

Some Exact Solutions in Moving Finite Elements

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It is shown that when the moving finite elements are used on a number of parabolic problems there are steady-state, stationary, similarity, or travelling-wave solutions that can be found numerically.

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1 Introduction

Donnelly [1] introduced a one parameter family of approximations to the diffusion term in the method of moving finite elements that included as special cases the original approximation of Miller [4] and that of Juarez-Romero and Sargent [3] and illustrated it with an application to a nonlinear diffusion problem. In this report the method is applied to a number of other problems with a linear diffusion term and which have steady state, stationary, similarity or travelling wave solutions. In all cases it is possible to solve the corresponding continuous equations to find the limiting distributions of the nodes.

The problems considered are convection-diffusion, the heat equation on finite and infinite intervals, the one-phase Stefan problem and the viscous Burgers' equation. For the third and fourth problems the time-dependent solution is examined. The viscous Burgers' equation using Miller's approximation was the subject of Donnelly [2] where the existence of an explicit solution to the discrete equations for a travelling wave solution was shown.

2 The Original Method

The most general problem we wish to consider is

$$u_t + cu_x + bu_x^2 = u_{xx}, \quad (2.1)$$

where the coefficients may in general depend on u .

We approximate u by $\sum a_i(t)\alpha_i(x, t)$, where $\alpha_i(x, t)$ is the usual ‘hat function’, piecewise linear, equal to one at $x = s_i(t)$, and vanishing at the other nodes. The position of the nodes $s_i(t)$ is allowed to vary in time. We shall not indicate a range for i , preferring to treat boundaries for specific problems. The range may in fact be finite, semi-infinite, or infinite. Using a dot to denote partial differentiation with respect to t we have

$$u_t = \sum_i (\dot{a}_i \alpha_i + a_i \dot{\alpha}_i). \quad (2.2)$$

It can be shown that $\alpha_i = \beta_i = -u_x \alpha_i$.

If we follow Wathen and Baines [5] in writing

$$\alpha_i = \phi_i^2 + \phi_{i+1}^1, \quad (2.3)$$

where the hat is split into two right-angled triangles such that ϕ_i^1, ϕ_i^2 have for support the interval $[s_{i-1}, s_i]$ and $\phi_i^1 + \phi_i^2$ is its characteristic function, and denote the value of u_x on the same interval by $m_{i-\frac{1}{2}} = (a_i - a_{i-1})/(s_i - s_{i-1})$, then we have

$$\beta_i = -m_{i-\frac{1}{2}} \phi_i^2 - m_{i+\frac{1}{2}} \phi_{i+1}^1, \quad (2.4)$$

We use the equivalent sets of test functions $\{\phi_i^1, \phi_i^2\}$ and $\{\alpha_i, \beta_i\}$. We will deal in order with the two sides of the equation. We start with the left. We take c and b to be

piecewise linear defined by their values at the nodes. Then the left-hand side is

$$\sum_i (w_{i+1}^1 \phi_{i+1}^1 + w_i^2 \phi_i^2), \quad (2.5)$$

where

$$w_i^1 = \dot{a}_{i-1} - m_{i-\frac{1}{2}}(\dot{s}_{i-1} - c_{i-1}) + m_{i-\frac{1}{2}}^2 b_{i-1} \quad (2.6)$$

$$w_i^2 = \dot{a}_i - m_{i-\frac{1}{2}}(\dot{s}_i - c_i) + m_{i-\frac{1}{2}}^2 b_i. \quad (2.7)$$

Using ϕ_i^1, ϕ_i^2 as test functions produces a symmetric matrix composed of 2 x 2 diagonal blocks

$$\frac{s_i - s_{i-1}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} w_i^1 \\ w_i^2 \end{bmatrix} \quad (2.8)$$

To apply the test functions to the right hand side of (2.1) we integrate by parts to find

$$(u_{xx}, \alpha_i) = m_{i+\frac{1}{2}} - m_{i-\frac{1}{2}} \quad (2.9)$$

and

$$(u_{xx}, \beta_i) = (m_{i+\frac{1}{2}} - m_{i-\frac{1}{2}})(m_{i+\frac{1}{2}} + m_{i-\frac{1}{2}} - u_x(s_i)), \quad (2.10)$$

with $u_x(s_i)$ is defined by

$$u_x(s_i) = \frac{1}{2}(1 - pr_i)m_{i-\frac{1}{2}} + (1 + pr_i)m_{i+\frac{1}{2}}, \quad (2.11)$$

where

$$r_i = \frac{\Delta s_{i+\frac{1}{2}} - \Delta s_{i-\frac{1}{2}}}{\Delta s_{i+\frac{1}{2}} + \Delta s_{i-\frac{1}{2}}} = \frac{\delta^2 s_i}{\Delta s_{i+\frac{1}{2}} + \Delta s_{i-\frac{1}{2}}} \quad (2.12)$$

with $\Delta s_{i-\frac{1}{2}} = s_i - s_{i-1}$, and when p is a parameter.

For $p = 0$ we have Miller's approximation [4] which implicitly takes $m_x(s_i) = \frac{1}{2}(m_{i+\frac{1}{2}} + m_{i-\frac{1}{2}})$, while for $p = -1$ we have the approximation of Juarez-Romero and Sargent [3] which takes $u_x(s_i)$ as the slope of the quadratic interpolant of u through s_{i-1}, s_i, s, s_{i+1} , i.e.

$$u_x(s_i) = \frac{\Delta s_{i+\frac{1}{2}} m_{i-\frac{1}{2}} + \Delta s_{i-\frac{1}{2}} m_{i+\frac{1}{2}}}{\Delta s_{i+\frac{1}{2}} + \Delta s_{i-\frac{1}{2}}} \quad (2.13)$$

In terms of the alternative basis functions we have

$$\begin{aligned} (u_{xx}, \phi_i^1) &= \frac{1}{2}(1 - pr_{i-1})\delta m_{i-1} \\ (u_{xx}, \phi_i^2) &= \frac{1}{2}(1 + pr_i)\delta m_i \end{aligned} \quad (2.14)$$

where $\delta m_i = m_{i+\frac{1}{2}} - m_{i-\frac{1}{2}}$.

3 Parabolic Decomposition

In Donnelly [1] we showed how to rearrange the equations (2.1) to allow implicit time-stepping to be implemented in a straight forward manner. We start with the basic equations for the interval (s_{i-1}, s_i) with the parameter p which are

$$\frac{1}{3}\Delta s_{i-\frac{1}{2}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} w_i^1 \\ w_i^2 \end{bmatrix} = \begin{bmatrix} (1 - pr_{i-1})\delta m_{i-1} \\ (1 + pr_i)\delta m_i \end{bmatrix} \quad (3.1)$$

Subtracting the two components we obtain

$$(\Delta s_{i-\frac{1}{2}})^2 (\dot{m}_{i-\frac{1}{2}} + c'_{i-\frac{1}{2}} m_{i-\frac{1}{2}}^2 + b'_{i-\frac{1}{2}} m_{i-\frac{1}{2}}^3) = 3\delta_p(\delta m)_{i-\frac{1}{2}}, \quad (3.2)$$

where c' and b' approximate the derivatives of c and b , and on the right we use the notation

$$\delta_p(\delta m)_{i-\frac{1}{2}} = (1 + pr_i)\delta m_i - (1 - pr_{i-1})\delta m_{i-1}. \quad (3.3)$$

Notice that (3.3) is a parabolic equation for $m_{i-\frac{1}{2}}$ with coefficients depending on s_i . To complete the system we need an equation for s_i . Inverting the matrix in (3.1) we have

$$\begin{aligned} \Delta s_{i-\frac{1}{2}} w_i^1 &= (1 - pr_{i-1})\delta m_{i-1} - \delta_p(\delta m)_{i-\frac{1}{2}} \\ \Delta s_{i-\frac{1}{2}} w_i^2 &= (1 + pr_i)\delta m_i + \delta_p(\delta m)_{i-\frac{1}{2}}. \end{aligned} \quad (3.4)$$

If we take $w_i^2 - w_i^1$ and use the definition of r_i , we find

$$\begin{aligned} \Delta s_{i-\frac{1}{2}} \Delta s_{i+\frac{1}{2}} \delta m_i (\dot{s}_i - c_i - (m_{i+\frac{1}{2}} + m_{i-\frac{1}{2}})b_i) \\ = (1 + p)\delta m_i \delta^2 s_i + \Delta s_{i-\frac{1}{2}} \delta_p(\delta m)_{i+\frac{1}{2}} + \Delta s_{i+\frac{1}{2}} \delta_p(\delta m)_{i-\frac{1}{2}}, \end{aligned} \quad (3.5)$$

which, for $p > -1$, is a parabolic equation for s_i . For $p = -1$ it reduces to an ordinary differential equation for s_i .

Remarks

1) The numerical experiments referred to in Sections 6-10 indicate stability for values of p above the upper limit and the onset of instability a little above $p = -1$.

2) A full solution of the linearized system is possible. It may be noted that (3.2) may be used to eliminate $\delta_p(\delta m)$ from (3.5) which may in turn be used to eliminate $\delta^2 s$ from (3.2). The resulting system has band width five.

4 Boundary Conditions

1. Symmetry at $s_0 = 0$. Since $m_{-\frac{1}{2}} = m_{\frac{1}{2}}$, it follows that (3.2) remains valid for $i = 1$, as does (3.4).
2. Anti-symmetry at $s_0 = 0$. This is equivalent to $a_0 = 0$. The test function ϕ_1^1 , is not used and (3.1) becomes a scalar equation.

$$\frac{2}{3}\Delta s_{\frac{1}{2}} w_i^2 = (1 + pr_1)\delta m, \quad (4.1)$$

which is equivalent to (3.2) with 3 replaced by $\frac{3}{2}$ on the right. Using (4.1) in (3.4) leads to (3.5) with $\delta_p(\delta m)\frac{1}{2}$ replaced by $\frac{1}{2}(1 + pr_1)\delta m_i$.

3. Infinite interval. We allow the end node s_{N+1} to go to infinity, so that $m_{N+\frac{1}{2}}$ tends to zero. Dividing (3.5) by $\Delta s_{N+\frac{1}{2}}$ and letting it become infinite results in

$$\Delta s_{N-\frac{1}{2}}(\dot{s}_N - c_N - m_{N-\frac{1}{2}}b_N) = \delta_p(\delta m)_{N-\frac{1}{2}} \quad (4.2)$$

A better boundary condition would use $U_x = (\partial/\partial t)^{\frac{1}{2}}U$ but this operator is not local and introduces a Volterra equation with an Abel kernel and has not been pursued.

5 A Continuous Analogue

If $s_i = x(\xi_i, \tau)$ and ξ_i are regarded as uniformly spaced we may write down the continuous limit of (3.3) (omitting c and b)

$$x_\xi^2 m_\tau = 3m_{\xi\xi} + 3pm_\xi x_{\xi\xi}/x_\xi, \quad (5.1)$$

or

$$x_\xi^{p+2} m_\tau = 3(x_\xi^p m_\xi)_\xi, \quad (5.2)$$

and that of (3.5) as

$$m_\xi x_\xi^2 x_\tau = 2x_\xi m_{\xi\xi} + (3p + 1)m_\xi x_{\xi\xi} \quad (5.3)$$

or

$$m_\xi x_\xi^{(3p+3)/2} x_\tau = 2(x_\xi^{(3p+1)/2} m_\xi)_\xi. \quad (5.4)$$

Note that the underlying equation for $m = u_x$ is $m_t = m_{xx}$, which in ξ and τ is

$$x_\xi^2 (x_\xi m_\tau - m_\xi x_\tau) = x_\xi m_{\xi\xi} - m_\xi x_{\xi\xi} \quad (5.5)$$

consistent with (5.2) and (5.4).

Suppose $x_\tau = 0$ so that we have a stationary distribution of nodes. Then from (5.4)

we have

$$x_\xi^{(3p+3)/2} m_x = x_\xi^{(3p+1)/2} m_\xi = \text{const}, \quad (5.6)$$

so that the nodes are distributed with spacing proportional to $(u_{xx})^{-(3p+3)/2}$ i.e. more closely where u_{xx} is large. The limit cases are $p = -1$ with all the nodes at the maximum of u_{xx} and $p = \infty$ with uniformly spaced nodes. This seems to justify the claim in [3] of greater accuracy but hints that the nodes may in fact become too close.

6 Convection-Diffusion

We consider the problem

$$u_t + cu_x = u_{xx} \quad \text{on } (0, 1), \quad u(0) = 0, u(1) = 1.$$

Here $b = 0$, $c' = 0$ so that (3.3) becomes

$$(\Delta s_{i-\frac{1}{2}})^2 \dot{m}_{i-\frac{1}{2}} = 3\delta_p \delta m_{i-\frac{1}{2}} \quad i = 2, \dots, n-1$$

in the interior and

$$(\Delta s_{\frac{1}{2}})^2 (\dot{m}_{\frac{1}{2}} + cm_{\frac{1}{2}}) = \frac{3}{2} \delta_p \delta m_1,$$

$$(\Delta s_{n-\frac{1}{2}})^2 (\dot{m}_{n-\frac{1}{2}} - cm_{n-\frac{1}{2}}) = \frac{3}{2} \delta_p \delta m_{n-1}$$

at the end of the interval with $s_0 = 0$, $s_n = 1$.

In the time-independent problem the time step is restricted by the size of c . If $c\Delta t$ is too large the solution of (3.5) becomes unstable. We shall present results only for the steady state. When we set $\dot{s}_i = 0$ in (3.4) we obtain

$$(1+p)\delta^2 s_i + c\Delta s_{i-\frac{1}{2}}\Delta s_{i+\frac{1}{2}} = 0 \quad (6.1)$$

which is a linear recurrence for $(\Delta s_{i-\frac{1}{2}})^{-1}$ with the solution $c(\gamma + i - 1)/(1 + p)$ for some constant γ . The details of the solution as given in Appendix 1. The limiting solution is easily obtained from (6.1) as $(p + 1)x_{\xi\xi} + cx_{\xi}^2 = 0$ or, changing the independent variable to x , $(p + 1)\xi_{xx} = c\xi_x$ which has the solution $\xi \propto \exp(cx/(p + 1)) - 1$. For $p = \infty$, as the number of nodes tends to infinity they become evenly spaced in the solution u .

7 The Heat Equation — Finite Interval

When we set $c = b = 0$ we have the heat equation

$$u_t = u_{xx}$$

which has a stationary solution $u = \sin(\pi x/2) \exp(-\pi^2 t/4)$ on $(0, 1)$ with boundary conditions $u = 0$ at $x = 0$ and $u_x = 0$ and $x = 1$.

The continuous analogue must yield this stationary solution, so that, if the stationary distribution of the nodes is given by $x = X(\xi)$, then

$$m_x = U_{xx} = -e^{-\pi^2 t/4} \pi^2 \sin X(\xi)$$

and substituting in (5.6) we find that X satisfies

$$X'(\sin \pi X)^2/(3 + 3p) = \text{const}, \quad X(0) = 0 \quad (7.1)$$

where the constant is chosen to make $X(1) = 1$. The solution may be found using a library routine for solving ordinary differential equations provided some care is taken

with the singularity at $X(0) = 0$.

The discrete stationary solution may be found by assuming that $\dot{a}_i = \sigma a_i$ but *not requiring the constant of proportionality to be that for the continuous solution* and again taking $\dot{s}_i = 0$. Equation (3.1) may be written

$$\begin{aligned}\sigma \Delta s_{1-\frac{1}{2}}(a_{i-1} + 2a_i) &= 3(1 + pr_1)\delta m_i \\ \sigma \Delta s_{i+\frac{1}{2}}(2a_i + a_{i+1}) &= 3(1 - pr_i)\delta m_i \quad i = 1, \dots, n-1\end{aligned}$$

with $a_0 = 0$ and $a_n = 1$.

If we assume a_1, \dots, a_{n-1} and σ are known we may take the quotient of these two equations to obtain a quadratic equation for $\Delta s_{i+\frac{1}{2}}/\Delta s_{i-\frac{1}{2}}$ which determines s_i with $s_0 = 0, s_n = 1$. The sum of the two equations then gives a nonlinear equation. These are $n-1$ of these which together with

$$\sigma \Delta s_{n-\frac{1}{2}}(a_{n-1} + 2a_n) = 3\delta m_n = -6m_{n-\frac{1}{2}}$$

gives the right number of equations to determine a_1, \dots, a_{n-1} and σ .

Table 2a shows the maximum error in the nodal values of the numerical stationary solution.

Table 2b shows the relative error in the ratio \dot{a}_i/a_i which is $-\pi^2/4$ in the continuous case.

8 The Heat Equation — Infinite Interval

The heat equation has a similarity solution on $(0, \infty)$ given by

$$u = \operatorname{erf}(x/\sqrt{2t})$$

If we take $x = \sqrt{2t}X(\xi)$ and use $m = u_x \propto (2t)^{-\frac{1}{2}} \exp(-X^2/2)$, we find $m_\xi = -XX'm, m_{\xi\xi} = ((XX')^2 - (X')^2 - XX'')m$ and $x_\xi = (2t)^{-\frac{1}{2}}X'$.

Substitution into (5.2) yields

$$(X')^2 + 3((XX')^2 - (X')^2 - (1+p)XX'') = 0,$$

which may be written

$$XX' - (1+p)X''/X' - (2/3)X'/X = 0$$

an exact differential of

$$\exp(-X^2/2)(X')^{1+p}X^{\frac{2}{3}} = \text{const}, \quad X(0) = 0.$$

The constant is given by $X(1) = \infty$ and is found as a gamma function. (Write $X^2 = 2(1+p)Y$ and reduce to an ODE).

To find the numerical similarity solution we notice that $\dot{a}_i = 0$ in (3.2) and that the equations will not make sense unless $s_i \dot{s}_i$ is independent of time and so equal to X^2 .

If we take $2t = 1$ we shall actually have $\dot{s}_i = s_i$ which allows us to write (3.1) as

$$-\Delta a_{\frac{1}{2}} = \frac{3}{2}(1 + pr_1)\delta m_1$$

$$-\Delta a_{i-\frac{1}{2}} = (1 - pr_{i-1})\delta m_{i-1} = (1 + pr_i)\delta m_i, \quad i = 1 \dots n - 1$$

$$-\Delta a_{n-\frac{1}{2}} = (1 - p)\delta m_{n-1}$$

which can be rewritten as

$$-\Delta a_{i-\frac{1}{2}} - \Delta a_{i+\frac{1}{2}} = 2\delta m_i$$

$$-\Delta a_{i-\frac{1}{2}} + \Delta a_{i+\frac{1}{2}} = 2pr_i\delta m_i, \quad i = 2, \dots, n - 1$$

If we start with the a_i then the first of this pair can be used to find $m_{i-\frac{1}{2}}$ and hence s_i and the second used in nonlinear equation solver to find a_i .

Table 3 shows the maximum error in the numerical nodal values of the similarity solution. The convergence is affected by the rather crude approximation used for the boundary condition at infinity.

Table 4 shows the error in the solution of the heat equation starting with the similarity solution with the nodes determined by the asymptotic solution X above. The solution is allowed to develop for time 0.1 and the tables show the quantity $-\text{sgn} |\text{error}| \log_2 |\text{error}|$. Unfortunately it does not seem possible to draw any firm conclusion about the rate of convergence.

9 One - Phase Stefan Problem

Here the similarity solution is

$$u = 1 - \operatorname{erf}(x/\sqrt{2t})/\operatorname{erf}\beta$$

with the moving boundary at $x = \beta\sqrt{2t}$ and β is determined by the Stefan boundary condition from

$$L\beta = \sqrt{(2/\pi)}e^{-\beta^2/2}/\operatorname{erf}(\beta\sqrt{2}).$$

The limiting position of the nodes is given by $x = \sqrt{2t}X(\xi)$ where X is the solution of the same differential equation as in the previous section with the boundary condition $X(1) = \beta$. The numerical solution requires an approximation to the Stefan boundary condition in the form

$$-L\dot{s}_n = m_{n-\frac{1}{2}} + \frac{1}{2}(1 - pr_{n-1})\delta m_{n-1},$$

where $p = -1$ gives the quadratically extrapolated slope at s_n . It is not obvious that the incorporation of p is necessary but it appears to give satisfactory results. Table 5a shows the error in the moving boundary. Table 6a shows the maximum nodal error.

In Table 6a we show the error in the solution in time starting with the nodes determined by the asymptotic solution, as the moving boundary moves from $x = 1$ to $x = 1.1$. The solutions show first order convergence in the time-step but the dependence on the number of nodes show no clear trend.

10 Viscous Burgers' Equation

We consider a stationary travelling wave $u = -\tanh(x/2)$ which satisfies $uu_x = u_{xx}$ and $u(\pm\infty) = \mp 1$. It is possible to find the limiting distribution of the nodes but the process is somewhat complicated and is given in Appendix B.

The numerical stationary solution satisfies the equations

$$(a_i - a_{i-1})(a_{i-1} + 2a_i) = 3(1 + pr_i)\delta m_i$$

$$(a_i - a_i)(2a_i + a_{i+1}) = 3(1 - pr_i)\delta m_i$$

derived from (3.1). As in Section 8 we may add and subtract these and starting from values for a_i obtain a set of equations which may be used in a nonlinear solution to find a solution for a_i . Table 7 shows the errors in the nodal values.

Elements	4	8	16	32	64
p					
-0.5	$2.50_{10} - 2$	$1.39_{10} - 2$	$6.51_{10} - 3$	$2.51_{10} - 3$	$8.25_{10} - 4$
0.0	$1.66_{10} - 2$	$6.00_{10} - 3$	$1.89_{10} - 3$	$5.42_{10} - 4$	$1.46_{10} - 4$
0.5	$1.33_{10} - 2$	$4.05_{10} - 3$	$1.14_{10} - 3$	$3.06_{10} - 4$	$7.92_{10} - 5$
1.0	$1.27_{10} - 2$	$3.52_{10} - 3$	$9.20_{10} - 4$	$2.35_{10} - 4$	$5.92_{10} - 5$
1.5	$1.26_{10} - 2$	$3.49_{10} - 3$	$9.20_{10} - 4$	$2.36_{10} - 4$	$5.97_{10} - 5$
2.0	$1.25_{10} - 2$	$3.48_{10} - 3$	$9.21_{10} - 4$	$2.37_{10} - 4$	$6.01_{10} - 5$

Table 1a: Convection-diffusion diffusion coefficient = 1.0

Appendix A

We use (3.2) to obtain

$$cm_{\frac{1}{2}}\Delta s_{\frac{1}{2}} = \frac{3}{2}(1 + pr_1)\delta m_1$$

$$cm_{n-\frac{1}{2}}\Delta s_{n-\frac{1}{2}} = \frac{3}{2}(1 - pr_{n-1})\delta m_{n-1}$$

$$cm_{i-\frac{1}{2}}\Delta s_{i-\frac{1}{2}} = (1 - pr_{i-1})\delta m_{i-1} = (1 + pr_i)\delta m_i, \quad i = 2, \dots, n-1$$

As already mentioned $\Delta s_{i-\frac{1}{2}} = (1 + p)/(c(\gamma + i - 1))$, $i = 2, \dots, n-1$.

For $i = 1$ we have

$$c\Delta s_{\frac{1}{2}}\Delta s_{\frac{3}{2}} + (1 + p)(\Delta s_{\frac{3}{2}} - \Delta s_{\frac{1}{2}}) + \frac{1}{2}(1 + pr_i)\Delta s_{\frac{3}{2}} = 0.$$

Elements	4	8	16	32	64
p					
-0.5	$2.01_{10} - 1$	$1.82_{10} - 1$	$1.55_{10} - 1$	$1.24_{10} - 1$	$9.55_{10} - 2$
0.0	$1.36_{10} - 1$	$7.63_{10} - 2$	$4.12_{10} - 2$	$2.12_{10} - 2$	$1.07_{10} - 2$
0.5	$5.72_{10} - 2$	$2.13_{10} - 2$	$8.07_{10} - 3$	$3.00_{10} - 3$	$1.09_{10} - 3$
1.0	$4.24_{10} - 2$	$1.31_{10} - 2$	$3.57_{10} - 3$	$9.32_{10} - 4$	$2.37_{10} - 4$
1.5	$3.57_{10} - 2$	$1.45_{10} - 2$	$4.44_{10} - 3$	$1.23_{10} - 3$	$3.22_{10} - 4$
2.0	$3.26_{10} - 2$	$1.66_{10} - 2$	$5.48_{10} - 3$	$1.56_{10} - 3$	$4.17_{10} - 4$

Table 1b: Convection-diffusion diffusion coefficient = 0.1

Elements	4	8	16	32	64
p					
-0.5	$2.56_{10} - 1$	$2.41_{10} - 1$	$2.09_{10} - 1$	$1.72_{10} - 1$	*
0.0	$1.38_{10} - 1$	$9.54_{10} - 2$	$5.23_{10} - 2$	$2.76_{10} - 2$	$1.44_{10} - 2$
0.5	$3.68_{10} - 2$	$1.78_{10} - 2$	$6.57_{10} - 3$	$2.43_{10} - 3$	$8.93_{10} - 4$
1.0	$2.44_{10} - 2$	$1.19_{10} - 2$	$3.45_{10} - 3$	$9.20_{10} - 4$	$2.37_{10} - 4$
1.5	$2.31_{10} - 2$	$1.42_{10} - 2$	$4.61_{10} - 3$	$1.28_{10} - 3$	$3.36_{10} - 4$
2.0	$2.62_{10} - 2$	$1.78_{10} - 2$	$6.02_{10} - 3$	$1.71_{10} - 3$	$4.53_{10} - 4$

Table 1c: Convection-diffusion diffusion coefficient = 0.01

Elements	4	8	16	32	64
p					
-0.5	$2.01_{10} - 1$	$1.82_{10} - 1$	$1.55_{10} - 1$	$1.24_{10} - 1$	$9.55_{10} - 2$
0.0	$1.64_{10} - 1$	$9.69_{10} - 2$	$5.31_{10} - 2$	$2.80_{10} - 2$	$1.46_{10} - 2$
0.5	$5.54_{10} - 2$	$1.74_{10} - 2$	$6.36_{10} - 3$	$2.35_{10} - 3$	$8.64_{10} - 4$
1.0	$3.01_{10} - 2$	$1.18_{10} - 2$	$3.44_{10} - 3$	$9.18_{10} - 4$	$2.37_{10} - 4$
1.5	$3.89_{10} - 2$	$1.42_{10} - 2$	$4.61_{10} - 3$	$1.28_{10} - 3$	$3.36_{10} - 4$
2.0	$3.78_{10} - 2$	$1.78_{10} - 2$	$6.03_{10} - 3$	$1.71_{10} - 3$	$4.54_{10} - 4$

Table 1d: Convection-diffusion diffusion coefficient = 0.001

If we let $\Delta s_{\frac{1}{2}} = (1 + p)/(cx)$ for some x this becomes

$$(1 + p)(1 + x - \gamma) + \frac{1}{2}x(1 + p(x - \delta)/(x + \gamma)) = 0$$

which allow x to be found from γ . A similar determination of $\Delta s_{n-\frac{1}{2}}$ from γ allows $s_n - s_0$ to be calculated from γ , and γ may be found so that $s_n - s_0 = 1$. Once γ is found, recalling that $m_{i-\frac{1}{2}}\Delta s_{i-\frac{1}{2}} = a_i - a_{i-1}$ we note that

$$\frac{a_{i+1} - a_i}{a_i - a_{i-1}} = \frac{1 - pr_i}{1 + pr_i}$$

determines a_i up to an arbitrary multiplier which is found from $a_n - a_0 = 1$.

Parameters	5	10	20	40	80
100.0	$8.89_{10} - 04$	$1.12_{10} - 04$	$1.40_{10} - 05$	$1.74_{10} - 06$	$2.25_{10} - 07$
50.0	$8.75_{10} - 04$	$1.09_{10} - 04$	$1.37_{10} - 05$	$1.71_{10} - 06$	$1.92_{10} - 07$
20.0	$8.38_{10} - 04$	$1.03_{10} - 04$	$1.24_{10} - 05$	$1.43_{10} - 06$	$1.54_{10} - 07$
10.0	$7.83_{10} - 04$	$9.40_{10} - 05$	$1.09_{10} - 05$	$1.16_{10} - 06$	$1.14_{10} - 07$
5.0	$6.96_{10} - 04$	$8.00_{10} - 05$	$8.53_{10} - 06$	$8.24_{10} - 07$	$2.46_{10} - 07$
2.0	$5.32_{10} - 04$	$5.47_{10} - 05$	$5.02_{10} - 06$	$-1.57_{10} - 06$	$-4.90_{10} - 07$
1.5	$4.77_{10} - 04$	$4.63_{10} - 05$	$-5.41_{10} - 06$	$-2.11_{10} - 06$	$-6.13_{10} - 07$
1.0	$4.03_{10} - 04$	$3.72_{10} - 05$	$-8.80_{10} - 06$	$-2.92_{10} - 06$	$-7.98_{10} - 07$
0.5	$2.97_{10} - 04$	$-4.12_{10} - 05$	$-1.50_{10} - 05$	$-4.23_{10} - 06$	$-1.09_{10} - 06$
0.0	$-2.63_{10} - 04$	$-1.08_{10} - 04$	$-3.05_{10} - 05$	$-8.05_{10} - 06$	$-2.06_{10} - 06$
-0.5	$-1.25_{10} - 03$	$-4.71_{10} - 04$	$-1.42_{10} - 04$	$-3.91_{10} - 05$	$-1.07_{10} - 05$
-0.9	$-3.71_{10} - 03$	$-3.73_{10} - 03$	$-3.04_{10} - 03$	$-2.26_{10} - 03$	$-1.60_{10} - 03$
0.0					

Table 2a: Heat equation on a finite interval. Maximum error in the solution

Elements parameters	5	10	20	40	80
100.0	$1.15_{10} - 02$	$2.45_{10} - 03$	$5.61_{10} - 04$	$1.34_{10} - 04$	$3.25_{10} - 05$
50.0	$1.14_{10} - 02$	$2.43_{10} - 03$	$5.57_{10} - 04$	$1.32_{10} - 04$	$3.25_{10} - 05$
20.0	$1.12_{10} - 02$	$2.38_{10} - 03$	$5.48_{10} - 04$	$1.31_{10} - 04$	$3.19_{10} - 05$
10.0	$1.08_{10} - 02$	$2.32_{10} - 03$	$5.32_{10} - 04$	$1.27_{10} - 04$	$3.09_{10} - 05$
5.0	$1.03_{10} - 02$	$2.21_{10} - 03$	$5.09_{10} - 04$	$1.21_{10} - 04$	$2.90_{10} - 05$
2.0	$9.38_{10} - 03$	$2.02_{10} - 03$	$4.67_{10} - 04$	$1.12_{10} - 04$	$2.73_{10} - 05$
1.5	$9.07_{10} - 03$	$1.96_{10} - 03$	$4.54_{10} - 04$	$1.09_{10} - 04$	$2.66_{10} - 05$
1.0	$8.67_{10} - 03$	$1.88_{10} - 03$	$4.36_{10} - 04$	$1.05_{10} - 04$	$2.55_{10} - 05$
0.5	$8.14_{10} - 03$	$1.78_{10} - 03$	$4.13_{10} - 04$	$9.86_{10} - 05$	$2.38_{10} - 05$
0.0	$7.54_{10} - 03$	$1.68_{10} - 03$	$3.96_{10} - 04$	$9.59_{10} - 05$	$2.35_{10} - 05$
-0.5	$8.48_{10} - 03$	$2.10_{10} - 03$	$5.31_{10} - 04$	$1.38_{10} - 04$	$3.54_{10} - 05$
-0.9	$5.33_{10} - 02$	$2.78_{10} - 02$	$1.52_{10} - 02$	$8.68_{10} - 03$	$5.09_{10} - 03$

Table 2b: Heat equation on a finite interval

Relative error in the rate of change of the solution

Elements parameters	5	10	20	40	80
10.0	$-1.30_{10} + 00$	$-4.08_{10} - 03$	$1.24_{10} - 03$	$-3.07_{10} - 02$	$-6.88_{10} - 04$
5.0	$-8.44_{10} - 03$	$-9.10_{10} - 03$	$-2.00_{10} + 00$	$-1.47_{10} - 03$	$-9.87_{10} - 03$
2.0	$3.30_{10} - 03$	$5.21_{10} - 04$	$-4.26_{10} - 03$	$-2.82_{10} - 03$	$-2.08_{10} - 03$
1.0	$7.02_{10} - 03$	$2.42_{10} - 03$	$7.40_{10} - 04$	$-1.23_{10} - 03$	$-1.49_{10} - 03$
0.5	$6.65_{10} - 03$	$3.18_{10} - 03$	$1.35_{10} - 03$	$5.56_{10} - 04$	$1.93_{10} - 04$
0.0	$-2.24_{10} - 03$	$-1.26_{10} - 03$	$-6.87_{10} - 04$	$-3.63_{10} - 04$	$1.19_{10} - 03$
-0.5	$-2.27_{10} - 02$	$-1.71_{10} - 02$	$-1.23_{10} - 02$	$-1.35_{10} - 02$	$-1.24_{10} - 02$
-0.9	$8.52_{10} - 02$	$8.15_{10} - 02$	$7.66_{10} - 02$	$7.13_{10} - 02$	$3.34_{10} - 02$

Table 3: Heat equation on the infinite interval. Maximum error in the solution

Elements	5	10	20	40	80
Timesteps					
1	-6.92	7.75	7.00	6.84	6.88
2	-6.66	9.40	8.35	8.22	8.35
4	-6.52	-8.27	9.93	9.58	9.64
8	-6.46	-7.87	-9.82	11.01	10.92
16	-6.43	-7.71	-9.25	-11.27	-12.22
32	-6.42	-7.63	-9.04	-10.65	-12.73
64	-6.41	-7.60	-8.94	-10.42	-12.07

Table 4a: Heat equation on the infinite interval

Maximum error in the solution at time 0.1

Parameter is 0.5

Elements	5	10	20	40	80
Timesteps					
1	-6.01	-7.45	-8.72	7.72	7.35
2	-6.06	-7.13	-8.86	-9.41	8.97
4	-6.08	-6.99	-8.16	-9.83	-10.22
8	-6.10	-6.93	-7.93	-9.10	-10.61
16	-6.11	-6.90	-7.83	-8.86	-10.01
32	-6.12	-6.89	-7.78	-8.76	-9.80
64	-6.12	-6.88	-7.76	-8.71	-9.70

Table 4b: Heat equation on the infinite interval

Maximum error in the solution at time 0.1

Parameter is 0.0

Elements	5	10	20	40	80
Timesteps					
1	-5.98	-6.29	-6.83	-7.62	-8.05
2	-6.13	-6.39	-6.82	-7.38	-8.08
4	-6.23	-6.46	-6.85	-7.33	-7.88
8	-6.28	-6.50	-6.88	-7.33	-7.83
16	-6.31	-6.53	-6.90	-7.34	-7.82
32	-6.33	-6.55	-6.91	-7.34	-7.82
64	-6.34	-6.55	-6.91	-7.35	-7.82

Table 4c: Heat equation on the infinite interval

Maximum error in the solution at time 0.1

Parameter is -0.5

Elements	5	10	20	40	80
Timesteps					
1	7.00	8.36	-9.13	-8.35	-7.93
2	6.61	7.65	9.78	-9.31	-8.19
4	6.26	7.13	8.67	-10.00	-8.23
8	6.00	6.74	8.05	-10.72	-8.11
16	5.82	6.48	7.65	-11.58	-8.01
32	5.72	6.33	7.41	12.20	-8.04
64	5.67	6.25	7.28	12.02	-8.18

Table 4d: Heat equation on the infinite interval

Maximum error in the solution at time 0.1

Parameter is -0.9

Appendix B

Consider the equation

$$g(u)u_x = u_{xx}$$

for which (3.2) becomes

$$x_\xi^2 g'(u) m^2 = 3(m_{\xi\xi} + px_{\xi\xi} m_\xi / x_\xi)$$

which on substitution into (3.4) yields

$$-x_\xi^2 g(u) = (1+p)x_{\xi\xi} + \frac{2}{3}x_\xi^3 g'(u) m^2 / m_\xi$$

which with a change of independent variable yields

$$(1+p) \frac{\xi_{xx}}{\xi_x} = \frac{2g'(u)}{3g(u)} u_x + \frac{m_x}{m}$$

where we have used $g(u)m = m_x$ in both terms on the right hand side. This equation integrates to give

$$\xi_x^{1+p} \propto (g(u))^{\frac{2}{3}} u_x$$

and in our particular case

$$\xi_x^{1+p} \propto |u|^{\frac{2}{3}} |u_x|$$

and, using the solution $u = \tanh(\frac{x}{2})$, $u_x = \frac{1}{2}(1 - u^2)$, we may set $v = u^2$ and obtain, writing $3(1+p) = q^{-1}$,

$$\xi_x \propto v^q (1-v)^{3q}$$

or, changing the independent variable to v ,

$$\xi_v \propto v^{q-\frac{1}{2}}(1-v)^{3q-1},$$

where the constant of proportionality to give $\xi = 1$ for $v = 1$ is a beta function. The actual computation of ξ as a function of v may be carried out numerically although because of the singularities at $v = 0$ and $v = 1$ the substitution $\xi = (1-v)^{3q}$ is recommended.

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