



# Weighted envy-freeness for submodular valuations

Luisa Montanari<sup>1</sup> · Ulrike Schmidt-Kraepelin<sup>2</sup> · Warut Suksompong<sup>3</sup> · Nicholas Teh<sup>4</sup>

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## Abstract

We investigate the fair allocation of indivisible goods to agents with possibly different entitlements represented by weights. Previous work has shown that guarantees for additive valuations with existing envy-based notions cannot be extended to the case where agents have matroid-rank (i.e., binary submodular) valuations. We propose two families of envy-based notions for matroid-rank and general submodular valuations, one based on the idea of transferability and the other on marginal values. We show that our notions can be satisfied via generalizations of rules such as picking sequences and maximum weighted Nash welfare. In addition, we introduce welfare measures based on harmonic numbers, and show that variants of maximum weighted harmonic welfare offer stronger fairness guarantees than maximum weighted Nash welfare under matroid-rank valuations.

## 1 Introduction

Fair division refers to the study of how to fairly allocate resources among agents with possibly differing preferences. Over the 75 years since Steinhaus (1948) initiated a mathematical framework of fair division, the field has given rise to numerous fairness notions and procedures for computing fair outcomes in a variety of scenarios (Brams and Taylor 1996; Robertson and Webb 1998; Moulin 2003, 2019). For instance, in the common scenario of allocating indivisible goods—such as furniture, office supplies, or electronic devices—the notion *envy-freeness up to one good (EF1)* has emerged

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✉ Warut Suksompong  
warut@comp.nus.edu.sg

- <sup>1</sup> Technische Universität Berlin, Berlin, Germany
- <sup>2</sup> TU Eindhoven, Eindhoven, The Netherlands
- <sup>3</sup> National University of Singapore, Singapore, Singapore
- <sup>4</sup> University of Oxford, Oxford, UK

as a standard benchmark. An allocation of the goods satisfies EF1 if any envy that an agent has toward another agent can be eliminated by removing some good in the latter agent's bundle. Even when agents have arbitrary monotone valuations over the goods, an EF1 allocation always exists and can be found in polynomial time (Lipton et al. 2004; Budish 2011). To see that the “up to one good” relaxation is necessary in the context of indivisible goods, observe that if there are two agents and only one valuable good to be allocated (and possibly other goods of no value), then no envy-free allocation exists.

The definitions of many fairness notions in the literature, including EF1, inherently assume that all agents have the same entitlement to the resource. In the past few years, several researchers have examined a more general model in which different agents may have different *weights* reflecting their entitlements to the goods (Farhadi et al. 2019; Aziz et al. 2020, 2023; Babaioff et al. 2021b, 2024; Chakraborty et al. 2021a, b, 2024; Suksompong and Teh 2022, 2023; Scarlett et al. 2023; Viswanathan and Zick 2023a; Hoefer et al. 2024).<sup>1</sup> This model allows us to capture settings such as inheritance division, in which relatives are typically entitled to unequal shares of the legacy, as well as resource allocation among groups or organizations of different sizes. Moreover, weighted fair division extends the well-studied setting of *apportionment*, wherein all goods are identical (e.g., seats in a parliament) (Balinski and Young 2001; Pukelsheim 2014). Chakraborty et al. (2021a) generalized EF1 to *weighted EF1 (WEF1)*: for instance, if Alice's weight is three times as high as Bob's, then WEF1 stipulates that, after removing some good from Bob's bundle, Alice should have at least three times as much value for her own bundle as for Bob's. The same authors demonstrated that if agents have *additive* valuations over the goods, a complete WEF1 allocation always exists and can be computed efficiently.<sup>2</sup> However, they provided the following example showing that existence is no longer guaranteed once we move beyond additivity.

**Example 1.1** (Chakraborty et al. 2021a) Consider an instance with  $n = 2$  agents whose weights are  $w_1 = 1$  and  $w_2 = 2$ , and  $m \geq 6$  goods. Agent 1 has an additive valuation with value 1 for every good, whereas agent 2 has value 0 for the empty bundle and 1 for any nonempty bundle.

If agent 1 is allocated more than one good, then agent 2 has weighted envy toward agent 1 even after removing any good from agent 1's bundle. Thus, assuming that all goods need to be allocated, agent 2 must obtain at least  $m - 1$  goods in a WEF1 allocation. Again, this causes weighted envy according to WEF1, this time from agent 1 toward agent 2. Hence, no complete WEF1 allocation exists in this instance.

The impossibility result illustrated in this example still holds even if WEF1 is relaxed to *weak WEF1 (WWEF1)*, whereby an agent is allowed to either remove a good from the other agent's bundle or copy one such good into her own bundle, and stands in contrast to the aforementioned EF1 guarantee in the unweighted setting (Lipton et al. 2004). In fact, the impossibility persists even with  $WWEF1_c$  for any constant  $c$  (Chakraborty et al. 2021a, Sec. 8). In light of these observations, Chakraborty et al. left open the direction of identifying appropriate envy-based notions for non-additive val-

<sup>1</sup> This line of work was summarized in a recent survey by Suksompong (2025).

<sup>2</sup> An allocation is called *complete* if it allocates all of the goods.

uations. We also remark that relaxing WEF1 using a multiplicative approximation studied in the unweighted setting by, e.g., Amanatidis et al. (2018) and Plaut and Roughgarden (2020) does not help circumvent this counterexample either.<sup>3</sup>

Note that the valuations in Example 1.1 are particularly simple: both agents have *binary submodular* valuations, that is, submodular valuations<sup>4</sup> in which the marginal gain from receiving any single good is either 0 or 1. Binary submodular valuations are also known as *matroid-rank valuations*, and have been studied in a number of recent fair division papers, mostly in the unweighted setting (Babaioff et al. 2021a; Barman and Verma 2021, 2022; Benabbou et al. 2021; Suksompong and Teh 2023; Viswanathan and Zick 2023a, b; Goko et al. 2024).<sup>5</sup> Such valuations arise in settings such as the allocation of course slots to students, or apartments in public housing estates to ethnic groups (Benabbou et al. 2021). General submodular valuations have likewise received interest among fair division researchers, for example in the context of a (non-envy-based) notion called maximin share fairness (Barman and Kumar 2020; Ghodsi et al. 2022; Ben Uziahu and Feige 2023).

In this paper, we explore weighted envy-freeness for both matroid-rank and general submodular valuations. We propose new envy-based notions and show that they can be satisfied in these settings, not only via extensions of existing algorithms, but also via new rules. For the sake of generality, we define our notions based on the notion  $\text{WEF}(x, 1-x)$  of Chakraborty et al. (2024). With additive valuations, given a parameter  $x \in [0, 1]$ ,  $\text{WEF}(x, 1-x)$  allows each agent  $i$  to subtract  $x$  times the value of some good in another agent  $j$ 's bundle from  $i$ 's value for this bundle, and add  $1-x$  times the value of this good to the value of  $i$ 's own bundle. WEF1 corresponds to  $\text{WEF}(1, 0)$ , and higher values of  $x$  yield notions that favor lower-weight agents. To obtain more intuition on  $\text{WEF}(x, 1-x)$ , consider the following example.

**Example 1.2** Let  $n \geq 2$ , and suppose that there are  $n$  identical goods with value 1 each, and  $n$  agents with additive valuations such that  $w_1 = \dots = w_{n-1} = 1$  and  $w_n = n + 1$ . If one wants to ensure that each agent receives nonzero value, the only way is to allocate one good to every agent—this is the only  $\text{WEF}(1, 0)$  allocation. However, agent  $n$  may reasonably object to this allocation, given that her weight is larger than the weight of all other agents combined. In particular, she may demand that all goods be allocated to her—this allocation is the unique one fulfilling  $\text{WEF}(0, 1)$ . Hence,  $\text{WEF}(x, 1-x)$  captures the (inevitable) trade-off between satisfying lower-weight agents and higher-weight agents.

Chakraborty et al. (2024) showed that for any instance with additive valuations and any  $x \in [0, 1]$ , a complete  $\text{WEF}(x, 1-x)$  allocation always exists; on the other hand, they proved that for any distinct  $x$  and  $x'$ , there is an instance with binary

<sup>3</sup> Specifically, for any  $r \in (0, 1]$ , if  $w_1 = r/2$ ,  $w_2 = 1$ ,  $m > 2 + 2/r^2$ , and the valuations of the two agents are as in Theorem 1.1, one can check that there is no  $r$ -WEF1 allocation.

<sup>4</sup> Submodular valuations exhibit diminishing returns from adding an extra good. See the formal definition in Sect. 2.

<sup>5</sup> Exceptions are the recent works of Suksompong and Teh (2023) and Viswanathan and Zick (2023a), which deal with the weighted setting. In fact, Viswanathan and Zick (2023a) pointed out that the “main limitation” of their approach is that it cannot be used to achieve envy-based fairness properties—this is precisely the issue that we address in our paper.

additive valuations and identical goods in which no complete allocation satisfies both  $\text{WEF}(x, 1 - x)$  and  $\text{WEF}(x', 1 - x')$ .

## 1.1 Our contributions

In Sect. 2, we introduce two new families of weighted envy-freeness notions. The first family,  $\text{TWEF}(x, 1 - x)$ , is based on the concept of *transferability*<sup>6</sup>: we consider the condition  $\text{TWEF}(x, 1 - x)$  from agent  $i$  to agent  $j$  to be violated only if the  $\text{WEF}(x, 1 - x)$  condition between  $i$  and  $j$  fails *and*  $i$ 's value for her own bundle increases if all goods from  $j$ 's bundle are transferred to  $i$ 's bundle.  $\text{TWEF}(x, 1 - x)$  handles instances such as the one in Example 1.1, where an agent could be unsatisfied with respect to  $\text{WEF}(x, 1 - x)$  even if she already receives her maximum possible utility. Our second family,  $\text{WMEF}(x, 1 - x)$ , is an extension of the notion *marginal EF1 (MEF1)* of Caragiannis et al. (2019) from the unweighted setting. The idea is that, instead of agent  $i$  considering her value for agent  $j$ 's bundle as in  $\text{WEF}(x, 1 - x)$ , agent  $i$  considers her *marginal* value of  $j$ 's bundle when added to  $i$ 's own bundle. While  $\text{TWEF}(x, 1 - x)$  is stronger than  $\text{WMEF}(x, 1 - x)$ , we show that the former notion is suitable primarily for matroid-rank valuations, whereas the latter can be guaranteed even for general submodular valuations. Note that when valuations are additive, both  $\text{TWEF}(x, 1 - x)$  and  $\text{WMEF}(x, 1 - x)$  reduce to  $\text{WEF}(x, 1 - x)$ , which in turn reduces to EF1 if all agents have equal weights.

In Sects. 3 and 4, we allow agents to have arbitrary submodular valuations. In Sect. 3, we investigate *picking sequences*, which let agents take turns picking a good according to a specified agent ordering until the goods run out. While previous work on picking sequences typically assumed that agents have additive valuations, this assumption may be violated in real-world applications of picking sequences, such as the allocation of ministries to political parties. We adjust picking sequences to submodular valuations by letting agents pick a good with the highest *marginal* gain in each of their turns. We show that for every  $x$ , the output of the adjusted version of the picking sequence proposed by Chakraborty et al. (2024) with parameter  $x$  satisfies  $\text{WMEF}(x, 1 - x)$ ; this generalizes their result from the weighted additive setting. As a corollary, in the unweighted submodular setting, the adjusted version of the commonly studied round-robin algorithm produces an MEF1 allocation. In Sect. 4, we consider the *maximum weighted Nash welfare (MWNW)* rule, which chooses an allocation that maximizes the weighted product of the agents' utilities. Although prior results rule out the possibility for each  $x$  that MWNW implies  $\text{WMEF}(x, 1 - x)$ , we show that an MWNW allocation always satisfies a relaxation of  $\text{WMEF}(x, 1 - x)$  called *weak weighted MEF1 (WWMEF1)*. This extends a corresponding result of Chakraborty et al. (2021a) from the weighted additive setting, which in turn generalizes the prominent result by Caragiannis et al. (2019) in the unweighted additive setting.

Next, in Sects. 5 and 6, we focus on agents with matroid-rank valuations—as we discussed earlier, this class of valuations has been studied in several recent papers. In Sect. 5, we extend the “transfer algorithm” of Benabbou et al. (2021) from the

<sup>6</sup> This concept has been discussed by Benabbou et al. (2021) and Chakraborty et al. (2021a).

unweighted setting, and prove that our algorithm returns a clean<sup>7</sup> TWEF( $x, 1 - x$ ) allocation that maximizes the unweighted utilitarian welfare. While Benabbou et al.'s potential function argument can be generalized to show that our algorithm terminates, it is insufficient for establishing polynomial-time termination in our setting with different weights; hence, we devise a more elaborate argument for this purpose. Finally, in Sect. 6, we introduce new welfare measures based on harmonic numbers and their variants.<sup>8</sup> Perhaps surprisingly, we demonstrate that under matroid-rank valuations, the maximum-welfare rules based on our measures offer stronger fairness guarantees than MWNW. In particular, while MWNW does not imply WEF( $x, 1 - x$ ) for any  $x$  even with binary additive valuations and identical goods (Chakraborty et al. 2024), we prove that a clean *maximum weighted harmonic welfare* allocation parameterized by  $x$  satisfies TWEF( $x, 1 - x$ ) for matroid-rank valuations (and therefore WEF( $x, 1 - x$ ) for binary additive valuations). We also characterize the maximum harmonic welfare rules within a class of rules that maximize certain welfare functions. To further exhibit the potential of harmonic welfare, we show in Appendix C that, in the unweighted additive setting, if each agent's value for each good is an integer, then a maximum harmonic welfare allocation always satisfies EF1.

## 1.2 Further related work

While we have already discussed the prior papers necessary to understand the context of our contributions, let us briefly describe additional work slightly further afield. Besides envy-freeness, another important fairness notion in the literature is *proportionality*. In the unweighted setting, a proportional allocation gives every agent at least  $1/n$  of her value for the entire set of goods, where  $n$  denotes the number of agents. To generalize this to the weighted setting, one can simply replace  $1/n$  with the corresponding agent's fraction of the total weight. Aziz et al. (2020) proposed an "up to one" relaxation of weighted proportionality and showed that it can always be satisfied. Other researchers extended *maximin share fairness*, which is a relaxation of proportionality in the unweighted setting, to accommodate weights (Farhadi et al. 2019; Babaioff et al. 2024; Chakraborty et al. 2024). All of these authors assumed additive valuations for their results. Note also that proportionality (and therefore maximin share fairness) is generally weaker than envy-freeness, since it only stipulates a threshold for each agent's utility rather than comparing each agent's bundle with that of every other agent.

The focus of our paper, and of most papers we have mentioned so far, is on the allocation of goods. In certain scenarios, one may wish to allocate *chores*, such as housework or teaching duties. Wu et al. (2023) and Springer et al. (2024) proved that, as is the case for goods, a WEF1 allocation is guaranteed to exist for chores under additive valuations. The aforementioned result of Aziz et al. (2020) holds for

<sup>7</sup> An allocation is *clean* if no good can be discarded from an agent's bundle without decreasing the agent's utility (Benabbou et al. 2021). The term *non-redundant* has also been used with the same meaning (Babaioff et al. 2021a).

<sup>8</sup> The harmonic welfare measure is the basis of the *proportional approval voting (PAV)* rule in the context of *approval-based committee voting* (see, e.g., the book by Lackner and Skowron (2023)). To the best of our knowledge, we are the first to consider this measure in the context of fair division.

combinations of goods and chores. Barman et al. (2023) studied chore allocation with *binary supermodular costs*, where the marginal cost of each chore is either 0 or 1 and does not decrease as the set of chores grows.

## 2 Preliminaries

Let  $N = [n]$  be the set of agents and  $G = \{g_1, \dots, g_m\}$  be the set of indivisible goods, where  $[k] := \{1, \dots, k\}$  for any positive integer  $k$ . A *bundle* refers to a subset of  $G$ . Each agent  $i \in N$  has a *weight*  $w_i > 0$  representing her entitlement, and a *valuation function* (or *utility function*)  $v_i : 2^G \rightarrow \mathbb{R}_{\geq 0}$ . The setting where all of the weights are equal is sometimes referred to as the *unweighted setting*. For convenience, we write  $v_i(g)$  instead of  $v_i(\{g\})$  for a single good  $g$ . We assume throughout the paper that  $v_i$  is

- *Monotone*:  $v_i(G') \leq v_i(G'')$  for all  $G' \subseteq G'' \subseteq G$ ;
- *Submodular*:  $v_i(G' \cup \{g\}) - v_i(G') \geq v_i(G'' \cup \{g\}) - v_i(G'')$  for all  $G' \subseteq G'' \subseteq G$  and  $g \in G \setminus G''$ ;
- *Normalized*:  $v_i(\emptyset) = 0$ .

The function  $v_i$  is said to be *matroid-rank* (or *binary submodular*) if it is submodular and  $v_i(G' \cup \{g\}) - v_i(G') \in \{0, 1\}$  for all  $G' \subseteq G$  and  $g \in G \setminus G'$ . Moreover,  $v_i$  is *additive* if  $v_i(G') = \sum_{g \in G'} v_i(g)$  for all  $G' \subseteq G$ , and *binary additive* if it is additive and  $v_i(g) \in \{0, 1\}$  for all  $g \in G$ . An *instance* consists of the set of agents  $N$ , the set of goods  $G$ , and the agents' weights  $(w_i)_{i \in N}$  and valuation functions  $(v_i)_{i \in N}$ .

An *allocation*  $\mathcal{A}$  is a list of bundles  $(A_1, \dots, A_n)$  such that no two bundles overlap, where bundle  $A_i$  is assigned to agent  $i$ . The allocation is *complete* if  $\bigcup_{i \in N} A_i = G$ . It is *Pareto-optimal (PO)* if there does not exist another allocation  $\mathcal{A}'$  such that  $v_i(A'_i) \geq v_i(A_i)$  for all  $i \in N$  and the inequality is strict for at least one  $i \in N$ ; such an allocation  $\mathcal{A}'$  is said to *Pareto-dominate*  $\mathcal{A}$ . We denote by  $N_{\mathcal{A}}^+ \subseteq N$  the subset of agents receiving positive utility from  $\mathcal{A}$ . The *unweighted utilitarian welfare* of  $\mathcal{A}$  is defined as  $\sum_{i \in N} v_i(A_i)$ .

For a bundle  $G' \subseteq G$ , we define the *marginal gain* of a good  $g \notin G'$  for agent  $i$  as  $\Delta_i^+(G', g) := v_i(G' \cup \{g\}) - v_i(G')$ . Similarly, the *marginal loss* of a good  $g \in G'$  for agent  $i$  is defined as  $\Delta_i^-(G', g) := v_i(G') - v_i(G' \setminus \{g\})$ .

**Definition 2.1** (*Cleanness*) An allocation  $\mathcal{A}$  is called *clean* (or *non-redundant*) if for any  $i \in N$  and any  $g \in A_i$ , it holds that  $\Delta_i^-(A_i, g) > 0$ .

For matroid-rank valuations,  $\mathcal{A}$  is clean if and only if  $v_i(A_i) = |A_i|$  for all  $i \in N$  (Benabbou et al. 2021, Prop. 3.3). Clean allocations are common in the study of matroid-rank valuations (Babaioff et al. 2021a; Benabbou et al. 2021; Barman and Verma 2022; Suksompong and Teh 2023; Viswanathan and Zick 2023a, b; Goko et al. 2024). While clean allocations may be incomplete, achieving completeness along with certain properties under matroid-rank valuations can be surprisingly challenging—we refer to the discussion by Benabbou et al. (2021, p. 21).

We now introduce our first family of fairness notions, TWEF( $x, y$ ).

**Definition 2.2** ( $TWEF(x, y)$ ) For  $x, y \in [0, 1]$ , an allocation  $\mathcal{A}$  is said to satisfy transferable  $WEF(x, y)$  ( $TWEF(x, y)$ ) if, for each pair of agents  $i, j \in N$ , either  $v_i(A_i) = v_i(A_i \cup A_j)$  or there exists  $g \in A_j$  such that

$$\frac{v_i(A_i) + y \cdot \Delta_i^+(A_i, g)}{w_i} \geq \frac{v_i(A_j) - x \cdot \Delta_i^-(A_j, g)}{w_j}.$$

By submodularity and monotonicity, the condition  $v_i(A_i) = v_i(A_i \cup A_j)$  is equivalent to the requirement that  $v_i(A_i) = v_i(A_i \cup \{g\})$  for every  $g \in A_j$ .

For any  $x$  and  $y$ , if valuations are additive, then  $TWEF(x, y)$  reduces to the notion  $WEF(x, y)$  of Chakraborty et al. (2024). Like Chakraborty et al., we will mostly be concerned with the case where  $y = 1 - x$ . As we will see,  $TWEF(x, 1 - x)$  is a useful notion for matroid-rank valuations. However, like  $WEF(x, 1 - x)$ , it can be too demanding for general submodular valuations. For instance, in Example 1.1, if agent 2 has value  $1 + (|G'| - 1) \cdot \varepsilon$  for any nonempty bundle  $G'$ , where  $\varepsilon > 0$  is a small constant, then the condition  $v_i(A_i) = v_i(A_i \cup A_j)$  becomes impotent and a complete  $TWEF(x, 1 - x)$  allocation does not exist for any  $x$ . The second family of notions that we propose, which is based on the marginal  $EF1$  ( $MEF1$ ) notion of Caragiannis et al. (2019),<sup>9</sup> does not suffer from this shortcoming.

**Definition 2.3** ( $WMEF(x, y)$ ) For  $x, y \in [0, 1]$ , an allocation  $\mathcal{A}$  is said to satisfy  $WMEF(x, y)$  if, for each pair of agents  $i, j \in N$ , either  $A_j = \emptyset$  or there exists  $g \in A_j$  such that

$$\frac{v_i(A_i) + y \cdot \Delta_i^+(A_i, g)}{w_i} \geq \frac{v_i(A_i \cup A_j) - v_i(A_i) - x \cdot \Delta_i^-(A_i \cup A_j, g)}{w_j}.$$

If valuations are additive,  $WMEF(x, y)$  reduces to  $WEF(x, y)$  for any  $x$  and  $y$ . On the other hand, if all agents have the same weight,  $WMEF(x, 1 - x)$  reduces to  $MEF1$  only if  $x = 1$ . The following proposition establishes an implication relationship between our two families of notions.

**Proposition 2.4** For all  $x, y \in [0, 1]$ , every  $TWEF(x, y)$  allocation is also  $WMEF(x, y)$ .

**Proof** Let  $\mathcal{A}$  be a  $TWEF(x, y)$  allocation, and consider any  $i, j \in N$ . By definition of  $TWEF(x, y)$ , either  $v_i(A_i) = v_i(A_i \cup A_j)$  or there exists  $g \in A_j$  such that

$$\frac{v_i(A_i) + y \cdot \Delta_i^+(A_i, g)}{w_i} \geq \frac{v_i(A_j) - x \cdot \Delta_i^-(A_j, g)}{w_j}.$$

Assume first that the latter holds. Since  $v_i$  is submodular, we have  $v_i(A_i) + v_i(A_j \setminus \{g\}) \geq v_i(A_i \cup A_j \setminus \{g\})$  and  $\Delta_i^-(A_j, g) \geq \Delta_i^-(A_i \cup A_j, g)$ . It follows that

$$\frac{v_i(A_i) + y \cdot \Delta_i^+(A_i, g)}{w_i} \geq \frac{v_i(A_j) - x \cdot \Delta_i^-(A_j, g)}{w_j}$$

<sup>9</sup> In the unweighted setting, an allocation satisfies  $MEF1$  if for all  $i, j \in N$ , either  $A_j = \emptyset$  or there exists  $g \in A_j$  such that  $v_i(A_i) \geq v_i(A_i \cup A_j \setminus \{g\}) - v_i(A_i)$ .

$$\begin{aligned}
 &= \frac{v_i(A_j \setminus \{g\}) + (1 - x) \cdot \Delta_i^-(A_j, g)}{w_j} \\
 &\geq \frac{v_i(A_i \cup A_j \setminus \{g\}) - v_i(A_i) + (1 - x) \cdot \Delta_i^-(A_i \cup A_j, g)}{w_j} \\
 &= \frac{v_i(A_i \cup A_j) - v_i(A_i) - x \cdot \Delta_i^-(A_i \cup A_j, g)}{w_j}.
 \end{aligned}$$

Hence, the WMEF( $x, y$ ) condition from agent  $i$  to agent  $j$  is fulfilled.

Next, assume that  $v_i(A_i) = v_i(A_i \cup A_j)$ . If  $A_j = \emptyset$ , then the WMEF( $x, y$ ) condition between agents  $i$  and  $j$  is automatically fulfilled. Otherwise, for any good  $g \in A_j$ , we have

$$\begin{aligned}
 &\frac{v_i(A_i) + y \cdot \Delta_i^+(A_i, g)}{w_i} \geq 0 \\
 &\geq \frac{-x \cdot \Delta_i^-(A_i \cup A_j, g)}{w_j} = \frac{v_i(A_i \cup A_j) - v_i(A_i) - x \cdot \Delta_i^-(A_i \cup A_j, g)}{w_j},
 \end{aligned}$$

and WMEF( $x, y$ ) from  $i$  to  $j$  is again fulfilled. □

Since the valuations that we consider in this paper are not necessarily additive, in order to reason about the running time of algorithms, we make the standard assumption that an algorithm can query the value of any agent  $i$  for any bundle  $G' \subseteq G$  in constant time.

### 3 Picking sequences

In this section, we investigate *picking sequences*, which are procedures wherein agents take turns picking a good according to a specified agent ordering until there are no more goods left. For brevity, we will say that a picking sequence satisfies a fairness notion if the allocation that it returns always satisfies that notion.

With additive valuations, Chakraborty et al. (2024) showed that for each  $x \in [0, 1]$ , a picking sequence that assigns each subsequent pick to an agent  $i \in N$  with the smallest ratio  $\frac{t_i + (1-x)}{w_i}$ , where  $t_i$  denotes the number of times agent  $i$  has picked so far, satisfies WEF( $x, 1 - x$ ). Our main result of this section extends their result to submodular valuations. We make the specification that, in each turn, the agent picks a good that yields the highest *marginal gain* with respect to the agent’s current bundle, breaking ties arbitrarily. More formally, if it is agent  $i$ ’s turn, then  $i$  chooses a good  $g$  that maximizes  $\Delta_i^+(A_i, g)$ , where  $A_i$  is the set of goods that  $i$  picked in previous turns.

**Theorem 3.1** *Let  $x \in [0, 1]$ . Consider a picking sequence  $\pi_x$  such that, in each turn, the pick is assigned to an agent  $i \in N$  with the smallest ratio  $\frac{t_i + (1-x)}{w_i}$ , where  $t_i$  denotes the number of times agent  $i$  has picked so far, and the agent picks a good that yields the highest marginal gain. Then, under submodular valuations,  $\pi_x$  satisfies WMEF( $x, 1 - x$ ).*

For any  $x$  and agents with equal weights,  $\pi_x$  encompasses the popular *round-robin algorithm* where the agents take turns in the order  $1, 2, \dots, n, 1, 2, \dots, n, 1, 2, \dots$ , and WMEF(1, 0) reduces to MEF1 of Caragiannis et al. (2019). We therefore have the following corollary in the unweighted setting, which is also new to the best of our knowledge.

**Corollary 3.2** *Assume that all agents have equal weights and submodular valuations. Suppose that in each turn of the round-robin algorithm, the picking agent picks a good with the highest marginal gain. Then, the algorithm returns a complete MEF1 allocation.*

As Corollary 3.2 admits a more direct proof, which also illustrates the ideas that we will use to show Theorem 3.1, we first present the (shorter) proof of Corollary 3.2.

**Proof of Corollary 3.2** Let  $\mathcal{A}$  be the allocation produced by the round-robin algorithm, and consider any  $i, j \in N$ . Assume without loss of generality that  $i < j$ .

We first establish the MEF1 condition from  $i$  toward  $j$ . Let  $k := |A_j| \leq |A_i|$ , and suppose that agent  $j$  picks the goods in the order  $c_1, c_2, \dots, c_k$ . Let  $b_1, b_2, \dots, b_k$  be the first  $k$  goods picked by agent  $i$  in this order. For  $0 \leq \ell \leq k$ , let  $B_\ell = \{b_1, \dots, b_\ell\}$  and  $C_\ell = \{c_1, \dots, c_\ell\}$  (so  $B_0 = C_0 = \emptyset$ ). For  $1 \leq \ell \leq k$ , since agent  $i$  picks  $b_\ell$  when  $c_\ell$  is also available, it must be that

$$v_i(B_\ell) - v_i(B_{\ell-1}) \geq v_i(B_{\ell-1} \cup \{c_\ell\}) - v_i(B_{\ell-1}).$$

Moreover, since  $B_{\ell-1} \subseteq A_i \subseteq A_i \cup C_{\ell-1}$ , submodularity implies that

$$v_i(B_{\ell-1} \cup \{c_\ell\}) - v_i(B_{\ell-1}) \geq v_i(A_i \cup C_{\ell-1} \cup \{c_\ell\}) - v_i(A_i \cup C_{\ell-1}).$$

Combining the previous two inequalities yields

$$v_i(B_\ell) - v_i(B_{\ell-1}) \geq v_i(A_i \cup C_{\ell-1} \cup \{c_\ell\}) - v_i(A_i \cup C_{\ell-1}).$$

Summing this over all  $\ell \in [k]$ , we get  $v_i(B_k) \geq v_i(A_i \cup C_k) - v_i(A_i)$ . Since  $B_k \subseteq A_i$  and  $C_k = A_j$ , it follows that  $v_i(A_i) \geq v_i(A_i \cup A_j) - v_i(A_i)$ , and the MEF1 condition from  $i$  to  $j$  is fulfilled.

The proof for the MEF1 condition from  $j$  toward  $i$  is almost identical: by ignoring the first good  $g$  picked by agent  $i$  and applying the same argument as before, we arrive at  $v_j(A_j) \geq v_j(A_j \cup (A_i \setminus \{g\})) - v_j(A_j)$ . Thus, the MEF1 condition is again satisfied.  $\square$

In Appendix A, we provide an example showing that the condition MEF1 in Corollary 3.2 cannot be strengthened to EF1, even when agents have matroid-rank valuations.

We now establish Theorem 3.1 by augmenting the proof of Chakraborty et al. (2024, Thm. 3.4) from the additive setting with the ideas from our proof of Corollary 3.2 and arguments involving submodularity.

**Proof of Theorem 3.1** Fix two agents  $i, j \in N$ . For convenience, we write  $\pi$  instead of  $\pi_x$ . For any prefix  $P$  of  $\pi$ , if  $i$  and  $j$  pick  $t_i$  and  $t_j$  times in  $P$ , respectively, then it must be that

$$\frac{(t_j + (1 - x)) - 1}{w_j} \leq \frac{t_i + (1 - x)}{w_i};$$

otherwise the  $t_j$ -th pick of  $j$  should have been assigned to  $i$  instead. That is, we have  $t_i + (1 - x) \geq \frac{w_i}{w_j} \cdot (t_j - x)$ . Using this property, we will show that the WMEF( $x, 1 - x$ ) condition from  $i$  to  $j$  is satisfied after every prefix of  $\pi$ .

We first prove a general claim that, for any  $x, y \in [0, 1]$ , if the WMEF( $x, y$ ) condition from  $i$  to  $j$  is satisfied with bundles  $A_i$  and  $A_j$ , and we add one good  $h \notin A_i \cup A_j$  to  $A_i$ , then the condition remains satisfied. To this end, we show that for every good  $g \in A_j$ ,

$$v_i(A_i \cup \{h\}) + y \cdot \Delta_i^+(A_i \cup \{h\}, g) \geq v_i(A_i) + y \cdot \Delta_i^+(A_i, g) \tag{1}$$

and

$$\begin{aligned} &v_i(A_i \cup A_j) - v_i(A_i) - x \cdot \Delta_i^-(A_i \cup A_j, g) \\ &\geq v_i(A_i \cup A_j \cup \{h\}) - v_i(A_i \cup \{h\}) - x \cdot \Delta_i^-(A_i \cup A_j \cup \{h\}, g). \end{aligned} \tag{2}$$

From the definition of WMEF( $x, y$ ), these two inequalities suffice to prove our claim. In order to prove inequality (1), we observe that

$$\begin{aligned} \Delta_i^+(A_i, h) + y \cdot \Delta_i^+(A_i \cup \{h\}, g) &\geq y \cdot \Delta_i^+(A_i, h) + y \cdot \Delta_i^+(A_i \cup \{h\}, g) \\ &= y \cdot (v_i(A_i \cup \{h, g\}) - v_i(A_i)) \\ &\geq y \cdot (v_i(A_i \cup \{g\}) - v_i(A_i)) \\ &= y \cdot \Delta_i^+(A_i, g). \end{aligned}$$

Since  $\Delta_i^+(A_i, h) = v_i(A_i \cup \{h\}) - v_i(A_i)$ , this implies (1). For (2), observe that

$$\begin{aligned} &v_i(A_i \cup A_j) - v_i(A_i) - x \cdot \Delta_i^-(A_i \cup A_j, g) \\ &= v_i(A_i \cup A_j \setminus \{g\}) - v_i(A_i) + (1 - x) \cdot \Delta_i^-(A_i \cup A_j, g) \\ &\geq v_i(A_i \cup A_j \cup \{h\} \setminus \{g\}) - v_i(A_i \cup \{h\}) + (1 - x) \cdot \Delta_i^-(A_i \cup A_j \cup \{h\}, g) \\ &= v_i(A_i \cup A_j \cup \{h\}) - v_i(A_i \cup \{h\}) - x \cdot \Delta_i^-(A_i \cup A_j \cup \{h\}, g), \end{aligned}$$

where the inequality follows from submodularity.

We are ready to prove Theorem 3.1. Let  $\rho = w_i/w_j$  and  $y = 1 - x$ . From the previous paragraph, it is sufficient to show that the WMEF( $x, 1 - x$ ) condition from  $i$  to  $j$  is fulfilled after every pick by agent  $j$ . Consider any pick by agent  $j$ , and suppose that it is the agent's  $t_j$ -th pick. We divide the sequence of picks up to this pick into *phases*, where each phase  $\ell \in [t_j]$  consists of the picks after agent  $j$ 's  $(\ell - 1)$ -th pick up to (and including) the agent's  $\ell$ -th pick. Let  $A_i$  and  $A_j$  be the bundle of agent  $i$  and  $j$  after phase  $t_j$ , respectively. If  $A_j = \emptyset$ , then the WMEF( $x, 1 - x$ ) condition from  $i$

to  $j$  holds trivially, so assume that  $A_j \neq \emptyset$ . In addition, we introduce the following notation:

- $\tau_\ell :=$  the number of times agent  $i$  picks in phase  $\ell$  (that is, between agent  $j$ 's  $(\ell - 1)$ -th and  $\ell$ -th picks);
- $\alpha_1^\ell, \alpha_2^\ell, \dots, \alpha_{\tau_\ell}^\ell :=$  agent  $i$ 's *marginal gain* for each good that she picks herself in phase  $\ell$  with respect to the goods that she has already picked (including those in phase  $\ell$ , if any);
- $\alpha_\ell := \alpha_1^\ell + \dots + \alpha_{\tau_\ell}^\ell =$  the total marginal gain of agent  $i$  in phase  $\ell$  with respect to the goods that she has picked in prior phases;
- $\beta_\ell :=$  agent  $i$ 's marginal gain for the good that agent  $j$  picks at the end of phase  $\ell$  with respect to  $A_i$ .

Note that  $\alpha_1^\ell, \alpha_2^\ell, \dots, \alpha_{\tau_\ell}^\ell$  (and therefore  $\alpha_\ell$ ) and  $\beta_\ell$  are defined differently than in the proof of Theorem 3.4 of Chakraborty et al. (2024), as the valuations that we consider may be non-additive.

For any integer  $s \in [t_j]$ , applying the condition in the first paragraph of our proof to the picking sequence up to and including phase  $s$ , we have

$$y + \sum_{\ell=1}^s \tau_\ell \geq \rho(s - x) \quad \forall s \in [t_j]. \tag{3}$$

Every time it is agent  $i$ 's turn, she picks a good with the highest marginal gain with respect to her current bundle among the available goods. In particular, in each phase  $\ell$ , she picks  $\tau_\ell$  goods each of which yields at least as high marginal gain to her as any good not yet picked by agent  $j$ . By submodularity, this implies

$$\alpha_\ell \geq \tau_\ell \cdot \max_{\ell \leq r \leq t_j} \beta_r \quad \forall \ell \in [t_j]. \tag{4}$$

Using (3) and (4), the same inductive argument as in Chakraborty et al.'s proof of their Theorem 3.4 yields

$$y \cdot \max_{1 \leq r \leq t_j} \beta_r + \sum_{\ell=1}^{t_j} \alpha_\ell \geq \rho \left( \sum_{\ell=1}^{t_j} \beta_\ell - x \beta_1 \right). \tag{5}$$

Let  $g^* \in \arg \max_{g \in A_j} \Delta_i^+(A_i, g)$ , and let  $g_1$  be the first good picked by agent  $j$  (possibly  $g_1 = g^*$ ). Using (5), we get

$$\begin{aligned} (1 - x) \cdot \Delta_i^+(A_i, g^*) + v_i(A_i) &\geq \frac{w_i}{w_j} \cdot \left( \sum_{g \in A_j} \Delta_i^+(A_i, g) - x \cdot \Delta_i^+(A_i, g_1) \right) \\ &\geq \frac{w_i}{w_j} \cdot \left( \sum_{g \in A_j} \Delta_i^+(A_i, g) - x \cdot \Delta_i^+(A_i, g^*) \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{w_i}{w_j} \cdot \left( \sum_{g \in A_j \setminus \{g^*\}} \Delta_i^+(A_i, g) + (1-x) \cdot \Delta_i^+(A_i, g^*) \right) \\
 &\geq \frac{w_i}{w_j} \cdot (v_i(A_i \cup A_j \setminus \{g^*\}) - v_i(A_i) + (1-x) \cdot \Delta_i^+(A_i, g^*)),
 \end{aligned}$$

where the second inequality follows from the definition of  $g^*$  and the last inequality from submodularity. Consequently, we have

$$\begin{aligned}
 \frac{v_i(A_i) + (1-x) \cdot \Delta_i^+(A_i, g^*)}{w_i} &\geq \frac{v_i(A_i \cup A_j \setminus \{g^*\}) - v_i(A_i) + (1-x) \cdot \Delta_i^+(A_i, g^*)}{w_j} \\
 &\geq \frac{v_i(A_i \cup A_j \setminus \{g^*\}) - v_i(A_i) + (1-x) \cdot \Delta_i^+(A_i \cup A_j \setminus \{g^*\}, g^*)}{w_j} \\
 &= \frac{v_i(A_i \cup A_j) - v_i(A_i) - x \cdot \Delta_i^-(A_i \cup A_j, g^*)}{w_j}.
 \end{aligned}$$

Here, the second inequality holds by submodularity. As a result, the  $WMEF(x, 1-x)$  condition between agents  $i$  and  $j$  is fulfilled, completing the proof.  $\square$

### 4 Nash welfare

In this section, we turn our attention to maximum weighted Nash welfare (MWNW), a weighted extension of the well-studied maximum Nash welfare (MNW). MWNW has been examined in several recent papers (Chakraborty et al. 2021a, b, 2024; Garg et al. 2021, 2022; Suksompong and Teh 2022; Viswanathan and Zick 2023a).

**Definition 4.1** (MWNW) Given an instance, an allocation  $\mathcal{A}$  is a *maximum weighted Nash welfare (MWNW)* allocation if it maximizes the *weighted Nash welfare*  $WNW(\mathcal{A}) := \prod_{i \in N} v_i(A_i)^{w_i}$ . If the highest possible weighted Nash welfare is 0, an MWNW allocation should maximize the number of agents receiving positive utility and, subject to that, maximize the weighted Nash welfare of these agents.

Chakraborty et al. (2024) showed that, for each  $x \in [0, 1]$ , there exists an instance with binary additive valuations and identical goods such that every MWNW allocation is not  $WEF(x, 1-x)$ . As a consequence, MWNW allocations cannot always satisfy  $WMEF(x, 1-x)$  for submodular valuations. On the other hand, Chakraborty et al. (2021a) proved that, under additive valuations, MWNW allocations satisfy *weak WEF1 (WWEF1)*, which is weaker than  $WEF(x, 1-x)$  for every  $x$  but still reduces to EF1 in the unweighted additive setting. We extend their result to the weighted submodular setting via a natural generalization of WWEF1.

**Definition 4.2** (WWMEF1) An allocation  $\mathcal{A}$  is said to satisfy *weak weighted marginal envy-freeness up to one good (WWMEF1)* if for each pair of agents  $i, j$  with  $A_j \neq \emptyset$ , there exists a good  $g \in A_j$  such that

$$\text{either } \frac{v_i(A_i)}{w_i} \geq \frac{v_i(A_i \cup A_j \setminus \{g\}) - v_i(A_i)}{w_j} \text{ or } \frac{v_i(A_i \cup \{g\})}{w_i} \geq \frac{v_i(A_i \cup A_j) - v_i(A_i)}{w_j}.$$

**Theorem 4.3** *Under submodular valuations, every MWNW allocation satisfies WWMEF1 and PO.*

**Proof** Let  $\mathcal{A}$  be an MWNW allocation, and recall that  $N_{\mathcal{A}}^+$  denotes the set of agents who receive positive utility from  $\mathcal{A}$ . We first prove the PO property. If  $\mathcal{A}$  were not PO, there would exist an allocation  $\widehat{\mathcal{A}}$  such that  $v_j(\widehat{A}_j) > v_j(A_j)$  for some  $j \in N$  and  $v_i(\widehat{A}_i) \geq v_i(A_i)$  for every  $i \in N \setminus \{j\}$ . If  $j \in N \setminus N_{\mathcal{A}}^+$ , we would have  $v_i(\widehat{A}_i) > 0$  for every  $i \in N_{\mathcal{A}}^+ \cup \{j\}$ , contradicting the assumption that  $N_{\mathcal{A}}^+$  is a largest subset of agents to whom it is possible to give positive utility simultaneously. On the other hand, if  $j \in N_{\mathcal{A}}^+$ , we would have  $\prod_{i \in N_{\mathcal{A}}^+} v_i(\widehat{A}_i)^{w_i} > \prod_{i \in N_{\mathcal{A}}^+} v_i(A_i)^{w_i}$ , which would mean that  $\mathcal{A}$  does not maximize the weighted Nash welfare of the agents in  $N_{\mathcal{A}}^+$ , again a contradiction. Therefore,  $\mathcal{A}$  is PO.

Next, we proceed to establish the WWMEF1 property. Following the approach of Caragiannis et al. (2019) and Chakraborty et al. (2021a), we first prove that  $\mathcal{A}$  is WWMEF1 for the scenario  $N_{\mathcal{A}}^+ = N$  and then address the case where  $N_{\mathcal{A}}^+ \neq N$ .

Assume that  $N_{\mathcal{A}}^+ = N$ , and suppose for contradiction that  $\mathcal{A}$  is not WWMEF1. This means that there exists a pair of agents  $i, j \in N$  such that the WWMEF1 condition from  $i$  to  $j$  is violated. That is,  $A_j \neq \emptyset$ , and for every good  $g \in A_j$  it holds that

$$\frac{v_i(A_i)}{w_i} < \frac{v_i(A_i \cup A_j \setminus \{g\}) - v_i(A_i)}{w_j} \tag{6}$$

and

$$\frac{v_i(A_i \cup \{g\})}{w_i} < \frac{v_i(A_i \cup A_j) - v_i(A_i)}{w_j}. \tag{7}$$

We will construct another allocation  $\mathcal{A}'$  obtained by transferring a good  $g^*$  (to be chosen later) from  $A_j$  to  $A_i$ , so that  $A'_i = A_i \cup \{g^*\}$  and  $A'_j = A_j \setminus \{g^*\}$ , and the bundles of all other agents remain unchanged. Then, we have

$$\begin{aligned} \frac{\text{WNW}(\mathcal{A}')}{\text{WNW}(\mathcal{A})} &= \left( \frac{v_i(A_i \cup \{g^*\})}{v_i(A_i)} \right)^{w_i} \left( \frac{v_j(A_j \setminus \{g^*\})}{v_j(A_j)} \right)^{w_j} \\ &= \left( \frac{v_i(A_i) + \Delta_i^+(A_i, g^*)}{v_i(A_i)} \right)^{w_i} \left( \frac{v_j(A_j) - \Delta_j^-(A_j, g^*)}{v_j(A_j)} \right)^{w_j} \\ &= \frac{(v_i(A_i) + \Delta_i^+(A_i, g^*))^{w_i} (v_j(A_j) - \Delta_j^-(A_j, g^*))^{w_j}}{v_i(A_i)^{w_i} v_j(A_j)^{w_j}}. \end{aligned} \tag{8}$$

By Lemma C.5 of Caragiannis et al. (2019) (which is a simple application of submodularity), it holds that

$$\sum_{g \in A_j} \Delta_j^-(A_j, g) \leq v_j(A_j). \tag{9}$$

Moreover, by submodularity, we have that for every  $g' \in A_j$ ,

$$\sum_{g \in A_j} \Delta_i^+(A_i, g) \geq v_i(A_i \cup A_j \setminus \{g'\}) - v_i(A_i) + \Delta_i^+(A_i, g') \tag{10}$$

and

$$\sum_{g \in A_j} \Delta_i^+(A_i, g) \geq v_i(A_i \cup A_j) - v_i(A_i) > 0, \tag{11}$$

where the last inequality follows from (7). By (11), we have that  $\Delta_i^+(A_i, g) > 0$  for at least one  $g \in A_j$ .

Let  $B = \{g \in A_j : \Delta_i^+(A_i, g) > 0\}$ , and let  $g^* \in \arg \min_{g \in B} \frac{\Delta_j^-(A_j, g)}{\Delta_i^+(A_i, g)}$ . Due to our choice of  $g^*$  and the definition of  $B$ , we have

$$\frac{\Delta_j^-(A_j, g^*)}{\Delta_i^+(A_i, g^*)} \leq \frac{\sum_{g \in B} \Delta_j^-(A_j, g)}{\sum_{g \in B} \Delta_i^+(A_i, g)} \leq \frac{\sum_{g \in A_j} \Delta_j^-(A_j, g)}{\sum_{g \in A_j} \Delta_i^+(A_i, g)}. \tag{12}$$

Note that  $v_i(A_i \cup A_j \setminus \{g^*\}) - v_i(A_i) + \Delta_i^+(A_i, g^*) \geq \Delta_i^+(A_i, g^*) > 0$ . Combining (12) with (9), (10), and (11), we get

$$\frac{\Delta_j^-(A_j, g^*)}{\Delta_i^+(A_i, g^*)} \leq \frac{v_j(A_j)}{v_i(A_i \cup A_j \setminus \{g^*\}) - v_i(A_i) + \Delta_i^+(A_i, g^*)} \tag{13}$$

and

$$\frac{\Delta_j^-(A_j, g^*)}{\Delta_i^+(A_i, g^*)} \leq \frac{v_j(A_j)}{v_i(A_i \cup A_j) - v_i(A_i)}. \tag{14}$$

We split our remaining argument into two cases. In each case, we will show that the expression in (8) is greater than 1, which results in a contradiction because  $\mathcal{A}$  is an MWNW allocation.

**Case 1:**  $w_i \geq w_j$ . Assume first that  $\Delta_j^-(A_j, g^*) > 0$ . By (6) and (13), we have

$$\frac{v_i(A_i)}{w_i} < \frac{v_i(A_i \cup A_j \setminus \{g^*\}) - v_i(A_i)}{w_j} \leq \frac{\frac{\Delta_i^+(A_i, g^*)}{\Delta_j^-(A_j, g^*)} \cdot v_j(A_j) - \Delta_i^+(A_i, g^*)}{w_j}.$$

Multiplying  $\Delta_j^-(A_j, g^*) \cdot w_i$  on both sides and manipulating, we get

$$\frac{w_i}{w_j} \cdot \Delta_i^+(A_i, g^*) \cdot v_j(A_j) - \frac{w_i}{w_j} \cdot \Delta_i^+(A_i, g^*) \cdot \Delta_j^-(A_j, g^*) - \Delta_j^-(A_j, g^*) \cdot v_i(A_i) > 0.$$

Since  $N_{\mathcal{A}}^+ = N$ , we can divide by  $v_i(A_i)v_j(A_j)$  on both sides to obtain

$$\frac{w_i}{w_j} \left( \frac{\Delta_i^+(A_i, g^*)}{v_i(A_i)} - \frac{\Delta_i^+(A_i, g^*)\Delta_j^-(A_j, g^*)}{v_i(A_i)v_j(A_j)} \right) - \frac{\Delta_j^-(A_j, g^*)}{v_j(A_j)} > 0.$$

Adding 1 to both sides yields

$$1 + \frac{w_i}{w_j} \left( \frac{\Delta_i^+(A_i, g^*)}{v_i(A_i)} - \frac{\Delta_i^+(A_i, g^*)\Delta_j^-(A_j, g^*)}{v_i(A_i)v_j(A_j)} \right) - \frac{\Delta_j^-(A_j, g^*)}{v_j(A_j)} > 1,$$

which can then be factorized as

$$\left( 1 + \frac{w_i}{w_j} \cdot \frac{\Delta_i^+(A_i, g^*)}{v_i(A_i)} \right) \left( 1 - \frac{\Delta_j^-(A_j, g^*)}{v_j(A_j)} \right) > 1. \tag{15}$$

If  $\Delta_j^-(A_j, g^*) = 0$ , then (15) holds trivially because  $\Delta_i^+(A_i, g^*) > 0$ . Hence, (15) always holds.

Now, since  $\frac{\Delta_i^+(A_i, g^*)}{v_i(A_i)} > 0$  and  $\frac{w_i}{w_j} \geq 1$ , Bernoulli’s inequality together with (15) implies that

$$\begin{aligned} & \left( 1 + \frac{\Delta_i^+(A_i, g^*)}{v_i(A_i)} \right)^{\frac{w_i}{w_j}} \left( 1 - \frac{\Delta_j^-(A_j, g^*)}{v_j(A_j)} \right) \\ & \geq \left( 1 + \frac{w_i}{w_j} \cdot \frac{\Delta_i^+(A_i, g^*)}{v_i(A_i)} \right) \left( 1 - \frac{\Delta_j^-(A_j, g^*)}{v_j(A_j)} \right) > 1, \end{aligned}$$

which means that

$$\left( \frac{(v_i(A_i) + \Delta_i^+(A_i, g^*))^{w_i} (v_j(A_j) - \Delta_j^-(A_j, g^*))^{w_j}}{v_i(A_i)^{w_i} v_j(A_j)^{w_j}} \right)^{\frac{1}{w_j}} > 1,$$

that is (from (8)),

$$\left[ \frac{\text{WNW}(\mathcal{A}')}{\text{WNW}(\mathcal{A})} \right]^{\frac{1}{w_j}} > 1,$$

a contradiction.

**Case 2:**  $w_i < w_j$ . Assume first that  $\Delta_j^-(A_j, g^*) > 0$ . By (7) and (14), we have

$$\begin{aligned} \frac{w_j}{w_i} \cdot (v_i(A_i) + \Delta_i^+(A_i, g^*)) &= \frac{w_j}{w_i} \cdot v_i(A_i \cup \{g^*\}) \\ &< v_i(A_i \cup A_j) - v_i(A_i) \leq \frac{\Delta_i^+(A_i, g^*)}{\Delta_j^-(A_j, g^*)} \cdot v_j(A_j). \end{aligned}$$

Multiplying  $\Delta_j^-(A_j, g^*)$  on both sides and manipulating, we get

$$\Delta_i^+(A_i, g^*) \cdot v_j(A_j) - \frac{w_j}{w_i} \cdot \Delta_j^-(A_j, g^*) \cdot v_i(A_i) - \frac{w_j}{w_i} \cdot \Delta_i^+(A_i, g^*) \cdot \Delta_j^-(A_j, g^*) > 0.$$

Since  $N_{\mathcal{A}}^+ = N$ , we can divide by  $v_i(A_i)v_j(A_j)$  on both sides to obtain

$$\frac{\Delta_i^+(A_i, g^*)}{v_i(A_i)} - \frac{w_j}{w_i} \left( \frac{\Delta_j^-(A_j, g^*)}{v_j(A_j)} + \frac{\Delta_i^+(A_i, g^*)\Delta_j^-(A_j, g^*)}{v_i(A_i)v_j(A_j)} \right) > 0.$$

Adding 1 to both sides yields

$$1 + \frac{\Delta_i^+(A_i, g^*)}{v_i(A_i)} - \frac{w_j}{w_i} \left( \frac{\Delta_j^-(A_j, g^*)}{v_j(A_j)} + \frac{\Delta_i^+(A_i, g^*)\Delta_j^-(A_j, g^*)}{v_i(A_i)v_j(A_j)} \right) > 1,$$

which can then be factorized as

$$\left( 1 + \frac{\Delta_i^+(A_i, g^*)}{v_i(A_i)} \right) \left( 1 - \frac{w_j}{w_i} \cdot \frac{\Delta_j^-(A_j, g^*)}{v_j(A_j)} \right) > 1. \tag{16}$$

If  $\Delta_j^-(A_j, g^*) = 0$ , then (16) holds trivially because  $\Delta_i^+(A_i, g^*) > 0$ . Hence, (16) always holds. Moreover, since  $w_j > w_i$ , it must be that  $\Delta_j^-(A_j, g^*) < v_j(A_j)$ .

Now, since  $\frac{\Delta_j^-(A_j, g^*)}{v_j(A_j)} < 1$  and  $\frac{w_j}{w_i} > 1$ , Bernoulli’s inequality together with (16) implies that

$$\begin{aligned} & \left( 1 + \frac{\Delta_i^+(A_i, g^*)}{v_i(A_i)} \right) \left( 1 - \frac{\Delta_j^-(A_j, g^*)}{v_j(A_j)} \right)^{\frac{w_j}{w_i}} \\ & \geq \left( 1 + \frac{\Delta_i^+(A_i, g^*)}{v_i(A_i)} \right) \left( 1 - \frac{w_j}{w_i} \cdot \frac{\Delta_j^-(A_j, g^*)}{v_j(A_j)} \right) > 1, \end{aligned}$$

which means that

$$\left( \frac{(v_i(A_i) + \Delta_i^+(A_i, g^*))^{w_i} (v_j(A_j) - \Delta_j^-(A_j, g^*))^{w_j}}{v_i(A_i)^{w_i} v_j(A_j)^{w_j}} \right)^{\frac{1}{w_i}} > 1,$$

that is (from (8)),

$$\left[ \frac{\text{WNW}(\mathcal{A}')}{\text{WNW}(\mathcal{A})} \right]^{\frac{1}{w_i}} > 1,$$

a contradiction. This completes the proof for the scenario where  $N_{\mathcal{A}}^+ = N$ .

Finally, we handle the scenario where  $N_{\mathcal{A}}^+ \subsetneq N$ . Let  $i, j \in N$  with  $A_j \neq \emptyset$ , and consider three cases.

- If  $i, j \in N_{\mathcal{A}}^+$ , we can show as in the scenario where  $N_{\mathcal{A}}^+ = N$  that the WWMEF1 condition between  $i$  and  $j$  is satisfied.

- Suppose that  $i \notin N_{\mathcal{A}}^+$  and  $j \in N_{\mathcal{A}}^+$ . This means that  $v_i(A_i) = 0$  and  $v_j(A_j) > 0$ . If there exists a good  $g \in A_j$  such that  $v_i(A_i \cup A_j \setminus \{g\}) = 0$ , the WWMEF1 condition from  $i$  to  $j$  is trivially satisfied. Thus, assume now that  $v_i(A_i \cup A_j \setminus \{g\}) > 0$  for every  $g \in A_j$ . Fix an arbitrary  $\widehat{g} \in A_j$ . Since  $v_i(A_i \cup A_j \setminus \{\widehat{g}\}) > 0$ , by submodularity, there exists  $g' \in A_j$  such that  $v_i(A_i \cup \{g'\}) > 0$ . Similarly, since  $v_i(A_i \cup A_j \setminus \{g'\}) > 0$ , there exists  $g'' \in A_j$  with  $g'' \neq g'$  (possibly  $g'' = \widehat{g}$ ) such that  $v_i(A_i \cup \{g''\}) > 0$ .  
 If  $v_j(A_j \setminus \{g'\}) > 0$ , we can transfer  $g'$  from  $A_j$  to  $A_i$  and obtain an allocation with more agents receiving positive utility than in  $\mathcal{A}$ , a contradiction. Therefore,  $v_j(A_j \setminus \{g'\}) = 0$ . Similarly,  $v_j(A_j \setminus \{g''\}) = 0$ . By submodularity, we must have  $v_j(A_j) = 0$  as well, contradicting the assumption that  $j \in N_{\mathcal{A}}^+$ .
- Suppose that  $j \notin N_{\mathcal{A}}^+$ . This means that  $v_j(A_j) = 0$ . If  $v_i(A_i \cup \{g\}) > v_i(A_i)$  for some  $g \in A_j$ , we can transfer  $g$  from  $A_j$  to  $A_i$  and obtain an allocation with the same number of agents receiving positive utility as in  $\mathcal{A}$  but a higher weighted Nash welfare of these agents than in  $\mathcal{A}$ , a contradiction. Hence,  $v_i(A_i \cup \{g\}) = v_i(A_i)$  for every  $g \in A_j$ . Submodularity implies that  $v_i(A_i \cup A_j) = v_i(A_i)$ . Therefore, the WWMEF1 condition from  $i$  to  $j$  is satisfied.

It follows that  $\mathcal{A}$  is WWMEF1 in all cases. □

Viswanathan and Zick (2023a) showed that if agents have matroid-rank valuations, an MWNW allocation can be found in polynomial time. On the other hand, with equal-weight agents and additive valuations, even approximating the maximum Nash welfare is computationally difficult (Lee 2017).

### 5 Transfer algorithm

Recall from Definition 2.1 that an allocation is *clean* if for every agent  $i$ , removing any item from  $i$ 's bundle would decrease  $i$ 's value for her bundle. For agents with equal weights and matroid-rank valuations, Benabbou et al. (2021, Algorithm 1) proposed a “transfer algorithm” that computes a clean, utilitarian welfare-maximizing EF1 allocation in polynomial time. In this section, we extend their algorithm to the weighted setting. Our algorithm is presented as Algorithm 1; we argue in the proof of Theorem 5.1 that the algorithm is well-defined.

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**Algorithm 1** For finding a clean TWEF( $x, 1 - x$ ) allocation maximizing  $\sum_{i \in N} v_i(A_i)$

---

Compute a clean allocation  $\mathcal{A}$  that maximizes the unweighted utilitarian welfare.  
**while** there exist  $i, j \in N$  such that TWEF( $x, 1 - x$ ) from  $i$  to  $j$  fails with respect to  $\mathcal{A}$  **do**  
     Find a good  $g \in A_j$  with  $\Delta_i^+(A_i, g) = 1$ .  
      $A_i \leftarrow A_i \cup \{g\}; A_j \leftarrow A_j \setminus \{g\}$ .  
**end while**

---

**Theorem 5.1** *Suppose that all agents have matroid-rank valuations, and let  $x \in [0, 1]$ . Algorithm 1 with parameter  $x$  returns a clean TWEF( $x, 1 - x$ ) (and therefore*

*WMEF*( $x, 1 - x$ ) allocation that maximizes the unweighted utilitarian welfare among all allocations in polynomial time.

Since any allocation maximizing the unweighted utilitarian welfare is PO, the allocation output by Algorithm 1 is also PO. In the unweighted setting, Benabbou et al. (2021) exhibited polynomial-time termination of their algorithm using the potential function  $\Phi(\mathcal{A}) := \sum_{i \in N} v_i(A_i)^2$ . As  $\Phi(\mathcal{A})$  is always an integer between 0 and  $m^2$  and decreases with every transfer, the number of transfers made by their algorithm is at most  $m^2$ . While we can also establish termination of our weighted algorithm by modifying the potential function to  $\Phi(\mathcal{A}) = \sum_{i \in N} \frac{v_i(A_i)^2 + (1-2x) \cdot v_i(A_i)}{w_i}$ , this argument does not yield a polynomial upper bound on the number of transfers, because the potential function may decrease by a very small amount depending on the weights. Therefore, we will instead employ a different, more refined, argument to show that our algorithm terminates in polynomial time as well.

**Proof of Theorem 5.1** By Proposition 2.4, it suffices to prove the statement for TWEF( $x, 1 - x$ ).

First, we claim that each transfer maintains the welfare optimality and cleanness of the allocation. Indeed,  $v_j(A_j)$  decreases by 1 because the previous allocation is clean, while  $v_i(A_i)$  increases by 1 due to the algorithm’s choice of the good  $g \in A_j$ . Hence,  $\sum_{k \in N} v_k(A_k)$  remains the same. Moreover, since  $v_k(A_k) = |A_k|$  for all  $k \in N$ , the allocation remains clean.

If the TWEF( $x, 1 - x$ ) condition from agent  $i$  to agent  $j$  fails at some point during the execution of the algorithm, it must be that  $v_i(A_i) < v_i(A_i \cup A_j)$  and for every  $g \in A_j$  we have

$$\begin{aligned} \frac{v_i(A_i) + (1 - x) \cdot \Delta_i^+(A_i, g)}{w_i} &< \frac{v_i(A_j) - x \cdot \Delta_i^-(A_j, g)}{w_j} \\ &= \frac{v_i(A_j \setminus \{g\}) + (1 - x) \cdot \Delta_i^-(A_j, g)}{w_j} \\ &\leq \frac{v_j(A_j \setminus \{g\}) + (1 - x) \cdot \Delta_j^-(A_j, g)}{w_j}, \end{aligned} \tag{17}$$

where the latter inequality follows from cleanness. Since  $v_i(A_i) < v_i(A_i \cup A_j)$ , by submodularity, there exists  $g^* \in A_j$  such that  $\Delta_i^+(A_i, g^*) = 1$ ; in particular, the algorithm is well-defined. Plugging this good  $g^*$  into (17) and using the cleanness of  $\mathcal{A}$ , we get

$$\frac{|A_i| + (1 - x)}{w_i} < \frac{|A_j| - x}{w_j}. \tag{18}$$

If the algorithm terminates, then TWEF( $x, 1 - x$ ) is satisfied for all pairs of agents  $i, j$ . We will show that the algorithm always terminates, and moreover does so in polynomial time. The initial clean allocation  $\mathcal{A}$  can be found in polynomial time by reducing to the matroid intersection problem (Benabbou et al. 2021, Thm. 3.7).

Checking whether  $TWEF(x, 1 - x)$  fails for some pair  $i, j$  (and, if so, finding a valid transfer) can be done in polynomial time. It therefore remains to argue that the number of transfers is polynomial. For ease of understanding, we will formulate this argument in terms of cupboards and balls.

Associate each  $k \in N$  with a cupboard consisting of  $m$  shelves at height  $\frac{1-x}{w_k}, \frac{2-x}{w_k}, \dots, \frac{m-x}{w_k}$ , respectively. For the clean allocation  $\mathcal{A}$  at the beginning of the algorithm and each  $k \in N$ , place one ball on each of the  $|A_k|$  lowest shelves<sup>10</sup> of cupboard  $k$ . Whenever a good is transferred from  $A_j$  to  $A_i$ , move the highest ball in cupboard  $j$  to the lowest shelf without a ball in cupboard  $i$ . This means that the ball is moved from height  $\frac{|A_j|-x}{w_j}$  to height  $\frac{|A_i|+1-x}{w_i}$ ; by (18), the height of the ball decreases. Since there are  $m$  balls and at most  $mn$  heights of the shelves, the number of transfers is at most  $m^2n$ , which is indeed polynomial.<sup>11</sup> This concludes the proof.  $\square$

We stress here that clean allocations are not necessarily complete; in fact, cleanness and completeness are sometimes incompatible with each other. For example, consider an instance with three goods and two agents where each agent has value 0 for the empty bundle and value 1 for any other bundle. In every complete allocation, one agent receives at least two items, so such an allocation cannot be clean.

## 6 Harmonic welfare

Recall from Sect. 4 that an MWNW allocation maximizes the product  $\prod_{i \in N} v_i(A_i)^{w_i}$ , or equivalently, the sum  $\sum_{i \in N} w_i \cdot \ln v_i(A_i)$ . Since  $\ln k$  is approximately the  $k$ -th harmonic number  $H_k := 1 + \frac{1}{2} + \dots + \frac{1}{k}$  for each positive integer  $k$ , one could also consider a *maximum weighted harmonic welfare (MWHW)* allocation, defined as an allocation that maximizes the sum  $\sum_{i \in N} w_i \cdot H_{v_i(A_i)}$ , where  $H_0 = 0$ . Interestingly, we show in this section that for agents with matroid-rank valuations, MWHW outperforms MWNW in terms of fairness. Specifically, even though for each  $x \in [0, 1]$  there exists an instance with binary additive valuations and identical goods in which every MWNW allocation fails  $WEF(x, 1 - x)$  (Chakraborty et al. 2024), we show that a clean MWHW allocation satisfies  $TWEF(0, 1)$  for matroid-rank valuations (and therefore  $WEF(0, 1)$  for binary additive valuations). More generally, we define a class of modified harmonic numbers parameterized by  $x$  such that a clean maximum-welfare allocation based on each  $x$  satisfies  $TWEF(x, 1 - x)$ .

**Definition 6.1** (*Modified harmonic numbers*) Let  $k \in \mathbb{Z}_{\geq 0}$ . For  $x \in [0, 1)$ , the number  $H_{k,x}$  is defined by

$$H_{k,x} = \begin{cases} \frac{1}{1-x} + \frac{1}{2-x} + \dots + \frac{1}{k-x} & \text{if } k \geq 1; \\ 0 & \text{if } k = 0, \end{cases}$$

<sup>10</sup> The sum of the heights of all balls is  $\sum_{i \in N} \frac{|A_i|^2 + (1-2x) \cdot |A_i|}{2w_i}$ , which is exactly half of the potential function mentioned before the proof.

<sup>11</sup> Note that if all agents have equal weights, the number of different shelf heights is only  $m$ . The number of transfers is then bounded by  $m^2$ , which matches the bound provided by Benabbou et al. (2021).

whereas for  $x = 1$ ,  $H_{k,x}$  is defined by

$$H_{k,1} = \begin{cases} 1 + \frac{1}{2} + \dots + \frac{1}{k-1} & \text{if } k \geq 2; \\ 0 & \text{if } k = 1; \\ -\infty & \text{if } k = 0. \end{cases}$$

Note that the numbers  $H_{k,0}$  correspond to the canonical harmonic numbers  $H_k$ , and for each  $x$  the sequence  $H_{0,x}, H_{1,x}, \dots$  is increasing. We define a maximum weighted harmonic welfare allocation parameterized by  $x$ . Recall that  $N_{\mathcal{A}}^+$  denotes the set of agents who receive positive utility from the allocation  $\mathcal{A}$ .

**Definition 6.2** ( $MWHW_x$ ) For  $x \in [0, 1)$ , given an instance with matroid-rank valuations, an allocation  $\mathcal{A}$  is an  $MWHW_x$  allocation if it maximizes the sum  $WHW_x(\mathcal{A}) := \sum_{i \in N} w_i \cdot H_{v_i(A_i),x}$ .

For  $x = 1$ ,  $\mathcal{A}$  is an  $MWHW_1$  allocation if it maximizes the number of agents receiving positive utility and, subject to that, maximizes the sum  $\sum_{i \in N_{\mathcal{A}}^+} w_i \cdot H_{v_i(A_i),1}$ .

The quantity  $H_{v_i(A_i),x}$  is well-defined because, for matroid-rank valuations,  $v_i(A_i)$  is always a non-negative integer. We now prove the efficiency and fairness guarantees of  $MWHW_x$  allocations, starting with efficiency.

**Theorem 6.3** Let  $x \in [0, 1]$ . Under matroid-rank valuations, every  $MWHW_x$  allocation is PO.

**Proof** Let  $\mathcal{A}$  be an  $MWHW_x$  allocation. For  $x < 1$ , if  $\mathcal{A}$  is Pareto-dominated by another allocation  $\mathcal{A}'$ , then  $\sum_{i \in N} w_i \cdot H_{v_i(A'_i),x} > \sum_{i \in N} w_i \cdot H_{v_i(A_i),x}$ , a contradiction.

Consider now the case  $x = 1$ . If  $\mathcal{A}$  were not PO, there would exist an allocation  $\widehat{\mathcal{A}}$  such that  $v_j(\widehat{A}_j) > v_j(A_j)$  for some  $j \in N$  and  $v_i(\widehat{A}_i) \geq v_i(A_i)$  for every  $i \in N \setminus \{j\}$ . If  $j \in N \setminus N_{\mathcal{A}}^+$ , we would have  $v_i(\widehat{A}_i) > 0$  for every  $i \in N_{\mathcal{A}}^+ \cup \{j\}$ , contradicting the assumption that  $N_{\mathcal{A}}^+$  is the largest subset of agents to whom it is possible to give positive utility simultaneously. On the other hand, if  $j \in N_{\mathcal{A}}^+$ , we would have  $\sum_{i \in N_{\widehat{\mathcal{A}}}^+} w_i \cdot H_{v_i(\widehat{A}_i),1} > \sum_{i \in N_{\mathcal{A}}^+} w_i \cdot H_{v_i(A_i),1}$ , again a contradiction. Therefore,  $\mathcal{A}$  is PO. □

For the fairness guarantee, we will make an assumption that the allocation is clean; we shall demonstrate later that this assumption is necessary. We also remark that given any  $MWHW_x$  allocation, one can easily obtain a clean  $MWHW_x$  allocation in which every agent receives the same utility as before by iteratively removing any good that does not contribute to its owner’s utility until no such good exists.

**Theorem 6.4** Let  $x \in [0, 1]$ . Under matroid-rank valuations, every clean  $MWHW_x$  allocation satisfies  $TWEF(x, 1 - x)$  (and therefore  $WMEF(x, 1 - x)$ ).

**Proof** By Proposition 2.4, it suffices to prove the statement for  $TWEF(x, 1 - x)$ .

Let  $\mathcal{A}$  be a clean  $MWHW_x$  allocation. Assume for contradiction that for some  $i, j \in N$ , the  $TWEF(x, 1 - x)$  condition from  $i$  to  $j$  is violated. This means that

$v_i(A_i) < v_i(A_i \cup A_j)$  and for every  $g \in A_j$  it holds that

$$\frac{v_i(A_i) + (1 - x) \cdot \Delta_i^+(A_i, g)}{w_i} < \frac{v_i(A_j) - x \cdot \Delta_i^-(A_j, g)}{w_j}.$$

By the same argument as in the proof of Theorem 5.1, this implies that

$$\frac{v_i(A_i) + (1 - x)}{w_i} < \frac{v_j(A_j) - x}{w_j}. \tag{19}$$

Also, since  $v_i(A_i) < v_i(A_i \cup A_j)$ , submodularity implies that there exists a good  $g^* \in A_j$  such that  $\Delta_i^+(A_i, g^*) = 1$ .

We now consider two cases depending on whether  $x = 1$ .

**Case 1:**  $0 \leq x < 1$ . If we transfer  $g^*$  from  $A_j$  to  $A_i$ , we obtain an allocation  $\mathcal{A}'$  in which  $v_i(A'_i) = v_i(A_i) + 1$ ,  $v_j(A'_j) = v_j(A_j) - 1$ , and  $v_k(A'_k) = v_k(A_k)$  for all  $k \in N \setminus \{i, j\}$ . Since  $\mathcal{A}$  is an  $MWHW_x$  allocation, it must be that

$$w_i \cdot H_{v_i(A_i),x} + w_j \cdot H_{v_j(A_j),x} \geq w_i \cdot H_{v_i(A_i)+1,x} + w_j \cdot H_{v_j(A_j)-1,x}.$$

This is equivalent to

$$w_j \cdot \frac{1}{v_j(A_j) - x} - w_i \cdot \frac{1}{v_i(A_i) + 1 - x} \geq 0.$$

Algebraic manipulation gives us

$$\frac{v_i(A_i) + 1 - x}{w_i} \geq \frac{v_j(A_j) - x}{w_j},$$

which contradicts (19).

**Case 2:**  $x = 1$ . From (19), we have that

$$\frac{v_i(A_i)}{w_i} < \frac{v_j(A_j) - 1}{w_j}. \tag{20}$$

Since  $v_i(A_i) \geq 0$  and  $v_j(A_j)$  is an integer, it must be that  $v_j(A_j) \geq 2$ . If  $v_i(A_i) = 0$ , we can transfer  $g^*$  from  $A_j$  to  $A_i$  and increase the number of agents with positive utility, contradicting the assumption that  $\mathcal{A}$  is an  $MWHW_1$  allocation. Hence,  $v_i(A_i) \geq 1$ .

The rest of the argument proceeds in a similar way as in Case 1. If we transfer  $g^*$  from  $A_j$  to  $A_i$ , we obtain an allocation  $\mathcal{A}'$  in which  $v_i(A'_i) = v_i(A_i) + 1$ ,  $v_j(A'_j) = v_j(A_j) - 1$ , and  $v_k(A'_k) = v_k(A_k)$  for all  $k \in N \setminus \{i, j\}$ . Note that the number of agents with positive utility is the same in  $\mathcal{A}$  and  $\mathcal{A}'$ . Since  $\mathcal{A}$  is an  $MWHW_1$  allocation, it must be that

$$w_i \cdot H_{v_i(A_i),1} + w_j \cdot H_{v_j(A_j),1} \geq w_i \cdot H_{v_i(A_i)+1,1} + w_j \cdot H_{v_j(A_j)-1,1}.$$

This is equivalent to

$$w_j \cdot \frac{1}{v_j(A_j) - 1} - w_i \cdot \frac{1}{v_i(A_i)} \geq 0.$$

Algebraic manipulation gives us

$$\frac{v_i(A_i)}{w_i} \geq \frac{v_j(A_j) - 1}{w_j},$$

which contradicts (20).  $\square$

We now exhibit the necessity of the cleanness condition in Theorem 6.4.

**Proposition 6.5** *There exists an instance and an allocation such that, for every  $x \in [0, 1]$ , the allocation is MWHW $_x$  but does not satisfy TWEF( $x, 1 - x$ ).*

**Proof** Consider an instance with  $n = 2$  agents whose weights are  $w_1 = 1$  and  $w_2 = 2$ , and  $m = 6$  goods. Agent 1 has an additive valuation with value 1 for  $g_1$  and 0 for the remaining goods. Agent 2's valuation  $v_2$  is given by

$$v_2(S) = \begin{cases} \min\{3, |S|\} & \text{if } g_1 \notin S; \\ \min\{4, |S|\} & \text{if } g_1 \in S, \end{cases}$$

for each bundle  $S \subseteq G$ .

First, we claim that  $v_2$  is matroid-rank. The marginal gain from adding  $g_1$  is always 1, while the marginal gain from adding any other good is either 0 or 1. To establish submodularity, let  $G' \subseteq G'' \subseteq G$  and  $g \in G \setminus G''$ , and assume that  $v_2(G' \cup \{g\}) = v_2(G')$ ; it suffices to prove that  $v_2(G'' \cup \{g\}) = v_2(G'')$  as well. From our earlier discussion, it must be that  $g \neq g_1$ . If  $g_1 \in G'$ , then  $|G''| \geq |G'| \geq 4$  and thus  $v_2(G'' \cup \{g\}) = v_2(G'')$ . Assume therefore that  $g_1 \notin G'$ , which means that  $|G'| \geq 3$ . If  $G'' = G'$ , then  $v_2(G'' \cup \{g\}) = v_2(G'')$  holds trivially. Otherwise, we have  $|G''| \geq |G'| + 1 \geq 4$ , and again  $v_2(G'' \cup \{g\}) = v_2(G'')$ .

Fix  $x \in [0, 1]$ . If  $x = 1$ , every MWHW $_x$  allocation must give  $g_1$  to agent 1, which leaves agent 2 with a utility of at most 3. Else, for  $x < 1$ , the maximum weighted harmonic welfare achievable by giving  $g_1$  to agent 1 is

$$\frac{1}{1-x} + 2 \cdot \left( \frac{1}{1-x} + \frac{1}{2-x} + \frac{1}{3-x} \right),$$

whereas the maximum achievable by giving  $g_1$  to agent 2 is

$$2 \cdot \left( \frac{1}{1-x} + \frac{1}{2-x} + \frac{1}{3-x} + \frac{1}{4-x} \right).$$

Since  $\frac{1}{1-x} = \frac{2}{2-2x} > \frac{2}{4-x}$ , every MWHW $_x$  allocation must again give  $g_1$  to agent 1. In particular, for every  $x \in [0, 1]$ , the allocation  $\mathcal{A} = (\{g_1, g_2, g_3\}, \{g_4, g_5, g_6\})$  is an MWHW $_x$  allocation.

To finish the proof, we show that  $\mathcal{A}$  violates the  $\text{TWEF}(x, 1 - x)$  condition from agent 2 toward agent 1. Note that  $v_2(A_2) = 3 < 4 = v_2(A_2 \cup A_1)$ . Moreover, for each  $g \in A_1$ , it holds that

$$\frac{v_2(A_2) + (1 - x) \cdot \Delta_2^+(A_2, g)}{w_2} \leq \frac{3 + (1 - x)}{2} < 3 - x = \frac{v_2(A_1) - x \cdot \Delta_2^-(A_1, g)}{w_1}.$$

Hence, the  $\text{TWEF}(x, 1 - x)$  condition from agent 2 to agent 1 is not satisfied.  $\square$

By applying results from the recent work of Viswanathan and Zick (2023a), we show in Appendix B that, for each  $x \in [0, 1]$ , an  $\text{MWHW}_x$  allocation (which additionally maximizes the unweighted utilitarian welfare across all allocations) can be found in polynomial time.

We also remark that it may be interesting to consider harmonic welfare beyond binary valuations. In Appendix C, we prove that for agents with equal weights and additive valuations, if the value of every agent for every good is an integer (in which case the harmonic welfare is well-defined), then an allocation maximizing the harmonic welfare is always EF1.

Both  $\text{MWHW}_x$  and  $\text{MWNW}$  maximize certain welfare functions. More formally, they belong to the class of *weighted additive welfarist rules*, previously studied by Suksompong and Teh (2023) and defined as follows.

**Definition 6.6** ( $\mathcal{R}_h$ ) Let  $h : \mathbb{Z}_{\geq 0} \rightarrow [-\infty, \infty)$  be a strictly increasing function. The *weighted additive welfarist rule with function  $h$* , denoted by  $\mathcal{R}_h$ , chooses an allocation  $\mathcal{A}$  that maximizes the weighted welfare  $\sum_{i \in N} w_i \cdot h(v_i(A_i))$ . If there are multiple such allocations,  $\mathcal{R}_h$  may choose an arbitrary one. The only exception is when the maximum possible weighted welfare is  $-\infty$ . In this case,  $\mathcal{R}_h$  first chooses a subset  $N_{\max} \subset N$  of maximum size such that all agents in  $N_{\max}$  can receive positive utility simultaneously. Subject to that, it maximizes the weighted welfare  $\sum_{i \in N_{\max}} w_i \cdot h(v_i(A_i))$ .

An allocation is said to be an  $\mathcal{R}_h$  allocation if it can be chosen by the rule  $\mathcal{R}_h$ .

$\text{MWNW}$  corresponds to taking  $h(k) = \ln k$ , while  $\text{MWHW}_x$  is equivalent to taking  $h(k) = H_{k,x}$ . This raises the question of whether, for each  $x$ ,  $\text{MWHW}_x$  is the only weighted additive welfarist rule such that every clean allocation output by the rule satisfies  $\text{TWEF}(x, 1 - x)$ . We will show that this is indeed the case.

**Theorem 6.7** Let  $x \in [0, 1]$ . Suppose that, under matroid-rank valuations, for some strictly increasing function  $h : \mathbb{Z}_{\geq 0} \rightarrow [-\infty, \infty)$ , every clean  $\mathcal{R}_h$  allocation satisfies  $\text{TWEF}(x, 1 - x)$ . Then, there exist constants  $c > 0$  and  $d \in \mathbb{R}$  such that  $h(k) = c \cdot H_{k,x} + d$  for all  $k \in \mathbb{Z}_{\geq 0}$ .

To prove Theorem 6.7, we will focus on a much more restricted class of valuations, where each good yields the same positive value (without loss of generality, 1) to every agent and the valuations are additive. This class of valuations is relevant in the well-studied setting of apportionment (Balinski and Young 2001; Pukelsheim 2014); for convenience, we refer to *additive valuations and identical valuable goods*. Note that when valuations are additive,  $\text{TWEF}(x, 1 - x)$  reduces to  $\text{WEF}(x, 1 - x)$ , and when goods are moreover identical and valuable, every allocation is clean. Hence, Theorem 6.7 is an immediate consequence of the following lemma.

**Lemma 6.8** *Let  $x \in [0, 1]$ . Suppose that, under additive valuations and identical valuable goods, for some strictly increasing function  $h : \mathbb{Z}_{\geq 0} \rightarrow [-\infty, \infty)$ , every  $\mathcal{R}_h$  allocation satisfies  $WEF(x, 1 - x)$ . Then, there exist constants  $c > 0$  and  $d \in \mathbb{R}$  such that  $h(k) = c \cdot H_{k,x} + d$  for all  $k \in \mathbb{Z}_{\geq 0}$ .*

We have studied picking sequences in Sect. 3 and weighted additive welfarist rules in Sect. 4 and this section. Interestingly, for additive valuations and identical valuable goods, there is a direct correspondence between the two class of rules, which will be helpful for establishing Lemma 6.8. We consider picking sequences that assign each subsequent pick to an agent  $i \in N$  with the smallest ratio  $f(t_i)/w_i$ , breaking ties arbitrarily, where  $t_i$  denotes the number of times agent  $i$  has picked so far and  $f : \mathbb{Z}_{\geq 0} \rightarrow [0, \infty)$  is a strictly increasing function. Note that the picking sequence in Theorem 3.1 has  $f(k) = k + (1 - x)$ .

**Proposition 6.9** *Let  $h : \mathbb{Z}_{\geq 0} \rightarrow [-\infty, \infty)$  be a strictly increasing function, and let  $\delta_h(k) = h(k + 1) - h(k)$  for all  $k \in \mathbb{Z}_{\geq 0}$ . Suppose that  $\delta_h$  is a strictly decreasing function. Then, in an instance with additive valuations and identical valuable goods, an allocation can be chosen by the rule  $\mathcal{R}_h$  if and only if it can be chosen by the picking sequence with function  $f$  where  $f(k) = 1/\delta_h(k)$  for all  $k \in \mathbb{Z}_{\geq 0}$ .*

**Proof** Consider an instance with additive valuations and identical valuable goods. Since  $\delta_h(k) \in (0, \infty]$  for all  $k \in \mathbb{Z}_{\geq 0}$ , we have  $f(k) \in [0, \infty)$  for all  $k$ , and since  $\delta_h$  is strictly decreasing,  $f$  is strictly increasing. Therefore, the picking sequence with function  $f$  is well-defined. If  $h(0) = -\infty$  and  $m \leq n$ , an allocation can be chosen by  $\mathcal{R}_h$  if and only if it assigns at most one good to each agent. Moreover, in this case we have  $\delta_h(0) = \infty$  and  $f(0) = 0$ , so an allocation can be chosen by the picking sequence with function  $f$  if and only if it assigns at most one good to each agent as well, so the desired statement holds. Hence, assume from now on that  $h(0) \neq -\infty$  or  $m > n$ .

( $\Rightarrow$ ) Suppose that an allocation  $\mathcal{A}$  can be chosen by the rule  $\mathcal{R}_h$ . Call an agent  $i \in N$  *saturated* if she has already received  $|A_i|$  goods in a procedure, and *non-saturated* otherwise. Run the picking sequence with function  $f$ , breaking ties in favor of non-saturated agents. Assume for contradiction that the procedure does not terminate with the allocation  $\mathcal{A}$ . Hence, at some point during the procedure, a saturated agent  $i$  is chosen ahead of a non-saturated agent  $j$ . By our tie-breaking convention and the fact that  $f$  is increasing, this implies that

$$\frac{f(|A_i|)}{w_i} < \frac{f(|A_j| - 1)}{w_j},$$

or equivalently,

$$[h(|A_j|) - h(|A_j| - 1)] \cdot w_j < [h(|A_i| + 1) - h(|A_i|)] \cdot w_i. \tag{21}$$

If  $h(0) \neq -\infty$ , then moving one good from  $j$  to  $i$  in  $\mathcal{A}$  would increase the weighted welfare with respect to  $h$ , contradicting the assumption that  $\mathcal{A}$  can be chosen by  $\mathcal{R}_h$ . Assume now that  $h(0) = -\infty$  and  $m > n$ , which means that every agent receives

at least one good in  $\mathcal{A}$ . Hence, the right-hand side of (21) is finite, so the left-hand side must be finite as well. Again, this implies that moving one good from  $j$  to  $i$  in  $\mathcal{A}$  would increase the weighted welfare with respect to  $h$ , a contradiction.

( $\Leftarrow$ ) Suppose that an allocation  $\mathcal{A}$  can be chosen by the picking sequence with function  $f$ . Assume first that  $h(0) \neq -\infty$ , which means that  $f(0) \neq 0$ . Observe that if an agent  $i$  has picked  $t_i$  goods, then picking the next good increases the weighted welfare with respect to  $h$  by  $[h(t_i + 1) - h(t_i)] \cdot w_i = \delta_h(t_i) \cdot w_i = w_i / f(t_i)$ . Hence, with each pick, the picking sequence increases the weighted welfare of the partial allocation by the largest possible amount. Moreover, since  $f$  is strictly increasing, the  $m$  picks must correspond to the  $m$  largest numbers in the multiset  $\{w_i / f(k)\}_{i \in N, k \in \mathbb{Z}_{\geq 0}}$  (with arbitrary tie-breaking). This means that the allocation  $\mathcal{A}$  maximizes the weighted welfare among all allocations, and can therefore be chosen by  $\mathcal{R}_h$ .

Finally, assume that  $h(0) = -\infty$  (which means that  $f(0) = 0$ ) and  $m > n$ . Since  $f(0) = 0$ , the first  $n$  picks must go to  $n$  different agents, and the weighted welfare of  $\mathcal{A}$  with respect to  $h$  is finite. For each subsequent pick, again, the picking sequence increases the weighted welfare of the partial allocation by the largest possible amount. A similar argument as in the previous paragraph then implies that  $\mathcal{A}$  can be chosen by the rule  $\mathcal{R}_h$ .  $\square$

With the relationship between weighted additive welfarist rules and picking sequences in hand, we will now concentrate on the latter. In particular, we examine the conditions on  $f$  such that the corresponding picking sequences guarantee  $WEF(x, 1 - x)$ . In fact, such conditions have been characterized by Chakraborty et al. (2024), although these authors restricted their attention to functions  $f$  such that  $f(k) \in [k, k + 1]$  for all  $k$ . Nevertheless, it is not difficult to extend their characterization to arbitrary functions  $f$  by adjusting their arguments.

**Proposition 6.10** *Let  $x \in [0, 1]$ , and let  $f : \mathbb{Z}_{\geq 0} \rightarrow [0, \infty)$  be a strictly increasing function. Suppose that, for additive valuations and identical valuable goods, every allocation chosen by the picking sequence with function  $f$  satisfies  $WEF(x, 1 - x)$ . Then, there exists a constant  $\widehat{c} > 0$  such that  $f(k) = \widehat{c} \cdot [k + (1 - x)]$  for all  $k \in \mathbb{Z}_{\geq 0}$ .*

**Proof** Assume that  $f$  satisfies the conditions in the proposition statement, and let  $y = 1 - x$  and  $\widehat{c} = \frac{f(1)}{1+y} > 0$ .

If  $y > 0$ , the same argument as in the proof of Chakraborty et al. (2024)<sup>12</sup> shows that  $\frac{f(a)}{f(b)} \leq \frac{a+y}{b+y}$  for all  $a, b \in \mathbb{Z}_{\geq 0}$ . In particular, by reversing the roles of  $a$  and  $b$ , we have  $\frac{f(a)}{f(b)} = \frac{a+y}{b+y}$  for all  $a, b \in \mathbb{Z}_{\geq 0}$ . Taking  $b = 1$ , we get  $f(k) = \widehat{c} \cdot (k + y)$  for all  $k \in \mathbb{Z}_{\geq 0}$ .

If  $y = 0$ , Chakraborty et al.’s argument shows that  $\frac{f(a)}{f(b)} \leq \frac{a+y}{b+y} = \frac{a}{b}$  for all  $a \in \mathbb{Z}_{\geq 0}$  and  $b \in \mathbb{Z}_{\geq 1}$ . In particular, by reversing the roles of  $a$  and  $b$ , we have  $\frac{f(a)}{f(b)} = \frac{a}{b}$  for all  $a, b \in \mathbb{Z}_{\geq 1}$ . Moreover, taking  $a = 0$  gives  $f(0) \leq 0$ , which means that  $f(0) = 0$ . Hence,  $f(k) = \widehat{c} \cdot k$  for all  $k \in \mathbb{Z}_{\geq 0}$ , as desired.  $\square$

<sup>12</sup> See Theorem 3.14 in their work, which holds for identical valuable goods. Their argument shows this separately for the cases  $b > 0$  and  $b = 0$ . Note that since they assume that  $f$  must satisfy  $f(k) \in [k, k + 1]$  for all  $k$ , their conclusion is that  $f(k) = k + y$  rather than  $f(k) = \widehat{c} \cdot (k + y)$ .

We now have all the necessary ingredients for proving Lemma 6.8, which, as we mentioned earlier, immediately implies Theorem 6.7.

**Proof of Lemma 6.8** Fix  $x \in [0, 1]$ , and assume that every  $\mathcal{R}_h$  allocation under additive valuations and identical valuable goods satisfies WEF( $x, 1-x$ ). To begin with, suppose for contradiction that  $\delta_h$  is *not* strictly decreasing. Then, there exists  $k \in \mathbb{Z}_{\geq 0}$  such that  $h(k+1) - h(k) \leq h(k+2) - h(k+1)$ . Consider an instance with  $n = 2$  equal-weight agents and  $2k+2$  identical valuable goods. Since the agents have equal weights, WEF( $x, 1-x$ ) reduces to EF1, and the only EF1 allocation gives exactly  $k+1$  goods to each agent. However, since  $2h(k+1) \leq h(k) + h(k+2)$ , the rule  $\mathcal{R}_h$  may choose an allocation different from this one, a contradiction. Therefore,  $\delta_h$  is strictly decreasing.

By Proposition 6.9, every allocation produced by the picking sequence with function  $f = 1/\delta_h$  satisfies WEF( $x, 1-x$ ). Hence, Proposition 6.10 implies that there exists a constant  $\widehat{c} > 0$  such that  $f(k) = \widehat{c} \cdot [k + (1-x)]$  for all  $k \in \mathbb{Z}_{\geq 0}$ . Applying the definitions of  $f$  and  $\delta_h$  and letting  $c = 1/\widehat{c} > 0$ , we get that

$$h(k+1) - h(k) = c \cdot \frac{1}{k + (1-x)} \quad (22)$$

for all  $k \geq 0$ .

First, assume that  $x < 1$ . It follows from (22) that

$$h(k) = h(0) + c \cdot \left( \frac{1}{1-x} + \frac{1}{2-x} + \cdots + \frac{1}{k-x} \right) = h(0) + c \cdot H_{k,x}$$

for all  $k \geq 0$ . Letting  $d = h(0)$  yields the desired result.

Next, assume that  $x = 1$ . The relation (22) with  $k = 0$  implies that  $h(0) = -\infty$ . Moreover, it follows from (22) that

$$h(k) = h(1) + c \cdot \left( 1 + \frac{1}{2} + \cdots + \frac{1}{k-1} \right) = h(1) + c \cdot H_{k,1}$$

for all  $k \geq 1$ . Let  $d = h(1)$ , and note that  $h(0) = -\infty = c \cdot H_{0,1} + d$ . This completes the proof.  $\square$

## 7 Conclusion

In this paper, we have embarked on a study of weighted envy-freeness beyond additive valuations by focusing on the important class of submodular valuations. We proposed two families of envy-based notions: TWEF( $x, 1-x$ ), which is suitable for matroid-rank (i.e., binary submodular) valuations, and WMEF( $x, 1-x$ ), which is useful even for arbitrary submodular valuations. We demonstrated that our notions can be satisfied via procedures ranging from picking sequences to welfare maximization, which means that envy-based guarantees are attainable even for non-additive valuations. To the best of our knowledge, these are the first notions that can always be satisfied in the weighted

setting under submodular (or matroid-rank) valuations and moreover reduce to EF1 in the unweighted additive setting. Some of our results concern harmonic welfare, an intriguing measure which, in our view, is worthy of further investigation in fair division contexts.

An interesting direction in light of our work is to consider weighted envy-freeness with respect to other valuation classes, for example, supermodular or subadditive valuations. In particular, for supermodular valuations, one could define  $WMEF(x, 1 - x)$  as follows:

$$\frac{v_i(A_i \cup A_j) - v_i(A_j) + (1 - x) \cdot \Delta_i^+(A_i, g)}{w_i} \geq \frac{v_i(A_j) - x \cdot \Delta_i^-(A_j, g)}{w_j};$$

since supermodularity implies that  $v_i(A_i \cup A_j) - v_i(A_j) \geq v_i(A_i)$ , this is less demanding than if we have  $v_i(A_i)$  instead of  $v_i(A_i \cup A_j) - v_i(A_j)$  on the left-hand side. However, the proof of Theorem 3.1 (and Corollary 3.2) does not appear to admit a supermodular analog, and it remains unclear whether the supermodular version of  $WMEF(x, 1 - x)$  can always be fulfilled.

### Appendix A: Round-robin algorithm and EF1

In the unweighted setting, it is well-known that if agents have additive valuations, then the round-robin algorithm always produces an EF1 allocation (e.g., Caragiannis et al. 2019, p. 7). A natural question is therefore whether the MEF1 condition in Corollary 3.2 can be strengthened to EF1. We show that the answer is negative, even for matroid-rank valuations.

Consider an instance with  $m = 12$  goods and  $n = 2$  agents with equal weights. Agent 1 has an additive valuation with value 1 for each of  $g_4, g_8,$  and  $g_{12}$ , and 0 for the remaining goods. The value of agent 2 for any bundle  $S$  is

$$v_2(S) := |S \cap \{g_4\}| + |S \cap \{g_8\}| + |S \cap \{g_{12}\}| + \min\{1, |S \cap \{g_1, g_2, g_3\}|\} \\ + \min\{1, |S \cap \{g_5, g_6, g_7\}|\} + \min\{1, |S \cap \{g_9, g_{10}, g_{11}\}|\}.$$

One can check that  $v_2$  is matroid-rank.<sup>13</sup>

Assume that the round-robin algorithm lets agent 1 pick first, and that the agents break ties in the goods lexicographically. Under these assumptions, agent 1 picks  $g_4$ , agent 2 picks  $g_1$ , agent 1 picks  $g_8$ , agent 2 picks  $g_5$ , agent 1 picks  $g_{12}$ , agent 2 picks  $g_9$ , agent 1 picks  $g_2$ , agent 2 picks  $g_3$ , agent 1 picks  $g_6$ , agent 2 picks  $g_7$ , agent 1 picks  $g_{10}$ , and agent 2 picks  $g_{11}$ . Hence,  $A_1 = \{g_2, g_4, g_6, g_8, g_{10}, g_{12}\}$  and  $A_2 = \{g_1, g_3, g_5, g_7, g_9, g_{11}\}$ , and so  $v_2(A_2) = 3 < 5 = v_2(A_1 \setminus \{g\})$  for every  $g \in A_1$ . This means that agent 2 is not EF1 toward agent 1. Note also that adding the condition  $v_i(A_i) = v_i(A_i \cup A_j)$  as in our definition of TWEF( $x, y$ ) does not help circumvent this negative result, as we have  $v_2(A_2) = 3 < 6 = v_2(A_2 \cup A_1)$ . Moreover, transferring any good from  $A_1$  to  $A_2$  cannot eliminate agent 2's envy, since

<sup>13</sup> In fact,  $v_2$  belongs to a subclass of matroid-rank valuations called  $(0, 1)$ -OXs (Benabbou et al. 2021).

agent 2 would still have value 5 for agent 1's bundle but value at most 4 for her own bundle.

## Appendix B: Computing MHHW<sub>x</sub> allocations

Recently, Viswanathan and Zick (2023a) introduced a framework for efficiently computing allocations that maximize a range of fairness objectives. Formally, a *fairness objective* is a function that maps each allocation  $\mathcal{A}$  (more precisely, the utility vector  $(v_1(A_1), \dots, v_n(A_n))$  induced by  $\mathcal{A}$ ) to a totally ordered space. A *gain function* maps each clean allocation and each agent  $i \in N$  to a real number.<sup>14</sup> These authors showed that if a fairness objective admits a gain function satisfying certain properties, then an allocation that maximizes the objective (along with the unweighted utilitarian welfare) can be computed in polynomial time.

**Lemma B.1** (Viswanathan and Zick 2023a) *Suppose that the fairness objective  $\Psi$  admits a gain function  $\phi$  such that the following conditions are satisfied:*

- (i) *For any two allocations  $\mathcal{A}$  and  $\mathcal{A}'$ , if  $\mathcal{A}$  Pareto-dominates  $\mathcal{A}'$ , then  $\Psi(\mathcal{A}) \geq \Psi(\mathcal{A}')$ .*
- (ii) *For any clean allocation  $\mathcal{A}$  and any agent  $i \in N$ , let  $\mathcal{A}^{+i}$  be an allocation resulting from giving a good  $g \notin \bigcup_{k \in N} A_k$  such that  $\Delta_i^+(A_i, g) = 1$  to  $i$ . Define  $\mathcal{A}^{+j}$  analogously. If  $\phi(\mathcal{A}, i) \geq \phi(\mathcal{A}, j)$ , then  $\Psi(\mathcal{A}^{+i}) \geq \Psi(\mathcal{A}^{+j})$ ; equality holds if and only if  $\phi(\mathcal{A}, i) = \phi(\mathcal{A}, j)$ .*
- (iii) *For any clean allocations  $\mathcal{A}$  and  $\mathcal{A}'$ , if  $|A_i| \leq |A'_i|$ , then  $\phi(\mathcal{A}, i) \geq \phi(\mathcal{A}', i)$ ; equality holds if and only if  $|A_i| = |A'_i|$ .*
- (iv) *The gain function  $\phi$  can be computed in polynomial time.*

*Then, there exists a polynomial-time algorithm that computes an allocation that maximizes the fairness objective  $\Psi$  as well as the unweighted utilitarian welfare.*

We now apply Lemma B.1 to MHHW<sub>x</sub>.

**Theorem B.2** *For each  $x \in [0, 1]$ , under matroid-rank valuations, an MHHW<sub>x</sub> allocation that maximizes the unweighted utilitarian welfare can be computed in polynomial time.*

**Proof** Fix  $x \in [0, 1]$ , and let

$$\Psi_x(\mathcal{A}) = \begin{cases} \sum_{i \in N} w_i \cdot H_{v_i(A_i), x} & \text{if } x < 1; \\ \left( |N_{\mathcal{A}}^+|, \sum_{i \in N_{\mathcal{A}}^+} w_i \cdot H_{v_i(A_i), 1} \right) & \text{if } x = 1, \end{cases}$$

where for  $x = 1$  we compare the ordered pairs  $\Psi(\mathcal{A})$  for different allocations  $\mathcal{A}$  lexicographically. By definition of MHHW<sub>x</sub>, an allocation  $\mathcal{A}$  that maximizes  $\Psi_x(\mathcal{A})$  is also an MHHW<sub>x</sub> allocation. It is clear that  $\Psi_x$  satisfies condition (i) of Lemma B.1.

<sup>14</sup> Viswanathan and Zick (2023a) also allow the output of a gain function to be a vector, but we do not need that.

Next, let  $w_{\max} = \max_{i \in N} w_i$ . We define the gain function  $\phi_x$  as follows:

$$\phi_x(\mathcal{A}, i) = \begin{cases} \frac{w_i}{|A_i|+1-x} & \text{if } |A_i| > 0 \text{ or } x < 1; \\ w_{\max} + 1 & \text{if } |A_i| = 0 \text{ and } x = 1. \end{cases}$$

Since  $\phi_x$  can be computed efficiently, condition (iv) of Lemma B.1 is satisfied. Condition (iii) is also trivially met.

It remains to show that condition (ii) is satisfied. Let  $\mathcal{A}$  be a clean allocation, so  $v_i(A_i) = |A_i|$  for all  $i \in N$ . Consider any  $i, j \in N$ . First, suppose that  $x < 1$ . We have

$$\begin{aligned} \phi_x(\mathcal{A}, i) \geq \phi_x(\mathcal{A}, j) &\iff \frac{w_i}{|A_i| + 1 - x} \geq \frac{w_j}{|A_j| + 1 - x} \\ &\iff w_i \cdot (H_{v_i(A_i)+1,x} - H_{v_i(A_i),x}) \geq w_j \cdot (H_{v_j(A_j)+1,x} - H_{v_j(A_j),x}) \\ &\iff \Psi_x(\mathcal{A}^{+i}) \geq \Psi_x(\mathcal{A}^{+j}). \end{aligned}$$

This reasoning also shows that  $\phi_x(\mathcal{A}, i) = \phi_x(\mathcal{A}, j)$  if and only if  $\Psi_x(\mathcal{A}^{+i}) = \Psi_x(\mathcal{A}^{+j})$ . Hence, (ii) is satisfied.

Next, suppose that  $x = 1$ . We consider four cases.

**Case 1:**  $|A_i| = |A_j| = 0$ . We have  $\phi_1(\mathcal{A}, i) = \phi_1(\mathcal{A}, j) = w_{\max} + 1$ . Also,

$$\Psi_1(\mathcal{A}^{+i}) = \Psi_1(\mathcal{A}^{+j}) = \left( \left( |N_{\mathcal{A}}^+| + 1, \sum_{k \in N_{\mathcal{A}}^+} w_k \cdot H_{v_k(A_k),1} \right) \right),$$

where the second coordinate is the same as in  $\Psi_1(\mathcal{A})$  because  $H_{1,1} = 0$ . Hence, (ii) holds.

**Case 2:**  $|A_j| > |A_i| = 0$ . We have  $\phi_1(\mathcal{A}, j) = w_{\max} + 1 > w_i \geq \phi_1(\mathcal{A}, i)$ , so (ii) holds trivially since the assumption  $\phi_1(\mathcal{A}, i) \geq \phi_1(\mathcal{A}, j)$  is not satisfied.

**Case 3:**  $|A_j| > |A_i| = 0$ . We have  $\phi_1(\mathcal{A}, i) = w_{\max} + 1 > w_j \geq \phi_1(\mathcal{A}, j)$ . Also, the first coordinate of  $\Psi_1(\mathcal{A}^{+i})$  is  $|N_{\mathcal{A}}^+| + 1$ , whereas that of  $\Psi_1(\mathcal{A}^{+j})$  is  $|N_{\mathcal{A}}^+|$ , which means that  $\Psi_1(\mathcal{A}^{+i}) > \Psi_1(\mathcal{A}^{+j})$ . Hence, (ii) holds.

**Case 4:**  $|A_i|, |A_j| > 0$ . In this case, the same argument as for  $x < 1$  applies.

Therefore, (ii) is satisfied in all cases, and Lemma B.2 follows readily from Lemma B.1. □

### Appendix C: Harmonic welfare in the unweighted additive setting

In this appendix, we assume that all agents have equal weights and additive valuations, and demonstrate some potential (and limits) of harmonic welfare. Recall that  $H_k = 1 + \frac{1}{2} + \dots + \frac{1}{k}$  for each positive integer  $k$  and  $H_0 = 0$ .

**Definition C.1** (MHW) Given an instance with equal weights and additive valuations in which each agent’s value for each good is an integer, an allocation  $\mathcal{A}$  is a *maximum*

harmonic welfare (MHW) allocation if it maximizes the harmonic welfare  $HW(\mathcal{A}) := \sum_{i \in N} H_{v_i(A_i)}$ .

Note that in the unweighted setting, MHW allocations are the same as  $MWHW_0$  allocations defined in Sect. 6. To establish the EF1 guarantee of MHW allocations, we will use the following lemma, which follows directly by inspecting the left and right Riemann sums of the function  $f(x) = 1/x$ .

**Lemma C.2** *For integers  $b \geq a \geq 1$ , it holds that  $\sum_{k=a}^b \frac{1}{k} > \ln\left(\frac{b+1}{a}\right)$ . Moreover, if  $a \geq 2$ , then  $\sum_{k=a}^b \frac{1}{k} < \ln\left(\frac{b}{a-1}\right)$ .*

**Theorem C.3** *Suppose that all agents have equal weights and additive valuations, and each agent’s value for each good is an integer. Then, any MHW allocation satisfies EF1.*

**Proof** Let  $\mathcal{A}$  be an MHW allocation, and assume for contradiction that for some pair of agents  $i, j \in N$ , it holds that  $A_j \neq \emptyset$  and  $v_i(A_i) < v_i(A_j) - v_i(g)$  for all  $g \in A_j$ . Since each agent’s value for each good is an integer, we have

$$v_i(A_i) \leq v_i(A_j) - v_i(g) - 1 \tag{23}$$

for all  $g \in A_j$ . In particular,  $v_i(A_j) \geq 1$ , and so there exists  $g \in A_j$  such that  $v_i(g) \geq 1$ . Plugging such a good  $g$  into (23), we get  $v_i(A_j) \geq 2$ . By (23) again,  $v_i(A_j) \geq v_i(g) + 1$  for each  $g \in A_j$ .

Let  $B = \{g \in A_j : v_i(g) > 0\}$ ; from the previous paragraph, we have  $|B| \geq 2$ . Since  $\mathcal{A}$  is an MHW allocation, moving any good  $g$  from  $A_j$  to  $A_i$  cannot increase the harmonic welfare. In particular,  $v_j(g) > 0$  for all  $g \in B$ , which also means that  $v_j(g) \leq v_j(A_j) - 1$  for each  $g \in B$ . We have

$$H_{v_i(A_i)} + H_{v_j(A_j)} \geq H_{v_i(A_i)+v_i(g)} + H_{v_j(A_j)-v_j(g)}$$

for all  $g \in B$ . Equivalently,

$$\frac{1}{v_j(A_j) - v_j(g) + 1} + \dots + \frac{1}{v_j(A_j)} \geq \frac{1}{v_i(A_i) + 1} + \dots + \frac{1}{v_i(A_i) + v_i(g)}.$$

Combining this with (23) yields

$$\frac{1}{v_j(A_j) - v_j(g) + 1} + \dots + \frac{1}{v_j(A_j)} \geq \frac{1}{v_i(A_j) - v_i(g)} + \dots + \frac{1}{v_i(A_j) - 1}.$$

Since  $v_i(A_j) - v_i(g) \geq 1$  and  $v_j(A_j) - v_j(g) + 1 \geq 2$  for every  $g \in B$ , applying Lemma C.2 to both sides, we get

$$\ln\left(\frac{v_j(A_j)}{v_j(A_j) - v_j(g)}\right) > \ln\left(\frac{v_i(A_j)}{v_i(A_j) - v_i(g)}\right)$$

for all  $g \in B$ . This implies that

$$\frac{v_j(A_j)}{v_j(A_j) - v_j(g)} > \frac{v_i(A_j)}{v_i(A_j) - v_i(g)},$$

which simplifies to  $v_i(g)/v_j(g) < v_i(A_j)/v_j(A_j)$  for all  $g \in B$ . As a result, we have

$$\frac{v_i(A_j)}{v_j(A_j)} = \frac{\sum_{g \in A_j} v_i(g)}{\sum_{g \in A_j} v_j(g)} \leq \frac{\sum_{g \in B} v_i(g)}{\sum_{g \in B} v_j(g)} < \frac{v_i(A_j)}{v_j(A_j)};$$

the first inequality holds because  $\sum_{g \in A_j} v_i(g) = \sum_{g \in B} v_i(g)$  by definition of  $B$  and  $\sum_{g \in A_j} v_j(g) \geq \sum_{g \in B} v_j(g)$  due to the relation  $B \subseteq A_j$ . This yields the desired contradiction.  $\square$

In spite of Theorem C.3, a disadvantage of MHW compared to MNW is that MHW is not *scale-invariant*—multiplying the value of a certain agent for every good by the same factor may change the MHW outcome. Moreover, MHW is well-defined only when all utilities are integers. Even though there is a natural extension of the harmonic numbers to the real domain given by  $H_x = \int_0^1 \frac{1-t^x}{1-t} dt$  for  $x \in \mathbb{R}$  (see, e.g., (Hintze 2019)), MHW defined via this extension is not guaranteed to satisfy EF1.<sup>15</sup>

**Proposition C.4** *There exists an instance with  $n = 2$  agents with equal weights and additive valuations such that every MHW allocation, where MHW is defined based on the extended harmonic numbers  $H_x = \int_0^1 \frac{1-t^x}{1-t} dt$ , does not satisfy EF1.*

**Proof** Let  $m = 3$ , and assume that the agents’ valuations are given by  $v_1(g_1) = 0$ ,  $v_1(g_2) = v_1(g_3) = 4$ ,  $v_2(g_1) = 1.9$ , and  $v_2(g_2) = v_2(g_3) = 2$ . Clearly, in any MHW allocation,  $g_1$  must be allocated to agent 2. We consider the three possibilities.

- If agent 1 receives both  $g_2$  and  $g_3$ , the harmonic welfare is  $H_8 + H_{1.9} > 2.717 + 1.459 = 4.176$ .
- If agent 1 receives one of  $g_2$  and  $g_3$ , the harmonic welfare is  $H_4 + H_{3.9} < 2.084 + 2.061 = 4.145$ .
- If agent 1 receives neither  $g_2$  nor  $g_3$ , the harmonic welfare is  $H_0 + H_{5.9} < 0 + 2.435 = 2.435$ .

Hence, the only MHW allocation gives  $g_1$  to agent 2 and both  $g_2$  and  $g_3$  to agent 1. However, agent 2 is not EF1 toward agent 1 with respect to this allocation.  $\square$

<sup>15</sup> Recently, Yuen and Suksompong (2023) showed that MNW is the only rule within the class of “welfarist rules” that guarantees EF1, which means that this shortcoming is far from being limited to MHW. This generalizes an earlier result on the class of additive welfarist rules, to which MHW also belongs (Suksompong 2023). Celine et al. (2025) further explored the fairness guarantees of additive welfarist rules based on (modified) harmonic numbers.

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**Data Availability** Not applicable.

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