THE MODEL THEORY OF CERTAIN INFINITE SOLUBLE GROUPS

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This thesis is concerned with aspects of the model theory of infinite soluble groups. The results proved lie on the border between group theory and model theory: the questions asked are of a model-theoretic nature but the techniques used are mainly group-theoretic in character.

We present a characterization of those groups contained in the universal closure of a restricted wreath product $U \operatorname{wr} G$, where U is an abelian group of zero or finite square-free exponent and G is a torsion-free soluble group with a bound on the class of its nilpotent subgroups. For certain choices of G we are able to use this characterization to prove further results about these groups; in particular, results related to the decidability of their universal theories.

The latter part of this work consists of a number of independent but related topics.

We show that if G is a finitely generated abelian-by-metanilpotent group and H is elementarily equivalent to G then the subgroups $\gamma_n(G)$ and $\gamma_n(H)$ are elementarily equivalent, as are the quotient groups $G/\gamma_n(G)$ and $G/\gamma_n(H)$.

We go on to consider those groups universally equivalent to $F_2(\mathfrak{V}\mathfrak{N}_c)$, where the free groups of the variety $\mathfrak{V}$ are residually finite p-groups for infinitely many primes p, distinguishing between the cases when $c = 1$ and when $c \geqslant 2$.

Finally, we address some important questions concerning the theories of free groups in product varieties $\mathfrak{V}_k \cdots \mathfrak{V}_1$, where $\mathfrak{V}_i$ is a nilpotent variety whose free groups are torsion-free; in particular we address questions about the decidability of the elementary and universal theories of such groups. Results mentioned in both of the previous two paragraphs have applications here.

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I thank my family and close friends for their support and encouragement.

Finally I would like to acknowledge the financial support of the EPSRC.

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NOTATION

		Page
G, H	groups	
g, h	group elements	
$o(g)$	the order of g	8
C_n	the cyclic group of exponent n	8
$\langle g_i \mid i \in I \rangle$	the group generated by the elements $\{g_i \mid i \in I\}$	
$\langle G_i \mid i \in I \rangle$	the group generated by the subgroups $\{G_i \mid i \in I\}$	
g^h	$h^{-1}gh$	8
$H \rtimes G$	the semi-direct product of H by G	8
$[g, h]$	$g^{-1}h^{-1}gh$	8
G'	the derived subgroup of G, namely $\langle [g, h] \mid g, h \in G \rangle$	9
$[G, H]$	$\langle [g, h] \mid g \in G, h \in H \rangle$	8
$Z(G)$	the centre of G, namely $\{g \in G \mid [g, h] = 1 \text{ for all } h \in G\}$	8
$G^{(d)}$	the dth term of the derived series of G	9
	$G^{(0)} = G$ and $G^{(d+1)} = \left[G^{(d)}, G^{(d)}\right]$	
$\gamma_c(G)$	the cth term of the lower central series of G	9
	$\gamma_1(G) = G$ and $\gamma_{c+1}(G) = [\gamma_c(G), G]$	
$\zeta_c(G)$	the cth term of the upper central series of G	9
	$\zeta_0(G) = 1$ and $\zeta_{c+1}(G)/\zeta_c(G) = Z(G/\zeta_c(G))$	
$\mathrm{Fitt}(G)$	the Fitting subgroup of G,	10
	generated by all nilpotent normal subgroups of G	
$\mathrm{Hom}(G, H)$	the set of homomorphisms from G to H	

X_∞	the set $\{x_1, x_2, \ldots\}$	10
F_∞	the free group on X_∞	10
u, v, w	words, elements of F_∞	10
$\mathfrak{U}, \mathfrak{V}, \mathfrak{W}$	varieties of groups	21
U, V, W	sets of words which define the varieties $\mathfrak{U}, \mathfrak{V}, \mathfrak{W}$	
$V(G)$	the verbal subgroup of G defined by the set of words V	21
	$\langle w\alpha \mid w \in V,\ \alpha \in \mathrm{Hom}(F_\infty, G)\rangle$	
$w(G)$	the verbal subgroup defined by the single word w	21
	$\langle w\alpha \mid \alpha \in \mathrm{Hom}(F_\infty, G)\rangle$	
$\mathfrak{O}$	the variety of all groups	22
$\mathfrak{E}$	the variety containing only the trivial group	22
$\mathfrak{A}$	the variety of abelian groups	22
$\mathfrak{A}_n$	the variety of abelian groups of exponent dividing n	22
$\mathfrak{N}_c$	the variety of nilpotent groups of class $\leqslant c$	22
$\mathfrak{S}_d$	the variety of soluble groups of derived length $\leqslant d$	22
$\mathfrak{P}_{\overline{c}}$	the variety of polynilpotent groups with class sequence $\leqslant \overline{c}$	22
$\overline{c}_i$	the subsequence $(c_1, \ldots, c_i)$ of $\overline{c} = (c_1, \ldots, c_k)$	22
$F_n(\mathfrak{V})$	the free group of rank n in $\mathfrak{V}$	22
$F_\infty(\mathfrak{V})$	the free group of countably infinite rank in $\mathfrak{V}$	22
$\mathfrak{U}\mathfrak{V}$	the product of the varieties $\mathfrak{U}$ and $\mathfrak{V}$	30
	the variety of all $\mathfrak{U}$-by-$\mathfrak{V}$ groups	
$\prod^C_{\lambda\in\Lambda} G_\lambda$	the Cartesian product of the groups G_λ	25
	the group of all functions $f : \Lambda \to \bigcup_{\lambda\in\Lambda} G_\lambda$ such that	
	$f(\lambda) \in G_\lambda$, multiplied componentwise	
G^Λ	the Cartesian power of G indexed by Λ	25
	the Cartesian product of groups G_λ with $G_\lambda \cong G$ for $\lambda \in \Lambda$	
g°	the constant function $g^\circ \in G^\Lambda$ such that $g^\circ(\lambda) = g$ for all $\lambda \in \Lambda$	25
G°	the diagonal subgroup of G^Λ, the set of all constant functions	25

$\prod_{\lambda\in\Lambda} G_\lambda$	the direct product of the groups G_λ	27
	$\{f \in \prod^C_{\lambda\in\Lambda} G_\lambda \mid f(\lambda) = 1$ for all but finitely many $\lambda \in \Lambda\}$	
$G^{(\Lambda)}$	the direct power of G indexed by Λ	27
	the direct product of groups G_λ with $G_\lambda \cong G$ for $\lambda \in \Lambda$	
$G \,\mathrm{Wr}\, H$	the unrestricted wreath product of G by H, namely $G^H \rtimes H$	25
$G \,\mathrm{wr}\, H$	the restricted wreath product of G by H, namely $G^{(H)} \rtimes H$	27
$\mathrm{wr}_{i=1}^n G_i$	the left-iterated restricted wreath product of the groups G_i	27
	$G_1 \,\mathrm{wr}(G_2 \,\mathrm{wr} \cdots (G_{n-1} \,\mathrm{wr}\, G_n)\cdots)$	
$\prod^*_{\lambda\in\Lambda} G_\lambda$	the free product of the groups G_λ	28
$[G_\lambda]^*$	the Cartesian subgroup of the free product	28
	the normal closure in the free product of	
	$\langle [G_\lambda, G_\mu] \mid \lambda, \mu \in \Lambda, \lambda \neq \mu\rangle$	
$\prod^{\mathfrak{V}}_{\lambda\in\Lambda} G_\lambda$	the $\mathfrak{V}$-verbal product of the groups G_λ	28
	$\dfrac{\prod^*_{\lambda\in\Lambda} G_\lambda}{[G_\lambda]^* \cap V(\prod^*_{\lambda\in\Lambda} G_\lambda)}$	
$G \,\mathrm{wr}_{\mathfrak{V}}\, H$	the $\mathfrak{V}$-verbal wreath product of G by H	28
	$\left(\prod^{\mathfrak{V}}_{h\in H} G_h\right) \rtimes H$ where $G_h \cong G$ for all $h \in H$	
$\mathcal{U}$	a non-principal ultrafilter over Λ	35
$\prod_{\lambda\in\Lambda} G_\lambda/\mathcal{U}$	the ultraproduct of the groups G_λ modulo $\mathcal{U}$	35
	the set of $\sim$-equivalence classes of $\prod^C_{\lambda\in\Lambda} G_\lambda$	
	$f_1 \sim f_2$ if and only if $\{\lambda \in \Lambda \mid f_1(\lambda) = f_2(\lambda)\} \in \mathcal{U}$	
$\mathbb{N}$	the natural numbers, does not contain 0	
$\aleph_0$	the cardinality of $\mathbb{N}$	
$\mathbb{Z}$	the ring of integers or the infinite cyclic group	
$\mathbb{Q}$	the field of rationals	
$\mathbb{F}_p$	$\mathbb{Z}/p\mathbb{Z}$, the field with p elements, p prime	
R_n	$\mathbb{Z}/n\mathbb{Z}$, the ring of integers modulo n	
Λ, Ω	sets	
$\mathbb{P}(\Lambda)$	the power set of Λ, namely $\{\Omega \mid \Omega \subseteq \Lambda\}$	
R, S	rings	

R^*	the non-zero elements of the ring R	41
$R[G]$	the group ring of G over R	41
	$\{\sum_{i=1}^n r_i g_i \mid g_i \in G, r_i \in R, n \in \mathbb{N}\}$	
$\mathrm{supp}(\alpha)$	the support of an element $\alpha \in R[G]$	41
	$\{g \in G \mid \alpha_g \neq 0\}$ where $\alpha = \sum_{g \in G} \alpha_g g$	
M, N	right R-modules	
$M \bigotimes_R N$	the tensor product of M and N	45
$\mathcal{L}_{\mathrm{gp}}$	the first order language of groups, namely $\{\cdot, ^{-1}, 1\}$	10
$\rightarrow$	"implies"	11
$\wedge$	"and"	11
$\vee$	"or"	11
$\neg$	"not"	11
$\forall$	"for all"	11
$\exists$	"there exists"	11
$\phi, \psi, \ldots$	first order sentences	11
$\phi(x), \psi(x), \ldots$	one-variable first-order formulae	
$G \models \phi$	ϕ holds in G	12
$\phi(G)$	the subset of G defined by $\phi(x)$	
	$\{g \in G \mid G \models \phi(g)\}$	12
$\mathrm{Th}(G)$	the (elementary) theory of G	12
$\mathrm{Th}_\forall(G)$	the universal theory of G	13
$\mathrm{Th}_\exists(G)$	the existential theory of G	13
$\mathrm{Th}_{\forall\exists}(G)$	the $\forall\exists$-theory of G	13
$\mathrm{Th}_+(G)$	the positive theory of G	14
$\mathrm{ucl}(G)$	the universal closure of G	
	the class of groups satisfying $\mathrm{Th}_\forall(G)$	13
ϕ_{gp}	the sentence $\forall xyz\,\big(x(yz) = (xy)z \wedge 1x = x \wedge xx^{-1} = 1\big)$	14

Part I

INTRODUCTION

AND

BACKGROUND

Chapter 1

INTRODUCTION

This thesis is concerned with aspects of the model theory of infinite soluble groups. We will give a characterization of those groups contained in the universal closure of a restricted wreath product $U \operatorname{wr} G$, where U is an abelian group of zero or finite square-free exponent and G is a torsion-free soluble group satisfying certain conditions. For certain choices of G we are able to use this characterization to prove further results about these groups. We will also address some important questions concerning the theories of free groups in product varieties $\mathfrak{V}_k \cdots \mathfrak{V}_1$, where each $\mathfrak{V}_i$ is a nilpotent variety whose free groups are torsion-free; in particular, we consider questions about the decidability of the elementary and universal theories of such groups. These results have close links with a number of major model-theoretic problems.

We begin with a discussion of these model-theoretic questions, which include some of the most significant mathematical problems of the twentieth century; in particular Hilbert's Tenth Problem, which took seventy years to be answered and the extensions of which are taxing mathematicians to this day: there is still no consensus on whether the version of Hilbert's Tenth Problem for the rational numbers has a positive or a negative answer. Another major set of problems which we discuss are those posed by Tarski in the 1940s which gave rise to the brand new subject of algebraic geometry over groups.

The results proved in this work lie on the border between group theory and model theory. The techniques used are mainly group-theoretic in character but the questions asked are of a model-theoretic nature and there is much model-theoretic notation and terminology. In order to make this thesis more accessible to group theorists with little knowledge of model theory and to model theorists with little knowledge of group theory, we include a brief introduction to each of these subjects, with particular reference to those areas of relevance to the content of the thesis. A list of notation used in this thesis can be found in the front matter together with a page reference guiding the reader to where this notation is explained in more detail. For coverage of the group-theoretic topics in greater depth see Robinson's book [53] and for coverage of the model-theoretic topics in greater depth see the book by Chang and Keisler [10].

We finish the introduction with an overview of our main results and an outline of the content of each chapter.

1.1 Background

Motivation for our work is provided by two very natural and long-standing problems relating to the elementary theory of groups, which were posed by Tarski in the 1940s and which were open until recently.

- Do free groups of distinct ranks have distinct elementary theories?
- Is the elementary theory of a free group decidable?

It is clear that the free group of rank 0 can be distinguished from those of higher rank by the sentence

$$\forall x\,(x = 1),$$

and that the free group of rank 1 can be distinguished by the sentence

$$\forall xy\,([x, y] = 1).$$

However there is no obvious sentence which allows us to differentiate between the free group of rank 2 and the free group of rank 3.

Efforts to solve these questions have led to a wide variety of interesting results. In particular, the whole theory of equations over groups grew out of the following question, posed by Vaught to test the elementary equivalence of free groups: does the sentence

$$\forall xyz\ \bigl(x^2y^2z^2 = 1 \rightarrow xy = yx\bigr)$$

hold in all free groups? This question was answered in the affirmative by Lyndon [29] using a combinatorial argument to show that the only solutions of the equation $x^2y^2z^2 = 1$ in a free group occur when x, y and z are powers of a common element. It was later shown by Lyndon and Schützenberger [30], again using combinatorial methods, that if elements x, y and z of a free group satisfy the equation $x^m y^n z^p = 1$ with $m, n, p \geqslant 2$ then $\langle x, y, z\rangle$ is cyclic. Since the inception of this subject, sets of solutions of equations over groups have been studied extensively.

Significant progress towards an answer to Tarski's questions was made by Makanin who showed that the universal and positive theories of a free group are decidable [36]. He also constructed an algorithm which determines whether a system of equations over a free group has a solution [35]. The development by Baumslag, Myasnikov and Remeslennikov of algebraic geometry over groups [7] made possible further progress and, using techniques from this blossoming subject, Kharlampovich and Myasnikov [23] and Sela [58] have addressed these questions.

It is natural to consider the corresponding questions for relatively free groups. It was shown by Szmielew [63] that free abelian groups of distinct finite ranks are not elementarily equivalent: consider, for example, the sentence

$$\exists xy\forall z\exists u\,(u^2 = z \vee u^2 = zx \vee u^2 = zy \vee u^2 = zxy).$$

Tarski observed that this sentence singles out the free abelian group of rank 2 from the free abelian groups of higher rank. It is also the case that free nilpotent groups of different finite ranks have different elementary theories: this was proved by Oger [45] using an argument involving ultrapowers. In [54], Rogers, Smith and Solitar show that free nilpotent groups of different finite ranks have different $\forall\exists\forall$-positive theories. To this end they construct positive universal sentences $s(n, c)$ such that if F is a free nilpotent group of class c and rank r then $F \models s(n, c)$ if and only if $\sum_{i=1}^{c} w(r, i) < n$, where $w(r, i)$ is the number of basic commutators of weight i. They show also that free soluble groups of different finite ranks have different $\forall\exists\forall$-positive theories: using the fact that the last non-trivial term of the derived series of a free soluble group is self-centralizing, they construct a positive universal formula defining this subgroup. In [13], Chapuis uses Szmeliew's result about free abelian groups to show that free soluble groups of different finite ranks have different ∀∃-theories by constructing existential formulae defining terms of the derived series. He comments that it is possible to strengthen this result for free soluble groups of derived length at most 3. In this case, groups of different finite ranks have different ∀∃-positive theories. Lawrence [26] shows that free groups of different finite ranks in a soluble variety have different ∀∃∀-positive theories: he constructs a normal subgroup which contains the derived subgroup and which is definable by a positive universal formula. Work in Chapter 10 will extend the results of Chapuis to free groups in product varieties $\mathfrak{V}_k \cdots \mathfrak{V}_1$, where each $\mathfrak{V}_i$ is a nilpotent variety whose free groups are torsion-free. We note also that it follows from a result of Timoshenko [66, Theorem 1] that all non-cyclic free soluble groups have the same universal theory as do all free nilpotent groups of class c and rank at least c.

The question of the decidability of the theories of relatively free groups has close links to one of the big mathematical questions of the twentieth century. Hilbert's Tenth Problem:

> Determination of the solvability of a diophantine equation.
>
> Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*

This problem was posed by Hilbert at the Second International Congress of Mathematicians in 1900 (his speech is translated into English in [20]) and it was not until 1970 that Matijasevich [39] showed that no such process exists. The analogous problem over the field $\mathbb{Q}$ of rational numbers is a major open problem. Many of the results in this area seem to point to the answer being a negative one, as over the ring of integers. However a number of mathematicians (mainly number theorists) conjecture a positive answer. For a more general and recent discussion of Hilbert's Tenth Problem see [48].

The elementary theory of a free abelian group is decidable [63]. However, as soon as we move away from abelian groups, things become more difficult. For example, the elementary theory of a finitely generated non-abelian free nilpotent group is undecidable [38]: this is proved by obtaining an interpretation of the ring of integers in a non-cyclic free nilpotent group of class 2, and the result then follows from the undecidability of the elementary theory of the ring of integers [64]. It is not known whether the universal theory of a non-cyclic free nilpotent group is decidable. In [55], Roman′kov showed that the decidability of the universal theory of the free nilpotent group of class and rank 2 would imply the decidability of the universal theory of the field of rationals. The problem of the decidability of the universal theory of $\mathbb{Q}$ is equivalent to Hilbert's Tenth Problem over $\mathbb{Q}$. Roman′kov's proof uses the fact that a free nilpotent group is universally equivalent to its $\mathbb{Q}$-completion: he associates to each universal sentence ϕ in the language of fields a universal sentence ϕ' in the language of groups such that ϕ holds in $\mathbb{Q}$ if and only if ϕ' holds in the $\mathbb{Q}$-completion of the free nilpotent group of class and rank 2.

In [37], Mal′cev proves that the elementary theory of a non-abelian free soluble group is undecidable. He uses the definability of the terms of the derived series to interpret the ring of integers in a non-abelian finitely generated free soluble group. His proof involves formulae with two constant symbols representing free generators. Chapuis [13] strengthens this result to give good bounds on the number of quantifiers: he shows that a non-abelian free soluble group F has undecidable $\forall\exists$-theory with no constants. Chapuis gives an algorithm which associates to each polynomial $P(\overline{X})$ in several variables with coefficients in $\mathbb{Z}$, a $\forall\exists$-sentence ψ such that $P(\overline{X})$ has a zero in $\mathbb{Z}$ if and only if $F \models \psi$. The negative answer to Hilbert's Tenth Problem over the integers [39] then gives the required result. Chapuis showed in [11] that the universal theory of a free metabelian group is decidable. As for nilpotent groups, it is unknown whether the universal theory of a non-cyclic free soluble group of derived length at least 3 is decidable. Using similar techniques to those above, Chapuis showed in [13] that the decidability of the universal theory of a non-cyclic free soluble group of derived length at least 3 would imply that there exists an algorithm to determine whether a homogeneous polynomial in several variables with coefficients in $\mathbb{Z}$ has a non-trivial zero in $\mathbb{Z}$. It is well known that such an algorithm exists if and only if the universal theory of the field of rationals is decidable [48, §9]. In Chapter 10, we extend these results about the decidability of elementary and universal theories to free groups in product varieties $\mathfrak{V}_k \cdots \mathfrak{V}_1$, where each $\mathfrak{V}_i$ is a nilpotent variety whose free groups are torsion-free.

During a problem day held at the First All-Union Symposium on Group Theory at Kourovka the following two problems were posed by Kargapolov [41, Problems 1.26 and 1.27]:

- Describe the universal theory of free groups.
- Describe the universal theory of free nilpotent groups of class n.

The first problem is still open. Makanin's work towards the second Tarski problem shows that the universal theory of a free group is decidable. Steps towards deriving an explicit description of the universal theory have been taken by Remeslennikov who gave a characterization of those groups with the same universal theory as a non-cyclic free group [49], [50]. The links between the decidability of the universal theory of a non-abelian free nilpotent group and Hilbert's Tenth Problem over the rationals, as discussed above, indicate that the second problem is very difficult. However there are varieties where progress can be made on questions of this type.

A group G is in the universal closure of a free abelian group F if and only if it is torsion-free and abelian. It is easy to write down a collection of universal sentences $\mathfrak{U}$ such that G is in the universal closure of F if and only if G satisfies the sentences $\mathfrak{U}$. We can, for example, take the collection of sentences

$$\mathfrak{U} = \{\forall xy\,([x,y]=1)\} \cup \{\forall x\ (x^n = 1 \rightarrow x = 1) \mid n \in \mathbb{N}\}\,.$$

In [12] Chapuis gives an explicit description of the universal theory of a non-cyclic free metabelian group and of those groups in its universal closure. These are precisely those groups such that a finitely generated subgroup can be embedded in a restricted wreath product $\mathbb{Z}^{(r)} \operatorname{wr} \mathbb{Z}^{(s)}$ for some non-negative integers r and s. Work in Part II provides similar descriptions in the case of restricted wreath products $U \operatorname{wr} G$, where U is an abelian group of zero or finite square-free exponent and G is a torsion-free soluble group satisfying certain conditions.

1.2 A brief introduction to group theory

Throughout this section let G be a group. The *order* of an element $g \in G$, denoted by $o(g)$, is the order of the cyclic subgroup generated by g. The group G is *periodic* if every element $g \in G$ has finite order. For a prime p, the group G is *p-group* if every element of G has order some power of p. The *exponent* of G is the least positive integer n such that $g^n = 1$ for all $g \in G$, if such an integer exists. If every non-trivial element of G has infinite order then G is *torsion-free*. We say that a torsion-free group has exponent zero. We denote by C_n the cyclic group of exponent n for any non-negative integer n.

The group G is an *extension* of a group A by a group H if G has a normal subgroup N such that $N \cong A$ and $G/N \cong H$. Let $\mathcal{P}$ and $\mathcal{Q}$ be properties of groups. The group G is a *$\mathcal{P}$-by-$\mathcal{Q}$* group if G is an extension of a group with property $\mathcal{P}$ by a group with property $\mathcal{Q}$.

For $g, h \in G$, the *conjugate* of g by h is the element $g^h := h^{-1}gh$. Let G and H be groups such that G acts on H by conjugation. The *semi-direct product* of H by G, denoted by $H \rtimes G$, is the set of ordered pairs $G \times H$ with multiplication defined by

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1^{g_1} h_2).$$

For $g, h \in G$, the *commutator* of g and h is the element $[g, h] := g^{-1}h^{-1}gh$ and we use the left-normed convention for commutators. We define a *simple commutator of weight n* recursively by the rule

$$[g_1, \ldots, g_n] := [[g_1, \ldots, g_{n-1}], g_n],$$

where $g_1, \ldots, g_n$ are elements of G. If X and Y are subsets of G then we define the *commutator subgroup of X and Y*, denoted by $[X, Y]$, to be the subgroup of G generated by all commutators $[x, y]$ with $x \in X$, $y \in Y$. The *derived subgroup* of G is the subgroup $G' := [G, G]$. The group G is abelian if and only if G' is trivial. The *centre* of G, denoted by $Z(G)$, is the normal subgroup of G defined by

$$Z(G) := \{g \in G \mid [g, h] = 1 \text{ for all } h \in G\}.$$

Soluble, nilpotent and polynilpotent groups

The group G is *soluble* if it has a subnormal series with abelian factors:

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_d = G$$

such that G_n/G_{n-1} is abelian for $n = 1, \ldots, d$. The length d of the shortest such series in G is the *derived length* of G. We define the terms of the *derived series* of G, denoted by $G^{(n)}$ for $n \geqslant 0$, as follows:

$$\begin{aligned} G^{(0)} &:= G, \\ G^{(1)} &:= G' = [G, G], \\ G^{(n+1)} &:= [G^{(n)}, G^{(n)}] \qquad \text{for } n \geqslant 1. \end{aligned}$$

If G is soluble then this series reaches the trivial subgroup after finitely many steps. The derived length of a soluble group G is equal to the length of the derived series of G. The group G is *metabelian* if it is soluble of derived length 2.

The group G is *nilpotent* if it has a normal series with central factors:

$$\{1\} = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_c = G$$

such that $G_n \lhd G$ and G_n/G_{n-1} is in the centre of G/G_{n-1} for $n = 1, \ldots, c$. The length c of the shortest such series is the *nilpotent class* of G. We define the terms of the *lower central series* of G, denoted by $\gamma_n(G)$ for $n \geqslant 1$, as follows:

$$\begin{aligned} \gamma_1(G) &:= G, \\ \gamma_{n+1}(G) &:= [\gamma_n(G), G] \qquad \text{for } n \geqslant 2. \end{aligned}$$

We note that $[\gamma_m(G), \gamma_n(G)] \leqslant \gamma_{m+n}(G)$ and that $\gamma_m(\gamma_n(G)) \leqslant \gamma_{mn}(G)$ for all positive integers m and n [53, 5.1.11]. We define the terms of the *upper central series* of G, denoted by $\zeta_n(G)$ for $n \geqslant 0$, as follows:

$$\begin{aligned} \zeta_0(G) &:= \{1\}, \\ \zeta_{n+1}(G)/\zeta_n(G) &:= Z(G/\zeta_n(G)) \qquad \text{for } n \geqslant 1. \end{aligned}$$

If G is nilpotent then the lower central series reaches the trivial subgroup after finitely many steps and the upper central series reaches the group G after finitely many steps;

in this case the lower central series and the upper central series have the same length and this length is equal to the nilpotent class of G. If G is nilpotent of class c then $\gamma_n(G) \leqslant \zeta_{c+1-n}(G)$ for $n = 1, \ldots, c$.

The *Fitting subgroup* of G, denoted by $\mathrm{Fitt}(G)$, is the subgroup of G generated by all the nilpotent normal subgroups of G. This subgroup may not be nilpotent but every finitely generated subgroup of $\mathrm{Fitt}(G)$ is nilpotent.

The group G is *polynilpotent* if it has a subnormal series with nilpotent factors:

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_k = G$$

such that G_n/G_{n-1} is nilpotent for $n = 1, \ldots, k$. If G_n/G_{n-1} is nilpotent of class at most c_n for $n = 1, \ldots, k$ then we say that G has class sequence at most $\overline{c} = (c_1, \ldots, c_k)$. The group G is *metanilpotent* if it is polynilpotent with class sequence (c_1, c_2) for some c_1, c_2.

Free groups

A group F is *free* on a subset X if, for each function $\alpha : X \to G$, where G is any group, there is a unique homomorphism $\beta : F \to G$ such that $x\beta = x\alpha$ for all $x \in X$. A group which is free on some set is an *(absolutely) free group*. If F is free on a set X of cardinality n then F is *free of rank* n. The countably infinite free group F_∞ is free on the set $X_\infty = \{x_1, x_2, \ldots\}$.

A *word* w is an element of F_∞. A word w can be written uniquely as a string $w = y_1 \cdots y_r$ with $y_i \in X_\infty \cup X_\infty^{-1}$ where $X_\infty^{-1} = \{x_1^{-1}, x_2^{-1}, \ldots\}$ with no occurrences of $x_i^{-1}x_i$ or $x_i x_i^{-1}$. A generator x_j occurs in w if $y_i = x_j$ or $y_i = x_j^{-1}$ for some i. If only the generators $x_1, \ldots, x_n$ occur in w then we say that w is a *word in the variables* $x_1, \ldots, x_n$ or a *word in* n*-variables.*

1.3 A brief introduction to first-order logic

First-order formulae in the language of groups $\mathfrak{L}_{\mathrm{gp}} := \{\cdot, {}^{-1}, 1\}$ are certain finite strings consisting of the following symbols:

- variables representing group elements;

- the symbols $\cdot, ^{-1}$ and 1 representing group multiplication, inverse and the identity element;
- the equality symbol $=$;
- the logical symbols $\to$ ("implies"), $\wedge$ ("and"), $\vee$ ("or"), $\neg$ ("not") and quantifiers $\forall$ ("for all") and $\exists$ ("there exists") ranging over all elements of G .

A first-order formula is constructed in a specific way: an *atomic formula* in $\mathcal{L}_{\text{gp}}$ is of the form $w(\overline{x}) = u(\overline{x})$ where $w(\overline{x})$ and $u(\overline{x})$ are words in variables $x_1, \ldots, x_n$ and a *formula* in $\mathcal{L}_{\text{gp}}$ is defined recursively as follows:

- an atomic formula is a formula;
- if ϕ is a formula then $\neg\phi$ is a formula;
- if ϕ and ψ are formulae then $\phi \wedge \psi$, $\phi \vee \psi$ and $\phi \to \psi$ are formulae;
- if ϕ is a formula then $\forall x\phi$ and $\exists x\phi$ are formulae for any variable x.

Abbreviations are permitted where the meaning is clear. We commonly use the following abbreviations:

- xy for $x \cdot y$;
- x^n for $\underbrace{x \cdot x \cdots x}_{n}$;
- x^y for $y^{-1}xy$;
- $[x, y]$ for $x^{-1}y^{-1}xy$;
- $x \neq y$ for $\neg(x = y)$.

A formula is *quantifier-free* if it contains no occurrences of $\forall$ or $\exists$. A variable occurs *freely* in a formula if it is not bound by a quantifier. A *sentence* is a formula with no free variables. For example,

- the formula $[x, y] = 1 \wedge z^7 = 1$ is quantifier-free;

- in the formula $\forall y\,([x,y]=1)$, the variable y is bound by the quantifier $\forall$ and the variable x occurs freely;

- the formula $\exists x \forall y\,([x,y]=1)$ is a sentence.

If $\phi(x_1,\ldots,x_n)$ is a formula in n free variables and $g_1,\ldots,g_n$ are elements of a group G then we write $G \models \phi(g_1,\ldots,g_n)$ if $\phi(g_1,\ldots,g_n)$ is true in G . If $\phi(x)$ is a formula in one free variable then $\phi(x)$ defines the subset $\phi(G)$ of G where

$$\phi(G) := \{g \in G \,|\, G \models \phi(g)\}.$$

For example, for any group G, the formula $\forall x\,([x,y]=1)$ defines the centre of G.

Sentences ϕ and ψ are *logically equivalent* if ϕ is true in a group G if and only if ψ is true in G. A quantifier-free formula in $\mathcal{L}_{\mathrm{gp}}$ is in *disjunctive normal form* if it is of the form

$$\bigvee_{i=1}^{s}\bigwedge_{j=1}^{t}(u_{ij}(\overline{x})=1 \wedge v_{ij}(\overline{x}) \neq 1),$$

where $u_{ij}(\overline{x})$ and $v_{ij}(\overline{x})$ are words in the variables $x_1,\ldots,x_r$. Every quantifier-free formula is logically equivalent to one in disjunctive normal form. A formula in $\mathcal{L}_{\mathrm{gp}}$ is in *prenex normal form* if it consists of a string of quantifiers (possibly empty) followed by a quantifier-free formula in disjunctive normal form. Every first-order formula is logically equivalent to one in prenex normal form and every sentence in $\mathcal{L}_{\mathrm{gp}}$ is logically equivalent to one of the form

$$Q_1x_1\ldots Q_rx_r\Bigg(\bigvee_{i=1}^{s}\bigwedge_{j=1}^{t}(u_{ij}(\overline{x})=1 \wedge v_{ij}(\overline{x}) \neq 1)\Bigg),$$

where Q_k is a quantifier and $u_{ij}(\overline{x})$ and $v_{ij}(\overline{x})$ are words in the variables $x_1,\ldots,x_r$.

The *elementary theory* of a group G, which we denote by $\mathrm{Th}(G)$, is the collection of sentences which hold in G. Two groups G and H are *elementarily equivalent* if they have the same elementary theory.

Universal and existential theory

A formula ϕ is *universal* (respectively *existential*) if the only quantifier which occurs when ϕ is written in prenex normal form is the universal quantifier $\forall$ (respectively the existential quantifier $\exists$). A universal sentence in $\mathcal{L}_{\text{gp}}$ is logically equivalent to one of the form

$$\forall x_1 \dots x_r \Big(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} (u_{ij}(\overline{x}) = 1 \wedge w_{ij}(\overline{x}) \neq 1) \Big),$$

and an existential sentence in $\mathcal{L}_{\text{gp}}$ is logically equivalent to one of the form

$$\exists x_1 \dots x_r \Big(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} (u_{ij}(\overline{x}) = 1 \wedge w_{ij}(\overline{x}) \neq 1) \Big),$$

where $u_{ij}(\overline{x})$ and $w_{ij}(\overline{x})$ are words in the variables $x_1, \dots, x_r$.

The *universal* (respectively *existential*) *theory* of a group G, denoted by $\text{Th}_\forall(G)$ (respectively $\text{Th}_\exists(G)$), is the collection of universal (respectively existential) sentences which hold in G. A group G satisfies the universal theory of a group H if and only if H satisfies the existential theory of G. Two groups G and H are *universally equivalent* if $\text{Th}_\forall(G) = \text{Th}_\forall(H)$. The universal closure of a group G, denoted by $\text{ucl}(G)$, is the class of all groups which satisfy the universal theory of G:

$$\text{ucl}(G) := \{H \mid H \models \psi \text{ for all } \psi \in \text{Th}_\forall(G)\}.$$

Two groups G and H are *existentially equivalent* if $\text{Th}_\exists(G) = \text{Th}_\exists(H)$. A group G is existentially equivalent to a group H if and only if G is universally equivalent to H. If H satisfies the universal theory and the existential theory of G then H is universally equivalent to G.

Other subsets of the elementary theory of a group are also of interest; in particular, those subsets determined by the number of alternations of quantifiers allowed in the sentences when in prenex normal form and those which contain no occurrences of the negation symbol when in prenex normal form.

A formula is a $\forall\exists$-*formula* if, when written in prenex normal form, it consists of a string of universal quantifiers, followed by a string of existential quantifiers, followed by a quantifier-free formula. A $\forall\exists$-sentence in $\mathcal{L}_{\text{gp}}$ is logically equivalent to one of the form

$$\forall x_1 \dots x_{r_1} \exists x_{r_1+1} \dots x_{r_2} \Big(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} (u_{ij}(\overline{x}) = 1 \wedge w_{ij}(\overline{x}) \neq 1) \Big),$$

where $u_{ij}(\overline{x})$ and $w_{ij}(\overline{x})$ are group words in the variables $x_1, \dots, x_{r_2}$. The $\forall\exists$-*theory* of a group G, denoted by $\text{Th}_{\forall\exists}(G)$, is the collection of $\forall\exists$-sentences which hold in G. We may extend these definitions to arbitrarily many alternations of quantifiers.

A formula is *positive* if it contains no occurrences of the negation symbol when written in prenex normal form. A positive sentence in $\mathcal{L}_{\mathrm{gp}}$ is logically equivalent to one of the form

$$Q_1 x_1 \dots Q_r x_r \left(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} u_{ij}(\overline{x}) = 1 \right),$$

where Q_k is a quantifier and $u_{ij}(\overline{x})$ is a word in the variables $x_1, \dots, x_r$. The *positive theory* of a group G, denoted by $\mathrm{Th}_+(G)$, is the collection of positive sentences which hold in G. We may combine the two concepts and consider the $\forall\exists$-positive theory of a group.

Finally in this section we define a universal sentence which we will require in each of the four subsequent chapters. We define ϕ_{gp} to be

$$\forall xyz \; \big(x(yz) = (xy)z \wedge 1x = x \wedge x^{-1}x = 1 \big) .$$

This sentence ensures that the assignment of the constant and function symbols from the language $\mathfrak{L}_{\mathrm{gp}}$ is consistent with the definition of a group.

1.4 Overview of thesis

In Chapter 2 we give a brief introduction to varieties of groups and recall some useful results. These results include the link between residual properties and discrimination, the Kaloujnine–Krasner, Magnus and Shmel′kin embeddings, conditions for a subgroup of a free group in a variety $\mathfrak{V}$ to itself be free in $\mathfrak{V}$ and information about the nilpotent subgroups of a free group in a product variety satisfying certain conditions. We also recall some known results about the universal theories of $\mathcal{L}$-structures for a language $\mathcal{L}$. Although these results apply more generally, we formulate them specifically in the case of groups as this is the situation in which we shall be applying them. Ultraproducts are a useful tool in dealing with elementary classes of groups and we take this opportunity to recall the definitions of ultrafilters, ultraproducts and ultrapowers and to state some standard results.

In Part II (Chapters 3 to 7) we consider the universal theory of restricted wreath products of abelian groups by certain soluble groups.

In Chapter 3 we prove some preliminary results which we need in the subsequent four chapters. These results split into two types: group-theoretic results about restricted wreath products and results related to the universal theory of restricted wreath products. Of the former type, Lemmas 3.3.1 and 3.3.2 will prove crucial in what follows.

Lemma 3.3.1. *Let A and B be torsion-free groups and suppose that A is abelian. Let G be a finitely generated group with a normal subgroup N such that $N \cong A$ and $G/N \cong B$. If $\mathbb{Z}[G/N]$ is an Ore domain and N is torsion-free as a $\mathbb{Z}[G/N]$-module then G embeds in $U \operatorname{wr} B$, for some torsion-free abelian group U.*

Lemma 3.3.2. *Let A be an abelian group of finite square-free exponent n and let B be a torsion-free group. Let G be a finitely generated group with a normal subgroup N such that $N \cong A$ and $G/N \cong B$. For each prime divisor p of n, suppose that $\mathbb{F}_p[G/N]$ is an Ore domain and that the submodule of N of elements of order p is torsion-free as an $\mathbb{F}_p[G/N]$-module. Then G embeds in $U \operatorname{wr} B$, for some abelian group U of exponent n.*

We begin the chapter by recalling the definitions of an integral domain and an Ore domain and some sufficient conditions for a group ring, over the integers or over a field, to be an Ore domain. The focus of the latter type of result is on showing that certain restricted wreath products are universally equivalent. We note that an easy adaptation of Timoshenko's proof that the restricted wreath product preserves universal equivalence yields the following result.

Lemma 3.4.1. *Let A_1, A_2, B_1 and B_2 be groups. If $A_1 \models \mathrm{Th}_\forall(A_2)$ and $B_1 \models \mathrm{Th}_\forall(B_2)$ then $A_1 \operatorname{wr} B_1 \models \mathrm{Th}_\forall(A_2 \operatorname{wr} B_2)$.*

If A is a non-trivial torsion-free abelian group then A is universally equivalent to $\mathbb{Z}$ by [16, Lemma 1]. Thus, if B_1 and B_2 are groups such that $B_1 \models \mathrm{Th}_\forall(B_2)$, then $A \operatorname{wr} B_1 \models \mathrm{Th}_\forall(\mathbb{Z} \operatorname{wr} B_2)$. If the group A has finite exponent then the corresponding result requires more work.

Lemma 3.4.2. *If B is a torsion-free group and A is an abelian group of exponent n for some $n \in \mathbb{N}$ then $A \operatorname{wr} B \models \mathrm{Th}_\forall(C_n \operatorname{wr} B)$.*

In Chapter 4 we give a complete description of the universal theory of a non-cyclic free group in the variety $\mathfrak{A}_n\mathfrak{A}$, where n is a square-free positive integer, and of the finitely generated groups in its universal closure.

Theorem 4.1.1. *If G is a group then the following conditions are equivalent:*

(1) *G satisfies the universal theory of a non-cyclic group free in $\mathfrak{A}_n\mathfrak{A}$;*

(2) *G satisfies the universal sentences $\mathfrak{U}_n$;*

(3) *G is a σ_n-group or G is torsion-free abelian;*

(4) *Every finitely generated subgroup of G can be embedded in $C_n^{(r)} \operatorname{wr} \mathbb{Z}^{(s)}$ for some $r, s \in \mathbb{N}$.*

The collection $\mathfrak{U}_n$ of universal sentences is defined in Section 4.2 (on pages 57 to 59) and the property of being a σ_n-group is defined on page 55. We use this description to prove that a free group in this variety has decidable universal theory. We also use the description to prove some further results about those groups universally equivalent to a non-cyclic free group in the variety $\mathfrak{A}_n\mathfrak{A}$.

In Chapter 5 we consider left-iterated wreath products of torsion-free abelian groups. Chapuis proved in [11] that such groups have decidable universal theory; however, no description of this theory exists in the literature. We give such a description and use it to give a characterization of those finitely generated groups contained in the universal closure of one of these groups.

Theorem 5.1.2. *Let S be a left-iterated wreath product of d non-trivial torsion-free abelian groups. If G is a finitely generated group then the following conditions are equivalent:*

(1) *$G \in \operatorname{ucl}(S)$;*

(2) *G can be embedded in $\operatorname{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}$ for some $r_1, \ldots, r_d \in \mathbb{N}$;*

(3) *G is a ρ_d-group.*

The property of being a ρ_d-group is defined on page 74.

In Chapter 6 we consider the universal theory of groups of the form $U \operatorname{wr} H$, where H is a finitely generated torsion-free nilpotent group and U is a non-trivial abelian group of exponent n, where n is a square-free positive integer or zero. We give an explicit description of the universal theory of the group $U \operatorname{wr} H$ modulo the universal theory of H and of the finitely generated groups in the universal closure of $U \operatorname{wr} H$.

Theorem 6.1.1. *Let H be a finitely generated torsion-free nilpotent group of class c and let U be a non-trivial abelian group of exponent n. If G is a group then the following conditions are equivalent:*

(1) $G \in \operatorname{ucl}\big(U \operatorname{wr} H\big)$;

(2) *G satisfies the universal sentences* $\mathfrak{W}_{n,H}$;

(3) *G is a $\rho_{n,H}$-group or* $G \models \operatorname{Th}_\forall(H)$;

(4) *Every finitely generated subgroup of G can be embedded in $C_n^{(r)} \operatorname{wr} K$ for some $r \in \mathbb{N}$ and some K such that $K \models \operatorname{Th}_\forall(H)$.*

The collection $\mathfrak{W}_{n,H}$ of universal sentences is defined in Section 6.2 (on pages 85 to 91) and the property of being a $\rho_{n,H}$-group is defined on page 84. This description provides a method of proof of the decidability of $U \operatorname{wr} H$, provided that there exists a recursive collection of sentences describing the universal and existential theory of H. We finish the chapter by considering some examples of groups H for which we can write down explicitly a collection of sentences $\mathfrak{V}_H$ such that if $K \in \operatorname{ucl}(H)$ then K is universally equivalent to H if and only if $K \models \mathfrak{V}_H$.

Proposition 6.7.1. *If G is in $\operatorname{ucl}(F_2(\mathfrak{N}_c))$ then G is universally equivalent to $F_2(\mathfrak{N}_c)$ if and only if G contains a 2-generator subgroup which is not in $\mathfrak{N}_{c-1}$.*

In Chapter 7 we generalize the results of Chapters 4, 5 and 6 to groups of the form $U \operatorname{wr} P$, where P is a torsion-free soluble group satisfying certain conditions and U is a non-trivial abelian group of exponent n, where n is a square-free positive integer or zero. We require that there exist a bound on the class of the nilpotent subgroups of P. We give an explicit description of the universal theory of the group $U \operatorname{wr} P$ modulo the universal theory of P and use this to give a characterization of those finitely generated groups in the universal closure of this group.

Theorem 7.1.2. *Let P be a finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$ and let U be a non-trivial abelian group of exponent n. If G is a finitely generated group then the following conditions are equivalent:*

(1) $G \in \operatorname{ucl}(U \operatorname{wr} P)$;

(2) *G can be embedded in $C_n^{(r)} \operatorname{wr} Q$ for some $r \in \mathbb{N}$ and some Q such that $Q \models \mathrm{Th}_\forall(P)$;*

(3) *Either G is a $\rho_{n,Q}$-group for some subgroup Q of P or $G \models \mathrm{Th}_\forall(P)$.*

The property of being a $\rho_{n,P}$-group is defined on page 110. If $F = \mathcal{F}/N'$ where $\mathcal{F}$ is an absolutely free group and $F/N \cong G$ satisfies the conditions above then $\mathrm{ucl}(F) \subseteq \mathrm{ucl}(C_n \operatorname{wr} G)$; this yields necessary conditions for a group to be in $\mathrm{ucl}(F)$. We show that in many cases this inclusion is strict.

In Part III (Chapters 8 to 10) we establish a collection of results concerning various aspects of the elementary and universal theory of certain infinite soluble groups.

We say that a group G is a C_n-group if there exists an integer t such that each element of $\gamma_n(G)$ can be written as a product of t commutators $[h_1, g_1] \cdots [h_t, g_t]$ with $g_i \in G$ and $h_i \in \gamma_{n-1}(G)$ for $i = 1, \ldots, t$. In Chapter 8 we extend a number results about commutator lengths in derived subgroups. It is known that finitely generated abelian-by-metanilpotent groups are C_2-groups, as are extensions of abelian groups by groups satisfying the maximal condition on normal subgroups (see [2], [19] and [52]). We extend these results to provide examples of groups which are C_n-groups for all $n \geqslant 2$ and in many cases give an explicit bound on the value of t. We show that G is a C_n-group for all $n \geqslant 2$ if G is in one of the following classes:

- finitely generated finite-by-abelian-by-metanilpotent groups;
- finite-by-soluble groups satisfying the maximal condition on normal subgroups.

We go on to consider the elementary theory of groups which are C_n-groups for all $n \geqslant 2$. We obtain one-variable first-order formulae defining the terms of the lower central series in such a group and use these formulae to prove the following result.

Theorem 8.2.1. *Fix a positive integer n and let H be a C_r-group for $2 \leqslant r \leqslant n$. If the group G is elementarily equivalent to H then $\gamma_n(G)$ is elementarily equivalent to $\gamma_n(H)$ and $G/\gamma_n(G)$ is elementarily equivalent to $H/\gamma_n(H)$.*

In [66], Timoshenko determines those 2-generator groups finitely presented in the variety of soluble groups of derived length d which are universally equivalent to a non-cyclic free soluble group of derived length d: there are precisely two such groups, namely the free soluble group of derived length d and rank 2, and the verbal wreath product $\mathbb{Z}\,\mathrm{wr}_{\mathfrak{S}_{d-1}}\,\mathbb{Z}$. His proof that these two groups are indeed universally equivalent appears to contain a small error. In Chapter 9 we correct this error and prove the following more general result.

Proposition 9.1.1. *Let $\mathfrak{V}$ be a variety containing $\mathfrak{A}$ such that the free groups of $\mathfrak{V}$ are residually finite p-groups for infinitely many primes p. Suppose further that $\mathfrak{V}$ is generated by its k-generator groups for some positive integer k. The free group of finite or infinite rank $r \geqslant 2$ in $\mathfrak{V}\mathfrak{A}$ is universally equivalent to $\mathbb{Z}\,\mathrm{wr}_{\mathfrak{V}}\,\mathbb{Z}$.*

Thus there are at least two isomorphism classes of 2-generator groups universally equivalent to $F_2(\mathfrak{V}\mathfrak{A})$. However, in a product variety $\mathfrak{V}\mathfrak{N}_c$, with $c \geqslant 2$, the situation is different, as can be seen by the following proposition.

Proposition 9.2.1. *Let $\mathfrak{V}$ be a variety whose free groups are residually finite p-groups for infinitely many primes p and let $c \geqslant 2$. Let G be a 2-generator group finitely presented in $\mathfrak{V}\mathfrak{N}_c$. Then G is universally equivalent to $F_2(\mathfrak{V}\mathfrak{N}_c)$ if and only if G is isomorphic to $F_2(\mathfrak{V}\mathfrak{N}_c)$.*

We also give the following necessary and sufficient condition for a group in the universal closure of a free group F in $\mathfrak{A}\mathfrak{N}_c$ to be universally equivalent to F.

Proposition 9.2.2. *Let G be in $\mathrm{ucl}(F_2(\mathfrak{A}\mathfrak{N}_c))$ for $c \geqslant 2$. Then G is universally equivalent to $F_2(\mathfrak{A}\mathfrak{N}_c)$ if and only if G contains a 2-generator subgroup which is not in $\mathfrak{A}\mathfrak{N}_{c-1}$.*

In Chapter 10 we prove some results about the theories of free groups in certain soluble varieties. Let $\mathfrak{V}_1, \ldots, \mathfrak{V}_k$ be non-trivial nilpotent varieties whose free groups are torsion-free and suppose that $k \geqslant 2$. We begin by obtaining one-variable formulae defining a number of verbal subgroups of a free group F in $\mathfrak{V}_k \cdots \mathfrak{V}_1$. We use the definability of the derived subgroup to extend the result of Romanovskii and Timoshenko [57] that if F is a free soluble group of rank n and G is an n-generator group elementarily equivalent to F then G is isomorphic to F.

Theorem 10.2.1. *Let F be the free group of rank n in $\mathfrak{V}_k \cdots \mathfrak{V}_1$. An n-generator group G is elementarily equivalent to F if and only if G is isomorphic to F.*

We also use our definability results to extend a number of results of Chapuis [13] about the elementary and universal theories of free soluble groups.

Proposition 10.3.1. *The free groups in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ of different finite ranks have different $\forall\exists$-theories.*

Theorem 10.4.1. *A free group F of finite rank $r \geqslant 2$ in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ has undecidable $\forall\exists$-theory.*

Theorem 10.5.1. *The decidability of the universal theory of the free group of finite rank $r \geqslant 2$ in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ with $c_1 \geqslant 2$ implies the decidability of the universal theory of the field of rationals.*

Theorem 10.5.2. *The decidability of the universal theory of a non-cyclic free group in $\mathfrak{V}_k \cdots \mathfrak{V}_2\mathfrak{V}_1$ with $k \geqslant 3$, $\mathfrak{V}_1 = \mathfrak{A}$ and $\mathfrak{V}_2 = \mathfrak{N}_{c_2}$ implies the decidability of the universal theory of the field of rationals.*

These results contrast with the positive answers to Tarski's questions about the elementary theory of absolutely free groups which have been announced by Kharlampovich and Myasnikov [23] and Sela [58].

Finally, in Chapter 11, we summarize the results proved in this thesis, highlighting those which we believe to be of particular interest. We also suggest areas where further work could be done.

Chapter 2

PRELIMINARY RESULTS ON VARIETIES AND UNIVERSAL THEORIES OF GROUPS

In this chapter we give a brief introduction to varieties of groups and recall some useful results. For a more general discussion of these topics see the book by Hanna Neumann [44]. We also recall some useful results about the universal theories of structures in a langauge; we formulate these results in the case of groups.

2.1 The basic facts

A *word* is an element of the free group F_∞ on the set $X_\infty = \{x_1, x_2, \ldots\}$. The image of w under a homomorphism from F_∞ to a group G is a *value* of w in G. If only the generators $x_1, \ldots, x_n$ occur in w then, for any n-tuple $g_1, \ldots, g_n$ in G, the map $x_i \mapsto g_i$ for $i = 1, \ldots, n$ and $x_i \mapsto 1$ for $i > n$ extends to a homomorphism $F_\infty \to G$. We write $w(g_1, \ldots, g_n)$ for the image of g under this map.

The word w is a *law* in the group G if the only value of w in G is 1. The *verbal subgroup* $V(G)$ of G corresponding to a set of words V is the subgroup generated by all values in G of words from V:

$$V(G) := \langle w\alpha \mid w \in V,\, \alpha \in \mathrm{Hom}(F_\infty, G)\rangle.$$

Thus w is a law in G for all $w \in V$ if and only if $V(G) = \{1\}$. If V consists of a single word w, we write $w(G)$ for $V(G)$.

The *variety* defined by a set of words V is the class of all groups G such that each word in V is a law in G. We list some examples of varieties which will be important in what follows.

Examples of varieties

- The variety $\mathfrak{O}$ of all groups, defined by the empty set of words.
- The variety $\mathfrak{E}$, which contains only the trivial group, defined by the word x.
- The variety $\mathfrak{A}$ of abelian groups, defined by the word $[x, y]$.
- The variety $\mathfrak{A}_n$ of abelian groups of exponent n, defined by the words $\{x^n, [x, y]\}$.
- The variety $\mathfrak{N}_c$ of nilpotent groups of class at most c, defined by the word $[x_1, \ldots, x_{c+1}]$.
- The variety $\mathfrak{S}_d$ of soluble groups of derived length at most d, defined by the word $s_d(\overline{x})$ where:

$$s_1(x, y) = [x, y],$$
$$s_{i+1}(\overline{x}) = [s_i(\overline{y}), s_i(\overline{z})].$$

  Here $\overline{x} = (\overline{y}, \overline{z})$, where $\overline{y}$ and $\overline{z}$ are each 2^{i-1}-tuples of elements of X_∞.

- The variety $\mathfrak{P}_{\overline{c}}$ of polynilpotent groups with class sequence at most $\overline{c} = (c_1, \ldots, c_k)$, defined by the word $s_{\overline{c}}(G)$, where $s_{\overline{c}}(\overline{x}) = s_{\overline{c}_k}(\overline{x})$ and:

$$s_{\overline{c}_1}(x_1, \ldots, x_{c_1+1}) = [x_1, \ldots, x_{c_1+1}],$$
$$s_{\overline{c}_i}(\overline{x}) = [s_{\overline{c}_{i-1}}(\overline{x}_1), \ldots, s_{\overline{c}_{i-1}}(\overline{x}_{c_i+1})].$$

  Here $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_{c_i+1})$, where $\overline{x}_1, \ldots, \overline{x}_{c_i+1}$ are each $(c_1 + 1) \cdots (c_{i-1} + 1)$-tuples of elements of X_∞, and the sequence $\overline{c}_i$ is the subsequence $(c_1, \ldots, c_i)$ of $\overline{c}$.

A group G is *relatively free* if $G \cong F/V(F)$ for some absolutely free group F and some set of words V. Let $\mathfrak{V}$ be a variety defined by a set of words V. The *free group of rank n in* $\mathfrak{V}$, denoted by $F_n(\mathfrak{V})$, is the relatively free group $F_n/V(F_n)$, where F_n is the absolutely free group of rank n. We denote by $F_\infty(\mathfrak{V})$ the free group of countably infinite rank in $\mathfrak{V}$. A group G belongs to a variety $\mathfrak{V}$ if and only if it is isomorphic to a quotient group of a free group of suitable rank in $\mathfrak{V}$.

Examples of relatively free groups

Let F_n be the absolutely free group of rank n.

- The free group of rank n in $\mathfrak{O}$ is F_n.
- The free abelian group of rank n is F_n/F_n'.
- The free nilpotent group of class c and rank n is $F_n/\gamma_{c+1}(F_n)$.
- The free soluble group of derived length d and rank n is $F_n/F_n^{(d)}$.
- The free polynilpotent group of rank n with class sequence $\overline{c}$ is $F_n/\gamma_{c_k+1}(\cdots\gamma_{c_1+1}(F_n))$.

2.2 Discrimination and residual properties

A set $\mathfrak{X}$ of groups is *discriminating* (as defined in [8]) if, for every finite set W of words in at most n variables with the property that, for each word $w \in W$, there is a homomorphism $f_w : F_n \to G$ with $G \in \mathfrak{X}$ such that $f_w(w) \neq 1$, there is a group $G \in \mathfrak{X}$ and a homomorphism $f : F_n \to G$ such that $f(w) \neq 1$ for all $w \in W$. A set $\mathfrak{X}$ of groups *discriminates the variety* $\mathfrak{V}$ if $\mathfrak{X} \subseteq \mathfrak{V}$ and, for every finite set W of words in at most n variables which are not laws in $\mathfrak{V}$, there exists a group $G \in \mathfrak{X}$ and a homomorphism $f : F_n \to G$ such that $f(w) \neq 1$ for all $w \in W$. A set $\mathfrak{X}$ is discriminating if and only if it discriminates the variety which it generates. The set of all free groups of finite rank in a variety $\mathfrak{V}$ discriminates $\mathfrak{V}$.

Example 2.2.1. *The infinite cyclic group $\mathbb{Z}$ is discriminating and, therefore, $\mathbb{Z}$ discriminates the variety of abelian groups $\mathfrak{A}$.*

Proof. This follows immediately from the fact that a finite set of linear equations can be simultaneously falsified. □

Let $\mathcal{P}$ be a property of groups. A group G is *residually* $\mathcal{P}$ if, for every non-trivial element $g \in G$, there is a normal subgroup N_g of G such that $g \notin N_g$ and G/N_g has the property $\mathcal{P}$. The following conditions are equivalent:

1. G is residually $\mathcal{P}$;

2. $\bigcap\limits_{N \in X} N = \{1\}$ where $X = \{N \lhd G \mid G/N \text{ has property } \mathcal{P}\}$;

3. G is a subcartesian product of groups with property $\mathcal{P}$.

The following result illuminates the link between residual properties and discrimination.

Theorem 2.2.1 ([44, 17.81]). *If $\mathcal{P}$ is a property inherited by subgroups and by finite direct products then the free groups of a variety $\mathfrak{V}$ are residually $\mathcal{P}$ if and only if the finitely generated groups with the property $\mathcal{P}$ in $\mathfrak{V}$ discriminate $\mathfrak{V}$.*

The property of being residually a finite p-group is of particular interest to us. This property is inherited by subgroups and by finite direct products. Thus, since the free groups of finite rank discriminate a variety, in order to show that the free groups in $\mathfrak{V}$ are residually $\mathcal{P}$, it is enough to show that the free groups of finite rank are residually $\mathcal{P}$.

Lemma 2.2.1. *If $\mathfrak{V} \subseteq \mathfrak{N}_c$ is a variety whose free groups are torsion-free then the free groups of $\mathfrak{V}$ are residually finite p-groups for every prime p.*

Proof. This result follows from a theorem of Gruenberg [18] which says that a finitely generated torsion-free nilpotent group is residually a finite p-group for every prime p. $\square$

The Nielsen–Schreier Theorem states that a subgroup of a free group is free. In general the same will not be true for relatively free groups: Shmel′kin [59] proves that a subgroup G of a free polynilpotent group F is itself a free polynilpotent group with the same class sequence as F if and only if G has a generating set whose elements are linearly independent modulo the derived subgroup F'. This result has been extended by Baumslag [4, Theorem 1] to a larger class of relatively free groups.

Theorem 2.2.2 (Baumslag, 1963). *Let F be a free group in the variety $\mathfrak{V}$ such that F is residually a finite p-group for infinitely many primes p. A subgroup G of F is free in $\mathfrak{V}$ if and only if G has a generating set whose elements are linearly independent modulo F'; moreover, such a generating set freely generates G.*

This next result of Baumslag, Neumann, Neumann and Neumann [8, Lemma 2.16] provides us with examples of finitely generated discriminating groups.

Theorem 2.2.3 (Baumslag, Neumann, Neumann and Neumann, 1964). *If G is residually a finite p-group for infinitely many primes p then G is discriminating.*

Proof. Let $W = \{w_i \mid i = 1, \dots, m\}$ be a set of words in at most n variables which are not laws in G. For each word $w_i \in W$ we may choose a prime p_i and a normal subgroup $N_i \lhd G$ such that $P_i \cong G/N_i$ is a p_i-group and w_i is not a law in P_i. We choose $g_1^{(i)}, \dots, g_n^{(i)} \in P_i$ such that $w_i\left(g_1^{(i)}, \dots, g_n^{(i)}\right)$ is non-trivial and define elements $\alpha_1, \dots, \alpha_n$ of the direct product $P_1 \times \cdots \times P_m$ as follows: $\alpha_j(i) = g_j^{(i)}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. Then $w_i(\alpha_1, \dots, \alpha_n) \neq 1$ for $i = 1, \dots, m$ and

$$G/\left(\bigcap_{i=1}^{m} N_i\right) \cong \prod_{i=1}^{m} G/N_i \cong \prod_{i=1}^{m} P_i.$$

The equations are simultaneously falsified in a quotient group of G and hence in G. $\square$

2.3 Wreath products

The unrestricted wreath product

The *Cartesian product* of a family of groups $\{G_\lambda \mid \lambda \in \Lambda\}$, denoted by $\prod_{\lambda\in\Lambda}^{C} G_\lambda$, is the group of all functions $f\colon \Lambda \to \bigcup_{\lambda\in\Lambda} G_\lambda$ with $f(\lambda) \in G_\lambda$, multiplied componentwise. The *Cartesian power* of G by Λ, denoted by G^Λ, is the Cartesian product of groups $\{G_\lambda \mid \lambda \in \Lambda\}$, where $G_\lambda \cong G$ for each $\lambda \in \Lambda$. In G^Λ we single out a special subgroup: for each $g \in G$ we define the constant function $g^\circ \in G^\Lambda$ by $g^\circ(\lambda) = g$ for all $\lambda \in \Lambda$; the *diagonal subgroup* G° is the set of all constant functions. The *unrestricted wreath product* of G by H, denoted by $G \operatorname{Wr} H$, is the semi-direct product of G^H by H, where the action of H on G^H is defined as follows: for $h_1 \in H$ and $u \in G^H$, we define u^{h_1} by

$$u^{h_1}(h) = u(hh_1^{-1})$$

for all $h \in H$. An element of $G \operatorname{Wr} H$ can be written as an ordered pair (h, d) with $h \in H$ and $d \in G^H$ and we say that G^H is the *base group* of $G \operatorname{Wr} H$.

The Kaloujnine–Krasner embedding

The Kaloujnine–Krasner embedding allows us to embed any group extension into an unrestricted wreath product. The ability to pass from a general extension to a split extension in this way is a very useful technique. The result was published in 1951 in one of a series of papers dealing with permutation groups [22]. Since we will use the details of this embedding in our applications we include a proof.

Theorem 2.3.1 (Kaloujnine and Krasner, 1951). *The unrestricted wreath product $A \operatorname{Wr} B$ contains an isomorphic copy of every group that is an extension of A by B.*

Proof. Let G be a group with normal subgroup N such that N is isomorphic to A and G/N is isomorphic to B. Let T be a transversal to N in G and let τ be the natural bijective map from B to T corresponding to the isomorphism $G/N \cong B$. Let $\mu : G \to B$ be the natural epimorphism such that $N = \ker \mu$ and let $\alpha : N \to A$ be an isomorphism. Let $D = A^B$ be the base group of the unrestricted wreath product $A \operatorname{Wr} B$. We write elements of $A \operatorname{Wr} B$ as ordered pairs (b, d) with $b \in B$ and $d \in D$ and define a map $\gamma : G \to A \operatorname{Wr} B$ as follows: for each element g of G we set $g\gamma = (g\mu, \phi_g)$ where, for each $b \in B$,

$$\phi_g(b) = \left(\left(b(g\mu)^{-1}\right) \tau g (b\tau)^{-1}\right) \alpha.$$

It is straightforward to check that this is a well-defined monomorphism. $\square$

In the situation of Theorem 2.3.1, suppose that A is abelian. If N is torsion-free, we may regard it as a $\mathbb{Z}[G/N]$-module and hence, since $G/N \cong B$, as a $\mathbb{Z}[B]$-module. We may also regard the base group D of the wreath product as a $\mathbb{Z}[B]$-module. The embedding $\gamma|_N$ of N into D is a $\mathbb{Z}[B]$-module homomorphism. Thus, if N is torsion-free as a $\mathbb{Z}[G/N]$-module, then the image of N in $A \operatorname{Wr} B$ is torsion-free as a $\mathbb{Z}[B]$-module. Suppose now that N has finite exponent n and that p is a prime divisor of n. We may regard the subgroup N_p of N of elements of order dividing p as an $\mathbb{F}_p[B]$-module. The embedding $\gamma|_{N_p}$ of N_p into the submodule of D of elements of order dividing p is an $\mathbb{F}_p[B]$-module homomorphism. Thus, if N_p is torsion-free as an $\mathbb{F}_p[G/N]$-module, then the image of N_p in $A \operatorname{Wr} B$ is torsion-free as an $\mathbb{F}_p[B]$-module.

The restricted wreath product

The *direct product* of a family of groups $\{G_\lambda \mid \lambda \in \Lambda\}$, denoted by $\prod_{\lambda\in\Lambda} G_\lambda$, is the subgroup of $\prod^C_{\lambda\in\Lambda} G_\lambda$ of functions f such that $f(\lambda) = 1$ for all but finitely many $\lambda \in \Lambda$. The *direct power* of G by Λ, denoted by $G^{(\Lambda)}$, is the direct product of groups $\{G_\lambda \mid \lambda \in \Lambda\}$ where $G_\lambda \cong G$ for each $\lambda \in \Lambda$. The *restricted wreath product* of G by H, denoted by $G \operatorname{wr} H$, is the semi-direct product of $G^{(H)}$ by H where the action of H on $G^{(H)}$ is defined as in the unrestricted wreath product. An element of $G \operatorname{wr} H$ can be written as a ordered pair (h, d) with $h \in H$ and $d \in G^{(H)}$ and we say that $G^{(H)}$ is the *base group* of $G \operatorname{wr} H$. The following lemma is a direct consequence of [3, Lemma 3].

Lemma 2.3.1 (Baumslag, 1959). *Let $G = U \operatorname{wr} H$, where H is torsion-free, and let A be the base group of G. Let a be a non-trivial element of A and let g be an element of G. If $[a, g] = 1$ then g is in A.*

Proof. Let $a = (1, d)$ and let $g = (h, c)$ with $h \in H$ and $d, c \in A$. Since $a \neq 1$, we may choose a non-trivial element u of H such that $d(u) \neq 1$. We have $[a, g] = (1, d(h - 1))$, so that $[a, g] = 1$ implies that $d(uh^{-1}) = d^h(u) = d(u) \neq 1$ and it follows that $d(uh^r) = d(u)$ for all $r \in \mathbb{Z}$. This implies that $\{uh^r | r \in \mathbb{Z}\}$ is in the support of d. If $h \neq 1$ then h is of infinite order and the set $\{uh^r | r \in \mathbb{Z}\}$ is infinite. But d is of finite support and hence we must have that $h = 1$ and $g \in A$. $\square$

The *left-iterated restricted wreath product* of the groups G_i for $i = 1, \ldots, n$, denoted by $\operatorname{wr}_{i=1}^n G_i$, is

$$G_1 \operatorname{wr} (G_2 \operatorname{wr} \cdots (G_{n-1} \operatorname{wr} G_n) \cdots).$$

The Magnus embedding

Let F be a free group of countable rank with free generating set $\{x_\lambda \,|\, \lambda \in \Lambda\}$, let R be a normal subgroup of F and let $\overline{x}_\lambda$ be the image of x_λ in F/R. Let A be a free abelian group of the same rank as F and let D be a free $\mathbb{Z}[F/R]$-module with free generating set $\{t_\lambda \mid \lambda \in \Lambda\}$. Magnus gave an embedding of F/R' into the matrix group

$$M(F/R, D) = \left\{ \begin{pmatrix} u & 0 \\ d & 1 \end{pmatrix} \mid u \in F/R, d \in D \right\} \cong A \operatorname{wr}(F/R).$$

Theorem 2.3.2 (Magnus, 1934 and 1939). *Let $\tilde{x}_\lambda$ be the image of x_λ in F/R'. An embedding of F/R' into $M(F/R, D)$ is given by setting*

$$\tilde{x}_\lambda \mapsto \begin{pmatrix} \overline{x}_\lambda & 0 \\ t_\lambda & 1 \end{pmatrix}.$$

Magnus proves the result when F/R is abelian [32] and when F/R is finite [33]. The result is a consequence of the Shmel′kin embedding (see Theorem 2.3.3 below).

Verbal wreath products

A group G is a *free product* of the subgroups $\{G_\lambda \mid \lambda \in \Lambda\}$ if

(i) $G = \langle G_\lambda \mid \lambda \in \Lambda \rangle$;

(ii) for any group H and any homomorphisms $\phi_\lambda : G_\lambda \to H$, there is a unique homomorphism $\phi : G \to H$ such that $\phi|_{G_\lambda} = \phi_\lambda$ for each $\lambda \in \Lambda$.

For every family of groups $\{G_\lambda \mid \lambda \in \Lambda\}$ there exists a group $\prod^*_{\lambda\in\Lambda} G_\lambda$ (unique up to isomorphism) which is a free product of subgroups isomorphic to the groups G_λ. The *Cartesian subgroup* of $\prod^*_{\lambda\in\Lambda} G_\lambda$, denoted by $[G_\lambda]^*$, is the normal closure in $\prod^*_{\lambda\in\Lambda} G_\lambda$ of $\langle [G_\lambda, G_\mu] \mid \lambda, \mu \in \Lambda, \lambda \neq \mu \rangle$. For more about free products see the book by Lyndon and Schupp [31, Chapter 4].

Let $\mathfrak{V}$ be a variety of groups defined by the set of words V. The $\mathfrak{V}$-*verbal product* of the family of groups $\{G_\lambda \mid \lambda \in \Lambda\}$, denoted by $\prod^{\mathfrak{V}}_{\lambda\in\Lambda} G_\lambda$, is the quotient group

$$\frac{\prod^*_{\lambda\in\Lambda} G_\lambda}{V\left(\prod^*_{\lambda\in\Lambda} G_\lambda\right) \cap [G_\lambda]^*}.$$

The $\mathfrak{O}$-verbal product of a family of groups is equal to their free product and the $\mathfrak{A}$-verbal product of a family of groups is equal to their direct product.

In [60], Shmel′kin introduces the notion of the $\mathfrak{V}$-verbal wreath product. Let G and H be groups and, for each $h \in H$, let G_h be an isomorphic copy of G, given by the isomorphism $g \mapsto g(h)$. The $\mathfrak{V}$-*verbal wreath product* of G by H, denoted by $G \operatorname{wr}_{\mathfrak{V}} H$, is the semidirect product of $D = \prod^{\mathfrak{V}}_{h\in H} G_h$ by H, where the action of H on D is defined as follows: for $h_1 \in H$ and $g \in G$, we set $g(h)^{h_1} = g(hh_1)$ for all $h \in H$; this extends to an action of H on D. An element of $G \operatorname{wr}_{\mathfrak{V}} H$ can be written as an ordered pair (h, d) with $h \in H$ and $d \in D$ and we say that D is the *base group* of $G \operatorname{wr}_{\mathfrak{V}} H$.

The Shmel′kin embedding

Shmel′kin [60, Theorem 1] generalized the Magnus embedding in the following way. Let F be a free group of countable rank with free generating set $\{x_\lambda \mid \lambda \in \Lambda\}$, let R be a normal subgroup of F and let $\overline{x}_\lambda$ be the image of x_λ in F/R. Let V be a set of words which defines a variety $\mathfrak{V}$ and let U be a free group in $\mathfrak{V}$ with free generating set $\{u_\lambda \mid \lambda \in \Lambda\}$.

Theorem 2.3.3 (Shmel′kin, 1964). *Let $\tilde{x}_\lambda$ be the image of x_λ in $F/V(R)$. An embedding of $F/V(R)$ into $U \operatorname{wr}_{\mathfrak{V}}(F/R)$ is given by setting*

$$\tilde{x}_\lambda \mapsto (\overline{x}_\lambda, u_\lambda(1)) \,.$$

Shmelkin's original proof [61] was computational and he later gave a brief and conceptual proof of the result [62]. Since the original proofs are in Russian, we include a proof which is based on that of Romanovskii for the generalized Shmel′kin embedding [56].

Proof. We define $\alpha : F \to U \operatorname{wr}_{\mathfrak{V}}(F/R)$ to be the homomorphism given by

$$x_\lambda \mapsto (\overline{x}_\lambda, u_\lambda(1)) \,.$$

Let N be the base group of $U \operatorname{wr}_{\mathfrak{V}}(F/R)$. Since $R\alpha \leqslant N$ and N is in $\mathfrak{V}$, we have $V(R) \leqslant \ker \alpha$. In order to prove the theorem it remains to prove the inverse inclusion.

The group $F/V(R)$ is an extension of $R/V(R)$ by F/R and the Kaloujnine–Krasner embedding (see Theorem 2.3.1) gives an embedding γ of $F/V(R)$ into

$$(R/V(R)) \operatorname{Wr}(F/R) = D \rtimes (F/R),$$

where $\tilde{x}_\lambda \gamma = (\overline{x}_\lambda, d_\lambda)$ with $d_\lambda \in D$. Since U is a free group in $\mathfrak{V}$ and D is in $\mathfrak{V}$, the map $\beta : U \to D : u_\lambda \mapsto d_\lambda$ is a homomorphism; β extends to a homomorphism $\overline{\beta} : N \to D$ which respects the action of F/R. We can define a homomorphism

$$\sigma : U \operatorname{wr}_{\mathfrak{V}}(F/R) \to (R/V(R)) \operatorname{Wr}(F/R) : (f, u) \mapsto (f, u\overline{\beta}).$$

By construction $\alpha\sigma = \gamma$ and hence $\ker \alpha$ is contained in $\ker \gamma = V(R)$. $\square$

Since the $\mathfrak{A}$-verbal product of a family of groups is equal to their direct product, $G \operatorname{wr}_{\mathfrak{A}} H$ is equal to $G \operatorname{wr} H$ and it follows that the Magnus embedding is a special case of the Shmel′kin embedding with $V = \{[x, y]\}$.

The Magnus and Shmel′kin embeddings provide a useful technique for dealing with free extensions of groups. The embeddings are particularly valuable owing to the greater ease of calculation in the wreath product.

2.4 Product varieties

For varieties $\mathfrak{U}$ and $\mathfrak{V}$ we define the product variety $\mathfrak{U}\mathfrak{V}$ to be the variety of all groups that are extensions of a group in $\mathfrak{U}$ by a group in $\mathfrak{V}$.

Examples of product varities

- The variety $\mathfrak{S}_d$ of soluble groups of derived length d is equal to the product variety $\underbrace{\mathfrak{A}\cdots\mathfrak{A}}_{d}$.
- The variety $\mathfrak{P}_{\overline{c}}$ of polynilpotent groups with class sequence at most $\overline{c} = (c_1, \ldots, c_k)$ is equal to the product variety $\mathfrak{N}_{c_k} \cdots \mathfrak{N}_{c_1}$.

The following results are useful in determining groups which discriminate a product variety. The first result, due to Baumslag [5, Theorem 1], illustrates the importance of the restricted wreath product in this context. The proof, based on an idea of Neumann, Neumann and Neumann [43, Theorem 6.5], involves the Kaloujnine–Krasner embedding as well as some very technical arguments.

Theorem 2.4.1 (Baumslag, 1963). *If the collection of groups $\mathfrak{C}$ discriminates $\mathfrak{U}$ and the collection of groups $\mathfrak{D}$ discriminates $\mathfrak{V}$ then the collection of groups*

$$\mathfrak{C} \operatorname{wr} \mathfrak{D} = \{C \operatorname{wr} D \mid C \in \mathfrak{C}, D \in \mathfrak{D}\}$$

discriminates $\mathfrak{U}\mathfrak{V}$.

In particular, if G discriminates $\mathfrak{U}$ and H discriminates $\mathfrak{V}$ then $G \operatorname{wr} H$ discriminates $\mathfrak{U}\mathfrak{V}$. Next we consider when a product variety can be discriminated by a group of finite rank [8, Theorem 3.3].

Theorem 2.4.2 (Baumslag, Neumann, Neumann and Neumann, 1964). *Let $\mathfrak{V} \neq \mathfrak{E}$ be a variety of exponent zero and suppose that $\mathfrak{V}$ can be discriminated by a k-*

generator group which is residually periodic. Then, for $k \geqslant 2$, the product variety $\mathfrak{U}\mathfrak{V}$ is discriminated by a k-generator group for any variety $\mathfrak{U}$.

The variety $\mathfrak{N}_c$ for $c \geqslant 3$ is generated by, and hence discriminated by, its $(c-1)$-generator groups (see, for exmple, [27], [24]). The variety of abelian groups is discriminated by the infinite cyclic group, and the variety of nilpotent groups of class 2 is generated by, and hence discriminated by, its 2-generator groups. It follows from Theorem 2.4.2 that the variety of polynilpotent groups with class sequence at most $(c_1, \ldots, c_k)$ is discriminated by its r-generator groups, where $r = \max\{2, c_1 - 1\}$. In particular, the variety of soluble groups of derived length d is generated by its 2-generator groups.

The following result, dealing with product varieties and the property of being residually a finite p-group, is an immediate corollary of a result of Baumslag [5, Theorem 3] dealing with a more general class of residual properties.

Lemma 2.4.1 (Baumslag, 1963). *Let p be a prime. If the free groups of the varieties $\mathfrak{U}$ and $\mathfrak{V}$ are residually finite p-groups then the free groups of $\mathfrak{U}\mathfrak{V}$ are residually finite p-groups.*

Proof. Let $\mathfrak{C} = \{F_n(\mathfrak{U}) \mid n \in \mathbb{N}\}$ and let $\mathfrak{D} = \{G \in \mathfrak{V} \mid G$ is a finitely generated p-group$\}$. The set $\mathfrak{C}$ discriminates $\mathfrak{U}$ and, by Theorem 2.2.1, the set $\mathfrak{D}$ discriminates $\mathfrak{V}$. Thus, by Theorem 2.4.1, the set $\mathfrak{C} \operatorname{wr} \mathfrak{D}$ discriminates the product variety $\mathfrak{U}\mathfrak{V}$. Every group in $\mathfrak{C} \operatorname{wr} \mathfrak{D}$ is residually a finite p-group by a result of Gruenberg [18]; it is also finitely generated. Thus the free groups of $\mathfrak{U}\mathfrak{V}$ are residually finite p-groups by Theorem 2.2.1. $\square$

It follows from this and Lemma 2.2.1 above, that free nilpotent groups, free soluble groups and free polynilpotent groups are residually finite p-groups for every prime p. Thus, by Theorem 2.2.3, each of these groups is discriminating.

For product varieties we have an alternative condition for a subgroup of a relatively free group to be relatively free. This is again a result of Baumslag [6, Corollary 1].

Theorem 2.4.3 (Baumslag, 1965). *Let F be a free group in the product variety $\mathfrak{U}\mathfrak{V}$, where $\mathfrak{U}$ and $\mathfrak{V}$ are varieties whose free groups are residually finite p-groups for the same prime p, and let V be the set of words defining $\mathfrak{V}$. A subgroup G of F is free in $\mathfrak{U}\mathfrak{W}$ if G has a generating set whose elements generate modulo $V(F)$ a free group in $\mathfrak{W}$.*

Mal′cev showed in [37] that any pair of elements in a free soluble group F which commute either belong to the last non-trivial term of the derived series of G or generate a cyclic subgroup. He used this fact to show that the terms of the derived series of F are definable. Shmel′kin [59] extended Mal′cev's result to free polynilpotent groups in the following way: if F is a free polynilpotent group with class sequence $(c_1, \dots, c_k)$ then any pair of elements which commute either generate a cyclic subgroup or belong to the subgroup $\gamma_{c_{k-1}}(\cdots(\gamma_{c_1}(F))\cdots)$. This result has also been extended, again by Baumslag (see [44, 42.52]), to a larger class of relatively free groups.

Theorem 2.4.4 (Baumslag). *Let $\mathfrak{U}$ and $\mathfrak{V}$ be non-trivial varieties whose free groups are residually finite p-groups for the same prime p. Let V be the set of words which define $\mathfrak{V}$ and suppose that the free groups of $\mathfrak{V}$ are torsion-free. Every non-cyclic nilpotent subgroup of a free group F in the product variety $\mathfrak{U}\mathfrak{V}$ is in the verbal subgroup $V(F)$.*

Proof. Let N be a non-cyclic nilpotent subgroup of F. Suppose to begin with that $N \cap V(F) = \{1\}$ and choose elements $g, h \in N$ such that $\langle g, h\rangle$ is abelian and non-cyclic. The elements g and h generate a free abelian group modulo $V(F)$ and, by Theorem 2.4.3, the group $\langle g, h\rangle$ is a free $\mathfrak{U}\mathfrak{A}$ group, contradicting the assumption that $\langle g, h\rangle$ is abelian.

Suppose now that N has non-trivial intersection with $V(F)$ but that $N \not\subseteq V(F)$. We choose an element $g \in N \setminus V(F)$. The intersection $N \cap V(F)$ is a non-trivial normal subgroup of N and hence intersects the centre non-trivially. (This is an elementary fact, see [53, 5.2.1].) We may choose $h \in N \cap V(F)$ such that $[g, h] = 1$. Since $F/V(F)$ is free in $\mathfrak{V}$, it is torsion-free and g has infinite order modulo $V(F)$. Consider the group $U \operatorname{wr} F_\infty(\mathfrak{V})$, where U generates $\mathfrak{U}$. For any finite set of non-trivial elements X in F, there exists a homomorphism $\alpha : F \to U \operatorname{wr} F_\infty(\mathfrak{V})$ satisfying the following conditions: $x\alpha \neq 1$ for all $x \in X$ and $g\alpha$ is in the base group of $U \operatorname{wr} F_\infty(\mathfrak{V})$ if and only if $g \in V(F)$ (see [8, Corollary 2.23]). The image of g in $U \operatorname{wr} F_\infty(\mathfrak{V})$ has infinite order modulo the base group. It follows, by Lemma 2.3.1, that $[g, h] = 1$ if and only if $g \in V(F)$, contradicting our choice of the elements g and h. $\square$

This result proves to be very useful in allowing us to give existential and universal sentences defining the verbal subgroup $V(F)$ of a free group F in a product variety $\mathfrak{U}\mathfrak{V}$ where the words V define the variety $\mathfrak{V}$.

2.5 Universal theory

The following lemma is a well-known fact. Since we do not know of a good reference, we include a proof in the case of groups.

Let $A \subseteq G$ and $B \subseteq H$ be finite subsets. In this context we define an isomorphism to be a bijective map $\nu : A \to B$ such that, for $g_1, g_2, g_3 \in A$,

(i) $g_1 g_2 = g_3$ if and only if $g_3\nu = g_1\nu g_2\nu$;

(ii) $g_1^{-1} = g_2$ if and only if $g_2\nu = (g_1\nu)^{-1}$.

Lemma 2.5.1. *A group G satisfies the universal theory of a group H if and only if every finite subset of G is isomorphic to a finite subset of H.*

Proof. Suppose to begin with that $G \models \mathrm{Th}_\forall(H)$ and let $A = \{a_1, \dots, a_r\}$ be a finite subset of G. Since A is a finite set, it has a finite theory which can be written as a single quantifier-free formula $\theta(a_1, \dots, a_r)$. It follows that G satisfies the existential sentence ϕ defined by

$$\exists x_1 \dots x_r \, \theta(x_1, \dots, x_r).$$

Since G satisfies the universal theory of H, we have $H \models \mathrm{Th}_\exists(G)$ and ϕ holds in H; we may choose elements $b_1, \dots, b_r \in H$ such that $H \models \theta(b_1, \dots, b_r)$. If $B = \{b_1, \dots, b_r\}$ then the map $A \to B : a_i \mapsto b_i$ is an isomorphism.

For the converse, suppose that every finite subset of G is isomorphic to a finite subset of H and let ϕ be a universal sentence such that $H \models \phi$. We may assume without loss of generality that ϕ is of the form

$$\forall x_1 \dots x_r \, \theta(x_1, \dots, x_r),$$

where $\theta(x_1, \dots, x_r)$ is a quantifier-free formula. Let $a_1, \dots, a_r$ be elements of G and let $A = \{a_1, \dots, a_r\}$. There is a finite subset $B = \{b_1, \dots, b_r\}$ of H and an isomorphism $\nu : A \to B : a_i \mapsto b_i$. We have $H \models \theta(b_1, \dots, b_r)$ and, since ν is an isomorphism, $G \models \theta(a_1, \dots, a_r)$. This holds for all tuples of elements from G and it follows that $G \models \phi$. Thus G satisfies the universal theory of H. □

Timoshenko [66] uses this result to show that if F is a free group in a variety $\mathfrak{V}$ and G is a subgroup of F then F and G have the same universal theory if and only they satisfy the same laws.

Theorem 2.5.1 (Timoshenko, 2000). *Let F be a free group in a variety $\mathfrak{V}$ such that F is residually a finite p-group for infinitely many primes p. If the subgroup G of F generates the same variety as F then the universal theories of G and F coincide.*

Proof. It is clear, since $G \leqslant F$, that G satisfies the universal theory of F. By Lemma 2.5.1, in order to prove that F satisfies the universal theory of G, it is enough to show that every finite subset of F is isomorphic to a finite subset of G.

By Lemma 2.2.3, the group G is discriminating and hence G discriminates $\mathfrak{V}$. Let S be a finite subset of F and let $\overline{S}$ be the extension of S defined by

$$\overline{S} = S \cup \{g_1 g_2^{-1} \mid g_1, g_2 \in S\} \cup \{g_1 g_2 g_3^{-1} \mid g_1, g_2, g_3 \in S\}.$$

Each element $s \in \overline{S}$ is the image of a word w_s and if $s \neq 1$ then w_s is not a law in $\mathfrak{V}$. Since G discriminates $\mathfrak{V}$, there exist elements $h_1, \ldots, h_n \in G$ such that $w_s(h_1, \ldots, h_n) \neq 1$ for all non-trivial $s \in \overline{S}$. Let $S_G = \{w_s(h_1, \ldots, h_n) \mid s \in S\} \subseteq G$. The map $\alpha : S \to S_G : s \mapsto w_s(h_1, \ldots, h_n)$ is the required isomorphism. $\square$

The following result is useful in allowing us to derive an embedding from the fact that one group is in the universal closure of another. In [66], Timoshenko proves the result for use in a limited context; we are able to apply it more generally (see Section 9.2).

Lemma 2.5.2 (Timoshenko, 2000). *Let G be a group which is finitely presented in some variety $\mathfrak{V}$ and let H be a group such that $G \in \mathrm{ucl}(H)$. There exists an embedding of G into the countable Cartesian power $H^{\mathbb{N}}$ of H.*

Proof. Let $\langle a_1, \ldots, a_s \mid r_1(\overline{a}), \ldots, r_t(\overline{a})\rangle$ be a presentation for G in $\mathfrak{V}$ and let g be a non-trivial element of G, so that g is the image of a word w. We define the existential sentence ψ_g by

$$\exists\, x_1 \ldots x_s \left(\bigwedge_{i=1}^{t} r_i(\overline{x}) = 1 \wedge w(\overline{x}) \neq 1 \right)$$

where $\overline{x}$ is the s-tuple $(x_1, \ldots, x_s)$. We have $G \models \psi_g$ and, since G is universally equivalent to H, it follows that $H \models \psi_g$. We may choose elements $b_1, \ldots, b_r \in H$ such that $r_i(\overline{b}) = 1$

for $i = 1, \ldots, r$ and $w(\bar{b}) \neq 1$. Then the map $\alpha_g : G \to H$ defined by $a_i \mapsto b_i$ for $i = 1, \ldots, r$ is a homomorphism by Von Dyck's theorem (see [53, 2.2.1]). We enumerate the non-identity elements of G to obtain a list $g_1, g_2, \ldots$ and define a map $\alpha : G \to H^{\mathbb{N}} : g \mapsto f_g$ where $f_g(i) = \alpha_{g_i}(g)$ for $i \in \mathbb{N}$. Then α is the desired embedding. □

2.6 Ultraproducts of groups

Let G be a group. The results in this next section establish the strong relationship between those groups in the universal closure of G and the ultrapowers of G. To begin with we define ultrafilters and ultrapowers and state some standard results.

Filters

Let Λ be a non-empty set. A *filter* over Λ is a set $D \subseteq \mathbb{P}(\Lambda)$ such that the following conditions hold: (i) $\Lambda \in D$ and $\emptyset \notin D$; (ii) $X, Y \in D \implies X \cap Y \in D$; (iii) $X \in D$ and $X \subseteq Y \in \mathbb{P}(\Lambda) \implies Y \in D$. An *ultrafilter* over Λ is a filter $\mathcal{U} \neq \mathbb{P}(\Lambda)$ such that, for all $X \in \mathbb{P}(\Lambda)$ either $X \in D$ or $\Lambda \setminus X \in D$. If F is the filter generated by some non-empty $S \subseteq \Lambda$ then F is a *principal* filter. A principal ultrafilter $\mathcal{U}$ is generated by a singleton.

Lemma 2.6.1. *If $S \subseteq \mathbb{P}(\Lambda)$ has the finite intersection property then there exists an ultrafilter over Λ containing S.*

Proof. The proof of this lemma is an easy consequence of Zorn's Lemma. □

Lemma 2.6.2. *An ultrafilter over Λ is non-principal if and only if $\mathcal{U}$ contains the cofinite filter.*

Proof. This is an easy exercise and we omit it. □

Ultraproducts of groups

Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a non-empty collection of groups indexed by Λ and let $A = \prod^C_{\lambda \in \Lambda} A_\lambda$. Let $\mathcal{U}$ be a non-principal ultrafilter over Λ. Define an equivalence relation $\sim$ on A as follows: $a \sim b$ if and only if $\{\lambda \in \Lambda \mid a(\lambda) = b(\lambda)\} \in \mathcal{U}$. Let $\tilde{a}$ be the $\sim$-equivalence class of $a \in A$. The *ultraproduct* of $\{A_\lambda \mid \lambda \in \Lambda\}$ modulo $\mathcal{U}$, denoted by $\prod_{\lambda \in \Lambda} A_\lambda / \mathcal{U}$, is the set of $\sim$-equivalence classes of A. If $A_\lambda \cong A$ for all $\lambda \in \Lambda$ then $\prod_{\lambda\in\Lambda} A_\lambda/\mathcal{U}$ is an *ultrapower* of A.

The elementary properties of a family of groups, that is those properties which can be expressed using first-order formulae, are preserved under taking ultraproducts. This follows from the fundamental theorem concerning ultraproducts:

Theorem 2.6.1 (Łoś). *Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a non-empty collection of groups indexed by Λ and let $\mathcal{U}$ be a non-principal ultrafilter over Λ. Then, for any formula $\phi(x_1, \ldots, x_k)$ in $\mathfrak{L}_{\mathrm{gp}}$ and any $a_1, \ldots, a_k \in A$, we have*

$$\prod_{\lambda\in\Lambda} A_\lambda/\mathcal{U} \models \phi(\tilde{a}_1, \ldots, \tilde{a}_k)$$

if and only if

$$\{\lambda \in \Lambda \mid A_\lambda \models \phi(a_1(\lambda), \ldots, a_k(\lambda))\} \in \mathcal{U}.$$

Proof. See [15, Theorem 3.1]. □

Theorem 2.6.2. *Let $\phi_\lambda : A_\lambda \to B_\lambda$, $\lambda \in \Lambda$ be a family of embeddings and let $\mathcal{U}$ be a non-principal ultrafilter over Λ. There is an embedding*

$$\phi : \prod_{\lambda\in\Lambda} A_\lambda/\mathcal{U} \to \prod_{\lambda\in\Lambda} B_\lambda/\mathcal{U}.$$

Proof. This follows from the fact that the ultraproduct is a functor; see [15, §4]. □

The following result is a counterpart to Frayne's Lemma (see [9, Lemmas 8.1.1 and 9.3.8]), which says that a structure $\mathcal{A}$ is elementarily equivalent to a structure $\mathcal{B}$ if and only if there is an elementary embedding of $\mathcal{A}$ into an ultrapower of $\mathcal{B}$, and the proof follows exactly the same lines. Let G and H be groups. We show that G satisfies the universal theory of H if and only if there is an embedding of G into some ultrapower of H. By the fundamental theorem, H is elementarily equivalent to any ultrapower of itself. Thus, if there is an embedding of G into some ultrapower $\overline{H}$ of H, then G satisfies the universal theory of $\overline{H}$ and hence G satisfies the universal theory of H. To prove the converse requires rather more work.

Lemma 2.6.3. *Let G and H be groups. If G is countable and satisfies the universal theory of H then there exists a set Λ and a non-principal ultrafilter $\mathcal{U}$ over Λ such that G embeds in $\prod_{\lambda\in\Lambda} H_\lambda/\mathcal{U}$ where $H_\lambda \cong H$ for all $\lambda \in \Lambda$.*

Proof. The group G is countable. We fix an enumeration $g_1, g_2, \ldots$ of the elements of G and an enumeration $x_1, x_2, \ldots$ of the variables in the language $\mathcal{L}_{\mathrm{gp}}$. We define $\mathcal{S}$ to be the collection of all quantifier-free formulae in $\mathcal{L}_{\mathrm{gp}}$. Let $\overline{g} = (g_{i_1}, \ldots, g_{i_k})$ be a k-tuple of elements from G. We denote by $\mathcal{I}_{\overline{g}}$ the set of formulae

$$\mathcal{I}_{\overline{g}} = \{\phi(x_{i_1}, \ldots, x_{i_k}) \in \mathcal{S} \mid G \models \phi(g_{i_1}, \ldots, g_{i_k})\},$$

and by $\mathcal{I}$ the union over all finite tuples $\overline{g}$ from G of the sets $\mathcal{I}_{\overline{g}}$. For each $\phi \in \mathcal{I}$, the group G satisfies the existential sentence θ defined by

$$\exists\, x_{i_1} \ldots x_{i_k}\, \phi(x_{i_1}, \ldots, x_{i_k}).$$

Since G satisfies the universal theory of H, the group H satisfies the existential theory of G and it follows that $H \models \theta$; thus there exist elements $h^{\phi}_{i_1}, \ldots, h^{\phi}_{i_k} \in H$ such that $H \models \phi(h^{\phi}_{i_1}, \ldots, h^{\phi}_{i_k})$. For a natural number $i \notin \{i_1, \ldots, i_k\}$ we set $h^{\phi}_{i} = 1$. We may construct a sequence of elements from H in this way for every formula $\phi \in \mathcal{I}$. We define a collection of formulae $\mathcal{J}_\phi$ by

$$\mathcal{J}_\phi = \{\psi \in \mathcal{I} \mid H \models \phi(h^{\psi}_{i_1}, \ldots, h^{\psi}_{i_k})\}.$$

For all formulae $\phi_1, \ldots, \phi_n \in \mathcal{I}$, the formula $\bigwedge_{i=1}^{n} \phi_i$ is in $\bigcap_{i=1}^{n} J_{\phi_i}$. The collection $\mathcal{J} = \{\mathcal{J}_\phi \mid \phi \in \mathcal{I}\}$ has the finite intersection property and can be extended to a non-principal ultrafilter $\mathcal{U}$ on the set $\mathcal{I}$ by Lemma 2.6.1.

Let H_ϕ be an isomorphic copy of H for each $\phi \in \mathcal{I}$. We define a map

$$\alpha : G \to \prod\nolimits_{\phi \in \mathcal{I}} H_\phi / \mathcal{U}$$

in the following way: $\alpha : g_\gamma \mapsto \tilde{h}_\gamma$ where $\tilde{h}_\gamma$ is the equivalence class of the element h_γ defined by $h_\gamma(\phi) = h^{\phi}_{\gamma}$. This map is clearly well defined. It remains to show that α is a monomorphism.

Let g_{i_1}, g_{i_2} and g_{i_3} be elements of G and consider the quantifier-free formulae

$$\phi_1(x_{i_1}, x_{i_2}, x_{i_3}) : (x_{i_1} x_{i_2} = x_{i_3}),$$

$$\phi_2(x_{i_1}, x_{i_2}) : (x_{i_1}^{-1} = x_{i_2}),$$

$$\phi_3(x_{i_1}, x_{i_2}) : (x_{i_1} = x_{i_2}).$$

Then $G \models \phi_1(g_{i_1}, g_{i_2}, g_{i_3})$ if and only if $\phi_1 \in \mathcal{I}$, and this is the case if and only if $\mathcal{J}_{\phi_1} \in \mathcal{U}$. Now $\mathcal{J}_{\phi_1} = \{\psi \in \mathcal{I} \mid H \models \phi_1(h^{\psi}_{i_1}, h^{\psi}_{i_2}, h^{\psi}_{i_3})\}$, and it follows by the fundamental theorem (Theorem 2.6.1 above) that $\mathcal{J}_{\phi_1} \in \mathcal{U}$ if and only if

$$\prod\nolimits_{\phi \in \mathcal{I}} H_\phi / \mathcal{U} \models \phi_1(\tilde{h}_{i_1}, \tilde{h}_{i_2}, \tilde{h}_{i_3}),$$

which holds if and only if

$$\prod\nolimits_{\phi \in \mathcal{I}} H_\phi / \mathcal{U} \models \phi_1(g_{i_1}\alpha, g_{i_2}\alpha, g_{i_3}\alpha).$$

Similarly, $G \models \phi_2(g_{i_1}, g_{i_2})$ if and only if

$$\prod\nolimits_{\phi \in \mathcal{I}} H_\phi / \mathcal{U} \models \phi_2(g_{i_1}\alpha, g_{i_2}\alpha),$$

and $G \models \phi_3(g_{i_1}, g_{i_2})$ if and only if

$$\prod\nolimits_{\phi \in \mathcal{I}} H_\phi / \mathcal{U} \models \phi_3(g_{i_1}\alpha, g_{i_2}\alpha).$$

Hence α is a monomorphism and G embeds in the ultrapower

$$\prod\nolimits_{\phi \in \mathcal{I}} H_\phi / \mathcal{U}.$$

$\square$

Part II

THE UNIVERSAL THEORY OF CERTAIN RESTRICTED WREATH PRODUCTS

Chapter 3

PRELIMINARY RESULTS

This chapter contains a range of preliminary results which will be important in the following four chapters. These results split into two types: group-theoretic results about restricted wreath products and results related to the universal theory of restricted wreath products.

In Section 3.1 we recall the definitions of an integral domain and an Ore domain. We recall also that the group ring over a field of a group, each of whose finitely generated subgroups has infinite cyclic image, is an integral domain and that, if the group ring over a field of a soluble group is an integral domain then it is an Ore domain.

In Section 3.2 we recall the definition of the tensor product of two modules.

In Section 3.3 we prove the crucial result that a finitely generated extension of an abelian group A, which is torsion-free or of finite square-free exponent n, by a torsion-free soluble group B can be embedded in a restricted wreath product $U \operatorname{wr} B$, where U is a finitely generated abelian group of exponent n. This relies on the result that, for a torsion-free soluble group B, the group rings $\mathbb{Z}[B]$ and $\mathbb{F}_p[B]$, where p is prime, are Ore domains. We also prove some results about finitely generated subgroups of restricted wreath products $U \operatorname{wr} G$, where U is a finitely generated abelian group which is torsion-free or of finite square-free exponent n and G is a torsion-free soluble group. In particular we show that every non-abelian nilpotent subgroup of $U \operatorname{wr} G$ embeds in G. We identify $U \operatorname{wr} G$ with the group of matrices

$$M(G,D) = \left\{ \begin{pmatrix} u & 0 \\ d & 1 \end{pmatrix} \mid u \in G \text{ and } d \in D \right\},$$

where D is the base group of the wreath product which we write additively. Using this identification we derive some useful identities which hold in $U \operatorname{wr} G$.

The focus of Section 3.4 is on showing that certain restricted wreath products are universally equivalent. An easy adaptation of Timoshenko's proof that the restricted wreath product preserves universal equivalence [65, Theorem 2] yields the result that if A_1, A_2, B_1, and B_2 are groups such that $A_1 \models \mathrm{Th}_\forall(A_2)$ and $B_1 \models \mathrm{Th}_\forall(B_2)$ then $A_1 \operatorname{wr} B_1$ satisfies the universal theory of $A_2 \operatorname{wr} B_2$. We also prove that if B is an abelian group of exponent n and G and H are torsion-free universally equivalent groups then $B \operatorname{wr} G$ is universally equivalent to $C_n \operatorname{wr} H$.

3.1 Ore domains

A *skew-field* is a ring R, not necessarily commutative, whose non-zero elements R^* form a group under multiplication. An element u of a ring R is a *left zero-divisor* if $u \neq 0$ and there exists $v \in R^*$ such that $uv = 0$. An element of R which is neither zero nor a left zero-divisor is a *left non-zero-divisor*. The analogous definitions hold with "right" in place of "left". A non-zero ring containing no left or right zero-divisors is an *integral domain*.

The *group ring* $R[G]$ of a group G is the set of all finite linear combinations of elements from G with multiplication defined by extending the multiplication on G. We may write an element α of $R[G]$ in the form

$$\alpha = \sum_{g \in G} \alpha_g g,$$

with each $\alpha_g \in R$ and where $\alpha_g = 0$ for all but finitely many $g \in G$. The *support* of an element α in $R[G]$, which we denote by $\mathrm{supp}(\alpha)$, is the set $\{g \in G \mid \alpha_g \neq 0\}$.

There is a long-standing conjecture that the group ring of a torsion-free group over a field (or over the integers) is an integral domain. The conjecture has been proved in a number of special cases: in particular, if G is a group such that every finitely generated subgroup of G has infinite cyclic image then $\mathbb{Z}[G]$ is an integral domain (Higman, see [53, 15.3.10]). We note that the same argument applies for group rings over fields.

Lemma 3.1.1. *Let F be a field. If G is a group such that every non-trivial finitely generated subgroup has an infinite cyclic image, then $F[G]$ is an integral domain.*

Proof. Suppose that $F[G]$ contains non-zero elements α and β with $\alpha\beta = 0$ and that $\mathrm{supp}(\alpha) = \{g_1, \ldots, g_m\}$ and $\mathrm{supp}(\beta) = \{h_1, \ldots, h_n\}$. We choose α and β in such a way that $m + n$ is minimal, subject to the condition that $\alpha\beta = 0$. It is clear that $m + n \neq 2$. Multiplying by group elements if necessary, we may assume that $1 \in \mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$.

Let $H = \langle g_1, \ldots, g_m, h_1, \ldots, h_n \rangle$. Then H has a normal subgroup K such that $H/K = \langle aK \rangle$ is an infinite cyclic group. We may write

$$\alpha = \gamma_1 a^{r_1} + \cdots + \gamma_s a^{r_s} \qquad \text{and} \qquad \beta = \delta_1 a^{q_1} + \cdots + \delta_t a^{q_t},$$

with $\gamma_i, \delta_j \in F[K]$ where $r_1 < \cdots < r_s$ and $q_1 < \cdots < q_t$. If $s = 1$ then $\alpha a^{-r_1} = \gamma_1 \in F[K]$ and, since $1 \in \mathrm{supp}(\alpha)$, the element a^{-r_1} is in $\mathrm{supp}(\gamma_1)$, and hence in K. Thus, since H/K is torsion-free, either a is in K and $H = K$, contradicting our assumption that H/K is infinite, or $r_1 = 0$ and $\alpha = \gamma_1 \in F[K]$. Similarly, if $t = 1$ then $\beta = \delta_1 \in F[K]$ and if $s = t = 1$ then $\mathrm{supp}(\alpha)$ and $\mathrm{supp}(\beta)$ are in K and $H = K$, once more contradicting our assumption that H/K is infinite. We may assume that either $r > 1$ or $s > 1$ and it follows that either $|\,\mathrm{supp}(\gamma_1)| < m$ or $|\,\mathrm{supp}(\delta_1)| < n$. The smallest power of a which occurs in the product $\alpha\beta$ is $a^{r_1+q_1}$, which has coefficient $\gamma_1\delta_1$. Since $\alpha\beta = 0$, we must have $\gamma_1\delta_1 = 0$ and, since $|\,\mathrm{supp}(\gamma_1)| + |\,\mathrm{supp}(\delta_1)| < m + n$, this contradicts the choice of α and β. □

In particular, if G is a finitely generated, torsion-free nilpotent group, then G has a subnormal series with infinite cyclic factors. Thus G satisfies the conditions of Lemma 3.1.1 and $F[G]$ is an integral domain.

For an arbitrary torsion-free soluble group G, Lemma 3.1.1 is not strong enough to imply that the groups rings $\mathbb{Z}[G]$ and $\mathbb{F}_p[G]$, where p is prime, are integral domains. The positive solution of the zero-divisor problem for torsion-free soluble groups is a very deep result:

Theorem 3.1.1 (Kropholler, Linnell and Moody, [25, Theorem 1.4]). *Let F be a field and let G be a torsion-free soluble group. If G is torsion-free then $F[G]$ is an integral domain.*

A *right Ore domain* is an integral domain R satisfying the *right Ore condition*: for each pair of elements $r, s \in R^*$ there exist elements $u, v \in R^*$ such that $ru = sv$. Analogously, a *left Ore domain* is an integral domain R satisfying the *left Ore condition*: for each pair of elements $r, s \in R^*$ there exist elements $u, v \in R^*$ such that $ur = vs$. A ring R which is both a left Ore domain and a right Ore domain is an *Ore domain*.

The following lemma follows easily from the fact that a skew polynomial ring over a right Ore domain is a right Ore domain [14, Proposition **0**.8.4]. See also Lewin's proof of this result [28].

Lemma 3.1.2. *Let G be a soluble group and let R be a field or the ring of integers. If the group ring $R[G]$ is an integral domain then $R[G]$ is an Ore domain.*

Proof. Let G be a soluble group and let R be a field or the ring of integers. We suppose that the group ring $R[G]$ is an integral domain. Let $\alpha = \sum_{i=1}^{n} a_i g_i$ and $\beta = \sum_{i=1}^{m} b_i h_i$ be elements of $R[G]^*$ with $a_i, b_i \in R$ and $g_i, h_i \in G$. Let $H = \langle g_1, \ldots, g_n, h_1, \ldots, h_m \rangle$; then $\alpha, \beta \in R[H]^*$. Thus it is enough to show that the group ring over R of every finitely generated subgroup of G is an Ore domain. By symmetry we need only consider the right Ore condition.

Let S be a ring which is not a right Ore domain, so that there exist $\alpha, \beta \in S$ such that $\alpha S \cap \beta S = \{0\}$. This implies that the elements $\alpha^n \beta$ (for $n = 0, 1, \ldots$) are right linearly independent over S and we have an infinite chain of right ideals

$$\{0\} = I_0 < \cdots < I_n < \cdots$$

where I_n is the ideal generated by the elements $\{\alpha^i \beta \mid i = 0, \ldots, n\}$; thus S is not right Noetherian.

We proceed by induction on the derived length d of G. Suppose that G is abelian and let H be a finitely generated subgroup of G; then H is polycyclic and, by a result of P. Hall (see [53, 15.3.3]), the ring $R[H]$ is Noetherian.

Suppose now that the hypothesis holds for every soluble group of derived length at most $d-1$ whose group ring over R is an integral domain and that G has derived length d. Let H be a finitely generated subgroup of G; then the derived subgroup H' of H has derived length at most $d-1$. The ring $R[H']$ is a subring of $R[G]$ and, therefore, an integral domain. By induction, $R[H']$ is an Ore domain. The quotient H/H' is a finitely generated abelian group; thus it is polycyclic and has a subnormal series with infinite cyclic factors

$$H' = H_0 \lhd H_1 \lhd \cdots \lhd H_{k-1} \lhd H_k = H.$$

We proceed now by induction on i to show that $R[H_i]$ is an Ore domain for $i = 0, \dots, k$. Suppose that $R[H_{i-1}]$ is an Ore domain and consider $R[H_i]$. The quotient H_i/H_{i-1} is an infinite cyclic group, $H_i/H_{i-1} = \langle aH_{i-1} \rangle$ say, and hence the ring $R[H_i]$ is a skew polynomial field in a over $R[H_{i-1}]$. It follows that $R[H_i]$ is an Ore domain [14, Proposition **0**.8.4] $\square$

A right Ore domain R can be embedded in a skew-field F. We define an equivalence relation $\sim$ on $R \times R^*$ as follows: $(r_1, s_1) \sim (r_2, s_2)$ if there exist $u, v \in R^*$ such that $s_1 u = s_2 v$ and $r_1 u = r_2 v$. The quotient of $R \times R^*$ by this equivalence relation, with the natural definitions of addition and multiplication, is the skew-field F. This is the *skew-field of right fractions* of R. Analogously, a left Ore domain R can by embedded in a skew-field, the *skew-field of left fractions* of R.

Since $R \leqslant F$, we have $R[G] \leqslant F[G]$ and, if $F[G]$ is an integral domain, then $R[G]$ is an integral domain. In particular, $\mathbb{Z}$ is an Ore domain with field of fractions $\mathbb{Q}$ and, if G is a soluble group then $\mathbb{Z}[G]$ is an integral domain by Theorem 3.1.1 and hence an Ore domain by Lemma 3.1.2.

For a discussion of the history of the zero-divisor problem see Passman's book [47, Chapter 13] and for a discussion of Ore domains see Cohn's book [14, Chapter 0].

3.2 Tensor products

Let R be a ring, let M be a right R-module and let N be a left R-module. The notion of the tensor product of R-modules is useful in allowing us to prove some important results.

Definition 3.2.1. Let A be an abelian group under addition. A map $\alpha : M \times N \to A$ is *R-balanced* if the following conditions hold:

(i) $(m_1 + m_2, n)\alpha = (m_1, n)\alpha + (m_2, n)\alpha$;

(ii) $(m, n_1 + n_2)\alpha = (m, n_1)\alpha + (m, n_2)\alpha$;

(iii) $(mr, n)\alpha = (m, rn)\alpha$.

Definition 3.2.2. The *tensor product* of M and N is an abelian group T, together with an R-balanced map $\otimes : M \times N \to T$ with the following property: for any R-balanced map $g : M \times N \to L$ to some abelian group L, there exists a unique group homomorphism $\tilde{g} : T \to L$ such that the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[rr, "g"] \arrow[dr, "\otimes"'] & & L \\ & T \arrow[ur, "\tilde{g}"'] & \end{tikzcd}$$

Let F be the additive free abelian group with free basis the set $M \times N$ and let $\iota : M \times N \to F$ be the inclusion map. Let K be the subgroup generated by the set

$$\begin{aligned} &\{(m_1 + m_2, n) - (m_1, n) - (m_2, n) \mid m_1, m_2 \in M, n \in N\} \\ &\qquad \cup \{(m, n_1 + n_2) - (m, n_1) - (m, n_2) \mid m \in M, n_1, n_2 \in N\} \\ &\qquad\qquad \cup \{(mr, n) - (m, rn) \mid m \in M, n \in N, r \in R\}. \end{aligned}$$

Let T be the quotient F/K, let $q : F \to T$ be the quotient map and let $\otimes : M \times N \to T$ be the composite map $\otimes = \iota q$. Then the pair $(T, \otimes)$ satisfies the conditions of Definition 3.2.2 (see [17, Theorem 3.4.2]). Thus the tensor product of M and N exists.

If $(T_1, \otimes_1)$ and $(T_2, \otimes_2)$ are both tensor products of M and N then, by Definition 3.2.2, there is an isomorphism $\phi : T_1 \to T_2$ such that the following diagram commutes:

$$\begin{tikzcd} & T_2 \\ M \times N \arrow[ur, "\otimes_2"] \arrow[dr, "\otimes_1"'] & \\ & T_1 \arrow[uu, "\phi"'] \end{tikzcd}$$

It follows that the tensor product of M and N is unique up to isomorphism. The image $\otimes(m, n)$ is written $m \otimes n$ and T is written $M \bigotimes_R N$.

If N is also a right R-module then there is a unique right R-module structure on $M \bigotimes_R N$ compatible with the map $\otimes$ (see [17, Theorem 3.4.3]).

For a further discussion of tensor products see Chapter 3 of the book on algebra by Goldhaber and Ehrlich [17].

3.3 Wreath products

In Lemmas 3.3.1 and 3.3.2 we prove two crucial embedding results.

Lemma 3.3.1. *Let A and B be torsion-free groups and suppose that A is abelian. Let G be a finitely generated group with a normal subgroup N such that $N \cong A$ and $G/N \cong B$. If $\mathbb{Z}[G/N]$ is an Ore domain and N is torsion-free as a $\mathbb{Z}[G/N]$-module then G embeds in $U \operatorname{wr} B$, for some finitely generated torsion-free abelian group U.*

Proof. The group G is an extension of A by B and therefore G embeds in the unrestricted wreath product $A \operatorname{Wr} B$ by the Kaloujnine–Krasner embedding (see Theorem 2.3.1). Let K be the image of G under this embedding and let H be the group generated by K and B. Then $H = D \rtimes B$ for some subgroup D which is finitely generated as a normal subgroup of H and which is in the base group of $A \operatorname{Wr} B$. We may regard D as a $\mathbb{Z}[B]$-module and we let T be the set of $\mathbb{Z}[B]$-torsion elements of D. Since $\mathbb{Z}[B]$ is an Ore domain, T is a submodule of D. By the definition of the Kaloujnine–Krasner embedding, only elements of N are mapped into D, and hence the intersection of D and K is in the image of N. As discussed in Section 2.3, the image of N in $A \operatorname{Wr} B$ is torsion-free as a $\mathbb{Z}[B]$-module and hence $K \cap T$ is trivial; it follows that G can be embedded in $(D/T) \rtimes B$.

The quotient D/T is finitely generated as a right $\mathbb{Z}[B]$-module. Let Q be the skew-field of right fractions of $\mathbb{Z}[B]$, which exists since $\mathbb{Z}[B]$ is a right Ore domain. Since D/T is torsion-free as a right $\mathbb{Z}[B]$-module, it can be embedded in the finitely generated Q-vector space $V = D/T \bigotimes_{\mathbb{Z}[B]} Q$ by [14, Corollary to Proposition **0**.6.1] for which we may choose a basis $e_1, \dots, e_r$. If $d_1, \dots, d_k$ generate D/T as a right $\mathbb{Z}[B]$-module then, for each i, there exist elements $a_{ij} \in Q$ such that $d_i = \sum_{i=1}^{r} e_j a_{ij}$. Since $\mathbb{Z}[B]$ is left Ore, we can find a common left denominator $b \in \mathbb{Z}[B]^*$ for the elements a_{ij}: let $a_{ij} = b^{-1} b_{ij}$ with $b_{ij} \in \mathbb{Z}[B]$ and $d_i = \sum_{i=1}^{r} (e_i b^{-1}) b_{ij}$. The quotient module D/T can be embedded in the finitely generated free right $\mathbb{Z}[B]$-module M with free generators $e_1b^{-1}, \ldots, e_rb^{-1}$.

It follows that G embeds in the semi-direct product $M \rtimes B$, which is the restricted wreath product of a finitely generated torsion-free abelian group and the group B. $\square$

Lemma 3.3.2. *Let A be an abelian group of finite square-free exponent n and let B be a torsion-free group. Let G be a finitely generated group with a normal subgroup N such that $N \cong A$ and $G/N \cong B$. For each prime divisor p of n, suppose that $\mathbb{F}_p[G/N]$ is an Ore domain and that the submodule of N of elements of order p is torsion-free as an $\mathbb{F}_p[G/N]$-module. Then G embeds in $U \operatorname{wr} B$, for some finitely generated abelian group U of exponent n.*

Proof. By the same argument as in Lemma 3.3.1, the group G embeds in the subgroup $H = D \rtimes B$ of $A \operatorname{Wr} B$ for some subgroup D which is finitely generated as a normal subgroup of H and which is in the base group of $A \operatorname{Wr} B$. Let $n = p_1 \cdots p_k$ where $p_1, \ldots, p_k$ are distinct primes and let D_i be the normal subgroup of D consisting of elements of order p_i so that $D = D_1 \times \cdots \times D_k$. We may regard D as an $R_n[B]$-module and each D_i as an $\mathbb{F}_{p_i}[B]$-module. By the definition of the Kaloujnine–Krasner embedding, only elements of N are mapped into D, and hence the intersection of D and K is in the image of N. Now, $N = N_1 \times \cdots \times N_k$, where N_i is of exponent p_i and is torsion-free as an $\mathbb{F}_{p_i}[G/N]$-module. As discussed in Section 2.3 we may regard the image of N_i in $A \operatorname{Wr} B$ as a torsion-free $\mathbb{F}_{p_i}[B]$-module for $i = 1, \ldots, k$. Let T_i be the set of $\mathbb{F}_{p_i}[B]$-torsion elements of D_i. Since $\mathbb{F}_{p_i}[B]$ is an Ore domain, the set T_i is an $\mathbb{F}_{p_i}[B]$-submodule of D_i. Moreover, $K \cap (T_1 \times \cdots \times T_k)$ is trivial. Hence G can be embedded in

$$\big((D_1/T_1) \times \cdots \times (D_k/T_k)\big) \rtimes B$$

and D_i/T_i is a finitely generated torsion-free $\mathbb{F}_{p_i}[B]$-module.

By the same argument as in Lemma 3.3.1, the quotient module D_i/T_i can be embedded in a finitely generated free $\mathbb{F}_{p_i}[B]$-module M_i. Suppose that M_i is freely generated as an $\mathbb{F}_{p_i}[B]$-module by r_i elements and let $r = \max\{r_1, \ldots, r_k\}$. For each i, we may embed M_i in a free $\mathbb{F}_{p_i}[B]$-module $\overline{M}_i$ of rank r and $M = \overline{M}_1 \times \cdots \times \overline{M}_k$ is a free $R_n[B]$-module of rank r. Then $(D_1/T_1) \times \cdots \times (D_k/T_k)$ embeds in M and G embeds in $M \rtimes B$ which is the restricted wreath product of a finitely generated abelian group of exponent n and the group B. $\square$

Suppose for the remainder of this section that n is a square-free positive integer or zero and that $\overline{c} = (c_1, \ldots, c_l)$ is a tuple of positive integers. Recall from Chapter 2 the definition of words $s_{\overline{c}_i}(\overline{x})$. A group G is in $\mathfrak{P}_{\overline{c}_i}$ if and only if the verbal subgroup $s_{\overline{c}_i}(G)$ is trivial. We prove some results about subgroups of restricted wreath products.

Lemma 3.3.3. *Let U be a finitely generated abelian group of exponent n and let P be a finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}}$. If G is a finitely generated subgroup of $U \operatorname{wr} P$ and G is in $\mathfrak{P}_{\overline{c}}$ then either G embeds in P or G embeds in $V \operatorname{wr} Q$, where V is a finitely generated abelian group of exponent n and Q is a finitely generated subgroup of P such that $Q \in \mathfrak{P}_{\overline{c}_{l-1}}$.*

Proof. Let A be the base group of the restricted wreath product $U \operatorname{wr} P$, so that $U \operatorname{wr} P = A \rtimes P$. We may write elements as pairs (u, d) with $u \in P$ and $d \in A$. Let $\phi : U \operatorname{wr} P \to P : (u, d) \mapsto u$ be the canonical map and let $\theta := \phi|_G$. There are two cases to consider:

(i) $\operatorname{im} \theta \in \mathfrak{P}_{\overline{c}_{l-1}}$;

(ii) $\operatorname{im} \theta \in \mathfrak{P}_{\overline{c}} \setminus \mathfrak{P}_{\overline{c}_{l-1}}$.

Let $Q = \operatorname{im} \theta$ and let $N = \ker \theta = G \cap A$, so that $Q \cong G/N$.

Case (i): Suppose to begin with that $n = 0$ and consider the action of $\mathbb{Z}[Q]$ on N: since A is torsion-free as a $\mathbb{Z}[P]$-module, the submodule N is torsion-free as a $\mathbb{Z}[Q]$-module. The group G is an extension of N by Q satisfying the conditions of Lemma 3.3.1 and it follows that G embeds in $V \operatorname{wr} Q$ where V is a finitely generated torsion-free abelian group.

Suppose now that $n \neq 0$ and let p be a prime divisor of n. The submodule of A of elements of order p is torsion-free as an $\mathbb{F}_p[P]$-module and hence the submodule of N of elements of order p is torsion-free as an $\mathbb{F}_p[Q]$-module. The group G is an extension of N by Q satisfying the conditions of Lemma 3.3.2 and it follows that G embeds in $V \operatorname{wr} Q$ where V is a finitely generated abelian group of exponent n.

Case (ii): Since $Q \notin P_{\overline{c}_{l-1}}$, the group G contains an element g such that $g \in s_{\overline{c}_{l-1}}(G)$ and $g \notin N$. Suppose for a contradiction that N is non-trivial, so that there is some $h' \in N$ with $h' \neq 1$. The element $[g, h']$ is in $N \cap s_{\overline{c}_{l-1}}(G)$ since $h' \in N \lhd G$ and $g \in s_{\overline{c}_{l-1}}(G) \lhd G$. If $[g, h'] = 1$ then, by Lemma 2.3.1, either $h' = 1$ or $g \in A \cap G = N$ contradicting our assumptions. Thus $h = [g, h']$ is a non-trivial element of $N \cap s_{\overline{c}_{l-1}}(G)$. The group $s_{\overline{c}_{l-1}}(G)$ is nilpotent of class c_l and contains both g and h. Thus $[h, \underbrace{g, \ldots, g}_{c_l}] = 1$ and, by repeated applications of Lemma 2.3.1, either $h = 1$ or $g \in A \cap G = N$ contradicting our choice of the elements h and g. Hence $N = \ker \theta$ is trivial and θ is an embedding of G into P. □

The following is a corollary of Lemma 3.3.3.

Lemma 3.3.4. *Let U be a finitely generated abelian group of exponent n and let P be a finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}}$. If G is a finitely generated subgroup of $U \operatorname{wr} P$ and G is in $\mathfrak{P}_{\overline{c}_i}$ for some $i \leqslant l$, then either G embeds in P or G embeds in $V \operatorname{wr} Q$, where V is a finitely generated abelian group of exponent n and Q is a finitely generated subgroup of P such that $Q \in \mathfrak{P}_{\overline{c}_{i-1}}$.*

Proof. Suppose that G does not embed in P. By Lemma 3.3.3, the group G embeds in $V_1 \operatorname{wr} Q_1$, where V_1 is a finitely generated abelian group of exponent n and Q_1 is a finitely generated subgroup of P such that $Q_1 \in \mathfrak{P}_{\overline{c}_{l-1}}$. Since G does not embed in P, it does not embed in Q_1 and, by repeated applications of Lemma 3.3.3, it follows that G embeds in $V \operatorname{wr} Q$ where V is a finitely generated abelian group of exponent n and Q is a finitely generated subgroup of P such that $Q \in \mathfrak{P}_{\overline{c}_i}$. □

We now consider what can be said about the nilpotent subgroups of restricted wreath products $U \operatorname{wr} H$. The next result is particularly useful when H is itself nilpotent.

Lemma 3.3.5. *If U is an abelian group of exponent n and H is a torsion-free group then a finitely generated nilpotent subgroup of $U \operatorname{wr} H$, which is not contained in the base group of the wreath product, can be embedded into H.*

Proof. Let A be the base group of $U \operatorname{wr} H$, so that $U \operatorname{wr} H = A \rtimes H$. Again we write elements as pairs (u, d) with $u \in H$ and $d \in A$. Let $\phi : U \operatorname{wr} H \to H : (u, d) \mapsto u$ be the canonical map and let $\theta := \phi|_G$. If $\ker\theta = 1$ then θ is an embedding of G into H. Suppose now that $\ker\theta \neq 1$. Let $N = \ker\theta = A \cap G$ and let y be a non-trivial element of N. The group G is nilpotent and hence has non-trivial centre; choose $z \in Z(G) \setminus \{1\}$. Since $y \neq 1$, this implies that z is in N by Lemma 2.3.1. Now let x be an arbitrary element of G. Then $[z, x] = 1$ and, again by Lemma 2.3.1, we have $x \in N$. Thus G is contained in $N \leqslant A$, the base group of $U \operatorname{wr} H$. □

Restricted wreath products

Let G be a group and let A be an abelian group. Let $W = A \operatorname{wr} G$ and let D be the base group of this wreath product, so that $W = D \rtimes G$. Then W is isomorphic to the group of matrices

$$M(G, D) = \left\{ \begin{pmatrix} u & 0 \\ d & 1 \end{pmatrix} \mid u \in G \text{ and } d \in D \right\}.$$

We write (u, d) for $\begin{pmatrix} u & 0 \\ d & 1 \end{pmatrix}$. Let $x = (u, d)$, $y = (v, c)$ and $t = (1, b)$. Let $y_i = (v_i, c_i)$ for $i = 1, \ldots, n$ and let $\overline{e} = (e_1, \ldots, e_n)$ be a finite sequence on $\{-1, 1\}$. It is easy to check that the following identities hold:

$$x^n = \big(u^n, d(u^{n-1} + \cdots + u + 1)\big), \tag{3.3.1}$$

$$[x, y] = \big([u, v], d(v - [u, v]) + c(1 - u[u, v])\big), \tag{3.3.2}$$

$$[t, \underbrace{x, \ldots, x}_{k}] = \big(1, b(u - 1)^k\big) \text{ for } k \geqslant 1, \tag{3.3.3}$$

$$t^{e_1 y_1} \cdots t^{e_n y_n} = \big(1, b(e_1 v_1 + \cdots + e_n v_n)\big). \tag{3.3.4}$$

3.4 Universal theory

We begin by noting that Timoshenko's proof that the restricted wreath product preserves universal equivalence [65, Theorem 2] can easily be adapted to give the following result.

Lemma 3.4.1. *Let A_1, A_2, B_1 and B_2 be groups. If $A_1 \models \mathrm{Th}_\forall(A_2)$ and $B_1 \models \mathrm{Th}_\forall(B_2)$ then $A_1 \operatorname{wr} B_1 \models \mathrm{Th}_\forall(A_2 \operatorname{wr} B_2)$.*

Proof. Let $C = \{c_1, \ldots, c_n\}$ be a finite subset of $A_1 \operatorname{wr} B_1$. Let $D_1 = A_1^{(B_1)}$ be the base group of $A_1 \operatorname{wr} B_1$. The elements of $A_1 \operatorname{wr} B_1$ can be expressed as ordered pairs (b, d) with $b \in B_1$ and $d \in D_1$. For each $c_i \in C$, let $c_i = (b_{ii}, d_i)$.

For each $d \in D_1$, let σ_d be the set of $b \in B_1$ such that $d(b) \neq 1$. For each d_i, the set σ_{d_i} is finite and there are only finitely many elements d_i. Hence we may list the elements of $\bigcup_{i=1}^{n} \sigma_{d_i}$:

$$\bigcup_{i=1}^{n} \sigma_{d_i} = \{b_1, \ldots, b_k\}.$$

Since $B_1 \models \mathrm{Th}_\forall(B_2)$, by Lemma 2.5.1, there is a monomorphism ν which maps the finite subset $\{b_1, \ldots, b_k, b_{11}, \ldots, b_{nn}\}$ of B_1 into B_2. We set $b'_i = b_i\nu$ and $b'_{ii} = b_{ii}\nu$.

Now consider the finite subset $\{d_i(b_j) \mid i = 1, \ldots, n; j = 1 \ldots, k\}$ of A_1. Again by Lemma 2.5.1, since $A_1 \models \mathrm{Th}_\forall(A_2)$, there is a monomorphism μ which maps this set into A_2. Let $d_i(b_j)'$ be the image of $d_i(b_j)$ under μ.

Define $\phi : C \to A_2 \operatorname{wr} B_2 : c_i \mapsto c'_i = (b'_{ii}, d'_i)$ where

$$d'_i(b'_j) = \begin{cases} d_i(b_j)' & \text{for } j = 1, \ldots, k, \\ 1 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that this is an embedding of C into $A_2 \operatorname{wr} B_2$. Hence, by Lemma 2.5.1, we have $A_1 \operatorname{wr} B_1 \models \mathrm{Th}_\forall(A_2 \operatorname{wr} B_2)$. $\square$

If A is a non-trivial torsion-free abelian group then A is universally equivalent to $\mathbb{Z}$ by [16, Lemma 1]. Thus, if B_1 and B_2 are groups such that $B_1 \models \mathrm{Th}_\forall(B_2)$, then $A \operatorname{wr} B_1 \models \mathrm{Th}_\forall(\mathbb{Z} \operatorname{wr} B_2)$. If the group A has finite exponent then the corresponding result requires more work.

Lemma 3.4.2. *If B is a torsion-free group and A is an abelian group of exponent n for some $n \in \mathbb{N}$ then $A \operatorname{wr} B \models \mathrm{Th}_\forall(C_n \operatorname{wr} B)$.*

If A is a non-cyclic abelian group of exponent p then A does not satisfy the universal theory of C_p. To illustrate this, consider the universal sentence θ defined by

$$\forall xy \left(x = 1 \vee \bigvee_{i=1}^{p} y = x^i \right).$$

Since every non-trivial element of C_p generates C_p, it is clear that $C_p \models \theta$. However θ does not hold in any non-cyclic group. Therefore we cannot use Lemma 3.4.1 to prove Lemma 3.4.2 when $n = p$.

Proof. We begin by showing that there is a group B_1 elementarily equivalent to B which contains a proper subgroup C such that C is elementarily equivalent to B.

Since B is torsion-free, it is not $\aleph_0$-categorical (see [21, Corollary 6.3.2]). Thus there exists a countable group D which is elementaily equivalent to B, but which is not isomorphic to B. Clearly D is universally equivalent to B. By Lemma 2.6.3, the group D embeds in an ultrapower $\overline{B}$ of B. We have two possibilities: either D is a proper subgroup of $\overline{B}$, and we may take $B_1 = \overline{B}$ and $C = D$, or $D = \overline{B}$. In the latter case $D \ncong B$ and B embeds in $\overline{B}$. We take $B_1 = \overline{B}$ and let C be the image of B in $\overline{B}$.

Since B is elementarily (and hence universally) equivalent to B_1, Lemma 3.4.1 implies that $A \operatorname{wr} B$ is universally equivalent to $A \operatorname{wr} B_1$ and $C_n \operatorname{wr} B$ is universally equivalent to $C_n \operatorname{wr} B_1$. Thus it suffices to show that $A \operatorname{wr} B_1 \models \mathrm{Th}_\forall(C_n \operatorname{wr} B_1)$.

If A is finitely generated then A is a finite direct product of cyclic groups of prime-power order dividing n (see [53, 4.2.10]). Let s be the number of cyclic factors in this direct product. Each of these cyclic factors embeds in C_n and hence A embeds into $C_n^{(s)}$. We proceed by induction on s. For $s = 1$ the result is immediate.

Suppose now that $C_n^{(s)} \operatorname{wr} B_1 \models \mathrm{Th}_\forall(C_n \operatorname{wr} B_1)$. Let C be a proper subgroup of B_1 such that C is universally equivalent to B_1 and let b be an element of $B_1 \setminus C$. Let $u_1, \ldots, u_s$ freely generate the base group of $C_n^{(s)} \operatorname{wr} B_1$ as an $R_n[B_1]$-module and consider the $R_n[C]$-module M generated by the elements $u_1, \ldots, u_s, u_s b$. Then M is a free module on $s+1$ generators and $C_n^{(s+1)} \operatorname{wr} C \cong M \rtimes C \leqslant C_n^{(s)} \operatorname{wr} B_1$. Thus $C_n^{(s+1)} \operatorname{wr} C$ satisfies the universal theory of $C_n^{(s)} \operatorname{wr} B_1$ and hence satisfies the universal theory of $C_n \operatorname{wr} B_1$.

Now let A be an arbitrary abelian group of exponent n. Suppose that $\phi \in \mathrm{Th}_\forall(C_n \operatorname{wr} B_1)$ and

$$\phi \sim \forall x_1 \ldots x_r \phi'(x_1, \ldots, x_r),$$

where ϕ' is quantifier-free. If $g_1, \ldots, g_r$ are elements of $A \operatorname{wr} B_1$ then $\langle g_1, \ldots, g_r \rangle \leqslant A_1 \operatorname{wr} B_1$ where A_1 is a finitely generated group of exponent n and $A_1 \operatorname{wr} B_1$ satisfies the universal theory of $C_n \operatorname{wr} B_1$. Thus $\phi'(g_1, \ldots, g_r)$ holds for all r-tuples of elements from $A \operatorname{wr} B_1$ and it follows that $A \operatorname{wr} B_1 \models \phi$. Hence $A \operatorname{wr} B_1$ satisfies the universal theory of $C_n \operatorname{wr} B_1$ as required. $\square$

The group A in Lemma 3.4.2 contains an element a of order n and so $C_n \operatorname{wr} B \cong \langle a \rangle \operatorname{wr} B \leqslant A \operatorname{wr} B$. Thus $C_n \operatorname{wr} B$ satisfies the universal theory of $A \operatorname{wr} B$. It follows that the two wreath products are universally equivalent.

Chapter 4

FREE GROUPS IN THE VARIETY $\mathfrak{A}_n\mathfrak{A}$ WHERE n IS A SQUARE-FREE POSITIVE INTEGER

Chapuis [12] characterizes those groups which satisfy the universal theory of a non-cyclic, free metabelian group from three distinct points of view:

- in terms of the module-theoretic properties of the Fitting subgroup;
- as those groups satisfying a certain collection of universal sentences;
- in terms of the embeddability of finitely generated subgroups in certain restricted wreath products.

As well as giving an explicit description of the universal theory of a non-cyclic free metabelian group, this characterization provides a proof of the decidability of the universal theory of a non-cyclic, free metabelian group which avoids the combinatorial arguments of his original proof in [11]. Chapuis calls groups universally equivalent to a non-cyclic, free metabelian group $\forall$-*free metabelian*. He gives a number of applications of his main theorem including a characterization of $\forall$-free metabelian groups in terms of residual properties, a proof that parafree metabelian groups are $\forall$-free metabelian and a study of 2-generator $\forall$-free metabelian groups. In particular, he shows that there are precisely two non-isomorphic 2-generator $\forall$-free metabelian groups, namely the free metabelian group of rank 2 and the restricted wreath product of two infinite cyclic groups.

In this chapter we obtain similar results to those of Chapuis in the case of free groups in the variety $\mathfrak{A}_n\mathfrak{A}$ where n is a square-free positive integer. Our proofs also are similar to those of Chapuis, the main new ingredients being some of those results proved in

Chapter 3. We characterize those groups contained in the universal closure of a non-cyclic free group in $\mathfrak{A}_n\mathfrak{A}$ and use this characterization to prove that the universal theory of a non-cyclic free group in this variety is decidable. We prove that a non-abelian finitely generated group G is universally equivalent to a non-cyclic free group in $\mathfrak{A}_p\mathfrak{A}$ for a prime p if and only if G is $\aleph_0$-approximable by groups universally equivalent to $F_2(\mathfrak{A}_p\mathfrak{A})$. Moreover we show that there are precisely two non-isomorphic 2-generator groups universally equivalent to a non-cyclic free group in $\mathfrak{A}_n\mathfrak{A}$.

4.1 Statement of theorem

In his paper [12], Chapuis defines the notion of ρ-groups in order to capture some of the module-theoretic properties of the Fitting subgroup of a non-cyclic free metabelian group. We shall require a similar notion for our study of groups satisfying the universal theory of a non-cyclic free group in $\mathfrak{A}_n\mathfrak{A}$.

Definition 4.1.1. (σ_n-groups). A group G is a σ_n-group if it satisfies the following conditions:

1. G is metabelian;
2. $\mathrm{Fitt}(G)$ is abelian, of exponent dividing n and isolated in G;
3. For each prime divisor p of n, the submodule of $\mathrm{Fitt}(G)$ of elements of order p is torsion-free as an $\mathbb{F}_p[G/\mathrm{Fitt}(G)]$-module;
4. Every abelian subgroup of G is either torsion-free or of exponent dividing n.

In Section 4.2 below we define the collection $\mathfrak{U}_n$ of universal sentences.

Theorem 4.1.1. *If G is a group then the following conditions are equivalent:*

(1) *G satisfies the universal theory of a non-cyclic free group in $\mathfrak{A}_n\mathfrak{A}$;*

(2) *G satisfies the universal sentences $\mathfrak{U}_n$;*

(3) *G is a σ_n-group or G is torsion-free abelian;*

(4) *Every finitely generated subgroup of G can be embedded in $C_n^{(r)} \operatorname{wr} \mathbb{Z}^{(s)}$ for some $r, s \in \mathbb{N}$.*

We show that, for a positive integer m, any two non-cyclic free groups in $\mathfrak{A}_m\mathfrak{A}$ have the same universal theory. Thus it makes sense to consider the universal theory of a non-cyclic free group in $\mathfrak{A}_m\mathfrak{A}$. In order to do this we first show that every non-cyclic free group in $\mathfrak{A}_m\mathfrak{A}$ is universally equivalent to the restricted wreath product $C_m \operatorname{wr} \mathbb{Z}$.

Lemma 4.1.1. *A non-cyclic free group in $\mathfrak{A}_m\mathfrak{A}$ satisfies the universal theory of the restricted wreath product $C_m \operatorname{wr} \mathbb{Z}$.*

Proof. Let $\mathcal{F}_r$ be the absolutely free group of finite or infinite rank r and let $F = F_r(\mathfrak{A}_m\mathfrak{A})$ be a free group of rank r in $\mathfrak{A}_m\mathfrak{A}$. Let V be the set of words $\{[x,y], x^m\}$ which define $\mathfrak{A}_m$, so that $\mathcal{F}_r/V(\mathcal{F}_r') \cong F$. By the Shmel′kin embedding (see Chapter 2), the group $\mathcal{F}_r/V(\mathcal{F}_r')$ can be embedded in $F_r(\mathfrak{A}_m) \operatorname{wr}_{\mathfrak{A}_m} (F_r/\mathcal{F}_r')$ and this is equal to $F_r(\mathfrak{A}_m) \operatorname{wr} (F_r/\mathcal{F}_r')$. Furthermore, $F_r(\mathfrak{A}_m) \cong C_m^{(r)}$ and $\mathcal{F}_r/\mathcal{F}_r' \cong F_r(\mathfrak{A}) \cong \mathbb{Z}^{(r)}$. Thus F can be embedded in $C_m^{(r)} \operatorname{wr} \mathbb{Z}^{(r)}$. By Lemma 3.4.2, the group $C_m^{(r)} \operatorname{wr} \mathbb{Z}^{(r)}$ satisfies the universal theory of $C_m \operatorname{wr} \mathbb{Z}$ and hence so does F. $\square$

Lemma 4.1.2. *The free group of rank 2 in $\mathfrak{A}_m\mathfrak{A}$ contains a copy of $C_m \operatorname{wr} \mathbb{Z}$.*

Proof. Let $F = F_2(\mathfrak{A}_m\mathfrak{A}) = \langle x, y\rangle$ be the free group of rank 2 in $\mathfrak{A}_m\mathfrak{A}$. As above, the group F can be embedded in $C_m^{(2)} \operatorname{wr} \mathbb{Z}^{(2)}$ by the map $x \mapsto (\overline{x}, t_x)$, $y \mapsto (\overline{y}, t_y)$, where $\overline{x}$ and $\overline{y}$ are the images of x and y in F/F' and t_x and t_y freely generate the base group of the wreath product as an $R_m[\mathbb{Z}^{(2)}]$-module. Consider the subgroup of F generated by $[x,y]$ and x. Identify F with its image in $C_m^{(2)} \operatorname{wr} \mathbb{Z}^{(2)}$, so that $x = (\overline{x}, t_x)$ and

$$[x,y] = (1, t_x(\overline{y}-1) + t_y(1-\overline{x})).$$

For $t \in \mathbb{Z}$, let $H_t = \langle [x,y]^{x^t}\rangle \cong C_m$ and let $H = \langle H_t \mid t \in \mathbb{Z}\rangle$. Suppose that $[x,y]^{\sum_{t=r}^{s} a_t x^t} = 1$ with $a_t \in R_m$; then

$$\begin{aligned}
[x,y]^{\sum_{t=r}^{s} a_t x^t} &= \left(1, (t_x(\overline{y}-1) + t_y(1-\overline{x}))\sum_{t=r}^{s} a_t \overline{x}^t\right) \\
&= \left(1, t_x(\overline{y}-1)\sum_{t=r}^{s} a_t\overline{x}^t + t_y(1-\overline{x})\sum_{t=r}^{s} a_t\overline{x}^t\right) = (1,0).
\end{aligned}$$

Since t_x and t_y are free generators, it follows that $(\overline{y}-1)\sum_{t=r}^{s} a_t\overline{x}^t = 0$ and it is clear from this that $\sum_{t=r}^{s} a_t\overline{x}^t = 0$; thus H is a direct product of the H_t. By definition $H \lhd \langle x, [x,y]\rangle$ and, since F/F' is a free abelian group, F' is isolated in F and the intersection $H \cap \langle x\rangle$ is trivial. Thus $\langle x, [x,y]\rangle$ is the semidirect product of H by $\langle x\rangle$ and it follows that $\langle x, [x,y]\rangle \cong C_m \operatorname{wr} \mathbb{Z}$. □

Corollary 4.1.1. *A non-cyclic free group in $\mathfrak{A}_m\mathfrak{A}$ is universally equivalent to $C_m \operatorname{wr} \mathbb{Z}$.*

Proof. Let F be a non-cyclic free group in $\mathfrak{A}_m\mathfrak{A}$. It contains a copy of $F_2(\mathfrak{A}_m\mathfrak{A})$ which in turn contains a copy of $C_m \operatorname{wr} \mathbb{Z}$ by Lemma 4.1.2. Thus $C_m \operatorname{wr} \mathbb{Z}$ satisfies the universal theory of F. By Lemma 4.1.1, the group F satisfies the universal theory of $C_m \operatorname{wr} \mathbb{Z}$. It follows that F is universally equivalent to $C_m \operatorname{wr} \mathbb{Z}$. □

It also follows immediately that

Corollary 4.1.2. *Any two non-cyclic free groups in $\mathfrak{A}_m\mathfrak{A}$ have the same universal theory.*

4.2 The sentences

We begin with some useful one-variable formulae. We define a universal formula

$$\mathrm{fitt}^{\forall}(x) : \forall\, t\, ([x, x^t] = 1)$$

and an existential formula

$$\mathrm{fitt}^{\exists}(x) : \exists\, t_1 t_2\ ([t_1, t_2, x] = 1 \wedge [t_1, t_2] \neq 1)\,.$$

The significance of the formulae $\mathrm{fitt}^{\forall}(x)$ and $\mathrm{fitt}^{\exists}(x)$ is as follows: if G is a model of $\mathfrak{U}_n$ then $\mathrm{Fitt}(G) = \mathrm{fitt}^{\forall}(G)$ and if G is a non-abelian model of $\mathfrak{U}_n$ then $\mathrm{Fitt}(G) \subseteq \mathrm{fitt}^{\exists}(G)$ (see Lemma 4.3.1 below).

For each positive integer m and each prime p, we denote by $\mathcal{I}_{m,p}$ the set of partitions of $\{1, \ldots, m\}$ into sets X of size a multiple of p.

Let $\mathfrak{U}_n$ be the union of the following six sets of universal sentences together with the sentence ϕ_{gp} as defined on page 14:

(1) The single sentence ϕ_1 defined by

$$\forall\, xyuv\, \big([[x,y],[u,v]] = 1\big).$$

A group G satisfies ϕ_1 if and only if G is metabelian.

(2) The countable collection of sentences $\{\phi_2^n(m) \mid m \in \mathbb{N}\}$, where $\phi_2^n(m)$ is defined by

$$\forall xy \Bigg([x,y] = 1 \to \\ \Big((x^n = 1 \wedge y^n = 1) \vee \big((x^m = 1 \to x = 1) \wedge (y^m = 1 \to y = 1)\big)\Big)\Bigg).$$

A group G satisfies $\phi_2^n(m)$ for all m if and only if every abelian subgroup of G is either torsion-free or of exponent n.

(3) The single sentence ϕ_3 defined by

$$\forall xy\ (([x,y,x] = 1 \wedge [x,y,y] = 1) \to [x,y] = 1)\,.$$

If a group G satisfies ϕ_3 then every nilpotent subgroup of G is abelian.

(4) The single sentence ϕ_4^n defined by

$$\forall x \left(\mathrm{fitt}^{\exists}(x) \to x^n = 1\right).$$

Let G be a non-abelian metabelian group such that every nilpotent subgroup of G is abelian. If G satisfies ϕ_4^n then $\mathrm{Fitt}(G)$ has exponent n.

(5) The countable collection of sentences $\{\phi_5(m), m \in \mathbb{N}\}$, where $\phi_5(m)$ is defined by

$$\forall x \left(\mathrm{fitt}^{\exists}(x^m) \to \mathrm{fitt}^{\forall}(x)\right).$$

Let G be a non-abelian metabelian group such that every nilpotent subgroup of G is abelian. If G satisfies $\phi_5(m)$ for all $m \in \mathbb{N}$ then $\mathrm{Fitt}(G)$ is isolated in G.

(6) For each prime divisor p of n, the countable collection of sentences $\{\phi_6^p(m) \mid m \in \mathbb{N}\}$, where

(a) if $m \not\equiv 0 \bmod p$ then $\phi_6^p(m)$ is defined by

$$\forall xy_1 \ldots y_m \left(\left(x^p = 1 \wedge x \neq 1 \wedge x^{\sum_{j=1}^m y_j} = 1\right) \to y_1 \neq y_1\right);$$

(b) if $m \equiv 0 \bmod p$ then $\phi_6^p(m)$ is defined by

$$\forall x y_1 \dots y_m \Bigg(\Big(x^p = 1 \wedge x \neq 1 \wedge x^{\sum_{j=1}^m y_j} = 1 \Big) \rightarrow \bigvee_{I \in \mathcal{I}_{m,p}} \bigwedge_{X \in\ I} \bigwedge_{j_1, j_2 \in X} \mathrm{fitt}^{\forall}(y_{j_1} y_{j_2}^{-1}) \Bigg).$$

Let G be a non-abelian metabelian group such that every nilpotent subgroup of G is abelian. If G satisfies $\phi_6^p(m)$ for every positive integer m and $G/\operatorname{Fitt}(G)$ is torsion-free then the elements of order p in $\operatorname{Fitt}(G)$ form a torsion-free $\mathbb{F}_p[G/\operatorname{Fitt}(G)]$-module.

Each of the sentences in $\mathfrak{U}_n$ contains only the universal quantifier $\forall$ when written in prenex normal form. In certain cases we have given the sentences in a more transparently comprehensible form than the prenex normal form.

Lemma 4.2.1.

$$C_n \operatorname{wr} \mathbb{Z} \models \mathfrak{U}_n$$

Proof. By Lemma 2.3.1, we have $\mathrm{fitt}^{\exists}(C_n \operatorname{wr} \mathbb{Z}) = \operatorname{Fitt}(C_n \operatorname{wr} \mathbb{Z})$ and it is clear that $\mathrm{fitt}^{\forall}(C_n \operatorname{wr} \mathbb{Z}) = \operatorname{Fitt}(C_n \operatorname{wr} \mathbb{Z})$. The base group D of $C_n \operatorname{wr} \mathbb{Z}$ is a direct product of groups D_i of exponent p_i, where the p_i are the prime divisors of n. The ring $\mathbb{F}_{p_i}[\mathbb{Z}]$ is an integral domain (for example by Lemma 3.1.1) and, since D_i is a free $\mathbb{F}_{p_i}[\mathbb{Z}]$-module, it is torsion-free. It follows easily from these facts that $C_n \operatorname{wr} \mathbb{Z} \models \mathfrak{U}_n$. □

4.3 σ_n-groups

In this section we establish the relationship between the σ_n-groups defined in Section 4.1 and the sentences $\mathfrak{U}_n$ defined in the previous section. This relationship plays a key role in our proof of Theorem 4.1.1.

In Lemma 4.3.1 we explain the significance of the universal formula $\mathrm{fitt}^{\forall}(x)$ and the existential formula $\mathrm{fitt}^{\exists}(x)$. These formulae are important in allowing us to describe, using universal sentences, the Fitting subgroup $\operatorname{Fitt}(G)$ of a group G and the quotient group $G/\operatorname{Fitt}(G)$.

Lemma 4.3.1. *Let G be a group such that $G \models \mathfrak{U}_n$; then* $\mathrm{Fitt}(G)$ *is abelian and* $\mathrm{Fitt}(G) = \mathrm{fitt}^{\forall}(G)$. *If G is not abelian then $g \in \mathrm{Fitt}(G)$ implies $G \models \mathrm{fitt}^{\exists}(g)$.*

Proof. By Fitting's theorem, every finitely generated subgroup of $\mathrm{Fitt}(G)$ is nilpotent. The validity of the sentence ϕ_3 in G implies that every nilpotent subgroup of G is abelian and it follows that $\mathrm{Fitt}(G)$ is abelian. If g is an element of $\mathrm{Fitt}(G)$ then the normal closure in G of g is in $\mathrm{Fitt}(G)$ which is abelian. Thus $[g^h, g] = 1$ for all $h \in G$ and $g \in \mathrm{fitt}^{\forall}(G)$. Now let g be an element of $\mathrm{fitt}^{\forall}(G)$, so that g commutes with all its conjugates in G. The normal closure in G of g is abelian and $g \in \mathrm{Fitt}(G)$.

Suppose now that G is not abelian and let g be an element of $\mathrm{Fitt}(G)$. Since G is metabelian, G' is an abelian normal subgroup of G and $G' \leqslant \mathrm{Fitt}(G)$. Since G is not abelian, there exist elements $h_1, h_2 \in G$ such that $[h_1, h_2] \neq 1$. The elements $[h_1, h_2]$ and g are in $\mathrm{Fitt}(G)$ which is abelian and hence $[h_1, h_2, g] = 1$. Therefore $g \in \mathrm{Fitt}(G)$ implies $G \models \mathrm{fitt}^{\exists}(g)$. □

In Lemmas 4.3.2 and 4.3.3 we show that, for a non-abelian group G, the statements that G is a σ_n-group and that G satisfies the sentences $\mathfrak{U}_n$ are equivalent.

Lemma 4.3.2. *If G is a non-abelian group such that $G \models \mathfrak{U}_n$ then G is a σ_n-group.*

Proof. The validity of the sentence ϕ_1 in G implies that G is metabelian. Since $G \models \phi_4^n$ and $G \models \phi_5(m)$ for all $m \in \mathbb{N}$, it follows that $\mathrm{Fitt}(G)$ is of exponent n and isolated in G. Let p be a prime divisor of n and let g be an element of $\mathrm{Fitt}(G)$ of order p. The sentences $\phi_6^p(m)$ for all $m \in \mathbb{N}$ hold in G; if $g^\alpha = 1$ for some $\alpha \in \mathbb{F}_p[G/\mathrm{Fitt}(G)]$ then $\alpha = 0$. Thus the submodule of $\mathrm{Fitt}(G)$ of elements of order p is torsion-free as an $\mathbb{F}_p[G/\mathrm{Fitt}(G)]$-module. Finally the validity of the sentence $\phi_2^n(m)$ for all $m \in \mathbb{N}$ implies that every abelian subgroup of G is either torsion-free or of exponent n. □

Lemma 4.3.3. *If G is a σ_n-group then $G \models \mathfrak{U}_n$.*

Proof. Let G be a σ_n-group. It is clear that G satisfies the sentences ϕ_1 and $\phi_2^n(m)$ for all $m \in \mathbb{N}$. Let g be an element of G such that $G \models \mathrm{fitt}^{\exists}(g)$. Then there exist elements $h_1, h_2 \in G$ such that $h = [h_1, h_2] \neq 1$ and $[h, g] = 1$. Since G is metabelian, $G' \leqslant \mathrm{Fitt}(G)$ and the element h is in $\mathrm{Fitt}(G)$. Suppose that h is of order m (where m divides n) and let p be a prime divisor of m. Then $h' = h^{\frac{m}{p}}$ is a non-trivial element of the submodule N_p of $\mathrm{Fitt}(G)$ of elements of order dividing p. Moreover, if $[h, g] = 1$ then

$$h'^{(-1+g)} = [h', g] = 1.$$

Thus, since N_p is torsion-free as an $\mathbb{F}_p[G/\mathrm{Fitt}(G)]$-module, the image of g in $G/\mathrm{Fitt}(G)$ is trivial and g is in $\mathrm{Fitt}(G)$. Hence $\mathrm{fitt}^{\exists}(G) \subseteq \mathrm{Fitt}(G)$. Moreover, since $\mathrm{Fitt}(G)$ is abelian, $\mathrm{fitt}^{\forall}(G) = \mathrm{Fitt}(G)$. It follows easily that the sentences ϕ_4^n, $\phi_5(m)$ for all $m \in \mathbb{N}$ and $\phi_6^p(m)$ for all prime divisors p of n and all $m \in \mathbb{N}$ hold in G. It remains to show that $G \models \phi_3$. Let g and h be elements of G such that $[g, h, g] = 1$ and $[g, h, h] = 1$. The subgroup $\mathrm{Fitt}(G)$ is abelian and so if g and h are elements of $\mathrm{Fitt}(G)$ then $[g, h] = 1$. We suppose, for a contradiction, that $[g, h] \neq 1$ and we may assume, without loss of generality, that g is not in $\mathrm{Fitt}(G)$. If $[g, h, g] = 1$ then $[g, h]^{-1+g} = 1$ and $-1+g$ is a non-trivial element of $\mathbb{F}_p[G/\mathrm{Fitt}(G)]$ for some prime divisor p of n. By the same argument as before, with $[g, h]$ in place of h, we show that g is an element of $\mathrm{Fitt}(G)$, contradicting our assumptions. Thus $G \models \phi_3$. □

4.4 Proof of theorem

For the reader's convenience we restate the theorem which will be proved in this section. Recall that n is a square-free positive integer.

Theorem 4.1.1 *If G is a group then the following conditions are equivalent:*

(1) *G satisfies the universal theory of a non-cyclic group free in $\mathfrak{A}_n\mathfrak{A}$;*

(2) *G satisfies the universal sentences $\mathfrak{U}_n$;*

(3) *G is a σ_n-group or G is torsion-free abelian;*

(4) *Every finitely generated subgroup of G can be embedded in $C_n^{(r)} \operatorname{wr} \mathbb{Z}^{(s)}$ for some $r, s \in \mathbb{N}$.*

Proof. We prove the implications $(1) \implies (2) \iff (3) \implies (4) \implies (1)$.

$(1) \implies (2)$**.** Let F be a non-cyclic free group in $\mathfrak{A}_n\mathfrak{A}$. By Corollary 4.1.1, the group F is universally equivalent to $C_n \operatorname{wr} \mathbb{Z}$. Thus $G \models \mathrm{Th}_\forall(F) = \mathrm{Th}_\forall(C_n \operatorname{wr} \mathbb{Z})$. By Lemma 4.2.1, we have $C_n \operatorname{wr} \mathbb{Z} \models \mathfrak{U}_n$ and hence $G \models \mathfrak{U}_n$.

$(2) \iff (3)$**.** Suppose to begin with that G is abelian. Since G satisfies $\phi_2^n(m)$ for all $m \in \mathbb{N}$, either G is torsion-free or G is of exponent dividing n. In the latter case G is a σ_n-group. This, together with Lemma 4.3.3, implies the equivalence of (2) and (3) when G is abelian. If G is non-abelian then the equivalence of (2) and (3) follows immediately from the results of Section 4.3.

$(3) \implies (4)$**.** Let G be a σ_n-group and let G_1 be a finitely generated subgroup of G. Since G satisfies $\mathfrak{U}_n$ which is a universal set of sentences, every subgroup of G satisfies $\mathfrak{U}_n$; in particular, we have $G_1 \models \mathfrak{U}_n$. The validity in G of the sentences $\phi_2^n(m)$ for $m \in \mathbb{N}$ implies that if G_1 is abelian then it is either torsion-free or of exponent dividing n. In the former case G_1 embeds in some power of $\mathbb{Z}$ and in the latter case $G_1 = C_{n_1} \times \cdots \times C_{n_r}$ where C_{n_j} is a cyclic group of exponent n_j and n_j divides n. The cyclic group C_{n_j} embeds in C_n for each j and hence G_1 embeds in the direct product of r copies of C_n.

Suppose now that G_1 is non-abelian, so that G_1 is a σ_n-group by Lemma 4.3.2. Then G_1 is an extension of $\mathrm{Fitt}(G_1)$ by $\overline{G}_1 = G_1/\mathrm{Fitt}(G_1)$ satisfying the conditions of Lemma 3.3.2, and hence G_1 embeds in $V \operatorname{wr} \overline{G}_1$, where V is a finitely generated abelian group of exponent dividing n. As in the abelian case above, V embeds in $C_n^{(r)}$ for some positive integer r and, since $\overline{G}_1$ is a finitely generated torsion-free abelian group, it isomorphic to $\mathbb{Z}^{(s)}$ for some positive integer s. It follows that G_1 embeds in $C_n^{(r)} \operatorname{wr} \mathbb{Z}^{(s)}$.

$(4) \implies (1)$**.** Let F be a non-cyclic free group in $\mathfrak{A}_n\mathfrak{A}$ and let ϕ be in $\mathrm{Th}_\forall(F)$. We may suppose that ϕ is of the form

$$\forall x_1 \dots x_k \, \varphi(x_1, \dots, x_k)$$

where $\varphi(x_1, \dots, x_k)$ is quantifier-free. We let $g_1 \dots, g_k$ be elements of G and let $G_1 = \langle g_1, \dots, g_k \rangle$, so that G_1 embeds in $C_n^{(r)} \operatorname{wr} \mathbb{Z}^{(s)}$ for some positive integers r and

s. By Lemmas 3.4.2 and 4.1.1, we have $C_n^{(r)} \operatorname{wr} \mathbb{Z}^{(s)} \models \mathrm{Th}_\forall(C_n \operatorname{wr} \mathbb{Z}) = \mathrm{Th}_\forall(F)$; in particular, G_1 satisfies ϕ and $\varphi(g_1, \ldots, g_k)$ holds. This is true for all k-tuples $g_1, \ldots, g_k \in G$ and hence G satisfies ϕ. It follows that $G \models \mathrm{Th}_\forall(F)$ and $G \in \mathrm{ucl}(F)$. $\square$

4.5 Some consequences of the theorem

In this section we consider some of the consequences of Theorem 4.1.1. Let F be a non-cyclic free group in $\mathfrak{A}_n\mathfrak{A}$. We establish necessary and sufficient conditions for a group in $\mathrm{ucl}(F)$ to be universally equivalent to F and we use these conditions to prove that the universal theory of F is decidable. We go on to construct lattice diagrams of those universally closed classes of groups in $\mathrm{ucl}(F)$ for selected values of n. We also construct such a lattice diagram for the universal closure of a non-cyclic free metabelian group; Chapuis did not do so in his paper [12]. As a further consequence of Theorem 4.1.1, we prove that a non-abelian finitely generated group G is universally equivalent to a non-cyclic free group in $\mathfrak{A}_p\mathfrak{A}$ for a prime p if and only if G is $\aleph_0$-approximable by groups universally equivalent to $F_2(\mathfrak{A}_p\mathfrak{A})$. Finally we determine the isomorphism classes of 2-generator groups universally equivalent to F.

Corollary 4.5.1. *A group G is universally equivalent to a non-cyclic free group in $\mathfrak{A}_n\mathfrak{A}$ if and only if it is non-abelian, contains an element of order n and satisfies the collection $\mathfrak{U}_n$ of universal sentences.*

Proof. Let G be a non-abelian group satisfying $\mathfrak{U}_n$ and containing an element of order n. Theorem 4.1.1 implies that G is a σ_n-group. Let g be an element of $G \setminus \mathrm{Fitt}(G)$ and let h be an element of G such that $o(h) = n$. Then h is a non-trivial element of $\mathrm{Fitt}(G)$ and the annihilator of h in $R_n[G/\mathrm{Fitt}(G)]$ is trivial. For $t \in \mathbb{Z}$, let $H_t = \langle h^{g^t} \rangle$ and let $H = \langle H_t \,|\, t \in \mathbb{Z} \rangle$. Then H is a direct product of the H_t and each H_t is isomorphic to C_n. By definition $H \lhd \langle g, h \rangle$ and, since A is isolated in G, the intersection $H \cap \langle g \rangle$ is trivial. Thus $\langle g, h \rangle$ is the semidirect product of H by $\langle g \rangle$ and $\langle g, h \rangle \cong C_n \operatorname{wr} \mathbb{Z}$. It follows that G is universally equivalent to $C_n \operatorname{wr} \mathbb{Z}$ and hence to $F_r(\mathfrak{A}_n\mathfrak{A})$ for $r \geqslant 2$ by Corollary 4.1.1.

Suppose now that G is universally equivalent to a non-cyclic group F free in $\mathfrak{A}_n\mathfrak{A}$.

Theorem 4.1.1 implies that $G \models \mathfrak{U}_n$. Let θ and θ_n be existential sentences defined as follows:

$$\theta : \exists xy\,([x,y] \neq 1),$$
$$\theta_n : \exists x \left(x^n = 1 \wedge \bigwedge_{i=1}^{k} x^{n/p_i} \neq 1 \right)$$

where $n = p_1 \cdots p_k$ and each p_i is prime. We have $C_n \operatorname{wr} \mathbb{Z} \models \theta$ and $C_n \operatorname{wr} \mathbb{Z} \models \theta_n$. Thus, since F is universally equivalent to $C_n \operatorname{wr} \mathbb{Z}$ and G is universally equivalent to F, it follows that $G \models \theta$ and $G \models \theta_n$. Hence G is non-abelian and contains an element of order n. $\square$

If n is a prime p then the condition that G contains an elements of order p follows from the fact that G is non-abelian: the subgroup G' is non-trivial and contained in $\mathrm{Fitt}(G)$; since $\mathrm{Fitt}(G)$ has exponent dividing p, it follows that any non-trivial element of G' is of order p.

Corollary 4.5.2. *A group G is universally equivalent to a non-cyclic free group in $\mathfrak{A}_p\mathfrak{A}$, where p is prime, if and only if it is non-abelian and satisfies the collection of universal sentences $\mathfrak{U}_p$.*

The necessary and sufficient conditions given in Corollary 4.5.1 for a group to be universally equivalent to a non-cyclic group F free in $\mathfrak{A}_n\mathfrak{A}$ yield a recursive collection of universal and existential sentences which completely distinguish those groups universally equivalent to F. Using this result we show that F has decidable universal theory.

Corollary 4.5.3. *The universal theory of a non-cyclic free group in $\mathfrak{A}_n\mathfrak{A}$ is decidable.*

Proof. Let $\overline{\mathfrak{U}}_n$ be the recursive set of sentences $\mathfrak{U}_n \cup \{\theta, \theta_n\}$. By Lemma 4.5.1, we have $G \models \overline{\mathfrak{U}}_n$ if and only if G is universally equivalent to $F = F_r(\mathfrak{A}_n\mathfrak{A})$ for $r \geqslant 2$. Let ϕ be a universal sentence; by the completeness theorem, $\overline{\mathfrak{U}}_n \vdash \phi$ or $\overline{\mathfrak{U}}_n \vdash \neg\phi$. We have $\phi \in \mathrm{Th}_\forall(F)$ if and only if $\overline{\mathfrak{U}}_n \vdash \phi$ and $\phi \notin \mathrm{Th}_\forall(F)$ if and only if $\overline{\mathfrak{U}}_n \vdash \neg\phi$. Thus, since $\overline{\mathfrak{U}}_n$ is recursive, the universal theory of F is decidable. $\square$

The results of Chapuis [12] show that a group is universally equivalent to a non-cyclic free metabelian group F if and only if it is non-abelian and contained in the universal closure of F. Moreover, a torsion-free abelian group A is universally equivalent to $\mathbb{Z}$ if and only if it is non-trivial [16, Lemma 1]. Using these facts we can construct a lattice diagram of the universally closed classes of groups in the universal closure of a non-cyclic free metabelian group (see Figure 4.1). Since finite groups of different orders can be distinguished using universal sentences[1], the picture becomes more complex when the bottom group is of finite exponent (see Figures 4.2 and 4.3). Whereas all non-trivial torsion-free abelian groups are universally equivalent, abelian groups of finite exponent, with finite generating sets of differing size, are not. When dealing with groups whose Fitting subgroup has non-prime exponent, we also have non-abelian universally closed classes of groups properly contained in the universal closure of a non-cyclic free group in $\mathfrak{A}_n\mathfrak{A}$. These are classes of groups which do not contain an element of order n.

In Figure 4.3 we have three sublattices $\mathcal{L}_p$, $\mathcal{L}_q$ and $\mathcal{L}_{pq}$, where p and q are distinct primes. These are the lattices of universally closed classes of abelian groups of exponent dividing p, q and pq respectively, such that each class in the lattice contains a group of exponent p, q and pq respectively. The lattice $\mathcal{L}_p$ also occurs as a sublattice in Figure 4.2. For a more explicit depiction of these lattices showing all universal closures of abelian groups of exponent dividing pq see Figure 4.4. We note that each class in $\mathcal{L}_p$ and each class in $\mathcal{L}_q$ is contained in some class in $\mathcal{L}_{pq}$. In Figure 4.3 we have omitted some of the points in the sublattice of abelian groups. These points represent the union of two or more universal closures of groups, at least one of which is contained in one of the sublattices $\mathcal{L}_p$, $\mathcal{L}_q$ and $\mathcal{L}_{pq}$. For example, we have omitted the class of groups which are abelian and either of order dividing p or of order dividing q. This class is the union of $\mathrm{ucl}(C_p)$ and $\mathrm{ucl}(C_q)$. These omissions simplify the presentation of the lattice greatly and do not have a significant impact on the information it contains. The two occurrences in Figure 4.3 of the following universally closed classes of groups should be thought of as single points in the lattice: the class of groups with Fitting subgroup of exponent p or q and the universal closure of $\mathbb{Z}$. They have been duplicated in order to simplify the presentation of the lattice.

[1] If G and H are groups of finite orders r and s with $r < s$ then the universal sentence

$$\forall x_1 \ldots x_{r+1} \left(\bigvee_{i \neq j} x_i = x_j \right)$$

holds in G but not in H.

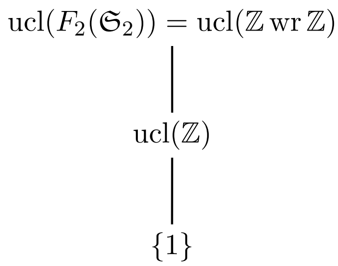

Figure 4.1: The lattice of universally closed classes contained in the universal closure of a non-cyclic free metabelian group.

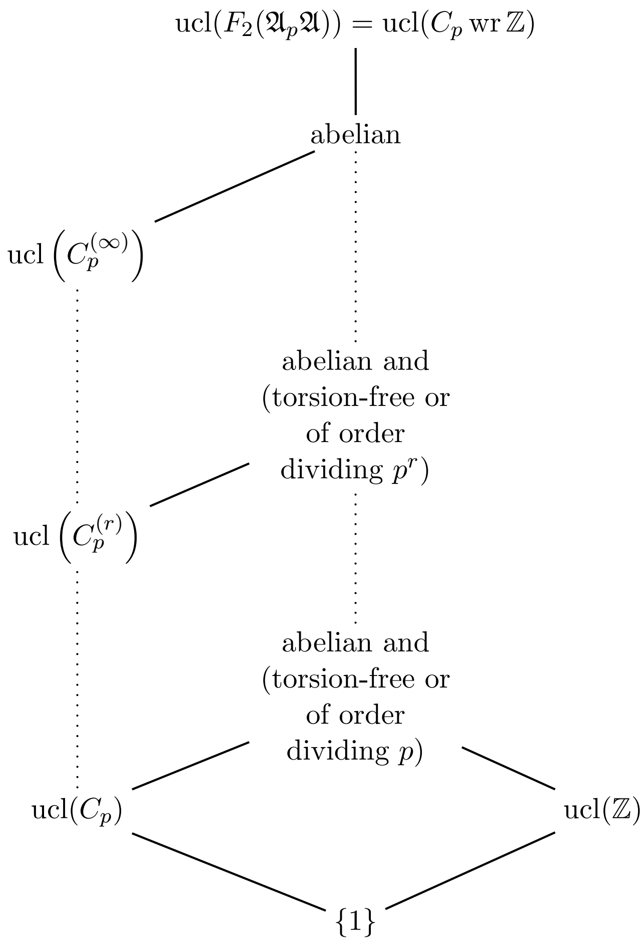

Figure 4.2: The lattice of universally closed classes contained in the universal closure of a non-cyclic free group in $\mathfrak{A}_p\mathfrak{A}$ where p is a prime.

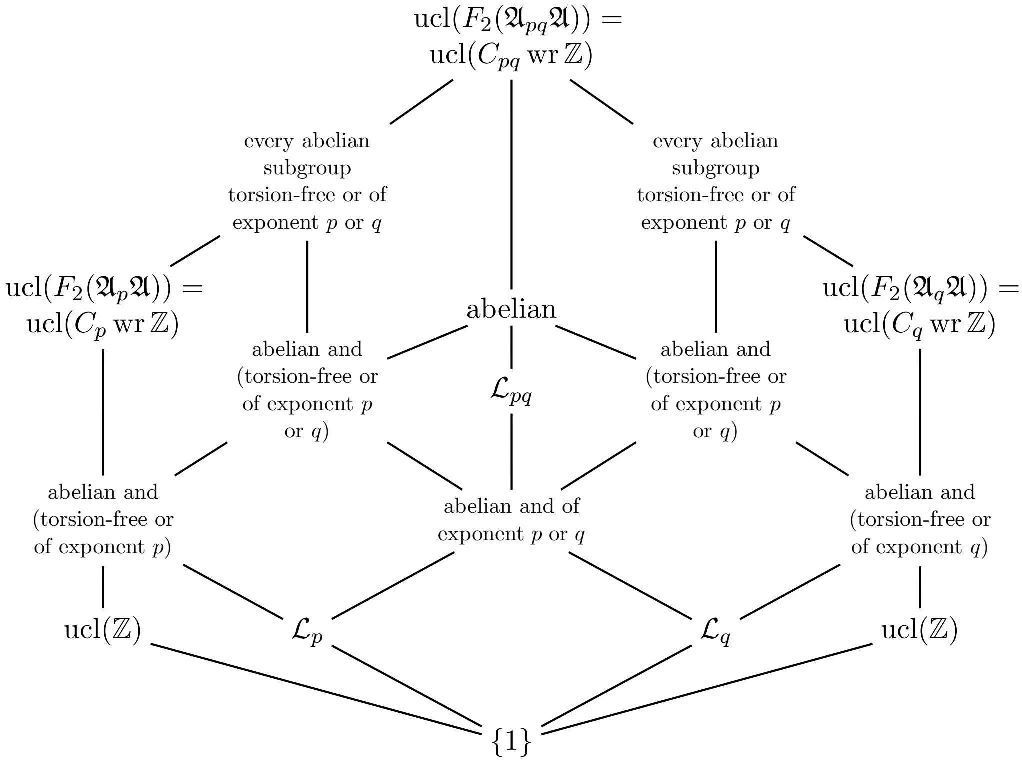

Figure 4.3: The lattice of universally closed classes contained in the universal closure of a non-cyclic free group in $\mathfrak{A}_{pq}\mathfrak{A}$ where p and q are primes.

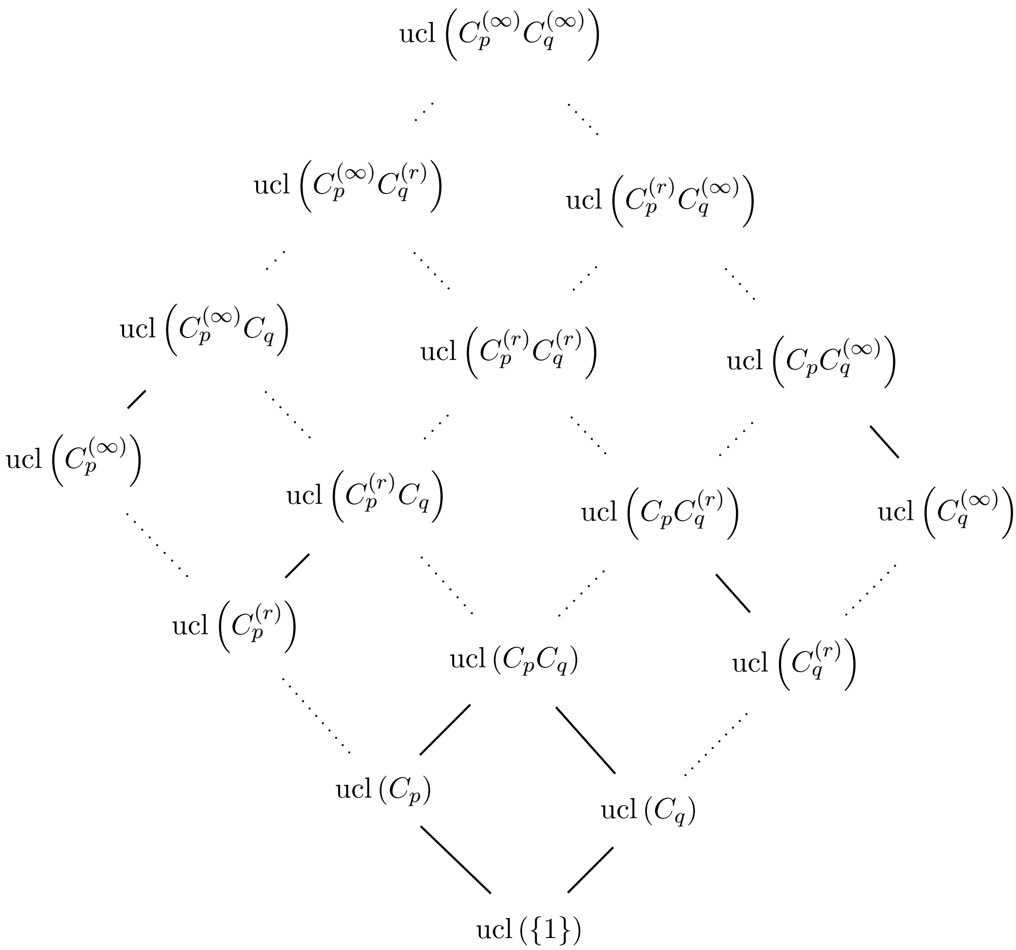

Figure 4.4: The lattice of universal closures of abelian groups of exponent dividing pq where p and q are primes.

Next we consider the relationship between universal equivalence and $\aleph_0$-approximability. Let $\mathfrak{X}$ be a non-trivial class of groups and let r be a positive integer. A group G is *r-approximable* by $\mathfrak{X}$-groups if, for all non-trivial $g_1, \ldots, g_r \in G$, there exists a group $H \in \mathfrak{X}$ and a homomorphism $f : G \to H$ such that $f(g_i) \neq 1$ for $i = 1, \ldots, n$. A group G is *$\aleph_0$-approximable* by $\mathfrak{X}$-groups if it is r-approximable by $\mathfrak{X}$-groups for all $r \in \mathbb{N}$. In [49] Remeslennikov showed that a finitely generated non-abelian group is universally equivalent to a non-cyclic free group if and only if it is $\aleph_0$-approximable by non-cyclic free groups and, in [12, Section 5], Chapuis showed that the same is true for free metabelian groups: a finitely generated non-abelian group is $\forall$-free metabelian if and only if it is $\aleph_0$-approximable by non-cyclic free metabelian groups. By Corollary 4.5.2, a non-abelian group which satisfies the universal theory of a non-cyclic free group in $\mathfrak{A}_p\mathfrak{A}$, where p is prime, is universally equivalent to it. We use this fact to prove a similar result about $\aleph_0$-approximability in the case of free groups in $\mathfrak{A}_p\mathfrak{A}$.

Corollary 4.5.4. *Let $\mathfrak{X}$ be a non-empty class of groups universally equivalent to a non-cyclic free group in $\mathfrak{A}_p\mathfrak{A}$, where p is prime. A finitely generated non-abelian group is universally equivalent to a non-cyclic free group in $\mathfrak{A}_p\mathfrak{A}$ if and only if it is $\aleph_0$-approximable by $\mathfrak{X}$-groups.*

Proof. Let $\mathfrak{X}$ be a non-empty class of groups universally equivalent to a non-cyclic free group in $\mathfrak{A}_p\mathfrak{A}$, where p is prime, and let G be a finitely generated non-abelian group.

We suppose initially that G is universally equivalent to a non-cyclic free group in $\mathfrak{A}_p\mathfrak{A}$, so that G is in $\mathfrak{A}_p\mathfrak{A}$. Finitely generated metabelian groups have the maximal condition on normal subgroups (P. Hall, see [53, 15.3.1]) and hence G has a finite presentation $\langle a_1, \ldots, a_s \,|\, r_1(\overline{a}), \ldots, r_t(\overline{a})\rangle$ in $\mathfrak{A}_p\mathfrak{A}$. Let $g_1, \ldots, g_r$ be non-trivial elements of G, so that $g_i = g_i(\overline{a})$ for $i = 1, \ldots, r$. We consider the existential sentence ψ defined by

$$\exists\, x_1 \ldots x_s \left(\bigwedge_{i=1}^{t} r_i(\overline{x}) = 1 \wedge \bigwedge_{i=1}^{r} g_i(\overline{x}) \neq 1 \right)$$

where $\overline{x}$ is the s-tuple $(x_1, \ldots, x_s)$. Each group in $\mathfrak{X}$ is universally equivalent to a non-cyclic free group in $\mathfrak{A}_p\mathfrak{A}$ and hence is universally equivalent to G. Since $G \models \psi$, it follows that $H \models \psi$ for each group $H \in \mathfrak{X}$. Fix $H \in \mathfrak{X}$ and choose $b_1, \ldots, b_s \in H$ such that $r_i(\overline{b}) = 1$ for $i = 1, \ldots, t$ and $g_i(\overline{b}) \neq 1$ for $i = 1, \ldots, r$. Then the map $f : G \to H$ defined by $a_i \mapsto b_i$ for $i = 1, \ldots, s$ is the required homomorphism by Von Dyck's Theorem (see [53, 2.2.1]).

Suppose now that G is $\aleph_0$-approximable by $\mathfrak{X}$-groups. By Corollary 4.5.2, it is sufficient to show that G satisfies the universal theory of a non-cyclic free group in $\mathfrak{A}_p\mathfrak{A}$. Equivalently, we show that a non-cyclic free group F in $\mathfrak{A}_p\mathfrak{A}$ satisfies the existential theory of G. We let ϕ be a sentence in $\mathrm{Th}_\exists(G)$ and we assume without loss of generality that ϕ is of the form

$$\exists x_1 \ldots x_r \left(\bigwedge_{i=1}^{s} u_i(\overline{x}) = 1 \wedge \bigwedge_{j=1}^{t} w_j(\overline{x}) \neq 1 \right)$$

where $\overline{x}$ is the r-tuple $(x_1, \ldots, x_r)$. Since G satisfies ϕ, we may choose $g_1, \ldots, g_r \in G$ such that $u_i(\overline{g}) = 1$ for $i = 1, \ldots, s$ and $w_j(\overline{g}) \neq 1$ for $j = 1, \ldots, t$. There is a group $H \in \mathfrak{X}$ and a homomorphism $f : G \to H$ such that $w_j(\overline{h}) = f(w_j(\overline{g})) \neq 1$ for $j = 1, \ldots, t$ where $\overline{h}$ is the r-tuple $(h_1, \ldots, h_r)$ and $h_i = f(g_i)$ for $i = 1, \ldots, r$. Since f is a homomorphism, $u_i(\overline{h}) = f(u_i(\overline{g})) = 1$ for $i = 1, \ldots, s$. Hence H satisfies ϕ and it follows, since H is universally equivalent to F, that F satisfies ϕ. $\square$

In particular we may choose $\mathfrak{X} = \{F_2(\mathfrak{A}_p\mathfrak{A})\}$. Thus a finitely generated non-abelian group G is universally equivalent to a non-cyclic free group in $\mathfrak{A}_p\mathfrak{A}$ if and only if it is $\aleph_0$-approximable by $F_2(\mathfrak{A}_p\mathfrak{A})$.

The final result in this section concerns 2-generator groups universally equivalent to a free group in $\mathfrak{A}_n\mathfrak{A}$. Our proof requires the following well-known result (see, for example, [12, Lemma 5.4]).

Lemma 4.5.1. *If G is a 2-generator group containing a normal subgroup N such that $G/N \cong \mathbb{Z}$ then there exist elements $u, v \in G$ such that $G = \langle u, v \rangle$ with $v \in N \setminus \{1\}$.*

Proof. Since G/N is cyclic we can choose $u \in G$ which generates G modulo N. If $H = \langle x, y \rangle$ then $x = u^r c_1$ and $y = u^s c_2$ for some integers r and s and some elements $c_1, c_2 \in N$. We choose x and y in such a way as to minimise $\max\{|r|, |s|\}$. Replacing x with x^{-1} and y with y^{-1} as required, we may assume without loss of generality that both r and s are non-negative. If $r = 0$ or $s = 0$ then we have the desired result. Suppose therefore that $r > 0$ and $s > 0$ and let $y_1 = x_2 x_1^{-1}$. Clearly G is generated by x and y_1 and

$y_1 = u^{s-r}c_3$ with $c_3 \in N$. If $r = s$ then $y_1 \in N$ and we have the desired result. Otherwise $\max\{|r|, |s-r|\} < \max\{|r|, |s|\}$ contradicting the choice of x and y. $\square$

We are now able to determine explicitly those 2 generator groups universally equivalent to a free group in $\mathfrak{A}_n\mathfrak{A}$.

Corollary 4.5.5. *A 2-generator group universally equivalent to a free group in $\mathfrak{A}_n\mathfrak{A}$ is isomorphic to either $F_2(\mathfrak{A}_n\mathfrak{A})$ or $C_n \operatorname{wr} \mathbb{Z}$.*

Proof. Let $H = \langle g, h\rangle$ be a 2-generator group universally equivalent to a free group in $\mathfrak{A}_n\mathfrak{A}$. Let $A = \mathrm{Fitt}(H)$. We have two cases:

(i) the elements g and h are linearly independent modulo A;

(ii) there exist $r, s \in \mathbb{N}$ such that $g^r h^s = 1$ modulo A.

Case (i): Let $F = F_2(\mathfrak{A}_n\mathfrak{A}) = \langle x, y\rangle$. Since $H \in \mathfrak{A}_n\mathfrak{A}$, there exists an epimorphism $f : F \to H$ with $f(x) = g$ and $f(y) = h$. Suppose that the kernel of f is non-trivial and let $u \in F \setminus \{1\}$ such that $f(u) = 1$. We can write u as $u = x^r y^s$ modulo F'. Thus, since $H' \leqslant A$, we have $1 = f(u) = g^r h^s$ modulo A. By our hypothesis this implies that $r = s = 0$ and $u \in F'$. The element $[x, y]$ generates F' as an $R_n[F/F']$-module and

$$u = \prod_{i=1}^{l} [x, y]^{n_j x^{r_i} y^{s_i}}$$

with $n_j \in R_n$, $r_i, s_i \in \mathbb{Z}$ and $(r_i, s_i) \neq (r_j, s_j)$ for $i \neq j$. Thus $1 = f(u) = [g, h]^{\sum_{i=1}^{l} n_j g^{r_i} h^{s_i}}$. Since g and h are linearly independent modulo A, the element $[g, h]$ generates A as a normal subgroup of H and hence $o([g, h]) = n$. This implies that $[g, h]$ has no $R_n[H/A]$-torsion and it follows that $\sum_{i=1}^{l} n_j g^{r_i} h^{s_i} = 0$ in $R_n[H/A]$. Since g and h are linearly independent modulo A, the elements $g^{r_i} h^{s_i}$ are distinct modulo A and hence $n_1 = \cdots = n_l = 0$ modulo n. Thus $u = 1$ and f is an isomorphism.

Case (ii): Since A is isolated in H, we may assume that r and s are non-zero and coprime. There exist $l, k \in \mathbb{Z}$ such that $lr + ks = 1$. If $\overline{g}$ and $\overline{h}$ are the images of g

and h in H/A then

$$\overline{g} = \overline{g}^{(lr+ks)} = (\overline{g}^r)^l(\overline{g}^s)^k = (\overline{g}^l\overline{h}^{(-k)})^r$$

and

$$\overline{h} = \overline{h}^{(lr+ks)} = (\overline{h}^r)^l(\overline{h}^s)^k = (\overline{g}^l\overline{h}^{(-k)})^{(-s)}.$$

Since $\overline{g}$ and $\overline{h}$ generate H/A and A is isolated in H, it follows that $H/A = \langle \overline{g}^l\overline{h}^{(-k)}\rangle$ is an infinite cyclic group. By Lemma 4.5.1, there exist elements u and v such that $H = \langle u, v\rangle$ with $u \in H \setminus A$ and $v \in A \setminus \{1\}$. It remains to check that $o(v) = n$. This follows immediately from the fact that v generates A as a normal subgroup of H. $\square$

Chapter 5

LEFT-ITERATED WREATH PRODUCTS OF TORSION-FREE ABELIAN GROUPS

Chapuis [11] proved using combinatorial arguments that a left-iterated restricted wreath product of torsion-free abelian groups has decidable universal theory. It is worth noting that, in contrast to the metabelian case, a non-cyclic free soluble group of derived length $d \geqslant 3$ is not universally equivalent to the left-iterated wreath product of d non-trivial torsion-free abelian groups; for $d = 3$ this can be seen as follows:

Remark 5.0.1. ([11, Remark 2.6]) The group $\mathrm{wr}_{i=1}^{3} A_i$ where A_i is a torsion-free abelian group can be distinguished from the group $F_2(\mathfrak{S}_3)$ by the sentence θ defined by

$$\exists x y_1 y_2 z_1 z_2 z_3 \Big(\big[[x, x^{z_1}], [x, x^{z_1}]^{z_2}\big] \neq 1 \wedge \\ \big[[y_1, y_2], [y_1, y_2]^{z_3}\big] \neq 1 \wedge \big[[y_1, y_2], [y_1, y_2]^{x}\big] = 1 \Big).$$

For completeness we include a proof of this remark.

Proof. Suppose for a contradiction that $F = F_2(\mathfrak{S}_3)$ satisfies θ, so that there exist elements $a \in F \setminus F'$ and $b = [b_1, b_2] \in F' \setminus F''$ such that $[b, b^a] = 1$. By Theorem 2.4.4, every pair of commuting elements in F which are not in F'' are in a cyclic subgroup. Thus there exist non-zero integers r and s such that $b^r = (b^a)^s$. From the Magnus embedding, we see that F'/F'' is torsion-free as a $\mathbb{Z}[F/F']$ module and it follows that b is in F'' or a is in F', contradicting our assumptions. Thus no such elements exist and we have the required contradiction.

A non-trivial torsion-free abelian group is universally equivalent to the infinite cyclic group $\mathbb{Z}$ [16, Lemma 1]. Since the restricted wreath product construction preserves universal equivalence [65, Theorem 2], a left-iterated wreath product of three non-trivial torsion-free abelian groups is universally equivalent to $\mathbb{Z}\operatorname{wr}(\mathbb{Z}\operatorname{wr}\mathbb{Z})$. It is straightforward to choose elements of $\mathbb{Z}\operatorname{wr}(\mathbb{Z}\operatorname{wr}\mathbb{Z})$ which demonstrate that $\mathbb{Z}\operatorname{wr}(\mathbb{Z}\operatorname{wr}\mathbb{Z}) \models \theta$. $\square$

Chapuis [12] comments that it would be interesting to obtain an explicit description of the universal theory of a left-iterated wreath product of torsion-free abelian groups. In this chapter we characterize those groups which satisfy the universal theory of a left-iterated wreath product of d non-trivial torsion-free abelian groups in two ways:

- in terms of the module-theoretic properties of the Fitting subgroup;
- in terms of the embeddability of finitely generated subgroups in certain left-iterated restricted wreath products.

We also recursively define a collection of universal sentences such that a group, which is not soluble of derived length $d-1$, satisfies the universal theory of a left-iterated wreath product of d non-trivial torsion-free abelian groups if and only if it satisfies these sentences.

5.1 Statement of theorems

The following definition is motivated by Chapuis' definition of ρ-groups in [12]. Our definition of ρ_2-groups is equivalent to that of ρ-groups.

Definition 5.1.1. (ρ_d-groups). A group G is a ρ_1-group if it is torsion-free and abelian. For $d \geqslant 2$, a group G is a ρ_d-group if the following conditions hold:

1. G is torsion-free and soluble of derived length at most d;
2. Every nilpotent subgroup of G is abelian;
3. $\operatorname{Fitt}(G)$ is torsion-free as a $\mathbb{Z}[G/\operatorname{Fitt}(G)]$-module;
4. $G/\operatorname{Fitt}(G)$ is a ρ_{d-1}-group.

In Section 5.2 below we define the collection $\mathfrak{W}_d$ of universal sentences.

Theorem 5.1.1. *Let S be a left-iterated wreath product of d non-trivial torsion-free abelian groups. If G is a group which is not in $\mathfrak{S}_{d-1}$ then the following conditions are equivalent:*

(1) $G \in \mathrm{ucl}(S)$;

(2) *G satisfies the universal sentences $\mathfrak{W}_d$;*

(3) *G is a ρ_d-group;*

(4) *Every finitely generated subgroup of G can be embedded in $\mathrm{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}$ for some $r_1, \ldots, r_d \in \mathbb{N}$.*

Theorem 5.1.2. *Let S be a left-iterated wreath product of d non-trivial torsion-free abelian groups. If G is a finitely generated group then the following conditions are equivalent:*

(1) $G \in \mathrm{ucl}(S)$;

(2) *G can be embedded in $\mathrm{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}$ for some $r_1, \ldots, r_d \in \mathbb{N}$;*

(3) *G is a ρ_d-group.*

We note to begin with that S is universally equivalent to $G_d = \mathrm{wr}_{i=1}^{d} A_i$ where each A_i is an infinite cyclic group: this is an immediate consequence of the fact that a torsion-free abelian group is universally equivalent to $\mathbb{Z}$ [16, Lemma 1] and the fact that the restricted wreath product preserves universal equivalence [65, Theorem 2].

5.2 The sentences

Let $G_d = \mathrm{wr}_{i=1}^{d} A_i$ be the left-iterated wreath product of d copies of the infinite cyclic group. We define recursively a set of universal sentences $\mathfrak{W}_d$ which hold in G_d. Recall from page 22 the definition of the words $s_i(\overline{x})$. We begin with some useful one-variable formulae.

We define a universal formula

$$\mathrm{fitt}^{\forall}(x) : \forall t\,([x, x^t] = 1)$$

and an existential formula

$$\mathrm{fitt}_d^{\exists}(x) : \exists t_1 \dots t_{2^{d-1}} \ \left(\left[s_{d-1}(t_1, \dots, t_{2^{d-1}}), x \right] = 1 \wedge s_{d-1}(t_1, \dots, t_{2^{d-1}}) \neq 1 \right).$$

The significance of the formulae $\mathrm{fitt}^{\forall}(x)$ and $\mathrm{fitt}_d^{\exists}(x)$ is as follows: if G is a model of $\mathfrak{W}_d$ then $\mathrm{Fitt}(G) = \mathrm{fitt}^{\forall}(G)$, and if $G \notin \mathfrak{S}_{d-1}$ is a model of $\mathfrak{W}_d$ then $\mathrm{Fitt}(G) \subseteq \mathrm{fitt}_d^{\exists}(G)$ (see Lemma 5.3.1 below).

For a finite sequence $\overline{e} = (e_1, \dots, e_m)$ on $\{-1, 1\}$ let $\mathcal{I}_{\overline{e}}$ be the set of partitions of $\{1, \dots, m\}$ into two-element sets $\{i, j\}$ such that $e_i = -e_j$.

Let $\mathfrak{W}_1$ denote the union of the following two sets of universal sentences:

(1) The single sentence ϕ_1^1 defined by

$$\forall xy \, ([x, y] = 1).$$

A group G satisfies ϕ_1^1 if and only if G is abelian.

(2) The countable collection of sentences $\{\phi_2(m) \mid m \in \mathbb{N}\}$, where $\phi_2(m)$ is defined by

$$\forall x \, (x^m = 1 \to x = 1).$$

A group G satisfies $\phi_2(m)$ for all $m \in \mathbb{N}$ if and only if G is torsion-free.

Let $\mathfrak{W}_d'$ denote the following four sets of universal sentences:

(1) The single sentence ϕ_1^d defined by

$$\forall x_1 \dots x_{2^d} \, (s_d(x_1, \dots, x_{2^d}) = 1).$$

A group G satisfies ϕ_1^d if and only if G is soluble of derived length at most d.

(2) The countable collection of sentences $\{\phi_2(m) \mid m \in \mathbb{N}\}$, where $\phi_2(m)$ is defined by

$$\forall x \, (x^m = 1 \to x = 1).$$

A group G satisfies $\phi_2(m)$ for all $m \in \mathbb{N}$ if and only if G is torsion-free.

(3) The single sentence ϕ_3 defined by

$$\forall xy \ \left(([x, y, x] = 1 \wedge [x, y, y] = 1) \to [x, y] = 1 \right).$$

If a group G satisfies ϕ_3 then every nilpotent subgroup of G is abelian.

(4) The countable collection of sentences $\{\phi_4^d(\overline{e}) \mid \overline{e} = (e_1, \ldots, e_m) \in \{-1,1\}^m, m \in \mathbb{N}\}$, where:

(a) if $\sum_{i=1}^m e_i \neq 0$ then $\phi_4^d(\overline{e})$ is defined by

$$\forall x y_1 \ldots y_m \left(\left(\mathrm{fitt}_d^{\exists}(x) \wedge x \neq 1 \wedge x^{\sum_{i=1}^m e_i y_i} = 1 \right) \to y_1 \neq y_1 \right);$$

(b) if $\sum_{i=1}^m e_i = 0$ then $\phi_4^d(\overline{e})$ is defined by

$$\forall x y_1 \ldots y_m \left(\left(\mathrm{fitt}_d^{\exists}(x) \wedge x \neq 1 \wedge x^{\sum_{i=1}^m e_i y_i} = 1 \right) \to \bigvee_{I \in \mathcal{I}_{\overline{e}}} \bigwedge_{\{i,j\} \in I} \mathrm{fitt}^{\forall}(y_i y_j^{-1}) \right).$$

Let G be a group in $\mathfrak{S}_d \setminus \mathfrak{S}_{d-1}$ such that every nilpotent subgroup of G is abelian. If G satisfies $\phi_4^d(\overline{e})$ for every finite sequence $\overline{e}$ on $\{-1,1\}$ and $G/\operatorname{Fitt}(G)$ is torsion-free then $\operatorname{Fitt}(G)$ is torsion-free as a $\mathbb{Z}[G/\operatorname{Fitt}(G)]$-module.

Next we consider the universal theory of the quotient of G_d by its Fitting subgroup; this quotient is isomorphic to G_{d-1}. We suppose that the collection of sentences $\mathfrak{W}_{d-1}$ has been defined and let ψ be in $\mathfrak{W}_{d-1}$. We may assume without loss of generality that ψ is of the form

$$\forall x_1 \ldots x_k \left(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} (u_{ij}(\overline{x}) = 1 \wedge w_{ij}(\overline{x}) \neq 1) \right),$$

where $u_{ij}(\overline{x})$ and $w_{ij}(\overline{x})$ are words in the variables $x_1, \ldots, x_k$. We define ψ_d^* by

$$\forall x_1 \ldots x_k \left(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} \left(\mathrm{fitt}^{\forall}(u_{ij}(\overline{x})) \wedge \neg\, \mathrm{fitt}_d^{\exists}(w_{ij}(\overline{x})) \right) \right).$$

Let $\mathfrak{W}_{d-1}^* = \{\psi_d^* | \psi \in \mathfrak{W}_{d-1}\}$ and let $\mathfrak{W}_d = \mathfrak{W}_d' \cup \{\phi_{\mathrm{gp}}\} \cup \mathfrak{W}_{d-1}^*$ (where ϕ_{gp} is the sentence defined on page 14).

Each of the sentences in $\mathfrak{W}_d$ contains only the universal quantifier $\forall$ when written in prenex normal form. As before we give some of the sentences in a more transparently comprehensible form than the prenex normal form. The sentences $\mathfrak{W}_2 \cup \{\phi_{\mathrm{gp}}\}$ correspond to the sentences $\mathfrak{A}$ used by Chapuis in dealing with $\forall$-free metabelian groups [12, §3]. There are some differences: we replace the sentence

$$\forall xyz\, (([x,y] = 1 \wedge [y,z] = 1 \wedge y \neq 1) \to ([x,z] = 1))$$

with the sentence ϕ_3 which allows us to use a slightly different existential formula for the Fitting subgroup; this formula adapts easily for use with higher values of d.

Lemma 5.2.1. *If $G_d = \mathrm{wr}_{i=1}^{d} A_i$, where $A_i \cong \mathbb{Z}$ for $i = 1, \ldots, d$, then*

$$G_d \models \mathfrak{W}_d.$$

Proof. We proceed by induction on d. For $d = 1$, the group $G_1 = \mathbb{Z}$ is a torsion-free abelian group and hence $G_1 \models \mathfrak{W}_1$.

Suppose that $G_{d-1} \models \mathfrak{W}_{d-1}$. It is easy to see that, for each positive integer k, there is a homomorphism from every finitely generated subgroup of G_k to an infinite cyclic group. By Lemma 3.1.1, the ring $\mathbb{Z}[G_{d-1}]$ is an integral domain and the base group D of the wreath product $G_d = \mathbb{Z} \operatorname{wr} G_{d-1}$ is torsion-free as a $\mathbb{Z}[G_{d-1}]$-module. Recall that G_d is isomorphic to $M(G_{d-1}, D)$ as defined in Section 3.3. It is easy to show, using the identities (3.3.1)–(3.3.4), that $M(G_{d-1}, D)$ satisfies $\mathfrak{W}'_d$ for each positive integer d. It remains to show that $G_d \models \mathfrak{W}^*_{d-1}$. The identities (3.3.1)–(3.3.4) also show that $\mathrm{fitt}^{\forall}(G_d) = D = \mathrm{fitt}^{\exists}_d(G_d)$. The result follows immediately from this and the fact that $G_d/D \cong G_{d-1} \models \mathfrak{W}_{d-1}$. □

5.3 ρ_d-groups

In this section we establish the relationship between the ρ_d-groups defined in Section 5.1 and the sentences $\mathfrak{W}_d$ defined in the previous section. This relationship plays a key role in our proof of Theorem 5.1.1.

In Lemma 5.3.1 we explain the significance of the universal formula $\mathrm{fitt}^{\forall}(x)$ and the existential formula $\mathrm{fitt}^{\exists}_d(x)$. These formulae are important in allowing us to describe, using universal sentences, the Fitting subgroup $\mathrm{Fitt}(G)$ of a group G and the quotient group $G/\mathrm{Fitt}(G)$.

Lemma 5.3.1. *If $G \notin \mathfrak{S}_{d-1}$ satisfies the sentences $\mathfrak{W}_d$ then $\mathrm{Fitt}(G) = \mathrm{fitt}^{\forall}(G)$ and $\mathrm{Fitt}(G) \subseteq \mathrm{fitt}^{\exists}_d(G)$.*

Proof. By the same argument as in Lemma 4.3.1, we have $\mathrm{Fitt}(G) = \mathrm{fitt}^{\forall}(G)$.

Let g be an element of $\mathrm{Fitt}(G)$. Since G is soluble of derived length d, the subgroup $G^{(d-1)} \lhd G$ is abelian, and hence $G^{(d-1)} \leqslant \mathrm{Fitt}(G)$. Since G is not in $\mathfrak{S}_{d-1}$, there exist $h_1, \ldots, h_{2^{d-1}} \in G$ such that $h = s_{d-1}(h_1, \ldots, h_{2^{d-1}}) \neq 1$. The elements h and g are in $\mathrm{Fitt}(G)$, which is abelian, and so $[h, g] = 1$. Hence $\mathrm{Fitt}(G) \subseteq \mathrm{fitt}^{\exists}_d(G)$. □

In Lemmas 5.3.2 and 5.3.3 we show that, for a group $G \notin \mathfrak{S}_{d-1}$, the statements that G is a ρ_d-group and that G satisfies the sentences $\mathfrak{W}_d$ are equivalent.

Lemma 5.3.2. *If $G \notin \mathfrak{S}_{d-1}$ satisfies the sentences $\mathfrak{W}_d$ then G is a ρ_d-group.*

Proof. We proceed by induction on d. If $G \models \mathfrak{W}_1$ then G is a torsion-free abelian group and hence a ρ_1-group. Suppose now that, whenever $G \notin \mathfrak{S}_{d-2}$ satisfies the sentences $\mathfrak{W}_{d-1}$, the group G is a ρ_{d-1}-group. Let $G \notin \mathfrak{S}_{d-1}$ satisfy the sentences $\mathfrak{W}_d$. The validity of the sentences ϕ_1^d and $\phi_2(m)$ for all $m \in \mathbb{N}$ implies that G is a torsion-free soluble group of derived length at most d. Since $\operatorname{Fitt}(G) \subseteq \operatorname{fitt}_d^{\exists}(G)$ and $\operatorname{Fitt}(G) = \operatorname{fitt}^{\forall}(G)$, the truth of the sentence $\phi_4^d(\overline{e})$ for every finite sequences $\overline{e}$ on $\{-1, 1\}$ implies that $\operatorname{Fitt}(G)$ is torsion-free as a $\mathbb{Z}[G/\operatorname{Fitt}(G)]$-module. The validity of the sentences $\mathfrak{W}_{d-1}^*$ implies that $G/\operatorname{Fitt}(G) \models \mathfrak{W}_{d-1}$. Since G is not in $\mathfrak{S}_{d-1}$, the quotient $G/\operatorname{Fitt}(G)$ is not in $\mathfrak{S}_{d-2}$ and it follows by induction that $G/\operatorname{Fitt}(G)$ is a ρ_{d-1}-group. □

Lemma 5.3.3. *If G is a ρ_d-group then $G \models \mathfrak{W}_d$.*

Proof. Once more we proceed by induction on d. A ρ_1-group is torsion-free and abelian and hence satisfies the sentences $\mathfrak{W}_1$. Suppose that, whenever G is a ρ_{d-1}-group, G satisfies $\mathfrak{W}_{d-1}$, and let G be a ρ_d-group. By the definition of a ρ_d-group, G is torsion-free soluble of derived length at most d and every nilpotent subgroup of G is abelian. Hence the sentences ϕ_1^d, $\phi_2(m)$ for all $m \in \mathbb{N}$, and ϕ_3 hold in G. Since every nilpotent subgroup of G is abelian it follows that $\operatorname{Fitt}(G)$ is abelian, and $\operatorname{Fitt}(G) = \operatorname{fitt}^{\forall}(G)$.

If g is an element of $\operatorname{fitt}_d^{\exists}(G)$ then there exist elements $h_1, \ldots, h_{2^{d-1}} \in G$ such that

$$h = s_{d-1}(h_1, \ldots, h_{2^{d-1}}) \neq 1$$

and $[h, g] = 1$. The element h is in $G^{(d-1)}$ and $G^{(d-1)} \leqslant \operatorname{Fitt}(G)$ which is torsion-free as a $\mathbb{Z}[G/\operatorname{Fitt}(G)]$-module. Since $h \neq 1$, the equality $1 = [h, g] = h^{(-1+g)}$ implies that $g \in \operatorname{Fitt}(G)$, and hence $\operatorname{fitt}_d^{\exists}(G) \subseteq \operatorname{Fitt}(G)$. It follows that $G \models \phi_4^d(\overline{e})$ for every finite sequence $\overline{e}$ on $\{-1, 1\}$.

The quotient $G/\operatorname{Fitt}(G)$ is a ρ_{d-1}-group and hence, by induction, $G/\operatorname{Fitt}(G) \models \mathfrak{W}_{d-1}$. Since $\operatorname{fitt}^{\forall}(G) = \operatorname{Fitt}(G) \supseteq \operatorname{fitt}^{\exists}(G)$, this implies that $G \models \mathfrak{W}_{d-1}^*$. □

5.4 Proof of theorems

In this section we prove the results stated in Section 5.1. For the reader's convenience we restate each theorem before proving it.

Theorem 5.1.1 *Let S be a left-iterated wreath product of d non-trivial torsion-free abelian groups. If G is a group which is not in $\mathfrak{S}_{d-1}$ then the following conditions are equivalent:*

(1) $G \in \mathrm{ucl}(S)$;

(2) *G satisfies the universal sentences $\mathfrak{W}_d$;*

(3) *G is a ρ_d-group;*

(4) *Every finitely generated subgroup of G can be embedded in $\mathrm{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}$ for some $r_1, \ldots, r_d \in \mathbb{N}$.*

Proof. We prove the implications (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2)**.** The group S is universally equivalent to $G_d = \mathrm{wr}_{i=1}^{d} A_i$ where each A_i is an infinite cyclic group; thus G is in $\mathrm{ucl}(S) = \mathrm{ucl}(G_d)$. By Lemma 5.2.1, we have $G_d \models \mathfrak{W}_d$ and hence $G \models \mathfrak{W}_d$.

(2) $\Longleftrightarrow$ (3)**.** The equivalence of (2) and (3) follows immediately from the results of Section 5.3.

(3) $\Longrightarrow$ (4)**.** We proceed by induction on d. The group G is a ρ_1-group if and only if it is torsion-free and abelian. Hence any finitely generated subgroup of G embeds in $\mathbb{Z}^{(r)}$ for some $r \in \mathbb{N}$.

Let G be a ρ_d-group, so that $G \models \mathfrak{W}_d$, and suppose that the hypothesis holds for ρ_{d-1}-groups. Since G is not in $\mathfrak{S}_{d-1}$, there exist elements $g_1, \ldots, g_{2^{d-1}} \in G$ such that $s_{d-1}(g_1, \ldots, g_{2^{d-1}}) \neq 1$. Let H be a finitely generated subgroup of G and let $H_1 = \langle H, g_1, \ldots, g_{2^{d-1}} \rangle$. Since $\mathfrak{W}_d$ is a set of universal sentences, every subgroup of G satisfies $\mathfrak{W}_d$; in particular, $H_1 \models \mathfrak{W}_d$ and H_1 is a ρ_d-group by Lemma 5.3.2. The quotient $\overline{H}_1 = H_1/\operatorname{Fitt}(H_1)$ is ρ_{d-1}-group and, by induction, $\overline{H}_1$ embeds in $K = \mathrm{wr}_{i=2}^{m} \mathbb{Z}^{(r_i)}$ for some $r_2, \ldots, r_m \in \mathbb{N}$. By the argument used in Lemma 5.2.1, the ring $\mathbb{Z}[K]$, and hence the ring $\mathbb{Z}[\overline{H}_1]$, are integral domains; thus $\mathbb{Z}[\overline{H}_1]$ is an Ore domain by Lemma 3.1.2. The group H_1 is an extension of $\mathrm{Fitt}(H_1)$ by $\overline{H}_1$ satisfying the conditions of Lemma 3.3.1; thus H_1 embeds in $U \operatorname{wr} \overline{H}_1$, where U is a finitely generated torsion-free abelian group. It follows that H_1 embeds in $\mathbb{Z}^{(r_1)} \operatorname{wr} \overline{H}_1$ for some $r_1 \in \mathbb{N}$, which in turn embeds in $\mathbb{Z}^{(r_1)} \operatorname{wr} K$. Finally, since $H \leqslant H_1$, it follows that H embeds in $\mathbb{Z}^{(r_1)} \operatorname{wr} K$ as required.

(4) $\implies$ (1). Let ϕ be in $\mathrm{Th}_\forall(S)$ and suppose without loss of generality that ϕ is of the form

$$\forall x_1 \dots x_m \, \varphi(x_1, \dots, x_m),$$

where $\varphi(x_1, \dots, x_m)$ is quantifier-free. We let $g_1 \dots, g_m$ be elements of G and let $H = \langle g_1, \dots, g_m \rangle$, so that H embeds in $\operatorname{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}$ for some $r_1, \dots, r_d \in \mathbb{N}$. Since the restricted wreath product preserves universal equivalence, $\mathrm{Th}_\forall\left(\operatorname{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}\right) = \mathrm{Th}_\forall(G_d) = \mathrm{Th}_\forall(S)$; in particular, $\operatorname{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}$ satisfies ϕ and $\varphi(g_1, \dots, g_m)$ holds. This is true for all m-tuples in G and hence G satisfies ϕ. Thus $G \models \mathrm{Th}_\forall(G_d)$ and $G \in \mathrm{ucl}(G_d)$. $\square$

Before proving Theorem 5.1.2, we state and prove the following consequence of Lemma 3.3.4 which we require.

Lemma 5.4.1. *Let G be a finitely generated subgroup of $\operatorname{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}$ with $d \geqslant 2$. If G is in $\mathfrak{S}_m$ for some $m \leqslant d$ then G embeds in $\operatorname{wr}_{i=1}^{m} \mathbb{Z}^{(s_i)}$ for some $s_1, \dots, s_m \in \mathbb{N}$.*

Proof. If $m = d$ then the result is trivial. Suppose therefore that $m \leqslant d-1$ and proceed by induction on d. If $d = 2$ and G is in $\mathfrak{S}_{d-1}$ then G is a finitely generated torsion-free abelian group and hence G is either trivial or isomorphic to $\mathbb{Z}^{(r)}$ for some $r \in \mathbb{N}$. Suppose now that the result holds for $d-1$ and let G be a finitely generated subgroup of $\operatorname{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}$ which is in $\mathfrak{S}_m$ for some $m < d$. By Lemma 3.3.4, we have two cases to consider: either G embeds in $\operatorname{wr}_{i=2}^{d} \mathbb{Z}^{(r_i)}$ and the result follows by induction, or G embeds in $\mathbb{Z}^{(s_1)} \operatorname{wr} K$, where K is a finitely generated subgroup of $\operatorname{wr}_{i=2}^{d} \mathbb{Z}^{(r_i)}$ such that $K \in \mathfrak{S}_{m-1}$. In the latter case, it follows by induction that K embeds in $\operatorname{wr}_{i=2}^{m} \mathbb{Z}^{(s_i)}$ and G embeds in $\mathbb{Z}^{(s_1)} \operatorname{wr} K \leqslant \operatorname{wr}_{i=1}^{m} \mathbb{Z}^{(s_i)}$ $\square$

Theorem 5.1.2 *Let S be a left-iterated wreath product of d non-trivial torsion-free abelian groups. If G is a finitely generated group then the following conditions are equivalent:*

(1) *$G \in \mathrm{ucl}(S)$;*

(2) *G can be embedded in $\mathrm{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}$ for some $r_1, \ldots, r_d \in \mathbb{N}$;*

(3) *G is a ρ_m-group for some $m \leqslant d$.*

Proof. (1) $\iff$ (2)**.** If G embeds in $W = \mathrm{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}$ for some $r_1, \ldots, r_d \in \mathbb{N}$ then W is universally equivalent to S and hence $G \models \mathrm{Th}_\forall(S)$; thus $G \in \mathrm{ucl}(S)$.

If G is in $\mathrm{ucl}(S)$ then G embeds in some ultrapower $\mathcal{G}$ of S by Lemma 2.6.3 and $\mathcal{G}$ is elementarily equivalent to S by Łoś' Theorem (see Theorem 2.6.1); thus $\mathcal{G}$ is not in $\mathfrak{S}_{d-1}$. It follows from Theorem 5.1.1 that every finitely generated subgroup of $\mathcal{G}$, and in particular G, can be embedded in a restricted wreath product $\mathrm{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}$ for some $r_1, \ldots, r_d \in \mathbb{N}$.

(2) $\iff$ (3)**.** Suppose that G embeds in $\mathrm{wr}_{i=1}^{d} \mathbb{Z}^{(r_i)}$ for some $r_1, \ldots, r_d \in \mathbb{N}$ and let $m \leqslant d$ be the least integer such that G is in $\mathfrak{S}_m$. By Lemma 5.4.1, the group G embeds in $\mathrm{wr}_{i=1}^{m} \mathbb{Z}^{(s_i)}$ for some $s_1, \ldots, s_m \in \mathbb{N}$ and hence Theorem 5.1.1 implies that G is a ρ_m-group.

Now let G be a ρ_m-group for some $m \leqslant d$. The quotient $\overline{G} = G/\operatorname{Fitt}(G)$ is a ρ_{m-1}-group and G is an extension of $\operatorname{Fitt}(G)$ by $\overline{G}$. As in the proof of Theorem 5.1.1, the group G can be embedded in a restricted wreath product as required. $\square$

Chapter 6

RESTRICTED WREATH PRODUCTS OF AN ABELIAN GROUP OF SQUARE-FREE OR ZERO EXPONENT BY A TORSION-FREE NILPOTENT GROUP

The universal theory of nilpotent groups is a difficult subject: Roman′kov [55] proved that the decidability of the universal theory of a free nilpotent group of class 2 and rank 2 is equivalent to the decidability of the universal theory of the field of rationals. As discussed in Chapter 1, the latter is a very hard open problem which seems likely to have a negative answer.

Let $U \operatorname{wr} H$ be a restricted wreath product of an abelian group U, which is either torsion-free or of finite square-free exponent, by a finitely generated torsion-free nilpotent group H. In this chapter we consider the universal theory of such a wreath product modulo the "difficult" universal theory of the group H. We characterize those groups which satisfy the universal theory of $U \operatorname{wr} H$ from three distinct points of view:

- in terms of the module-theoretic properties of the Fitting subgroup and the universal theory of the quotient group by the Fitting subgroup;
- as those groups satisfying a certain collection of universal sentences;
- in terms of the embeddability of the finitely generated subgroups in certain restricted wreath products.

This characterization provides a method of proving that $U \operatorname{wr} H$ has decidable universal theory, provided that there exists a recursive collection of sentences describing the universal and existential theory of H. In [11] Chapuis proved that if H has decidable universal theory and U is torsion-free then $U \operatorname{wr} H$ has decidable universal theory. Our method, which uses ultraproducts, avoids the combinatorial arguments of his proof. Our method also has the advantage that it is applicable when the bottom group U is of finite square-free exponent as well as when U is torsion-free. Our results here extend those of Chapuis in [12] for the restricted wreath product $\mathbb{Z} \operatorname{wr} \mathbb{Z}$.

Finally in this chapter we consider some examples where we can explicitly determine when a group in the universal closure of H is universally equivalent to H, and hence when a group in the universal closure of $U \operatorname{wr} H$ is universally equivalent to $U \operatorname{wr} H$. For example, if $H = F_2(\mathfrak{N}_2)$ then, for $G \in \operatorname{ucl}(H)$, the group G is universally equivalent to H if and only if G contains a non-abelian nilpotent subgroup. Using this information we are able to construct the lattice of universally closed classes contained in $\operatorname{ucl}(U \operatorname{wr} H)$.

Throughout the remainder of this chapter let n be a positive square-free integer or zero and let c be a non-negative integer.

6.1 Statement of theorem

We begin by defining the notion of a $\rho_{n,H}$-group for a torsion-free nilpotent group H of class c. When $H = \mathbb{Z}$ this reduces to the notion of ρ-groups in the terminology of [12] when $n = 0$ and to the notion of σ_n-groups in the terminology of Chapter 4 when $n \neq 0$.

Definition 6.1.1. ($\rho_{n,H}$-groups). Let H be a finitely generated torsion-free nilpotent group of class c. If $c = 0$ then a group G is a $\rho_{n,H}$-group if G is abelian and of exponent n. If $c \geqslant 1$ then a group G is a $\rho_{n,H}$-group if the following conditions hold:

1. $\gamma_{c+1}(G)$ is abelian;
2. $\operatorname{Fitt}(G)$ is abelian and
   (a) if $n = 0$ then G is torsion-free;
   (b) if $n \neq 0$ then $\operatorname{Fitt}(G)$ is of exponent dividing n;
3. $G/\operatorname{Fitt}(G)$ satisfies the universal theory of H;
4. If G_1 is a nilpotent subgroup of G then either G_1 is abelian of exponent n or G_1 satisfies the universal theory of H;

5. (a) If $n = 0$ then $\mathrm{Fitt}(G)$ is torsion-free as a $\mathbb{Z}[G/\mathrm{Fitt}(G)]$-module;

   (b) If $n \neq 0$ then, for each prime divisor p of n, the submodule of $\mathrm{Fitt}(G)$ of elements of order p is torsion-free as an $\mathbb{F}_p[G/\mathrm{Fitt}(G)]$-module.

In Section 6.2 below we shall define the collection $\mathfrak{W}_{n,H}$ of universal sentences.

Theorem 6.1.1. *Let H be a torsion-free nilpotent group of class c and let U be a non-trivial abelian group of exponent n. If G is a group then the following conditions are equivalent:*

(1) $G \in \mathrm{ucl}\big(U \operatorname{wr} H\big)$;

(2) *G satisfies the universal sentences $\mathfrak{W}_{n,H}$;*

(3) *G is a $\rho_{n,H}$-group or $G \models \mathrm{Th}_{\forall}(H)$;*

(4) *Every finitely generated subgroup of G can be embedded in $C_n^{(r)} \operatorname{wr} K$ for some $r \in \mathbb{N}$ and some K such that $K \models \mathrm{Th}_{\forall}(H)$.*

6.2 The sentences

Let G be a group and let H be a torsion-free nilpotent group of class c. We construct sets of universal sentences $\mathfrak{W}_{n,H}$ in a number of stages.

We begin with some useful one-variable formulae. We define a universal formula

$$\mathrm{fitt}^{\forall}(x) : \forall t\,([x, x^t] = 1)$$

and an existential formula

$$\mathrm{fitt}_c^{\exists}(x) : \exists t_1 \ldots t_{c+1} \Big(\big[[t_1, \ldots, t_{c+1}], \underbrace{x, \ldots, x}_{c}\big] = 1 \wedge [t_1, \ldots, t_{c+1}] \neq 1 \Big).$$

The significance of the formulae $\mathrm{fitt}^{\forall}(x)$ and $\mathrm{fitt}_c^{\exists}(x)$ is as follows: if G is a model of $\mathfrak{W}_{n,H}$ then $\mathrm{Fitt}(G) = \mathrm{fitt}^{\forall}(G)$ and if G is a non-nilpotent model of $\mathfrak{W}_{n,H}$ then $\mathrm{Fitt}(G) \subseteq \mathrm{fitt}_c^{\exists}(G)$ (see Lemmas 6.3.2 and 6.3.3 below).

For each $l, m \in \mathbb{N}$, we define an l-variable quantifier-free formula,

$$\beta_l^m(x_1, \ldots, x_l) : \bigwedge_{\substack{y_1,\ldots,y_{m+1} \\ \in\{x_1,\ldots,x_l\}}} [y_1, \ldots, y_{m+1}] = 1.$$

Let $g_1, \ldots, g_l$ be group elements and let $G = \langle g_1, \ldots, g_l \rangle$. The group G is nilpotent of class at most m if and only if $\beta_l^m(g_1, \ldots, g_l)$ holds.

For each finite sequence $\overline{e} = (e_1, \ldots, e_m)$ on $\{-1, 1\}$, we denote by $\mathcal{I}_{\overline{e}}$ the set of partitions of $\{1, \ldots, m\}$ into two-element sets $\{i, j\}$ such that $e_i = -e_j$. For each positive integer m and each prime p, we denote by $\mathcal{I}_{m,p}$ the set of partitions of $\{1, \ldots, m\}$ into sets X of size a multiple of p.

We give sentences $\mathfrak{W}_{n,c}$ which hold in $C_n \operatorname{wr} H$, where H is any finitely generated torsion-free nilpotent group of class c (see Lemma 6.2.1 below).

$c = 0$**.** We denote by $\mathfrak{W}_{n,0}$ the union of the following two sets of universal sentences together with the sentence ϕ_{gp} as defined on page 14:

(1) The single sentence ϕ_1^0 defined by

$$\forall xy\,([x, y] = 1).$$

A group G satisfies ϕ_1^0 if and only if G is abelian.

(2) (a) If $n = 0$, the countable collection of sentences $\{\phi_2^0(m) \mid m \in \mathbb{N}\}$, where $\phi_2^0(m)$ is defined by

$$\forall x\,(x^m = 1 \to x = 1).$$

A group G satisfies $\phi_2^0(m)$ for all $m \in \mathbb{N}$ if and only if G is torsion-free.

(b) If $n \neq 0$, the single sentence $\phi_2^{n,0}$ defined by

$$\forall x\,(x^n = 1).$$

A group G satisfies $\phi_2^{n,0}$ if and only if G is of exponent dividing n.

$c \geqslant 1$. We denote by $\mathfrak{W}_{n,c}$ the union of the following five sets of universal sentences together with the sentence ϕ_{gp} as defined on page 14:

(1) The single sentence ϕ_1^c defined by

$$\forall x_1 \dots x_{c+1}, y_1 \dots y_{c+1} \left(\big[[x_1, \dots, x_{c+1}], [y_1, \dots, y_{c+1}]\big] = 1 \right).$$

A group G satisfies ϕ_1^c if and only if $\gamma_{c+1}(G)$ is abelian.

(2) The single sentence ϕ_2^c defined by

$$\forall x_1 \dots x_{c+1} \left(\left(\bigwedge_{j=1}^{c+1} [x_1, \dots, x_{c+1}, x_j] = 1 \right) \to [x_1, \dots, x_{c+1}] = 1 \right).$$

If a group G satisfies ϕ_2^c then every nilpotent subgroup of G has class $\leqslant c$.

(3) (a) If $n = 0$, the countable collection of sentences $\left\{\phi_3^0(m) \mid m \in \mathbb{N}\right\}$, where $\phi_3^0(m)$ is defined by

$$\forall x \, (x^m = 1 \to x = 1).$$

A group G satisfies $\phi_3^0(m)$ for all $m \in \mathbb{N}$ if and only if G is torsion-free.

(b) If $n \neq 0$, the single sentence $\phi_3^{n,c}$ defined by

$$\forall x \, (\mathrm{fitt}_c^{\exists}(x) \to x^n = 1).$$

Let G be a non-nilpotent group such that $\gamma_{c+1}(G)$ is abelian and every nilpotent subgroup of G has class at most c. If G satisfies $\phi_3^{n,c}$ then $\mathrm{Fitt}(G)$ has exponent dividing n.

(4) The single sentence ϕ_4^c defined by

$$\forall xy \left(\left(\mathrm{fitt}_c^{\exists}(x) \wedge \mathrm{fitt}_c^{\exists}(y) \right) \to [x, y] = 1 \right).$$

Let G be a non-nilpotent group such that $\gamma_{c+1}(G)$ is abelian and every nilpotent subgroup of G has class at most c. If G satisfies ϕ_4^c then $\mathrm{Fitt}(G)$ is abelian.

(5) (a) If $n = 0$, the countable collection of sentences

$$\left\{ \phi_5^{0,c}(\overline{e}) \mid \overline{e} = (e_1, \dots, e_m) \in \{-1, 1\}^m, m \in \mathbb{N} \right\},$$

where

i. if $\sum_{j=1}^m e_j \neq 0$ then $\phi_5^{0,c}(\overline{e})$ is defined by

$$\forall x y_1 \dots y_m \left(\left(\mathrm{fitt}_c^{\exists}(x) \wedge x \neq 1 \wedge x^{\sum_{j=1}^m e_j y_j} = 1 \right) \to y_1 \neq y_1 \right);$$

ii. if $\sum_{j=1}^{m} e_j = 0$ then $\phi_5^{0,c}(\overline{e})$ is defined by

$$\forall x y_1 \dots y_m \Bigg(\Big(\mathrm{fitt}_c^{\exists}(x) \wedge x \neq 1 \wedge x^{\sum_{j=1}^{m} e_j y_j} = 1 \Big) \rightarrow \bigvee_{I \in \mathcal{I}_{\overline{e}}} \bigwedge_{\{j_1, j_2\} \in I} \mathrm{fitt}^{\forall}(y_{j_1} y_{j_2}^{-1}) \Bigg).$$

Let G be a non-nilpotent group such that $\gamma_{c+1}(G)$ is abelian and every nilpotent subgroup of G has class at most c. If G satisfies $\phi_5^{0,c}(\overline{e})$ for every finite sequence $\overline{e}$ on $\{-1, 1\}$ and $G/\operatorname{Fitt}(G)$ is torsion-free, then $\operatorname{Fitt}(G)$ is torsion-free as a $\mathbb{Z}[G/\operatorname{Fitt}(G)]$-module.

(b) If $n \neq 0$, for each prime divisor p of n, the countable collection of sentences $\{\phi_5^{p,c}(m) \mid m \in \mathbb{N}\}$, where

i. if $m \not\equiv 0 \bmod p$ then $\phi_5^{p,c}(m)$ is defined by

$$\forall x y_1 \dots y_m \left(\left(x^p = 1 \wedge x \neq 1 \wedge x^{\sum_{j=1}^{m} y_j} = 1 \right) \rightarrow y_1 \neq y_1 \right);$$

ii. if $m \equiv 0 \bmod p$ then $\phi_5^{p,c}(m)$ is defined by

$$\forall x y_1 \dots y_m \Bigg(\Big(x^p = 1 \wedge x \neq 1 \wedge x^{\sum_{j=1}^{m} y_j} = 1 \Big) \rightarrow \bigvee_{I \in \mathcal{I}_{m,p}} \bigwedge_{X \in I} \bigwedge_{j_1, j_2 \in X} \mathrm{fitt}^{\forall}(y_{j_1} y_{j_2}^{-1}) \Bigg).$$

Let G be a non-nilpotent group such that $\gamma_{c+1}(G)$ is abelian and every nilpotent subgroup of G has class at most c. If G satisfies $\phi_5^{p,c}(m)$ for all $m \in \mathbb{N}$ and $G/\operatorname{Fitt}(G)$ is torsion-free then the submodule of $\operatorname{Fitt}(G)$ of elements of order p is a torsion-free $\mathbb{F}_p[G/\operatorname{Fitt}(G)]$-module.

The required sets of sentences have become much more complicated than in previous chapters: as well as taking into account the possibly finite exponent of the Fitting subgroup, we also have to deal with the fact that, although all nilpotent normal subgroups of $C_n \operatorname{wr} H$ are abelian, when $c \geqslant 2$ the wreath product $C_n \operatorname{wr} H$ does contain non-abelian nilpotent subgroups, in particular the group H itself.

Lemma 6.2.1. *If H is a finitely generated torsion-free nilpotent group of class c then*

$$C_n \operatorname{wr} H \models \mathfrak{W}_{n,c}.$$

Proof. Let D be the base group of the wreath product. It follows from Lemma 2.3.1 that $\operatorname{fitt}^{\exists}(C_n \operatorname{wr} H) = \operatorname{Fitt}(C_n \operatorname{wr} H) = D$ and it is trivial to show that $\operatorname{fitt}^{\forall}(C_n \operatorname{wr} H) = D$.

Since H is a finitely generated, torsion-free, nilpotent group, the group rings $\mathbb{Z}[H]$ and $\mathbb{F}_p[H]$, where p is a prime, are integral domains by Lemma 3.1.1. If $n = 0$ then, since D is a free $\mathbb{Z}[H]$-module, D is torsion-free as a $\mathbb{Z}[H]$-module. If $n \neq 0$ then, since D is an abelian group of square-free exponent n, the base group D is a direct product of groups D_p of exponent p, where each p is a prime divisor of n. Since D is a free $R_n[H]$-module, D_p is a free $\mathbb{F}_p[H]$-module, for each prime p, and hence D_p is torsion-free as an $\mathbb{F}_p[H]$-module. It is easy to show, using the identities (3.3.1)–(3.3.4), that $M(H, D)$ satisfies $\mathfrak{W}_{n,c}$. $\square$

Next we consider the universal theory of a particular group H: let H be a finitely generated torsion-free nilpotent group of class c and let $\mathfrak{U}_H \subseteq \mathrm{Th}_{\forall}(H)$ be a set of sentences such that G satisfies the universal theory of H if and only if $G \models \mathfrak{U}_H$. If H is abelian then $G \models \mathrm{Th}_{\forall}(H)$ if and only if it is torsion-free and abelian; thus the collection of sentences $\mathfrak{U}_H$ can be easily written down. This is the situation dealt with by Chapuis [12] when $n = 0$ and in Chapter 4 when $n \neq 0$. We define a collection of universal sentences $\mathfrak{U}^+_{n,H}$ such that if G_1 is any nilpotent subgroup of G and $G \models \mathfrak{W}_{n,c} \cup \mathfrak{U}^+_{n,H}$ then either $G_1 \models \mathfrak{U}_H$ or G_1 is abelian of exponent dividing n (see Lemma 6.3.4 below).

For each $\psi \in \mathfrak{U}_H$, of the form $\forall x_1 \ldots x_m \, \varphi(x_1, \ldots, x_m)$, where $\varphi(x_1, \ldots, x_m)$ is quantifier-free, we define ψ^+_n in the following way:

(a) if $n = 0$ we define ψ^+_0 by

$$\forall x_1 \ldots x_m \left(\beta^c_m(x_1, \ldots, x_m) \to \varphi(x_1, \ldots, x_m)\right);$$

(b) if $n \neq 0$ we define ψ^+_n by

$$\forall x_1 \ldots x_m y_1 y_2 \Bigg(\beta^c_{m+2}(x_1, \ldots, x_m, y_1, y_2) \to \Big(\big([y_1, y_2] = 1 \wedge y_1^n = 1 \wedge y_2^n = 1 \big) \vee \varphi(x_1, \ldots, x_m) \Big) \Bigg).$$

Let

$$\mathfrak{U}^+_{n,H} = \{ \psi^+_n \mid \psi \in \mathfrak{U}_H \}.$$

Lemma 6.2.2. *If H is a non-trivial finitely generated torsion-free nilpotent group of class c then*

$$C_n \operatorname{wr} H \models \mathfrak{U}^+_{n,H}.$$

Proof. Suppose to begin with that $n = 0$. Let ψ^+_n be a sentence in $\mathfrak{U}^+_{0,H}$, so that ψ^+_n is of the form

$$\forall x_1 \dots x_m \left(\beta^c_m(x_1, \dots, x_m) \to \varphi(x_1, \dots, x_m)\right).$$

Let $g_1, \dots, g_m$ be elements of $\mathbb{Z} \operatorname{wr} H$ such that $\beta^c_m(g_1, \dots, g_m)$ holds and let $G = \langle g_1, \dots, g_m \rangle$. Thus G is nilpotent, and it follows from Lemma 3.3.5 that either G embeds in H or G is a torsion-free abelian group. If G embeds in H then G satisfies the universal theory of H. Otherwise G is torsion-free abelian and it satisfies the universal theory of $\mathbb{Z}$ (see [16, Lemma 1]) which can be embedded in any non-trivial torsion-free group. Again G satisfies the universal theory of H. Since $H \models \mathfrak{U}_H$, it follows that $G \models \mathfrak{U}_H$ and $\varphi(g_1, \dots, g_m)$ holds. Hence $\mathbb{Z} \operatorname{wr} H$ satisfies ψ^+ for each $\psi^+ \in \mathfrak{U}^+_H$.

Suppose now that $n \neq 0$. Let ψ^+_n be a sentence in $\mathfrak{U}^+_{n,H}$, so that ψ^+_n is of the form

$$\forall x_1 \dots x_m y_1 y_2 \Bigg(\beta^c_{m+2}(x_1, \dots, x_m, y_1, y_2) \to \\ \Big(\Big([y_1, y_2] = 1 \wedge y_1^n = 1 \wedge y_2^n = 1\Big) \vee \varphi(x_1, \dots, x_m)\Big)\Bigg).$$

Let $g_1, \dots, g_m, h_1, h_2$ be elements of $C_n \operatorname{wr} H$ such that $\beta^c_{m+2}(g_1, \dots, g_m, h_1, h_2)$ holds and let $G = \langle g_1, \dots, g_m, h_1, h_2 \rangle$. Thus G is nilpotent, and it follows from Lemma 3.3.5 that either G embeds in H or G is an abelian group of exponent n. In the former case G satisfies the universal theory of H. Since $H \models \mathfrak{U}_H$, it follows that $G \models \mathfrak{U}_H$ and $\varphi(g_1, \dots, g_m)$ holds. In the latter case $[h_1, h_2] = 1$, $h_1^n = 1$ and $h_2^n = 1$. Hence $C_n \operatorname{wr} H$ satisfies ψ^+_n for each $\psi^+_n \in \mathfrak{U}^+_{n,H}$. □

Finally we define a collection of universal sentences $\mathfrak{U}^*_H$ such that if G is not nilpotent and $G \models \mathfrak{W}_{n,c} \cup \mathfrak{U}^*_H$ then $G/\operatorname{Fitt}(G) \models \mathfrak{U}_H$ (see Lemma 6.3.5 below). For each ψ in $\mathfrak{U}_H$ we may assume, without loss of generality, that ψ is of the form

$$\forall x_1 \dots x_k \left(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} (u_{ij}(\overline{x}) = 1 \wedge w_{ij}(\overline{x}) \neq 1)\right),$$

where $u_{ij}(\overline{x})$ and $w_{ij}(\overline{x})$ are words in the variables $x_1, \ldots, x_k$. We define ψ_c^* by

$$\forall x_1 \ldots x_k \left(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} \left(\mathrm{fitt}^{\forall}(u_{ij}(\overline{x})) \wedge \neg\, \mathrm{fitt}_c^{\exists}(w_{ij}(\overline{x})) \right) \right)$$

and let

$$\mathfrak{U}_H^* = \{\psi_c^* | \psi \in \mathfrak{U}\}.$$

Lemma 6.2.3. *If H is a finitely generated torsion-free nilpotent group of class c then*

$$C_n \operatorname{wr} H \models \mathfrak{U}_H^*.$$

Proof. Let D be the base group of the restricted wreath product $C_n \operatorname{wr} H$; thus $C_n \operatorname{wr} H = D \rtimes H$ and $(C_n \operatorname{wr} H)/D \cong H$. As in the previous lemma, $D = \mathrm{fitt}^{\forall}(C_n \operatorname{wr} H)$ and $D = \mathrm{fitt}_c^{\exists}(C_n \operatorname{wr} H)$. Let $v(x_1, \ldots, x_m)$ be a word in the variables $x_1, \ldots, x_m$, let $g_1, \ldots, g_m$ be elements of $C_n \operatorname{wr} H$ and let $\overline{g}_1, \ldots, \overline{g}_m$ be their images in H. We have $v(\overline{g}_1, \ldots, \overline{g}_m) = 1$ if and only if $v(g_1, \ldots, g_m) \in D$ and this is the case if and only if $\mathrm{fitt}^{\forall}(v(g_1, \ldots, g_m))$ and $\mathrm{fitt}_c^{\exists}(v(g_1, \ldots, g_m))$ hold in $C_n \operatorname{wr} H$. Hence, if the universal sentence ψ given by

$$\forall x_1 \ldots x_m \left(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} (u_{ij}(\overline{x}) = 1 \wedge w_{ij}(\overline{x}) \neq 1) \right)$$

holds in H, where $u_{ij}(\overline{x})$ and $v_{ij}(\overline{x})$ are words in the variables $x_1, \ldots, x_m$, then the universal sentence ψ_c^* given by

$$\forall x_1 \ldots x_m \left(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} \left(\mathrm{fitt}^{\forall}(u_{ij}(\overline{x})) \wedge \neg\, \mathrm{fitt}_c^{\exists}(w_{ij}(\overline{x})) \right) \right).$$

holds in $C_n \operatorname{wr} H$. □

Let H be a finitely generated torsion-free nilpotent group of class c. If $c = 0$ (so that H is trivial) let

$$\mathfrak{W}_{n,H} = \mathfrak{W}_{n,c} \cup \mathfrak{U}_H^*;$$

otherwise let

$$\mathfrak{W}_{n,H} = \mathfrak{W}_{n,c} \cup \mathfrak{U}_{n,H}^+ \cup \mathfrak{U}_H^*.$$

Lemma 6.2.4. *If H is a finitely generated torsion-free nilpotent group of class c then*

$$C_n \operatorname{wr} H \models \mathfrak{W}_{n,H}.$$

Proof. This follows directly from Lemmas 6.2.1, 6.2.2 and 6.2.3. □

Each of the sentences in $\mathfrak{W}_{n,H}$ contains only the universal quantifier $\forall$ when written in prenex normal form. As usual we have given the sentences in a more transparently comprehensible form than the prenex normal form.

We may compare the sentences $\mathfrak{W}_{0,\mathbb{Z}}$ with the sentences $\mathfrak{A}$ given by Chapuis in [12, §3]. As in Chapter 5 we replace his sentence

$$\forall xyz\,(([x,y]=1 \wedge [y,z]=1 \wedge y \neq 1) \rightarrow ([x,z]=1))$$

with the sentence ϕ_2^1 which allows us to use a slightly different existential formula for the Fitting subgroup; this formula adapts easily for use when c is greater than one.

6.3 The universal closure of $C_n \operatorname{wr} H$

Fix a finitely generated torsion-free nilpotent group H of class $c \geqslant 1$ and suppose throughout this section that G is a group in which the universal sentences $\mathfrak{W}_{n,H}$ hold. Our aim is to describe some of the properties of G that follow from this supposition.

In Lemmas 6.3.1, 6.3.2 and 6.3.3 we establish the significance of the universal formula $\mathrm{fitt}^{\forall}(x)$ and the existential formula $\mathrm{fitt}_c^{\exists}(x)$. These formulae are important in allowing us to describe, using universal sentences, the Fitting subgroup $\mathrm{Fitt}(G)$ and the quotient group $G/\mathrm{Fitt}(G)$.

Lemma 6.3.1. *The Fitting subgroup of G is nilpotent of class at most c.*

Proof. By Fitting's theorem, every finitely generated subgroup of $\mathrm{Fitt}(G)$ is nilpotent. The validity in G of the sentence ϕ_2^c implies that every nilpotent subgroup has class at most c and it follows that $\mathrm{Fitt}(G)$ is nilpotent of class at most c. □

Lemma 6.3.2. *If G is a non-nilpotent group and $g \in \mathrm{Fitt}(G)$ then G satisfies $\mathrm{fitt}_c^{\exists}(g)$.*

Proof. Let g be an element of $\mathrm{Fitt}(G)$. Since G is not nilpotent, we may choose elements $h_1, \dots, h_{c+1} \in G$ such that $h = [h_1, \dots, h_{c+1}] \neq 1$. Since $\gamma_{c+1}(G)$ is an abelian normal subgroup of G, it follows that h is in $\mathrm{Fitt}(G)$. Thus $\langle g, h\rangle \leqslant \mathrm{Fitt}(G)$ is nilpotent of class at most c and $[h, \underbrace{g, \dots, g}_{c}] = 1$. □

Lemma 6.3.3. *If G is a non-nilpotent group then* $\mathrm{Fitt}(G)$ *is abelian and* $\mathrm{Fitt}(G) = \mathrm{fitt}^{\forall}(G)$.

Proof. Let g and h be elements of $\mathrm{Fitt}(G)$. By Lemma 6.3.2, we have $G \models \mathrm{fitt}_c^{\exists}(g)$ and $G \models \mathrm{fitt}_c^{\exists}(h)$. Thus, since $G \models \phi_4^c$, the elements g and h commute and $\mathrm{Fitt}(G)$ is abelian.

If g is an element of $\mathrm{Fitt}(G)$ then the normal closure in G of g is in $\mathrm{Fitt}(G)$, which is abelian, and so $g \in \mathrm{fitt}^{\forall}(G)$. If g is in $\mathrm{fitt}^{\forall}(G)$ then the normal closure in G of g is abelian and so $g \in \mathrm{Fitt}(G)$. Thus $\mathrm{Fitt}(G) = \mathrm{fitt}^{\forall}(G)$. □

We next describe the nilpotent subgroups of G. Although every nilpotent normal subgroup of G is abelian, G may contain non-abelian nilpotent subgroups. All such non-abelian nilpotent subgroups satisfy the universal theory of H.

Lemma 6.3.4. *If G_1 is a nilpotent subgroup of G then either G_1 is abelian of exponent dividing n or $G_1 \models \mathfrak{U}_H$.*

Proof. Suppose to begin with that $n = 0$. Let ψ be a sentence in $\mathfrak{U}_H$; we may assume without loss of generality that ψ is of the form $\forall x_1 \ldots x_m \varphi(x_1, \ldots x_m)$, where $\varphi(x_1, \ldots, x_m)$ is quantifier-free. If $g_1, \ldots, g_m$ are elements of G_1 then, since G_1 is nilpotent of class at most c, we have $G \models \beta_m^c(g_1, \ldots, g_m)$. It follows, since $G \models \psi_n^+$, that $\varphi(g_1, \ldots, g_m)$ holds. This is true for all m-tuples in G_1 and hence $G_1 \models \psi$.

Suppose now that $n \neq 0$ and that G_1 is not abelian of exponent dividing n, so that G_1 contains elements h_1 and h_2 such that $[h_1, h_2] \neq 1$ or $h_1^n \neq 1$ or $h_2^n \neq 1$. Let ψ be a sentence in $\mathfrak{U}_H$; we may suppose without loss of generality that ψ is of the form $\forall x_1 \ldots x_m \varphi(x_1, \ldots, x_m)$ where $\varphi(x_1, \ldots, x_m)$ is quantifier-free. If $g_1, \ldots, g_m$ are elements of G_1 then, since G_1 is nilpotent of class at most c, we have

$$G \models \beta_{m+2}^c(g_1, \ldots, g_m, h_1, h_2).$$

It follows, since $G \models \psi_n^+$, that $\varphi(g_1, \ldots, g_m)$ holds. This is true for all m-tuples in G_1 and hence $G_1 \models \psi$. □

In Lemma 6.3.5 we describe the universal theory of the quotient of G by its Fitting subgroup when G is not nilpotent. Clearly when G is nilpotent this quotient is trivial.

Lemma 6.3.5. *If G is not nilpotent then $G/\operatorname{Fitt}(G)$ satisfies $\mathfrak{U}_H$.*

Proof. Let $v(x_1,\dots,x_m)$ be a word in the variables $x_1,\dots,x_m$, let $g_1,\dots,g_m$ be elements of G and let $\overline{g}_1,\dots,\overline{g}_m$ be their images in $G/\operatorname{Fitt}(G)$. We have $v(\overline{g}_1,\dots,\overline{g}_m)=1$ if and only if $v(g_1,\dots,g_m)\in \operatorname{Fitt}(G)$. This is the case if and only if $\operatorname{fitt}^{\forall}(v(g_1,\dots,g_m))$ holds in G and only if $\operatorname{fitt}^{\exists}_c(v(g_1,\dots,g_m))$ holds in G. Hence if the sentence ψ_c^*, given by

$$\forall x_1\dots x_m\left(\bigvee_{i=1}^{s}\bigwedge_{j=1}^{t}(\operatorname{fitt}^{\forall}(u_{ij}(\overline{x}))\wedge\neg\operatorname{fitt}^{\exists}_c(w_{ij}(\overline{x}))\right),$$

holds in G, where $u_{ij}(\overline{x})$ and $v_{ij}(\overline{x})$ are words in the variables $x_1,\dots,x_m$, then the sentence ψ, given by

$$\forall x_1\dots x_m\left(\bigvee_{i=1}^{s}\bigwedge_{j=1}^{t}(u_{ij}(\overline{x})=1\wedge w_{ij}(\overline{x})\neq 1)\right),$$

holds in $G/\operatorname{Fitt}(G)$. □

6.4 $\rho_{n,H}$-groups

Fix a finitely generated torsion-free nilpotent group H of class c. In this section we establish the relationship between the $\rho_{n,H}$-groups defined in Section 6.1 and the sentences $\mathfrak{W}_{n,H}$. This relationship plays a key role in our proof of Theorem 6.1.1.

Lemma 6.4.1. *If G is a non-nilpotent group such that $G\models\mathfrak{W}_{n,H}$ then G is a $\rho_{n,H}$-group.*

Proof. There does not exist a non-nilpotent group such that $G\models\mathfrak{W}_{n,H}$ where H is trivial; thus we may assume that $c\geqslant 1$. The validity in G of the sentences ϕ_1^c and ϕ_2^c implies that $\gamma_{c+1}(G)$ is abelian and that every nilpotent subgroup of G is nilpotent of class at most c. By Lemmas 6.3.3 and 6.3.2, the subgroup $\operatorname{Fitt}(G)$ is abelian, $\operatorname{fitt}^{\forall}(G)=\operatorname{Fitt}(G)$ and $\operatorname{Fitt}(G)\subseteq\operatorname{fitt}^{\exists}_c(G)$. If $n=0$, the truth of the sentences $\phi_3^0(m)$ for all $m\in\mathbb{N}$ implies that G is torsion-free and, if $n\neq 0$, the truth of the sentence $\phi_3^{n,c}$ implies that $\operatorname{Fitt}(G)$ is of exponent dividing n. By Lemma 6.3.5, we have $G/\operatorname{Fitt}(G)\models\mathfrak{U}_H$ and hence $G/\operatorname{Fitt}(G)\models\operatorname{Th}_{\forall}(H)$. If G_1 is a nilpotent subgroup of G then $G_1\models\mathfrak{U}_H$ by Lemma 6.3.4 and it follows that G_1 satisfies the universal theory of H.

Since $G/\operatorname{Fitt}(G)$ satisfies the universal theory of H, it is a torsion-free nilpotent group and hence the group rings $\mathbb{Z}[G/\operatorname{Fitt}(G)]$ and $\mathbb{F}_p[G/\operatorname{Fitt}(G)]$, where p is prime, are integral domains by Lemma 3.1.1. If $n = 0$, the truth of the sentences $\phi_5^{0,c}(\overline{e})$ for every finite sequence $\overline{e}$ on $\{-1, 1\}$ implies that if $g^\alpha = 1$ for some $\alpha \in \mathbb{Z}[G/\operatorname{Fitt}(G)]$ then $\alpha = 0$. Thus $\operatorname{Fitt}(G)$ is torsion-free as a $\mathbb{Z}[G/\operatorname{Fitt}(G)]$-module. Suppose now that $n \neq 0$ and let g be a non-trivial element of $\operatorname{Fitt}(G)$ of order p, where p is a prime divisor of n. Since $G \models \phi_5^{p,c}(m)$ for all $m \in \mathbb{N}$, if $g^\alpha = 1$ for some $\alpha \in \mathbb{F}_p[G/\operatorname{Fitt}(G)]$ then $\alpha = 0$. Thus the submodule of $\operatorname{Fitt}(G)$ of elements of order dividing p is torsion-free as an $\mathbb{F}_p[G/\operatorname{Fitt}(G)]$-module. $\square$

Lemma 6.4.2. *If G is a $\rho_{n,H}$-group then G satisfies $\mathfrak{W}_{n,c}$.*

Proof. If $c = 0$ then a $\rho_{n,H}$-group G is an abelian group of exponent n and hence $G \models \mathfrak{W}_{n,0}$. Suppose now that H is non-trivial and let G be a $\rho_{n,H}$-group. Clearly G satisfies the sentence ϕ_1^c. Since any nilpotent subgroup of G either is abelian or satisfies $\operatorname{Th}_\forall(H)$, the sentence ϕ_2^c holds in G. Hence every nilpotent subgroup of G has class at most c.

Since G has abelian Fitting subgroup, $\operatorname{Fitt}(G) = \operatorname{fitt}^\forall(G)$. In order to show that G satisfies the remaining sentences in $\mathfrak{W}_{n,c}$, it is enough to check that $\operatorname{fitt}_c^\exists(G) \subseteq \operatorname{Fitt}(G)$. Note that, since $\gamma_{c+1}(G)$ is an abelian normal subgroup of G, it is contained in $\operatorname{Fitt}(G)$. Let g be an element of $\operatorname{fitt}_c^\exists(G)$, so that there is a non-trivial element $h = [h_1, \ldots, h_{c+1}] \in \gamma_{c+1}(G) \leqslant \operatorname{Fitt}(G)$ such that $\big[h, \underbrace{g, \ldots, g}_{c}\big] = 1$. If $n = 0$ then $\operatorname{Fitt}(G)$ is torsion-free as a $\mathbb{Z}[G/\operatorname{Fitt}(G)]$-module. Since $h \neq 1$ and

$$h^{(-1+g)^c} = \big[h, \underbrace{g, \ldots, g}_{c}\big] = 1,$$

it follows that g is in $\operatorname{Fitt}(G)$ and $\operatorname{fitt}_c^\exists(G) \subseteq \operatorname{Fitt}(G)$. If $n \neq 0$ then we obtain the required containment by an argument almost identical to that of Lemma 4.3.3. $\square$

Lemma 6.4.3. *If G is a $\rho_{n,H}$-group then G satisfies $\mathfrak{U}^+_{n,H}$.*

Proof. Suppose to begin with that $n = 0$ and let ψ_0^+ be in $U^+_{0,H}$, so that ψ_0^+ is of the form

$$\forall x_1 \ldots x_m \left(\beta_m^c(x_1, \ldots, x_m) \to \varphi(x_1, \ldots, x_m)\right),$$

where $\varphi(x_1, \ldots, x_m)$ is quantifier-free. If $g_1, \ldots, g_m$ are elements of G such that $\beta_m^c(g_1, \ldots, g_m)$ holds then $G_1 = \langle g_1, \ldots, g_m \rangle$ is nilpotent and hence $G_1 \models \mathfrak{U}_H$. The sentence

$$\forall x_1 \ldots x_m \, \varphi(x_1, \ldots, x_m)$$

is in $\mathfrak{U}_H$ and it follows that $\varphi(g_1, \dots, g_m)$ holds. This is true for all m-tuples in G and G satisfies $\mathfrak{U}^+_{0,H}$.

Suppose now that $n \neq 0$ and let ψ^+_n be in $U^+_{n,H}$, so that ψ^+_n is of the form

$$\forall x_1 \dots x_m y_1 y_2 \Bigg(\beta^c_{m+2}(x_1, \dots, x_m, y_1, y_2) \\ \rightarrow \bigg(\Big([y_1, y_2] = 1 \wedge y_1^n = 1 \wedge y_2^n = 1 \Big) \vee \varphi(x_1, \dots, x_m) \bigg) \Bigg),$$

where $\varphi(x_1, \dots, x_m)$ is quantifier-free. If $g_1, \dots, g_m, h_1, h_2$ are elements of G such that $\beta^c_m(g_1, \dots, g_m)$ holds then $G_1 = \langle g_1, \dots, g_m, h_1, h_2 \rangle$ is nilpotent; thus either G_1 is abelian of exponent dividing n and $[h_1, h_2] = 1$, $h_1^n = 1$ and $h_2^n = 1$, or G_1 satisfies $\mathfrak{U}_H$. The sentence

$$\forall x_1 \dots x_m \, \varphi(x_1, \dots, x_m)$$

is in $\mathfrak{U}_H$ and hence, in the latter case, $\varphi(g_1, \dots, g_m)$ holds. This is true for all m-tuples in G and G satisfies $\mathfrak{U}^+_{n,H}$. $\square$

Lemma 6.4.4. *If G is a $\rho_{n,H}$-group then G satisfies $\mathfrak{U}^*_H$.*

Proof. From Lemma 6.4.2, we have $\mathrm{fitt}^\forall(G) = \mathrm{Fitt}(G)$ and $\mathrm{fitt}^\exists_c(G) \subseteq \mathrm{Fitt}(G)$. Let $v(x_1, \dots, x_m)$ be a word in the variables $x_1, \dots, x_m$, let $g_1, \dots, g_m$ be elements of G and let $\overline{g}_1, \dots, \overline{g}_m$ be their images in $G/\mathrm{Fitt}(G)$. Then $v(\overline{g}_1, \dots, \overline{g}_m) = 1$ if and only if $v(g_1, \dots, g_m) \in \mathrm{Fitt}(G)$. This is the case if $\mathrm{fitt}^\exists_c(v(g_1, \dots, g_m))$ holds in G, and if and only if $\mathrm{fitt}^\forall(v(g_1, \dots, g_m))$ holds in G. Hence, if the universal sentence ψ given by

$$\forall x_1 \dots x_m \left(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} (u_{ij}(\overline{x}) = 1 \wedge w_{ij}(\overline{x}) \neq 1) \right),$$

holds in $G/\mathrm{Fitt}(G)$, where $u_{ij}(\overline{x})$ and $w_{ij}(\overline{x})$ are words in the variables $x_1, \dots, x_m$, then the universal sentence ψ^*_c given by

$$\forall x_1 \dots x_m \left(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} (\mathrm{fitt}^\forall(u_{ij}(\overline{x})) \wedge \neg \, \mathrm{fitt}^\exists_m(w_{ij}(\overline{x})) \right)$$

holds in G. Since $G/\mathrm{Fitt}(G) \models \mathfrak{U}_H$, it follows that $G \models \mathfrak{U}^*_H$. $\square$

Combining Lemmas 6.4.2, 6.4.3 and 6.4.4 we have:

Lemma 6.4.5. *If G is a $\rho_{n,H}$-group then G satisfies $\mathfrak{W}_{n,H}$.*

6.5 Proof of theorem

For the reader's convenience we restate the theorem which will be proved in this section. Recall that n is a square-free positive integer or zero.

Theorem 6.1.1 *Let H be a finitely generated torsion-free nilpotent group of class c and let U be a non-trivial abelian group of exponent n. If G is a group then the following conditions are equivalent:*

(1) $G \in \operatorname{ucl}\big(U \operatorname{wr} H\big)$;

(2) *G satisfies the universal sentences $\mathfrak{W}_{n,H}$;*

(3) *G is a $\rho_{n,H}$-group or $G \models \mathrm{Th}_\forall(H)$;*

(4) *Every finitely generated subgroup of G can be embedded in $C_n^{(r)} \operatorname{wr} K$ for some $r \in \mathbb{N}$ and some K such that $K \models \mathrm{Th}_\forall(H)$.*

Proof. We prove the implications (1) $\implies$ (2) $\iff$ (3) $\implies$ (4) $\implies$ (1).

(1) $\implies$ (2). The group G is in $\operatorname{ucl}(U \operatorname{wr} H)$ and so $G \models \mathrm{Th}_\forall(U \operatorname{wr} H)$. By Lemmas 3.4.1 and 3.4.2, we have $U \operatorname{wr} H \models \mathrm{Th}_\forall(C_n \operatorname{wr} H)$; thus $G \models \mathrm{Th}_\forall(C_n \operatorname{wr} H)$ and it follows, by Lemma 6.2.4, that $G \models \mathfrak{W}_{n,H}$.

(2) $\iff$ (3). If G is nilpotent and $G \models \mathfrak{W}_{n,H}$ then either G is abelian of exponent n or $G \models \mathrm{Th}_\forall(H)$ by Lemma 6.3.4 and in the former case G is a $\rho_{n,H}$-group. Since H is a subgroup of $C_n \operatorname{wr} H$, we have $\mathfrak{W}_{n,H} \subseteq \mathrm{Th}_\forall(C_n \operatorname{wr} H) \subseteq \mathrm{Th}_\forall(H)$; thus if $G \models \mathrm{Th}_\forall(H)$ then $G \models \mathfrak{W}_{n,H}$. The equivalence of (2) and (3) now follows immediately from the results of Section 6.4.

(3) $\implies$ (4). If G satisfies $\mathrm{Th}_\forall(H)$ then every finitely generated subgroup $G_1 \leqslant G$ also satisfies $\mathrm{Th}_\forall(H)$. If G is a $\rho_{n,H}$-group and G_1 is a finitely generated nilpotent subgroup of G then $G_1 \models \mathrm{Th}_\forall(H)$ or G is abelian of exponent dividing n. In the former case $G_1 \leqslant C_n \operatorname{wr} G_1$ as required and in the latter case $G_1 = C_{n_1} \times \cdots \times C_{n_r}$ where each n_j is a divisor of n (see [53, 4.2.6]). Each cyclic group C_{n_j} embeds in C_n and hence G_1 embeds in the direct product of r copies of C_n. It follows that G_1 embeds in $C_n^{(r)} \operatorname{wr} H$ for some positive integer r.

Now suppose that G is a $\rho_{n,H}$-group, so that $G \models \mathfrak{W}_{n,H}$ by Lemma 6.4.5, and that G_1 is a finitely generated non-nilpotent subgroup of G. Since the sentences $\mathfrak{W}_{n,H}$ are universal, every subgroup of G satisfies $\mathfrak{W}_{n,H}$; in particular, $G_1 \models \mathfrak{W}_{n,H}$ and, since G_1 is not nilpotent, G_1 is a $\rho_{n,H}$-group by Lemma 6.4.1. The quotient group $\overline{G}_1 = G_1/\operatorname{Fitt}(G_1)$ satisfies the universal theory of H; thus $\overline{G}_1$ is a finitely generated torsion-free nilpotent group and hence the group rings $\mathbb{Z}[\overline{G}_1]$ and $\mathbb{F}_p[\overline{G}_1]$, where p is a prime, are Ore domains by Lemma 3.1.2.

If $n = 0$, the group G_1 is an extension of $\operatorname{Fitt}(G_1)$ by $\overline{G}_1$ satisfying the conditions of Lemma 3.3.1 and G_1 embeds in $U \operatorname{wr} \overline{G}_1$, where U is a finitely generated torsion-free abelian group. It follows that G_1 embeds in $\mathbb{Z}^{(r)} \operatorname{wr} \overline{G}_1$ for some $r \in \mathbb{N}$. If $n \neq 0$, the group G_1 is an extension of $\operatorname{Fitt}(G_1)$ by $\overline{G}_1$ satisfying the conditions of Lemma 3.3.2 and G_1 embeds in $U \operatorname{wr} \overline{G}_1$, where U is some finitely generated abelian group of exponent dividing n. As above, U embeds in $C_n^{(r)}$ for some $r \in \mathbb{N}$ and hence G_1 embeds in $C_n^{(r)} \operatorname{wr} \overline{G}_1$. In both cases we have $\overline{G}_1 \models \operatorname{Th}_\forall(H)$ as required.

(4) $\implies$ (1). Let ϕ be in $\operatorname{Th}_\forall\big(U \operatorname{wr} H\big)$ and suppose without loss of generality that ϕ is of the form

$$\forall x_1 \dots x_m\, \varphi(x_1, \dots, x_m),$$

where $\varphi(x_1, \dots, x_m)$ is quantifier-free. We let $g_1 \dots, g_m$ be elements of G and let $G_1 = \langle g_1, \dots, g_m \rangle$, so that G_1 embeds in $C_n^{(r)} \operatorname{wr} K$ for some $r \in \mathbb{N}$ and some K such that $K \models \operatorname{Th}_\forall(H)$. By Lemmas 3.4.2 and 3.4.1, the restricted wreath product $C_n^{(r)} \operatorname{wr} K$ satisfies the universal theory of $U \operatorname{wr} H$; in particular, $C_n^{(r)} \operatorname{wr} K \models \phi$ and $\varphi(g_1, \dots, g_m)$ holds. This is true for all m-tuples in G and hence $G \models \phi$. Thus G satisfies $\operatorname{Th}_\forall\big(U \operatorname{wr} H\big)$ and G is in $\operatorname{ucl}\big(U \operatorname{wr} H\big)$. $\square$

6.6 Universal equivalence and decidability

Fix a torsion-free nilpotent group H of class $c \geqslant 1$. Our aims in this section are to give conditions for a group to be universally equivalent to $C_n \operatorname{wr} H$ and to show that the universal theory of this group is decidable. We start by proving that if G is a non-nilpotent group which satisfies the sentences $\mathfrak{W}_{n,H}$, and G contains a subgroup universally equivalent to H and an element of order n, then G is universally equivalent to $C_n \operatorname{wr} H$.

Lemma 6.6.1. *Suppose that $c \geqslant 2$. If G is a non-nilpotent group such that $G \models \mathfrak{W}_{n,H}$ and G_1 is a non-abelian nilpotent subgroup of G then $G_1 \cap \operatorname{Fitt}(G) = \{1\}$.*

Proof. By Lemma 3.3.5, if G_1 is a non-abelian nilpotent subgroup of $C_n \operatorname{wr} H$ then the canonical map $\theta : C_n \operatorname{wr} H \to H$ embeds G_1 in H. The kernel of $\theta|_{G_1}$ is $G_1 \cap \operatorname{Fitt}(C_n \operatorname{wr} H)$. Since $\theta|_{G_1}$ is an embedding, this kernel is trivial and hence G_1 has trivial intersection with $\operatorname{Fitt}(C_n \operatorname{wr} H)$. This can be expressed in a universal sentence

$$\forall xyz \left(\left(\beta_3^c(x, y, z) \wedge [y, z] \neq 1 \wedge \operatorname{fitt}_c^{\exists}(x) \right) \to x = 1 \right).$$

The group G satisfies the sentences $\mathfrak{W}_{n,H}$, which implies, by Theorem 6.1.1, that G satisfies the universal theory of $C_n \operatorname{wr} H$. In particular,

$$G \models \forall xyz \left(\left(\beta_3^c(x, y, z) \wedge [y, z] \neq 1 \wedge \operatorname{fitt}_c^{\exists}(x) \right) \to x = 1 \right).$$

It follows from this that any non-abelian nilpotent subgroup of G has trivial intersection with $\operatorname{Fitt}(G)$. $\square$

Lemma 6.6.2. *Let G be a non-nilpotent group such that $G \models \mathfrak{W}_{n,H}$. Suppose that G contains a subgroup universally equivalent to H. For $n \neq 0$, suppose in addition that G contains an element of order n. Then G is universally equivalent to $C_n \operatorname{wr} H$.*

Proof. For $c = 1$, the result follows from [12, Lemma 4.9] if $n = 0$ and from Lemma 4.5.1 if $n \neq 0$. We suppose that $c \geqslant 2$.

Let K be a subgroup of G such that K is universally equivalent to H. The group G is not nilpotent and, since $c \geqslant 2$, the group K is not abelian. We apply Lemma 6.6.1 to obtain $K \cap \operatorname{Fitt}(G) = \{1\}$. If $n = 0$, we choose h to be a non-trivial element of $\operatorname{Fitt}(G)$ and, if $n \neq 0$, we choose $h \in G$ such that $o(h) = n$; again h is a non-trivial element of $\operatorname{Fitt}(G)$. For each $g \in K$, we let $H_g = \langle h^g \rangle$ and $L = \langle H_g \mid g \in K \rangle$. If $n = 0$ then, since $\operatorname{Fitt}(G)$ is torsion-free as a $\mathbb{Z}[G/\operatorname{Fitt}(G)]$-module and $K \cap \operatorname{Fitt}(G) = \{1\}$, the group L is a direct product of the cyclic groups H_g. If $n \neq 0$ then, since $o(h) = n$, the element h has trivial annihilator in $R_n[G/\operatorname{Fitt}(G)]$ and L is again a direct product of the groups H_g. In both cases L is abelian and, by definition, $L \lhd \langle K, h \rangle$. Also $K \cap L \leqslant K \cap \operatorname{Fitt}(G) = \{1\}$ and hence $\langle K, h\rangle = L \rtimes K \cong C_n \operatorname{wr} K$. Since $\langle K, h\rangle \leqslant G$, it follows that $G \models \mathrm{Th}_\exists(C_n \operatorname{wr} K) = \mathrm{Th}_\exists(C_n \operatorname{wr} H)$. Thus $G \models \mathrm{Th}_\forall(C_n \operatorname{wr} H) \cup \mathrm{Th}_\exists(C_n \operatorname{wr} H)$ and G is universally equivalent to $C_n \operatorname{wr} H$. □

If n is a prime p then the condition that G contains an elements of order p follows from the fact that G is not nilpotent: the subgroup $\gamma_{c+1}(G)$ is non-trivial and contained in $\mathrm{Fitt}(G)$; since $\mathrm{Fitt}(G)$ has exponent dividing p, it follows that any non-trivial element of $\gamma_{c+1}(G)$ is of order p.

Let G be a group. We introduce existential sentences θ_m and π_m for each $m \geqslant 0$. The sentence θ_0 says that G is non-trivial, and G satisfies θ_m for $m \geqslant 1$ if G is not nilpotent of class m:

$$\begin{aligned} &\theta_0 : \exists x\, (x \neq 1), \\ &\theta_m : \exists x_1 \ldots x_{m+1}([x_1, \ldots, x_{m+1}] \neq 1) \qquad \text{for } m \geqslant 1. \end{aligned}$$

The sentence π_0 is true in any group and G satisfies π_m for $m \geqslant 1$ if G contains an element of order m:

$$\begin{aligned} &\pi_0 : \exists x\, (x = x), \\ &\pi_m : \exists x \left(x^m = 1 \wedge \bigwedge_{i=1}^{r} x^{\frac{m}{q_i}} \neq 1 \right) \qquad \text{for } m \geqslant 1. \end{aligned}$$

Here $q_1, \ldots, q_r$ are the prime divisors of m.

Let $\mathfrak{V}_H \subseteq \mathrm{Th}_\exists(H)$ be a set of existential sentences such that a group $K \in \mathrm{ucl}(H)$ satisfies the existential theory of H if and only if $K \models \mathfrak{V}_H$. We introduce existential sentences ψ_Y for each finite subset $Y = \{\psi_1, \ldots, \psi_r\}$ of $\mathfrak{V}_H$: we may suppose that ψ_i is of the form $\exists x_1 \ldots x_{k_i} \varphi_i(x_1, \ldots, x_{k_i})$ where $\varphi_i(x_1, \ldots, x_{k_i})$ is quantifier-free and we define ψ_Y to be the sentence

$$\exists x_{11} \ldots x_{1k_1} x_{21} \ldots x_{rk_r} \left(\beta^c_{k_1+\cdots+k_r}(x_{11}, \ldots, x_{rk_r}) \wedge \bigwedge_{i=1}^{r} \varphi_i(x_{i1}, \ldots, x_{ik_i}) \right).$$

Let $\mathfrak{V}_H^+ = \{\psi_Y \mid Y \in \mathfrak{V}_H, Y \text{ finite}\}$. It is clear that $C_n \operatorname{wr} H \models \mathfrak{V}_H^+ \cup \{\theta_c, \pi_n\}$.

Lemma 6.6.3. *If G is a finitely generated group such that $G \models \mathfrak{W}_{n,H} \cup \mathfrak{V}_H^+ \cup \{\theta_c, \pi_n\}$ then G is universally equivalent to $C_n \operatorname{wr} H$.*

Proof. If H is non-trivial and abelian then, by the results of Chapuis [12, Lemma 4.9] for $n = 0$ and by Lemma 4.5.1 for $n \neq 0$, the group $C_n \operatorname{wr} \mathbb{Z}$ embeds in G and the result follows. Suppose therefore that H is not abelian, so that $c \geqslant 2$.

For each finite subset Y of $\mathfrak{V}_H$, we have $G \models \psi_Y$; thus we may choose elements $g_{11}, \ldots, g_{rk_r} \in G$ which satisfy this sentence. The subgroup $K_Y = \langle g_{11}, \ldots, g_{rk_r} \rangle$ is nilpotent and $K_Y \models Y$.

Suppose first that $\mathfrak{V}_H$ is infinite and let $\Upsilon = \{Y \subseteq \mathfrak{V}_H \mid Y \text{ finite }\}$. For a sentence ψ in $\mathfrak{V}_H$, we define the subset P_ψ of Υ by

$$P_\psi = \{Y \in \Upsilon \mid \psi \in Y\}.$$

The subset $\{\psi_1, \ldots, \psi_r\}$ of $\mathfrak{V}_H$ is contained in $\bigcap_{i=1}^r P_{\psi_i}$; thus the set $\{P_\psi \mid \psi \in \mathfrak{V}_H\}$ has the finite intersection property and is contained in some ultrafilter $\mathcal{U}$ on Υ by Lemma 2.6.1. Let X be a finite subset of Υ, so that X contains finitely many Y each of which contains finitely many sentences. Since $\mathfrak{V}_H$ is infinite, there is some $\psi \in \mathfrak{V}_H$ such that, for all $Y \in X$, we have $\psi \notin Y$. Now, $P_\psi \in \mathcal{U}$ and $P_\psi \subseteq \Upsilon \setminus X$, so that $\Upsilon \setminus X \in \mathcal{U}$. Thus $\mathcal{U}$ contains the cofinite filter and, by Lemma 2.6.2, the ultrafilter $\mathcal{U}$ is non-principal.

Let $K = \prod K_Y/\mathcal{U}$. Since each K_Y is nilpotent, K is nilpotent by Łoś' Theorem (see Theorem 2.6.1). Also, for each $\psi \in \mathfrak{V}_H$, we have $K_Y \models \psi$ for each $Y \in P_\psi \in \mathcal{U}$. Thus $\{Y \in \Upsilon \mid K_Y \models \psi\} \in \mathcal{U}$ and, by Łoś' Theorem, $K \models \psi$. Hence K satisfies the existential theory of H.

For each $Y \in \Upsilon$, let G_Y be an isomorphic copy of G. Since K_Y embeds in G_Y for each $Y \in \Upsilon$, the ultraproduct K embeds in $G_1 = \prod G_Y/\mathcal{U}$ (see Theorem 2.6.2). It follows from Łoś' Theorem that taking ultrapowers preserves elementary equivalence, and hence G_1 is elementarily equivalent to G. The group K satisfies the existential theory of H; in particular, since H is non-abelian, K is a non-abelian nilpotent subgroup of G_1 and hence $K \models \mathfrak{U}_H$ by Lemma 6.3.4. Thus $K \models \mathfrak{U}_H \cup \mathfrak{V}_H$ and K is universally equivalent to H.

If $\mathfrak{V}_H$ is finite, let $G_1 = G$ and let $K = K_Y$, where $Y = \mathfrak{V}_H$. Certainly, G_1 is elementarily equivalent to G and K is universally equivalent to H.

The group G_1 satisfies the elementary theory of G and $G \models \mathfrak{W}_{n,H} \cup \mathfrak{V}_H^+ \cup \{\theta_c, \pi_n\}$;

thus G_1 satisfies the sentences $\mathfrak{W}_{n,H} \cup \mathfrak{V}_H^+ \cup \{\theta_c, \pi_n\}$. The validity in G_1 of the sentence θ_c implies that G_1 is not nilpotent of class c and hence that G_1 is not nilpotent. The validity in G_1 of the sentence π_n implies that G_1 contains an element of order n. Moreover, the subgroup K of G_1 is universally equivalent to H. Thus G_1 is universally equivalent to $C_n \operatorname{wr} H$ by Lemma 6.6.2 and, since G is elementarily equivalent to G_1, it follows that G is universally equivalent to $C_n \operatorname{wr} H$. □

Corollary 6.6.1. *Suppose that H is non-trivial and that there exist a recursive set of universal sentences $\mathfrak{U}_H$ and a recursive set of existential sentences $\mathfrak{V}_H$ such that a group K is universally equivalent to H if and only if $K \models \mathfrak{U}_H \cup \mathfrak{V}_H$. Then, for any abelian group U of exponent n, the group $U \operatorname{wr} H$ has decidable universal theory.*

Proof. Let $\overline{\mathfrak{W}}_{n,H}$ be the recursive collection of sentences $\mathfrak{W}_{n,H} \cup \mathfrak{V}_H^+ \cup \{\theta_c, \pi_n\}$. By Lemma 6.6.3, a group G satisfies $\overline{\mathfrak{W}}_{n,H}$ if and only if G is universally equivalent to $C_n \operatorname{wr} H$. Let ϕ be a universal sentence; by the completeness theorem, $\overline{\mathfrak{W}}_{n,H} \vdash \phi$ or $\overline{\mathfrak{W}}_{n,H} \vdash \neg\phi$. Hence $\phi \in \mathrm{Th}_\forall\big(U \operatorname{wr} H\big)$ if and only if $\overline{\mathfrak{W}}_{n,H} \vdash \phi$, and $\phi \notin \mathrm{Th}_\forall\big(U \operatorname{wr} H\big)$ if and only if $\overline{\mathfrak{W}}_{n,H} \vdash \neg\phi$. Since $\overline{\mathfrak{W}}_{n,H}$ is recursive, the universal theory of $U \operatorname{wr} H$ is decidable. □

6.7 Free nilpotent groups

In this section we consider necessary and sufficient conditions for a group in the universal closure of a free nilpotent group to be universally equivalent to it. We present examples of torsion-free nilpotent groups H for which we can write down explicitly a collection of sentences $\mathfrak{V}_H$ such that if $K \in \mathrm{ucl}(H)$ then K is universally equivalent to H if and only if $K \models \mathfrak{V}_H$. Proposition 6.7.1 gives conditions for a group in $\mathrm{ucl}(F)$, where F is a free nilpotent group of rank 2, to be universally equivalent to F.

Proposition 6.7.1. *If G is in $\mathrm{ucl}(F_2(\mathfrak{N}_c))$ then G is universally equivalent to $F_2(\mathfrak{N}_c)$ if and only if G contains a 2-generator subgroup which is not in $\mathfrak{N}_{c-1}$.*

Proof. Let $H = F_2(\mathfrak{N}_c)$ and let g and h be elements of G such that $\langle g, h\rangle \notin \mathfrak{N}_{c-1}$. A finitely generated nilpotent group is finitely presented (P. Hall, see [53, 2.2.4]), and hence

$\langle g, h\rangle$ embeds in the countable Cartesian power $H^{\mathbb{N}}$ of H by Lemma 2.5.2. Let $\overline{g}$ and $\overline{h}$ be the images of g and h in $H^{\mathbb{N}}$. Let $\overline{g}(i)$ and $\overline{h}(i)$ be the ith coordinates and let $G_i = \langle \overline{g}(i), \overline{h}(i)\rangle$. For some i and j, suppose that there is a homomorphism from G_i to G_j such that $\overline{g}(i) \mapsto \overline{g}(j)$ and $\overline{h}(i) \mapsto \overline{h}(j)$. If $\overline{g}'$ and $\overline{h}'$ are the elements obtained from $\overline{g}$ and $\overline{h}$ by replacing the jth coordinates with identity elements then $\langle \overline{g}, \overline{h}\rangle$ is isomorphic to $\langle \overline{g}', \overline{h}'\rangle$.

If $\overline{g}(i)$ and $\overline{h}(i)$ are linearly dependent modulo H' then, since H/H' is torsion-free, G_i is cyclic modulo $H' \cap G_i$. By Lemma 4.5.1, there are elements $a \in G_i$ and $b \in H' \cap G_i$ such that $G_i = \langle a, b\rangle$. We have $G_i' = \langle [a,b]\rangle^{G_i} \leqslant \gamma_3(H)$ and $\gamma_c(G_i) \leqslant \gamma_{c+1}(H) = \{1\}$, so that $G_i \in \mathfrak{N}_{c-1}$. If G_i is in $\mathfrak{N}_{c-1}$ for all i then $\prod_{i\in\mathbb{N}}^{C} G_i$ is in $\mathfrak{N}_{c-1}$ and, since $\langle g, h\rangle$ embeds in $\prod_{i\in\mathbb{N}}^{C} G_i$, we have $\langle g, h\rangle \in \mathfrak{N}_{c-1}$, contradicting our assumptions. We fix $i \in \mathbb{N}$ such that the elements $\overline{g}(i)$ and $\overline{h}(i)$ are linearly independent modulo H'.

The group G_i is free in $\mathfrak{N}_c$ by Lemma 2.2.2 and hence $G_i \cong H$; moreover, G_i is freely generated by $\overline{g}(i)$ and $\overline{h}(i)$. For each j, the group $G_j \leqslant H$ is a 2-generator group in $\mathfrak{N}_c$ and there is a homomorphism $G_i \to G_j$ such that $\overline{g}(i) \mapsto \overline{g}(j)$ and $\overline{h}(i) \mapsto \overline{h}(j)$. It follows that $\langle g, h\rangle$ is isomorphic to $G_i \cong H$. Thus G contains an isomorphic copy of H and hence G satisfies the existential theory of H. Since G is in $\mathrm{ucl}(H)$, it follows that G is universally equivalent to H.

Suppose now that G is universally equivalent to H and let g and h be free generators of H. Then $[g, \underbrace{h, \ldots, h}_{c-1}] \neq 1$ and H satisfies the existential sentence θ defined by

$$\exists xy\,([x, \underbrace{y, \ldots, y}_{c-1}] \neq 1).$$

Thus G satisfies θ and G contains a 2-generator group which is not in $\mathfrak{N}_{c-1}$. $\square$

This result has important consequences in determining those groups universally equivalent to $C_n \operatorname{wr} H$, where $H = F_2(\mathfrak{N}_c)$ with $c \geqslant 2$. Using Proposition 6.7.1, we can give explicitly the collections of existential sentences $\mathfrak{V}_H$ used in the proof of Lemma 6.6.3. Let G be a group in the universal closure of $C_n \operatorname{wr} H$. Since a non-abelian nilpotent subgroup of G satisfies the universal theory of H, a 2-generator nilpotent subgroup of G which is not in $\mathfrak{N}_{c-1}$ is universally equivalent to H. We set

$$\mathfrak{V}_H = \{\exists xy([x, \underbrace{y, \ldots, y}_{c-1}] \neq 1)\}.$$

It follows that

$$\mathfrak{V}_H^+ = \{\exists xy(\beta_2^c(x,y) \wedge [x, \underbrace{y, \ldots, y}_{c-1}] \neq 1)\}.$$

Now let $c = 2$ or 3, so that $F_2(\mathfrak{N}_c)$ generates $\mathfrak{N}_c$ (see [27]); then $F_k(\mathfrak{N}_c)$ is universally equivalent to $F_2(\mathfrak{N}_c)$ for all $k \geqslant 2$ by Theorem 2.5.1. For $c = 2$, if G is non-abelian then G contains elements g and h such that $[g,h] \neq 1$; thus $\langle g, h\rangle$ is a non-abelian 2-generator subgroup of G. Suppose now that $c = 3$. If G is a torsion-free group which is not in $\mathfrak{N}_{c-1}$ then G contains elements u, v and w such that $[u,v,w] \neq 1$ and hence $[u,v,w]^3 \neq 1$. It follows that there exist elements $g, h \in G$ such that $[g,h,g] \neq 1$ (Levi, see [44, 34.31]) and G contains a 2-generator subgroup $\langle g, h\rangle$ which is not in $\mathfrak{N}_{c-1}$. We have the following corollary to Proposition 6.7.1:

Corollary 6.7.1. *If G is in $\mathrm{ucl}(F_k(\mathfrak{N}_c))$, with $c = 2$ or 3 and $k \geqslant 2$, then G is universally equivalent to H if and only if G is not in $\mathfrak{N}_{c-1}$.*

We have the following sets of sentences $\mathfrak{V}_H$ in Lemma 6.6.3:

- $\mathfrak{V}_H = \{\exists\, xy([x,y] \neq 1)\}$ where $H = F_k(\mathfrak{N}_2)$, $k \geqslant 2$;
- $\mathfrak{V}_H = \{\exists\, xyz([x,y,z] \neq 1)\}$ where $H = F_k(\mathfrak{N}_3)$, $k \geqslant 2$.

Using Proposition 6.7.1, we are able to draw the lattice of universally closed classes of groups contained in the universal closure of $F_2(\mathfrak{N}_2)$ (see Figure 6.1). We also obtain much information about the lattices of universally closed classes of groups contained in the universal closures of $\mathbb{Z} \,\mathrm{wr}\, F_2(\mathfrak{N}_2)$ (see Figure 6.2) and $C_p \,\mathrm{wr}\, F_2(\mathfrak{N}_2)$ when p is prime (see Figure 6.3). When n is neither prime nor zero, the lattice of universally closed classes of groups contained in the universal closure of $C_n \,\mathrm{wr}\, F_2(\mathfrak{N}_2)$ becomes much more complicated to draw. We construct instead the lattice of universal closures of groups in $C_n \,\mathrm{wr}\, F_2(\mathfrak{N}_2)$ (see Figure 6.4). Every other universally closed class of groups contained in the universal closure of $C_n \,\mathrm{wr}\, F_2(\mathfrak{N}_2)$ is a union of two or more of these classes. To

$$\begin{tikzcd}
\mathrm{ucl}(F_2(\mathfrak{N}_2)) \arrow[d, dash] \\
\mathrm{ucl}(\mathbb{Z}) \arrow[d, dash] \\
\{1\}
\end{tikzcd}$$

Figure 6.1: The lattice of universally closed classes contained in the universal closure of a non-cyclic free nilpotent group of class 2.

include all of these classes would give a total of at least 90 distinct points in the lattice and it would be difficult to meaningfully present all the containments.

If G is a non-nilpotent metabelian group in the universal closure of $\mathbb{Z} \operatorname{wr} F_2(\mathfrak{N}_2)$ then, by the definition of $\rho_{0,F_2(\mathfrak{N}_2)}$-groups, G is a non-abelian $\rho_{0,\mathbb{Z}}$-group[1] and hence G is universally equivalent to $F_2(\mathfrak{S}_2)$ and $\mathbb{Z} \operatorname{wr} \mathbb{Z}$. Similarly, by the results of Chapter 4, if G is a non-nilpotent metabelian group in the universal closure of $C_n \operatorname{wr} F_2(\mathfrak{N}_2)$ with $n \neq 0$ then G is universally equivalent to $F_2(\mathfrak{A}_n\mathfrak{A})$ and $C_n \operatorname{wr} \mathbb{Z}$. We were able to draw in full the sublattices of universally closed classes of groups contained in the universal closures of these metabelian wreath products: $\mathbb{Z} \operatorname{wr} \mathbb{Z}$, $C_p \operatorname{wr} \mathbb{Z}$ and $C_{pq} \operatorname{wr} \mathbb{Z}$ where p and q are primes (Figures 4.1, 4.2 and 4.3). Every non-abelian nilpotent group contained in the universal closure of $C_n \operatorname{wr} F_2(\mathfrak{N}_2)$ satisfies the universal theory of $F_2(\mathfrak{N}_2)$ and the abelian groups are either torsion-free or of exponent dividing n. Thus the sublattices of nilpotent groups can be drawn explicitly. The only classes of groups in the lattice about which our information is incomplete are those non-metabelian groups which are not universally equivalent to $C_n \operatorname{wr} F_2(\mathfrak{N}_2)$. These groups satisfy the condition that all of their nilpotent subgroups are abelian.

The sublattice $\mathfrak{L}$ in Figure 6.2, the sublattice $\mathcal{K}_p$ in Figures 6.3 and 6.4 and the sublattices $\mathcal{K}_q$ and $\mathcal{K}_{pq}$ in Figure 6.4 consist of universally closed classes of groups satisfying the following two conditions: each nilpotent subgroup of a group in the class is abelian and the class contains a non-metabelian group. The sublattice $\mathcal{K}_{pq}$ satisfies the additional condition that in each class in $\mathcal{K}_{pq}$, there is a group which contains an element of order pq. We note that each class in either $\mathcal{K}_p$ or $\mathcal{K}_q$ is contained in some class in $\mathcal{K}_{pq}$. The sublattice $\mathcal{L}$ contains the universal closure of $F_2(\mathfrak{A}\mathfrak{N}_2)$ and the sublattices $\mathcal{K}_p$, $\mathcal{K}_q$ and

[1]Or equivalently a ρ-group in the terminology of [12].

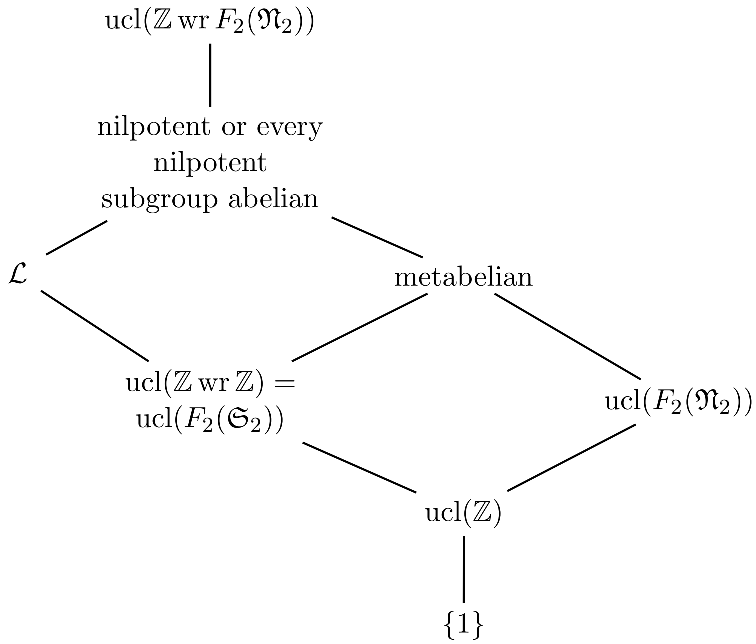

Figure 6.2: The lattice of universally closed classes contained in the universal closure of the restricted wreath product $\mathbb{Z} \operatorname{wr} F_2(\mathfrak{N}_2)$.

$\mathcal{K}_{pq}$ contain the universal closures of $F_2(\mathfrak{A}_p\mathfrak{N}_2)$, $F_2(\mathfrak{A}_q\mathfrak{N}_2)$ and $F_2(\mathfrak{A}_{pq}\mathfrak{N}_2)$ respectively. It would be interesting to find examples of non-metabelian groups which satisfy the universal theory of $\mathbb{Z} \operatorname{wr} F_2(\mathfrak{N}_2)$, and all of whose nilpotent subgroups are abelian, but which are not universally equivalent to $F_2(\mathfrak{A}\mathfrak{N}_2)$, should any such exist. Similarly, it would be interesting to find examples of non-metabelian groups which satisfy the universal theory of $C_n \operatorname{wr} F_2(\mathfrak{N}_2)$ with $n \neq 0$, all of whose nilpotent subgroups are abelian and which contain an element of order n, but which are not universally equivalent to $F_2(\mathfrak{A}_n\mathfrak{N}_2)$. Again it is unclear whether any such groups exist. In Figure 6.4, we have three further sublattices $\mathcal{L}_p$, $\mathcal{L}_q$ and $\mathcal{L}_{pq}$: these are the lattices of universally closed classes of abelian groups of exponent dividing p, q and pq respectively such that each class in the lattice contains a group of exponent p, q and pq respectively. For a more explicit depiction of the lattice $\mathcal{L}_{pq}$ and its sublattices see Figure 4.4 on page 68. The two occurrences in Figure 6.4 of both $\mathrm{ucl}(F_2(\mathfrak{N}_2))$ and $\mathrm{ucl}(\mathbb{Z})$ should be thought of as single points in the lattice. They have been duplicated in order to simplify presentation. One way of avoiding this duplication would be to draw the lattice on a cylinder.

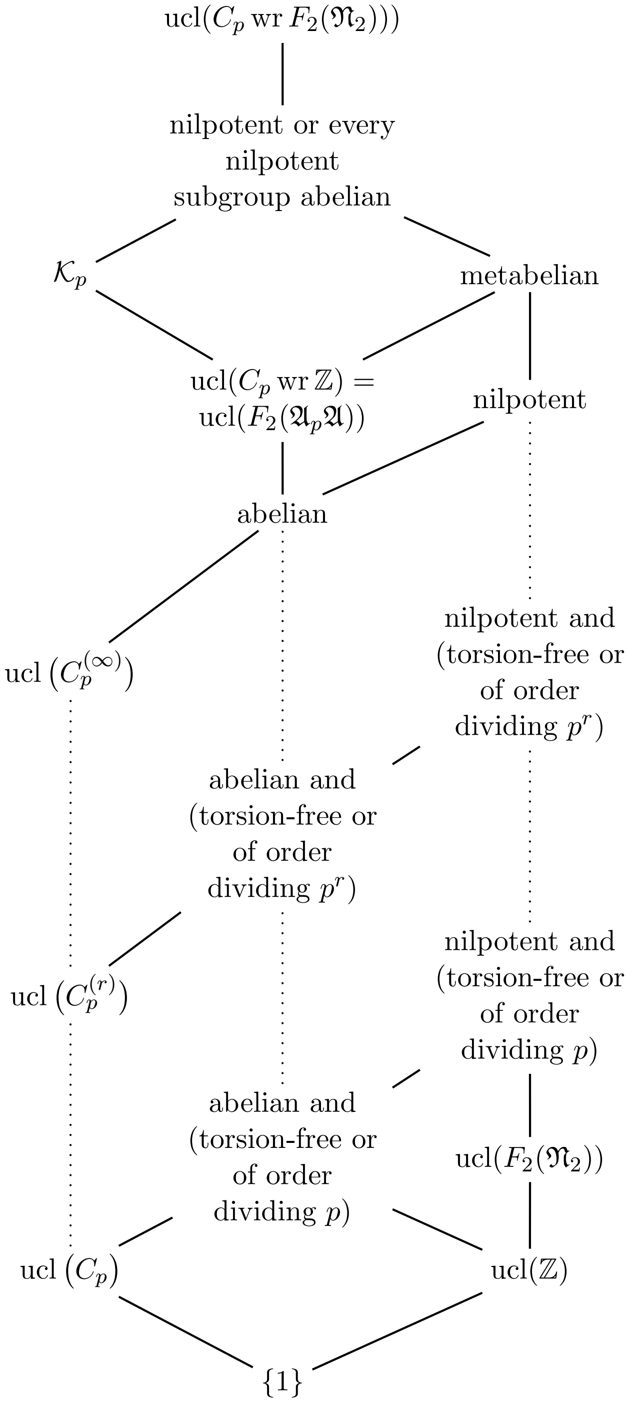

Figure 6.3: The lattice of universally closed classes contained in the universal closure of the restricted wreath product $C_p \,\mathrm{wr}\, F_2(\mathfrak{N}_2)$ where p is a prime.

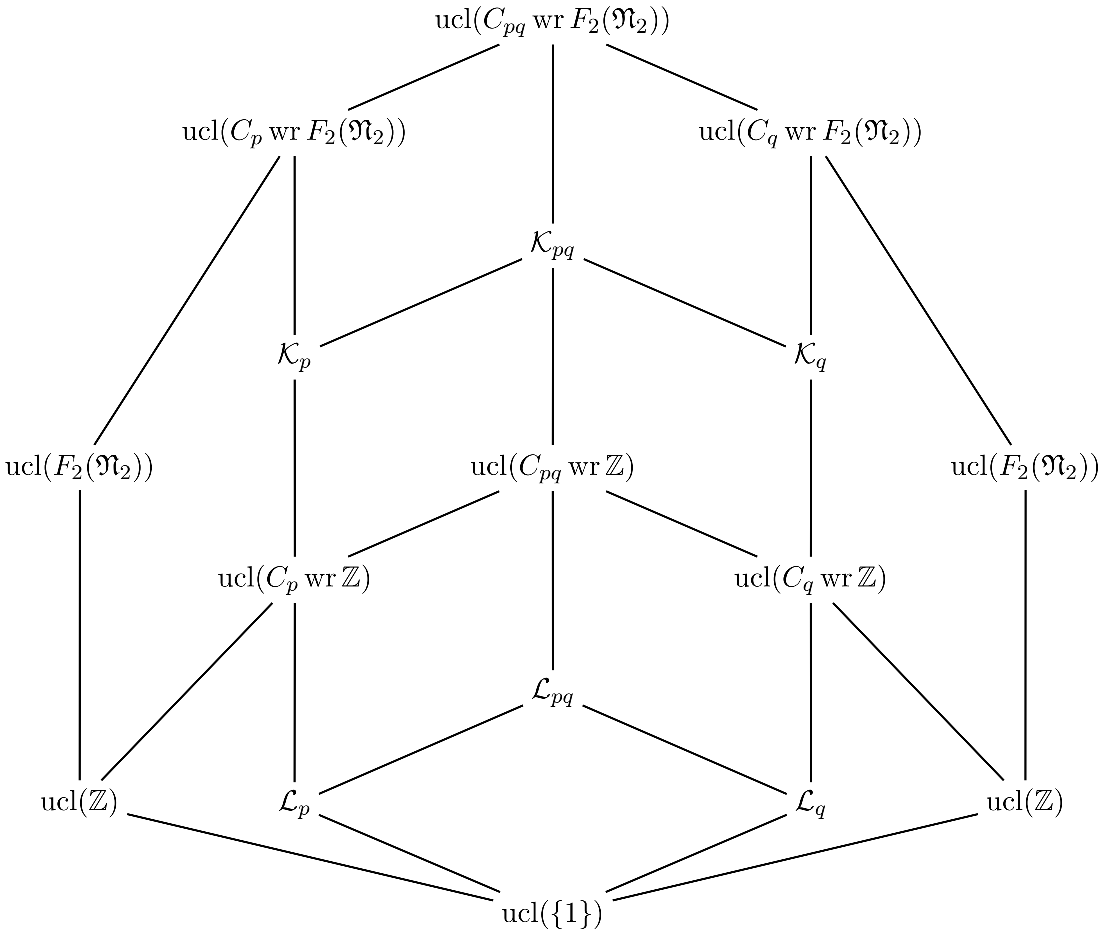

Figure 6.4: The lattice of universal closures of groups contained in the universal closure of the restricted wreath product $C_{pq}\,\mathrm{wr}\,F_2(\mathfrak{N}_2)$ where p and q are primes.

Chapter 7

RESTRICTED WREATH PRODUCTS OF AN ABELIAN GROUP OF SQUARE-FREE OR ZERO EXPONENT BY A TORSION-FREE SOLUBLE GROUP

We define a class of groups $\mathcal{B}_k$ as follows: a group G is in $\mathcal{B}_k$ if and only if every nilpotent subgroup of G has class at most k. Let P be a finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$. If, for example, P is a free soluble group of derived length $d \geqslant 3$ then the question of whether the universal theory of P is decidable is a very hard problem related to Hilbert's Tenth Problem over the rationals. See Chapter 1 for a discussion of this difficult open problem. In this chapter we consider the universal theory of groups $U \operatorname{wr} P$, where U is abelian and either torsion-free or of finite square-free exponent, modulo the potentially difficult universal theory of P. We characterize those groups which satisfy the universal theory of $U \operatorname{wr} P$ in two ways:

- in terms of the module-theoretic properties of the Fitting subgroup and the universal theory of the quotient group by the Fitting subgroup;
- in terms of the embeddability of finitely generated subgroups in certain restricted wreath products.

We also define a collection of universal sentences such that a group which is not in $\mathfrak{P}_{\overline{c}}$ satisfies the universal theory of $U \operatorname{wr} P$ if and only if it satisfies these sentences. We give some examples of finitely generated polynilpotent groups which are in the class $\mathcal{B}_k$ for some positive integer k; these are groups for which the theorem holds. We also give an example of a finitely generated torsion-free soluble group which is not in $\mathcal{B}_k$ for any positive integer k.

Let G be a finitely generated group with an abelian normal subgroup A such that G/A is a torsion-free polynilpotent group. If A is torsion-free as a $\mathbb{Z}[G/A]$-module then G embeds in $\mathbb{Z}^{(r)} \operatorname{wr}(G/A)$ for some positive integer r and hence $\operatorname{ucl}(G) \subseteq \operatorname{ucl}(\mathbb{Z} \operatorname{wr}(G/A))$. If A is of finite square-free exponent n and A is torsion-free as an $\mathbb{F}_p[G/A]$-module for each prime divisor p of n then G embeds in $C_n^{(r)} \operatorname{wr}(G/A)$ for some positive integer r and hence $\operatorname{ucl}(G) \subseteq \operatorname{ucl}(C_n \operatorname{wr}(G/A))$. We consider some examples of groups where this inclusion is strict.

Throughout this chapter let n be a positive square-free integer or zero and let k, l and $c_1, \ldots, c_l$ be positive integers. Let $\overline{c} = (c_1, \ldots, c_l)$ and let $C = (c_1 + 1) \cdots (c_l + 1)$.

7.1 Statement of theorems

We begin by defining the notion of a $\rho_{n,P}$-group for a torsion-free group in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$. This extends Definition 6.1.1 of $\rho_{n,H}$-groups where H is a finitely generated torsion-free nilpotent group. In this wider context we are unable to make a general statement about the universal theory of a nilpotent subgroup of a $\rho_{n,P}$-group but we are able to place a bound on the class of a nilpotent subgroup of a $\rho_{n,P}$-group.

Definition 7.1.1. ($\rho_{n,P}$-groups). Let P be a finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$. A group G is a $\rho_{n,P}$-group if the following conditions hold:

1. G is in $\mathfrak{A}\mathfrak{P}_{\overline{c}}$;
2. $\operatorname{Fitt}(G)$ is abelian and
   (a) if $n = 0$ then G is torsion-free;
   (b) if $n \neq 0$ then $\operatorname{Fitt}(G)$ is of exponent dividing n;
3. $G/\operatorname{Fitt}(G)$ satisfies the universal theory of P;
4. If G_1 is a nilpotent subgroup of G then G_1 has class at most k;
5. (a) If $n = 0$ then $\operatorname{Fitt}(G)$ is torsion-free as a $\mathbb{Z}[G/\operatorname{Fitt}(G)]$-module;

(b) If $n \neq 0$ then, for each prime divisor p of n, the submodule of $\mathrm{Fitt}(G)$ of elements of order p is torsion-free as an $\mathbb{F}_p[G/\mathrm{Fitt}(G)]$-module.

In Section 7.2 below we shall define the collection $\mathfrak{W}_{n,P}$ of sentences.

Theorem 7.1.1. *Let P be a finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$ and let U be a non-trivial abelian group of exponent n. If G is a group which is not in $\mathfrak{P}_{\overline{c}}$ then the following conditions are equivalent:*

(1) $G \in \mathrm{ucl}(U \,\mathrm{wr}\, P)$;

(2) *G satisfies the universal sentences $\mathfrak{W}_{n,P}$;*

(3) *G is a $\rho_{n,P}$-group;*

(4) *Every finitely generated subgroup of G can be embedded in $C_n^{(r)} \,\mathrm{wr}\, Q$ for some $r \in \mathbb{N}$ and some Q such that $Q \models \mathrm{Th}_\forall(P)$.*

Theorem 7.1.2. *Let P be a finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$ and let U be a non-trivial abelian group of exponent n. If G is a finitely generated group then the following conditions are equivalent:*

(1) $G \in \mathrm{ucl}(U \,\mathrm{wr}\, P)$;

(2) *G can be embedded in $C_n^{(r)} \,\mathrm{wr}\, Q$ for some $r \in \mathbb{N}$ and some Q such that $Q \models \mathrm{Th}_\forall(P)$;*

(3) *Either G is a $\rho_{n,Q}$-group for some subgroup Q of P or $G \models \mathrm{Th}_\forall(P)$.*

We begin with some examples of polynilpotent groups which satisfy the conditions of the above theorems. Consider the group

$$P_{\overline{c}} = H_l \mathrm{wr}_{\mathfrak{N}_{c_l}} (H_{l-1} \mathrm{wr}_{\mathfrak{N}_{c_{l-1}}} (\cdots (H_2 \mathrm{wr}_{\mathfrak{N}_{c_2}} H_1) \cdots)),$$

where each H_i is a finitely generated torsion-free group in $\mathfrak{N}_{c_i}$.

Lemma 7.1.1. *Let N be the base group of $G = H \,\mathrm{wr}_{\mathfrak{N}_c}\, K$, where H is in $\mathfrak{N}_c$ and K is a torsion-free group. If $a \in N \setminus \{1\}$ and $g \in G$ then $[g,a] = 1$ implies $g \in N$.*

Proof. We proceed by induction on c. If $c=1$ then $G = H \operatorname{wr} K$ and the result holds by Lemma 2.3.1. Suppose now that the result holds for $c-1$ and let N be the base group of $G = H \operatorname{wr}_{\mathfrak{N}_c} K$, so that $G = N \rtimes K$. We may write elements of G as ordered pairs (u,d) with $u \in K$ and $d \in N$: let $g=(v,c)$ and $a=(1,d)$ with $v \in K$ and $c,d \in N$. We have $[a,g] = (1, d^{-1}c^{-1}d^v c))$ and if $[a,g]=1$ then $d^v = d^{c^{-1}}$. Since N is nilpotent of class at most c, if d is in $\gamma_c(N)$ then $[d,v] = [d,c^{-1}] = 1$ and a similar argument to that in Lemma 2.3.1 implies that $v=1$ as required. Suppose that d is not in $\gamma_c(N)$. Recall that

$$N = \frac{\prod^*_{u\in K} H_u}{\gamma_{c+1}\left(\prod^*_{u\in K} H_u\right)},$$

where $H_u \cong H$, so that

$$\frac{N}{\gamma_c(N)} \cong \frac{\prod^*_{u\in K} H_u}{\gamma_c\left(\prod^*_{u\in K} H_u\right)} \cong \frac{\prod^*_{u\in K} \frac{H_u}{\gamma_c(H_u)}}{\gamma_c\left(\prod^*_{u\in K} \frac{H_u}{\gamma_c(H_u)}\right)}.$$

This is the $\mathfrak{N}_{c-1}$-verbal product of the quotient groups $H_u/\gamma_c(H_u)$ and hence

$$(G/\gamma_c(N)) \cong H/\gamma_c(H) \operatorname{wr}_{\mathfrak{N}_{c-1}} K.$$

The image of a in $G/\gamma_c(N)$ is non-trivial and, by induction, g is in N. $\square$

It follows from this, by the same argument as in Lemma 3.3.5, that every nilpotent subgroup of G either is contained in N or embeds in K.

We show, also by induction, that every nilpotent subgroup of $P_{\overline{c}}$ is nilpotent of class at most $c = \max\{c_1, \dots, c_l\}$. If $l=1$ then $P_{\overline{c}}$ is nilpotent of class at most c_1 and the result holds. Suppose that the result holds for $P_{\overline{c}_{l-1}}$ which is a torsion-free group. If K is a nilpotent subgroup of $P_{\overline{c}} = H_l \operatorname{wr}_{\mathfrak{N}_{c_l}} P_{\overline{c}_{l-1}}$ then either K is contained in the base group of this verbal wreath product or K embeds in $P_{\overline{c}_{l-1}}$. In the former case K is nilpotent of class at most c_l and in the latter case K is nilpotent of class at most $\max\{c_1, \dots, c_{l-1}\}$ by induction; thus K is nilpotent of class at most $c = \max\{c_1, \dots, c_l\}$.

If G is any finitely generated subgroup of $P_{\overline{c}}$ then there is a positive integer l such that G is in $\mathcal{B}_l$. It follows that, if U is a torsion-free abelian group of exponent n, then the conclusions of Theorems 7.1.1 and 7.1.2 hold for $U \operatorname{wr} G$.

Now let F be the free group of rank r in $\mathfrak{P}_{\overline{c}}$. By the Shmel′kin embedding (see Chapter 2), the group F can be embedded in $F_r(\mathfrak{N}_{c_l}) \operatorname{wr}_{\mathfrak{N}_{c_l}} F_r(\mathfrak{P}_{\overline{c}_{l-1}})$. Applying the

Shmel'kin embedding repeatedly, we obtain the result that F can be embedded into the left-iterated verbal wreath product

$$P_{\overline{c}} = F_r(\mathfrak{N}_{c_l} \operatorname{wr}_{\mathfrak{N}_{c_l}} \left(F_r(\mathfrak{N}_{c_{l-1}}) \operatorname{wr}_{\mathfrak{N}_{c_{l-1}}} \left(\cdots \left(F_r(\mathfrak{N}_{c_2}) \operatorname{wr}_{\mathfrak{N}_{c_2}} F_r(\mathfrak{N}_{c_1})\right) \cdots\right)\right).$$

Thus, by the argument above, every subgroup of F is nilpotent of class at most $c = \max\{c_1, \ldots, c_l\}$. In fact there is a better bound on the class of nilpotent subgroups of a free polynilpotent group P with class sequence $\overline{c}$: every nilpotent subgroup of P is either cyclic or contained in the verbal subgroup $\gamma_{c_{l-1}}(\gamma_{c_{l-2}}(\cdots \gamma_1(P))$ (Baumslag, see Lemma 2.4.4) and this verbal subgroup is nilpotent of class c_l. Thus every nilpotent subgroup of P is either nilpotent of class at most c_l or abelian and c_l gives the bound we require.

Remark 7.1.1. The condition that there is a bound on the class of the nilpotent subgroups of P is a necessary one: without this bound we are unable to construct an existential formula defining the Fitting subgroup. We give an example of a finitely generated torsion-free soluble group P of derived length at most 4 such that, for every positive integer c, the group P contains a nilpotent subgroup which is not nilpotent of class $\leqslant c$. Let Q be the direct product

$$Q = \prod\nolimits_{c \in \mathbb{N}} F_2(\mathfrak{N}_c \wedge \mathfrak{S}_2).$$

Then Q is a direct product of countably many countable metabelian groups and hence it is itself a countable metabelian group. In [42], Neumann and Neumann showed that every countable soluble group of derived length d can be embedded in a 2-generator soluble group of derived length at most $d+2$. Their proof allows us to construct a 2-generator torsion-free soluble group P of derived length 4 which contains Q.

The group Q is generated by countably many elements and we may write $Q = \langle g_i \mid i \in \mathbb{N}\rangle$. We let $C = \langle c\rangle$ and $B = \langle b\rangle$ be infinite cyclic groups and we form the unrestricted wreath product $G = Q \operatorname{Wr} C$. For each $i \in \mathbb{N}$, we define an element u_i of Q^C by

$$u_i(c^r) = g_i^{-r}$$

for each $r \in \mathbb{Z}$, so that $[u_i, c] = g_i^{\circ}$. We now form a second unrestricted wreath product $H = G \operatorname{Wr} B$. We set $b_i = b^{2i-1}$ for each $i \in \mathbb{N}$ and we let a be the element of G^B defined as follows:

$$a(u) = \begin{cases} c & u = 1, \\ u_i & u = b_i^{-1} \text{ for } i \in \mathbb{Z}, \\ 1^\circ & \text{for all other } u \in B. \end{cases}$$

If we define elements g_i^* of G^B by $g_i^* = [a^{b_i}, a]$ then

$$g_i^*(u) = \begin{cases} 1 & \text{for } u \in B \setminus \{1\}, \\ g_i^\circ & u = 1. \end{cases}$$

Thus the group $Q^* = \langle g_i^* \mid i \in \mathbb{N} \rangle$ is isomorphic to the diagonal subgroup Q° of Q^C and hence to Q. It is contained in the 2-generator group $P = \langle a, b \rangle$ and P is a subgroup of H which is torsion-free and soluble of derived length at most 4. Thus P is an example of a finitely generated torsion-free soluble group which does not satisfy the conditions of the theorems.

The same argument can be used to construct further examples of groups for which the theorems do not hold by giving an embedding of the group

$$Q = \prod_{c \in \mathbb{N}} F_r(\mathfrak{N}_c \wedge \mathfrak{S}_d),$$

where d and r are any positive integers, into a 2-generator group which is torsion-free and soluble of derived length at most $d + 2$.

7.2 The sentences

Let P be a finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$. We construct a set of universal sentences $\mathfrak{W}_{n,P}$ which hold in $C_n \operatorname{wr} P$.

Recall from page 22 the definition of words $s_{\overline{c}_i}(\overline{x})$. We define a universal formula

$$\operatorname{fitt}^{\forall}(x) : \forall t \, ([x, x^t] = 1)$$

and an existential formula

$$\operatorname{fitt}^{\exists}_{\overline{c},k}(x) : \exists t_1 \dots t_C \Big(\big[s_{\overline{c}}(t_1, \dots, t_C), \underbrace{x, \dots, x}_{k} \big] = 1 \wedge s_{\overline{c}}(t_1, \dots, t_C) \neq 1 \Big).$$

The significance of the formulae $\mathrm{fitt}^{\forall}(x)$ and $\mathrm{fitt}^{\exists}_{\overline{c},k}(x)$ is as follows: if G is a model of $\mathfrak{W}_{n,P}$ then $\mathrm{Fitt}(G) = \mathrm{fitt}^{\forall}(G)$, and if $G \notin \mathfrak{P}_{\overline{c}}$ is a model of $\mathfrak{W}_{n,P}$ then $\mathrm{Fitt}(G) \subseteq \mathrm{fitt}^{\exists}_{\overline{c},k}(G)$ (see Lemmas 7.3.2 and 7.3.3 below).

As in previous chapters, we define sets of partitions of the set $\{1, \dots, m\}$ for each positive integer m. For each finite sequence $\overline{e} = (e_1, \dots, e_m)$ on $\{-1, 1\}$, we denote by $\mathcal{I}_{\overline{e}}$ the set of partitions of $\{1, \dots, m\}$ into two-element sets $\{i, j\}$ such that $e_i = -e_j$. For each positive integer m and each prime p, we denote by $\mathcal{I}_{m,p}$ the set of partitions of $\{1, \dots, m\}$ into sets X of size a multiple of p.

We begin with sentences which hold in $C_n \operatorname{wr} P$, where P is any finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$ (see Lemma 7.2.1 below). Let $\mathfrak{W}_{n,\overline{c},k}$ denote the union of the following five sets of universal sentences together with the sentence ϕ_{gp} as defined on page 14:

(1) The single sentence $\phi_1^{\overline{c}}$ defined by

$$\forall x_1 \dots x_C, y_1, \dots, y_C \left([s_{\overline{c}}(x_1, \dots, x_C), s_{\overline{c}}(y_1, \dots, y_C)] = 1\right).$$

A group G satisfies $\phi_1^{\overline{c}}$ if and only if G is in $\mathfrak{A}\mathfrak{P}_{\overline{c}}$.

(2) The single sentence ϕ_2^k defined by

$$\forall x_1 \dots x_{k+1} \left(\left(\bigwedge_{j=1}^{k+1} [x_1, \dots, x_{k+1}, x_j] = 1 \right) \to [x_1, \dots, x_{k+1}] = 1 \right).$$

If a group G satisfies ϕ_2^k then G is in $\mathcal{B}_k$.

(3) (a) If $n = 0$, the countable collection of sentences $\left\{\phi_3^0(m) \mid m \in \mathbb{N}\right\}$, where $\phi_3^0(m)$ is defined by

$$\forall x \left(x^m = 1 \to x = 1\right).$$

A group G satisfies $\phi_3^0(m)$ for all $m \in \mathbb{N}$ if and only if G is torsion-free.

(b) For $n \neq 0$, the single sentence $\phi_3^{n,\overline{c},k}$ defined by

$$\forall x \left(\mathrm{fitt}^{\exists}_{\overline{c},k}(x) \to x^n = 1\right).$$

Let G be a group in $(\mathfrak{A}\mathfrak{P}_{\overline{c}} \setminus \mathfrak{P}_{\overline{c}}) \cap \mathcal{B}_k$. If G satisfies $\phi_3^{n,\overline{c},k}(m)$ then $\mathrm{Fitt}(G)$ is of exponent dividing n.

(4) The single sentence $\phi_4^{\overline{c},k}$ defined by

$$\forall xy \left(\left(\mathrm{fitt}^{\exists}_{\overline{c},k}(x) \wedge \mathrm{fitt}^{\exists}_{\overline{c},k}(y) \right) \to [x,y]=1 \right).$$

Let G be a group in $(\mathfrak{AP}_{\overline{c}} \setminus \mathfrak{P}_{\overline{c}}) \cap \mathcal{B}_k$. If G satisfies $\phi_4^{\overline{c}}$ then $\mathrm{Fitt}(G)$ is abelian.

(5) (a) If $n = 0$, the countable collection of sentences

$$\left\{ \phi_5^{0,\overline{c},k}(\overline{e}) \mid \overline{e} = (e_1, \ldots, e_m) \in \{-1,1\}^m, m \in \mathbb{N} \right\},$$

where

i. if $\sum_{j=1}^m e_j \neq 0$ then $\phi_5^{0,\overline{c},k}(\overline{e})$ is defined by

$$\forall xy_1 \ldots y_m \left(\left(\mathrm{fitt}^{\exists}_{\overline{c},k}(x) \wedge x \neq 1 \wedge x^{\sum_{j=1}^m e_j y_j} = 1 \right) \to y_1 \neq y_1 \right);$$

ii. if $\sum_{j=1}^m e_j = 0$ then $\phi_5^{0,\overline{c},k}(\overline{e})$ is defined by

$$\forall xy_1 \ldots y_m \Bigg(\left(\mathrm{fitt}^{\exists}_{\overline{c},k}(x) \wedge x \neq 1 \wedge x^{\sum_{j=1}^m e_j y_j} = 1 \right) \\ \to \bigvee_{I \in \mathcal{I}_{0,\overline{e}}} \bigwedge_{X \in I} \bigwedge_{j_1, j_2 \in X} \mathrm{fitt}^{\forall}(y_{j_1} y_{j_2}^{-1}) \Bigg).$$

Let G be a group in $(\mathfrak{AP}_{\overline{c}} \setminus \mathfrak{P}_{\overline{c}}) \cap \mathcal{B}_k$. If $G/\mathrm{Fitt}(G)$ is torsion-free and G satisfies $\phi_5^{0,\overline{c},k}(\overline{e})$ for all finite sequences $\overline{e}$ on $\{-1,1\}$ then $\mathrm{Fitt}(G)$ is torsion-free as a $\mathbb{Z}[G/\mathrm{Fitt}(G)]$-module.

(b) If $n \neq 0$, for each prime divisor p of n, the countable collections of sentences, $\left\{ \phi_5^{p,\overline{c},k}(m) \mid m \in \mathbb{N} \right\}$, where

i. if $m \not\equiv 0 \bmod p$ then $\phi_5^{p,\overline{c},k}(m)$ is defined by

$$\forall xy_1 \ldots y_m \left(\left(x^p = 1 \wedge x \neq 1 \wedge x^{\sum_{j=1}^m y_j} = 1 \right) \to y_1 \neq y_1 \right);$$

ii. if $m \equiv 0 \bmod p$ then $\phi_5^{p,\overline{c},k}(m)$ is defined by

$$\forall xy_1 \ldots y_m \Bigg(\left(x^p = 1 \wedge x \neq 1 \wedge x^{\sum_{j=1}^m y_j} = 1 \right) \\ \to \bigvee_{I \in \mathcal{I}_{p,m}} \bigwedge_{X \in I} \bigwedge_{j_1, j_2 \in X} \mathrm{fitt}^{\forall}(y_{j_1} y_{j_2}^{-1}) \Bigg).$$

Let G be a group in $(\mathfrak{AP}_{\overline{c}} \backslash \mathfrak{P}_{\overline{c}}) \cap \mathcal{B}_k$. If $G/\operatorname{Fitt}(G)$ is torsion-free and G satisfies $\phi_5^{p,\overline{c},k}(m)$ for all $m \in \mathbb{N}$ then the submodule of $\operatorname{Fitt}(G)$ of elements of order p is a torsion-free $\mathbb{F}_p[G/\operatorname{Fitt}(G)]$-module.

Next we consider the universal theory of a particular group P. Let P be a finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$ and let $\mathfrak{U}_P \subseteq \operatorname{Th}_\forall(P)$ be a set of sentences such that a group Q satisfies the universal theory of P if and only if $Q \models \mathfrak{U}_P$. Let ψ be a sentence in $\mathfrak{U}_P$. We may assume without loss of generality that ψ is of the form

$$\forall x_1 \ldots x_m \left(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} (u_{ij}(\overline{x}) = 1 \wedge w_{ij}(\overline{x}) \neq 1) \right),$$

where $u_{ij}(\overline{x})$ and $w_{ij}(\overline{x})$ are words in the variables $x_1, \ldots, x_m$. We define $\psi^*_{\overline{c},k}$ by

$$\forall x_1 \ldots x_m \left(\bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} \left(\operatorname{fitt}^\forall(u_{ij}(\overline{x})) \wedge \neg \operatorname{fitt}^\exists_{\overline{c},k}(w_{ij}(\overline{x})) \right) \right).$$

We let $\mathfrak{U}^*_P = \{\psi^*_{\overline{c},k} | \psi \in \mathfrak{U}_P\}$ and let

$$\mathfrak{W}_{n,P} = \mathfrak{W}_{n,\overline{c},k} \cup \mathfrak{U}^*_P.$$

Lemma 7.2.1. *If P is a finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$ then*

$$C_n \operatorname{wr} P \models \mathfrak{W}_{n,P}.$$

Proof. Let D be the base group of the wreath product. It follows from Lemma 2.3.1 that $\operatorname{fitt}^\exists(C_n \operatorname{wr} P) = \operatorname{Fitt}(C_n \operatorname{wr} P) = D$ and it is clear that $\operatorname{fitt}^\forall(C_n \operatorname{wr} P) = D$.

Since P is a finitely generated torsion-free soluble group, the group rings $\mathbb{Z}[P]$ and $\mathbb{F}_p[P]$, where p is a prime, are integral domains by Theorem 3.1.1. If $n = 0$ then, since D is a free $\mathbb{Z}[P]$-module, D is torsion-free as a $\mathbb{Z}[P]$-module. If $n \neq 0$ then D is a direct product of groups D_p of exponent p, where p is a prime divisor of n, and each group D_p is torsion-free as an $\mathbb{F}_p[P]$-module. It follows easily from these facts and the identities (3.3.1)–(3.3.4) that $C_n \operatorname{wr} P \models \mathfrak{W}_{n,\overline{c},k}$.

It remains to show that $C_n \operatorname{wr} P \models \mathfrak{U}^*_P$; the result follows immediately from the fact that $(C_n \operatorname{wr} P)/D \cong P \models \mathfrak{U}_P$. $\square$

As in previous chapters, each of the sentences in $\mathfrak{W}_{n,P}$ contains only the universal quantifier $\forall$ when written in prenex normal form. As usual we have given the sentences in a more transparently comprehensible form than the prenex normal form.

7.3 The universal closure of $U \operatorname{wr} P$

Fix a finitely generated torsion-free group P in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$ and suppose throughout this section that G is a group in which the universal sentences $\mathfrak{W}_{n,P}$ hold. Our aim is to describe some of the properties of G that follow from this supposition.

In Lemmas 7.3.1, 7.3.2 and 7.3.3 we establish the significance of the universal formula $\operatorname{fitt}^{\forall}(x)$ and the existential formula $\operatorname{fitt}^{\exists}_{\overline{c},k}(x)$. These formulae will allow us to describe, using universal sentences, the subgroup $\operatorname{Fitt}(G)$ and the quotient group $G/\operatorname{Fitt}(G)$. The arguments used in the proofs of Lemmas 7.3.1 and 7.3.3 are almost identical to those of their counterparts in Chapter 6.

Lemma 7.3.1. *The Fitting subgroup of G is nilpotent of class at most c.*

Proof. See Lemma 6.3.1. $\square$

Lemma 7.3.2. *If G is not in $\mathfrak{P}_{\overline{c}}$ and g is an element of* $\operatorname{Fitt}(G)$ *then* $G \models \operatorname{fitt}^{\exists}_{\overline{c},k}(g)$.

Proof. Let g be an element of $\operatorname{Fitt}(G)$. Since $G \notin \mathfrak{P}_{\overline{c}}$, we may choose elements $h_1, \dots, h_C \in G$ such that $h = s_{\overline{c}}(h_1, \dots, h_C) \neq 1$. The element h is in the abelian normal subgroup $s_{\overline{c}}(G)$ of G and it follows that $h \in \operatorname{Fitt}(G)$. Thus $\langle g, h\rangle \leqslant \operatorname{Fitt}(G)$ is nilpotent of class at most k and $[h, \underbrace{g, \dots, g}_{k}] = 1$. $\square$

Lemma 7.3.3. *If G is not in $\mathfrak{P}_{\overline{c}}$ then* $\operatorname{Fitt}(G)$ *is abelian and* $\operatorname{Fitt}(G) = \operatorname{fitt}^{\forall}(G)$.

Proof. See Lemma 6.3.3. $\square$

In Lemma 7.3.4 we describe the universal theory of the quotient of G by its Fitting subgroup provided that $G \notin \mathfrak{P}_{\overline{c}}$. Again the proof is almost identical to the proof of the corresponding lemma in Chapter 6.

Lemma 7.3.4. *If G is not in $\mathfrak{P}_{\overline{c}}$ then $G/\operatorname{Fitt}(G)$ satisfies the sentences $\mathfrak{U}_P$.*

Proof. See Lemma 6.3.5. $\square$

7.4 $\rho_{n,P}$-groups

Fix a finitely generated torsion-free group P in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$. In this section we establish the relationship between the $\rho_{n,P}$-groups defined in Section 7.1 and the sentences $\mathfrak{W}_{n,P}$. This relationship plays a key role in our proof of Theorem 7.1.1.

Lemma 7.4.1. *If G satisfies $\mathfrak{W}_{n,P}$ and G is not in $\mathfrak{P}_{\overline{c}}$ then G is a $\rho_{n,P}$-group.*

Proof. The validity in G of the sentences $\phi_1^{\overline{c}}$ and ϕ_2^k implies that $G \in \mathfrak{A}\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$. By Lemmas 7.3.3 and 7.3.2, the Fitting subgroup of G is abelian, $\operatorname{fitt}^{\forall}(G) = \operatorname{Fitt}(G)$ and $\operatorname{Fitt}(G) \subseteq \operatorname{fitt}^{\exists}_{\overline{c},k}(G)$. If $n = 0$, the truth of the sentences $\phi_3^0(m)$ for all $m \in \mathbb{N}$ implies that G is torsion-free and, if $n \neq 0$, the truth of the sentence $\phi_3^{n,\overline{c},k}$ implies that $\operatorname{Fitt}(G)$ is of exponent dividing n. Lemma 7.3.4 implies that $G/\operatorname{Fitt}(G) \models \mathfrak{U}_H$ and hence $G/A \models \operatorname{Th}_{\forall}(P)$.

The same argument as in Lemma 6.4.1 completes the proof: if $n = 0$ then $\operatorname{Fitt}(G)$ is torsion-free as a $\mathbb{Z}[G/\operatorname{Fitt}(G)]$-module, and if $n \neq 0$ then, for each prime divisor p of n, the submodule of $\operatorname{Fitt}(G)$ of elements of order p is torsion-free as an $\mathbb{F}_p[G/\operatorname{Fitt}(G)]$-module. $\square$

Lemma 7.4.2. *If G is a $\rho_{n,P}$-group then G satisfies $\mathfrak{W}_{n,\overline{c},k}$.*

Proof. Since G is in $\mathfrak{A}\mathfrak{P}_{\overline{c}} \cup \mathcal{B}_k$, the sentences $\phi_1^{\overline{c}}$ and ϕ_2^k hold in G. Also, since $\operatorname{Fitt}(G)$ is abelian, $\operatorname{Fitt}(G) = \operatorname{fitt}^{\forall}(G)$.

In order to show that G satisfies the remaining sentences in $\mathfrak{W}_{n,\overline{c},k}$, it is enough to check that $\operatorname{fitt}^{\exists}_{\overline{c},k}(G) \subseteq \operatorname{Fitt}(G)$. Note that, since $s_{\overline{c}}(G)$ is abelian, it is in $\operatorname{Fitt}(G)$. Let g be an element of $\operatorname{fitt}^{\exists}_{\overline{c},k}(G)$, so that there is an element $h = s_{\overline{c}}(h_1, \ldots, h_C) \neq 1$ with $h_1, \ldots, h_C \in G$ such that $\big[h, \underbrace{g, \ldots, g}_{k}\big] = 1$. Now $h \in s_{\overline{c}}(G) \leqslant \operatorname{Fitt}(G)$ and the result follows by the same argument as in the proof of Lemma 6.4.2. $\square$

Lemma 7.4.3. *If G is a $\rho_{n,P}$-group then G satisfies $\mathfrak{U}_P^*$.*

Proof. The argument is almost identical to that of the proof of Lemma 6.4.4. $\square$

Combining Lemmas 7.4.2 and 7.4.3 we have:

Lemma 7.4.4. *If G is a $\rho_{n,P}$-group then G satisfies $\mathfrak{W}_{n,P}$.*

7.5 Proof of theorems

In this section we prove the results stated in Section 7.1. For the reader's convenience we restate each theorem before proving it. Recall that n is a square-free positive integer or zero.

Theorem 7.1.1 *Let P be a finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$ and let U be a non-trivial abelian group of exponent n. If G is a group which is not in $\mathfrak{P}_{\overline{c}}$ then the following conditions are equivalent:*

(1) $G \in \mathrm{ucl}(U \operatorname{wr} P)$;

(2) *G satisfies the universal sentences $\mathfrak{W}_{n,P}$;*

(3) *G is a $\rho_{n,P}$-group;*

(4) *Every finitely generated subgroup of G can be embedded in $C_n^{(r)} \operatorname{wr} Q$ for some $r \in \mathbb{N}$ and some Q such that $Q \in \mathrm{Th}_{\forall}(P)$.*

Proof. We prove the implications (1) $\implies$ (2) $\iff$ (3) $\implies$ (4) $\implies$ (1).

(1) $\implies$ (2)**.** The group G is in $\mathrm{ucl}(U \operatorname{wr} P)$ and so $G \models \mathrm{Th}_{\forall}(U \operatorname{wr} H)$. By Lemma 3.4.1 and Lemma 3.4.2, we have $U \operatorname{wr} P \models \mathrm{Th}_{\forall}(C_n \operatorname{wr} P)$; thus $G \models \mathrm{Th}_{\forall}(C_n \operatorname{wr} P)$ and it follows, by Lemma 7.2.1, that $G \models \mathfrak{W}_{n,P}$.

(2) $\iff$ (3)**.** The equivalence of (2) and (3) follows immediately from the results of Section 7.4.

(3) $\implies$ (4)**.** Suppose now that G is a $\rho_{n,P}$-group, so that $G \models \mathfrak{W}_{n,P}$ by Lemma 7.4.4, and let G_1 be a finitely generated subgroup of G. Since G is not in $\mathfrak{P}_{\overline{c}}$, there exist elements $h_1, \ldots, h_C \in G$ such that $s_{\overline{c}}(h_1, \ldots, h_C) \neq 1$ and we let $G_2 = \langle G_1, h_1, \ldots, h_C \rangle$. The subgroup $s_{\overline{c}}(G_2)$ is non-trivial and G_2 is not in $\mathfrak{P}_{\overline{c}}$. The sentences $\mathfrak{W}_{n,P}$ are universal and so every subgroup of G satisfies $\mathfrak{W}_{n,P}$; in particular G_2 satisfies $\mathfrak{W}_{n,P}$, and hence G_2 is a $\rho_{n,P}$-group by Lemma 7.4.1. The quotient $\overline{G}_2 = G_2/\operatorname{Fitt}(G_2)$ satisfies the universal theory of P; thus $\overline{G}_2$ is a finitely generated torsion-free soluble group and hence the group rings $\mathbb{Z}[\overline{G}_2]$ and $\mathbb{F}_p[\overline{G}_2]$, where p is a prime, are integral

domains by Theorem 3.1.1. It follows, by Lemma 3.1.2, that $\mathbb{Z}[\overline{G}_2]$ and $\mathbb{F}_p[\overline{G}_2]$ are Ore domains.

If $n = 0$ then G_2 is an extension of $\mathrm{Fitt}(G_2)$ by $\overline{G}_2$ satisfying the conditions of Lemma 3.3.1, and G_2 embeds in $V \operatorname{wr} \overline{G}_2$, where V is a finitely generated torsion-free abelian group. It follows that G_2 embeds in $\mathbb{Z}^{(r)} \operatorname{wr} \overline{G}_2$ for some $r \in \mathbb{N}$. If $n \neq 0$ then G_2 is an extension of $\mathrm{Fitt}(G_2)$ by $\overline{G}_2$ satisfying the conditions of Lemma 3.3.2. Thus G_2 embeds in $V \operatorname{wr} \overline{G}_2$, where V is a finitely generated abelian group of exponent dividing n. We may write $V = C_{n_1} \times \cdots \times C_{n_r}$ where C_{n_j} is a cyclic group of exponent n_j which divides n (see [53, 4.2.1]). Each cyclic group C_{n_j} embeds in C_n and hence V embeds in the direct product of r copies of C_n. It follows that G_2 embeds in $C_n^{(r)} \operatorname{wr}(\overline{G}_2)$. In both cases we have $\overline{G}_2 \models \mathrm{Th}_\forall(P)$ and, since $G_1 \leqslant G_2$, the subgroup G_1 embeds in $C_n^{(r)} \operatorname{wr} \overline{G}_2$ as required.

(4) $\Longrightarrow$ (1)**.** Let ϕ be in $\mathrm{Th}_\forall\big(U \operatorname{wr} P\big)$ and suppose without loss of generality that ϕ is of the form

$$\forall x_1 \ldots x_m \, \varphi(x_1, \ldots, x_m),$$

where $\varphi(x_1, \ldots, x_m)$ is quantifier-free. We let $g_1 \ldots, g_m$ be elements of G and let $G_1 = \langle g_1, \ldots, g_m \rangle$, so that G_1 embeds in $V \operatorname{wr} Q$, where V is a finitely generated abelian group of exponent n and $Q \models \mathfrak{U}_P$. By Lemmas 3.4.1 and 3.4.2, the restricted wreath product $V \operatorname{wr} Q$ satisfies the universal theory of $U \operatorname{wr} P$; in particular $V \operatorname{wr} Q \models \phi$ and hence $\varphi(g_1, \ldots, g_m)$ holds. This is true for all m-tuples in G and hence $G \models \phi$. Thus G satisfies $\mathrm{Th}_\forall\big(U \operatorname{wr} P\big)$ and G is in $\mathrm{ucl}\big(U \operatorname{wr} P\big)$. $\square$

Theorem 7.1.2 *Let P be a finitely generated torsion-free group in $\mathfrak{P}_{\overline{c}} \cap \mathcal{B}_k$ and let U be a non-trivial abelian group of exponent n. If G is a finitely generated group then the following conditions are equivalent:*

(1) $G \in \mathrm{ucl}(U \operatorname{wr} P)$*;*

(2) *G can be embedded in some restricted wreath product $C_n^{(r)} \operatorname{wr} Q$ with $r \in \mathbb{N}$ where $Q \models \mathrm{Th}_\forall(P)$;*

(3) *Either G is a $\rho_{n,Q}$-group for some subgroup Q of P or $G \models \mathrm{Th}_\forall(P)$.*

Proof.

(1) $\iff$ (2). If G embeds in $W = C_n^{(r)} \operatorname{wr} Q$ for some $r \in \mathbb{N}$ and some Q such that $Q \models \mathrm{Th}_\forall(P)$ then W satisfies $\mathrm{Th}_\forall(U \operatorname{wr} P)$ by Lemmas 3.4.1 and 3.4.2. Hence G satisfies $\mathrm{Th}_\forall(U \operatorname{wr} P)$ and G is in $\mathrm{ucl}(U \operatorname{wr} P)$.

If G is in $\mathrm{ucl}(U \operatorname{wr} P)$ then G embeds in some ultrapower $\mathcal{G}$ of $U \operatorname{wr} P$ by Lemma 2.6.3 and $\mathcal{G}$ is elementarily equivalent to $U \operatorname{wr} P$ by Łoś' Theorem (see Theorem 2.6.1); thus $\mathcal{G}$ is not in $\mathfrak{P}_{\overline{c}}$. It follows from Theorem 7.1.1 that G can be embedded in $C_n^{(r)} \operatorname{wr} Q$ for some $r \in \mathbb{N}$ and some Q such that $Q \models \mathrm{Th}_\forall(P)$.

(2) $\iff$ (3). If G satisfies $\mathrm{Th}_\forall(P)$ then G embeds in $C_n \operatorname{wr} G$ and the result holds. Now let G be a $\rho_{n,Q}$-group for some subgroup Q of P. The quotient $\overline{G} = G/\operatorname{Fitt}(G)$ satisfies the universal theory of Q and G is an extension of $\operatorname{Fitt}(G)$ by $\overline{G}$ satisfying the conditions of Lemma 3.3.1 or Lemma 3.3.2. As in the proof of Theorem 7.1.1, the group G can be embedded in $C_n^{(r)} \operatorname{wr} \overline{G}$ for some $r \in \mathbb{N}$ and $\overline{G} \models \mathrm{Th}_\forall(Q)$. Since $Q \leqslant P$, we have $\mathrm{Th}_\forall(P) \subseteq \mathrm{Th}_\forall(Q)$ and $\overline{G} \models \mathrm{Th}_\forall(P)$.

Now suppose that G embeds in $V \operatorname{wr} Q$, where V is a finitely generated abelian group of exponent n and $Q \models \mathrm{Th}_\forall(P)$. If G is not in $\mathfrak{P}_{\overline{c}}$ then the result follows from Theorem 7.1.1. Now let $l' \leqslant l$ be the least integer such that G is in $\mathfrak{P}_{\overline{c}_{l'}}$. By Lemma 3.3.4 there are two cases to consider: if G embeds in P then G satisfies $\mathrm{Th}_\forall(P)$; otherwise G embeds in $V_1 \operatorname{wr} Q_1$, where V_1 is a finitely generated abelian group of exponent n and $Q_1 \in \mathfrak{P}_{\overline{c}_{l'-1}}$ is a subgroup of Q. Since G is not in $\mathfrak{P}_{\overline{c}_{l'-1}}$, Theorem 7.1.1 implies that G is a $\rho_{n,Q}$-group. $\square$

7.6 Non-split extensions

Let G be a finitely generated group with an abelian normal subgroup A such that G/A is a torsion-free soluble group. If A is torsion-free as a $\mathbb{Z}[G/A]$-module then, by Lemma 3.3.1, the group G embeds in $U \operatorname{wr}(G/A)$, where U is a finitely generated torsion-free abelian group, and hence G is in $\mathrm{ucl}(\mathbb{Z} \operatorname{wr}(G/A))$. If A has exponent n and the submodule of A of elements of order p is torsion-free as an $\mathbb{F}_p[G/A]$-module for each prime divisor p of n then, by Lemma 3.3.2, the group G embeds in $U \operatorname{wr}(G/A)$, where U is a finitely generated abelian group of exponent n, and hence G is in $\mathrm{ucl}(C_n \,\mathrm{wr}(G/A))$. We construct examples of such groups using the Magnus and Shmel′kin embeddings. See Chapter 2 for more about these embeddings.

Let H be an r-generator torsion-free polynilpotent group and let F be an absolutely free group of rank r; then H is isomorphic to F/R for some $R \lhd F$. By the Magnus embedding, $G = F/R'$ can be embedded in $(F/F') \,\mathrm{wr}(F/R)$. The quotient F/F' is a non-trivial torsion-free abelian group and hence, by [16, Lemma 1], is universally equivalent to $\mathbb{Z}$. Thus the group G is in $\mathrm{ucl}(\mathbb{Z} \,\mathrm{wr}\, H)$ and it follows that $\mathrm{ucl}(G) \subseteq \mathrm{ucl}(\mathbb{Z} \,\mathrm{wr}\, H)$.

Now let H be a free group of rank r in $\mathfrak{P}_{\overline{c}}$, so that G is a free group of rank r in $\mathfrak{A}\mathfrak{P}_{\overline{c}}$; the universal closure of G is contained in $\mathrm{ucl}(\mathbb{Z}\mathrm{wr}F_r(\mathfrak{P}_{\overline{c}}))$. By a result of Baumslag (see Lemma 2.4.4), every nilpotent subgroup of G is either cyclic or contained in the verbal subgroup $s_{\overline{c}}(G)$, which is abelian; thus every nilpotent subgroup of G is abelian. Let ϕ denote the sentence

$$\forall xy\big(([x, y, x] = 1 \wedge [x, y, y] = 1) \rightarrow [x, y] = 1\big).$$

Then ϕ holds in G and it follows that every nilpotent group in $\mathrm{ucl}(G)$ is abelian. The verbal subgroup $s_{\overline{c}_{l-1}}(H)$ is a nilpotent subgroup of H and hence of $\mathbb{Z} \,\mathrm{wr}\, H$. Since universal closures are closed under taking subgroups, we have $s_{\overline{c}'}(H) \in \mathrm{ucl}(\mathbb{Z} \,\mathrm{wr}\, H)$. If $c_l \neq 1$ then $s_{\overline{c}_{l-1}}(H)$ is not abelian and $s_{\overline{c}_{l-1}}(H)$ is not in $\mathrm{ucl}(G)$; thus the inclusion

$$\mathrm{ucl}(F_r(\mathfrak{A}\mathfrak{P}_{\overline{c}})) \subseteq \mathrm{ucl}(\mathbb{Z} \,\mathrm{wr}\, F_r(\mathfrak{P}_{\overline{c}}))$$

is strict.

Let V be the set of words $\{[x, y], x^p\}$. By the Shmel′kin embedding, the group $G_p = F/V(R)$ can be embedded in $(F/V(F)) \,\mathrm{wr}_{\mathfrak{V}}(F/R) = (F/V(F)) \,\mathrm{wr}(F/R)$. Since the quotient $F/V(F)$ is an abelian group of exponent p, Lemma 3.4.2 implies that $(F/V(F)) \,\mathrm{wr}\, F/R$ is universally equivalent to $C_p \,\mathrm{wr}\, H$ and hence $G_p \in \mathrm{ucl}(C_p \,\mathrm{wr}\, H)$: for example, $\mathrm{ucl}(F_r(\mathfrak{A}_p\mathfrak{P}_{\overline{c}}))$ is contained in $\mathrm{ucl}(C_p \,\mathrm{wr}\, F_r(\mathfrak{P}_{\overline{c}}))$. By the same argument as above, this inclusion is strict when $c_l \neq 1$.

Part III

THE
THEORIES OF
CERTAIN
SOLUBLE GROUPS

Chapter 8

COMMUTATOR LENGTHS AND THE THEORY OF CERTAIN SOLUBLE GROUPS

We say that a group G is a C_n*-group* if there exists an integer t such that each element of $\gamma_n(G)$ can be written as a product of t commutators $[h_1, g_1] \cdots [h_t, g_t]$ with $g_i \in G$ and $h_i \in \gamma_{n-1}(G)$ for $i = 1, \ldots, t$.

In the first part of the chapter we consider some examples of C_n-groups. It is known that a group G is a C_2-group if it is in one of the following classes of soluble groups:

- finitely generated nilpotent groups (Stroud, see [2]);
- finitely generated abelian-by-nilpotent groups (Akhavan-Malayeri and Rhemtulla [2]);
- finitely generated metanilpotent groups (Hartley [19]);
- finitely generated abelian-by-metanilpotent groups (Rhemtulla [52]);
- groups with an abelian normal subgroup such that the quotient satisfies the maximal condition for normal subgroups (Rhemtulla [52]);
- groups of finite Prüfer rank (Akhavan-Malayeri [1]).

For a number of these classes we can extend these results to all terms of the lower central series. In the following pages we show that G is a C_n-group for all $n \geqslant 2$ if G is in one of the following classes:

- finitely generated finite-by-abelian-by-metanilpotent groups;

- finite-by-soluble groups satisfying the maximal condition on normal subgroups.

In the case of finitely generated abelian-by-metanilpotent groups we obtain an explicit bound.

We go on to give one-variable existential formulae defining the terms of the lower central series in a group which is a C_n-group for all $n \geqslant 2$. These formulae allow us to say something about the elementary theory of such groups; in particular we show that elementary equivalence is preserved under taking quotients by the terms of the lower central series.

The corresponding result for universal equivalence does not hold: Timoshenko [67, Proposition 2] constructs a group universally equivalent to a non-cyclic free metabelian group whose abelianization contains elements of finite order.

8.1 The lower central series

In this section we obtain some examples of C_n-groups. We note to begin with that a finite-by-C_n group is a C_n-group. Let G be a group with a finite normal subgroup A such that G/A is a C_n-group and let t be a positive integer such that every element of $\gamma_n(G/A)$ can be written as a product of t commutators. For $a \in A \cap \gamma_n(G)$, we suppose that $a = [h_1, g_1] \cdots [h_{t_a}, g_{t_a}]$ with $h_i \in \gamma_{n-1}(G)$ and $g_i \in G$ and we let $t' = \max\{t_a \,|\, a \in A \cap \gamma_n(G)\}$; every element of $\gamma_n(G)$ can be written as a product of $t + t'$ commutators.

In his Ph.D. thesis Peter Stroud proved the following result (see [2]).

Lemma 8.1.1. *Let $G = \langle g_1, \ldots, g_n \rangle$ be a nilpotent group. Every element of the commutator subgroup G' is a product of n commutators $[g_1, h_1] \cdots [g_n, h_n]$, for suitable elements h_i in G.*

Using the same argument, we extend the result to all terms of the lower central series. Our result differs from that of Oger [46, Lemma 1.1] for finite-by-nilpotent groups in that we obtain an explicit bound on the number of commutators needed.

Lemma 8.1.2. *Let $G = \langle g_1, \ldots, g_r \rangle$ be a nilpotent group. For $n \geqslant 2$, an element g*

of $\gamma_n(G)$ can be written as a product of r commutators $g = [h_1, g_1] \cdots [h_r, g_r]$ with $h_i \in \gamma_{n-1}(G)$ for $i = 1, \ldots, r$.

Proof. Suppose that G is nilpotent of class $c \geqslant 1$ and proceed by induction on c. If $c = 1$ then G is abelian and $\gamma_n(G) = \{1\}$ for all $n \geqslant 2$. Thus the result holds vacuously.

Now let $c \geqslant 2$ and suppose that the result holds for $c - 1$. If $n \geqslant c + 1$ then $\gamma_n(G) \leqslant \gamma_{c+1}(G) = 1$ and the result holds vacuously. Suppose therefore that $n \leqslant c$ and let g be an element of $\gamma_n(G)$. The quotient $G/\gamma_c(G)$ is nilpotent of class at most $c-1$ and the image of g in $G/\gamma_c(G)$ is in $\gamma_n(G/\gamma_c(G))$. By induction, we may write $g = [h_1, g_1] \cdots [h_r, g_r]k$ with $h_i \in \gamma_{n-1}(G)$ and $g_i \in G$ for $i = 1, \ldots, r$ and $k \in \gamma_c(G)$. Now $\gamma_c(G) = [\gamma_{c-1}(G), G]$ and $\gamma_{c-1}(G)$ is contained in $\zeta_2(G)$. Thus, by [2, Lemma 3], we have $k = [a_1, g_1] \cdots [a_r, g_r]$ with $a_i \in \gamma_{c-1}(G)$ for $i = 1, \ldots, r$. Since $[a_i, g_i] \in \gamma_c(G) \leqslant Z(G)$ for $i = 1, \ldots, r$, we have

$$\begin{aligned} g &= [h_1, g_1] \cdots [h_r, g_r]k \\ &= [h_1, g_1] \cdots [h_r, g_r][a_1, g_1] \cdots [a_r, g_r] \\ &= [h_1, g_1][a_1, g_1] \cdots [h_r, g_r][a_r, g_r] \\ &= [a_1 h_1, g_1] \cdots [a_r h_r, g_r] \end{aligned}$$

with $a_i \in \gamma_{c-1}(G) \leqslant \gamma_{n-1}(G)$ and $a_i h_i \in \gamma_{n-1}(G)$ as required. □

We also extend the result of Akhavan-Malayeri and Rhemtulla [2, Theorem 2] about commutator lengths in abelian-by-nilpotent groups.

Lemma 8.1.3. *Let G be an abelian-by-nilpotent group generated by r elements. For $n \geqslant 2$, an element g of $\gamma_n(G)$ can be written as a product of r commutators $g = [u_1, v_1] \cdots [u_r, v_r]$ with $u_i \in \gamma_{n-1}(G)$ and $v_i \in G$ for $i = 1, \ldots, r$.*

Proof. Let $g_1, \ldots, g_r$ generate G. Since G is abelian-by-nilpotent, the subgroup $\gamma_{c+1}(G)$ is abelian for some positive integer c. We let $d = \max\{n-2, c\}$, so that $\gamma_{d+1}(G) \leqslant \gamma_{c+1}(G)$ is abelian. By [2, Lemma 3], we have

$$[\gamma_{d+1}(G), G] = \{[a_1, g_1] \cdots [a_r, g_r] \mid a_i \in \gamma_{d+1}(G), i = 1, \ldots, r\}.$$

Let $\overline{G} = G/\gamma_{d+2}(G)$, so that $\overline{G}$ is nilpotent and, by Lemma 8.1.2,

$$\gamma_n(\overline{G}) = \{[\overline{h}_1, \overline{g}_1] \cdots [\overline{h}_r, \overline{g}_r] \mid \overline{h}_i \in \gamma_{n-1}(\overline{G}), i = 1, \ldots, r\},$$

where $\overline{g}_i$ is the image of g_i in $\overline{G}$. Let g be an element of $\gamma_n(G)$, so that the image of g in $\overline{G}$ is in $\gamma_n(\overline{G})$ and $g = [h_1, g_1] \cdots [h_r, g_r]w$ with $h_i \in \gamma_{n-1}(G)$ for $i = 1, \ldots, r$ and $w \in \gamma_{d+2}(G) = [\gamma_{d+1}(G), G]$. It follows that

$$g = [h_1, g_1] \cdots [h_r, g_r][a_1, g_1] \cdots [a_r, g_r],$$

with $h_i \in \gamma_{n-1}(G)$ and $a_i \in \gamma_{d+1}(G) \leqslant \gamma_{n-1}(G)$ for $i = 1, \ldots, r$. Thus we have

$$g = \prod_{i=1}^{r} [h_i, g_i]^{d_i} [a_i, g_i]$$

for suitable $d_i \in \gamma_{d+2}(G) \leqslant \gamma_{d+1}(G)$ and, since $a_i, d_i, [a_i, g_i] \in \gamma_{d+1}(G)$ for $i = 1, \ldots, r$, we have

$$g = \prod_{i=1}^{r} [h_i a_i, g_i]^{d_i a_i^{-1}} = \prod_{i=1}^{r} \left[(h_i a_i)^{d_i a_i^{-1}}, g_i^{d_i a_i^{-1}} \right].$$

By setting $u_i = (h_i a_i)^{d_i a_i^{-1}}$ and $v_i = g_i^{d_i a_i^{-1}}$, we obtain $g = [u_1, v_1] \cdots [u_r, v_r]$ for $i = 1, \ldots, r$ with $u_i \in \gamma_{n-1}(G)$ since $a_i, h_i \in \gamma_{n-1}(G) \lhd G$ and $v_i \in G$. $\square$

The bound in Lemma 8.1.3 depends only on the number of generators of G. In metanilpotent groups the bound we obtain depends on the number of generators, the nilpotent class of the top group and the term of the lower central series that we are considering. This next lemma extends Hartley's result about the derived subgroup of a metanilpotent group [19, Corollary 2].

Lemma 8.1.4. *Let G be a metanilpotent group with class sequence (c_1, c_2) generated by r elements. For $n \geqslant 2$, an element g of $\gamma_n(G)$ can be written as a product of at most t commutators $g = [u_1, v_1] \cdots [u_t, v_t]$ with $u_i \in \gamma_{n-1}(G)$ and $v_i \in G$ for $i = 1, \ldots, t$ where the value of t depends only on n, r and c_1.*

Proof. Let $g_1, \ldots, g_r$ generate G and let $d = \max\{n-2, c_1\}$. We claim that the result holds with $t = 2r + r^{d+1}$.

The subgroup $\gamma_{d+1}(G) \leqslant \gamma_{c_1+1}(G)$ is nilpotent of class at most c_2 and $\gamma_{d+1}(G)$ is generated as a normal subgroup of G by $\{[g_{i_1}, \ldots, g_{i_{d+1}}] \mid 1 \leqslant i_j \leqslant r\} = \{y_1, \ldots, y_s\}$ where $s = r^{d+1}$. By [19, Lemma 3], we have

$$[\gamma_{d+1}(G), G] = \{[h_1, g_1] \cdots [h_r, g_r][k_1, y_1] \cdots [k_s, y_s] \mid h_i, k_j \in \gamma_{d+1}(G),$$
$$i = 1, \ldots, r, j = 1, \ldots, s\}.$$

Let $\overline{G} = G/\gamma_{d+2}(G)$, so that $\overline{G}$ is nilpotent. By Lemma 8.1.2,

$$\gamma_n(\overline{G}) = \{[\overline{u}_1, \overline{g}_1] \cdots [\overline{u}_r, \overline{g}_r] \mid \overline{u}_i \in \gamma_{n-1}(\overline{G}), i = 1, \ldots, r\},$$

where $\overline{g}_i$ is the image of g_i in $\overline{G}$. Let g be an element of $\gamma_n(G)$, so that the image of g in $\overline{G}$ is in $\gamma_n(\overline{G})$ and $g = [u_1, g_1] \cdots [u_r, g_r]w$ with $u_i \in \gamma_{n-1}(G)$ for $i = 1, \ldots, r$ and $w \in \gamma_{d+2}(G) = [\gamma_{d+1}(G), G]$. It follows that

$$g = [u_1, g_1] \cdots [u_r, g_r][h_1, g_1] \cdots [h_r, g_r][k_1, y_1] \cdots [k_s, y_s]$$

with $u_i \in \gamma_{n-1}(G)$ and $h_i, k_j \in \gamma_{d+1}(G) \leqslant \gamma_{n-1}(G)$ for $i = 1, \ldots, r$, $j = 1, \ldots, s_1$. Thus $g = [u_1, v_1] \cdots [u_t, v_t]$ with $u_i \in \gamma_{n-1}(G)$ and $v_i \in G$ for $i = 1, \ldots, t$ where $t = 2r + s$. □

As a corollary to Lemma 8.1.4, we obtain the following result for abelian-by-metanilpotent groups.

Corollary 8.1.1. *Let G be an abelian-by-metanilpotent group with class sequence $(c_1, c_2, 1)$ generated by r elements. For $n \geqslant 2$, an element g of $\gamma_n(G)$ can be written as a product of at most s commutators $g = [u_1, v_1] \cdots [u_s, v_s]$ with $u_i \in \gamma_{n-1}(G)$ and $v_i \in G$ for $i = 1, \ldots, s$, where the value of s depends only on n, r and c_1.*

Proof. Let $d = \max\{n - c_2 - 2, c_1\}$ and let

$$A = \gamma_{c_2+1}(\gamma_{d+1}(G)) \leqslant \gamma_{c_2+1}(\gamma_{c_1+1}(G)) \cap \gamma_{n-1}(G).$$

The subgroup A is abelian and

$$\gamma_{c_2+2}(\gamma_{d+1}(G)) = [A, \gamma_{d+1}(G)] \leqslant [A, G].$$

Thus $G/[A, G]$ is a finitely generated metanilpotent group with class sequence $(d, c_2 + 1)$ and by Lemma 8.1.4, each element of $\gamma_n(G/[A, G])$ can be written as a product of $t = 2r + r^{d+1}$ commutators. If g is an element of $\gamma_n(G)$ then g can be written as a product

$$g = [u_1, v_1] \cdots [u_t, v_t]k$$

with $u_i \in \gamma_{n-1}(G)$, $v_i \in G$ for $i = 1, \ldots, t$ and $k \in [A, G]$. Since A is abelian, it follows from [2, Lemma 3] that $k = [k_1, g_1] \cdots [k_r, g_r]$ with $k_i \in A \leqslant \gamma_{n-1}(G)$. Thus g can be written as a product of $s = t + r = 3r + r^{d+1}$ commutators $g = [u_1, v_1] \cdots [u_s, v_s]$ with $u_i \in \gamma_{n-1}(G)$ and $v_i \in G$. □

Finally we show that a bound exists for soluble groups with the maximal condition on normal subgroups (cf. [52, Corollary 1]). We recall that a group G satisfies the *maximal condition on normal subgroups* (or *max-n*) if, for every series of normal subgroups of G

$$1 = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n \leqslant \cdots ,$$

there is some natural number N such that $G_n = G_N$ for all $n \geqslant N$. We require the following intermediate result.

Lemma 8.1.5. *Let G be a group satisfying max-n and let H be a normal subgroup of G. For $n \geqslant 2$, the quotient $G/\gamma_n(H)$ is a C_n-group if and only if $G/\gamma_n(\gamma_n(H))$ is a C_n-group.*

Proof. The group $G/\gamma_n(H)$ is a quotient of $G/\gamma_n(\gamma_n(H))$. Thus the result is clear in one direction.

Suppose now that $G/\gamma_n(H)$ is a C_n-group. We may assume without loss of generality that $\gamma_n(\gamma_n(H))$ is trivial, so that $\gamma_n(H)$ is a nilpotent normal subgroup of G. Since G satisfies max-n, $\gamma_n(H)$ is finitely generated as a normal subgroup of G. By a result of Hartley [19, Lemma 3], there is a positive integer s such that every element of $[\gamma_n(H), G]$ can be written as a product of commutators $[h_1, g_1] \cdots [h_s, g_s]$ with $g_i \in G$ and $h_i \in \gamma_n(H) \leqslant \gamma_{n-1}(H)$. It is enough to show that $G/[\gamma_n(H), G]$ is a C_n-group and we may assume without loss of generality that $[\gamma_n(H), G]$ is trivial, so that $\gamma_n(H) \leqslant Z(G)$. Again since G satisfies max-n, H is finitely generated as a normal subgroup of G. We suppose that $H = \langle y_1, \ldots, y_r \rangle^G$ and define a sequence of maps ϕ_i for $i = 1, \ldots, r$ by

$$\phi_i : \gamma_{n-1}(H) \to \gamma_n(H) : h \mapsto [h, y_i].$$

The map ϕ_i is a homomorphism and

$$H_i = \operatorname{im} \phi_i = \{[h, y_i] \mid h \in \gamma_{n-1}(H)\} \leqslant \gamma_n(H)$$

is a normal subgroup of G. Let $K = \prod_{i=1}^{r} H_i \lhd G$ and consider

$$C_H(\gamma_{n-1}(H)/K) = \{h \in H \mid [\gamma_{n-1}(H), h] \leqslant K\},$$

which is a normal subgroup of G. For each i, we have $y_i \in C_H(\gamma_{n-1}(H)/K)$ and hence $H \leqslant C_H(\gamma_{n-1}(H)/K)$. Thus $\gamma_n(H) = [\gamma_{n-1}(H), H] \leqslant K \leqslant \gamma_n(H)$ and we have $\gamma_n(H) =$

K. Since $G/\gamma_n(H)$ is a C_n-group, there is a positive integer t such that each element of $\gamma_n(G/\gamma_n(H))$ can be written as a product $[\overline{h}_1, \overline{g}_1] \cdots [\overline{h}_t, \overline{g}_t]$, with $\overline{g}_i \in G/\gamma_n(H)$ and $\overline{h}_i \in \gamma_{n-1}(G/\gamma_n(H))$. Thus each element $g \in \gamma_n(G)$ can be written as a product $g = [h_1, g_1] \cdots [h_t, g_t]k$, with $g_i \in G$, $h_i \in \gamma_{n-1}(G)$ and $k \in \gamma_n(H) = K$. It follows that

$$g = [h_1, g_1] \cdots [h_t, g_t][k_1, y_1] \cdots [k_r, y_r],$$

with $k_i \in \gamma_{n-1}(H) \leqslant \gamma_{n-1}(G)$ for $i = 1, \ldots, r$ and G is a C_n-group. □

Let c and k be positive integers. We say that a group G has *c-nilpotent length k* if G is polynilpotent with class sequence $(\underbrace{c, \ldots, c}_{k})$.

Lemma 8.1.6. *A soluble group satisfying max-n is a C_n-group for all $n \geqslant 2$.*

Proof. Let G be a soluble group satisfying max-n so that G is finitely generated [53, 5.4.21]. Since G is soluble, it has finite m-nilpotent length for any $m \geqslant 1$. Fix $n \geqslant 2$ and proceed by induction on the $(n-1)$-nilpotent length of G.

If G is nilpotent then G is a C_n-group by Lemma 8.1.2. Suppose now that every group of $(n-1)$-nilpotent length k which satisfies the maximal condition for normal subgroups is a C_n-group and that G has $(n-1)$-nilpotent length $k+1$. Let $N = \underbrace{\gamma_n(\gamma_n(\cdots \gamma_n}_{k-1}(G) \cdots))$, so that $\gamma_n(\gamma_n(N)) = 1$. The quotient $G/\gamma_n(N)$ has $(n-1)$-nilpotent length k and hence, by induction, $G/\gamma_n(N)$ is a C_n-group. Lemma 8.1.5 implies that $G = G/\gamma_n(\gamma_n(N))$ is a C_n-group. □

Combining each of these results with our remark at the beginning of this section that a finite-by-C_n group is a C_n-group, we have shown that G is a C_n-group for all $n \geqslant 2$ if G is in one of the following classes:

- finitely generated finite-by-abelian-by-metanilpotent groups;
- finite-by-soluble groups satisfying the maximal condition on normal subgroups.

8.2 Elementary theory

In Section 8.1 we gave a number of examples of groups which are C_n-groups for all $n \geqslant 2$. We now go on to consider the implications of being a C_n-group for the elementary theory of a group.

Lemma 8.2.1. *Fix a positive integer n and let H be a C_r-group for $2 \leqslant r \leqslant n$. There is an existential formula in one variable $\phi_n(x)$ such that, for any group G which is elementarily equivalent to H, the formula $\phi_n(x)$ defines $\gamma_n(G)$.*

Proof. We proceed by induction on n. For $n = 1$, we have $\gamma_1(G) = G$ and we define $\phi_1(x)$ to be the formula $x = x$.

Suppose now that the result holds for $n-1$. For each $k \geqslant 1$, we define an existential formula $\alpha_k(x)$ by

$$\exists y_1 \dots y_k z_1 \dots z_k \left(\bigwedge_{i=1}^{k} \phi_{n-1}(y_i) \wedge x = [y_1, z_1] \cdots [y_k, z_k] \right).$$

By assumption there exists a positive integer t such that every element h of $\gamma_n(H)$ can be written as a product $h = [h_1, g_1] \cdots [h_t, g_t]$ with $g_i \in H$ and $h_i \in \gamma_{n-1}(H)$ for $i = 1, \dots, t$. The sentence $\forall x\, (\alpha_{t+1}(x) \to \alpha_t(x))$ holds in H, and hence in G, since G is elementarily equivalent to H. If $\phi_n(x)$ is the formula $\alpha_t(x)$ then $\phi_n(x)$ defines $\gamma_n(G)$. $\square$

We come now to the main result of this section.

Theorem 8.2.1. *Fix a positive integer n and let H be a C_r-group for $2 \leqslant r \leqslant n$. If the group G is elementarily equivalent to H then $\gamma_n(G)$ is elementarily equivalent to $\gamma_n(H)$ and $G/\gamma_n(G)$ is elementarily equivalent to $H/\gamma_n(H)$.*

Proof. Let $\phi(x)$ be the formula $\phi_n(x)$ as defined in Lemma 8.2.1 and let ψ be a first-order sentence such that $H/\gamma_n(H) \models \psi$. We may assume without loss of generality that ψ is in prenex normal form; thus ψ is of the form

$$Q_1 x_1 \dots Q_k x_k \left(\bigvee_{i=1}^{s_1} \bigwedge_{j=1}^{s_2} (u_{ij}(\overline{x}) = 1 \wedge w_{ij}(\overline{x}) \neq 1) \right),$$

where $u_{ij}(\overline{x})$ and $w_{ij}(\overline{x})$ are words in the variables $x_1, \dots, x_k$ and each Q_l is a quantifier. We define ψ^* by

$$Q_1 x_1 \dots Q_k x_k \left(\bigvee_{i=1}^{s_1} \bigwedge_{j=1}^{s_2} (\phi(u_{ij}(\overline{x})) \wedge \neg\phi(w_{ij}(\overline{x}))) \right).$$

For any group Γ such that $\phi(x)$ defines $\gamma_n(\Gamma)$, we have $\Gamma \models \psi^*$ if and only if $\Gamma/\gamma_n(\Gamma) \models \psi$; thus $H \models \psi^*$. Since G is elementarily equivalent to H, we have $G \models \psi^*$ and hence $G/\gamma_n(G) \models \psi$. Thus $G/\gamma_n(G)$ is elementarily equivalent to $G/\gamma_n(G)$.

Now let θ be a first-order sentence such that $\gamma_n(H) \models \theta$ and suppose without loss of generality that θ is in prenex normal form. Thus θ is of the form

$$Q_1 x_1 \ldots Q_k x_k \sigma(x_1, \ldots, x_k),$$

where $\sigma = \sigma(x_1, \ldots, x_k)$ is quantifier-free. We let $\sigma_0 = \sigma$. Suppose that σ_i has been defined. We define σ_{i+1} in the following way:

$$\sigma_{i+1} : \begin{cases} \forall x_{k-i}(\phi(x_i) \to \sigma_i) & \text{if} \quad Q_{k-i} = \forall, \\ \exists x_{k-i}(\phi(x_i) \wedge \sigma_i) & \text{if} \quad Q_{k-i} = \exists, \end{cases}$$

and we take θ^* to be σ_k. For any group Γ such that $\phi(x)$ defines $\gamma_n(\Gamma)$, we have $\Gamma \models \theta^*$ if and only if $\gamma_n(\Gamma) \models \theta$; thus $H \models \theta^*$. Since G is elementarily equivalent to H, we have $G \models \theta^*$ and hence $\gamma_n(G) \models \theta$. Thus $\gamma_n(G)$ is elementarily equivalent to $\gamma_n(H)$. □

The results of Section 8.1 give us examples of groups which are C_n-groups for all integers $n \geqslant 2$. We can combine these results with Theorem 8.2.1 to obtain the following statements:

- If H is a finitely generated finite-by-abelian-by-metanilpotent group and G is elementarily equivalent to H then, for all $n \geqslant 2$, the subgroup $\gamma_n(G)$ is elementarily equivalent to $\gamma_n(H)$ and the quotient $G/\gamma_n(G)$ is elementarily equivalent to $H/\gamma_n(H)$.

- If H is a finite-by-soluble group satisfying the maximal condition on normal subgroups and G is elementarily equivalent to H then, for all $n \geqslant 2$, the subgroup $\gamma_n(G)$ is elementarily equivalent to $\gamma_n(H)$ and the quotient $G/\gamma_n(G)$ is elementarily equivalent to $H/\gamma_n(H)$.

Chapter 9

THE UNIVERSAL THEORY OF FREE GROUPS IN CERTAIN SOLUBLE VARIETIES

Chapuis [12, Corollary 5.3] determines those 2-generator groups which are universally equivalent to a non-cyclic free metabelian group: there are precisely two isomorphism classes of such groups, namely the free metabelian group of rank 2, and the restricted wreath product of two infinite cyclic groups. In Chapter 4 we obtained a similar result for the variety $\mathfrak{A}_n\mathfrak{A}$, where n is a square-free positive integer: there are precisely two non-isomorphic 2-generator groups universally equivalent to a non-cyclic free group in $\mathfrak{A}_n\mathfrak{A}$, namely the free group of rank 2 in $\mathfrak{A}_n\mathfrak{A}$, and the restricted wreath product of a cyclic group of exponent n by an infinite cyclic group. In [66], Timoshenko determines those 2-generator groups finitely presented in the variety of soluble groups of derived length d which are universally equivalent to a non-cyclic free soluble group of derived length d: again there are precisely two isomorphism classes of such groups, namely the free soluble group of derived length d and rank 2, and the $\mathfrak{S}_{d-1}$-verbal wreath product of two infinite cyclic groups. Since finitely generated metabelian groups satisfy the maximal condition on normal subgroups, they are finitely presented in the variety of metabelian groups and so Timoshenko's result implies that of Chapuis.

Timoshenko's proof that $F_2(\mathfrak{S}_d)$ and $\mathbb{Z}\,\mathrm{wr}_{\mathfrak{S}_{d-1}}\,\mathbb{Z}$ are universally equivalent appears to contain a small error. We correct this error and extend the result to free groups in certain other soluble varieties. We deal with product varieties $\mathfrak{V}\mathfrak{A}$ where $\mathfrak{V}$ satisfies the following conditions: $\mathfrak{V}$ contains the variety of abelian groups, the free groups of $\mathfrak{V}$ are residually finite p-groups for infinitely many primes p and $\mathfrak{V}$ is generated by its k-generator groups for some positive integer k. Under these conditions, we prove that the free group of rank 2 in $\mathfrak{V}\mathfrak{A}$ is universally equivalent to the $\mathfrak{V}$-verbal wreath product of two infinite cyclic groups.

Consider now the variety $\mathfrak{V}\mathfrak{N}_c$ for $c \geqslant 2$, where $\mathfrak{V}$ is a variety whose free groups are residually finite p-groups for infinitely many primes p. We show that a 2-generator group finitely presented in this variety is universally equivalent to the free group of rank 2 in $\mathfrak{V}\mathfrak{N}_c$ if and only if it is isomorphic to it. This contrasts with the earlier result for $\mathfrak{V}\mathfrak{A}$ where there are at least two non-isomorphic such groups. We also consider conditions for a group in the universal closure of $F_2(\mathfrak{V}\mathfrak{N}_c)$ to be universally equivalent to $F_2(\mathfrak{V}\mathfrak{N}_c)$.

9.1 Universally equivalent 2-generator groups in $\mathfrak{V}\mathfrak{A}$

Corollary 4 of Timoshenko's paper [66] (the proof that $F_2(\mathfrak{S}_d)$ and $\mathbb{Z} \operatorname{wr}_{\mathfrak{S}_{d-1}} \mathbb{Z}$ are universally equivalent) appears to contain a small error in the definition of the normal subgroup N. In Proposition 9.1.1 we correct this error. In Lemma 9.1.1 and Proposition 9.1.1 we extend the results of Lemma 1 and Corollary 4 of [66] to a more general class of groups.

Lemma 9.1.1. *Let $\mathfrak{V}$ be a variety containing $\mathfrak{A}$ such that the free groups of $\mathfrak{V}$ are residually finite p-groups for infinitely many primes p. Let $F = F_2(\mathfrak{V}\mathfrak{A})$ and let g and h be elements of F such that $g \in F \setminus F'$ and $h \in F' \setminus \gamma_2(F')$. The group $\langle g, h\rangle$ is isomorphic to $\mathbb{Z} \operatorname{wr}_{\mathfrak{V}} \mathbb{Z}$.*

Proof. Let $H_t = \langle h^{g^t}\rangle$ for $t \in \mathbb{Z}$, so that H_t is isomorphic to $\mathbb{Z}$, and let $H = \langle H_t \mid t \in \mathbb{Z}\rangle$. The $\mathfrak{V}$-verbal product of infinite cyclic groups $\langle h_\lambda\rangle$ with $\lambda \in \Lambda$ is the free group of rank $|\Lambda|$ in $\mathfrak{V}$ (see [44, 18.43]). Thus, to show that $\langle g, h\rangle = H \rtimes \langle g\rangle$ is isomorphic to $\mathbb{Z} \operatorname{wr}_{\mathfrak{V}} \mathbb{Z}$, it suffices to show that H is a free group in $\mathfrak{V}$. By Theorem 2.2.2, elements of a free group G in a variety $\mathfrak{V}$ themselves generate a free group in $\mathfrak{V}$ if and only if they are linearly independent modulo G'. The derived subgroup F' of F is a free group in $\mathfrak{V}$ and it contains H. Therefore it remains only to prove that the elements h^{g^t} are linearly independent modulo $\gamma_2(F')$. Suppose that

$$h^{a_k g^k + \cdots + a_1 g + a_0} \equiv 1 \qquad \operatorname{mod} \gamma_2(F').$$

By the Magnus embedding (see Theorem 2.3.2), $F/\gamma_2(F')$ embeds in the restricted wreath product $\mathbb{Z}^{(2)} \operatorname{wr} \overline{F}$, where $\overline{F} \cong F/F'$ is the quotient of $F/\gamma_2(F')$ by its derived subgroup. The element h is mapped to a non-trivial element of the base group and the element $h^{a_k g^k + \cdots + a_1 g + a_0}$ to the identity. Thus, since the base group of the restricted wreath product is torsion-free as a $\mathbb{Z}[\overline{F}]$-module, it follows that $a_k g^k + \cdots + a_1 g + a_0 = 0$ in $\mathbb{Z}[\overline{F}]$. Since $g \notin F'$, this implies that $a_k = \cdots = a_0 = 0$ and the elements are linearly independent modulo $\gamma_2(F')$. □

Proposition 9.1.1. *Let $\mathfrak{V}$ be a variety containing $\mathfrak{A}$ such that the free groups of $\mathfrak{V}$ are residually finite p-groups for infinitely many primes p. Suppose further that $\mathfrak{V}$ is generated by its k-generator groups for some positive integer k. The free group of finite or infinite rank $r \geqslant 2$ in $\mathfrak{V}\mathfrak{A}$ is universally equivalent to $\mathbb{Z} \operatorname{wr}_{\mathfrak{V}} \mathbb{Z}$.*

Proof. By Theorem 2.4.2, the product variety $\mathfrak{V}\mathfrak{A}$ can be discriminated by, and hence generated by, a 2-generator group; thus $F_2(\mathfrak{V}\mathfrak{A})$ generates $\mathfrak{V}\mathfrak{A}$. For $r \geqslant 1$, the group $F_r(\mathfrak{V}\mathfrak{A})$ is residually a finite p-group for infinitely many primes p. By Theorem 2.5.1, the group $F_r(\mathfrak{V}\mathfrak{A})$ is universally equivalent to $F_2(\mathfrak{V}\mathfrak{A})$ for all $r \geqslant 2$. It is therefore enough to show that $\mathbb{Z} \operatorname{wr}_{\mathfrak{V}} \mathbb{Z}$ is universally equivalent to $F = F_2(\mathfrak{V}\mathfrak{A})$.

Let $G = \mathbb{Z} \operatorname{wr}_{\mathfrak{V}} \mathbb{Z}$, so that G is isomorphic to $F_\infty(\mathfrak{V}) \rtimes \langle b \rangle$. The group $F_\infty(\mathfrak{V})$ has basis $\{y_i \mid i \in \mathbb{Z}\}$ and $y_i^b = y_{i+1}$ for each $i \in \mathbb{Z}$. Let G_1 be the subgroup of G generated by b^k and $F_\infty(\mathfrak{V})$ and let N be the normal closure in G_1 of the elements

$$\Big\{[y_{ks+r}, y_{ks+n}] \mid s, n \in \mathbb{Z}, n \notin \{0, \ldots, k-1\}, r \in \{0, \ldots, k-1\}\Big\}.$$

Then $F_\infty(\mathfrak{V})/N$ is isomorphic to the direct product of countably infinitely many copies of $F_k(\mathfrak{V})$ and

$$G_1/N \cong F_k(\mathfrak{V}) \operatorname{wr} \langle b^k \rangle.$$

The group $F_k(\mathfrak{V})$ discriminates $\mathfrak{V}$ and $\langle b^k \rangle$ discriminates $\mathfrak{A}$. By Theorem 2.4.1, the quotient G_1/N discriminates and hence generates $\mathfrak{V}\mathfrak{A}$ and it follows that G generates $\mathfrak{V}\mathfrak{A}$.

By Lemma 9.1.1, the group G is a subgroup of $F = F_2(\mathfrak{V}\mathfrak{A})$ and both F and G generate $\mathfrak{V}\mathfrak{A}$. Thus, since F is residually a finite p-group for all primes p, the group G is universally equivalent to F, again by Theorem 2.5.1. □

Special cases:

- $\mathfrak{V} = \mathfrak{S}_d$. The non-cyclic free soluble group of derived length $d+1$ is universally equivalent to the verbal wreath product $\mathbb{Z} \operatorname{wr}_{\mathfrak{S}_d} \mathbb{Z}$ (cf. [66, Corollary 4]).

- $\mathfrak{V} = \mathfrak{P}_{\overline{c}}$. The non-cyclic free (polynilpotent with class sequence $\overline{c}$)-by-abelian group is universally equivalent to the verbal wreath product $\mathbb{Z} \operatorname{wr}_{\mathfrak{P}_{\overline{c}}} \mathbb{Z}$.

9.2 Conditions for universal equivalence in $\mathfrak{V}\mathfrak{N}_c$ for $c \geqslant 2$

Throughout this section fix a positive integer $c \geqslant 2$. The proof of the following lemma uses the same arguments as Proposition 6.7.1.

Lemma 9.2.1. *Let $\mathfrak{V}$ be a variety whose free groups are residually finite p-groups for infinitely many primes p. Let G be a 2-generator group finitely presented in $\mathfrak{V}\mathfrak{N}_c$. If $G \in \operatorname{ucl}(F_2(\mathfrak{V}\mathfrak{N}_c))$ then G is isomorphic to $F_2(\mathfrak{V}\mathfrak{N}_c)$ if and only if $G \notin \mathfrak{V}\mathfrak{N}_{c-1}$.*

Proof. One direction is clear: if G is isomorphic to $F_2(\mathfrak{V}\mathfrak{N}_c)$ then $G \notin \mathfrak{V}\mathfrak{N}_{c-1}$.

Let $F = F_2(\mathfrak{V}\mathfrak{N}_c)$. By Lemma 2.5.2, the group G embeds in $F^{\mathbb{N}}$. Let $\overline{g}$ and $\overline{h}$ be elements of $F^{\mathbb{N}}$ which generate G. Let $\overline{g}(i)$ and $\overline{h}(i)$ be ith coordinates and let $G_i = \langle \overline{g}(i), \overline{h}(i) \rangle$. For some i and j, suppose that there is a homomorphism from G_i to G_j such that $\overline{g}(i) \mapsto \overline{g}(j)$ and $\overline{h}(i) \mapsto \overline{h}(j)$. If $\overline{g}'$ and $\overline{h}'$ are the elements obtained from $\overline{g}$ and $\overline{h}$ by replacing the jth coordinates with identity elements then G is isomorphic to $\langle \overline{g}', \overline{h}' \rangle$.

If, for some i, the elements $\overline{g}(i)$ and $\overline{h}(i)$ are linearly independent modulo F' then, by Theorem 2.2.2, the group G_i is free in $\mathfrak{V}\mathfrak{N}_c$ and hence $G_i \cong F$; moreover, G_i is freely generated by $\overline{g}(i)$ and $\overline{h}(i)$. For each j, the group $G_j \leqslant F$ is a 2-generator group in $\mathfrak{V}\mathfrak{N}_c$. Thus there is a homomorphism $G_i \to G_j$ such that $\overline{g}(i) \mapsto \overline{g}(j)$ and $\overline{h}(i) \mapsto \overline{h}(j)$. It follows that G is isomorphic to $G_i \cong F$.

Suppose now that, for all i, the elements $\overline{g}(i)$ and $\overline{h}(i)$ are linearly dependent modulo F'. It follows, since F/F' is torsion-free, that G_i is cyclic modulo F'. By Lemma 4.5.5 there are elements $g \in G$ and $c \in F' \cap G$ such that $G_i = \langle g, c \rangle$. We have $G_i' = \langle [g,c] \rangle_i^G \leqslant \gamma_3(F)$ and $\gamma_c(G_i) \leqslant \gamma_{c+1}(F)$; thus G_i is in $\mathfrak{V}\mathfrak{N}_{c-1}$. This holds for all i and, since G embeds in $\prod_{i \in \mathbb{N}}^C G_i$, we have $G \in \mathfrak{V}\mathfrak{N}_{c-1}$. We conclude that, if G is not in $\mathfrak{V}\mathfrak{N}_{c-1}$, then there exists some i such that $G \cong G_i \cong F$. $\square$

As an immediate consequence of Lemma 9.2.1 we have the following proposition.

Proposition 9.2.1. *Let $\mathfrak{V}$ be a variety whose free groups are residually finite p-groups for infinitely many primes p and let G be a 2-generator group which is finitely presented in $\mathfrak{V}\mathfrak{N}_c$. Then G is universally equivalent to $F_2(\mathfrak{V}\mathfrak{N}_c)$ if and only if G is isomorphic to $F_2(\mathfrak{V}\mathfrak{N}_c)$.*

A finitely generated group in $\mathfrak{A}\mathfrak{N}_c$ is finitely presented in $\mathfrak{A}\mathfrak{N}_c$. Taking $\mathfrak{V} = \mathfrak{A}$ in Proposition 9.2.1, we have the following result.

Corollary 9.2.1. *If G is a 2-generator group universally equivalent to $F_2(\mathfrak{A}\mathfrak{N}_c)$ then G is isomorphic to $F_2(\mathfrak{A}\mathfrak{N}_c)$.*

Thus there is precisely one isomorphism class of 2-generator groups universally equivalent to $F_2(\mathfrak{A}\mathfrak{N}_c)$. This contrasts with the free soluble case, where there are at least two isomorphism class of 2-generator groups universally equivalent to $F_2(\mathfrak{S}_d)$ and, in particular, with the metabelian case where there are precisely two such isomorphism classes.

Taking $\mathfrak{V} = \mathfrak{E}$ in Proposition 9.2.1, we have the following result.

Corollary 9.2.2. *If G is a 2-generator group universally equivalent to $F_2(\mathfrak{N}_c)$ then G is isomorphic to $F_2(\mathfrak{N}_c)$.*

Next we consider the implications of Lemma 9.2.1 for an arbitrary group in the universal closure of $F_2(\mathfrak{A}\mathfrak{N}_c)$.

Proposition 9.2.2. *A group $G \in \operatorname{ucl}(F_2(\mathfrak{A}\mathfrak{N}_c))$ is universally equivalent to $F_2(\mathfrak{A}\mathfrak{N}_c)$ if and only if G contains a 2-generator subgroup which is not in $\mathfrak{A}\mathfrak{N}_{c-1}$.*

Proof. Let H be a 2-generator subgroup of G such that $H \notin \mathfrak{A}\mathfrak{N}_{c-1}$. Since G is in $\operatorname{ucl}(F_2(\mathfrak{A}\mathfrak{N}_c))$ and universally closed classes are closed under taking subgroups, H is in $\operatorname{ucl}(F_2(\mathfrak{A}\mathfrak{N}_c))$. By Lemma 9.2.1, the group H is isomorphic to $F_2(\mathfrak{A}\mathfrak{N}_c)$. Thus G contains an isomorphic copy of $F_2(\mathfrak{A}\mathfrak{N}_c)$ and hence G satisfies the existential theory of $F_2(\mathfrak{A}\mathfrak{N}_c)$. Since G is in $\operatorname{ucl}(F_2(\mathfrak{A}\mathfrak{N}_c))$, it follows that G is universally equivalent to $F_2(\mathfrak{A}\mathfrak{N}_c)$.

Suppose now that G is universally equivalent to $F_2(\mathfrak{A}\mathfrak{N}_c)$ and let g and h be free generators of $F_2(\mathfrak{A}\mathfrak{N}_c)$. Then $\big[[g, \underbrace{h, \ldots, h}_{c-1}], [g, \underbrace{h, \ldots, h}_{c-1}]^g\big] \neq 1$ and $F_2(\mathfrak{A}\mathfrak{N}_c) \models \theta$ where θ

is defined by

$$\exists xy\left[[x,\underbrace{y,\ldots,y}_{c-1}],[x,\underbrace{y,\ldots,y}_{c-1}]^x\right]\neq 1.$$

Thus $G\models\theta$ and G contains a 2-generator group which is not in $\mathfrak{AN}_{c-1}$. □

For $c=2$ or 3, the variety $\mathfrak{N}_c$ is discriminated by its 2-generator free group and it follows, by Theorem 2.4.2, that $\mathfrak{AN}_c$ is discriminated by, and hence generated by, its 2-generator groups. Thus $F_2(\mathfrak{AN}_c)$ generates $\mathfrak{AN}_c$ and, by Theorem 2.5.1, the group $F_k(\mathfrak{AN}_c)$ is universally equivalent to $F_2(\mathfrak{AN}_c)$ for all $k\geqslant 2$. We show that, for $G\in\operatorname{ucl}(F_2(\mathfrak{AN}_c))$ with $c=2$ or 3, the group G is in $\mathfrak{AN}_{c-1}$ if and only if every 2-generator subgroup of G is in $\mathfrak{AN}_{c-1}$.

Lemma 9.2.2.

(1) *Let $G\in\operatorname{ucl}(F_2(\mathfrak{AN}_2))$. Then G is metabelian if and only if every 2-generator subgroup of G is metabelian.*

(2) *Let $G\in\operatorname{ucl}(F_2(\mathfrak{AN}_3))$. Then G is abelian-by-(nilpotent of class 2) if and only if every 2-generator subgroup of G is abelian-by-(nilpotent of class 2).*

Proof. One direction is clear: if G is metabelian (respectively abelian-by-(nilpotent of class 2)) then every 2-generator subgroup of G is metabelian (respectively abelian-by-(nilpotent of class 2)).

Recall, from Section 7.6, that $\operatorname{ucl}(F_2(\mathfrak{AN}_c))$ is contained in $\operatorname{ucl}(\mathbb{Z}\operatorname{wr}F_2(\mathfrak{N}_c))$. Thus, by Theorem 6.1.1, either G is nilpotent of class at most c or the Fitting subgroup A of G is abelian and torsion-free as a $\mathbb{Z}[G/A]$-module. By Theorem 2.4.4, every non-cyclic nilpotent subgroup of $F_2(\mathfrak{AN}_c)$ is contained in $\gamma_{c+1}(F_2(\mathfrak{AN}_c))$, which is abelian. Hence every nilpotent subgroup of $F_2(\mathfrak{AN}_c)$ is abelian and $F_2(\mathfrak{AN}_c)\models\theta$, where θ is defined by

$$\forall xy\,([x,y,x]=1\wedge[x,y,y]=1\to[x,y]=1).$$

Since G is in $\operatorname{ucl}(F_2(\mathfrak{AN}_c))$, we have $G\models\theta$ and every nilpotent subgroup of G is abelian; thus if G is nilpotent, it is abelian and there is nothing to prove. Suppose therefore that $A=\operatorname{Fitt}(G)$ is abelian and torsion-free as a $\mathbb{Z}[G/A]$-module.

Let $c=2$. For elements $g,h\in G$ such that $\langle g,h\rangle$ is metabelian we have the following identities:

$$[g,h,g]^{(-[g,h]+1)} = \big[[g,h],[g,h,g]\big] = 1,$$
$$[g,h,h]^{(-[g,h]+1)} = \big[[g,h],[g,h,h]\big] = 1.$$

Since G is abelian-by-(nilpotent of class 2), the elements $[g,h,g]$ and $[g,h,h]$ are in A. Thus, since A is torsion-free as a $\mathbb{Z}[G/A]$-module, either $[g,h]\in A$ or $[g,h,g]=[g,h,h]=1$. In the latter case the group $\langle g,h\rangle$ is nilpotent, and hence abelian, and $[g,h]=1\in A$.

If every 2-generator subgroup of G is metabelian then, for all $g,h\in G$, we have $[g,h]\in A$. It follows that $G'\leqslant A$, which is abelian, and hence G is metabelian.

Now let $c=3$. For elements $g,h\in G$ such that $\langle g,h\rangle$ is abelian-by-(nilpotent of class 2) we have the following identities:

$$[g,h,g,g]^{(-[g,h,g]+1)} = \big[[g,h,g],[g,h,g,g]\big] = 1,$$
$$[g,h,g,h]^{(-[g,h,g]+1)} = \big[[g,h,g],[g,h,g,h]\big] = 1,$$
$$[g,h,h,g]^{(-[g,h,g]+1)} = \big[[g,h,g],[g,h,h,g]\big] = 1,$$
$$[g,h,h,h]^{(-[g,h,g]+1)} = \big[[g,h,g],[g,h,h,h]\big] = 1.$$

Since G is abelian-by-(nilpotent of class 3), the elements $[g,h,g,g]$, $[g,h,g,h]$, $[g,h,h,g]$ and $[g,h,h,h]$ are in A. Thus, since A is torsion-free as a $\mathbb{Z}[G/A]$-module, either $[g,h,g]\in A$ or $[g,h,g,g]=[g,h,g,h]=[g,h,h,g]=[g,h,h,h]=1$. In the latter case $\langle g,h\rangle$ is nilpotent, and hence abelian, and $[g,h,g]=1\in A$.

If every 2-generator subgroup of G is abelian-by-(nilpotent of class 2) then, for all $g,h\in G$, we have $[g,h,g]\in A$. It follows that G/A satisfies the law $[x,y,x]$ and, since G/A is torsion-free, G/A is nilpotent of class 2 (Levi, see [44, 34.31]). Thus $\gamma_3(G)\leqslant A$ which is abelian and hence G is abelian-by-(nilpotent of class 2). $\square$

By combining Lemma 9.2.2 and Proposition 9.2.2, we obtain a necessary and sufficient condition for a group in $\mathrm{ucl}(F_k(\mathfrak{AN}_c))$ with $k\geqslant 2$ to be universally equivalent to $F_k(\mathfrak{AN}_c)$ when $c=2$ or 3.

$$\begin{array}{c} \mathrm{ucl}(F_2(\mathfrak{AN}_2)) \\ | \\ \mathrm{ucl}(F_2(\mathfrak{S}_2)) = \mathrm{ucl}(\mathbb{Z} \operatorname{wr} \mathbb{Z}) \\ | \\ \mathrm{ucl}(\mathbb{Z}) \\ | \\ \{1\} \end{array}$$

Figure 9.1: The lattice of universally closed classes of groups contained in the universal closure of a non-cyclic free metabelian group.

Proposition 9.2.3.

1. *Let* $G \in \mathrm{ucl}(F_k(\mathfrak{AN}_2))$ *for* $k \geqslant 2$*. Then* G *is universally equivalent to* $F_k(\mathfrak{AN}_2)$ *if and only if* G *is not metabelian.*

2. *Let* $G \in \mathrm{ucl}(F_k(\mathfrak{AN}_3))$ *for* $k \geqslant 2$*. Then* G *is universally equivalent to* $F_k(\mathfrak{AN}_3)$ *if and only if* G *is not abelian-by-(nilpotent of class* 2*).*

We can now describe explicitly the lattice of universally closed classes of groups in $\mathrm{ucl}(F_k(\mathfrak{AN}_2))$ for $k \geqslant 2$ (see Figure 9.1). If G is a non-abelian metabelian group in $\mathrm{ucl}(F_k(\mathfrak{AN}_2)) \subseteq \mathrm{ucl}(\mathbb{Z} \operatorname{wr} F_2(\mathfrak{N}_2))$ then G is a non-nilpotent $\rho_{0,\mathbb{Z}}$-group[1]. Thus G is universally equivalent to $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ (by [12, Lemma 4.9]) and hence to $F_2(\mathfrak{S}_2)$. This lattice occurs as a sublattice in the lattice of universally closed classes of groups in $\mathrm{ucl}(\mathbb{Z} \operatorname{wr} F_2(\mathfrak{N}_2))$ (see Figure 6.2 on page 106).

We do not have complete information about the lattice of universally closed classes of groups in the universal closure of a non-cyclic free group in $\mathfrak{AN}_3$. The metabelian groups in $\mathrm{ucl}(F_2(\mathfrak{AN}_3)$ are once again those in $\mathrm{ucl}(F_2(\mathfrak{S}_2)) = \mathrm{ucl}(\mathbb{Z} \operatorname{wr} \mathbb{Z})$. It would be interesting to obtain examples of non-metabelian groups which are in $\mathrm{ucl}(F_2(\mathfrak{AN}_3)) \cap \mathfrak{AN}_2$ but not in $\mathrm{ucl}(F_2(\mathfrak{AN}_2))$. It is not clear whether any such groups exist.

[1] Or equivalently a non-abelian ρ-group in the terminology of Chapuis [12].

Chapter 10

THE THEORIES OF SOME RELATIVELY FREE SOLUBLE GROUPS

Throughout this chapter, let $\mathfrak{V}_1, \ldots, \mathfrak{V}_k$ be non-trivial nilpotent varieties whose free groups are torsion-free. Let V_i be any set of words which defines $\mathfrak{V}_i$ and suppose that $\mathfrak{V}_i \subseteq \mathfrak{N}_{c_i}$ and $\mathfrak{V}_i \not\subseteq \mathfrak{N}_{c_i-1}$. Let G be a group. For $i = 0, \ldots, k-1$, we define $U_i(G)$ to be the verbal subgroup $V_{k-i}(\cdots(V_1(G))\cdots)$ of G.

As a consequence of the results discussed in Chapter 2, the free groups in $\mathfrak{V}_i$ for $i = 1, \ldots, k$ are residually finite p-groups for every prime p (by Lemma 2.2.1) and hence the free groups in the product variety $\mathfrak{V}_k \cdots \mathfrak{V}_1$ are residually finite p-groups for every prime p (by Lemma 2.4.1).

We begin by obtaining one-variable formulae defining a number of verbal subgroups of a free group F in $\mathfrak{V}_k \cdots \mathfrak{V}_1$. In particular we are able to find universal and existential formulae which define the verbal subgroups $U_i(F)$ for $i = 0, \ldots, k-1$. We use these formulae to extend results of Romanovskiĭ and Timoshenko in [57] and Chapuis in [13] about the theories of free soluble groups.

In [57, Theorem 3], Romanovskiĭ and Timoshenko prove that if F is a free soluble group of rank n and G is an n-generator group elementarily equivalent to F then G is isomorphic to F. We use the definability of the derived subgroup to extend this result to free groups in $\mathfrak{V}_k \cdots \mathfrak{V}_1$.

In Chapter 1 we discussed the questions posed by Tarski about the elementary theory of absolutely free groups. Here we ask, and answer, the corresponding questions for the free groups in the product variety $\mathfrak{V}_k \cdots \mathfrak{V}_1$.

- Do free groups in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ of distinct ranks have distinct elementary theories?

- Is the elementary theory of a free group in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ decidable?

To these we add the question of whether the universal theory of a free group in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ is decidable.

We show that two free groups in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ of different finite ranks have different $\forall\exists$-theories by considering the abelianizations of the groups and using the results of Szmielew [63] about free abelian groups.

We go on to use the negative answer to Hilbert's Tenth Problem over the integers to show that a free group F in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ of finite rank has undecidable $\forall\exists$-theory. To each polynomial $P(\overline{X})$ in several variables with coefficients in $\mathbb{Z}$, we associate a $\forall\exists$-sentence ψ such that $P(\overline{X})$ has a zero in $\mathbb{Z}$ if and only if $F \models \psi$. By Matiyasevich's theorem [40], no algorithm exists to determine whether a polynomial $P(\overline{X})$ in several variables with coefficients in $\mathbb{Z}$ has a zero in $\mathbb{Z}$. It follows that there does not exist an algorithm to determine whether a $\forall\exists$-sentence holds in F.

The problem of determining whether a non-cyclic free soluble group of derived length $d \geqslant 3$ has decidable universal theory is a very difficult one. Chapuis [13, Theorem C1] showed that an answer in the affirmative would imply the existence of an algorithm to determine whether a homogeneous polynomial in several variables with coefficients in $\mathbb{Z}$ has a non-trivial zero in $\mathbb{Z}$. For fixed d the proof associates to each homogeneous polynomial $P(\overline{X})$ an existential sentence ψ in the language of groups such that $P(\overline{X})$ has a non-trivial zero in $\mathbb{Z}$ if and only if ψ holds in a non-cyclic free soluble group of derived length d. The existence of such an algorithm is equivalent to the decidability of the universal theory of the field of rationals [48, §9] which in turn is equivalent to Hilbert's Tenth Problem over the rationals (see Chapter 1). We extend Chapuis results to show that the decidability of the universal theory of a free group of finite rank $\geqslant 2$ in certain varieties $\mathfrak{V}_k \cdots \mathfrak{V}_1$ with $k \geqslant 2$ would imply a positive answer. We impose the following conditions to expedite our proofs: either $c_1 \geqslant 2$, or $k \geqslant 3$ and $\mathfrak{V}_2 = \mathfrak{N}_{c_2}$.

Throughout the remainder of this chapter let k be a positive integer such that $k \geqslant 2$.

10.1 The definability of certain verbal subgroups

We begin by showing that a number of verbal subgroups of a non-cyclic free group in the product variety $\mathfrak{V}_k \cdots \mathfrak{V}_1$ are definable. In Lemmas 10.1.1 and 10.1.2 we obtain, for each i, a universal formula $\theta_i^{\forall}(x)$ and an existential formula $\theta_i^{\exists}(x)$ which define the verbal subgroup $U_i(F)$.

Lemma 10.1.1. *There exist one-variable universal formulae $\theta_i^{\forall}(x)$ which define the verbal subgroups $U_i(F)$ of F, where F is a free group of rank $r \geqslant 2$ in $\mathfrak{V}_k \cdots \mathfrak{V}_1$.*

Proof. We define one-variable formulae $\theta_i^{\forall}(x)$ recursively. Let $\theta_0^{\forall}(x)$ be the formula $x = 1$. Now let $i \geqslant 0$ and suppose that $\theta_i^{\forall}(x)$ has been defined. We let $c = c_{k-i}$ and define $\theta_{i+1}^{\forall}(x)$ by

$$\forall y_1 \ldots y_{c+1} \left(\bigwedge_{\substack{z_1,\ldots,z_{c+1} \in \\ \{x^{y_1},\ldots,x^{y_{c+1}}\}}} \theta_i^{\forall}\big([z_1, \ldots, z_{c+1}]\big) \right).$$

We now prove by induction on i that the formula $\theta_i^{\forall}(x)$ defines the verbal subgroup $U_i(F)$ for $i = 0, \ldots, k-1$. It is clear that $U_0(F) = \{1\}$ and the result holds for $i = 0$.

Suppose now that $\theta_i^{\forall}(x)$ defines $U_i(F)$ and let g be an element of $U_{i+1}(F)$. The normal closure of g in F is contained in $U_{i+1}(F)$. Moreover, the quotient $U_{i+1}(F)/U_i(F)$ is nilpotent of class $c = c_{k-i}$ and hence $[g^{h_1}, \ldots, g^{h_{c+1}}] \in U_i(F)$ for all $h_1, \ldots, h_{c+1} \in F$; thus $F \models \theta_{i+1}^{\forall}(g)$.

For the converse, let g be an element of $F \setminus U_{i+1}(F)$ and let $\overline{g}$ be the image of g in $\overline{F} = F/U_i(F)$, so that $\overline{g}$ is not in $U_{i+1}(\overline{F})$. Suppose for a contradiction that F satisfies $\theta_{i+1}^{\forall}(g)$. Then the normal closure of $\overline{g}$ in $\overline{F}$ is nilpotent and hence, by Theorem 2.4.4, the normal closure of $\overline{g}$ in $\overline{F}$ is cyclic. Let h be an element of $U_{i+1}(F)$ and let $\overline{h}$ be the image of h in $\overline{F}$. If $[\overline{g}, \overline{h}] \neq 1$ then $\langle \overline{g} \rangle^{\overline{F}} \cap U_{i+1}(\overline{F})$ is non-trivial and, since $\overline{F}/U_{i+1}(\overline{F})$ is torsion-free, we have $\overline{g} \in U_{i+1}(\overline{F})$; thus $[\overline{g}, \overline{h}] = 1$ for all $h \in U_{i+1}(F)$. Let h be an element of $U_{i+1}(F) \setminus U_i(F)$, so that $\overline{h}$ is non-trivial. The group $\langle \overline{g}, \overline{h} \rangle$ is abelian and, since $\overline{g} \notin U_{i+1}(\overline{F})$, there exists $\overline{k} \in \overline{F}$ such that $\langle \overline{g}, \overline{h} \rangle = \langle \overline{k} \rangle$. Thus, since $\overline{F}/U_{i+1}(\overline{F})$ is torsion-free, $\overline{k}$ is in $U_{i+1}(\overline{F})$, and hence $\overline{g}$ is in $U_{i+1}(\overline{F})$, contradicting our assumption.

We conclude that g is in $U_{i+1}(F)$ if and only if F satisfies $\theta_{i+1}^{\forall}(g)$. □

Lemma 10.1.2. *There exist one-variable existential formulae $\theta_i^{\exists}(x)$ which define the verbal subgroups $U_i(F)$ of F, where F is the free group of rank $r \geqslant 2$ in $\mathfrak{V}_k \cdots \mathfrak{V}_1$.*

Proof. Recall from page 22 the definition of the words $s_{\overline{c}_i}(\overline{x})$. We define one-variable formulae $\theta_i^{\exists}(x)$ recursively. Let $\theta_0^{\exists}(x)$ be the formula $x = 1$. Now let $i \geqslant 0$ and suppose that $\theta_i^{\exists}(x)$ has been defined. Let $c_i' = c_{k-i}$, $\overline{c}_i' = \overline{c}_{k-i-1}$ and $C_i = (c_1+1)\cdots(c_{k-i-1}+1)$. We define $\theta_{i+1}^{\exists}(x)$ by

$$\exists y_1 \cdots y_{C_i} \left(\bigwedge_{\substack{z_1,\ldots,z_{c_i'+1} \in \\ \{x, s_{\overline{c}_i'}(y_1,\ldots,y_{C_i})\}}} \theta_i^{\exists}\big([z_1,\ldots,z_{c_i'+1}]\big) \wedge \neg\theta_i^{\forall}(s_{\overline{c}_i'}(y_1,\ldots,y_{C_i})) \right).$$

We now prove by induction on i that the formula $\theta_i^{\exists}(x)$ defines the verbal subgroup $U_i(F)$ for $i = 0, \ldots, k-1$. Since $U_0(F) = \{1\}$, the result holds for $i = 0$.

Suppose now that $\theta_i^{\exists}(x)$ defines $U_i(F)$ and let g be an element of $U_{i+1}(F)$. The quotient $F/U_i(F)$ is not in $\mathfrak{V}_{k-i-1}\cdots\mathfrak{V}_1$ and hence there exist elements $h_1, \ldots, h_{C_i} \in F$ such that $h = s_{\overline{c}}(h_1, \ldots, h_{C_i}) \in U_{i+1}(F) \setminus U_i(F)$; thus $F \models \neg\theta_i^{\forall}(h)$. The quotient $U_{i+1}(F)/U_i(F)$ is nilpotent of class c_i' and hence $\langle g, h\rangle$ is nilpotent of class c_i' modulo $U_i(F)$; thus $F \models \theta_{i+1}^{\exists}(g)$.

For the converse, let g be an element of $F \setminus U_{i+1}(F)$ and let $\overline{g}$ be the image of g in $\overline{F} = F/U_i(F)$, so that $\overline{g}$ is not in $U_{i+1}(\overline{F})$. Suppose for a contradiction that F satisfies $\theta_{i+1}^{\exists}(g)$. Then there exist $h_1, \ldots, h_{C_i} \in F$ such that $\overline{h} = s_{\overline{c}_i'}(\overline{h}_1, \ldots, \overline{h}_{C_i}) \neq 1$, where $\overline{h}_i$ is the image of h_i in $\overline{F}$, and $\langle \overline{g}, \overline{h}\rangle$ is nilpotent. Thus, by Theorem 2.4.4, the group $\langle \overline{g}, \overline{h}\rangle$ is cyclic and there exists $\overline{k} \in \overline{F}$ such that $\langle \overline{g}, \overline{h}\rangle = \langle \overline{k}\rangle$. Since $\overline{F}/U_{i+1}(\overline{F})$ is torsion-free, $\overline{k}$ is in $U_{i+1}(\overline{F})$, and hence $\overline{g}$ is in $U_{i+1}(\overline{F})$, contradicting our assumption.

We conclude that g is in $U_{i+1}(F)$ if and only if F satisfies $\theta_{i+1}^{\exists}(g)$. □

Let F_r be the free group of finite rank $r \geqslant 2$ in the product variety $\mathfrak{V}_k \cdots \mathfrak{V}_1$. Our next objective is to give useful one-variable formulae defining terms of the upper and lower central series.

By Lemma 8.1.1, every element of the derived subgroup of an r-generator nilpotent group can be written as a product of r commutators. We use this result to give existential one-variable formulae $\phi_r(x)$ such that $\phi_r(x)$ defines F_i' for $i = 1, \ldots, r$. We define $\phi_r(x)$

to be the one-variable formula

$$\exists u_1 \ldots u_r v_1 \ldots v_r w \left(x = [u_1, v_1] \cdots [u_r, v_r] w \wedge \theta^{\exists}_{k-1}(w)\right).$$

Consideration of the proof of Lemma 10.1.1, shows that, for a group G elementarily equivalent to F_r, if g is in $U_i(G)$ then G satisfies $\theta^{\forall}_i(g)$. We give one-variable formulae $\mu_r(x)$ such that $\mu_r(x)$ defines F'_r and, for any r-generator group G elementarily equivalent to F_r, if $g \in G'$ then $G \models \mu_r(g)$. We define $\mu_r(x)$ to be the one-variable formula

$$\exists u_1 \ldots u_r v_1 \ldots v_r w \left(x = [u_1, v_1] \cdots [u_r, v_r] w \wedge \theta^{\forall}_{k-1}(w)\right).$$

In Lemma 8.1.4 we showed that, for an r-generator metanilpotent group G with class sequence (c_1, c_2), there exists a positive integer t, whose value depends only on c_1 and r, such that if $g \in \gamma_3(G)$ then g can be expressed as a product $g = [u_1, v_1] \cdots [u_t, v_t]$ with $u_i \in G'$ and $v_i \in G$ for $i = 1, \ldots, t$. We use this result to give existential one-variable formulae $\nu_r(x)$ such that $\nu_r(x)$ defines $\gamma_3(F_r)$. We define $\nu_r(x)$ to be the one-variable formula

$$\exists y_1 \ldots y_t z_1 \ldots z_t w \left(x = [y_1, z_1] \cdots [y_t, z_t] w \wedge \bigwedge_{i=1}^{t} \phi_r(y_i) \wedge \theta^{\exists}_{k-2}(w)\right).$$

We also define recursively one-variable universal formulae $\rho_i(x)$ for $i = 0, \ldots, c_1$ such that, for $g \in F$, we have $F \models \rho_i(g)$ if and only if the image of g in $\overline{F} = F/U_{k-1}(F)$ is in $\zeta_i(\overline{F})$. Let $\rho_0(x)$ be the formula $\theta^{\forall}_{k-1}(x)$. Now let $i \geqslant 1$ and suppose that $\rho_i(x)$ has been defined. We define $\rho_{i+1}(x)$ to be the formula $\forall y \rho_i([x, y])$.

Finally we give a one-variable existential formula $\sigma^{\exists}_i(x)$ and one-variable universal formulae $\sigma^{\forall}_i(x)$ which define the subgroups $\gamma_{c+1-i}(U_{k-1}(F))$ when $\mathfrak{V}_2 = \mathfrak{N}_{c_2}$. In order to do this we need to obtain an alternative description of the centre of a non-abelian free nilpotent group. The outline proof of the following result was suggested by Michael Vaughan-Lee. It uses the correspondence between the free nilpotent group of rank n and a suitable quotient of free associative algebra of rank n; for further details of this correspondence see [34, §5.4].

Lemma 10.1.3. *Let F be a free nilpotent group of class $c \geqslant 3$ and let h be an element of $\gamma_{c-1}(F) \setminus \gamma_c(F)$. The centralizer of h in F is the derived subgroup F'.*

Proof. We suppose for a contradiction that g is an element of $C_F(h)$ which is not in F'. The element g can be written as a product

$$g = x_1^{\alpha_1} \cdots x_n^{\alpha_n} c,$$

where $x_1, \ldots, x_n$ are free generators of F, $\alpha_1, \ldots, \alpha_n$ are non-zero integers and $c \in F'$. Since $g \notin F'$, we may assume that $n \neq 0$. Now

$$1 = [h, g] = [h, x_1^{\alpha_1} \cdots x_n^{\alpha_n} c] = [h, c][h, x_1^{\alpha_1} \cdots x_n^{\alpha_n}]^c.$$

The commutator $[h, c]$ is in $[\gamma_{c-1}(F), \gamma_2(F)] \leqslant \gamma_{c+1}(F)$ [53, 5.1.11(i)] and hence $[h, c] = 1$. Thus $[h, x_1^{\alpha_1} \cdots x_n^{\alpha_n}] = 1$ and $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is in $C_F(h)$; we may assume that $c = 1$. If $\gcd\{\alpha_1, \ldots, \alpha_n\} = 1$ then there exist integers $\beta_1, \ldots, \beta_n$ such that $\alpha_1\beta_1 + \cdots + \alpha_n\beta_n = 1$. We may assume without loss of generality that $\beta_1 \neq 0$. We have

$$\begin{array}{rclcl} x_i & = & x_i^{\alpha_1\beta_1 + \cdots + \alpha_n\beta_n} & = & \prod_{j \neq i} x_i^{\alpha_j\beta_j} \cdot \left(g \prod_{j \neq i} x_j^{-\alpha_j}\right)^{\beta_i} \qquad \bmod F' \\ & & & = & \prod_{j \neq 1} \left(x_i^{\beta_j} x_j^{-\beta_i}\right)^{\alpha_j} \cdot g^{\beta_i} \qquad \bmod F'. \end{array}$$

Moreover

$$x_i^{\beta_j} x_j^{-\beta_i} = \left(x_1^{\beta_i} x_i^{-\beta_1}\right)^{\frac{-\beta_j}{\beta_1}} \left(x_1^{\beta_j} x_j^{-\beta_1}\right)^{\frac{-\beta_i}{\beta_1}} \quad \bmod F'.$$

Thus the set of elements $\{gF', x_1^{\beta_2} x_2^{-\beta_1} F', \ldots, x_1^{\beta_n} x_n^{-\beta_1} F'\}$ freely generate the same subgroup of F/F' as $\{x_1 F', \ldots, x_n F'\}$. We may extend this to a free generating set of F/F' which contains gF' and hence, by Theorem 2.2.2, there is a free generating set of F which contains g. Suppose now that $\gcd\{\alpha_1, \ldots, \alpha_n\} = d \neq 1$. Then $g = k^d c'$ where $k = x_1^{\beta_1} \cdots x_n^{\beta_n}$, the element c' is in F', the integers $\alpha_i = \beta_i d$ and $\gcd\{\beta_1, \ldots, \beta_n\} = 1$. By the same argument as above, there is a free generating set of F which contains k. Moreover

$$1 = [h, g] = [h, k^d c'] = [h, c'][h, k^d]^{c'}.$$

As before, the commutator $[h, c']$ is in $\gamma_{c+1}(F)$ and hence $[h, c'] = 1$. Since $[h, k]$ is in $\gamma_c(F) \leqslant Z(F)$,

$$1 = [h, k^d] = [h, k]^d$$

and it follows, since F is torsion-free, that k is in $C_F(h)$. In both cases we obtain a free generator of F in $C_F(h)$.

Let $x_1, x_2, \dots$ be free generators of F. Suppose that $[h, x_1] = 1$ and that h lies in the subgroup $G = \langle x_1, \dots, x_n \rangle$. (The generator x_1 may or may not occur in h.) Consider the free associative algebra $\mathcal{A}$ over $\mathbb{Z}$ generated by non-commuting indeterminates $X_1, \dots, X_n$ and let $\mathcal{I}$ be the ideal of $\mathcal{A}$ generated by all products $X_{i_1} \cdots X_{i_{c+1}}$. The subgroup of $\mathcal{A}/\mathcal{I}$ generated by the elements $1 + X_1, \dots, 1 + X_n$ is a free nilpotent group of class c and rank n. Thus the map $G \to \mathcal{A}/\mathcal{I} : x_i \mapsto 1 + X_i$ for $i = 1, \dots, n$ is an embedding; we associate G with its image in $\mathcal{A}/\mathcal{I}$. The element h is in $\gamma_{c-1}(G) \setminus \gamma_c(G)$ and it expands as $1 + P + Q$, where $P \neq 0$ is homogeneous of weight $c - 1$ and Q is homogeneous of weight c. Then $[h, x_1]$ expands as $1 + PX_1 - X_1 P$ and $[h, x_1] = 1$ implies that $PX_1 - X_1 P = 0$. The element P of $\mathcal{A}/\mathcal{I}$ is homogeneous of weight $c - 1$ and hence we may write it as a sum of monomials in the X_i of weight $c - 1$ where at least two variables occur in each monomial. We have

$$P = \sum_{i=1}^{r} \delta_i m_i,$$

where the δ_i are non-zero integers and the m_i are monomials. We order the indeterminates $X_1 > X_2 > \cdots > X_n$ and extend this to order the monomials lexicographically. We may suppose, after suitable reordering, that $m_1 > m_2 > \cdots > m_r$. Then $X_1 m_1 > \cdots > X_1 m_r$ and $X_1 m_1 > m_1 X_1 > \cdots > m_r X_1$. It follows that the leading term of $PX_1 - X_1 P$ is $-\delta_1 X_1 m_1$ which is non-zero and this contradicts our assumption that $PX_1 - X_1 P = 0$.

It follows that there is no element g in $C_F(h)$ which is not in F'. A consideration of commutator weights shows that $F' \leqslant C_F(h)$. Thus $C_F(h) = F'$. $\square$

Using the above result we obtain a description of the centre of a non-abelian free nilpotent group which will allow us to define suitable universal formulae.

Lemma 10.1.4. *Let F be a non-abelian free nilpotent group. If g is an element of F then $g \in Z(F)$ if and only if there exist $h_1, h_2 \in F$ such that $[h_1, h_2] \notin \gamma_3(F)$ and $[g, h_1] = [g, h_2] = 1$.*

Proof. The "only if" direction is clear. We suppose for the converse that $g, h_1, h_2 \in F$ with $[h_1, h_2] \notin \gamma_3(F)$ and $[g, h_i] = 1$ for $i = 1, 2$ and we proceed by induction on c.

Suppose to begin with that F is nilpotent of class and rank 2; then F has presentation $\langle x, y, z | [x, y] = z, [x, z] = [y, z] = 1 \rangle$. Every element of F can be written as a word of the form $x^r y^s z^t$ with $r, s, t \in \mathbb{Z}$ and the only trivial such word is the empty one. If $g = x^r y^s z^t$ and $h_i = x^{r_i} y^{s_i} z^{t_i}$ for $i = 1, 2$ then

$$[h_1, h_2] = [x^{r_1} y^{s_1} z^{t_1}, x^{r_2} y^{s_2} z^{t_2}] = z^{r_1 s_2 - r_2 s_1}$$

and

$$[g, h_i] = [x^r y^s z^t, x^{r_i} y^{s_i} z^{t_i}] = z^{r s_i - r_i s}$$

for $i = 1, 2$. Thus $[g, h_i] = 1$ implies that $rs_i - r_i s = 0$ for $i = 1, 2$ and

$$(r_1 s_2 - r_2 s_1)s = (rs_2 - r_2 s)s_1 - (rs_1 - r_1 s)s_2 = 0,$$

$$(r_1 s_2 - r_2 s_1)r = (rs_2 - r_2 s)r_1 - (rs_1 - r_1 s)r_2 = 0.$$

Since $[h_1, h_2] \neq 1$, we have $r_1 s_2 - r_2 s_1 \neq 0$ and it follows that $r = s = 0$. Thus $g = z^t \in Z(F)$. The universal sentence θ defined by

$$\forall x \Big(\big(\exists y_1 y_2 [y_1, y_2] \neq 1 \wedge [x, y_1] = 1 \wedge [x, y_2] = 1 \big) \to (\forall z [x, z] = 1) \Big)$$

holds in F. Suppose now that $\mathcal{F}$ is a non-cyclic free nilpotent group of class 2 and any rank. By Lemma 2.2.1, the group $\mathcal{F}$ is residually a finite p-group for every prime p. Moreover, the group F generates the variety $\mathfrak{N}_2$. Thus, by Theorem 2.5.1, the group $\mathcal{F}$ is universally equivalent to F and hence $\mathcal{F} \models \theta$. The result holds for all non-cyclic free nilpotent groups of class 2.

Suppose now that $c \geqslant 3$ and that the result holds for $c-1$. Let $\overline{F} = F/\gamma_c(F)$, so that $\overline{F}$ is a free nilpotent group of class $c-1$. Let $\overline{g}$, $\overline{h}_1$ and $\overline{h}_2$ be the images of g, h_1 and h_2 in $\overline{F}$. Since $[h_1, h_2] \notin \gamma_3(F)$, the commutator $[\overline{h}_1, \overline{h}_2]$ is not in $\gamma_3(\overline{F})$. Moreover $[\overline{g}, \overline{h}_i] = 1$ in $\overline{F}$ for $i = 1, 2$. By induction, we have $\overline{g} \in Z(\overline{F})$. In a free nilpotent group the terms of the upper and lower central series coincide [68, Theorem 13]. Thus $\gamma_c(F) = Z(F)$ and hence $\overline{g} \in Z(F/Z(F)) = \zeta_2(F)/Z(F)$. The element g is in $\zeta_2(F) = \gamma_{c-1}(F)$ and we may apply Lemma 10.1.3. Since $[h_1, h_2] \notin \gamma_3(F)$, the element h_1 does not lie in F' and it follows that g lies in $Z(F)$. □

We are now in a position to define the required formulae. Suppose that $\mathfrak{V}_2 = \mathfrak{N}_{c_2}$ and let g be an element of F. Firstly we recursively define one variable universal formulae

$\sigma_i^{\forall}(x)$ for $i = 0, \dots, c_2$. Let $\sigma_0^{\forall}(x)$ be the formula $\theta_{k-2}^{\forall}(x)$. Now let $i \geqslant 1$ and suppose that $\sigma_i^{\forall}(x)$ has been defined. We define $\sigma_{i+1}^{\forall}(x)$ to be the formula

$$\forall y\bigl(\theta_{k-1}^{\exists}(y) \rightarrow \sigma_i^{\forall}([x, y])\bigr).$$

Then, for $g \in F$, we have $F \models \sigma_i^{\forall}(g)$ if and only if the image of g in $\overline{F} = F/U_{k-2}(F)$ is in $\zeta_i(U_{k-1}(\overline{F}))$. Since $U_{k-1}(\overline{F})$ is a free nilpotent group, $\zeta_i(U_{k-1}(\overline{F})) = \gamma_{c_2+1-i}(U_{k-1}(\overline{F}))$. Hence $F \models \sigma_i^{\forall}(g)$ if and only if $g \in \gamma_{c_2+1-i}(U_{k-1}(F))$.

Next we recursively define one-variable existential formulae $\sigma_i^{\exists}(x)$ for $i = 0, \dots, c_2$. Let $\sigma_0^{\exists}(x)$ be the formula $\theta_{k-2}^{\exists}(x)$. Now let $i \geqslant 0$ and suppose that $\sigma_i^{\exists}(x)$ has been defined. We define $\sigma_{i+1}^{\exists}(x)$ by

$$\exists y_1 y_2\bigl(\theta_{k-1}^{\exists}(y_1) \wedge \theta_{k-1}^{\exists}(y_2) \wedge \neg\sigma_{c_2-2}^{\forall}([y_1, y_2]) \wedge \sigma_i^{\exists}([x, y_1]) \wedge \sigma_i^{\exists}([x, y_2])\bigr).$$

Applying Lemma 10.1.4, it is clear that $F \models \sigma_i^{\exists}(g)$ if and only if the image of g in $\overline{F} = F/U_{k-2}(F)$ is in $\zeta_i(U_{k-1}(\overline{F}))$. Again this implies that $F \models \sigma_i^{\exists}(g)$ if and only if $g \in \gamma_{c_2+1-i}(U_{k-1}(F))$.

10.2 Elementary equivalence and isomorphism

Romanovskiĭ and Timoshenko [57, Theorem 3] showed that an n-generator group elementary equivalent to a free soluble group F of rank n is isomorphic to F. We prove the analogous result for free groups in the product variety $\mathfrak{V}_k \cdots \mathfrak{V}_1$.

Theorem 10.2.1. *Let F be a free group of rank n in $\mathfrak{V}_k \cdots \mathfrak{V}_1$. An n-generator group G is elementarily equivalent to F if and only if G is isomorphic to F.*

Proof. Since elementary equivalence is preserved by isomorphism, one of the implications is trivial. Suppose that G is elementarily equivalent to F. If $n = 1$ then G is a torsion-free abelian group of rank 1 and hence isomorphic to $\mathbb{Z} \cong F$. Suppose therefore that $n \geqslant 2$.

Let $g_1, \dots, g_n$ generate G and let $x_1, \dots, x_n$ be free generators of F. The group G is non-trivial (since F is) and therefore at least one of the generators differs from the identity. Assume without loss of generality that $g_1 \neq 1$. Replacing any generator equal to the identity with g_1 we may assume that all the generators are non-trivial. Since G

is elementarily equivalent to F, it is in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ and the map $\pi : F \to G : x_i \mapsto g_i$ for $i = 1, \dots, n$ extends to a surjective homomorphism. Let R be the kernel of π, so that $G \cong F/R$. Suppose that R is non-trivial: we show that this contradicts the elementary equivalence of F and G. Let $r = r(x_1, \dots, x_n)$ be a non-trivial element of R, so that $r(g_1, \dots, g_n) = 1$.

By the results of Section 10.1, there is a formula $\mu_n(x)$ defining the derived subgroup of F such that if $g \in G'$ then $G \models \mu_n(g)$. We define a sentence θ as follows:

$$\exists y_1 \dots y_n \forall x \exists u_1 \dots u_n v \Big(\bigwedge_{i=1}^{n} y_i \neq 1 \wedge \bigwedge_{i=1}^{n} [y_i, u_i] = 1 \\ \wedge\, r(y_1, \dots, y_n) = 1 \wedge \mu_n(v) \wedge x = u_1 \cdots u_n v \Big).$$

Every element $g \in G$ can be written as a product $g = h_1 \cdots h_n k$ where h_i is a power of g_i for $i = 1, \dots, n$ and k is an element of G'. Thus G satisfies θ and, since G is elementarily equivalent to F, we have $F \models \theta$.

Let $f_1, \dots, f_n \in F$ play the role of the y_i in the sentence. We may suppose that $f_1, \dots, f_l \notin U_1(F)$ and that $f_{l+1} \dots, f_n \in U_1(F)$ for some l. Now let f be an element of F and pick $h_1, \dots, h_n \in F$ and $k \in F'$ such that $[f_i, h_i] = 1$ for $i = 1, \dots, n$ and $f = h_1 \cdots h_n k$. By Theorem 2.4.4, every non-cyclic nilpotent subgroup of F is contained in $U_1(F)$. Thus the elements f_i and h_i are in the same cyclic subgroup for $i = 1, \dots, l$ and $h_{l+1}, \dots, h_n \in U_1(F)$. The sentence θ implies that F/F' is generated by l elements; however, F/F' is the free abelian group of rank n. Thus $l = n$ and the elements $f_1, \dots, f_n$ are linearly independent modulo F'. It follows, by Theorem 2.2.2, that $f_1, \dots, f_n$ are free generators of the free group of rank n in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ and the identity $r(f_1, \dots, f_n) = 1$ cannot hold. □

As a special case of Theorem 10.2.1 with $\mathfrak{V}_i = \mathfrak{N}_{c_i}$ we have the following corollary.

Corollary 10.2.1. *Let G be an n-generator group and let F be a free polynilpotent group of rank n with class sequence $\overline{c} = (c_1, \dots, c_k)$. Then G is elementarily equivalent to F if and only if G is isomorphic to F.*

10.3 Relatively free groups of different finite ranks

We show that the rank of a free group in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ has an effect on the elementary theory.

Proposition 10.3.1. *The free groups in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ of different finite ranks have different $\forall\exists$-theories.*

Proof. Let F_1 and F_2 be two free groups in the product variety $\mathfrak{V}_k \ldots \mathfrak{V}_1$ of different finite ranks r_1 and r_2 respectively. If one of the groups is of rank 1 then it is abelian and therefore not universally equivalent to the other which must be of rank strictly greater than 1 and hence non-abelian. We may suppose without loss of generality that $2 \leqslant r_1 < r_2$.

The quotients F_1/F_1' and F_2/F_2' are free abelian groups of rank r_1 and r_2 respectively. It follows from results of Szmielew [63] that there exists a sentence ψ of the form

$$\forall x_1 \ldots x_s \exists y_1 \ldots y_t \bigvee_{i=1}^{l} w_i(\overline{x}, \overline{y}) = 1,$$

where the $w_i(\overline{x}, \overline{y})$ are words in the variables $x_1, \ldots, x_s, y_1, \ldots, y_t$, such that $F_1/F_1' \models \psi$ and $F_2/F_2' \models \neg\psi$. Consider, for example, the sentence ψ defined by

$$\forall x_1 \ldots x_{2^{r_1}} \exists y \left(\bigvee_i x_i = y^2 \vee \bigvee_{i \neq j} x_i x_j = y^2 \right).$$

Let $\phi(x)$ be the one-variable formula $\phi_{r_2}(x)$ as defined in Section 10.1; the formula $\phi(x)$ defines F_i' for $i = 1, 2$. We define ψ^* to be the sentence

$$\forall x_1 \ldots x_{2^{r_1}} \exists y \left(\bigvee_i \phi(x_i y^2) \vee \bigvee_{i \neq j} \phi(x_i x_j y^2) \right).$$

Since $\phi(x)$ is existential, ψ^* is a $\forall\exists$-sentence. Moreover $F_1 \models \psi^*$ and $F_2 \models \neg\psi^*$ and hence F_1 and F_2 have different $\forall\exists$-theories. $\square$

As a special case with $\mathfrak{V}_i = \mathfrak{N}_{c_i}$ we have the following corollary.

Corollary 10.3.1. *Two free polynilpotent groups with class sequence $\overline{c} = (c_1, \ldots, c_k)$ of different finite ranks do not have the same $\forall\exists$-theory.*

We note that Lawrence [26, Theorem 5] showed that, in a soluble variety, free groups of different finite ranks have distinct $\forall\exists\forall$-positive theories. In the case where $k = 2$, or $k = 3$ and $c_3 = 1$, we may use the results of Chapter 8 to obtain a positive existential formula defining the derived subgroups of F_1 and F_2. In this case the same proof as that of Lemma 10.3.1 shows that F_1 and F_2 do not have the same $\forall\exists$-positive theory.

10.4 The undecidability of $\forall\exists$-theory

We show that the elementary theory of a non-abelian finitely generated free group in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ is undecidable.

Theorem 10.4.1. *A free group F of finite rank $r \geqslant 2$ in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ has undecidable $\forall\exists$-theory.*

Proof. Let $\phi_r(x)$ and $\nu_r(x)$ be the one-variable formulae defined in Section 10.1. We note to begin with that there is a two-variable universal formula $\phi(x,y)$ such that, for $g,h \in F$, we have $F \models \phi(g,h)$ only if g and h are linearly independent modulo F': we define $\phi(x,y)$ to be the formula $\neg\nu_r([x,y])$. Let g and h be elements of F and let $G = \langle g,h\rangle$; then g and h are linearly dependent modulo F' if and only if G is cyclic modulo F' (since F/F' is torsion-free). By Lemma 4.5.1, this implies that G is generated by elements $a \in F$ and $c \in F'$ and it follows that $G' \leqslant \langle [a,c]\rangle^G \leqslant \gamma_3(F)$ and $[g,h] \in \gamma_3(F)$. Thus if $F \models \phi(g,h)$ then g and h are linearly independent modulo F'.

In order to prove our theorem it is sufficient to associate to each polynomial $P(\overline{X})$ in several variables with coefficients in $\mathbb{Z}$ a $\forall\exists$-sentence ψ in the language of groups such that $P(\overline{X})$ has a zero in $\mathbb{Z}$ if and only if $F \models \psi$. Let $P(X_1,\ldots,X_m)$ be such a polynomial. Following the proof of [13, Theorem C2] we set

$$\begin{aligned}
+(x,y,z) &: \phi_r(xyz^{-1}),\\
\times(x,y,z,a,b) &: \exists uvw\big([u,xb] = 1 \wedge [v,b] = 1\\
&\qquad\qquad \wedge [w,ab] = 1 \wedge \phi_r(yvw^{-1}) \wedge \phi_r(uv^{-1}z^{-1})\big),\\
1(x,a) &: (x = a),\\
0(x) &: (x = 1).
\end{aligned}$$

These are all existential formulae. Now let g and h be elements of F such that $F \models \phi(g,h)$ and let x_1,x_2,x_3 be elements of F such that $[g,x_i] = 1$ for $i = 1,2,3$. By Theorem 2.4.4, every non-cyclic nilpotent subgroup of F is contained in $U_1(F)$. Since $g \notin F' \geqslant U_1(F)$, the abelian subgroup generated by g and x_i is cyclic for $i = 1,2,3$ and it follows that there exist non-zero integers s_1,s_2,s_3 and integers t_1,t_2,t_3 such that $x_i^{s_i} = g^{t_i}$ for $i = 1,2,3$.

Using these equalities and the fact that g and h are linearly independent modulo F', it is easy to show that the following hold:

(i) $F \models +(x_1, x_2, x_3)$ if and only if $\dfrac{t_1}{s_1} + \dfrac{t_2}{s_2} = \dfrac{t_3}{s_3}$;

(ii) if $F \models \times(x_1, x_2, x_3, g, h)$ then $\dfrac{t_1}{s_1} \cdot \dfrac{t_2}{s_2} = \dfrac{t_3}{s_3}$;

(iii) if $\dfrac{t_1}{s_1} \cdot \dfrac{t_2}{s_2} = \dfrac{t_3}{s_3}$ and s_i divides t_i for $i = 1, 2, 3$ then $F \models \times(x_1, x_2, x_3, g, h)$.

We construct, by induction on the complexity of P and using the formulae defined above, an existential formula $\psi'(a, b, x_1, \ldots, x_m)$ corresponding to P and we define ψ to be the $\forall\exists$-sentence

$$\forall xy \left(\phi(x, y) \to \left(\exists x_1 \ldots x_m \left(\bigwedge_{i=1}^{m} [x, x_i] = 1 \wedge \psi'(x, y, x_1, \ldots, x_m) \right) \right) \right).$$

Since $\phi(x, y)$ is a universal formula and $\psi'(x, y, x_1, \ldots, x_m)$ is an existential formula, ψ is a $\forall\exists$-sentence.

Let g and h be elements of F such that $F \models \phi(g, h)$ and suppose that P has a zero $(t_1, \ldots, t_m)$ in $\mathbb{Z}$. It follows from the construction of $\psi'(x, y, x_1, \ldots, x_m)$ that $\psi'(g, h, g^{t_1}, \ldots, g^{t_m})$ holds in F and hence F satisfies ψ.

Suppose now that $F \models \psi$ and let g and h be distinct free generators of F. It is clear that $F \models \phi(g, h)$ and hence

$$F \models \exists x_1 \ldots x_m \left(\bigwedge_{i=1}^{m} [g, x_i] = 1 \wedge \psi'(g, h, x_1, \ldots, x_m) \right).$$

Since g is not in F', the group generated by g and x_i is cyclic. For some $f \in F$, some non-zero integer p_i and some integer q_i, we have $g = f^{p_i}$ and $x_i = f^{q_i}$. However g is a free generator of F and hence $p_i = 1$ and $g = f$. Thus $x_i = g^{q_i}$ for $i = 1, \ldots, k$. It follows from the construction of $\psi'(x, y, x_1, \ldots, x_m)$ that $(q_1, \ldots, q_m)$ is a zero of P in $\mathbb{Z}$. $\square$

Special cases

- Abelian-by-nilpotent. If F is a free group of finite rank in $\mathfrak{A}\mathfrak{N}_{c_1}$ then F has undecidable $\forall\exists$-theory.

- Polynilpotent. If F is a free polynilpotent group of finite rank with class sequence $\overline{c} = (c_1, \ldots, c_k)$ then F has undecidable $\forall\exists$-theory.

10.5 The (un)decidability of universal theory

In this section we show that the decidability of the universal theory of a non-cyclic free group in the product variety $\mathfrak{V}_k \cdots \mathfrak{V}_2\mathfrak{V}_1$ with $c_1 \geqslant 2$, or $k \geqslant 3$, $\mathfrak{V}_1 = \mathfrak{A}$ and $\mathfrak{V}_2 = \mathfrak{N}_{c_2}$, would imply the decidability of the universal theory of the field of rationals. We divide the result into two cases which require slightly different treatment and explain why our methods do not work for free abelian-by-nilpotent groups.

Theorem 10.5.1. *The decidability of the universal theory of the free group of finite rank $r \geqslant 2$ in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ with $c_1 \geqslant 2$ implies the decidability of the universal theory of the field of rationals.*

One approach would be to use Romankov's result [55] that the decidability of universal theory of the free nilpotent group of class and rank 2 would imply the decidability of the universal theory of the field of rationals, together with the definability of the third term of the lower central series of a free group of finite rank $r \geqslant 2$ in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ with $c_1 \geqslant 2$. The following approach seems more direct.

Proof. We begin by defining $\phi(x, y)$ to be the existential formula $\neg\rho_{c_1-2}([x, y])$. The argument used in Theorem 10.4.1 shows that if $F \models \phi(g, h)$ then g and h are linearly independent modulo F'.

The field of rationals has decidable universal theory if and only if there exists an algorithm to determine whether a homogeneous polynomial in several variables with coefficients in $\mathbb{Z}$ has a non-trivial zero in $\mathbb{Z}$ (see [48, §9]). In order to prove our theorem, it is sufficient to associate to each such polynomial $P(\overline{X})$ an existential sentence ψ in $\mathcal{L}_{\text{gp}}$ such that $P(\overline{X})$ has a non-trivial zero in $\mathbb{Z}$ if and only if $F \models \psi$. Let $P(X_1, \ldots, X_m)$ be such a polynomial. We construct, by induction on the complexity of P, an existential formula $\psi'(a, b, x_1, \ldots, x_m)$ corresponding to P exactly as in Theorem 10.4.1. We define ψ to be the existential sentence

$$\exists xyx_1 \ldots x_m \left(\phi(x, y) \wedge \left(\bigwedge_{i=1}^{m} ([x, x_i] = 1 \wedge x_i \neq 1) \wedge \psi'(x, y, x_1, \ldots, x_m) \right) \right).$$

Since $\phi(x, y)$ and $\psi'(x, y, x_1, \ldots, x_m)$ are existential formulae, ψ is an existential sentence.

Suppose that P has a non-trivial zero $(t_1, \ldots, t_m)$ in $\mathbb{Z}$ and let g and h be distinct free generators of F. It follows from the construction of $\psi'(x, y, x_1, \ldots, x_m)$ that $\psi'(g, h, g^{t_1}, \ldots, g^{t_m})$ holds in F and hence F satisfies ψ.

Suppose now that F satisfies ψ. Let $g, h, x_1, \ldots, x_m$ be elements of F such that

$$F \models \phi(g, h) \wedge \left(\bigwedge_{i=1}^{m} ([g, x_i] = 1 \wedge x_i \neq 1) \wedge \psi'(x, y, x_1, \ldots, x_m) \right).$$

The elements g and h are linearly independent modulo F' and there are non-zero integers s_i and t_i such that $g^{t_i} = x_i^{s_i}$ for $i = 1, \ldots, k$. It follows from the construction of $\psi'(x, y, x_1, \ldots, x_m)$ that P has a non-trivial zero in $\mathbb{Q}$, namely $(\frac{t_1}{s_1}, \ldots, \frac{t_m}{s_m})$. Since P is homogeneous, we may conclude that P has a non-trivial zero in $\mathbb{Z}$. $\square$

Theorem 10.5.2. *The decidability of the universal theory of a non-cyclic free group in* $\mathfrak{V}_k \cdots \mathfrak{V}_2\mathfrak{V}_1$ *with* $k \geqslant 3$, $\mathfrak{V}_1 = \mathfrak{A}$ *and* $\mathfrak{V}_2 = \mathfrak{N}_{c_2}$ *implies the decidability of the universal theory of the field of rationals.*

Proof. We begin by observing that $F' = U_{k-1}(F)$ and $F'' = \gamma_{c_2+1}(U_{k-1}(F))$ are definable. We note also that there is a two-variable existential formula $\phi(x, y)$ such that, for $g, h \in F$, we have $F \models \phi(g, h)$ if and only if g and g^h are linearly independent modulo F'' with $g \in F' \setminus F''$ and $h \in F \setminus F'$. We define $\phi(x, y)$ by

$$\left(\theta^{\exists}_{k-1}(g) \wedge \neg\sigma^{\forall}_{c_2-1}(g) \wedge \neg\theta^{\forall}_{k-1}(h) \right).$$

Let g and h be elements of F. It is clear that $F \models \phi(g, h)$ if and only if $g \in F' \setminus F''$ and $h \in F \setminus F'$. It remains to show that this implies the linear independence of g and g^h modulo F''. Suppose we have integers r and s such that $g^r = (g^h)^s$. The quotient F'/F'' is torsion-free as a $\mathbb{Z}[F/F']$ module and hence $r = sh$ in $\mathbb{Z}[F/F']$. Since $h \notin F'$, we have $r = s = 0$ and g and g^h are linearly independent modulo F''.

As in Theorem 10.5.1, it is sufficient to associate to each such polynomial $P(\overline{X})$ an existential sentence ψ in the language of groups such that $P(\overline{X})$ has a non-trivial zero in $\mathbb{Z}$ if and only if $F \models \psi$. Let $P(X_1, \ldots, X_r)$ be such a polynomial. Following the proof of

[13, Theorem C1] we set

$$
\begin{aligned}
+(x,y,z) &: \sigma^{\exists}_{c_2-1}(xyz^{-1}),\\
\times(x,y,z,a,b) &: \exists uvw\big([u,xa^b]=1 \wedge [v,a^b]=1\\
&\qquad\qquad \wedge [w,aa^b]=1 \wedge \sigma^{\exists}_{c_2-1}(yvw^{-1}) \wedge \sigma^{\exists}_{c_2-1}(uv^{-1}z^{-1})\big),\\
1(x,a) &: (x=a),\\
0(x) &: (x=1).
\end{aligned}
$$

These are all existential formulae. Now let g and h be elements of F such that $F \models \phi(g,h)$ and let x_1, x_2, x_3 be elements of F such that $[g, x_i] = 1$ and $x_i \neq 1$ for $i = 1, 2, 3$. Every non-cyclic nilpotent subgroup of F is contained in $\gamma_{c_2+1}(F')$ (see Theorem 2.4.4). Since $g \notin F''$, the abelian group $\langle g, x_i \rangle$ is cyclic for $i = 1, 2, 3$ and it follows that there exist non-zero integers s_1, s_2, s_3 and t_1, t_2, t_3 such that $x_i^{s_i} = g^{t_i}$ for $i = 1, 2, 3$. Using these equalities and the fact that g and h are linearly independent modulo F' we construct, by induction on the complexity of P, an existential formula $\psi'(a, b, x_1, \ldots, x_r)$ corresponding to P exactly as in Theorem 10.4.1 and Theorem 10.5.1. We define ψ to be the existential sentence

$$\exists xyx_1 \ldots x_r \left(\phi(x,y) \wedge \left(\bigwedge_{i=1}^{r} ([x,x_i]=1 \wedge x_i \neq 1) \wedge \psi'(x,y,x_1,\ldots,x_r) \right) \right).$$

By the argument used in Theorem 10.5.1, we have $F \models \psi$ if and only if P has a non-trivial zero in $\mathbb{Z}$. □

It is not possible to use these ideas to show that the decidability of the universal theory of a non-cyclic free nilpotent-by-abelian group implies the decidability of the universal theory of the field of rationals. Let F be a non-cyclic free metabelian group. Chapuis [11, Proposition 4.4] shows that there is no existential formula $\phi(x,y)$ such that $F \models \exists xy\phi(x,y)$ and, for $a, b \in F$, if $F \models \phi(a,b)$ then a and b are linearly independent modulo F'. We extend this result to the product variety $\mathfrak{V}\mathfrak{A}$ where $\mathfrak{V}$ satisfies the following conditions: the free groups of $\mathfrak{V}$ are residually finite p-groups for infinitely many primes p, the variety $\mathfrak{V}$ is generated by its k-generator groups for some positive integer k and $\mathfrak{V}$ contains the variety of all abelian groups.

Proposition 10.5.1. *Let F be a non-cyclic free group in $\mathfrak{V}\mathfrak{A}$. There is no existential formulae $\phi(x, y)$ such that $F \models \exists xy\phi(x, y)$ and, for $g, h \in F$, if $F \models \phi(g, h)$ then g and h are linearly independent modulo F'.*

Proof. Suppose for a contradiction that such a formula $\phi(x, y)$ exists. Let a and b be free generators of F, so that $a \notin F'$ and $[a, b] \in F' \setminus \gamma_2(F')$. By Lemma 9.1.1, the group $G = \langle a, [a, b]\rangle$ is isomorphic to the verbal wreath product $\mathbb{Z} \operatorname{wr}_{\mathfrak{V}} \mathbb{Z}$. Moreover, by Proposition 9.1.1, the group F is universally equivalent to $\mathbb{Z} \operatorname{wr}_{\mathfrak{V}} \mathbb{Z}$. Thus $G \models \exists xy\phi(x, y)$ and there exist elements $g, h \in G \leqslant F$ such that $G \models \phi(g, h)$. Since $\phi(x, y)$ is existential, $F \models \phi(g, h)$ and g and h are linearly independent modulo F'. However, G is cyclic modulo F' and we obtain the required contradiction. ∎

In particular, let $\mathfrak{V}$ be a nilpotent variety of class c whose free groups are torsion-free; then $\mathfrak{V}$ is generated by its free group of rank c. Since there is no formula which says that two elements are linearly independent modulo the derived subgroup of a non-cyclic free group our methods cannot be applied here.

Part IV

CONCLUSION

Chapter 11

CONCLUDING REMARKS

In this thesis we have been concerned with aspects of the model theory of infinite soluble groups, with many of the results lying on the border between group theory and model theory. A necessary foundation for readers unfamiliar with either of these branches of mathematics was provided in Part I.

In Part II, we gave explicit descriptions of the universal theory of restricted wreath products of abelian groups of zero or square-free exponent by certain soluble groups, and of those groups in the universal closure of such groups. The groups in the universal closures of these wreath products were described in a number of ways: in terms of the module-theoretic properties of the Fitting subgroup and the universal theory of the quotient by the Fitting subgroup; in terms of the universal sentences which they satisfy; and in terms of the embeddability of finitely generated subgroups into certain restricted wreath products.

In Chapter 4, we were concerned with non-cyclic free groups in the variety $\mathfrak{A}_n\mathfrak{A}$, where n is a square-free positive integer; these groups are universally equivalent to restricted wreath products $U \operatorname{wr} A$, where U is an abelian group of exponent n and A is a finitely generated torsion-free abelian group. In Chapter 5, we dealt with left iterated wreath products of torsion-free abelian groups. Our results in this chapter were in part a response to Chapuis' comment [11] that it would be interesting to obtain an explicit description of the universal theory of such groups. In Chapter 6, we considered groups of the form $U \operatorname{wr} H$, where U is an abelian group of zero or finite square-free exponent and H is a finitely generated, torsion-free nilpotent group. Finally in Part II, Chapter 7 extended the results of the preceding three chapters, addressing groups of the form $U \operatorname{wr} P$, where U is an abelian group of zero or finite square-free exponent and P is a finitely generated torsion-free soluble group with a bound on the class of its nilpotent subgroups.

We used the descriptions obtained to prove a number of further results about these groups and about their universal theory: for example, in Chapter 4, we showed that the universal theory of a free group in the variety $\mathfrak{A}_n\mathfrak{A}$ is decidable.

The results of Chapters 4, 6 and 7 were marred by the unnatural restriction of the finite exponent n to square-free positive integers only. It would be a very natural extension of these results to deal with the case where n is any positive integer. An obvious starting point for any attempt would be to consider the case of a non-cyclic free group in $\mathfrak{A}_{p^2}\mathfrak{A}$ or, equivalently, the restricted wreath product $C_{p^2} \operatorname{wr} \mathbb{Z}$. The difficulty arises from the fact that the Fitting subgroup of a group G in $\operatorname{ucl}(F_2(\mathfrak{A}_{p^2}\mathfrak{A})) = \operatorname{ucl}(C_{p^2} \operatorname{wr} \mathbb{Z})$ does not split into a direct product of torsion-free $\mathbb{F}_p[G/\operatorname{Fitt}(G)]$-modules. Instead, all we are able to say is that $\operatorname{Fitt}(G)$ is an extension of a torsion-free $\mathbb{F}_p[G/\operatorname{Fitt}(G)]$-module by a torsion-free $\mathbb{F}_p[G/\operatorname{Fitt}(G)]$-module. It is not clear whether such a module embeds into a free $\mathbb{F}_{p^2}[G/\operatorname{Fitt}(G)]$-module. The structure of $\operatorname{Fitt}(G)$ is potentially complex: for example, the group $(C_p \times C_{p^2}) \operatorname{wr} \mathbb{Z}$ is universally equivalent to the group $C_{p^2} \operatorname{wr} \mathbb{Z}$.

In [51], Remeslennikov and Stöhr consider the quasivariety generated by a non-cyclic free metabelian group. They characterize the groups in this quasivariety in a number of ways: in terms of the module-theoretic properties of the Fitting subgroup; in terms of the quasi-identities which they satisfy; and in terms of the embeddability of finitely generated subgroups into direct products of wreath products of free abelian groups. It would be interesting to obtain similar characterizations of the quasivarieties generated by the restricted wreath products whose universal closures we have considered in Part II. As the work of Remeslennikov and Stöhr uses results from commutative algebra, it would be sensible to begin by considering the quasivariety generated by a non-cyclic free group in the variety $\mathfrak{A}_p\mathfrak{A}$. This presents a possible area for future work.

Part III brought together a collection of results concerning various aspects of the elementary and universal theory of certain infinite soluble groups.

In Chapter 8, we gave a number of examples of C_n-groups (as defined on page 125) and obtained first-order formulae which allowed us to show that, for certain classes of groups, elementary equivalence is preserved under taking quotients by the terms of the lower central series. The restriction to abelian-by-metanilpotent groups seems an artificial one and we know of no example of a finitely generated soluble group which is not a C_n-group for all $n \geqslant 2$.

In Chapter 9 we extended some of the results of Timoshenko [66]. There is significant potential for further work in this area: in particular, to extend the major result of his paper to show that there are at most two non-isomorphic groups, finitely presented in some variety, which are universally equivalent to the free group of rank 2 in $\mathfrak{V}\mathfrak{A}$. In this chapter, we also proved a number of results about groups in $\mathrm{ucl}(F_2(\mathfrak{V}\mathfrak{N}_c))$ for $c \geqslant 2$ under certain conditions on $\mathfrak{V}$. These results related to free nilpotent groups are especially pleasing because of the difficult nature of the universal theory of such groups, as emphasized by the links between this universal theory and Hilbert's Tenth Problem over the rationals.

The results in Chapter 10 relate to the elementary and universal theories of free groups in product varieties $\mathfrak{V}_k \cdots \mathfrak{V}_1$ with $k \geqslant 2$, where each $\mathfrak{V}_i$ is nilpotent and the free groups of $\mathfrak{V}_i$ are torsion-free. We showed that the free groups in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ of different finite ranks have different $\forall\exists$-theories and that the $\forall\exists$-theory of a non-cyclic free group of finite rank in $\mathfrak{V}_k \cdots \mathfrak{V}_1$ is undecidable. We also showed that the decidability of the universal theory of a non-cyclic free group of finite rank in certain of the varieties $\mathfrak{V}_k \cdots \mathfrak{V}_1$ would imply a positive answer to Hilbert's Tenth Problem over the rationals. It was necessary to impose restrictive conditions for our methods to work. The justification for some of these limitations has been discussed. In other cases, it seems reasonable to hope that these limitations can be removed and that the same result can be proved when $\mathfrak{V}_2$ is any nilpotent variety whose free groups are torsion-free. This may provide an additional avenue for future work.

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