

Particle-Hole Duality in the Lowest Landau Level

Dung Xuan Nguyen,¹ Tankut Can,² and Andrey Gromov³

¹*Department of Physics, University of Chicago, Chicago, Illinois 60637, USA*

²*Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY 11794, USA*

³*Kadanoff Center for Theoretical Physics and Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637*

(Dated: March 29, 2017)

We derive a number of exact relations between response functions of holomorphic, chiral fractional quantum Hall states and their particle-hole (PH) conjugates. These exact relations allow one to calculate the Hall conductivity, Hall viscosity, various Berry phases, and the static structure factor of PH-conjugate states from the corresponding properties of the original states. These relations establish a precise duality between chiral quantum Hall states and their PH-conjugates. The key ingredient in the proof of the relations is a generalization of Girvin's construction of PH-conjugate states to inhomogeneous magnetic field and curvature. Finally, we make several non-trivial checks of the relations, including for the Jain states and their PH-conjugates.

Introduction. Particle-hole (PH) transformation for fractional quantum Hall (FQH) states was introduced by Girvin [1]. This transformation relates a FQH state at filling fraction ν to a FQH state at filling fraction $1-\nu$. In the absence of Landau level mixing the projected lowest Landau level (LLL) Hamiltonian is PH-symmetric and, therefore, two states related by a PH transformation have the same energy (up to a shift in the chemical potential). Despite the physical clarity of PH-symmetry, the PH-transformed wave functions look quite complicated and are difficult to work with. PH-transformed states contain a different number of particles, have different transport properties and different topological order. In this Letter we will explain that all of the information about PH-transformed state is encoded in the original state, so that both states are a different representation for essentially the same physics. For this reason we feel it is more appropriate to refer to the PH-transformation as a *particle-hole duality* (PHD).

Recent years have also brought the rise of interest in the role of PHD in the problem of the half-filled Landau level. To resolve the issue of the apparent absence of the PH-invariance in the Halperin-Lee-Read [2] theory, Son has proposed a manifestly PH-invariant effective theory of composite fermions with π Berry phase around the composite Fermi surface [3]. This theory can successfully be used to describe Jain states at fillings close to $\nu = 1/2$ and a PH-invariant (or self-dual) version of the Pfaffian state [3, 4], which is a viable candidate for the observed $\nu = 5/2$ plateau [5].

PH-transformation, as defined by Girvin [1], works in flat space and homogeneous magnetic field. It was recently appreciated that placing a FQH state in inhomogeneous background magnetic field and curved geometry allows one to extract considerable information about the flat space properties of the state [6–25]. For example, the projected static structure factor (SSF) [26] in leading and sub-leading order in momentum, and long-wave

corrections to Hall conductivity and Hall viscosity can be calculated from the properties that become apparent in curved space [11, 19, 27–29]. Integer quantum Hall states in curved geometry are available in (synthetic) photonic systems [30].

In this Letter, we will use the approach of [9, 19] to extend Girvin's construction to inhomogeneous magnetic field and curved geometry. Next, we will derive several exact relations between Hall conductivity, Hall viscosity, Berry phases, and the SSF of the holomorphic, chiral FQH states and their PH-duals. These relations establish the PHD quantitatively and show that properties of the PH-dual state are completely determined by the original state. The duality is non-trivial since the calculations can be easily done before the PH-transformation, but are difficult to do after.

Under certain assumptions, the long-wave corrections to Hall conductivity, Hall viscosity and the SSF are determined by topological quantum numbers [11, 19, 20, 27, 28]: filling fraction ν , shift $\mathcal{S} = 2\bar{s}$ [6], chiral central charge c_- [31], and the orbital spin variance $\text{var}(s)$ [14, 16]. We will explain how the topological quantum numbers transform under the PHD and prove that the aforementioned long-wave corrections are still determined by the (transformed) topological quantum numbers, albeit via different relations. We will check the derived relations against the explicit computation of the corresponding quantities for Jain states done in Son's theory of composite fermions and find complete agreement.

FQH states in inhomogeneous background. We start with a brief review of the construction [9, 19, 20] of a LLL FQH state in inhomogeneous magnetic field and on a surface with nonuniform spatial curvature. Consider a holomorphic FQH state $\Psi_\nu(\{\xi\})$, where $\{\xi\} = \xi_1, \dots, \xi_N$ denotes the collection of particle coordinates. The complex coordinate $\xi = x + iy$ will be used to label the particle position in the plane. We will assume that the magnetic

field B is inhomogeneous and the background geometry is curved. Then the unnormalized wavefunction $\Psi_\nu(\{\xi\})$ takes the following form [9, 19]

$$\Psi_\nu(\{\xi\}) = f_\nu(\{\xi\}) e^{\frac{1}{2} \sum_{i=1}^N \mathcal{Q}(\xi_i, \bar{\xi}_i)}, \quad (1)$$

where $f_\nu(\{\xi\})$ is holomorphic in complex coordinates ξ_i , and \mathcal{Q} is the magnetic potential [32] defined by

$$\Delta_g \mathcal{Q} = -2B, \quad (2)$$

where Δ_g is the Laplace operator for the metric g_{ij} . Throughout the Letter we will fix the coordinates so that $g_{ij} = \sqrt{g} \delta_{ij}$. In these coordinates (also known as the “conformal gauge”) the Laplacian is given by $\Delta_g = \frac{4}{\sqrt{g}} \partial_z \partial_{\bar{z}}$. When the magnetic field is homogeneous, but the space is curved, the magnetic potential is given by

$$\mathcal{Q} = -\frac{\mathcal{K}}{2\ell^2}, \quad (3)$$

where \mathcal{K} is the Kähler potential satisfying $\partial_z \partial_{\bar{z}} \mathcal{K} = \sqrt{g}$, and $\ell = B^{-1/2}$ is the magnetic length. It is of crucial importance that $f_\nu(\{\xi\})$ *does not depend on* \mathcal{Q} or the metric \sqrt{g} . This will not be the case for PH-dual states. Such states comprise a proper subset of chiral FQH states, i.e. FQH states with fully chiral effective edge theories [33].

The central object of interest is the logarithm of the normalization factor

$$\mathcal{Z}_\nu[W] = \int [d\xi] |f_\nu(\{\xi\})|^2 e^{\sum_i W(\xi_i, \bar{\xi}_i)}, \quad (4)$$

where $[d\xi] = d^2\xi_1 \cdots d^2\xi_N$ with $d^2\xi = dx dy$, and $W = \mathcal{Q} + \log \sqrt{g}$. We assume that for constant magnetic field and flat space, when $W = -|z|^2/2\ell^2$, the state is normalized and $\mathcal{Z}_\nu = 1$. It is not hard to see that $\log \mathcal{Z}_\nu[W]$ is the generating functional of the density correlation functions [19]

$$\langle \rho_\nu(\zeta) \rangle \equiv \langle \Psi_\nu | \rho_\nu(\zeta) | \Psi_\nu \rangle = \frac{1}{\sqrt{g}} \frac{\delta \log \mathcal{Z}_\nu[W]}{\delta W(\zeta)}, \quad (5)$$

where $\rho_\nu(\zeta) = \frac{1}{\sqrt{g}} \sum_{i=1}^{N_\nu} \delta(\zeta - \xi_i)$ is the density operator, and N_ν is the number of particles in the state Ψ_ν . In writing $\langle \rho_\nu \rangle$ we will always implicitly assume that the expectation value is taken in the state with the filling factor ν .

The second variation produces the connected two-point function [19]

$$\langle \rho_\nu(\zeta) \rho_\nu(\zeta') \rangle_c = \frac{1}{\sqrt{g(\zeta')}} \frac{\delta}{\delta W(\zeta')} \langle \rho_\nu(\zeta) \rangle, \quad (6)$$

where $\langle \rho_\nu(\zeta) \rho_\nu(\zeta') \rangle_c = \langle \rho_\nu(\zeta) \rho_\nu(\zeta') \rangle - \langle \rho_\nu(\zeta) \rangle \langle \rho_\nu(\zeta') \rangle$. The static structure factor (SSF) is defined as the Fourier transform (q is the dimensionless momentum)

$$s_\nu(q) = \frac{1}{\rho_\nu} \langle \rho_\nu(q) \rho_\nu(-q) \rangle_c, \quad (7)$$

where $\bar{\rho}_\nu = \nu/(2\pi\ell^2)$ is the mean electron density in the homogeneous limit (in the bulk of the FQH droplet).

It follows from (2) that in flat space, derivatives w.r.t. W and B can be traded with each other. Going to momentum space we recover [19]

$$s_\nu(q) = \frac{q^2}{2} \frac{2\pi}{\nu} \frac{\delta \langle \rho_\nu(q) \rangle}{\delta B(-q)} = \frac{q^2}{2} \frac{\sigma_\nu^H(q)}{\sigma_\nu^H(0)}, \quad (8)$$

where we used the Středa formula $\delta \langle \rho_\nu \rangle / \delta B = \sigma_\nu^H$ [34], and the DC Hall conductance $\sigma_\nu^H(0) = \nu/2\pi$.

We will also need to know how the electron density depends on the spatial curvature. This dependence is captured by the function $\eta_\nu(q) = \frac{2\pi}{\nu} \delta \langle \rho_\nu \rangle / \delta R$ [19]. In general, $\eta_\nu(q)$ has the following momentum expansion

$$\eta_\nu(q) = \frac{\mathcal{S}}{4} - \frac{b}{4\nu} q^2 + O(q^4), \quad (9)$$

where the constant b is an *a priori* non-universal parameter. However in the LLL it is determined by the topological quantum numbers [11]

$$b = \nu \bar{s} (1 - \bar{s}) + \frac{\tilde{c}}{12}, \quad (10)$$

where $\tilde{c} = c_- - 12\nu \text{var}(s)$. It is also known to control the Berry curvature on the moduli space of higher genus surfaces [18]. Note that the kinematic Hall viscosity [35] follows from the zero momentum limit of the curvature response $\eta_\nu^H / \bar{\rho}_\nu = \eta_\nu(0)$ [20]. At the same time using the expression for the scalar curvature $R = -\Delta_g \log \sqrt{g}$, and the general relation, valid for any metric-independent operator \mathcal{O} [19]

$$-\frac{\ell^2}{2} \Delta_g \frac{\delta \langle \mathcal{O} \rangle}{\delta \sqrt{g}} = \left(1 - \frac{\ell^2}{2} \Delta_g\right) \langle \mathcal{O} \rho_\nu \rangle_c \quad (11)$$

it is possible to show that [19]

$$s_\nu(q) = \frac{q^2/2}{1 + q^2/2} (q^2 \eta_\nu(q) + 1). \quad (12)$$

These relations imply

$$s_\nu(q) = \frac{1}{2} q^2 + \frac{\mathcal{S} - 2}{8} q^4 - \left[\frac{b}{8\nu} + \frac{(\mathcal{S} - 2)}{16} \right] q^6 + \dots \quad (13)$$

This relation, together with Eq.(10), was derived for the Laughlin states in [19], and conjectured to hold for states of the form (1) with $\text{var}(s) = 0$ in [20]. We conjecture that it holds generally for chiral FQH states.

Finally, combining (8) and (12) we establish an exact relation between Hall conductivity and $\eta_\nu(q)$

$$\frac{\sigma_\nu^H(q)}{\sigma_\nu^H(0)} = \frac{1}{1 + q^2/2} (q^2 \eta_\nu(q) + 1). \quad (14)$$

Relations (8), (12) and (14) hold to all orders in q , under the assumption of the absence of Landau level mixing

and long-range interactions. Together with (10) these relations imply that first 3 terms in the momentum expansion of $\sigma_\nu^H(q)$ and $s_\nu(q)$ are completely determined by the topological quantum numbers.

Particle-hole transformation in inhomogeneous background. Following Girvin [1], we define $\Psi_{1-\nu}(\{z\})$ as a state of *holes* at filling ν , which, when viewed as a state of electrons, has filling $1 - \nu$. Let z_1, \dots, z_M be the coordinates of electrons and ξ_1, \dots, ξ_N be the coordinates of holes. Then the PH dual state is defined as

$$\Psi_{1-\nu}(\{z\}) = \sqrt{\frac{(N+M)!}{N!M!}} \int [d\xi] \Psi_1(\{z\}, \{\xi\}) \Psi_\nu^*(\{\xi\}), \quad (15)$$

Property (16) is ensured by the following identity. First, we define an n -particle reduced density matrix [36]

$$\mathcal{P}_\eta^{(n)}(\xi_1, \dots, \xi_n; \xi'_1, \dots, \xi'_n) = \frac{N_\eta!}{n!(N_\eta - n)!} \frac{1}{\mathcal{Z}_\eta[W]} \int [d\hat{\xi}] \Psi_\eta(\hat{\xi}_{n+1}, \dots, \hat{\xi}_{N_\eta}, \xi_1, \dots, \xi_n) \Psi_\eta^*(\hat{\xi}_{n+1}, \dots, \hat{\xi}_{N_\eta}, \xi'_1, \dots, \xi'_n). \quad (17)$$

For $\eta = 1$, this density matrix is a projector to the LLL satisfying

$$\Psi_\nu(\{\xi\}) = \int [d\xi'] \mathcal{P}_1^{(N_\nu)}(\{\xi\}, \{\xi'\}) \Psi_\nu(\{\xi'\}). \quad (18)$$

Eq. (16) follows by applying PH conjugation to $\Psi_{1-\nu}$ and expressing the resulting convolution integral for $\Psi_{1-(1-\nu)}$ in terms of (18). We make extensive use of the following formula relating the 2-particle reduced density matrices between PH-dual states [37]

$$\begin{aligned} \mathcal{P}_{1-\nu}^{(2)}(\xi_1, \xi_2; \xi_1, \xi_2) &= \mathcal{P}_1^{(2)}(\xi_1, \xi_2; \xi_1, \xi_2) + \mathcal{P}_\nu^{(2)}(\xi_1, \xi_2; \xi_1, \xi_2) + \frac{1}{2} \mathcal{P}_1^{(1)}(\xi_1; \xi_2) \mathcal{P}_\nu^{(1)}(\xi_2; \xi_1) + \frac{1}{2} \mathcal{P}_1^{(1)}(\xi_2; \xi_1) \mathcal{P}_\nu^{(1)}(\xi_1; \xi_2) \\ &\quad - \frac{1}{2} \langle \rho_1(\xi_1) \rangle \langle \rho_\nu(\xi_2) \rangle - \frac{1}{2} \langle \rho_1(\xi_2) \rangle \langle \rho_\nu(\xi_1) \rangle. \end{aligned} \quad (19)$$

Integrating over position ξ_2 reduces this to a simple formula relating the electron density (in inhomogeneous background)

$$\langle \rho_\nu \rangle + \langle \rho_{1-\nu} \rangle = \langle \rho_1 \rangle. \quad (20)$$

Equations (19) and (20) reveal the PHD, and are the central results of the present Letter. Next we will discuss the physical consequences of the duality.

Particle-hole duality. The Hall conductivity and curvature response in the PH-dual state can be found using (20). Taking a derivative w.r.t. the magnetic field $B(q)$, and applying the Středa formula we obtain an exact relation between the Hall conductivities

$$\sigma_\nu^H(q) + \sigma_{1-\nu}^H(q) = \sigma_1^H(q). \quad (21)$$

Similarly we find

$$\nu \eta_\nu(q) + (1 - \nu) \eta_{1-\nu}(q) = \eta_1(q). \quad (22)$$

These are exact relations connecting the linear response functions of PH conjugate pair states, and hold

where $\Psi_\nu(\{\xi\})$ is given by (1) and Ψ_1 is the $\nu = 1$ state. The overall factor is required to ensure that the PH-dual state is normalized to 1 in constant magnetic field and flat space. A defining property of the PH transformation is that it is an involution

$$\Psi_{1-(1-\nu)}(\{\xi\}) = \pm \Psi_\nu(\{\xi\}). \quad (16)$$

for *all* LLL wave functions connected by (15). They are new results of the present Letter.

Next, we turn to the normalization factor. It follows directly from the definition of the reduced density matrix and the reproducing formula (18), as well as the definition of the generating functional (4) that

$$\frac{\mathcal{Z}_{1-\nu}}{\mathcal{Z}_\nu} = \mathcal{Z}_1 \quad \Rightarrow \quad \log \mathcal{Z}_{1-\nu} - \log \mathcal{Z}_\nu = \log \mathcal{Z}_1, \quad (23)$$

where we have dropped the argument of \mathcal{Z}_ν for brevity. Eq.(23) is an *exact* relation between the generating functionals for a pair of PH-dual states. Eq.(23) clearly illustrates the duality, and is a new result of the present Letter.

Assuming now that Ψ_ν has the form (1), variation of $\log \mathcal{Z}_{1-\nu}$ over $W(\zeta, \bar{\zeta})$ is given by

$$\frac{\delta}{\delta W(\zeta)} \log \mathcal{Z}_{1-\nu} = \langle \rho_1 \rangle + \langle \rho_\nu \rangle = \langle \rho_{1-\nu} \rangle + 2 \langle \rho_\nu \rangle. \quad (24)$$

We emphasize that since (24) does not have the same form as (5), the wavefunction $\Psi_{1-\nu}$ does not have the

form (1). More precisely, we see that $f_{1-\nu}$ has to depend on W . In other words, the dual states couple differently to inhomogeneous magnetic field. It appears that the condition for f_ν to be independent of W has to do with the chirality of a state [38]. For instance, all conformal block trial states share this property. This complication indicates that identity (23) is not sufficient to extract all of the observables in a PH-dual state in terms of the observables in the original state, because the relationship between the observables and variations of $\log \mathcal{Z}_{1-\nu}$ is more complicated for $\Psi_{1-\nu}$ states.

Now we will derive an analogue of (8) for the dual states. We utilize the fact that, by definition, the two-point density correlation function is related to the 2-particle density matrix via

$$\langle \rho_\nu(\zeta) \rho_\nu(\zeta') \rangle = \langle \rho_\nu(\zeta) \rangle \delta(\zeta - \zeta') + 2\mathcal{P}_\nu^{(2)}(\zeta, \zeta'; \zeta, \zeta').$$

This allows us to gain insight using the exact formula (19). Then the one-particle density matrix for the ν state is known in the translation-invariant limit to be [36]

$$\mathcal{P}_\nu^{(1)}(\zeta; \zeta') = \bar{\rho}_\nu e^{\zeta \bar{\zeta}' / 2\ell^2 - |\zeta'|^2 / 2\ell^2}. \quad (25)$$

Using this in (19), we find

$$\langle \rho_{1-\nu}(\zeta) \rho_{1-\nu}(\zeta') \rangle_c = \langle \rho_\nu(\zeta) \rho_\nu(\zeta') \rangle_c + \frac{\bar{\rho}_1 - 2\bar{\rho}_\nu}{\bar{\rho}_1} \langle \rho_1(\zeta) \rho_1(\zeta') \rangle_c. \quad (26)$$

Taking the Fourier transform, we find a beautiful exact relation between the projected static structure factors $\bar{s}_\nu = s_\nu - s_1$ for a pair of PH-dual states [39]

$$\bar{\rho}_\nu \bar{s}_\nu(q) = \bar{\rho}_{1-\nu} \bar{s}_{1-\nu}(q). \quad (27)$$

This relation is another novel result of the Letter. Eq. (27) also follows from the relation between the pair-correlation functions [40] for the dual states [41]. We stress here that (27) does not require any assumptions such as (1) on the wave function other than holomorphicity.

Now we are in position to relate the Hall conductivity to the SSF of the PH-dual state

$$\sigma_{1-\nu}^H(q) = \sigma_{1-\nu}^H(0) \frac{2}{q^2} (s_1(q) - \bar{s}_{1-\nu}(q)). \quad (28)$$

The simplest way to obtain (28) is to use (21) and (27), and assume (8).

Next we will derive an analogue of (12) for PH-dual states. Using (21), (22) and (14) we find

$$\bar{s}_{1-\nu}(q) = s_1(q) - \frac{q^2/2}{1 + q^2/2} (q^2 \eta_{1-\nu}(q) + 1). \quad (29)$$

Excluding $s_{1-\nu}(q)$ from (28)-(29) we come to a surprising conclusion – the relation between $\sigma_{1-\nu}^H(q)$ and $\eta_{1-\nu}(q)$ is precisely the same as before the PH-transformation (14), up to replacing ν by $1 - \nu$. In other words, Eq. (14) is invariant under PH transformation.

Berry curvature. Next we turn to the dependence of the PH-dual states on parameters such as adiabatically varying fluxes of magnetic field or the modular parameter of a torus τ . Denote any of these parameters in complex coordinates as x and \bar{x} . Berry curvature can be computed under the assumption that the state Ψ_ν is holomorphic in the coordinates on the parameter space, except for the real-analytic normalization factor. The *normalized* states have the form [16, 18, 42]

$$\psi_\nu(\{\xi, \bar{\xi}\}; x, \bar{x}) = \frac{1}{\sqrt{\mathcal{Z}_\nu[x, \bar{x}]}} \Psi_\nu(\{\xi, \bar{\xi}\}; x). \quad (30)$$

Then the holomorphic component of the Berry connection is determined entirely by the normalization factor

$$A_x \equiv i \langle \psi_\nu | \partial_x | \psi_\nu \rangle = \frac{i}{2} \partial_x \log \mathcal{Z}_\nu, \quad (31)$$

which follows by using the identity $\partial_x \langle \psi_\nu | \psi_\nu \rangle = 0$ to trade derivatives of Ψ_ν for derivatives of \mathcal{Z}_ν . Thus, for such holomorphic states the Berry curvature is a Kähler form with the Kähler potential $\mathcal{U}_\nu = \log \mathcal{Z}_\nu$, and is given by

$$\Omega_\nu = \frac{i}{2} (\partial_x \partial_{\bar{x}} \mathcal{U}_\nu) dx \wedge d\bar{x}. \quad (32)$$

This structure is nearly preserved for the PH-dual state. A straightforward calculation shows that the Kähler potential is $\mathcal{U}_{1-\nu} = \log(\mathcal{Z}_1/\mathcal{Z}_\nu)$, which is *not* the logarithm of the normalization as before. Thus, in contrast to the formula (23), the Berry curvature obeys

$$\Omega_\nu + \Omega_{1-\nu} = \Omega_1. \quad (33)$$

PHD and Chern-Simons terms. The first few terms in the long wave expansion of $\sigma_\nu^H(q)$, $s_\nu(q)$ and $\eta_\nu(q)$ are determined by the topological quantum numbers, which appear as the coefficients of the Chern-Simons terms in the effective action [11, 19, 27].

The Chern-Simons part of the effective action is given by [14, 16]

$$\mathcal{W}_{\text{CS}}^\nu = \frac{\nu}{4\pi} \int (A + \bar{s}\omega) d(A + \bar{s}\omega) - \frac{\tilde{c}}{48\pi} \int \omega d\omega, \quad (34)$$

where $\tilde{c} = c_- - 12\nu \text{var}(s)$ and other coefficients are the topological quantum numbers discussed in the introduction. We have also introduced ω_μ – a spatial part of the spin connection satisfying $\partial_1 \omega_2 - \partial_2 \omega_1 = \frac{1}{2} \sqrt{g} R$. This effective action encodes the linear response functions. Notably, the Hall conductance, shift, and the Hall viscosity, averaged over the sample, are given by

$$\sigma_\nu^H = \frac{\nu}{2\pi}, \quad \mathcal{S} = \frac{\nu^{-1}N - N_\phi}{\chi/2}, \quad \eta_\nu^H = \frac{\bar{s}}{2\bar{\rho}_\nu} + \frac{\tilde{c}}{24} \frac{\chi}{A}, \quad (35)$$

where A and χ are respectively the area and the Euler characteristic of the sample, N is the number of electrons

and N_ϕ is the total magnetic flux in units of the flux quantum. When a FQH state is constructed as a single conformal block in a conformal field theory [16, 43] $\tilde{c} = c_-$. However, in general (and notably for Jain states) $\text{var}(s)$ does not vanish [14].

The action of PH-transformation on the Chern-Simons part of the effective action is

$$\mathcal{W}_{\text{CS}}^\nu + \mathcal{W}_{\text{CS}}^{1-\nu} = \mathcal{W}_{\text{CS}}^1. \quad (36)$$

This can be seen as a consequence of the formula for the Berry curvature (33) following the arguments of [16]. In addition to $\nu^{PH} = 1 - \nu$ it implies

$$\mathcal{S}^{PH} = \frac{1 - \nu\mathcal{S}}{1 - \nu}, \quad \text{var}(s)^{PH} = \frac{\nu}{\nu - 1} \left(\frac{(1 - \mathcal{S})^2}{4(1 - \nu)} + \text{var}(s) \right). \quad (37)$$

PHD also transforms the chiral central charge according to $c_-^{PH} = 1 - c_-$, and $\tilde{c}^{PH} = c_-^{PH} - 12\nu^{PH}\text{var}(s)^{PH}$. Curiously, if the initial state had $\text{var}(s) = 0$, then $\text{var}(s)^{PH} \neq 0$, unless $\mathcal{S} = 1$.

These relations are in agreement with the results derived above. For example, using the transformation laws (37), combined with (10) we can check that relations (21) and (22) hold at the order q^4 and q^2 correspondingly.

As another example we provide an explicit formula for first two terms in the long-wave expansion of the projected SSF of a PH-dual state

$$\bar{s}_{1-\nu}(q) = \frac{\nu(\mathcal{S} - 1)}{8(1 - \nu)}q^4 + \frac{(-6b + 5\nu - 3\nu\mathcal{S})}{48(1 - \nu)}q^6 + \dots \quad (38)$$

where b is given by (10) and all of the topological quantum numbers are known for a large variety of states [14]. Eq. (38) follows from using (13) in (27), and is another new result of the present Letter. For example, it can be readily used to determine the projected SSF for the $\nu = \frac{q-1}{q}$ states, related to the Laughlin states by the PHD. Note that all of the quantum numbers are taken from the state at filling ν . We are not aware of this type of general result in the literature.

Finally, we can ask about the values of the topological quantum numbers for a *self-dual* state which satisfies $\Omega_\nu = \Omega_{1-\nu}$. This state has been previously discussed by Son [3] and is known as the PH-Pfaffian. We find that it must have $\nu = 1/2$, $\mathcal{S} = 1$, $c_- = 1/2$ and $\text{var}(s) = 0$.

Jain states. We apply our relations to the PH-duals of $\nu = \frac{N}{2N+1}$ Jain states with the relevant topological quantum numbers given by

$$\mathcal{S} = N + 2, \quad c_- = N, \quad \nu \text{var}(s) = \frac{N(N^2 - 1)}{12}. \quad (39)$$

Then the projected SSF of the PH-dual state is given by

$$\bar{s}_{\frac{N+1}{2N+1}}(q) = \frac{N}{8}q^4 + \frac{N^4 + 2N^3 - 2N^2 - 2N}{48(N + 1)}q^6 + \dots \quad (40)$$

To the best of our knowledge (40) is a new result.

Hall conductivity of Jain's state at the filling factor $\nu = \frac{N}{2N+1}$ and its PH-dual state at the filling factor $(1 - \nu) = \frac{N+1}{2N+1}$ can be calculated exactly, in the large N limit using the Dirac composite fermion theory and the results of [44]. In the regime $z = q(2N + 1) \sim 1$ [45] the result is

$$\sigma_\nu^H(q) = \frac{((4N + 2)^2 - z^2) \left(8N + \frac{2zJ_2(z)}{J_1(z)} \right)}{64\pi(2N + 1)^3}, \quad (41)$$

$$\sigma_{1-\nu}^H(q) = \frac{((4N + 2)^2 - z^2) \left(8N + 8 - \frac{2zJ_2(z)}{J_1(z)} \right)}{64\pi(2N + 1)^3}. \quad (42)$$

where $J_\alpha(z)$ is the Bessel function. The correction to this is order $O(N^{-4})$. We have also assumed absence of long-range interactions.

The projected SSF can also be derived exactly using Dirac composite fermion theory [3]

$$\bar{s}_\nu(q) = \frac{z^3((4N + 2)^2 - z^2)J_2(z)}{32N(2N + 1)^4J_1(z)}, \quad (43)$$

$$\bar{s}_{1-\nu}(q) = \frac{z^3((4N + 2)^2 - z^2)J_2(z)}{32(N + 1)(2N + 1)^4J_1(z)}. \quad (44)$$

With these expressions at hand we can check that (21) holds up to order N^{-2} . It also follows from (41)-(44) that (8) and (28) hold in the large N limit at leading and sub-leading orders in N . We emphasize that these are quite non-trivial checks that probe the relations we derived in *all* orders in the momentum expansion.

Long-range interactions can be included in the computation of [45], which leads to the breakdown of relations (12) and (29) at $O(q^7)$ order and of (14) at $O(q^3)$ order.

Conclusion. We have presented arguments for the particle-hole duality in the lowest Landau level. This duality implies several exact, non-perturbative relations between the observables in the pair of PH-dual states such as static structure factor, Hall conductivity and response of the electron density to curvature. Our results for the coefficients of the small momentum expansion of these response functions likely do not hold for PH-dual states which are both non-chiral (as defined in [46], see also [8]) – a notable example of such a state is the PH-Pfaffian. However, Eq. (27) should be applicable to such states as well. We leave the investigation of general non-chiral states (and in particular the PH-Pfaffian) to future work.

Acknowledgements It is a pleasure to thank A. Abanov, S. Klevtsov, D.T. Son and P. Wiegmann for very enlightening discussions and correspondence.

D.X.N. is supported by the Chicago MRSEC, which is funded by NSF through grant DMR-1420709. A.G. was supported by the Leo Kadanoff Fellowship.

-
- [1] S. Girvin, Physical Review B **29**, 6012 (1984).
- [2] B. I. Halperin, P. A. Lee, and N. Read, Physical Review B **47**, 7312 (1993).
- [3] D. T. Son, Physical Review X **5**, 031027 (2015).
- [4] X. Chen, L. Fidkowski, and A. Vishwanath, Physical Review B **89**, 165132 (2014).
- [5] P. Zucker and D. Feldman, Physical Review Letters **117**, 096802.
- [6] X. Wen and A. Zee, Phys. Rev. Lett. **69**, 953 (1992).
- [7] D.-H. Lee and X.-G. Wen, Physical Review B **49**, 11066 (1994).
- [8] F. Haldane, arXiv preprint arXiv:0906.1854 (2009).
- [9] S. Klevtsov, Journal of High Energy Physics **2014**, 1 (2014).
- [10] A. G. Abanov and A. Gromov, Phys. Rev. B **90**, 014435 (2014).
- [11] A. Gromov and A. G. Abanov, Phys. Rev. Lett. **113**, 266802 (2014).
- [12] A. Gromov and A. G. Abanov, Phys. Rev. Lett. **114**, 016802 (2015).
- [13] A. Gromov, K. Jensen, and A. G. Abanov, Physical review letters **116**, 126802 (2016).
- [14] A. Gromov, G. Y. Cho, Y. You, A. G. Abanov, and E. Fradkin, Phys. Rev. Lett. **114**, 016805 (2015).
- [15] B. Bradlyn and N. Read, Phys. Rev. B **91**, 125303 (2015).
- [16] B. Bradlyn and N. Read, Phys. Rev. B **91**, 165306 (2015).
- [17] F. Ferrari and S. Klevtsov, Journal of High Energy Physics **86** **2014** (2014).
- [18] S. Klevtsov and P. Wiegmann, Phys. Rev. Lett. **115**, 086801 (2015).
- [19] T. Can, M. Laskin, and P. Wiegmann, Phys. Rev. Lett. **113**, 046803 (2014).
- [20] T. Can, M. Laskin, and P. B. Wiegmann, Annals of Physics **362**, 752 (2015).
- [21] M. Laskin, T. Can, and P. Wiegmann, Physical Review B **92**, 235141 (2015).
- [22] T. Can, Y. Chiu, M. Laskin, and P. Wiegmann, Phys. Rev. Lett. **117**, 266803 (2016).
- [23] S. Klevtsov, X. Ma, G. Marinescu, and P. Wiegmann, Communications in Mathematical Physics **349**, 819.
- [24] G. Y. Cho, Y. You, and E. Fradkin, Phys. Rev. B **90**, 115139 (2014).
- [25] T. L. Hughes, R. G. Leigh, and E. Fradkin, Physical Review Letters **107**, 075502 (2011).
- [26] S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Phys Rev B **33**, 2481 (1986).
- [27] B. Bradlyn, M. Goldstein, and N. Read, Physical Review B **86**, 245309 (2012).
- [28] C. Hoyos and D. T. Son, Phys. Rev. Lett. **108**, 066805 (2012).
- [29] D. T. Son, arXiv:1306.0638 (2013).
- [30] N. Schine, A. Ryou, A. Gromov, A. Sommer, and J. Simon, Nature **534**, 671 (2016).
- [31] A. Kitaev, Annals of Physics **321**, 2 (2006).
- [32] We have chosen to use the natural units $e = c = \hbar = 1$.
- [33] More precisely, we call a FQH wave function chiral if it describes a bulk FQH state which possesses a chiral effective edge theory.
- [34] P. Streda, Journal of Physics C: Solid State **15**, L717 (1982).
- [35] J. E. Avron, R. Seiler, and P. G. Zograf, Phys Rev Lett **75**, 697 (1995).
- [36] A. MacDonald and S. Girvin, Physical Review B **38**, 6295 (1988).
- [37] For a proof of this formula, and further discussion of the reduced density matrix, see the Supplemental Material.
- [38] We speculate that this property of f_ν is a sufficient condition for a state to be chiral, but it is not necessary (Jain states being a notable exception).
- [39] Levin and Son have obtained the same relation using a different set of arguments [47].
- [40] A. C. Balram, C. Tóke, and J. Jain, Physical review letters **115**, 186805 (2015).
- [41] We thank A.C. Balram for bringing [40] to our attention.
- [42] N. Read, Phys Rev B **79**, 045308 (2009).
- [43] G. Moore and N. Read, Nuclear Physics B **360**, 362 (1991).
- [44] D. X. Nguyen and A. Gromov, Physical Review B.
- [45] D. X. Nguyen and D. T. Son, In prepatarion.
- [46] D. X. Nguyen, D. T. Son, and C. Wu, arXiv preprint arXiv:1411.3316 (2014).
- [47] M. Levin and D. T. Son, arXiv preprint arXiv:1612.06402 (2016).

Supplementary Material

Lowest Landau level states

The orthonormal complete basis set of single particle wave functions in the LLL is

$$\phi_m(z) = \frac{z^m}{\sqrt{2\pi 2^m m!}} e^{-\frac{1}{4\ell^2}|z|^2}. \quad (45)$$

We will take $\ell = 1$ to simplify the formulas. We have the projected annihilation field operator

$$\psi_L(z) = \sum_{m=0}^{\infty} \hat{a}_m \phi_m(z) = \sum_{m=0}^{\infty} \hat{a}_m \frac{z^m}{\sqrt{2\pi 2^m m!}} e^{-\frac{1}{4}|z|^2}, \quad (46)$$

where \hat{a}_m and \hat{a}_m^\dagger annihilate and create normalized LLL states and obey the Fermi canonical relation

$$\{\hat{a}_n, \hat{a}_m^\dagger\} = \delta_{mn}, \quad \{\hat{a}_n^\dagger, \hat{a}_m^\dagger\} = \{\hat{a}_n, \hat{a}_m\} = 0. \quad (47)$$

Thus equal time anti-commutators of projected electron field operators are

$$\{\psi_L^\dagger(z_1), \psi_L(z_2)\} = \sum_{m=0}^{\infty} \phi_m^*(z_1) \phi_m(z_2) = \frac{1}{2\pi} e^{-\frac{1}{4}|z_1 - z_2|^2} e^{\frac{1}{4}(\bar{z}_1 z_2 - \bar{z}_2 z_1)} \equiv \{z_1|z_2\}. \quad (48)$$

Since the LLL single-particle states are not complete in the *full* Hilbert space $\{z_1|z_2\} \neq \delta^2(z_1 - z_2)$. In fact, $\{z_1|z_2\}$ acts as the δ -function in the LLL

$$\int d^2 z_1 F(z_1) \{z_1|z_2\} = F(z_2), \quad (49)$$

where $F(z)$ is any function of the form $F(z) = f(z) e^{-\frac{1}{4}|z|^2}$ and $f(z)$ is holomorphic. Further more,

$$\int d^2 z_2 \{z_1|z_2\} G(\bar{z}_2) = G(\bar{z}_1), \quad (50)$$

where $G(\bar{z})$ is any function of the form $G(\bar{z}) = g(\bar{z}) e^{-\frac{1}{4}|\bar{z}|^2}$ and $g(\bar{z})$ is anti-holomorphic.

Finally, $\{z_1|z_2\}$ satisfies the composition rule

$$\int d^2 z_2 \{z_1|z_2\} \{z_2|z_3\} = \{z_1|z_3\}, \quad \{z_1|z_2\}^* = \{z_2|z_1\}. \quad (51)$$

LLL at finite number of particles

Next we assume that only N orbitals are available in the LLL. The finite N version of the LLL δ -function is

$$K_N(z_1; z_2) = \sum_{m=0}^{N-1} \phi_m(z_1) \phi_m^*(z_2), \quad (52)$$

which is known as the reproducing kernel for the Hilbert space of states spanning the LLL (also known as the Bergman kernel), which satisfies the composition rule

$$K_N(z; z') = \int d^2 \xi K_N(z; \xi) K_N(\xi; z'), \quad (53)$$

and the reproducing formula

$$\int d^2 z_2 K_N(z_1; z_2) F_N(z_2) = F_N(z_1), \quad (54)$$

where $F_N(z)$ is any linear combination of $\phi_m(z)$ with m running from 0 to $N - 1$. Similarly,

$$\int d^2 z_1 G_N(\bar{z}_1) K_N(z_1; z_2) = G_N(\bar{z}_2), \quad (55)$$

where $G_N(\bar{z})$ is any linear combination of $\phi_m^*(z)$ with m running from 0 to $N - 1$.

The normalized wave function of the $\nu = 1$ integer quantum Hall state is

$$\Psi_1(z_1, \dots, z_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_0(z_1) & \phi_0(z_2) & \cdots & \phi_0(z_N) \\ \phi_1(z_1) & \phi_1(z_2) & \cdots & \phi_1(z_N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N-1}(z_1) & \phi_{N-1}(z_2) & \cdots & \phi_{N-1}(z_N) \end{vmatrix}. \quad (56)$$

We define the n -particle density matrix for this state by

$$\mathcal{P}_1^{(n)}(\{\xi\}, \{\xi'\}) = \frac{N!}{(N-n)!n!} \int [dz] \Psi_1(\{z\}, \{\xi\}) \Psi_1^*(\{z\}, \{\xi'\}). \quad (57)$$

where we used the shorthand notation

$$\{z\} = z_1, \dots, z_M \quad \{\xi\} = z_{M+1}, \dots, z_N \quad \{\xi'\} = z'_{M+1}, \dots, z'_N. \quad (58)$$

with $n = N - M$. It can be shown that

$$\mathcal{P}_1^{(n)}(\xi_1, \dots, \xi_n; \xi'_1, \dots, \xi'_n) = \frac{1}{n!} \det [K_N(\xi_i; \xi'_j)]_{1 \leq i, j \leq n}. \quad (59)$$

For example, $\mathcal{P}_1^{(1)}(\xi; \xi') = K_N(\xi; \xi')$, which at equal points coincides with the mean density of the integer state $\langle \rho_1(\xi) \rangle = K_N(\xi; \xi)$. Furthermore, it's clear that

$$\mathcal{P}_1^{(2)}(\xi_1, \xi_2; \xi'_1, \xi'_2) = \frac{1}{2} (K_N(\xi_1; \xi'_1) K_N(\xi_2; \xi'_2) - K_N(\xi_1; \xi'_2) K_N(\xi_2; \xi'_1)). \quad (60)$$

On the diagonal, $\mathcal{P}^{(2)}(\xi_1, \xi_2; \xi_1, \xi_2)$ is related to the two-particle distribution function $\mathcal{P}^{(2)}(\xi_1, \xi_2; \xi_1, \xi_2) = \frac{1}{2} n^{(2)}(\xi_1, \xi_2)$, which can be expressed in terms of the pair distribution function $n^{(2)}(\xi_1, \xi_2) = \langle \rho(\xi_1) \rangle \langle \rho(\xi_2) \rangle g(\xi_1, \xi_2)$. Similar to the reproducing kernel, the reduced density matrix has the property that

$$\int [d\xi'] \mathcal{P}_1^{(n)}(\{\xi\}; \{\xi'\}) F_N(\{\xi'\}) = F_N(\{\xi\}). \quad (61)$$

for any *antisymmetric* function $F_N(\{\xi\}) = F_N(\xi_1, \dots, \xi_n)$ built by linear superposition of the N LLL orbitals. In particular, for a fermionic M -particle wave function Ψ_ν , it clearly follows that

$$\int [d\xi'] \mathcal{P}_1^{(M)}(\{\xi\}; \{\xi'\}) \Psi_\nu(\{\xi'\}) = \Psi_\nu(\{\xi\}). \quad (62)$$

Then defining PH conjugation by

$$\Psi_{1-\nu}(\{z\}) = \sqrt{\frac{N!}{(N-M)!M!}} \int [d\xi] \Psi_1(\{z\}, \{\xi\}) \Psi_\nu^*(\{\xi\}). \quad (63)$$

we find

$$\Psi_{1-\nu}^{PH} = \Psi_{1-(1-\nu)} = (-1)^{M(N-M)} \Psi_\nu. \quad (64)$$

Results of this Section remain true in inhomogeneous magnetic field and curvature.

PHD for the density matrix

Here we obtain a relation between the density matrices computed for different filling fractions. Specifically

$$\mathcal{P}_\nu^{(n)}(\{\xi\}; \{\xi'\}) = \frac{N_\nu!}{(N_\nu - n)!n!} \int [dz] \Psi_\nu(\{z\}, \{\xi\}) \Psi_\nu^*(\{z\}, \{\xi'\}), \quad (65)$$

where $N_\nu = M$ is the total number of particles in the state Ψ_ν . It is similarly defined for the PH-conjugate state $\Psi_{1-\nu}$, with $N_{1-\nu} = N - M$. We start with the one-particle density matrix

$$\mathcal{P}_{1-\nu}^{(1)}(z; z') \equiv \frac{(M+1)!}{M!} \int [d\xi][d\xi'] \mathcal{P}_1^{(M+1)}(z, \{\xi\}; z', \{\xi'\}) \Psi_\nu(\{\xi'\}) \Psi_\nu^*(\{\xi\}), \quad (66)$$

$$= \mathcal{P}_1^{(1)}(z; z') - \mathcal{P}_\nu^{(1)}(z; z'), \quad (67)$$

where the second equality follows directly from the following expansion of the determinant in the definition of the density matrix

$$\mathcal{P}_1^{(M+1)}(z, \{\xi\}; z', \{\xi'\}) = \frac{1}{M+1} \left[K_N(z; z') \mathcal{P}_1^{(M)}(\{\xi\}, \{\xi'\}) - \sum_{k=1}^M K_N(z; \xi'_k) \mathcal{P}_1^{(M)}(\xi_k, \{\xi\}_k; z', \{\xi'\}_k) \right], \quad (68)$$

where $\{\xi\}_k = \{\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_M\}$ is the ordered array of coordinates with ξ_k excluded. On the diagonal, $\mathcal{P}^{(1)} = \langle \rho \rangle$, and (67) implies the duality relation for the mean density $\langle \rho_{1-\nu} \rangle = \langle \rho_1 \rangle - \langle \rho_\nu \rangle$.

Next, we consider the two-particle reduced density matrix, which can be written as a convolution involving the $(M+2)$ -particle density matrix

$$\mathcal{P}_{1-\nu}^{(2)}(z_1, z_2; z_1, z_2) = \frac{(M+2)!}{2(M!)} \int [d\xi][d\xi'] \mathcal{P}_1^{(M+2)}(z_1, z_2, \{\xi\}; z_1, z_2, \{\xi'\}) \Psi_\nu(\{\xi'\}) \Psi_\nu^*(\{\xi\}) \quad (69)$$

We next utilize the expansion

$$\begin{aligned} \mathcal{P}_1^{(M+2)}(z_1, z_2, \{\xi\}; z_1, z_2, \{\xi'\}) &= \frac{1}{M+2} \left[K_N(z_1; z_1) \mathcal{P}_1^{(M+1)}(z_2, \{\xi\}; z_2, \{\xi'\}) \right. \\ &\quad - K_N(z_1; z_2) \mathcal{P}_1^{(M+1)}(z_2, \{\xi\}; z_1, \{\xi'\}) \\ &\quad \left. - \sum_{k=1}^M (-1)^k K_N(z_1; \xi'_k) \mathcal{P}_1^{(M+1)}(z_2, \{\xi\}; z_1, z_2, \{\xi'\}_k) \right] \end{aligned} \quad (70)$$

Inserting this into the definition and using (66), we get

$$\begin{aligned} \mathcal{P}_{1-\nu}^{(2)}(z_1, z_2; z_1, z_2) &= \frac{1}{2} \left[K_N(z_1; z_1) \mathcal{P}_{1-\nu}^{(1)}(z_2; z_2) - K_N(z_1; z_2) \mathcal{P}_{1-\nu}^{(1)}(z_2; z_1) \right. \\ &\quad \left. - \frac{1}{2} (M+1) \sum_k (-1)^k \int [d\xi'] [d\xi] K_N(z_1; \xi'_k) \mathcal{P}_1^{(M+1)}(z_2, \{\xi\}; z_1, z_2, \{\xi'\}_k) \Psi_\nu(\xi'_k, \{\xi'\}) \Psi_\nu^*(\xi_k, \{\xi\}_k) \right]. \end{aligned} \quad (71)$$

In order to evaluate the sum in the second line, we use the expansion

$$\begin{aligned} \mathcal{P}_1^{(M+1)}(z_2, \{\xi\}; z_1, z_2, \{\xi'\}_k) &= \frac{1}{(M+1)} \left[K_N(z_2; z_1) \mathcal{P}_1^{(M)}(\{\xi\}; z_2, \{\xi'\}_k) \right. \\ &\quad - K_N(z_2; z_2) \mathcal{P}_1^{(M)}(\{\xi\}; z_1, \{\xi'\}_k) \\ &\quad \left. + \sum_{j \neq k=1}^M (-1)^{j+1} K_N(z_2, \xi'_j) \mathcal{P}_1^{(M)}(\{\xi\}; z_1, z_2, \{\xi'\}_{j,k}) \right]. \end{aligned} \quad (72)$$

Plugging this in and using (61) gives

$$\begin{aligned}
\mathcal{P}_{1-\nu}^{(2)}(z_1, z_2; z_1, z_2) &= \frac{1}{2} \left[K_N(z_1; z_1) \mathcal{P}_{1-\nu}^{(1)}(z_2; z_2) - K_N(z_1; z_2) \mathcal{P}_{1-\nu}^{(1)}(z_2; z_1) \right. \\
&\quad - \frac{1}{2} \sum_k (-1)^k \int [d\xi'] K_N(z_1; \xi'_k) K_N(z_2; z_1) \Psi_\nu(\{\xi'\}) \Psi_\nu^*(z_2, \{\xi'\}_k) \\
&\quad + \frac{1}{2} \sum_k (-1)^k \int [d\xi'] K_N(z_1; \xi'_k) K_N(z_2; z_2) \Psi_\nu(\{\xi'\}) \Psi_\nu^*(z_1, \{\xi'\}_k) \\
&\quad \left. - \frac{1}{2} \sum_{k \neq j} (-1)^{k+j+1} \int [d\xi'] K_N(z_1; \xi'_k) K_N(z_2; \xi'_j) \Psi_\nu(\{\xi'\}) \Psi_\nu^*(z_1, z_2, \{\xi'\}_{j,k}) \right]. \quad (73)
\end{aligned}$$

which can be expressed in terms of the reduced density matrix of the Ψ_ν state as

$$\begin{aligned}
\mathcal{P}_{1-\nu}^{(2)}(z_1, z_2; z_1, z_2) &= \frac{1}{2} \left[K_N(z_1; z_1) \mathcal{P}_{1-\nu}^{(1)}(z_2; z_2) - K_N(z_1; z_2) \mathcal{P}_{1-\nu}^{(1)}(z_2; z_1) \right. \\
&\quad + \frac{1}{2} \int d^2 \xi'_1 K_N(z_1; \xi'_1) K_N(z_2; z_1) \mathcal{P}_\nu^{(1)}(\xi'_1; z_2) \\
&\quad - \frac{1}{2} \int d^2 \xi'_1 K_N(z_1; \xi'_1) K_N(z_2; z_2) \mathcal{P}_\nu^{(1)}(\xi'_1; z_1) \\
&\quad \left. + \int d^2 \xi'_1 d^2 \xi'_2 K_N(z_1; \xi'_1) K_N(z_2; \xi'_2) \mathcal{P}_\nu^{(2)}(\xi'_1, \xi'_2; z_1, z_2) \right] \quad (74)
\end{aligned}$$

using the reproducing property of the kernel, and the determinant expression for the 2-particle density matrix $\mathcal{P}_1^{(2)}$, this can be brought into the form presented in the paper.

Two-point functions of density from the generating functional

Next we explain how to obtain connected two-point functions of density $\rho_{1-\nu}$ from the generating functional $\log \mathcal{Z}_{1-\nu}$. Indeed, differentiating (23) w.r.t. $W(\zeta')$

$$\frac{\delta}{\delta W(\zeta')} \langle \rho_{1-\nu}(\zeta) \rangle = \langle \rho_{1-\nu}(\zeta') \rho_{1-\nu}(\zeta) \rangle_c + [\rho_\nu(\zeta'); \rho_{1-\nu}(\zeta)], \quad (75)$$

where we have introduced a “mixed” correlator

$$[\rho_\nu(\zeta'); \rho_{1-\nu}(\zeta)] = 2 \frac{(N+M)!}{N!M!} \int [d\xi] [d\xi'] \Psi_\nu(\{\xi\}) \Psi_\nu^*(\{\xi'\}) \rho_\nu(\zeta') \int [dz] \Psi_1(\{z\}, \{\xi'\}) \Psi_1^*(\{z\}, \{\xi\}) \rho_{1-\nu}(\zeta), \quad (76)$$

which signals the failure of the formula (6) for PH-dual states. To evaluate this term we combine (75) together with (20) and (24) to obtain

$$\begin{aligned}
\frac{\delta^2 \ln \mathcal{Z}_{1-\nu}}{\delta W(\zeta') \delta W(\zeta)} &= \langle \rho_{1-\nu}(\zeta) \rho_{1-\nu}(\zeta') \rangle_c + 2 \langle \rho_\nu(\zeta) \rho_\nu(\zeta') \rangle_c + [\rho_\nu(\zeta'); \rho_{1-\nu}(\zeta)] \\
&= 2 \langle \rho_1(\zeta) \rho_1(\zeta') \rangle_c - \langle \rho_{1-\nu}(\zeta) \rho_{1-\nu}(\zeta') \rangle_c - [\rho_\nu(\zeta'); \rho_{1-\nu}(\zeta)]. \quad (77)
\end{aligned}$$

Which implies

$$[\rho_\nu(\zeta'); \rho_{1-\nu}(\zeta)] = \langle \rho_1(\zeta) \rho_1(\zeta') \rangle_c - \langle \rho_{1-\nu}(\zeta) \rho_{1-\nu}(\zeta') \rangle_c - \langle \rho_\nu(\zeta) \rho_\nu(\zeta') \rangle_c. \quad (78)$$

In this form, one can see that if the connected two-point functions were related in a manner similar to the one-point functions (20) of the main text, then the mixed correlator would vanish. In the translation-invariant state, we can use (26) to evaluate this in momentum space, and combined with (27) we find a simple expression for the mixed correlator

$$[\rho_\nu; \rho_{1-\nu}](q) = -2\bar{\rho}_\nu \bar{s}_\nu = -2\bar{\rho}_{1-\nu} \bar{s}_{1-\nu}. \quad (79)$$