (\theta R, R) : F'_\varepsilon(r) \leq \frac{2\eta}{1-\theta} \log \frac{R}{\varepsilon} \right\}.$$

The assumption (88) and an average argument give

$$\mathcal{H}^1(\tilde{D}^\varepsilon) \geq \frac{(1-\theta)R}{2}. \quad (89)$$

On the other hand, for any radius $r \in \tilde{D}^\varepsilon$ we have

$$\begin{aligned} E_\varepsilon(Q_\varepsilon, \partial(B_r \cap \Omega)) &= F'_\varepsilon(r) + E_\varepsilon(Q_\varepsilon, B_r \cap \partial\Omega) \\ &\stackrel{(H_\Gamma)}{\leq} \frac{2\eta}{1-\theta} \left(\log \frac{r}{\varepsilon} - \log \theta \right) + C, \end{aligned}$$

where C is a constant depending on x_0 and R . Therefore, by choosing η small enough we obtain

$$E_\varepsilon(Q_\varepsilon, \partial(B_r \cap \Omega)) \leq \eta_0 \log \frac{r}{\varepsilon}$$

for $0 < \varepsilon \leq \varepsilon_0 R$, where η_0 and ε_1 are given by Proposition 33. With the help of this estimate, and since $B_r \cap \Omega$ is bilipschitz equivalent to a ball, we can repeat the proof of Lemma 45. We deduce that

$$F_\varepsilon(r) \leq CR \left(E_\varepsilon(Q_\varepsilon, \partial(B_r \cap \Omega))^{1/2} + 1 \right)$$

for any $r \in \tilde{D}^\varepsilon$ and $0 < \varepsilon \leq \varepsilon_0 R$. Then, using the elementary inequality $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$ and (H_Γ) again, we infer

$$\begin{aligned} F_\varepsilon(r) &\leq CR \left\{ (F'_\varepsilon(r) + E_\varepsilon(Q_\varepsilon, B_r \cap \partial\Omega))^{1/2} + 1 \right\} \\ &\leq CR \left(F'_\varepsilon(r)^{1/2} + 1 \right) \end{aligned} \quad (90)$$

for any $r \in \tilde{D}^\varepsilon$ and $0 < \varepsilon \leq \varepsilon_0 R$. Thanks to (89) and (90), we can apply Lemma 46 to $y := F_\varepsilon$. This yields the conclusion of the proof. \square

5 Structure of the singular set: Proof of Proposition 2

5.1 The limit measure is a stationary varifold

The aim of this section is to prove Proposition 2. We start by showing that $\mu_0 \llcorner \Omega$ is a stationary varifold. These objects, introduced by Almgren [4], can be thought as weak counterparts of manifolds with vanishing mean curvature. For more details, the reader is referred to the paper by Allard [2] or the book by Simon [63].

Before stating the following proposition, let us recall some basic facts. The rectifiability of $\mu_0 \llcorner \Omega$ (Proposition 51), together with [63, Remarks 1.9 and 11.5, Theorem 11.6], implies that for μ_0 -a.e. $x \in \Omega$ there exists a unique 1-dimensional subspace $L_x \subseteq \mathbb{R}^n$ such that

$$\lim_{\lambda \rightarrow 0} \int_{\mathbb{R}^d} \lambda^{-1} \varphi \left(\frac{z-x}{\lambda} \right) d\mu_0(z) = \Theta(x) \int_{L_x} \varphi(y) d\mathcal{H}^1(y) \quad (91)$$

for all $\varphi \in C_c(\mathbb{R}^3)$. Such line is called the *approximate tangent line* of μ_0 at x , and noted $\text{Tan}(\mu_0, x)$. Now, let $\mathbf{G}_{1,3} \subseteq \text{M}_3(\mathbb{R})$ be the set of matrices representing orthogonal projections on 1-subspaces of \mathbb{R}^3 . Let $A(x) \in \mathbf{G}_{1,3}$ denote the orthogonal projection on $\text{Tan}(\mu_0, x)$, for a.e. $x \in \Omega$. A varifold is a Radon measure on $\Omega \times \mathbf{G}_{1,3}$. The varifold associated with $\mu_0 \llcorner \Omega$ is defined as the push-forward measure $\mathbf{V}_0 := (\text{Id}, A)_\#(\mu_0 \llcorner \Omega)$, i.e. the measure $\mathbf{V}_0 \in \mathcal{M}(\Omega \times \mathbf{G}_{1,3})$ given by

$$\mathbf{V}_0(E) := \mu_0 \{x \in \Omega : (x, A(x)) \in E\} \quad \text{for any Borel set } E \subseteq \Omega \times \mathbf{G}_{1,3}.$$

The varifold \mathbf{V}_0 is stationary (see [2, § 4.2]) if and only if there holds

$$\int_{\Omega} A_{ij}(x) \partial_j \mathbf{X}_i(x) d\mu_0(x) = 0 \quad \text{for any } \mathbf{X} \in C_c^1(\Omega, \mathbb{R}^3). \quad (92)$$

Proposition 55 *The varifold \mathbf{V}_0 associated with $\mu_0 \llcorner \Omega$ is stationary.*

Proof. The proposition follows by adapting Ambrosio and Sonner's analysis in [5]. For the convenience of the reader, we give here the proof. Define the matrix-valued map $A^\varepsilon = (A_{ij}^\varepsilon)_{i,j} : \Omega \rightarrow \text{M}_3(\mathbb{R})$ by

$$A_{ij}^\varepsilon := \frac{1}{|\log \varepsilon|} (e_\varepsilon(Q_\varepsilon) \delta_{ij} - \partial_i Q_\varepsilon \cdot \partial_j Q_\varepsilon)$$

for $i, j \in \{1, 2, 3\}$. Then A^ε is a symmetric matrix, such that

$$\operatorname{tr} A^\varepsilon = \frac{1}{|\log \varepsilon|} \left(3e_\varepsilon(Q_\varepsilon) - |\nabla Q_\varepsilon|^2 \right) \geq \mu_\varepsilon \quad (93)$$

and

$$|A^\varepsilon| \leq C\mu_\varepsilon. \quad (94)$$

For any vector $v \in \mathbb{S}^2$, there holds

$$A_{ij}^\varepsilon v_i v_j = \frac{1}{|\log \varepsilon|} \left(e_\varepsilon(Q_\varepsilon) - |v_i \partial_i Q_\varepsilon|^2 \right) \leq \mu_\varepsilon, \quad (95)$$

so the eigenvalues of A^ε are less than or equal to μ_ε . Moreover, by integrating by parts the stress-energy identity (Lemma 22) we obtain

$$\int_{\Omega} A_{ij}^\varepsilon(x) \partial_j \mathbf{X}_i(x) \, dx = 0 \quad (96)$$

for any $\mathbf{X} \in C_c^1(\Omega, \mathbb{R}^3)$. In view of (94), and extracting a subsequence if necessary, we have that $A^\varepsilon \rightharpoonup^* A^0$ in the weak- \star topology of $\mathcal{M}(\Omega, M_3(\mathbb{R})) := C_c(\Omega, M_3(\mathbb{R}))'$. The limit measure A^0 satisfies $|A^0| \leq C(\mu_0 \llcorner \Omega)$, in particular is absolutely continuous with respect to $\mu_0 \llcorner \Omega$. Therefore, there exists a matrix-valued function $A^* \in L^1(\Omega, \mu_0; M_3(\mathbb{R}))$ such that

$$dA^0 = A^*(x) \, d(\mu_0 \llcorner \Omega) \quad \text{as measures in } \mathcal{M}(\Omega, M_3(\mathbb{R})).$$

Passing to the limit in (93), (95) and (96), for μ_0 -a.e. x we obtain that $A^*(x)$ is a symmetric matrix, with $\operatorname{tr} A^*(x) \geq 1$ and eigenvalues less than or equal to 1, such that

$$\int_{\Omega} A_{ij}^*(x) \partial_j \mathbf{X}_i(x) \, d\mu_0(x) = 0 \quad (97)$$

for any $\mathbf{X} \in C_c^1(\Omega, \mathbb{R}^3)$. Now, fix a Lebesgue point x for A^* (with respect to μ_0) and $0 < \lambda < \operatorname{dist}(x, \partial\Omega)$. Condition (97) implies

$$\lambda^{-1} \int_{\mathbb{R}^3} A^*(z) \cdot \nabla \mathbf{X} \left(\frac{z-x}{\lambda} \right) \, d\mu_0(z) = 0 \quad (98)$$

for any $\mathbf{X} \in C_c^1(B_1, \mathbb{R}^3)$. Then,

$$\begin{aligned} & \left| \lambda^{-1} \int_{\mathbb{R}^3} (A^*(z) - A^*(x)) \cdot \nabla \mathbf{X} \left(\frac{z-x}{\lambda} \right) \, d\mu_0(z) \right| \\ & \leq \underbrace{\frac{\mu_0(\overline{B}_\lambda(x))}{\lambda}}_{\rightarrow \Theta(x)/2} \|\nabla \mathbf{X}\|_{L^\infty(B_1)} \int_{B_\lambda(x)} |A^*(z) - A^*(x)| \, d\mu_0(z) \rightarrow 0 \end{aligned}$$

as $\lambda \rightarrow 0$. Combined with (91) and (98), this provides

$$\begin{aligned} & \Theta(x) A^*(x) \cdot \int_{\operatorname{Tan}(\mu_0, x)} \nabla \mathbf{X} \, d\mathcal{H}^1 \\ & = \lim_{\lambda \rightarrow 0} \lambda^{-1} \int_{\mathbb{R}^3} A^*(x) \cdot \nabla \mathbf{X} \left(\frac{z-x}{\lambda} \right) \, d\mu_0(x) = 0 \end{aligned}$$

for any $\mathbf{X} \in C_c^1(B_1, \mathbb{R}^3)$. Since $\Theta(x) > 0$ by Lemma 50, applying [5, Lemma 3.9] we deduce that at least two eigenvalues of $A^*(x)$ vanish, for μ_0 -a.e. x . On the other hand, we know already that $\text{tr } A^*(x) = 1$ with eigenvalues ≤ 1 . Therefore, the eigenvalues of $A^*(x)$ are $(1, 0, 0)$ and $A^*(x)$ represents the orthogonal projection on a line.

The push-forward measure $\mathbf{V} := (\text{Id}, A^*)_{\#}(\mu_0 \llcorner \Omega)$ is a varifold, and (97) means that \mathbf{V} is stationary. A classical result by Allard (see [2, Rectifiability Theorem, § 5.5] or [5, Theorem 3.3]) asserts that every varifold with locally bounded first variation and positive density is rectifiable. In our case, \mathbf{V} has vanishing first variation, and the density is bounded from below by Lemma 50. Therefore, by Allard's theorem \mathbf{V} is rectifiable. In particular $A^*(x)$ is the orthogonal projection on $\text{Tan}(\mathcal{S}_{\text{line}}, x)$ for μ_0 -a.e. $x \in \Omega$, so $\mathbf{V} = \mathbf{V}_0$ and the proposition follows. \square

Remark 56 In general, we cannot expect that μ_0 is associated with a stationary varifold, i.e. stationarity may fail on the boundary of the domain (see Section 5.4). Indeed, stationarity is deduced by taking the limit in the Euler-Lagrange system associated to the energy, and such a system is not satisfied on the boundary.

Stationary varifolds of dimension 1 are essentially the sum of straight line segments (see Allard and Almgren, [3]). However, the sum can be locally infinite. To rule out this possibility, in the rest of the section we prove that the 1-dimensional density of $\mu_0 \llcorner \Omega$ is constant a.e. As a consequence, we obtain that $\mathcal{S}_{\text{line}} \cap \Omega$ is essentially a locally finite union of line segments [3, Theorem p. 89]. In order to compute the density of $\mu_0 \llcorner \Omega$, we apply an argument by Lin and Rivière (see [48, Section III.1]). Essentially, by scaling we reduce to an auxiliary problem defined on a cylinder, for which we prove refined energy estimates. This requires, once again, interpolation and extension arguments. For the convenience of the reader, we work out this argument, which is sketched in [48], in detail.

5.2 An auxiliary problem: energy bounds on a cylinder

We consider the following auxiliary problem. Given some (small) parameters $0 < \delta, \epsilon < 1$, we consider the closed cylinder $A_\delta := \bar{B}_\delta^2 \times [-1, 1]$ with lateral surface $\Gamma_\delta := \partial B_\delta^2 \times [-1, 1]$. Let $g_{\delta, \epsilon} \in H^1(\partial A_\delta, \mathbf{S}_0)$ be a boundary datum which satisfies the following conditions:

$$\|g_{\delta, \epsilon}\|_{L^\infty(A_\delta)} \leq M \tag{99}$$

$$E_\epsilon(g_{\delta, \epsilon}, B_\delta^2 \times \{-1, 1\}) \leq M \log \frac{\delta}{\epsilon} \tag{100}$$

$$E_\epsilon(g_{\delta, \epsilon}, \Gamma_\delta) \leq \eta \log \frac{\delta}{\epsilon}, \tag{101}$$

for some positive constants M and η . Let $u_{\delta, \epsilon}$ be a minimizer of the Landau-de Gennes energy (LG_ϵ) in the class $H_{g_{\delta, \epsilon}}^1(A_\delta, \mathbf{S}_0)$.

Lemma 57 *For any $M > 0$, there exists $\eta_0 > 0$ and for any $0 < \eta \leq \eta_0$, $0 < \delta < 1$ there exist positive numbers ϵ_0 , C and $\alpha(M, \eta, \delta)$ with the following properties. If $0 < \epsilon \leq \epsilon_0$ and $g_{\delta, \epsilon}$ satisfies (99)–(101), then either*

$$E_\epsilon(u_{\delta, \epsilon}, A_\delta) \leq \alpha(M, \eta, \delta) \log \frac{\delta}{\epsilon} \quad (102)$$

or

$$\begin{aligned} (2\kappa_* - \alpha(M, \eta, \delta)) \log \frac{\delta}{\epsilon} - C &\leq E_\epsilon(u_{\delta, \epsilon}, A_\delta) \\ &\leq (2\kappa_* + \alpha(M, \eta, \delta)) \log \frac{\delta}{\epsilon} + C. \end{aligned} \quad (103)$$

Moreover, we can choose the number $\alpha(M, \eta, \delta)$ in such a way that

$$\alpha(M, \eta, \delta) \leq C (\delta M + \delta^2 \eta + \delta \eta + \eta + \delta^{-1} \eta).$$

Again, the key step in the proof is to approximate $u_{\delta, \epsilon}$ with an \mathcal{N} -valued map, defined on the lateral surface of the cylinder. This is possible, because the energy on Γ_δ is small compared to $|\log \epsilon|$, by (101). Set $h(\epsilon) := \epsilon^{1/2} |\log \epsilon|$ and

$$A_{\delta, \epsilon} := (\bar{B}_\delta^2 \setminus B_{\delta - \delta h(\epsilon)}^2) \times [-1, 1], \quad D_{\delta, \epsilon} := \partial B_{\delta - \delta h(\epsilon)}^2 \times [-1, 1].$$

Then, by arguing exactly as in the proof of Proposition 33, we obtain

Lemma 58 *For any $M > 0$, there exist positive numbers η_0 and C and, for any $0 < \eta \leq \eta_0$ and $0 < \delta < 1$, there exists $\epsilon_0 > 0$ with the following property. If $0 < \epsilon \leq \epsilon_0$ and $g_{\delta, \epsilon}$ satisfies (99)–(101), then there exist maps $v_{\delta, \epsilon} \in H^1(D_{\delta, \epsilon}, \mathcal{N})$ and $\varphi_{\delta, \epsilon} \in H^1(A_{\delta, \epsilon}, \mathbf{S}_0)$ which satisfy*

$$\varphi_{\delta, \epsilon} = g_{\delta, \epsilon} \quad \mathcal{H}^2\text{-a.e. on } \partial A_{\delta, \epsilon} \setminus D_{\delta, \epsilon}, \quad \varphi_{\delta, \epsilon} = v_{\delta, \epsilon} \quad \mathcal{H}^2\text{-a.e. on } D_{\delta, \epsilon} \quad (104)$$

$$\frac{1}{2} \int_{D_{\delta, \epsilon}} |\nabla v_{\delta, \epsilon}|^2 \, d\mathcal{H}^2 \leq C \eta \log \frac{\delta}{\epsilon}, \quad (105)$$

$$E_\epsilon(\varphi_{\delta, \epsilon}, A_{\delta, \epsilon}) \leq C \eta h(\epsilon) \log \frac{\delta}{\epsilon}. \quad (106)$$

Sketch of the proof. In Proposition 33, the datum $g_{\delta, \epsilon}$ is defined on a sphere of fixed radius. Here, in contrast, the domain is the lateral surface of a cylinder of variable radius δ . To overcome these issues, we first rescale the domain so that we work in the cylinder $B_1^2 \times [-\delta^{-1}, \delta^{-1}]$. Then, we construct a good grid of size $h(\epsilon)$, in the sense of Definition 36. By an average argument, the constant $C_{\mathcal{G}}$ in Definition 36 behaves as $O(\delta)$, and in particular is uniformly bounded. Although the geometry is different, we can repeat the proof of Proposition 33 because the construction used in the proof is local, that is, the behaviour of $v_{\delta, \epsilon}$ and $\varphi_{\delta, \epsilon}$ on a cell of the grid only depends on the behaviour of $u_{\delta, \epsilon}$ on the same cell. By Remark 43, the constants in (105)–(106) only depend on the shape of a given cell (i.e. on $C_{\mathcal{G}}$), not on the size of the whole domain. Therefore, they are uniformly bounded with respect to δ . \square

Since $v_{\delta,\epsilon}$ is an \mathcal{N} -valued H^1 -map, it is possible to define its homotopy class (see Lemma 29). If such homotopy class is trivial, there is no topological obstruction, therefore the energy of a minimizer is small compared to $|\log \epsilon|$, that is, the upper bound (102) holds. Otherwise, we prove that (103) holds. The lower bound follows by the Jerrard-Sandier type estimate (Corollary 16), while the upper bound is obtained via a comparison argument.

The homotopy class of $v_{\delta,\epsilon}$ is trivial: proof of (102)

We assume now that the homotopy class of $v_{\delta,\epsilon}$ is trivial, and we construct a competitor which satisfies the energy bound (102). Using the properties (105), (106) and a comparison argument, we find $z_- \in (-1 + \delta, -1 + 2\delta)$ and $z_+ \in (1 - 2\delta, 1 - \delta)$ such that

$$\frac{1}{2} \int_{\partial B_{\delta-\delta h(\epsilon)}^2 \times \{z_-, z_+\}} |\nabla v_{\delta,\epsilon}|^2 d\mathcal{H}^1 \leq \frac{C\eta}{\delta} \log \frac{\delta}{\epsilon} \quad (107)$$

and

$$E_\epsilon(\varphi_{\delta,\epsilon}, (B_\delta^2 \setminus B_{\delta-\delta h(\epsilon)}^2) \times \{z_-, z_+\}) \leq \frac{C\eta}{\delta} h(\epsilon) \log \frac{\delta}{\epsilon}. \quad (108)$$

Since the homotopy class of $v_{\delta,\epsilon}$ is trivial, with the help of (107) and of Lemma 28 we find a map $w_{\delta,\epsilon} \in H^1(B_{\delta-\delta h(\epsilon)}^2 \times \{z_-, z_+\}, \mathcal{N})$ such that

$$\frac{1}{2} \int_{B_{\delta-\delta h(\epsilon)}^2 \times \{z_-, z_+\}} |\nabla w_{\delta,\epsilon}|^2 d\mathcal{H}^2 \leq C\eta \log \frac{\delta}{\epsilon}. \quad (109)$$

We consider now four subdomains:

$$\Lambda_\delta^- := B_\delta^2 \times (-1, z_-), \quad \Lambda_{\delta,\epsilon}^0 := B_{\delta-\delta h(\epsilon)}^2 \times (z_-, z_+), \quad \Lambda_\delta^+ := B_\delta^2 \times (z_+, 1)$$

and

$$\Lambda'_{\delta,\epsilon} := (B_\delta^2 \setminus B_{\delta-\delta h(\epsilon)}^2) \times [z_-, z_+].$$

We are going to apply Lemma 25 to Λ_δ^- , $\Lambda_{\delta,\epsilon}^0$ and Λ_δ^+ (these subdomains are convex, so they are star-shaped with respect to each of their points). We first consider Λ_δ^+ , and we assign the boundary datum

$$g_{\delta,\epsilon}^+ := \begin{cases} g_{\delta,\epsilon} & \text{on } \partial \Lambda_\delta^+ \cap \partial \Lambda_\delta \\ \varphi_{\delta,\epsilon} & \text{on } (B_\delta^2 \setminus B_{\delta-\delta h(\epsilon)}^2) \times \{z_+\} \\ w_{\delta,\epsilon} & \text{on } B_{\delta-\delta h(\epsilon)}^2 \times \{z_+\}. \end{cases}$$

Let $u_{\delta,\epsilon}^+$ be a minimizer of (LG_ϵ) on Λ_δ^+ , subject to the boundary condition $u = g_{\delta,\epsilon}^+$ on $\partial \Lambda_\delta^+$. By applying Lemma 25 and (100), (101), (108), (109), we obtain

$$E_\epsilon(u_{\delta,\epsilon}^+, \Lambda_\delta^+) \leq C\delta E_\epsilon(g_{\delta,\epsilon}^+, \partial \Lambda_\delta^+) \leq C(\delta M + \delta\eta + \eta h(\epsilon)) \log \frac{\delta}{\epsilon}. \quad (110)$$

We define a function $u_{\delta,\epsilon}^-$ in Λ_δ^- a similar way. Finally, in $\Lambda_{\delta,\epsilon}^0$ we consider the boundary datum

$$g_{\delta,\epsilon}^0 := \begin{cases} v_{\delta,\epsilon} & \text{on } \partial B_{\delta-\delta h(\epsilon)}^2 \times (z_-, z_+) \\ w_{\delta,\epsilon} & \text{on } B_{\delta-\delta h(\epsilon)}^2 \times \{z_-, z_+\}. \end{cases}$$

and denote by $u_{\delta,\epsilon}^0$ the corresponding minimizer. By applying Lemma 25, (105) and (109), we deduce

$$E_\epsilon(u_{\delta,\epsilon}^0, \Lambda_{\delta,\epsilon}^0) \leq CE_\epsilon(g_{\delta,\epsilon}^0, \partial \Lambda_{\delta,\epsilon}^0) \leq C\eta \log \frac{\delta}{\epsilon}. \quad (111)$$

The boundary conditions we have defined on the boundaries of Λ_δ^+ , $\Lambda_{\delta,\epsilon}^0$, Λ_δ^- and $\Lambda'_{\delta,\epsilon}$ match. Therefore, we can define an admissible comparison map by pasting $u_{\delta,\epsilon}^+$, $u_{\delta,\epsilon}^0$, $u_{\delta,\epsilon}^-$ and $\varphi_{\delta,\epsilon}$ restricted to $\Lambda'_{\delta,\epsilon}$. Combining (106), (110) and (111), we obtain

$$E_\epsilon(u_{\delta,\epsilon}, \Lambda_\delta) \leq C(\delta M + \delta\eta + \eta h(\epsilon)) \log \frac{\delta}{\epsilon}.$$

Since $h(\epsilon) = \epsilon^{1/2} |\log \epsilon| \leq 2e^{-1}$ for $0 < \epsilon < 1$, we conclude that (102) holds if

$$\alpha(M, \eta, \delta) \geq C_1(\delta M + \delta\eta + \eta), \quad (112)$$

for some universal constant C_1 .

The homotopy class of $v_{\delta,\epsilon}$ is non-trivial: proof of the upper bound in (103)

We suppose now that the homotopy class of $v_{\delta,\epsilon}$ is non-trivial and we prove the upper bound in (103), again by a comparison argument. To construct the comparison map, we consider the same decomposition of Λ_δ into four subdomains as before. We first construct a map $u_{\delta,\epsilon}^0 \in H^1(\Lambda_{\delta,\epsilon}^0, \mathbf{S}_0)$ by applying Lemma 29 with the choice $g = v_{\delta,\epsilon}$. Thanks to (105), we obtain that

$$E_\epsilon(u_{\delta,\epsilon}^0, \Lambda_{\delta,\epsilon}^0) \leq (2\kappa_* + C(\delta^{-1} + \delta)\eta) \log \frac{\delta}{\epsilon} + C \quad (113)$$

and

$$E_\epsilon(u_{\delta,\epsilon}^0, B_{\delta-h(\epsilon)\delta}^2 \times \{z_-, z_+\}) \leq (2\kappa_* + C(\delta^{-1} + \delta)\eta) \log \frac{\delta}{\epsilon} + C \quad (114)$$

Next, we consider Λ_δ^+ and we assign the boundary datum

$$g_{\delta,\epsilon}^+ := \begin{cases} g_{\delta,\epsilon} & \text{on } \partial \Lambda_\delta^+ \cap \partial \Lambda_\delta \\ \varphi_{\delta,\epsilon} & \text{on } (B_\delta^2 \setminus B_{\delta-\delta h(\epsilon)}^2) \times \{z_+\} \\ u_{\delta,\epsilon}^0 & \text{on } B_{\delta-\delta h(\epsilon)}^2 \times \{z_+\}. \end{cases}$$

Because of Lemma 25, (100), (101), (108) and (114), a minimizer $u_{\delta,\epsilon}^+$ corresponding to the boundary condition $u = g_{\delta,\epsilon}^+$ on $\partial\Lambda_\delta^+$ satisfies

$$E_\epsilon(u_{\delta,\epsilon}^+, \Lambda_\delta^+) \leq C\delta E_\epsilon(g_{\delta,\epsilon}^+, \partial\Lambda_\delta^+) \leq C(\delta M + \delta^2\eta + \delta\eta + \eta) \log \frac{\delta}{\epsilon} \quad (115)$$

(we have also used that $h(\epsilon) \leq C$). In the subdomain Λ_δ^- , we define $u_{\delta,\epsilon}^-$ in a similar way. As before, pasting $u_{\delta,\epsilon}^-$, $u_{\delta,\epsilon}^0$, $u_{\delta,\epsilon}^+$ and $\varphi_{\delta,\epsilon}$ restricted to $A'_{\delta,\epsilon}$ we obtain an admissible comparison map. Therefore, combining (113), (115) and (106), we deduce that the upper bound in (103) holds, provided that

$$\alpha(M, \eta, \delta) \geq C_2(\delta M + \delta^2\eta + \delta\eta + \eta + \delta^{-1}\eta). \quad (116)$$

The homotopy class of $v_{\delta,\epsilon}$ is non-trivial: proof of the lower bound in (103)

Finally, we need to prove the lower bound in (103), again assuming that the homotopy class of $v_{\delta,\epsilon}$ is non-trivial. The essential tool, here, is the Jerrard-Sandier type estimate (Corollary 16). However, in order to be able to apply such an estimate, once again we need to take care of the boundary conditions by means of an interpolation argument. Using cylindric coordinates $(\rho, \theta, z) \in [0, \delta] \times [0, 2\pi) \times [-1, 1]$, we define the map $\tilde{u}_{\delta,\epsilon}: \Lambda_\delta \rightarrow \mathbf{S}_0$ by

$$\tilde{u}_{\delta,\epsilon}(\rho e^{i\theta}, z) := \begin{cases} u_{\delta,\epsilon}\left(\frac{\rho e^{i\theta}}{1-h(\epsilon)}, \theta, z\right) & \text{if } \rho \leq \delta - \delta h(\epsilon), |z| \leq 1 \\ \varphi_{\delta,\epsilon}((2\delta - \delta h(\epsilon) - \rho)e^{i\theta}, z) & \text{if } \delta - \delta h(\epsilon) \leq \rho \leq \delta, |z| \leq 1. \end{cases}$$

This map belongs to H^1 , satisfies

$$\tilde{u}_{\delta,\epsilon}(\delta e^{i\theta}, z) = v_{\delta,\epsilon}(e^{i\theta}, z) \quad \text{for } \mathcal{H}^2\text{-a.e. } (\theta, z) \in [0, 2\pi) \times [-1, 1] \quad (117)$$

and

$$\begin{aligned} E_\epsilon(\tilde{u}_{\delta,\epsilon}, \Lambda_\delta) &\leq E_\epsilon(u_{\delta,\epsilon}, \Lambda_\delta) + E_\epsilon(\varphi_{\delta,\epsilon}, A_{\delta,\epsilon}) \\ &\stackrel{(106)}{\leq} E_\epsilon(u_{\delta,\epsilon}, \Lambda_\delta) + C\eta h(\epsilon) \log \frac{\delta}{\epsilon}. \end{aligned} \quad (118)$$

By Fubini theorem and (117), for a.e. $z \in [-1, 1]$ the map $\tilde{u}_{\delta,\epsilon}$ restricted to $\partial B_\delta^2 \times \{z\}$ belongs to H^1 , is \mathcal{N} -valued and has a nontrivial homotopy class. Moreover, we can always assume WLOG that $\epsilon \leq \epsilon_0 < \delta/2$. Let us apply Corollary 16 to the function $\tilde{u}_{\delta,\epsilon}$ restricted to $B_\delta^2 \times \{z\}$. This yields

$$E_\epsilon(\tilde{u}_{\delta,\epsilon}, B_\delta^2 \times \{z\}) + C\delta E_\epsilon(\tilde{u}_{\delta,\epsilon}, \partial B_\delta^2 \times \{z\}) \geq \kappa_* \log \frac{\delta}{\epsilon} - C.$$

By integrating with respect to $z \in [-1, 1]$, and using (117) again, we deduce that

$$E_\epsilon(\tilde{u}_{\delta,\epsilon}, \Lambda_\delta) + C\delta \int_{D_{\delta,\epsilon}} |\nabla v_{\delta,\epsilon}|^2 \, d\mathcal{H}^2 \geq 2\kappa_* \log \frac{\delta}{\epsilon} - C.$$

Then, thanks to (105), we obtain

$$E_\epsilon(\tilde{u}_{\delta,\epsilon}, \Lambda_\delta) \geq (2\kappa_* - C\delta\eta) \log \frac{\delta}{\epsilon} - C,$$

Finally, combining this inequality with (118), we conclude that

$$E_\epsilon(u_{\delta,\epsilon}, \Lambda_\delta) \geq (2\kappa_* - C\delta\eta - C\eta h(\epsilon)) \log \frac{\delta}{\epsilon} - C$$

and, since $h(\epsilon) := \epsilon^{1/2} |\log \epsilon|$ is bounded, the lower bound in (103) is satisfied if

$$\alpha(M, \eta, \delta) \geq C_3 (\delta\eta + \eta). \quad (119)$$

Thanks to (112), (116) and (119), Lemma 57 is satisfied if we set

$$\alpha(M, \eta, \delta) := \max\{C_1, C_2, C_3\} (\delta M + \delta^2\eta + \delta\eta + \eta + \delta^{-1}\eta).$$

5.3 The singular measure has constant density

Aim of this subsequence is to prove the following

Proposition 59 *For \mathcal{H}^1 -a.e. $x \in \mathcal{S}_{\text{line}} \cap \Omega$, there holds $\Theta(x) = \kappa_*$.*

This property is of crucial importance, because it allow us to describe the structure of the singular set in the interior of the domain and to prove Proposition 2.

Proof of Proposition 59. Because $\mu_0 \llcorner \Omega$ is rectifiable, Θ is approximately continuous and μ_0 has an approximate tangent line (i.e., (91) holds) at \mathcal{H}^1 -a.e. point $x_0 \in \mathcal{S}_{\text{line}} \cap \Omega$. Fix such a point x_0 . By (91), there exists a line L such that the measures $(\nu_\lambda)_{\lambda>0}$ defined by

$$\nu_\lambda(A) := \lambda^{-1} \mu_0(\lambda A \cap \Omega) \quad \text{for } A \in \mathcal{B}(\mathbb{R}^3)$$

satisfy

$$\nu_\lambda \rightharpoonup^* \nu_0 := \Theta(x_0) \mathcal{H}^1 \llcorner L \quad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega) \text{ as } \lambda \rightarrow 0. \quad (120)$$

Up to rotations and translations, we can assume WLOG that $x_0 = 0$ and $L = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$. Let

$$M' := \frac{2\sqrt{2}M}{\text{dist}(x_0, \partial\Omega)}, \quad (121)$$

where M is given by assumption (H), and let $\eta_0 = \eta_0(M')$ be the corresponding number given by Lemma 57. Let $0 < \eta \leq \eta_0$ and $0 < \delta < 1$ be two small parameters, to be chosen later. We consider again the cylinder $\Lambda_\delta := B_\delta^2 \times [-1, 1]$, with lateral surface $\Gamma_\delta := \partial B_\delta^2 \times [-1, 1]$. Since $\nu_0(\Gamma_\delta) = 0$, because

of (120) there exists a positive number $\lambda_0 = \lambda_0(\eta, \delta) < \text{dist}(x_0, \partial\Omega)/(2\sqrt{2})$ such that

$$\mu_0(\lambda\Gamma_\delta) \leq \frac{\lambda\eta}{2} \quad \text{for } 0 < \lambda \leq \lambda_0.$$

Then, for a fixed $0 < \lambda \leq \lambda_0$, thanks to (84) we find a positive number $n_0 = n_0(M', \eta, \delta, \lambda)$ such that

$$E_{\varepsilon_n}(Q_{\varepsilon_n}, \lambda\Gamma_\delta) \leq \lambda\eta \log \frac{\lambda\delta}{\varepsilon_n} \quad \text{for any integer } n \geq n_0. \quad (122)$$

Moreover, the cylinder $\lambda\Lambda_\delta$ is contained in a ball centered at x_0 with radius $\sqrt{2}\lambda < r_0 := \text{dist}(x_0, \partial\Omega)/2$. Then, because of the monotonicity formula (Lemma 24) and (H) we have

$$E_{\varepsilon_n}(Q_{\varepsilon_n}, \lambda\Lambda_\delta) \leq \frac{\sqrt{2}\lambda}{r_0} E_{\varepsilon_n}(Q_{\varepsilon_n}, B_{r_0}) \stackrel{(H)-(121)}{\leq} \lambda M' \log \frac{\lambda\delta}{\varepsilon_n}$$

for any integer $n \geq n_0$. Thanks to Fatou lemma, we deduce

$$\int_{-\lambda}^{\lambda} \liminf_{n \rightarrow +\infty} E_{\varepsilon_n}(Q_{\varepsilon_n}, B_{\lambda\delta}^2 \times \{z\}) dz \leq \lambda M' \log \frac{\lambda\delta}{\varepsilon_n}$$

so, by an average argument, we can find two numbers $z_- \in [-\lambda, -3\lambda/4]$, $z_+ \in [3\lambda/4, \lambda]$ and a subsequence (still denoted ε_n) such that

$$E_{\varepsilon_n}(Q_{\varepsilon_n}, B_{\lambda\delta}^2 \times \{z_-, z_+\}) \leq M' \log \frac{\lambda\delta}{\varepsilon_n} \quad (123)$$

for any integer $n \geq n_0$. To avoid notation, we will assume that $z_- = -\lambda$ and $z_+ = \lambda$. Reducing the value of n_0 if necessary, we can assume that $\varepsilon_n \leq \lambda\epsilon_0$, where $\epsilon_0 = \epsilon_0(M', \eta, \delta)$ is given by Lemma 57.

Now, fix an integer $n \geq n_0$. We set $\epsilon := \varepsilon_n/\lambda$ (notice that $\epsilon \leq \epsilon_0$) and

$$u_{\delta, \epsilon}(y) := Q_{\varepsilon_n}(\lambda y) \quad \text{for } y \in \Lambda_\delta.$$

From (122) and (123) we deduce that

$$E_\epsilon(u_{\delta, \epsilon}, \Gamma_\delta) \stackrel{\lambda \leq 1}{\leq} \eta \log \frac{\delta}{\epsilon}, \quad E_\epsilon(u_{\delta, \epsilon}, B_\delta^2 \times \{-1, 1\}) \leq M' \log \frac{\delta}{\epsilon}$$

so the conditions (100) and (101) are satisfied; moreover, $u_{\delta, \epsilon}$ satisfies (99) because of (H). Therefore, we can apply Lemma 57. Scaling back to Q_{ε_n} , we conclude that either it holds

$$E_{\varepsilon_n}(Q_{\varepsilon_n}, \lambda\Lambda_\delta) \leq \lambda\alpha \log \frac{\lambda\delta}{\varepsilon_n} \quad (124)$$

or it holds

$$\lambda(2\kappa_* - \alpha) \log \frac{\lambda\delta}{\varepsilon_n} - \lambda C \leq E_{\varepsilon_n}(Q_{\varepsilon_n}, \lambda\Lambda_\delta) \leq \lambda(2\kappa_* + \alpha) \log \frac{\lambda\delta}{\varepsilon_n} + \lambda C, \quad (125)$$

where $\alpha = \alpha(M', \eta, \delta)$ is a positive number which satisfies

$$\alpha \leq C (\delta M' + \delta^2 \eta + \delta \eta + \eta + \delta^{-1} \eta).$$

At this point, we choose $\delta := \eta^{1/2}$, so that $\alpha \rightarrow 0$ when $\eta \rightarrow 0$.

Suppose that the inequality (124) holds. then, passing to the limit as $n \rightarrow +\infty$, thanks to (84) we find that

$$\mu_0(\text{interior of } \lambda A_\delta) \leq \lambda \alpha.$$

Passing to the limit as $\lambda \rightarrow 0$, with the help of (120) we obtain

$$2\Theta(x_0) = \nu_0(A_\delta) \leq \alpha$$

and finally, letting $\eta \rightarrow 0$ (so that $\alpha \rightarrow 0$) we conclude that $\Theta(x_0) = 0$. This is a contradiction, because x_0 is supposed to be an approximate continuity point for Θ and Θ is bounded away from 0 on $\mathcal{S}_{\text{line}} \cap \Omega$ (Lemma 50). Therefore, (124) does not hold, and so (125) must be satisfied instead. Passing to the limit as $n \rightarrow +\infty$ and $\lambda \rightarrow 0$, and using (84) and (120) again, we deduce that

$$2\kappa_* - \alpha \leq \nu_0(A_\delta) \leq 2\kappa_* + \alpha$$

or equivalently

$$\kappa_* - \frac{\alpha}{2} \leq \Theta(x_0) \leq \kappa_* + \frac{\alpha}{2}.$$

Letting $\eta \rightarrow 0$, we conclude that $\Theta(x_0) = \kappa_*$. \square

Remark 60 As a byproduct of the previous proof, we obtain a topological information about Q_0 . Let $x_0 \in \mathcal{S}_{\text{line}} \cap \Omega$ be as in the previous proof (i.e., Θ is approximately continuous at x_0 and $\text{Tan}(\mu_0, x)$ exists). If $D \subset\subset \Omega$ is a disk and $D \cap \mathcal{S}_{\text{line}} = \{x_0\}$, then the homotopy class of Q_0 restricted to ∂D is non-trivial. For we know by the previous proof that (125) must be satisfied, so we are in the case $v_{\delta, \epsilon}$ has a non-trivial class (we are using the notation of Section 5.2). This means that $\varrho \circ u_{\delta, \epsilon}$, where is well-defined, has a non-trivial class too. Taking the limit as $\epsilon \rightarrow 0$, we conclude that Q_0 has a non-trivial class because $u_{\delta, \epsilon}$ converge locally uniformly to Q_0 , away from the singular set $\mathcal{S}_{\text{line}} \cup \mathcal{S}_{\text{pts}}$ (Theorem 1).

Proposition 2 now follows quite easily from a result by Allard and Almgren, which is a structure theorem for stationary varifolds of dimension 1.

Proof of Proposition 2. Let \mathcal{S}' be the set of points $x \in \mathcal{S}_{\text{line}} \cap \Omega$ such that $\Theta(x) = \kappa_*$. Since $\mu_0 \llcorner \Omega$ is a stationary varifold, [3, Theorem and Remark p. 89] imply that \mathcal{S}' is a relatively open subset of $\mathcal{S}_{\text{line}} \cap \Omega$, such that

$$\mathcal{H}^1((\mathcal{S}_{\text{line}} \cap \Omega) \setminus \mathcal{S}') = 0, \quad (126)$$

and $\mathcal{S}' \cap K$ is a finite union of straight segments, for any open set $K \subset\subset \Omega$. Moreover, the set \mathcal{S}' must be dense in $\mathcal{S}_{\text{line}} \cap \Omega$. Indeed, suppose that there exists a point $x_0 \in \mathcal{S}_{\text{line}} \cap \Omega$ and an open neighborhood $B \subseteq \Omega$ of x_0 , such

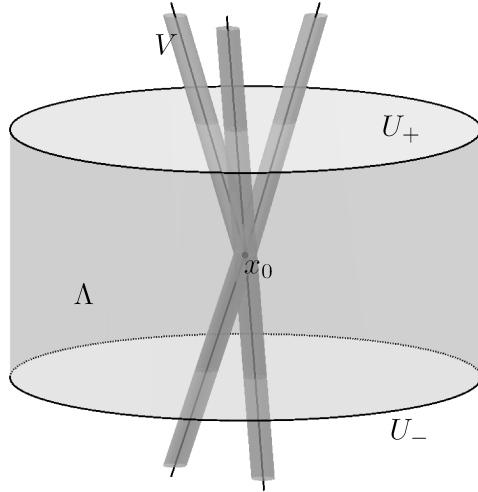


Fig. 4 A branching point for the singular set $\mathcal{S}_{\text{line}}$, surrounded by a cylinder Λ . The homotopy class of Q_0 restricted to the boundary of a disk which crosses transversely $\mathcal{S}_{\text{line}}$ is determined by the number of intersections between the disk and $\mathcal{S}_{\text{line}}$, modulo 2. A configuration such as the one represented in this figure cannot occur, otherwise Q_0 restricted to ∂U_+ and ∂U_- would be in different homotopy classes.

that B does not intersect the closure of \mathcal{S}' . Then, we have $\mathcal{H}^1(\mathcal{S}_{\text{line}} \cap B) = 0$ because of (126), therefore $\mu_0(B) = 0$ by Proposition 51 and so $\mathcal{S}_{\text{line}} \cap B = \emptyset$, which is a contradiction. It follows that, for any open set $K \subset \subset \Omega$, $\mathcal{S}_{\text{line}} \cap \bar{K}$ is a finite union of closed line segments, L_1, \dots, L_p . By subdividing the segments, if necessary, we can assume WLOG that, for each $i \neq j$, either L_i and L_j are disjoint or their intersection is a common endpoint.

Property (i) now follows directly from Remark 60. We still have to show Property (ii). Suppose that $x_0 \in K$ is an endpoint of exactly $q \leq p$ line segments, say L_1, \dots, L_q . We claim that q is even. Let V be the δ -neighborhood of $\mathcal{S}_{\text{line}}$, for a small, positive number δ . Pick a cylinder $\Lambda \subseteq K$ which contains x_0 , such that the lateral surface of Λ does not intersect V (see Figure 4). Since $(\mathcal{S}_{\text{pts}} \setminus V) \cap K$ is finite, modifying Λ if necessary we can assume that $\partial \Lambda \setminus V$ does not contain any singular point \mathcal{S}_{pts} , so Q_0 is well-defined and continuous on $\partial \Lambda \setminus V$. In particular, if we denote by U_- , U_+ the two bases of the cylinder, the maps $Q_0|_{\partial U_+}$ and $Q_0|_{\partial U_-}$ are homotopic to each other. Assume now, by contradiction, that q is odd. Then one of the bases — say U_+ — must intersect an even number of segments, and the other must intersect an odd number of segments. Therefore, due to Remark 60, the homotopy class of $Q_0|_{\partial U_+}$ must be trivial, and the homotopy class of $Q_0|_{\partial U_-}$ must be non-trivial. This is a contradiction, hence q is even. \square

Remark 61 For $k \in \{1, \dots, q\}$, let $\nu^{(k)}$ be the direction vector associated with L_k which has unit norm and points outward from x_0 . Then, the station-

arity of $\mathcal{S}_{\text{line}}$ implies that

$$\sum_{k=1}^q \boldsymbol{\nu}^{(k)} = 0.$$

Indeed, taking an arbitrary vector field \mathbf{X} supported in a small neighborhood of x_0 , thanks to (92) we have

$$0 = \kappa_* \sum_{k=1}^q \int_0^{\text{length}(L_k)} \boldsymbol{\nu}_i^{(k)} \boldsymbol{\nu}_j^{(k)} \partial_j \mathbf{X}_i(x_0 + t\boldsymbol{\nu}^{(k)}) dt = -\kappa_* \sum_{k=1}^q \boldsymbol{\nu}^{(k)} \cdot \mathbf{X}(x_0).$$

5.4 Concentration of the energy at the boundary: an example

The arguments we presented in this section may not be extended to the analysis of the singular set near the boundary of the domain. In particular, the stationarity of μ_0 may fail. We discuss now an example where the boundary datum is independent of ε and smooth, yet the geometry of the domain forces the energy of the minimizers to concentrate at the boundary. As a result, $\mathcal{S}_{\text{line}}$ is non-empty but $\mathcal{S}_{\text{line}} \subseteq \partial\Omega$, and $\mathcal{S}_{\text{line}}$ is *not* a locally finite union of straight line segments.

Let $\mathbf{r}: (0, +\infty) \times \mathbb{R} \times [0, 2\pi] \rightarrow \mathbb{R}^3$ be the cylindrical change of coordinates, given by

$$\mathbf{r}(\rho, z, \theta) := (\rho \cos \theta, \rho \sin \theta, z)^\top.$$

Let D be the disk $B_1(2, 0)$ in the (ρ, z) -plane, and let Ω be the solid torus generated by the revolution of D , that is

$$\Omega := \mathbf{r}(D \times [0, 2\pi)) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \left(\sqrt{x_1^2 + x_2^2} - 2 \right)^2 + x_3^2 < 1 \right\}.$$

We consider the boundary datum $g \in C^1(\partial\Omega, \mathcal{N})$ given by $g = h \circ \mathbf{r}^{-1}$, where

$$h(\rho, z) := s_* \left\{ \left(\mathbf{e}_1 \cos \frac{\varphi(\rho, z)}{2} + \mathbf{e}_3 \sin \frac{\varphi(\rho, z)}{2} \right)^{\otimes 2} - \frac{1}{3} \text{Id} \right\},$$

and $\varphi(\rho, z)$ is the oriented angle between the ray starting at $(2, 0)$ and passing through (ρ, z) and the positive ρ -axis. Notice that, for each $\theta \in [0, 2\pi)$, the restriction of g to the slice $\mathbf{r}(\partial D \times \{\theta\})$ has a non-trivial homotopy class. We will prove the following

Proposition 62 *For this choice of the domain and the boundary datum, we have*

$$\mathcal{S}_{\text{line}} = \mathbf{r}(\{(1, 0)\} \times [0, 2\pi)) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, x_3 = 0\}.$$

In particular, $\mathcal{S}_{\text{line}} \subseteq \partial\Omega$.

Because of the cylindrical symmetry, the problem is essentially bidimensional. Indeed, given any map $Q \in H_g^1(\Omega, \mathbf{S}_0)$, by a change of variable we obtain

$$\begin{aligned} E_\varepsilon(Q, \Omega) &= \int_0^{2\pi} \int_D \left\{ \frac{1}{2} |\partial_\rho(Q \circ \mathbf{r})|^2 + \frac{1}{2} |\partial_z(Q \circ \mathbf{r})|^2 + \frac{1}{\varepsilon^2} f(Q \circ \mathbf{r}) \right\} \rho \, d\rho \, dz \, d\theta \\ &\quad + \int_0^{2\pi} \int_D \frac{1}{2\rho} |\partial_\theta(Q \circ \mathbf{r})|^2 \, d\rho \, dz \, d\theta. \end{aligned} \quad (127)$$

Therefore, a map Q_ε is a minimizer for (LG_ε) in the class $H_g^1(\Omega, \mathbf{S}_0)$ if and only if $Q_\varepsilon = P_\varepsilon \circ \mathbf{r}^{-1}$, where P_ε only depend on (ρ, z) and is a minimizer for the weighted functional

$$F_\varepsilon(P, D) := 2\pi \int_D \left\{ \frac{1}{2} |\nabla P|^2 + \frac{1}{\varepsilon^2} f(P) \right\} \rho \, d\rho \, dz \quad (128)$$

in the class

$$H_h^1(D, \mathbf{S}_0) := \{P \in H^1(D, \mathbf{S}_0) : P = h \text{ on } \partial D\}.$$

Lemma 63 *For any $\delta > 0$, there exists a constant C_δ such that, for any $0 < \varepsilon < \delta/4$, there holds*

$$F_\varepsilon(P_\varepsilon, D) \leq 2\pi\kappa_*(1 + \delta)|\log \varepsilon| + C_\delta.$$

Proof. Fix $\delta > 0$, $0 < \varepsilon < \delta/4$ and let $\rho_\delta := 1 + 3\delta/4$, $D_\delta := B_{\delta/4}^2(\rho_\delta, 0)$. We define a map $\tilde{P}_{\delta,\varepsilon} : D_\delta \rightarrow \mathbf{S}_0$ by setting

$$\tilde{P}_{\delta,\varepsilon}(\rho, z) := \eta_\varepsilon \left(\sqrt{(\rho - \rho_\delta)^2 + z^2} \right) h \left(2 + \frac{4}{\delta}(\rho - \rho_\delta), \frac{4z}{\delta} \right)$$

if

$$(\rho - \rho_\delta)^2 + z^2 \leq \frac{\delta^2}{16},$$

where η_ε is defined by (40). The restriction of $\tilde{P}_{\delta,\varepsilon}$ to ∂D_δ is a \mathcal{N} -valued loop which depends on δ but not on ε , and has a non-trivial homotopy class. Therefore, there exists a map $\tilde{P}_\delta : D \setminus D_\delta \rightarrow \mathcal{N}$ such that

$$\tilde{P}_\delta = h \quad \text{on } \partial D, \quad \tilde{P}_\delta = \tilde{P}_{\delta,\varepsilon} \quad \text{on } \partial D_\delta.$$

We extend $\tilde{P}_{\delta,\varepsilon}$ to a new map, still denoted $\tilde{P}_{\delta,\varepsilon}$, by setting $\tilde{P}_{\delta,\varepsilon} := \tilde{P}_\delta$ on $D \setminus D_\delta$. Then $\tilde{P}_{\delta,\varepsilon} \in H_h^1(D, \mathbf{S}_0)$ and, through a straightforward computation, we obtain

$$\begin{aligned} F_\varepsilon(P_\varepsilon, D) &\leq F_\varepsilon(\tilde{P}_{\delta,\varepsilon}, D) = F_\varepsilon(\tilde{P}_{\delta,\varepsilon}, D_\delta) + \pi \int_{D \setminus D_\delta} |\nabla \tilde{P}_\delta|^2 \, d\mathcal{H}^2 \\ &\leq 2\pi(1 + \delta)E_\varepsilon(\tilde{P}_{\delta,\varepsilon}, D_\delta) + C_\delta \\ &\leq 2\pi(1 + \delta) \log \frac{\delta/4}{\varepsilon} + C_\delta = 2\pi(1 + \delta)|\log \varepsilon| + C_\delta, \end{aligned}$$

where the symbol C_δ denotes several constants which only depend on δ , D and h . \square

This lemma has an immediate consequence on the limit measure of the three-dimensional minimization problem.

Corollary 64 *There holds*

$$\mu_0(\overline{\Omega}) \leq 2\pi\kappa_*.$$

Proof. Since we have $E_\varepsilon(Q_\varepsilon, \Omega) = F_\varepsilon(P_\varepsilon, D)$ because of (127), the corollary follows by taking the limit in Lemma 63 first as $\varepsilon \rightarrow 0$, then as $\delta \rightarrow 0$. \square

Now, since the 3D-minimizers Q_ε satisfy $Q_\varepsilon = P_\varepsilon \circ \mathbf{r}^{-1}$ and P_ε is independent of the θ -variable, the singular set $\mathcal{S}_{\text{line}}$ must be of the form $\mathcal{S}_{\text{line}} = \mathbf{r}(\Sigma \times [0, 2\pi))$ for some $\Sigma \subset \overline{D}$. Notice that Σ is non-empty, because the homotopy class of the boundary datum h is non-trivial. Moreover, Σ is finite because $\mathcal{H}^1(\mathcal{S}_{\text{line}}) < \infty$.

Lemma 65 *Suppose that $(\rho_0, z_0) \in \Sigma$. Then*

$$\mu_0(\overline{\Omega}) \geq 2\pi\rho_0\kappa_*.$$

Combining Corollary 64 and Lemma 65, we immediately deduce that $\rho_0 = 1$, that is, $\Sigma = \{(1, 0)\}$, whence Proposition 62 follows.

Proof of Lemma 65. Fix a positive number $\delta < 1/2$. It suffices to prove the inequality

$$F_\varepsilon(P_\varepsilon, D) \geq 2\pi\kappa_*(\rho_0 - \delta)|\log \varepsilon| - C_\delta \quad (129)$$

for any small ε and for some positive constant C_δ which depends only on δ , D , h . The lemma will follow by using (127) and passing to the limit as $\varepsilon \rightarrow 0$, $\delta \rightarrow 0$.

Consider the disk $D' := B_{3/2}^2(2, 0) \supset D$, as well as the associated torus of revolution $\Omega' := \mathbf{r}(D' \times [0, 2\pi))$. We extend each minimizer P_ε to a new map defined on D' , still denoted by P_ε , by setting $P_\varepsilon := h$ on $D' \setminus D$. In the same way, we extend Q_ε by setting $Q_\varepsilon := g$ on $\Omega' \setminus \Omega$. Moreover, we set

$$D'_\delta := B_{\delta/2}^2(\rho_0, z_0).$$

Since Σ is finite, reducing the value of δ if necessary we can assume that $D'_\delta \cap \Sigma = \{(\rho_0, z_0)\}$. Therefore, Remark 60 implies that Q_ε restricted to $\partial D'_\delta$ has a non-trivial homotopy class, provided that ε is small enough. Thus, we can apply Corollary 16 to bound from below the energy on D'_δ . We have

$$\begin{aligned} F_\varepsilon(P_\varepsilon, D'_\delta) &\geq 2\pi \left(\rho_0 - \frac{\delta}{2} \right) E_\varepsilon(P_\varepsilon, D'_\delta) \\ &\geq 2\pi\kappa_* \left(\rho_0 - \frac{\delta}{2} \right) \phi_0(P_\varepsilon, \partial D'_\delta) \log \frac{\delta}{\varepsilon} - C \left(E_\varepsilon(P_\varepsilon, \partial D'_\delta) + 1 \right). \end{aligned} \quad (130)$$

Thanks to Theorem 1 and Corollary 54, we have $Q_\varepsilon \rightarrow Q_0$ weakly in H^1 , in a small neighborhood of $\mathbf{r}(\partial D'_\delta \times [0, 2\pi)) \subset \Omega'$. Therefore, modifying again the value of δ if necessary, we can assume that

$$Q_\varepsilon \rightharpoonup Q_0 \quad \text{weakly in } H^1(\mathbf{r}(\partial D'_\delta \times [0, 2\pi)), \mathbf{S}_0)$$

(this property may not be satisfied for any value δ , but it is satisfied for almost every value thanks to Fubini theorem). By Fubini theorem again and compact Sobolev embedding, this yields

$$P_\varepsilon \rightharpoonup P_0 \quad \text{weakly in } H^1(\partial D'_\delta, \mathbf{S}_0) \text{ and uniformly,}$$

where $P_0 := Q_0 \circ \mathbf{r}$ is an \mathcal{N} -valued map. In particular, there holds

$$\phi_0(P_\varepsilon, \partial D'_\delta) \rightarrow \phi_0(P_0, \partial D'_\delta) = 1 \quad \text{and} \quad E_\varepsilon(P_\varepsilon, \partial D'_\delta) \leq C_\delta. \quad (131)$$

Combining (130) and (131), we deduce that for (almost every) small enough δ , there exist positive constants C_δ and ε_δ such that (129) is satisfied for any $0 < \varepsilon \leq \varepsilon_\delta$. This is enough to conclude the proof of the lemma. \square

6 Sufficient conditions for (H)

6.1 Proof of Proposition 3

In this section, we analyze the role of the domain and the boundary data in connection with (H), and prove sufficient conditions for (H) to hold true. We prove first Proposition 3, namely, we assume that Ω is a bounded, Lipschitz domain and the boundary datum is bounded in $H^{1/2}(\partial\Omega, \mathcal{N})$, and we show that the inequalities

$$\|Q_\varepsilon\|_{L^\infty(\Omega)} \leq M \quad (132)$$

and

$$E_\varepsilon(Q_\varepsilon) \leq M(|\log \varepsilon| + 1) \quad (133)$$

are satisfied for some positive M . The L^∞ -bound (132) is easily obtained by a comparison argument.

Lemma 66 *Minimizers Q_ε of (LG $_\varepsilon$) satisfy*

$$\|Q_\varepsilon\|_{L^\infty(\Omega)} \leq \max \left\{ \sqrt{\frac{2}{3}} s_*, \|g_\varepsilon\|_{L^\infty(\partial\Omega)} \right\}.$$

Proof. Set

$$M := \max \left\{ \sqrt{\frac{2}{3}} s_*, \|Q_\varepsilon\|_{L^\infty(\partial\Omega)} \right\},$$

and define $\pi: \mathbf{S}_0 \rightarrow \mathbf{S}_0$ by $\pi(Q) := M|Q|^{-1}Q$ if $|Q| \geq M$, $\pi(Q) := Q$ otherwise. We have

$$\text{Df}(Q) \cdot Q = -a|Q|^2 - b \text{tr} Q^3 + c|Q|^4 > 0 \quad \text{when } |Q| > \sqrt{\frac{2}{3}} s_*$$

(this follows from the inequality $\sqrt{6}|\operatorname{tr} Q^3| \leq |Q|^3$; see [50, Lemma 1]). We deduce that $f(\pi(Q)) \geq f(Q)$ for any Q . Moreover, π is the projection on a convex set, so it is 1-Lipschitz continuous. Thus, the map $\pi \circ Q_\varepsilon$ belongs to $H_{g_\varepsilon}^1(\Omega, \mathbf{S}_0)$, satisfies $|\nabla(\pi \circ Q_\varepsilon)| \leq |\nabla Q_\varepsilon|$ a.e. and $E_\varepsilon(\pi \circ Q_\varepsilon) \leq E_\varepsilon(Q_\varepsilon)$, with strict inequality if $|Q_\varepsilon| > M$ on a set of positive measure. By minimality of Q_ε , we conclude that $|Q_\varepsilon| \leq M$ a.e. \square

The logarithmic energy bound (133) is more delicate. The proof is adapted from an argument by Rivière [59, Proposition 2.1], which involves Hardt, Kinderlehrer and Lin's re-projection trick (see [37, Lemma 2.3]) in an essential way.

Proof of Proposition 3. Since the boundary data are supposed to be \mathcal{N} -valued, they are bounded in L^∞ , so (132) follows by Lemma 66. We prove (133) by constructing a suitable comparison function, whose energy is bounded by the right-hand side of (133). For any $0 < \varepsilon < 1$, let $u_\varepsilon \in H^1(\Omega, \mathbf{S}_0)$ be the harmonic extension of g_ε , i.e. the unique solution of

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega \\ u_\varepsilon = g_\varepsilon & \text{on } \partial\Omega. \end{cases}$$

Since $\{g_\varepsilon\}_\varepsilon$ is bounded in $H^{1/2} \cap L^\infty$, the sequence $\{u_\varepsilon\}_\varepsilon$ is bounded in $H^1 \cap L^\infty$. Let $\delta > 0$ be a small parameter to be chosen later. For any $A \in \mathbf{S}_0$ with $|A| \leq \delta$ and any ε , we define

$$u_\varepsilon^A := (\eta_\varepsilon \circ \phi)(u_\varepsilon - A) \varrho(u_\varepsilon - A)$$

where $\phi: \mathbf{S}_0 \rightarrow \mathbb{R}$ and $\varrho: \mathbf{S}_0 \setminus \mathcal{C} \rightarrow \mathcal{N}$ are defined respectively in Lemmas 13, 12, and $\eta_\varepsilon \in C(\mathbb{R}^+, \mathbb{R})$ is given by

$$\eta_\varepsilon(r) := \varepsilon^{-1}r \quad \text{if } 0 \leq r < \varepsilon, \quad \eta_\varepsilon(r) = 1 \quad \text{if } r \geq \varepsilon.$$

By Lemma 12 and Corollary 18, we have $u_\varepsilon^A \in (H^1 \cap L^\infty)(\Omega, \mathbf{S}_0)$. We differentiate u_ε^A and, taking advantage of the Lipschitz continuity of ϕ (Lemma 13), we deduce

$$|\nabla u_\varepsilon^A|^2 \leq C \left\{ (\eta'_\varepsilon \circ \phi)^2(u_\varepsilon - A) |\nabla u_\varepsilon|^2 + (\eta_\varepsilon \circ \phi)^2(u_\varepsilon - A) |\nabla(\varrho(u_\varepsilon - A))|^2 \right\}.$$

We apply Corollary 18 to bound the derivative of $\varrho(u_\varepsilon - A)$:

$$|\nabla u_\varepsilon^A|^2 \leq C \left\{ (\eta'_\varepsilon \circ \phi)^2(u_\varepsilon - A) + \frac{(\eta_\varepsilon \circ \phi)^2(u_\varepsilon - A)}{\phi^2(u_\varepsilon - A)} \right\} |\nabla u_\varepsilon|^2.$$

On the other hand, there holds

$$f(u_\varepsilon^A) \leq C \mathbb{1}_{\{\phi(u_\varepsilon - A) \leq \varepsilon\}},$$

so

$$E_\varepsilon(u_\varepsilon^A) \leq C \int_\Omega \left\{ \left(\frac{\mathbb{1}_{\{\phi(u_\varepsilon - A) \geq \varepsilon\}}}{\phi^2(u_\varepsilon - A)} + \varepsilon^{-2} \mathbb{1}_{\{\phi(u_\varepsilon - A) \leq \varepsilon\}} \right) |\nabla u_\varepsilon|^2 + \varepsilon^{-2} \mathbb{1}_{\{\phi(u_\varepsilon - A) \leq \varepsilon\}} \right\}. \quad (134)$$

Now, fix a bounded subset $K \subseteq \mathbf{S}_0$, so large that $u_\varepsilon(x) + B_\delta^{\mathbf{S}_0} \subseteq K$ for a.e. $x \in \Omega$ and any ε (we denote by $B_\delta^{\mathbf{S}_0}$ the set of $Q \in \mathbf{S}_0$ with $|Q| \leq \delta$). We set $K_\varepsilon := K \cap \{\phi \leq \varepsilon\}$. We integrate (134) with respect to $A \in B_\delta^{\mathbf{S}_0}$. We apply Fubini-Tonelli theorem and introduce the new variable $B := u_\varepsilon(x) - A$. We obtain

$$\int_{B_\delta^{\mathbf{S}_0}} E_\varepsilon(u_\varepsilon^A) \, d\mathcal{H}^5(A) \leq C \int_\Omega \left\{ \left(\int_{K \setminus K_\varepsilon} \frac{d\mathcal{H}^5(B)}{\phi^2(B)} + \varepsilon^{-2} \mathcal{H}^5(K_\varepsilon) \right) |\nabla u_\varepsilon|^2 + \varepsilon^{-2} \mathcal{H}^5(K_\varepsilon) \right\} dx.$$

We claim that

$$\mathcal{H}^5(K_\varepsilon) \leq C\varepsilon^2 \quad \text{and} \quad \int_{K \setminus K_\varepsilon} \frac{d\mathcal{H}^5(B)}{\phi^2(B)} \leq C(|\log \varepsilon| + 1). \quad (135)$$

To simplify the presentation, we postpone the proof of this claim. With the help of (135), we obtain

$$\int_{B_\delta^{\mathbf{S}_0}} E_\varepsilon(u_\varepsilon^A) \, d\mathcal{H}^5(A) \leq C \left\{ (|\log \varepsilon| + 1) \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 + 1 \right\} \leq C(|\log \varepsilon| + 1).$$

Therefore, we can choose $A_0 \in \mathbf{S}_0$ such that $|A_0| \leq \delta$ and

$$E_\varepsilon(u_\varepsilon^{A_0}) \leq C(|\log \varepsilon| + 1). \quad (136)$$

The map $u_\varepsilon^{A_0}$ satisfies the desired energy estimate, but it does not satisfy the boundary condition, since

$$u_\varepsilon^{A_0} = \varrho(g_\varepsilon - A_0) \quad \text{on } \partial\Omega. \quad (137)$$

To correct this, we consider the maps $\{\varrho_A\}_{A \in B_\delta^{\mathbf{S}_0}}$ defined by

$$\varrho_A: Q \in \mathcal{N} \mapsto \varrho(Q - A).$$

This is a continuous family of mappings in $C^1(\mathcal{N}, \mathcal{N})$ and $\varrho_0 = \text{Id}_\mathcal{N}$. Therefore, we can choose δ so small that the map $\varrho_A: \mathcal{N} \rightarrow \mathcal{N}$ is a diffeomorphism for any $A \in B_\delta^{\mathbf{S}_0}$ (in particular for $A = A_0$). Let \mathcal{U} be the set defined by

$$\mathcal{U} := \{\lambda Q: \lambda \in \mathbb{R}^+, Q \in \mathcal{N}\}.$$

We extend $\varrho_{A_0}^{-1}$ to a Lipschitz function $\mathcal{F}: \mathcal{U} \rightarrow \mathcal{U}$ by setting

$$\mathcal{F}(\lambda Q) := \lambda \varrho_{A_0}^{-1}(Q) \quad \text{for any } \lambda \in \mathbb{R}^+, Q \in \mathcal{N}.$$

Remark that any $P \in \mathcal{U} \setminus \{0\}$ can be uniquely written in the form $P = \lambda Q$ for $\lambda \in \mathbb{R}^+$ and $Q \in \mathcal{N}$, so \mathcal{F} is well-defined. Also, $f \circ \mathcal{F}(P) = f(P)$ because $\mathcal{F}(P)$ and P have the same scalar invariants. The map $P_\varepsilon := \mathcal{F} \circ u_\varepsilon^{A_0}$ is well-defined, because $u_\varepsilon^{A_0} \in \mathcal{U}$. Moreover, P_ε belongs to $H_{g_\varepsilon}^1(\Omega, \mathbf{S}_0)$ thanks to (137), and satisfies

$$E_\varepsilon(P_\varepsilon) \leq C(|\log \varepsilon| + 1)$$

due to (136). By comparison, the minimizers satisfy (133). \square

The claim (135) follows by this

Lemma 67 *For any $R > 0$, there exist positive constants C_R, M_R such that, for any non increasing, non negative function $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, there holds*

$$\int_{B_R^{\mathbf{S}_0}} (g \circ \phi)(Q) \, d\mathcal{H}^5(Q) \leq C_R \int_0^{M_R} (s + s^4) g(s) \, ds.$$

Assuming that the lemma holds true, choose R so large that $K \subseteq B_R^{\mathbf{S}_0}$. Then, the two assertions of Claim (135) follow by taking $g = \mathbb{1}_{(0, \varepsilon)}$ and $g(s) = \varepsilon^{-2} \mathbb{1}_{(0, \varepsilon)}(s) + s^{-2} \mathbb{1}_{[\varepsilon, +\infty)}(s)$, respectively. For the sake of clarity, we split the proof of Lemma 67 into a few steps. For $r > 0$, we let dist_r denote the geodesic distance in $\partial B_r^{\mathbf{S}_0}$, that is

$$\begin{aligned} \text{dist}_r(Q, A) := \inf \left\{ \int_0^1 |\gamma'(t)| \, dt : \gamma \in C^1([0, 1], \partial B_1^{\mathbf{S}_0}), \right. \\ \left. \gamma(0) = Q, \gamma(1) \in A \right\} \end{aligned} \quad (138)$$

for any $Q \in \partial B_r^{\mathbf{S}_0}$ and $A \subseteq \partial B_r^{\mathbf{S}_0}$, and set $\mathcal{N}'_r := \mathcal{C} \cap \partial B_r^{\mathbf{S}_0}$. Notice that there exists a positive constant C such that

$$\text{dist}_{|Q|}(Q, P) \leq C |Q - P| \quad (139)$$

for any $Q, P \in \mathbf{S}_0$ with $|Q| = |P|$.

Lemma 68 *There exists a positive constant α such that*

$$\phi(Q) \geq \alpha \text{dist}_{|Q|}(Q, \mathcal{N}'_{|Q|}) \quad \text{for any } Q \in \mathbf{S}_0.$$

Proof. Fix $Q \in \mathbf{S}_0$ and $P \in \mathcal{N}'_{|Q|}$. By Lemmas 10 and 12, we can write

$$Q = s \left(\mathbf{n}^{\otimes 2} - \frac{1}{3} \text{Id} \right) + sr \left(\mathbf{m}^{\otimes 2} - \frac{1}{3} \text{Id} \right), \quad P = -s' \left(\mathbf{p}^{\otimes 2} - \frac{1}{3} \text{Id} \right)$$

for some orthonormal pair (\mathbf{n}, \mathbf{m}) , some unit vector \mathbf{p} , s , some positive numbers s, s' and $0 \leq r \leq 1$. Through simple algebra, we obtain

$$\begin{aligned} |Q - P|^2 = \frac{2}{3} s^2 (r^2 - r + 1) - \frac{2}{3} s s' (1 - r) + \frac{2}{3} s'^2 \\ + 2 s s' \left\{ (\mathbf{n} \cdot \mathbf{p})^2 + r (\mathbf{m} \cdot \mathbf{p})^2 \right\}. \end{aligned} \quad (140)$$

Moreover, we have $|P| = |Q|$ if and only if

$$s'^2 = s^2 (r^2 - r + 1). \quad (141)$$

By minimizing (with respect to s' , \mathbf{p}) the right-hand side in (140), subject to the constraint (141), we find

$$\begin{aligned} \text{dist}^2(Q, \mathcal{N}'_1) &= \frac{2}{3} s^2 \sqrt{r^2 - r + 1} \left\{ (1 - r)^2 - \left(\sqrt{r^2 - r + 1} - 1 \right)^2 \right\} \\ &\leq \frac{2}{3} s^2 (1 - r)^2 = \frac{2}{3} s_*^2 \phi^2(Q). \end{aligned}$$

Combining this inequality with (139), the lemma follows. \square

Lemma 69 *Let \mathcal{N}' be a compact n -submanifold of a smooth Riemann m -manifold \mathcal{M} , and let*

$$U_\delta := \{x \in \mathcal{M} : \text{dist}_{\mathcal{M}}(x, \mathcal{N}') \leq \delta\}$$

be the δ -neighborhood of \mathcal{N}' in \mathcal{M} , for $\delta > 0$ (here $\text{dist}_{\mathcal{M}}$ stands for the geodesic distance in \mathcal{M}). There exist $\delta_ > 0$ and, for any $\delta \in (0, \delta_*)$, a constant $C = C(\mathcal{M}, \mathcal{N}', \delta) > 0$ such that for any decreasing function $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ there holds*

$$\int_{U_\delta} h(\text{dist}_{\mathcal{M}}(x, \mathcal{N}')) \, d\mathcal{H}^m(x) \leq C \int_0^{C\delta} s^{m-n-1} h(s) \, ds.$$

Proof. We identify $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}$, and call the variable $y = (y', z) \in \mathbb{R}^n \times \mathbb{R}^{m-n}$. For a small $\delta_* > 0$, the δ_* -neighborhood U_{δ_*} can be covered with finitely many open sets $\{V_j\}_{1 \leq j \leq K}$ and, for each j , there exists a bilipschitz homeomorphism $\varphi_j: V_j \rightarrow W_j \subseteq \mathbb{R}^m$ which maps $\mathcal{N}' \cap V_j$ onto $\mathbb{R}^n \cap W_j$. Due to the bilipschitz continuity of the φ_j 's, there exist two constants γ_1, γ_2 such that, for any j and any $y = (y', z) \in W_j$, there holds

$$\gamma_1 |z| \leq \text{dist}_{\mathcal{M}}(\varphi_j^{-1}(y), \mathcal{N}') \leq \gamma_2 |z|.$$

Therefore, if $0 < \delta < \delta_*$ the change of variable $x = \varphi_j^{-1}(y)$ implies

$$\begin{aligned} \int_{U_\delta} h(\text{dist}_{\mathcal{M}}(x, \mathcal{N}')) \, d\mathcal{H}^m(x) &\leq \sum_{j=1}^K \int_{\varphi_j^{-1}(V_j)} h(\gamma_1 |z|) |J\varphi_j^{-1}(y)| \, d\mathcal{H}^m(y) \\ &\leq M \int_{B^{m-n}(0, \gamma_2 \delta)} h(\gamma_1 |z|) \, d\mathcal{H}^{m-n}(z) \end{aligned}$$

where M is an upper bound for the norm of the Jacobians $J\varphi_j^{-1}$. Then, passing to polar coordinates,

$$\begin{aligned} \int_{U_\delta} h(\text{dist}_{\mathcal{M}}(x, \mathcal{N}')) \, d\mathcal{H}^m(x) &\leq M \int_0^{\gamma_2 \delta} \rho^{m-n-1} h(\gamma_1 \rho) \, d\rho \\ &\leq M \gamma_1^{1+n-m} \int_0^{\gamma_1 \gamma_2 \delta} s^{m-n-1} h(s) \, ds. \quad \square \end{aligned}$$

Proof of Lemma 67. By Lemma 13, the function ϕ is positively homogeneous of degree 1. Then,

$$\int_{B_R^{S_0}} (g \circ \phi)(Q) \, d\mathcal{H}^5(Q) = \int_0^R \rho^4 \int_{\partial B_1^{S_0}} g(\rho\phi(Q)) \, d\mathcal{H}^4(Q) \, d\rho.$$

By applying Lemma 68, and since g is a decreasing function,

$$\int_{B_R^{S_0}} (g \circ \phi)(Q) \, d\mathcal{H}^5(Q) \leq \int_0^R \rho^4 \int_{\partial B_1^{S_0}} g(\alpha\rho \operatorname{dist}_1(Q, \mathcal{N}'_1)) \, d\mathcal{H}^4(Q) \, d\rho.$$

Now, we apply Lemma 69 with $\mathcal{M} = \partial B_1^{S_0}$, $\mathcal{N}' = \mathcal{N}'_1$ and $h: s \mapsto g(\alpha\rho s)$. We find constants δ and C such that, letting U_δ be the δ -neighborhood of \mathcal{N}'_1 in $\partial B_1^{S_0}$ and $V_\delta := \partial B_1^{S_0} \setminus U_\delta$, we have

$$\begin{aligned} & \int_{B_R^{S_0}} (g \circ \phi)(Q) \, d\mathcal{H}^5(Q) \\ &= \int_0^R \rho^4 \left\{ \int_{U_\delta} g(\alpha\rho \operatorname{dist}_1(Q, \mathcal{N}'_1)) \, d\mathcal{H}^4(Q) \right. \\ & \quad \left. + \int_{V_\delta} g(\alpha\rho \operatorname{dist}_1(Q, \mathcal{N}'_1)) \, d\mathcal{H}^4(Q) \right\} d\rho \\ &\leq C \int_0^R \rho^4 \left\{ \int_0^{C\delta} sg(\alpha\rho s) \, ds + g(\alpha\rho\delta) \mathcal{H}^4(V_\delta) \right\} d\rho \end{aligned}$$

(to bound the integral on V_δ , we use again that g is decreasing). Now, the two terms can be easily handled by changing the variables and using Fubini-Tonelli theorem:

$$\begin{aligned} \int_{B_R^{S_0}} (g \circ \phi)(Q) \, d\mathcal{H}^5(Q) &\leq \alpha^{-2} C \int_0^R \rho^2 \int_0^{\alpha\rho\delta C} tg(t) \, dt \, d\rho \\ &\quad + (\alpha\delta)^{-5} C \mathcal{H}^4(V_\delta) \int_0^{\alpha\delta R} t^4 g(t) \, dt \\ &\leq C_{\alpha,\delta,R} \int_0^{C_{\alpha,\delta,R}} (t + t^4) g(t) \, dt. \end{aligned}$$

Since α, δ depend only on ϕ and \mathcal{N}'_1 , the lemma is proved. \square

6.2 Proof of Propositions 4 and 6

Condition (H) may be satisfied even if the boundary datum is not \mathcal{N} -valued. As we show in this section, if the domain satisfies a topological condition (H₂) and if boundary datum is smooth (at least $H^1 \cap L^\infty$), satisfying a uniform L^∞ -bound and a logarithmic energy estimate (H₃), then (132)–(133) are satisfied

for some constant $M = M(\Omega, M_0) > 0$. This proves Proposition 4. In particular, we prove the logarithmic energy bound (133) by constructing an admissible comparison function, whose energy is controlled by the right-hand side of (133). If Ω is a ball, it suffices to extend homogeneously the boundary data. Since Ω is bilipschitz equivalent to a handlebody by (H_2) , we reduce to the case of a ball by cutting each handle of Ω along a meridian disk. This technique was used already in [37, Lemma 1.1]. At the end of the section, we also prove Proposition 6.

Lemma 70 *Assume that (H_2) and (H_3) hold. There exists a finite number of properly embedded disks³ $D_1, D_2, \dots, D_k \subseteq \Omega$ such that $\Omega \setminus \cup_{i=1}^k D_i$ is diffeomorphic to a ball,*

$$E_\varepsilon(g_\varepsilon, \partial D_i) \leq C(|\log \varepsilon| + 1) \quad (142)$$

and

$$\text{dist}(g_\varepsilon(x), \mathcal{N}) \rightarrow 0 \quad \text{uniformly in } x \in \bigcup_{i=1}^k \partial D_i. \quad (143)$$

Proof. For each handle i of Ω , there is an open set U_i such that $\partial\Omega \cap U_i$ is foliated by

$$\partial\Omega \cap U_i = \coprod_{-a_0 < a < a_0} \partial D_i^a,$$

where the generic D_i^a is a properly embedded disk, which cross transversely a generator of $\pi_1(\Omega)$ at some point. Then, Fatou's lemma implies that

$$\int_{-a_0}^{a_0} \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(g_\varepsilon, \partial D_i^a) da \leq \liminf_{\varepsilon \rightarrow 0} \int_{-a_0}^{a_0} E_\varepsilon(g_\varepsilon, \partial D_i^a) da \stackrel{(H_3)}{\leq} C(|\log \varepsilon| + 1),$$

so, by an average argument, we can choose the parameter a in such a way that $D_i := D_i^a$ satisfies (142). Then, (143) is obtained by the same arguments as Lemma 38. (As in the lemma, we apply Sobolev embedding inequality not on ∂D_i directly, but on 1-cells $K \subseteq \partial D_i$ of size comparable to $\varepsilon^\alpha |\log \varepsilon|$.) Furthermore, by construction $\Omega \setminus \cup_{i=1}^k D_i$ is a ball, since we have removed a meridian disk for each handle of Ω . \square

Proof of Proposition 4. The L^∞ -bound (132) holds by virtue of Lemma 66, so we only need to prove (133). Assume for a moment that $\Omega = B_1$. In this case, define the function

$$P_\varepsilon(x) := g_\varepsilon\left(\frac{x}{|x|}\right) \quad \text{for } x \in B_1. \quad (144)$$

Then $P_\varepsilon \in H_{g_\varepsilon}^1(B_1, \mathbf{S}_0)$ and we easily compute

$$E_\varepsilon(P_\varepsilon) = \int_0^1 \int_{\partial B_1} \left(|\nabla_\top g_\varepsilon|^2 + \varepsilon^{-2} \rho^2 f(g_\varepsilon) \right) d\mathcal{H}^2 d\rho \stackrel{(H_3)}{\leq} C(|\log \varepsilon| + 1),$$

³ By saying that D_i is *properly* embedded, we mean that $\partial D_i = D_i \cap \partial\Omega$ and D_i is transverse to $\partial\Omega$ at each point of ∂D_i .

so the lemma holds true when $\Omega = B_1$.

Now, arguing as in [37, Lemma 1.1], we prove that the general case can be reduced to the previous one. Let Ω be any domain satisfying (H_2) , and let D_1, \dots, D_k be the disks given by Lemma 70. By (143), there exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon \leq \varepsilon_0$ and any $x \in \cup_i \partial D_i$,

$$\text{dist}(g_\varepsilon(x), \mathcal{N}) \leq \delta_0.$$

For ease of notation, for a fixed $i \in \{1, \dots, k\}$ we assume, up to a bilipschitz equivalence, that $D_i = B_1^2$. Then, we define $\hat{g}_{\varepsilon,i}: B_1^2 \rightarrow \mathbf{S}_0$ by

$$\hat{g}_{\varepsilon,i}(x) := \begin{cases} \frac{\delta_0 + |x| - 1}{\delta_0} g_\varepsilon\left(\frac{x}{|x|}\right) + \frac{1 - |x|}{\delta_0} (\varrho \circ g_\varepsilon)\left(\frac{x}{|x|}\right) & \text{if } |x| \geq 1 - \delta_0 \\ v_\varepsilon\left(\frac{x}{1 - \delta_0}\right) & \text{if } |x| \leq 1 - \delta_0, \end{cases}$$

where $v_\varepsilon \in H^1(B_1^2, \mathbf{S}_0)$ is the extension of $\varrho \circ g_\varepsilon|_{\partial B_1^2}$ given by Lemma 28. By a straightforward computation, one checks that

$$E_\varepsilon(\hat{g}_{\varepsilon,i}, D_i) \leq C(E_\varepsilon(g_\varepsilon, \partial D_i) + |\log \varepsilon| + 1). \quad (145)$$

Now, consider two copies D_i^+ and D_i^- of each disk D_i . Let Ω' be a smooth domain such that

$$\Omega' \simeq (\Omega \setminus \cup_i D_i) \cup_i D_i^+ \cup_i D_i^-,$$

and let $\varphi: \Omega' \rightarrow \Omega$ be the smooth map which identifies each D_i^+ with the corresponding D_i^- (see Figure 5). This new domain is simply connected, and in fact is diffeomorphic to a ball. Up to a bilipschitz equivalence, we will assume that Ω' is a ball. We define a boundary datum g'_ε for Ω' by setting $g'_\varepsilon := g_\varepsilon$ on $\Omega \setminus \cup_i D_i$, and $g'_\varepsilon := g_{\varepsilon,i}$ on $D_i^+ \cup D_i^-$. Then, (145), (142) and (H_3) imply

$$E_\varepsilon(g'_\varepsilon, \partial \Omega) \leq C(E_\varepsilon(g_\varepsilon, \partial \Omega) + |\log \varepsilon| + 1) \leq C(|\log \varepsilon| + 1).$$

Then Formula (144) gives a map $P'_\varepsilon \in H_{g'_\varepsilon}^1(\Omega', \mathbf{S}_0)$ which satisfies

$$E_\varepsilon(P'_\varepsilon, \Omega') \leq C(|\log \varepsilon| + 1).$$

Since $P'_{\varepsilon|D_i^+} = P'_{\varepsilon|D_i^-}$ for every i , the map P'_ε factorizes through φ , and defines a new function $P_\varepsilon \in H_{g_\varepsilon}^1(\Omega, \mathbf{S}_0)$ such that

$$E_\varepsilon(P_\varepsilon, \Omega) \leq C(|\log \varepsilon| + 1).$$

By comparison, we conclude that (133) holds for any $0 < \varepsilon \leq \varepsilon_0$. Now, fix $\varepsilon_0 < \varepsilon < 1$ and consider the $(\mathbf{S}_0\text{-valued})$ harmonic extension \tilde{P}_ε of g_ε . There holds

$$\begin{aligned} \|\nabla \tilde{P}_\varepsilon\|_{L^2(\Omega)}^2 &\leq C \|\nabla g_\varepsilon\|_{L^2(\partial \Omega)}^2 \stackrel{(H_3)}{\leq} C(|\log \varepsilon| + 1), \\ \|\tilde{P}_\varepsilon\|_{L^\infty(\Omega)}^2 &\leq C \|g_\varepsilon\|_{L^\infty(\partial \Omega)}^2 \stackrel{(H_3)}{\leq} C \end{aligned}$$

and so

$$E_\varepsilon(P_\varepsilon) \leq \frac{1}{2} \|\nabla \tilde{P}_\varepsilon\|_{L^2(\Omega)}^2 + C\varepsilon_0^{-2} \leq C(1 + \varepsilon_0^2)(|\log \varepsilon| + 1).$$

Also in this case, the lemma follows by comparison. \square

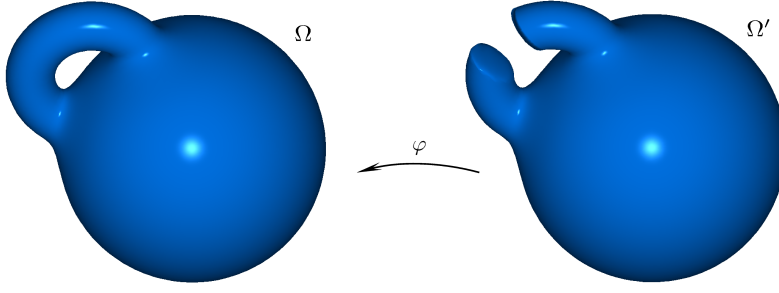


Fig. 5 On the left, a ball with one handle. On the right, the corresponding domain Ω' : the handle has been cut along a disk. The map $\varphi: \Omega' \rightarrow \Omega$ identifies the opposite disks in the handle cut.

Proof of Proposition 6. Up to rotations and translations, we can assume that the x_3 -axis $\{x_1 = x_2 = 0\}$ crosses transversely $\partial\Omega$ at one point x_0 at least. Let $\eta_\varepsilon \in C^\infty(\mathbb{R}^+, \mathbb{R})$ be a cut-off function satisfying

$$\eta_\varepsilon(0) = \eta'_\varepsilon(0) = 0, \quad \eta_\varepsilon(r) = s_* \text{ for } r \geq \varepsilon, \quad 0 \leq \eta_\varepsilon \leq s_*, \quad |\eta'_\varepsilon| \leq C\varepsilon^{-1}.$$

Set

$$g_\varepsilon(x) := \eta_\varepsilon(|x'|) \left\{ \left(\frac{x'}{|x'|} \right)^{\otimes 2} - \frac{1}{3} \text{Id} \right\} \quad \text{for } x \in \partial\Omega,$$

where $x' := (x_1, x_2, 0)$. Computing as in Lemma 28, one sees that $\{g_\varepsilon\}$ satisfies (H_3) . It remains to prove that the energy of minimizers Q_ε satisfies a logarithmic lower bound. Take a ball $B_r(x_0)$. If the radius r is small enough, the set $\Omega \cap B_r(x_0)$ can be mapped diffeomorphically onto the half-ball

$$U := \{x \in \mathbb{R}^3: |x| \leq 1, x_3 \geq 0\},$$

so we can assume WLOG that $\Omega \cap B_r(x_0) = U$. Let $U_s := \{x \in U: x_3 = s\}$, for $r/2 \leq s \leq r$. The map $Q_\varepsilon|_{\partial U_s}: \partial U_s \rightarrow \mathcal{N}$ is a homotopically non-trivial loop, which satisfies $E_\varepsilon(Q_\varepsilon, \partial U_s) \leq C$. Then, by applying Corollary 16 we deduce

$$E_\varepsilon(Q_\varepsilon, U_s) \geq \kappa_* \log \frac{s}{\varepsilon} - C$$

for a constant C depending on r, Ω . By integrating this bound for $s \in (r/2, r)$, the proposition follows. \square

7 Coexistence of line and point singularities: an example

In this section, we show through an example that both the set of line singularities $\mathcal{S}_{\text{line}}$ and the set of point singularities \mathcal{S}_{pts} may be non-empty. We consider the following domain. For two fixed positive numbers

$$L > 0 \quad \text{and} \quad 0 < r < \frac{1}{2}, \quad (146)$$

define

$$p_{\pm} := (\pm(L+1), 0, 0)$$

$$\Omega_{\pm} := B_1(p_{\pm}), \quad \Omega_0 := ([-L-1, L+1] \times B_r^2(0)) \setminus (\Omega_- \cup \Omega_+)$$

and $\Omega := \Omega_- \cup \Omega_0 \cup \Omega_+$. In other words, the domain consists of two balls joined by a cylinder about the x_1 -axis (see Figure 1). This is a Lipschitz domain; however, one could consider a smooth domain Ω' , obtained from Ω by “smoothing the corners”, and the arguments of this section could be easily adapted to Ω' .

We write $\partial\Omega = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$, where $\Gamma_{\pm} := \partial\Omega_{\pm} \setminus \Omega_0$ and $\Gamma_0 := \partial\Omega_0 \setminus (\overline{\Omega}_+ \cup \overline{\Omega}_-)$. We define the auxiliary functions $\chi \in H^1(0, 2\pi)$, $\eta_{\varepsilon} \in H^1(0, \pi)$ and $\xi_r \in H^1(0, \pi)$ by

$$\chi(\theta) := \begin{cases} \pi/2 - 3\theta/5 & \text{for } 0 \leq \theta \leq 5\pi/6 \\ 0 & \text{for } 5\pi/6 < \theta < 7\pi/6 \\ 7\pi/10 - 3\theta/5 & \text{for } 7\pi/6 \leq \theta \leq 2\pi, \end{cases}$$

$$\eta_{\varepsilon}(\varphi) := \begin{cases} \varepsilon^{-1}\varphi & \text{for } 0 \leq \varphi \leq \varepsilon \\ 1 & \text{for } \varepsilon < \varphi < \pi - \varepsilon \\ \varepsilon^{-1}(\pi - \varphi) & \text{for } \pi - \varepsilon \leq \varphi \leq \pi \end{cases}$$

and

$$\xi_r(\tilde{\varphi}) := \begin{cases} 0 & \text{if } 0 \leq \tilde{\varphi} \leq \arcsin r \\ \frac{\arcsin 2r (\tilde{\varphi} - \arcsin r)}{\arcsin 2r - \arcsin r} & \text{if } \arcsin 2r < \tilde{\varphi} < \arcsin 2r \\ \tilde{\varphi} & \text{if } \arcsin 2r \leq \tilde{\varphi} \leq \pi. \end{cases}$$

Notice that

$$|\eta'_{\varepsilon}| \leq \varepsilon^{-1} \quad \text{and} \quad |\xi'_r| \leq 2. \quad (147)$$

The boundary datum g_{ε} is defined as follows. We parametrize Γ_+ using spherical coordinates $(\theta, \varphi) \in [0, 2\pi] \times [0, \pi]$ centered at p_+ :

$$x_1 = L + 1 + \sin \varphi \cos \theta, \quad x_2 = \sin \varphi \sin \theta, \quad x_3 = \cos \varphi$$

and define g_{ε} on Γ_+ by

$$g_{\varepsilon}(x_1, x_2, x_3) := s_* \eta_{\varepsilon}(\varphi) \left\{ \left(\mathbf{e}_1 \cos \chi(\theta) + \mathbf{e}_2 \sin \chi(\theta) \right)^{\otimes 2} - \frac{1}{3} \text{Id} \right\}. \quad (148)$$

On Γ_- , we use spherical coordinates about the x_1 -axis, i.e. $(\tilde{\theta}, \tilde{\varphi}) \in [0, 2\pi] \times [0, \pi]$ given by

$$x_1 = -L - 1 + \cos \tilde{\varphi}, \quad x_2 = \sin \tilde{\varphi} \cos \tilde{\theta}, \quad x_3 = \sin \tilde{\varphi} \sin \tilde{\theta}$$

and set

$$g_\varepsilon(x_1, x_2, x_3) := s_* \left\{ \left(\mathbf{e}_1 \cos \xi_r(\tilde{\varphi}) + \mathbf{e}_2 \sin \xi_r(\tilde{\varphi}) \cos \tilde{\theta} + \mathbf{e}_3 \sin \xi_r(\tilde{\varphi}) \sin \tilde{\theta} \right)^{\otimes 2} - \frac{1}{3} \text{Id} \right\}. \quad (149)$$

Finally, we set

$$g_\varepsilon := s_* \left(\mathbf{e}_1^{\otimes 2} - \frac{1}{3} \text{Id} \right) \quad \text{on } \Gamma_0.$$

This defines a map $g_\varepsilon \in H^1(\partial\Omega, \mathbf{S}_0)$ which is non-orientable with two point disclinations on Γ_+ , is constant on Γ_0 and has a hedgehog-type behaviour on Γ_- . In Figure 1, we represent the direction of the eigenvector associated with the leading eigenvalue of $g_\varepsilon(x)$, for $x \in \partial\Omega$. One could regularize the functions χ , η_ε and ξ_r so that the map g_ε is smooth; this would not affect our arguments. Pick a subsequence $\varepsilon_n \searrow 0$ so that the measures μ_{ε_n} defined by (84) converge weakly* in $C(\overline{\Omega})'$ to a measure μ_0 , and let $\mathcal{S}_{\text{line}} \subseteq \overline{\Omega}$ be the support of μ_0 . Let $\mathcal{S}_{\text{pts}} \subseteq \Omega \setminus \mathcal{S}_{\text{line}}$ be a set such that the sequence $\{Q_{\varepsilon_n}\}_{n \in \mathbb{N}}$ is compact in $C^0(\Omega \setminus (\mathcal{S}_{\text{line}} \cup \mathcal{S}_{\text{pts}}), \mathbf{S}_0)$. By Theorem 1, such a set exists and is locally finite in $\Omega \setminus \mathcal{S}_{\text{line}}$. We will show the following result, which implies Proposition 7.

Proposition 71 *There exists L^* such that, if*

$$L \geq L^*$$

then $\emptyset \neq \mathcal{S}_{\text{line}} \subseteq \{x_1 \geq 0\}$ and $\mathcal{S}_{\text{pts}} \cap \{x_1 \leq -L/2\} \neq \emptyset$.

Remark 72 The presence of a point defect is *not* forced by a topological obstruction. In other words, there exists maps $P_\varepsilon: \Omega \rightarrow \mathbf{S}_0$ which satisfy $P_\varepsilon = g_\varepsilon$ on $\partial\Omega$ and converge to a map with a line singularity but no point singularity. Indeed, let $\varphi: \overline{\Omega} \rightarrow \overline{B}_1$ be a bilipschitz homeomorphism such that $\varphi(L, 0, \pm 1) = (0, 0, \pm 1)$. Then, the functions

$$P_\varepsilon(x) := g_\varepsilon \circ \varphi^{-1} \left(\frac{\varphi(x)}{|\varphi(x)|} \right)$$

converge a.e. to a map with a line singularity $\mathcal{S}_{\text{line}} := \varphi^{-1}\{x_1 = x_2 = 0\}$, but no point singularities. The convergence also holds in $H_{\text{loc}}^1(\Omega \setminus \mathcal{S}_{\text{line}}, \mathbf{S}_0)$.

We split the proof of Proposition 71 into some lemmas. Throughout the section, we use the symbol C to denote a generic constant, which does not depend on ε , L and r .

Lemma 73 *There exists a constant M , independent of L and r , such that*

$$E_\varepsilon(Q_\varepsilon, \Omega) \leq M(|\log \varepsilon| + 1) \quad \text{and} \quad \|Q_\varepsilon\|_{L^\infty(\Omega)} \leq M$$

for any $0 < \varepsilon < 1$.

Proof. The L^∞ -bound follows by Lemma 66, since $|g_\varepsilon(x)| \leq (2/3)^{1/2}s_*$ for a.e. $x \in \partial\Omega$ and any $0 < \varepsilon < 1$. The energy bound follows by a comparison argument. We define a map P_ε on Ω_+ and Ω_- by homogeneous extension:

$$P_\varepsilon(x) := g_\varepsilon \left(p_+ + \frac{x - p_\pm}{|x - p_\pm|} \right) \quad \text{if } x \in \Omega_\pm,$$

and we set

$$P_\varepsilon(x) := s_* \left(\mathbf{e}_1^{\otimes 2} - \frac{1}{3} \text{Id} \right) \quad \text{if } x \in \Omega_0.$$

(Here we assume that g_ε is defined also on $\partial\Omega_\pm \setminus \Gamma_\pm$, by the same formulae (148)–(149).) Then, the map P_ε is continuous and belongs to $H^1(\Omega, \mathbf{S}_0)$. Moreover, $E_\varepsilon(P_\varepsilon, \Omega_0) = 0$ since $P_\varepsilon|_{\Omega_0}$ is constant and takes values in \mathcal{N} . A simple computation, based on (147), concludes the proof. \square

For any $s \in \mathbb{R}$, let $U_s := \Omega \cap \{x_1 < s\}$ and $G_\varepsilon(s) := E_\varepsilon(Q_\varepsilon, U_s)$. Fubini-Tonelli theorem entails that $G'_\varepsilon(s) = E_\varepsilon(Q_\varepsilon, \Omega \cap \{x_1 = s\})$ for a.e. s .

Lemma 74 *There exist positive constants L_* and M such that, for any $L \geq L_*$ and any $0 < \varepsilon \leq 1/2$, there holds*

$$G_\varepsilon(0) \leq M.$$

In particular, if $L \geq L_$ then $\mathcal{S}_{\text{line}} \subseteq \overline{\Omega} \setminus U_0$.*

Proof. This proof is based on the same arguments as Proposition 8. Define the set

$$D^\varepsilon := \left\{ s \in (0, L) : G'_\varepsilon(s) \leq \frac{2M}{L} (|\log \varepsilon| + 1) \right\}.$$

By Lemma 73 and an average argument, we know that

$$\mathcal{H}^1(D^\varepsilon) \geq \frac{L}{2}. \quad (150)$$

Moreover, there exists $L_* > 0$ such that, for any $L \geq L_*$, any $0 < \varepsilon \leq 1/2$ and any $s \in D^\varepsilon$, there holds

$$G'_\varepsilon(s) \leq \eta_0 |\log \varepsilon|$$

where η_0 is the constant given by Proposition 33. Therefore, for a fixed $s \in D^\varepsilon$ we can apply Proposition 33 to the map $u_\varepsilon := Q_\varepsilon|_{\{s\} \times B_r^2}$. Notice that u_ε is defined on a disk, while the maps we consider in Proposition 33 are defined over a sphere. However, since u_ε takes a constant value on the boundary, it can be identified with a map defined on a sphere by collapsing $\{s\} \times \partial B_r^2$ into a point. Setting $h(\varepsilon) := \varepsilon^{1/2} |\log \varepsilon|$ and $A_\varepsilon := (s - h(\varepsilon), s) \times B_r^2$, we find maps $v_\varepsilon : \{s - h(\varepsilon)\} \times B_r^2 \rightarrow \mathcal{N}$ and $\varphi_\varepsilon : A_\varepsilon \rightarrow \mathbf{S}_0$ such that

$$\frac{1}{2} \int_{\{s-h(\varepsilon)\} \times B_r^2} |\nabla v_\varepsilon|^2 \, d\mathcal{H}^2 \leq G'_\varepsilon(s), \quad E_\varepsilon(\varphi_\varepsilon) \leq Ch(\varepsilon) |\log \varepsilon|. \quad (151)$$

Now, consider the set $V_s := [s - h(\varepsilon) - r, s - h(\varepsilon)] \times B_r^2$ (we assume that $L_* > 2$, so that $s - h(\varepsilon) - r > -L$ for $\varepsilon \leq 1/2$) and the map $\tilde{v}_\varepsilon \in H^1(\partial V_s, \mathcal{N})$ given by $\tilde{v}_\varepsilon := v_\varepsilon$ on $\{s\} \times B_r^2$,

$$\tilde{v}_\varepsilon := s_* \left(\mathbf{e}_1^{\otimes 2} - \frac{1}{3} \text{Id} \right) \quad \text{on } \partial V_s \setminus (\{s\} \times B_r^2).$$

Thanks to (151), we have

$$\frac{1}{2} \int_{\partial V_s} |\nabla \tilde{v}_\varepsilon|^2 \, d\mathcal{H}^2 = \frac{1}{2} \int_{\{s\} \times B_r^2} |\nabla v_\varepsilon|^2 \, d\mathcal{H}^2 \leq G'_\varepsilon(s).$$

Then, by applying Lemma 26 (which is possible because V_s is bilipschitz equivalent to a ball), we find a map $w_\varepsilon \in H^1(V_s, \mathcal{N})$ such that $w_\varepsilon = \tilde{v}_\varepsilon$ on ∂V_s and

$$\frac{1}{2} \int_{V_s} |\nabla w_\varepsilon|^2 \leq C G'_\varepsilon(s)^{1/2}, \quad (152)$$

for a constant C independent of ε , L , r . (Here we have used that $r < 1$.) Finally, we define a map $\tilde{w}_\varepsilon \in H^1(U_s, \mathbf{S}_0)$ as follows. We set $\tilde{w}_\varepsilon := \varphi_\varepsilon$ on $[s - h(\varepsilon), s] \times B_r^2$ and $\tilde{w}_\varepsilon := w_\varepsilon$ on V_s ,

$$\tilde{w}_\varepsilon := s_* \left(\mathbf{e}_1^{\otimes 2} - \frac{1}{3} \text{Id} \right) \quad \text{on } \Omega_0 \setminus ([s - h(\varepsilon) - r, s] \times B_r^2)$$

and use an homogeneous extension to construct \tilde{w}_ε on Ω_- :

$$\tilde{w}_\varepsilon(x) := g_\varepsilon \left(p_- + \frac{x - p_-}{|x - p_-|} \right) \quad \text{for } x \in \Omega_-.$$

The map \tilde{w}_ε is continuous, satisfies $E_\varepsilon(\tilde{w}_\varepsilon) = 0$ on $\Omega_0 \setminus ([s - h(\varepsilon) - r, s] \times B_r^2)$ and $E_\varepsilon(\tilde{w}_\varepsilon, \Omega_-) \leq C$ because of (147). Thus, from (151) and (152) we infer

$$E_\varepsilon(\tilde{w}_\varepsilon, U_s) \leq C G'_\varepsilon(s)^{1/2} + C.$$

Moreover, \tilde{w}_ε is an admissible competitor for Q_ε , because $\tilde{w}_\varepsilon = Q_\varepsilon$ on ∂U_s . Then, the minimality of Q_ε yields

$$G_\varepsilon(s) \leq C G'_\varepsilon(s)^{1/2} + C \quad (153)$$

for a.e. $s \in D^\varepsilon$ and every $0 < \varepsilon \leq 1/2$. Thanks to (150) and (153), we apply Lemma 46 to $y := G_\varepsilon$ and obtain

$$G_\varepsilon(0) \leq \left(1 + \frac{2}{L} \right) C \leq \left(1 + \frac{2}{L_*} \right) C =: M$$

for every $0 < \varepsilon \leq 1/2$. Therefore, $\mu_\varepsilon \llcorner U_0 \rightarrow 0$ in $\mathcal{M}_b(U_0) := C_0(U_0)'$ and hence $\mathcal{S}_{\text{line}} \subseteq \overline{\Omega} \setminus U_0$. \square

Before concluding the proof of Proposition 71, we recall a well-known fact. Given $r > 0$ and a continuous map $\mathbf{n}: B_r^2 \rightarrow \mathbb{S}^2$ that takes a constant value on ∂B_r^2 , it is possible to define the topological degree of \mathbf{n} . Indeed, the topological space which is obtained by collapsing the boundary of ∂B_r^2 into a point is homeomorphic to a sphere. Then, since $\mathbf{n}|_{\partial B_r^2}$ is a constant, \mathbf{n} induces a continuous map $\mathbb{S}^2 \rightarrow \mathbb{S}^2$ whose homotopy class is characterized by an integer number d called the degree of \mathbf{n} . We will write $d =: \deg(\mathbf{n}, B_r^2)$. In case $\mathbf{n} \in H^1(B_r^2, \mathbb{S}^2)$ takes a constant value at the boundary, the degree of \mathbf{n} can still be defined (for instance, one can apply the VMO-theory by Brezis and Nirenberg [19, 20]).

Lemma 75 *For any $r > 0$ and any $\mathbf{n} \in H^1(B_r^2, \mathbb{S}^2)$ with constant value at the boundary, if*

$$\frac{1}{2} \int_{B_r^2} |\nabla \mathbf{n}|^2 \, d\mathcal{H}^2 < 4\pi$$

then $\deg(\mathbf{n}, B_r^2) = 0$.

Proof. By applying the area formula, we obtain

$$\int_{B_r^2} |\partial_{x_1} \mathbf{n} \times \partial_{x_2} \mathbf{n}| \, d\mathcal{H}^2 = \int_{\mathbf{n}(B_r^2)} \mathcal{H}^0(\mathbf{n}^{-1}(y)) \, d\mathcal{H}^2(y) \geq \mathcal{H}^2(\mathbf{n}(B_r^2)).$$

On the other hand, we have $|\partial_{x_1} \mathbf{n} \times \partial_{x_2} \mathbf{n}| \leq |\partial_{x_1} \mathbf{n}| |\partial_{x_2} \mathbf{n}| \leq |\nabla \mathbf{n}|^2 / 2$. Therefore, there holds

$$\frac{1}{2} \int_{B_r^2} |\nabla \mathbf{n}|^2 \, d\mathcal{H}^2 \geq \mathcal{H}^2(\mathbf{n}(B_r^2)).$$

If the left-hand side is $< 4\pi$, then \mathbf{n} is not surjective and so $\deg(\mathbf{n}, B_r^2) = 0$ (see e.g. [19, Property 1]). \square

Proof of Proposition 71. Arguing as in the proof of Proposition 6, and using that the boundary conditions $g_\varepsilon|_{U_+}$ are non orientable, one shows that

$$E_\varepsilon(Q_\varepsilon) \geq C(|\log \varepsilon| - 1)$$

for any ε, L and r , so $\mathcal{S}_{\text{line}} \neq \emptyset$. By Lemma 74, there exists L_* such that $\mathcal{S}_{\text{line}} \subseteq \Omega \setminus U_0$ if $L \geq L_*$. Set

$$L^* := \max \left\{ L_*, \frac{M}{\pi s_*} \right\} \quad (154)$$

where M is given by Lemma 74, and let $L \geq L^*$. The proposition follows once we show that $\mathcal{S}_{\text{pts}} \cap U_{-L/2} \neq \emptyset$.

By applying Lemma 74, Theorem 1 and Corollary 54, we deduce that Q_{ε_n} converges to Q_0 in $H^1(U_{-\delta}, \mathbf{S}_0)$ for every $\delta > 0$, where $Q_0: U_0 \rightarrow \mathcal{N}$ is a locally minimizing harmonic map. Passing to the limit as $\varepsilon \rightarrow 0$ in Lemma 74, we see that

$$\frac{1}{2} \int_{U_0} |\nabla Q_0|^2 \leq M \quad (155)$$

for an L -independent constant M . In particular, $Q_0 \in H^1(U_0, \mathcal{N})$. An average argument, combined with Lemma 155, shows that there exists $-L < s < -L/2$ such that $Q_0 \in H^1(\{s\} \times B_r^2, \mathcal{N})$ and

$$\frac{1}{2} \int_{\{s\} \times B_r^2} |\nabla Q_0|^2 \, d\mathcal{H}^2 \leq \frac{4M}{L}.$$

Due to Lemma 31, we find a lifting of $Q_0|_{\partial U_s}$, i.e. a map $\mathbf{n}_0 \in H^1(\partial U_s, \mathbb{S}^2)$ which satisfies (35) and $|\nabla Q_0|^2 = 2s_* |\nabla \mathbf{n}_0|^2$ a.e. Then, we have

$$\frac{1}{2} \int_{\{s\} \times B_r^2} |\nabla \mathbf{n}_0|^2 \, d\mathcal{H}^2 \leq \frac{2M}{s_* L}. \quad (156)$$

Combining (156) with (154), we deduce

$$\frac{1}{2} \int_{\{s\} \times B_r^2} |\nabla \mathbf{n}_0|^2 \, d\mathcal{H}^2 \leq 2\pi.$$

Moreover, \mathbf{n}_0 takes a constant value on the boundary of $\{s\} \times B_r^2$, since Q_0 does. Then, by Lemma 75, $\deg(\mathbf{n}_0, \{s\} \times B_r^2) = 0$. On the other hand, $\deg(\mathbf{n}_0, \partial U_s \cap \Gamma_0) = 0$ since \mathbf{n}_0 takes a constant value on $\partial U_s \cap \Gamma_0$, and $\deg(\mathbf{n}_0, \Gamma_-)$ can be computed explicitly thanks to (149). This yields

$$\deg(\mathbf{n}_0, \partial U_s) = \deg(\mathbf{n}_0, \Gamma_-) = \pm 1,$$

so the map $Q_0|_{\partial U_s}$ is homotopically non-trivial and $\mathcal{S}_{\text{pts}} \cap U_s \neq \emptyset$. \square

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