Essays on Unconventional Monetary Policy and Long-Term Government Debt

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Abstract

This thesis studies the optimal conduct of unconventional monetary policy in the form of purchases of long-term government debt by the central bank and, motivated by this policy tool, the evolution of long-term government debt holdings in household portfolios over the course of the life cycle. It is comprised of three self-contained chapters.

The first chapter investigates whether it can be beneficial for central banks to use the unconventional tool even when the main policy rate is not constrained by the zero lower bound. A friction in the interaction between households and banks allows central bank purchases of long-term government debt to reduce long-term interest rates and thus to stimulate economic activity. If debt purchases and conventional short-term interest rate policy are coordinated in an appropriate way, the central bank is able to reduce the volatility of output and inflation.

In the second chapter, the role that unconventional monetary policy can play in a currency union is analysed. A model is laid out, in which two countries form a currency union with a common central bank but separate and uncoordinated fiscal policy institutions. When monetary policy is implemented only through the common short-term interest rate, the central bank is unable to respond effectively to country-specific shocks. Due to segmentation in the market for long-term government debt, the yield on long-term debt can differ across countries. As a result, a monetary policy authority that can rely on bond purchases is able to address idiosyncratic shocks reflected in volatility of the natural terms of trade more effectively and to achieve higher welfare than one that cannot make use of this instrument.

The final chapter studies the long-term government bond share in household portfolios over the course of the life cycle. US data from the Survey of Consumer Finances suggests that participation in the market for long-term government debt first increases and later decreases as agents approach the retirement age. The portfolio share conditional on participation is non-decreasing over the working life. These stylised facts can be explained by means of a portfolio choice model in which agents are subject to aggregate risk through asset returns as well as idiosyncratic risk through labour income and the stochastic events of retirement and death.

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Preface

This thesis was written between October 2011 and January 2015, a time in which economic crises have shaken the world economy and a time that forced the economics profession to re-evaluate its framework for policy analysis. Beginning in mid-2007, uncertainty about the quality of structured assets and counterparty balance sheets arose, causing turmoil in financial markets and eventually the collapse of Lehman Brothers, which became the largest filing of bankruptcy in US history, in September 2008. Major central banks reacted to the ensuing sharp declines in inflation and growth prospects with cuts in their respective policy rates. The Federal Reserve cut the target range for the federal funds rate to 0-0.25% in late 2008. Similarly, the Bank of England cut its short-term interest rate target incrementally from 5.75% in mid-2007 to 0.5% in March 2009. The European Central Bank followed more gradually, reaching 0.05% in September 2014. All three central banks have kept their respective policy rates at historically low values until the time of writing. With the main policy rate effectively constrained by the zero lower bound and faced by the threat of falling into a liquidity trap, central banks resorted to "unconventional" measures including the purchases of long-term government debt aimed at stimulating the economy by lowering long-term interest rates.

The aim of this thesis is to improve our understanding of unconventional monetary policy. It includes three self-contained chapters that have previously been circulated as independent papers. The first two chapters are concerned with the optimal conduct of unconventional monetary policy in a closed economy and a currency union, respectively. The final chapter explores how the incentives of agents to hold long-term government bonds, the class of assets many central banks have purchased as a part of their unconventional monetary policy programmes, vary over the life cycle. While short-term interest rate policy has been studied extensively in the past, unconventional monetary policy received less attention during the years of the Great Moderation. Until the outbreak of the Financial Crisis, the marked decline in cyclical fluctuations since the early 1980s was seen by many as a result of improved policy making (Blanchard et al., 2010). The general view was that short-term interest rate policy was the only tool that central banks required to steer inflation close to target and output close to its natural rate. The experiences from the recent crises have shown that this view was too optimistic. A logical consequence is to study additional policy tools and their potential applicability in the future. The papers collected in this thesis should be viewed as a step in this direction.

In recent years, a multitude of unconventional measures have been adopted across different central banks. They included, for example, the widening of eligible collateral and counterparties, the switch to a system with full allotment of banks' liquidity demands at a fixed rate, the provision of foreign exchange liquidity, the adoption of forward guidance, changes to reserve requirements and purchases of financial assets. The motives for and effects of asset purchases differed by the type of assets acquired. Borio and Disyatat (2009) distinguish between bank-reserves policy, credit policy and quasi-debt-management policy. Bank-reserves policy describes a regime in which central banks set a target for bank reserves, that is, for the amount of liquidity that it aims to inject into the economy, irrespective of how this affects the asset side of the central bank's balance sheet. Credit policy includes all operations that change the central bank's exposure profile to private sector claims. This includes "credit easing", a type of policy through which the central bank aims to ease the repercussions of a distressed banking sector, one that is unable to intermediate funds in a satisfying way, by extending credit directly to the private sector. Equivalently, quasi-debt-management policy includes all operations that change the central bank's exposure profile to public claims. Policies that fall into this last category are the subject of study here.

Central banks have made heavy use of quasi-debt-management policies in recent years. Depending on the way in which purchases of government debt were financed, these interventions also included elements of bank-reserves policy. The Federal Reserve announced Large-Scale Asset Purchases (LSAPs) of up to $100 billion of US agency debt in November 2008, a target which was doubled in March 2009. At the same meeting of the Federal Open Market Committee

(FOMC) in March 2009, purchases of up to \$300 billion of longer term Treasury securities were decided. Maturing assets were re-invested in long-term Treasuries and further purchases of \$600 billion of long-term Treasuries took place in a second round of LSAPs starting in November 2010. In September 2011, the Maturity Extension Program was initiated under which the Fed replaced a total of \$627 billion of short-term Treasuries with long-term Treasuries on its balance sheet. In December 2012, it was announced that the Fed would continue to purchase \$45 billion of longer term Treasuries per month after the expiration of the Maturity Extension Program until asset purchases were decided to slowly taper off starting in December 2013. The Bank of England started its Quantitative Easing (QE) programme in March 2009. Until July 2012, a total of £375 billion of assets were purchased through the Bank of England's Asset Purchase Facility, a large fraction of which was longer term government debt.

Despite of these programmes of unprecedented volume, little is known about the optimal conduct of this tool and its optimal combination with short-term interest rate policy. There are at least two reasons for this circumstance. First, as mentioned above, quasi-debt-management policy was not in the focus of economic research during the years of the Great Moderation. Second, the theoretical literature struggles to reconcile the empirical evidence that is largely in favour of significant effects of asset purchases on long-term interest rates with popular models of the term structure. Under the expectations theory of the term structure, long-term interest rates are fully determined by the expected path of future short-term rates, implying that a withdrawal of long-term government debt from private sector balance sheets does not affect long-term interest rates as long as it leaves expectations about future short-term rates unchanged. Eggertsson and Woodford (2003) show that a comparable result holds in a general equilibrium model with frictionless financial markets. A model that captures the empirically observed effects therefore has to contain frictions that break the irrelevance proposition contained in Eggertsson and Woodford (2003).

The first chapter introduces a friction in the banking sector of a DSGE model that allows central bank purchases of government bonds to affect long-term interest rates and thus output and inflation through a "preferred habitat" channel. This chapter is co-authored by Martin Ellison. In the model, households can transfer income between periods only with the help of banks, which allocate deposits into short-term and long-term government bonds according to a perception that savers differ with respect to their preferred investment horizons. Purchases of long-term government debt by the central bank reduce the supply of long-term debt available to the private sector, which increases the marginal willingness of banks to pay for it and thus cause the yield curve to flatten. The central bank is assumed to follow simple rules for short-term interest rate policy and long-term bond purchases. The optimised policy, when conventional and unconventional policy is used, improves over the one that is optimal if only the standard instrument is employed. The benefits of unconventional monetary policy are small if the central bank's rule for the short-term interest rate is not adjusted to account for purchases of long-term debt. However, when both policies are jointly optimised, the ability of the central bank to stabilise inflation and output is significantly increased.

The second chapter investigates the role that central bank purchases of government debt can play in a currency union, such as the Euro Area, when financial markets are fragmented. The analysis is based on a limited-arbitrage model of two countries with a common monetary policy authority but separate and uncoordinated fiscal policy makers. Purchases of long-term government debt by the central bank lower the yield of the bonds purchased, decrease debt servicing costs and increase aggregate demand and inflation. Due to market segmentation, asset purchases have asymmetric effects across both countries, allowing the central bank to stimulate economic activity at different intensities in both member countries. A second-order approximation of union-wide welfare shows that inflation in both countries, deviations of consumption and the terms of trade from their respective natural rates as well as usage of unconventional monetary policy is associated with welfare costs. Benefits from using unconventional monetary policy arise from its ability to respond to shocks that lead to volatility in the natural terms of trade. The chapter gives a detailed description of the optimal mix of the conventional and the unconventional tool in this environment and derives welfare implications.

In the final chapter, the portfolio choice problem of households becomes the main subject of analysis. Motivated by the importance of long-term government debt in the transmission of quasi-debt management policies, the chapter studies the decision of households to participate in the market for long-term government debt and portfolio shares conditional on participation over the course of the life cycle. The first part contains an empirical analysis based on a quasi-panel constructed from seven waves of the Survey of Consumer Finances. The participation profile in the long-term government debt market, similar to that in the equity market, takes an inverse U-shape. Furthermore, the data suggests that participating households reduce the portfolio share of equity and increase the share of long-term debt over their working lives. In the second part, a heterogeneous-agent model is set up to explain the key features of the data. Agents can invest in three types of financial assets, a risk-free asset, equity and long-term bonds. They face idiosyncratic risk through labour income and fixed probabilities of retirement and death as well as aggregate risk through asset returns. Agents' policy functions reflect their participation decision and their desire for portfolio rebalancing in the wake and anticipation of wealth accumulation, retirement and death.

Unconventional monetary policy has been a subject of controversial debate over the last years. Critics have pointed out, for example, that central bank asset purchases may reduce the political pressure to introduce necessary structural reforms, that they may lead to counter-productive outcomes when they are phased out and that they may force investors to seek an inefficiently high level of risk exposure in a search for yield. These concerns reinforce the need for research on the subject of unconventional monetary policy. In the view of Blanchard et al. (2010), the Financial Crisis has "made clear that [policy makers] have potentially many more tools at their disposal than they used before the crisis. The challenge is to learn how to use these instruments in the best way" (p. 213). I would be glad if this thesis could make a contribution to this process.

Chapter 1

Unconventional Government Debt Purchases as a Supplement to Conventional Monetary Policy

Chapter Summary

In response to the Great Financial Crisis, the Federal Reserve, the Bank of England and many other central banks have adopted unconventional monetary policy instruments. We investigate if one of these, purchases of long-term government debt, could be a valuable addition to conventional short-term interest rate policy even if the main policy rate is not constrained by the zero lower bound. To do so, we add a stylised financial sector and central bank asset purchases to an otherwise standard New Keynesian DSGE model. Asset quantities matter for interest rates through a preferred habitat channel. If conventional and unconventional monetary policy instruments are coordinated appropriately then the central bank is better able to stabilise both output and inflation.[1]

[1]This chapter is co-authored by Martin Ellison.

1.1 Introduction

The Great Financial Crisis has seen the emergence of monetary policy instruments that are often described as "unconventional". The events of 2007-2008 forced monetary policy authorities to adopt new tools, even though they had little previous experience with them and there was considerable uncertainty about their likely impact. The general belief was that unconventional policy was an emergency response that would be phased out once the crisis was over. However, if it is designed carefully it may help a central bank reach its objectives even in non-crisis times.

This paper investigates whether the unconventional policy of central banks purchasing long-term government debt could be useful even after the Great Financial Crisis has passed. We obtain our results in a New Keynesian DSGE model with a stylised financial sector and a Taylor-type policy rule for central bank asset purchases. Asset quantities matter because of the behaviour of banks and the incompleteness of financial markets. Households can only transfer income between periods with the help of banks, who invest the deposits they receive into government bonds. Banks allocate deposits into government bonds of different maturities according to a perception that savers are heterogeneous with respect to their preferred investment horizons. Central bank purchases of long-term government bonds reduce the supply of long-term debt available to the private sector, which increases the marginal willingness of banks to pay for it. This reduces yields on long-term debt, discourages saving, and hence increases output and inflation. When the policy parameters are chosen optimally, a combination of conventional and unconventional policies leads to significantly lower losses compared to when the central bank uses only conventional policies.

Central bank purchases of government bonds in our model have an effect through a "preferred habitat" channel of the type identified by Modigliano and Sutch (1966), and later developed by Vayanos and Vila (2009). The idea is that investors see government bonds of different maturities as imperfect substitutes and so are willing to pay a premium on bonds of their preferred maturity. The quantities of assets available then matter for prices and returns; if the central bank purchases government debt of a particular maturity then the supply of that asset to the private sector is reduced, its price rises and its return falls. The preferred habitat channel operates in our model as banks hold government debt of different maturities in response to a perception that savers have heterogeneous investment horizons. The closest models to ours are Andrés et al. (2004) and Chen et al. (2012), although they consider different mechanisms and have less emphasis on implications for optimal policy coordination.

The main focus of the current unconventional monetary policy literature is on credit easing, i.e. central bank purchases of private financial assets.[2] An important exception is Eggertsson and Woodford (2003), who examine central bank purchases of government debt. They argue that unconventional monetary policy works by acting as a signal for the future path of short-term interest rates, so will be especially useful when the main policy rate is constrained by the zero lower bound. In their model, though, the risk-premium component of long-term interest rates is unaffected by any reallocation of assets between the central bank and the private sector. This is because risks are ultimately born by the private sector, even if government debt is purchased by the central bank. If the central bank makes losses on government debt then government revenue falls and taxes on the private sector have to rise to satisfy the government budget constraint.[3] This view is not supported by the empirical evidence in Bernanke et al. (2004), D'Amico et al. (2012), D'Amico and King (2013), Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2011) and Neely (2011). Instead, these studies find strong evidence that central bank purchases of government debt have small but significant effects via the term premium component of long-term interest rates. The evidence supporting a preferred habitat mechanism comes from the "scarcity channel" in D'Amico et al. (2012) and the "safety channel" in Krishnamurthy and Vissing-Jorgensen (2011).

1.2 Model

The model economy consists of households, monopolistically competitive firms, banks, a treasury and a central bank. There is price stickiness, wages are assumed to be fully flexible, and firms use labour as the only input in the production of consumption goods.[4]

[2] See Cúrdia and Woodford (2011), Del Negro et al. (2011), Gertler and Karadi (2011), Gertler and Kiyotaki (2010), Gertler et al. (2012) and Kiyotaki and Moore (2012).

[3] This intuition is given on page 5 of Cúrdia and Woodford (2010).

[4] The model incorporates elements from Benigno and Woodford (2005), Gali (2008) and Woodford (2003).

1.2.1 Households

The preferences of the representative household are given by

$$U_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left(\chi_t^C \frac{C_t^{1-\delta}}{1-\delta} - \chi_t^L \frac{L_t^{1+\psi}}{1+\psi} \right)$$

where $C_t \equiv \left[\int_0^1 C_t(i)^{\frac{\theta_t - 1}{\theta_t}} di \right]^{\frac{\theta_t}{\theta_t - 1}}$ is a CES consumption index composed to minimise cost and L_t is the time devoted to market employment. χ_t^C and χ_t^L are exogenous preference shock processes that evolve according to

$$\begin{aligned} \ln(\chi_t^C) &= \rho_C \ln(\chi_{t-1}^C) + \varepsilon_t^C \quad \text{ with } \varepsilon_t^C \sim N(0, \sigma_C^2) &(1.1)\\ \ln(\chi_t^L) &= \rho_L \ln(\chi_{t-1}^L) + \varepsilon_t^L \quad \text{ with } \varepsilon_t^L \sim N(0, \sigma_L^2) &(1.2)\end{aligned}$$

The household maximises expected utility subject to the budget constraint:

$$P_t C_t + T_t + P_t^S S_{t,t+1} = S_{t-1,t} + W_t L_t + (1 - t_\pi)(P_t Y_t - W_t L_t) \tag{1.3}$$

$P_t \equiv \left[\int_0^1 P_t(i)^{1-\theta_t} di \right]^{\frac{1}{1-\theta_t}}$ is the price of the composite consumption good, T_t is a lump-sum tax paid to the government and $S_{t,t+1}$ is the quantity of a savings device purchased from perfectly competitive banks at unit price $P_t^S < 1$ in period t. The household can secure a payment of $S_{t,t+1}$ in period $t+1$ by saving $P_t^S S_{t,t+1}$ in period t. W_t is the nominal market wage. Households own firms that produce and sell consumption goods, so they receive dividend income $(P_t Y_t - W_t L_t)$ which is subject to tax at the rate t_π.[5] All prices are measured in units of the numeraire good "money", which is not modelled.

The first-order conditions of the household's optimisation problem are

$$\begin{aligned} 1 &= \beta E_t \left[\frac{\chi_{t+1}^C}{\chi_t^C} \left(\frac{C_{t+1}}{C_t} \right)^{-\delta} \frac{1}{\Pi_{t+1}} \right] \frac{1}{P_t^S} &(1.4)\\ \frac{W_t}{P_t} &= \frac{\chi_t^L}{\chi_t^C} L_t^\psi C_t^\delta &(1.5)\end{aligned}$$

[5] The representative household views dividends as lump-sum income and does not internalise the effect of labour supply on profits.

with $\Pi_{t+1} \equiv \frac{P_{t+1}}{P_t}$. Equation (1.4) is the intertemporal Euler equation that characterises the optimal consumption-savings decision. Equation (1.5) is the intratemporal condition describing optimality between labour supply and consumption.

1.2.2 Firms

There is a continuum of monopolistically competitive firms indexed by $i \in [0,1]$. Firm i's production function is $Y_t(i) = A_t L_t(i)^{\frac{1}{\phi}}$, where $L_t(i)$ is labour employed and A_t is an exogenous technology shock process that evolves according to

$$\ln(A_t) = \rho_A \ln(A_{t-1}) + \varepsilon_t^A \quad \text{with } |\rho_A| < 1 \text{ and } \varepsilon_t^A \sim N(0, \sigma_A^2) \tag{1.6}$$

As in Calvo (1983), only a fraction $1-\alpha$ of firms can adjust the price of their respective good in any given period. Let $P_t^*(i)$ be the price chosen by a firm that is able to reset its price in period t. The evolution of the aggregate price level is then described by

$$P_t = \left[(1-\alpha)P_t^*(i)^{1-\theta_t} + \alpha P_{t-1}^{1-\theta_t}\right]^{\frac{1}{1-\theta_t}}$$

θ_t is the time-varying elasticity of substitution between consumption goods. All firms that can change their price in period t choose $P_t(i)$ to maximise the expected discounted stream of their future profits

$$E_t \sum_{T=t}^{\infty} \alpha^{T-t} M_{t,T} \left[P_t(i)Y_T(i) - W_T L_T(i)\right]$$

subject to the relevant demand constraints. $M_{t,T} \equiv \beta^{T-t} \frac{\chi_T^C}{\chi_t^C} \frac{C_T^{-\delta}}{C_t^{-\delta}} \frac{P_t}{P_T}$ is the firm's stochastic discount factor, derived from the consumption Euler equation (1.4).

The first-order condition for price setting and the assumed price adjustment process imply that equilibrium inflation is given by[6]

$$\frac{1-\alpha\Pi_t^{\theta_t-1}}{1-\alpha} = \left(\frac{F_t}{K_t}\right)^{\frac{\theta_t-1}{\theta_t(\phi-1)+1}} \tag{1.7}$$

[6]See Section 1.5.1 in the appendix for the derivation.

where

$$F_t = \chi_t^C C_t^{-\delta} Y_t + \alpha\beta E_t \Pi_{t+1}^{\theta_t - 1} F_{t+1} \tag{1.8}$$

$$K_t = \frac{\theta_t \phi}{\theta_t - 1} \chi_t^L L_t^{\psi} \left(\frac{Y_t}{A_t}\right)^{\phi} + \alpha\beta E_t \Pi_{t+1}^{\theta_t \phi} K_{t+1} \tag{1.9}$$

F_t and K_t are auxiliary variables and

$$\ln\left(\frac{\theta_t}{\theta}\right) = \rho_\theta \ln\left(\frac{\theta_{t-1}}{\theta}\right) + \varepsilon_t^\theta \quad \text{with } \varepsilon_t^\theta \sim N(0, \sigma_\theta^2) \tag{1.10}$$

is the law of motion for the time-varying elasticity of substitution between consumption goods.

1.2.3 Banks

The role of the representative bank is to determine the maturity composition of the aggregate savings device offered to households. Perfect competition in the banking sector ensures that all bank revenues are returned to households, but it is the bank that decides how deposits are invested into short-term and long-term government bonds. We assume when doing so that banks perceive households as heterogeneous with regards to their desired investment horizons, an assumption consistent with the preferred habitat literature where investors value characteristics of assets other than just their payoff. In the context of our model, this means that investors are perceived as having a preference for assets with maturities that match their preferred investment horizon. Assets of other maturities are viewed as only imperfect substitutes. As a result, the price of an asset with a given maturity is influenced by supply and demand effects local to that maturity, see Vayanos and Vila (2009), and the central bank can use purchases of government bonds to influence prices and yields at different maturities.[7]

Formally, in every period t the representative bank collects nominal deposits $P_t^S S_{t,t+1}$ from households, which it uses to purchase $B_{t,t+1}$ units of a short-term government bond and $Q_{t,t+\tau}$

[7]The assumption that households are *perceived* as having heterogeneous preferences over different investment horizons is made so the model remains within a standard representative agent framework. If households differ with respect to their *actual* preferences then a full heterogeneous agent model would be needed.

units of a long-term government bond. The flow budget constraint of the representative bank is

$$P_t^S S_{t,t+1} = P_t^B B_{t,t+1} + P_t^Q Q_{t,t+\tau} \tag{1.11}$$

A unit of the short-term bond can be purchased at price P_t^B in t and has a payoff of one in the next period $t+1$. The long-term bond is traded at the price P_t^Q in t; a unit of this bond yields a payoff of $1/\tau$ in every period between $t+1$ and $t+\tau$.

The bank constructs the savings device S offered to households on the basis of a function V that reflects its perception of the different investment horizons preferred by heterogenous households. The allocation problem of the representative bank is

$$\max_{B_{t,t+1}, Q_{t,t+\tau}} V\left(\frac{B_{t,t+1}}{P_t}, \frac{Q_{t,t+\tau}}{P_t}\right)$$

subject to the flow constraint (1.11).[8] The asset demand schedules that solve the allocation problem depend on the functional form chosen for V. We prefer not to impose restrictive properties on the demand curves, so borrow from the empirical literature on demand estimation and adopt the flexible functional form of the Generalised Translog (GTL) model introduced by Pollak and Wales (1980). Rather than specifying V directly, the GTL model specifies the indirect utility function $V^* \equiv V(\frac{B^*}{P}, \frac{Q^*}{P})$:

$$\begin{aligned} \log(V_t^*) \quad = \quad & a_0 + \sum_k a_1^k \log\left(\frac{P_t^k}{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}\right) + \\ & \frac{1}{2}\sum_k \sum_l a_2^{kl} \log\left(\frac{P_t^k}{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}\right) \log\left(\frac{P_t^l}{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}\right) \end{aligned}$$

with $k, l \in \{B, Q\}$, $a_2^{kl} = a_2^{lk}$ and $s_t \equiv \frac{S_{t,t+1}}{P_t}$. The asset shares $as^B \equiv \frac{P^B B}{P^S S}$ and $as^Q \equiv \frac{P^Q Q}{P^S S} = 1 - as^B$ can be derived using the logarithmic form of Roy's identity, see Barnett and Serletis

[8]The optimisation problem is equivalent to that of a bank in a model where V aggregates the heterogeneous preferences of households over bonds of different maturities.

(2008) and Section 1.5.2 in the appendix for the details.

$$as_t^k = \frac{P_t^k g^k}{P_t^S s_t} + \left(1 - \frac{P_t^B g^B + P_t^Q g^Q}{P_t^S s_t}\right) \frac{a_1^k + \sum_l a_2^{kl} \log\left(\frac{P_t^l}{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}\right)}{\sum_l a_1^l + \sum_k \sum_l a_2^{kl} \log\left(\frac{P_t^l}{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}\right)}$$

To ensure that the above asset shares are the result of the bank's optimisation problem, the parameter space has to be restricted such that four "integrability conditions" hold. Under these conditions, quasi-homotheticity, $a_1 \equiv a_1^B$ and $a_2 \equiv a_2^{BB}$, the asset demands are:

$$\frac{B_{t,t+1}}{P_t} = g^B + \frac{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}{P_t^B}\left[a_1 + a_2 \log\left(\frac{P_t^B}{P_t^Q}\right)\right] \tag{1.12}$$

$$\frac{Q_{t,t+\tau}}{P_t} = g^Q + \frac{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}{P_t^Q}\left[1 - a_1 - a_2 \log\left(\frac{P_t^B}{P_t^Q}\right)\right] \tag{1.13}$$

The income of the representative bank from its holdings of long-term bonds in period t is

$$\frac{1}{\tau}\sum_{j=1}^{\tau} Q_{t-j,t+\tau-j}$$

Perfect competition in the banking sector requires that banks return all their revenues from government bonds to depositors, so the return to a household depositing $P_{t-1}^S S_{t-1,t}$ is

$$S_{t-1,t} = B_{t-1,t} + \frac{1}{\tau}\sum_{j=1}^{\tau} Q_{t-j,t+\tau-j} \tag{1.14}$$

The prices P_t^B and P_t^Q of the short-term and long-term government bonds are known with certainty in period t but not in advance. The gross nominal interest rate paid on the one-period bond is

$$1 + i_t = \frac{1}{P_t^B} \tag{1.15}$$

For comparison purposes, it is possible to calculate an interest rate associated with the τ-period bond traded in t. It is implicitly defined by

$$\begin{aligned} P_t^Q &= \frac{\frac{1}{\tau}}{1+i_t^Q} + \frac{\frac{1}{\tau}}{\left(1+i_t^Q\right)^2} + \frac{\frac{1}{\tau}}{\left(1+i_t^Q\right)^3} + \ldots + \frac{\frac{1}{\tau}}{\left(1+i_t^Q\right)^\tau} \\ &= \frac{1}{\tau}\frac{1}{1+i_t^Q}\frac{1-\left(\frac{1}{1+i_t^Q}\right)^\tau}{1-\frac{1}{1+i_t^Q}} \end{aligned} \tag{1.16}$$

where i_t^Q is the per-period interest rate that equates the unit price P_t^Q with the value of the discounted payoff stream from a unit investment in the long-term bond.

An important feature of the bank behaviour is that the closed-form asset demand schedules are simple and intuitive, in contrast with previous contributions to the literature on unconventional monetary policy. Both demand curves are downward sloping for small enough values of a_2. Low asset quantities are associated with high asset prices, which allows unconventional monetary policy instruments to work through a scarcity channel. For example, central bank purchases of long-term bonds reduce the quantity of those bonds available to the banks and thus increase their price.

1.2.4 Government

The government's economic policy is implemented by the actions of a treasury and a central bank. We model the operations of the treasury in a simplified way to maintain the focus of our paper on monetary policy.

The treasury issues both short and long-term government debt. Short-term bonds are issued in a quantity consistent with the central bank's setting of the short-term nominal interest rate. The quantity of long-term bonds issued in period t is $\bar{Q}_{t,t+\tau}$, determined by a simple rule

$$\frac{\bar{Q}_{t,t+\tau}}{P_t} = fY \tag{1.17}$$

to maintain constant real issuance of long-term bonds each period. $f > 0$ and Y is steady-state output. Long-term bonds can only be purchased from the treasury in the period they are issued and then have to be held until maturity, i.e. we follow Sargent (1987, pp. 102-105) and Andrés et al. (2004) in assuming that there is no secondary market for long-term government debt.[9]

The treasury uses lump-sum taxes to finance government spending. The government consumption good G_t has the same CES aggregator as the composite household consumption good and is given exogenously by

$$\ln\left(\frac{G_t}{G}\right) = \rho_G \ln\left(\frac{G_{t-1}}{G}\right) + \varepsilon_t^G \quad \text{with } \varepsilon_t^G \sim N(0, \sigma_G^2) \tag{1.18}$$

with G being the steady state value of government spending on consumption goods. Lump-sum taxes satisfy $T_t = P_t G_t$.

The central bank sets the short-term nominal interest rate according to the following rule:

$$\frac{1+i_t}{1+i} = \left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi} \left(\frac{Y_t}{Y}\right)^{\gamma_Y} \nu_t \tag{1.19}$$

Variables without time subscript denote steady-state values and ν_t is an interest rate shock term that evolves according to

$$\ln(\nu_t) = \rho_\nu \ln(\nu_{t-1}) + \varepsilon_t^\nu \quad \text{with } \varepsilon_t^\nu \sim N(0, \sigma_\nu^2) \tag{1.20}$$

Asset purchases are carried out according to a Taylor-type rule:

$$\frac{\bar{Q}_{t,t+\tau} - Q_{t,t+\tau}^{CB}}{\bar{Q}_{t,t+\tau}} = \left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi^{QE}} \left(\frac{Y_t}{Y}\right)^{\gamma_Y^{QE}} \xi_t \tag{1.21}$$

where $Q_{t,t+\tau}^{CB}$ is the quantity of long-term bonds purchased by the central bank in period t.[10] The logarithm of the shock ξ_t follows an AR(1) process.

$$\ln(\xi_t) = \rho_\xi \ln(\xi_{t-1}) + \varepsilon_t^\xi \quad \text{with } \varepsilon_t^\xi \sim N(0, \sigma_\xi^2) \tag{1.22}$$

[9]If there is a secondary market in long-term government debt then long-term and short-term bonds become perfect substitutes and it is difficult to maintain the assumption of a preferred habitat across the two different types of bond.

[10]Only the net supply of long-term bonds matters in the model, so it would be possible to abstract from the supply of bonds by the treasury and work instead with an equation for the net supply of long-term bonds. The distinction is retained to avoid confusion between questions of issuance by the treasury and asset purchases by the central bank. Note that such a net bond supply rule can be obtained by combining (1.17) and (1.21). However, all fluctuations in the real supply of the long-term bond result from central bank operations.

The central bank is a net buyer of long-term bonds of the type issued in t if $Q^{CB}_{t,t+\tau} > 0$ and is a net supplier of those to the market if $Q^{CB}_{t,t+\tau} < 0$.[11] For example, if inflation and output are below their respective steady-state values then with $\gamma^{QE}_{\Pi} > 0$, $\gamma^{QE}_{Y} > 0$ and $\xi_t = \xi = 1$ the central bank stimulates the economy by purchasing long-term bonds. Conversely, if inflation and output lie above their steady-state values then asset sales are used to depress prices and productive activity.

The treasury and central bank are subject to a consolidated budget constraint that equates government revenue to government expenditures:

$$P^B_t B_{t,t+1} + P^Q_t \bar{Q}_{t,t+\tau} + T_t + t_\pi (P_t Y_t - W_t L_t) + \pi^{CB}_t = P_t G_t + B_{t-1,t} + \frac{1}{\tau} \sum_{j=1}^{\tau} \bar{Q}_{t-j,t+\tau-j}$$

π^{CB}_t are central bank profits (or losses) from asset purchases. The combination of perfect competition in the banking sector and goods markets together with lump-sum funding of current government expenditure means that the model's consolidated government budget constraint is always satisfied. Section 1.5.3 of the appendix gives full details.

1.2.5 Market Clearing

The model is completed by conditions for clearing in bond, goods and labour markets. Demand and supply in the market for long-term bonds are equated for

$$\bar{Q}_{t,t+\tau} = Q_{t,t+\tau} + Q^{CB}_{t,t+\tau} \tag{1.23}$$

The market for short-term bonds clears and the aggregate resource constraint in goods markets is

$$Y_t = C_t + G_t \tag{1.24}$$

Labour market clearing requires that hours supplied by the representative household L_t are equal to aggregate hours demanded by the firms, $\int_0^1 L_t(i)di$, implying that aggregate production

[11] Since long-term bonds are only traded in the period in which they are issued, unconventional policy in our model should not be viewed as changing the existing stock of long-term bonds held by the central bank. It should instead be seen as affecting the supply of an asset available for purchase at a particular point in time.

is

$$Y_t = A_t \left(\frac{L_t}{D_t} \right)^{\frac{1}{\phi}} \tag{1.25}$$

where $D_t \equiv \int_0^1 \left[\frac{P_t(i)}{P_t} \right]^{-\theta_t \phi} di$ is a measure of price dispersion. The dynamics of the price dispersion term are given by[12]

$$D_t = (1-\alpha) \left(\frac{1-\alpha \Pi_t^{\theta_t - 1}}{1-\alpha} \right)^{\frac{\theta_t \phi}{\theta_t - 1}} + \alpha \Pi_t^{\theta_t \phi} D_{t-1} \tag{1.26}$$

Price dispersion is a source of inefficiency in the model. It acts in addition to the mark-up distortions created by the market power of firms.

1.2.6 Model Summary and Calibration

Equations (1.1)–(1.26) are sufficient to describe the behaviour of the endogenous variables in rational expectations equilibrium. Real variables are stationary, but for $\Pi > 1$ there is a positive trend in all nominal variables. Section 1.5.6 of the appendix shows how we take a first-order numerical approximation of a trend-stationary version of the model and simulate it around the steady state described in Section 1.5.7 of the appendix. In what follows we define $s_t \equiv \frac{S_{t,t+1}}{P_t}$, $b_t \equiv \frac{B_{t,t+1}}{P_t}$ and $q_t \equiv \frac{Q_{t,t+\tau}}{P_t}$.

Table 1.1 gives an overview of the model calibration. The calibration of the parameters in the household and firm problems is standard and in line with Gali (2008) and Smets and Wouters (2003, 2007). The household discount factor is 0.99. The inverse of the intertemporal substitution elasticities in consumption and labour supply are 2 and 0.5 respectively. The intratemporal elasticity of substitution between consumption goods equals 6, which implies a steady-state mark-up of 20 per cent. The production function exhibits decreasing returns to scale. Price rigidity is calibrated to a relatively high value, however wages are fully flexible in the model. Steady-state inflation corresponds to an annual inflation rate of close to 2 per cent. τ is 20 so the long-term bond has a maturity of 5 years. The government spending share of GDP is set to 40 per cent. This is at the upper end for the US and at the lower end for most European countries. Half of firm profits are paid to the households, the other half to the

[12] See Section 1.5.4 in the appendix for derivations.

government, reflecting corporate taxes as well as public ownership of goods producing firms.

Parameter	Value	Description
β	0.99	Household discount factor
δ	2	Inverse elasticity of intertemporal substitution in consumption
ψ	0.5	Inverse Frisch elasticity of labour supply
θ	6	Steady-state value of intratemporal elasticity of substitution
ϕ	1.1	Inverse of returns to scale in production
α	0.85	Degree of price rigidity
Π	1.005	Steady-state inflation
τ	20	Horizon of long-term bond
$\bar{g}$	0.4	Steady-state ratio of government spending to GDP
$t_{\mathcal{P}}$	0.5	Share of firm profits received by the government
f	0.66	Parameter in long-term bond supply rule
a_1	0.95	Asset demand
a_2	0	Asset demand
g^B	10.21	Asset demand (subsistence level of B)
g^Q	0.59	Asset demand (subsistence level of Q)
ρ_ν	0.1	Persistence of shock to interest rate rule
ρ_ξ	0.1	Persistence of shock to asset purchase rule
ρ_C	0.1	Persistence of consumption preference shock
ρ_L	0.7	Persistence of labour supply preference shock
ρ_G	0.1	Persistence of government spending
ρ_A	0.7	Persistence of technology shock
ρ_θ	0.95	Persistence of shock to elasticity of substitution
σ_ν	0.0025	Standard deviation of shock to interest rate rule
σ_ξ	0.0025	Standard deviation of shock to asset purchase rule
σ_C	0.0025	Standard deviation of consumption preference shock
σ_L	0.0025	Standard deviation of labour supply preference shock
σ_G	0.005	Standard deviation of government spending shock
σ_A	0.01	Standard deviation of technology shock
σ_θ	0.06	Standard deviation of shock to elasticity of substitution

Table 1.1: Calibration

The calibration of the asset supply and demand equations generates a steady state in bond markets and a response to central bank asset purchases that is in line with empirical estimates. We target steady-state asset prices P^B and P^Q that translate into annualised interest rates on 1-period and 20-period bonds of 4.5 and 5.5 per cent respectively, in line with the average interest paid on 3-month and 5-year US Treasury Bills between the start of the Greenspan era in 1987 and the end of 2007. The only parameterisation of the issuance equation for long-term bonds that is consistent with these steady-state bond prices and parameters already calibrated is f equal to 0.66. This parameterisation means that long-term government debt accounts for an acceptable 36 per cent share of the government's total steady-state outstanding debt obligations.

A realistic response to central bank asset purchases is achieved through the calibration of the parameters a_1, a_2, g^B and g^Q in the asset demand equations. We set $a_2 = 0$ without loss of generality because the logarithmic term in the asset demand equations is redundant when the model is solved to a first-order approximation. With this, the asset demand equation for the long-term bond can be written as

$$q_t = g^Q + \frac{1-a_1}{a_1}\frac{P_t^B}{P_t^Q}(b_t - g^B)$$

and g^B and g^Q can be interpreted as the parts of demand that do not react to relative prices, sometimes referred to as the "subsistence levels" of demand in consumer theory. a_1 determines the relative marginal gain in the objective of the bank from holding the short-term rather than the long-term bond, similar to the parameter in first-order homogeneous Cobb-Douglas preferences. The values of a_1, g^B, g^Q are chosen such that the yield on the long-term bond falls by 100 basis points (bps) if the central bank makes asset purchases that reduce the present discounted value of long-term payment obligations to the private sector by 1.5 per cent. In quantitative terms, this translates to the yield on long-term debt falling by 7 bps after central bank asset purchases that remove \$100 billion of long-term payment obligations to the private sector.[13] This is consistent with the empirical estimates of a 3-15 bps fall in long-term yields per \$100 billion of asset purchases reported in various contributions, summarised in Table 1 of Chen et al. (2012).

The calibration of the shock processes emphasises the significance of shocks to government spending and the elasticity of substitution, creating a reasonable amount of volatility in macroeconomic aggregates. The standard deviation of nominal US GDP growth 1987:4-2007:4 was 0.022, which compares favourably to a standard deviation of 0.028 from simulations of the model. The model somewhat understates output volatility and overstates inflation volatility compared to US data. In Section 1.3.3, we report the contribution of each shock to total volatility in output, inflation and interest rates.

[13]We arrive at our estimate of 7 bps as follows. Asset purchases that reduce the present discounted value of coupon payments on long-term debt payable to households in the model by 0.013 correspond to a 10 bps decrease in the annualised long-term yield. GDP in the US in 2007 was approximately \$14 trillion. In our model, steady-state GDP is 1.3 so if 0.013/1.3 corresponds to 10 bps then 100/14,000 corresponds to 7 bps.

1.3 Conventional and Unconventional Monetary Policy

The possibility of purchasing long-term government debt presents the central bank with an additional tool to help stabilise the economy. To understand the implications of this, it is useful to begin with a discussion of how conventional and unconventional monetary policy actions are transmitted in the model.

1.3.1 Transmission Mechanisms

The response of the economy to an expansionary shock in the rule for the short-term nominal interest rate (1.19) is depicted in Figure 1.1; the response to an expansionary shock in the rule for long-term bond purchases (1.21) is shown in Figure 1.2. Both figures are obtained under the calibration $\gamma_\Pi = 1.01$, $\gamma_Y = 0.3$ and $\gamma_\Pi^{QE} = 0$, $\gamma_Y^{QE} = 60$ for conventional and unconventional monetary policies. A value of 60 for γ_Y^{QE} implies in steady state that a one per cent decrease in output leads to the central bank buying approximately 45 per cent of the new long-term bonds issued in that period. This amounts to less than five per cent of all outstanding long-term government debt.

The shock to the interest rate rule immediately decreases the short-term nominal interest rate and equivalently increases the price of the short-term bond. Demand for the short-term bond therefore falls. The subsequent rise in P^B also increases the price of the composite savings device available to households, leading to a decrease in savings $P^S s$ and a rise in both output and consumption. The rise in output is accompanied by inflation above its steady-state level. The shock has a less persistent impact on output than inflation because inflation induces relative price distortions that mitigate the rise in output.

The effects on the market for long-term bonds work through two channels. A decrease in savings shifts down the demand curve for the long-term bond, which puts downward pressure on its price P^Q. In addition, output and inflation rise above their respective steady-state levels so the central bank uses unconventional monetary policy to sell long-term government bonds. This increases the quantity of long-term bonds available to the private sector, further decreasing their price.

Prices quickly return to their steady-state values once the shock to the short-term nominal interest rate has passed. The temporary decline in the yield of the short-term bond gives rise to a rebalancing of the representative household's bond portfolio in favour of long-term debt. This causes the return from deposits s to decline immediately and to only gradually recover as the coupon payments from the higher level of long-term debt are collected and the share of short-term debt slowly rises again. The recovery is supported by an increase in hours worked.

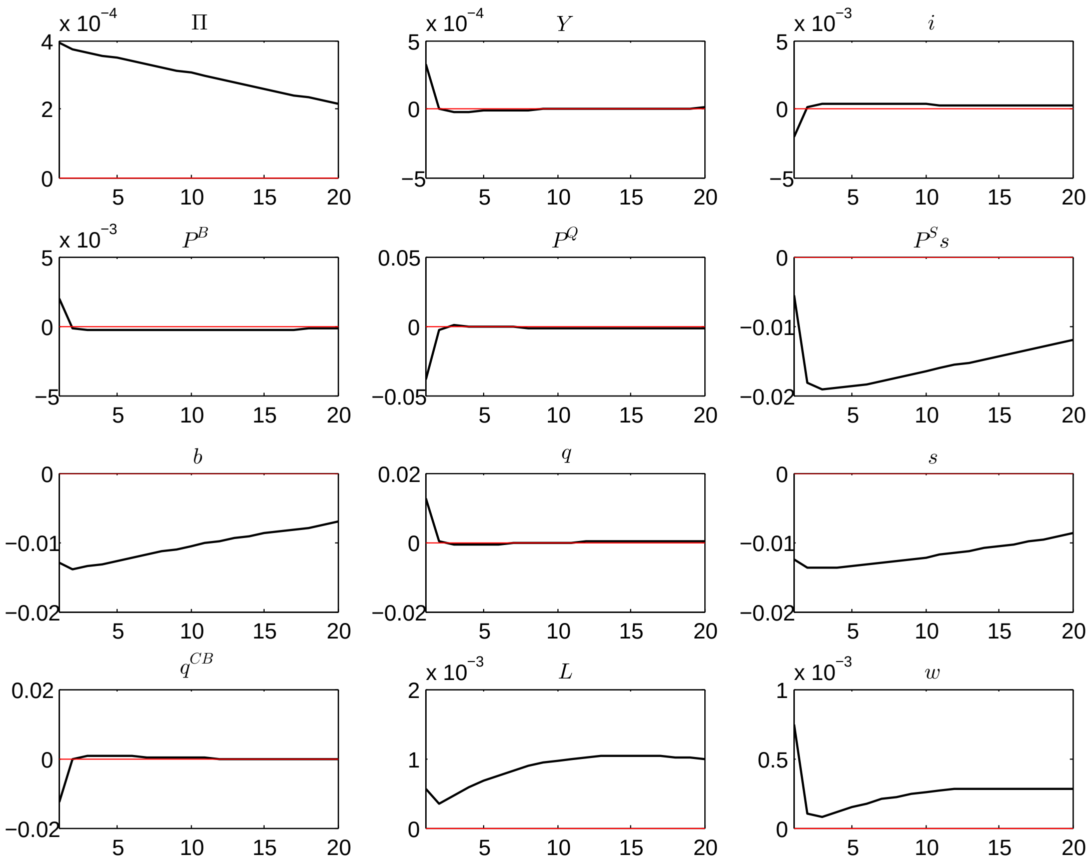

Figure 1.1: Response to an expansionary short-term nominal interest rate shock

Figure 1.2 shows the response of the economy to an expansionary shock in the central bank's long-term bond purchase rule (1.21). The unanticipated rise in central bank purchases of long-term bonds q^{CB} reduces the supply of this asset to the private sector and pushes up its price. The increase in P^Q similarly increases the price of the composite savings device P^S, causing households to save less, consume more and supply more labour to firms. Output then increases and prices grow at a faster rate. The rise in the price P^Q of the long-term bond is associated with a decline in the long-term interest rate, as can be seen from equation (1.16) defining i^Q. This corresponds to a “flattening of the yield curve” effect generally attributed to purchases of long-term government securities.

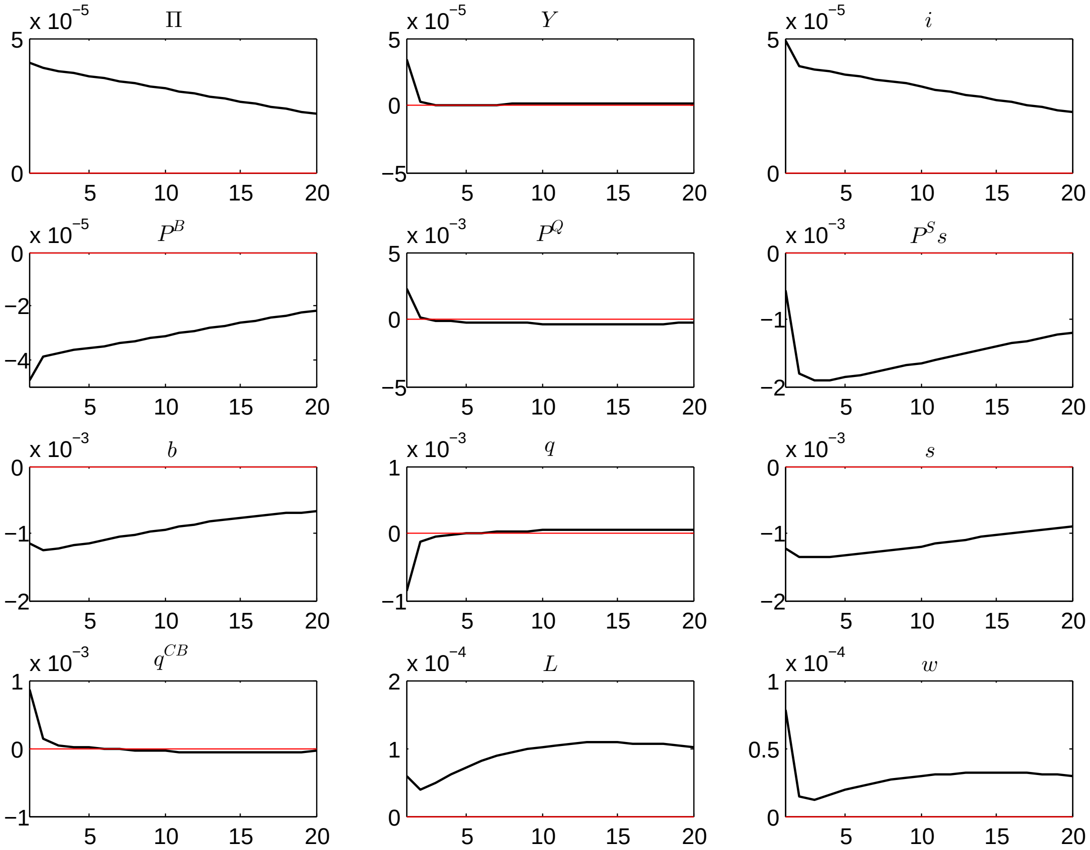

Figure 1.2: Response to an expansionary shock to purchases of the long-term bond

The decline in savings shifts the demand curves for both types of bonds inwards, leading to a fall in the quantity of the short-term bond held by the private sector. The fall in holdings is though limited because the central bank subsequently follows its interest rate rule and increases the short-term nominal interest rate, thereby reducing the price of the short-term bond and restoring some of its demand. Savings $P^S s$ only gradually return to their steady-state level after the unexpected purchases of long-term bonds. Labour supply similarly remains elevated for a number of periods, which helps bring savings back to their steady-state level and ameliorates the negative output effects that inflation causes through price dispersion.

The impulse response functions suggest that both conventional and unconventional monetary policies may be effective at stabilising the economy. In what follows, the focus is on the benefits of allowing the central bank to purchase long-run government bonds as an unconventional policy that acts as a supplement to conventional short-term interest rate policy.

1.3.2 Optimised Policy Rules

Let the central bank loss function be

$$\Omega = \omega_\Pi \mathrm{Var}(\Pi_t - \Pi) + \omega_Y \mathrm{Var}(Y_t - Y) + \omega_i \mathrm{Var}(i_t - i) + \omega_Q \mathrm{Var}(i_t^Q - i^Q)$$

where ω_Π, ω_Y, ω_i and ω_Q are positive weights. Losses are therefore a weighted sum of the variances of inflation, output and interest rates.[14] This specification allows for the volatility of both short-term and long-term interest rates to enter the central bank's considerations, although in the baseline case we set ω_i and ω_Q to zero.

The central bank sets the interest rate rule parameters γ_Π and γ_Y together with the asset purchase rule parameters γ_Π^{QE} and γ_Y^{QE}, subject to the constraints imposed by the model and assumed feasibility constraints $\gamma_\Pi \in [1, 6]$, $\gamma_Y \in [0, 6]$ and $\gamma_\Pi^{QE}, \gamma_Y^{QE} \in [0, 75]$. We allow for a high upper bound on the set of feasible interest rate rule parameters so that the benefits of unconventional monetary model do not arise solely when the feasibility constraints for γ_Π and γ_Y are binding. A lower upper bound on the set of interest rate rule parameters would likely make unconventional policy even more effective, as the central bank would then find it more difficult to stabilise the economy through conventional policy.

Table 1.2 shows optimised policy rules for different weights on inflation and output stabilisation in the central bank's loss function, with ω_i and ω_Q set to zero so the central bank is not concerned about stabilising interest rates. For each set of weights, the first row presents results when the central bank uses only conventional monetary policy. In this baseline case, coefficients on the asset purchase rule are set to zero and only the coefficients on the interest rate rule are optimised. For the next case, the central bank fixes the coefficients of the interest rate rule

[14] Woodford (2003) shows that, in the presence of transactions frictions, a welfare-theoretic loss function derived from a second-order approximation to household utility contains an interest-rate-volatility term.

at their baseline values, but optimises over the coefficients in the asset purchase rule. This experiment shows the improvements attainable if the central bank starts purchasing long-term government bonds without internalising the potential effect this has on optimal interest rate policy. In the final case, the central bank acknowledges the complementarity of conventional and unconventional policies by jointly optimising over the parameters in the interest rate and asset purchase rules.

Weights (%) (ω_Π, ω_Y, ω_i, ω_Q)	Interest rate rule (γ_Π, γ_Y)	Asset purchase rule (γ_Π^{QE}, γ_Y^{QE})	Var(Π_t) $\times 10^{-5}$	Var(Y_t) $\times 10^{-5}$	Var(i_t) $\times 10^{-4}$	Var(i_t^Q) $\times 10^{-4}$	Loss	$\Delta\pi^*$ equ. (%)
70, 30, 0, 0	1.48, 2.22	0(f), 0(f)	9.33	1.43	1.59	0.85	6.96	
	1.48(f), 2.22(f)	0.05, 1.87	9.36	1.16	1.62	0.60	6.90	
	1.66, 0.00	0.00, 18.59	7.71	1.35	2.16	1.16	5.80	1.64
80, 20, 0, 0	1.49, 2.16	0(f), 0(f)	9.30	1.53	1.58	0.88	7.75	
	1.49(f), 2.16(f)	0.04, 1.78	9.29	1.23	1.61	0.62	7.68	
	1.67, 0.00	0.00, 18.22	7.68	1.42	2.18	1.14	6.43	1.63
90, 10, 0, 0	1.49, 2.11	0(f), 0(f)	9.28	1.62	1.58	0.91	8.52	
	1.49(f), 2.11(f)	0.04, 1.70	9.24	1.30	1.60	0.64	8.44	
	1.67, 0.00	0.00, 17.92	7.67	1.48	2.20	1.13	7.05	1.63

(f): fixed, i.e. not chosen optimally

Table 1.2: Optimised policy rules without interest rate stabilisation

The benefits of unconventional monetary policy are small if the central bank's interest rate rule is not adjusted to account for purchases of long-term government bonds. However, when both policies are jointly optimised then the ability of the central bank to stabilise inflation and output is greatly improved. The use of unconventional monetary policy reduces losses by more than 17 per cent compared to the baseline case, provided conventional monetary policy is adjusted accordingly. The optimal policy mix is dichotomised, assigning conventional short-term interest rate policy to react to inflation and unconventional asset purchasing policy to react to output. An important prerequisite for this result is that conventional and unconventional policies differ in their impact on inflation and output, so interventions on one margin are not exactly offset at another margin.

The final column of Table 1.2 provides an interpretation of welfare losses in terms of an equivalent change in annualised steady-state inflation, using a metric due to Jensen (2002). For example, comparing the losses in the third row to those in the first row shows that not allowing the central bank to make asset purchases causes welfare losses equivalent to a rise in steady-state inflation of 1.64 percentage points. The derivation of the equivalent steady-state inflation measure is explained in Section 1.5.5 of the appendix.

Table 1.3 suggests that unconventional monetary policy is more effective when the central bank is also concerned about the volatility of interest rates. If only variability in the short-term interest rate is costly then the unconventional policy of purchasing long-term government bonds almost completely replaces conventional short-term interest rate policy. The optimal interest rate rule parameters are at the respective lower bounds of the feasibility sets. Large gains arise because long-term interest rate volatility from asset purchases is costless and substituting conventional policy with asset purchases avoids the losses that arise from movements in the short-term interest rate. The final part of the table shows that the gains from the unconventional tool are still significant when there is a positive weight on the variance of the long-term interest rate in the loss function. These gains are slightly smaller than in the previous case, but still larger than in Table 2 when the central bank is not concerned with interest rate volatility.

Weights (%) (ω_Π, ω_Y, ω_i, ω_Q)	Interest rate rule (γ_Π, γ_Y)	Asset purchase rule (γ_Π^{QE}, γ_Y^{QE})	Var(Π_t) $\times 10^{-5}$	Var(Y_t) $\times 10^{-5}$	Var(i_t) $\times 10^{-4}$	Var(i_t^Q) $\times 10^{-4}$	Loss	$\Delta\pi^*$ equ. (%)
70, 20, 10, 0	1.55, 2.26	0(f), 0(f)	9.31	1.58	1.55	0.91	8.38	
	1.55(f), 2.26(f)	1.78, 4.33	9.50	1.60	1.18	1.07	8.15	
	1.00, 0.00	4.34, 16.14	8.45	1.31	0.89	1.53	7.07	1.74
70, 10, 10, 10	1.51, 2.27	0(f), 0(f)	9.34	1.43	1.58	0.86	9.12	
	1.51(f), 2.27(f)	0.69, 4.07	9.46	1.10	1.48	0.61	8.82	
	1.36, 0.40	1.10, 11.66	8.24	1.50	1.46	0.54	7.91	1.67

(f): fixed, i.e. not chosen optimally

Table 1.3: Optimised policy rules with interest rate stabilisation

1.3.3 Loss Decomposition

To show the effect of the different policy regimes on the losses due to each type of shock, we focus on the case where the weights in the loss function are given by $(\omega_\Pi, \omega_Y, \omega_i, \omega_Q) = (80, 20, 0, 0)$. In Table 1.4 the variances of all but one shock are set to zero. The shock of interest is indicated in the first column and has its variance calibrated as in Table 1.1. Policy parameters, when not set to zero, take the value that is optimal when all shocks are operating jointly.

A large proportion of the overall welfare gains arise because combining conventional and unconventional policy is particularly effective at stabilising the effects of shocks to government spending and the elasticity of substitution. Table 1.1 shows these shocks make a significant contribution to volatility in the model. Unconventional policy has smaller but noticeable effects on volatility due to shocks to the interest rate rule, consumption preferences and productivity. It does not perform significantly better in the face of labour supply and bond purchase shocks. In general, the volatility of both output and inflation falls when unconventional policy is introduced alongside conventional policy. The exception is if there are only shocks to the interest rate rule and technology, in which case the fall in inflation volatility comes at the cost of a small rise in the volatility of output.

Shock	Interest rate rule (γ_Π, γ_Y)	Asset purchase rule (γ_Π^{QE}, γ_Y^{QE})	Var(Π_t) $\times 10^{-5}$	Var(Y_t) $\times 10^{-5}$	Var(i_t) $\times 10^{-4}$	Var(i_t^Q) $\times 10^{-4}$	Loss
ν	1.49, 2.16	0, 0	0.511	0.057	0.065	0.050	0.420
	1.49, 2.16	0.04, 1.78	0.485	0.047	0.067	0.039	0.397
	1.67, 0.00	0.00, 18.22	0.433	0.065	0.145	0.047	0.359
ξ	1.49, 2.16	0, 0	0.005	0.000	0.001	0.005	0.004
	1.49, 2.16	0.04, 1.78	0.005	0.000	0.001	0.005	0.004
	1.67, 0.00	0.00, 18.22	0.005	0.000	0.001	0.003	0.004
ε^C	1.49, 2.16	0, 0	0.218	0.077	0.028	0.021	0.190
	1.49, 2.16	0.04, 1.78	0.243	0.071	0.032	0.018	0.209
	1.67, 0.00	0.00, 18.22	0.144	0.075	0.041	0.095	0.130
ε^L	1.49, 2.16	0, 0	0.015	0.002	0.002	0.002	0.012
	1.49, 2.16	0.04, 1.78	0.015	0.001	0.002	0.001	0.012
	1.67, 0.00	0.00, 18.22	0.014	0.002	0.004	0.001	0.012
ε^G	1.49, 2.16	0, 0	1.551	0.544	0.196	0.152	1.350
	1.49, 2.16	0.04, 1.78	1.729	0.502	0.226	0.128	1.484
	1.67, 0.00	0.00, 18.22	1.026	0.532	0.288	0.678	0.927
ε^A	1.49, 2.16	0, 0	0.663	0.068	0.070	0.072	0.544
	1.49, 2.16	0.04, 1.78	0.642	0.052	0.075	0.054	0.524
	1.67, 0.00	0.00, 18.22	0.615	0.077	0.173	0.023	0.508
θ	1.49, 2.16	0, 0	6.337	0.781	1.222	0.578	5.226
	1.49, 2.16	0.04, 1.78	6.170	0.561	1.209	0.374	5.048
	1.67, 0.00	0.00, 18.22	5.445	0.668	1.529	0.293	4.489

Table 1.4: Losses by type of shock

The benefits from unconventional policy can be seen in Figure 1.3, which compares the impulse responses of a shock to the elasticity of substitution under the baseline policy (dashed line) and the optimal combined conventional and unconventional policy (solid line). Inflation reacts less to the shock when the central bank purchases government bonds. Output initially reacts more, but overall is less volatile.

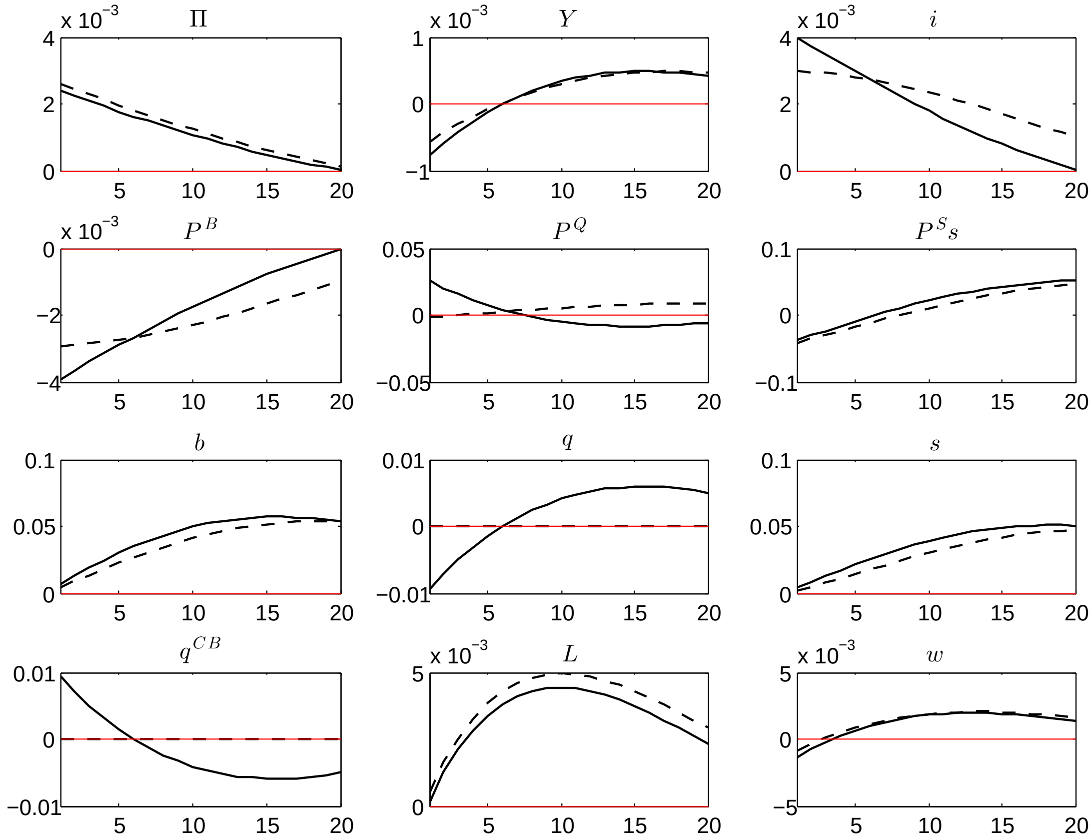

Figure 1.3: Response to a shock to the elasticity of substitution for short-term interest policy only (dashed) and with conventional and unconventional policy (solid)

1.4 Conclusion

Our results call for central banks to consider purchasing long-term government bonds, even when the economy is not in financial crisis. Unconventional asset purchases of this type are a valuable addition to the tool kit of a central bank trying to stabilise inflation and output, whatever the state of the economy and not just when the short-term nominal interest rate is constrained by the zero lower bound. In order to reap the full benefit, though, it is important to coordinate unconventional and conventional monetary policy. In our model, this means that the short-term interest rate should respond to inflation while the central bank's purchases of long-term debt should respond to output. Unconventional monetary policy plays an even more important role if the central bank is additionally concerned about interest rate volatility. The 'Maturity Extension Program and Reinvestment Policy" announced by the Federal Reserve in September 2011 is an example of the type of unconventional monetary policy measure we advocate.[15] Under this programme, the Federal Reserve lengthens the average maturity of its government bond portfolio by selling short-term treasuries and re-investing the proceeds in long-term treasuries. Our results suggest that such operations may become increasingly important even when financial crisis has passed.

The benefit of unconventional monetary policy in our model is indicative of more general gains available from the coordination of monetary policy and the debt management of fiscal authorities. In our model, central bank purchases of long-term government bonds are equivalent to a reduced issuance of these bonds by the fiscal authority. Whilst our results abstract from active debt management by the treasury, Borio and Disyatat (2009) argues that fiscal authorities may be tempted to reduce debt financing costs by increasing the maturity of new debt at a time when long-term interest rates are relatively low. This is problematic, because it implies that any attempt by the central bank to use unconventional asset purchases to stimulate the economy may be unwound by the treasury issuing more long-term government debt.

[15] In reference to a comparable programme from the 1960s, this policy is sometimes referred to as "Operation Twist" or "Operation Twist Again."

1.5 Appendix

1.5.1 Firms

The profit maximisation problem of a firm i that can adjust its price in period t is

$$\max_{P_t(i)} \; E_t \sum_{T=t}^{\infty} \alpha^{T-t} M_{t,T} \left[P_t(i) Y_T(i) - W_T L_T(i) \right] \tag{1.27}$$

subject to

$$Y_T(i) = Y_T \left[\frac{P_t(i)}{P_T} \right]^{-\theta_t} \tag{1.28}$$

Equation (1.28) is the aggregate demand for firm i's product.[16] Firm i's production function implies that $L_T(i) = \left[\frac{Y_T(i)}{A_T} \right]^{\phi}$ which, together with (1.28), can be used to rewrite i's optimisation problem as

$$\max_{P_t(i)} \; E_t \sum_{T=t}^{\infty} \alpha^{T-t} M_{t,T} \left\{ P_t(i) Y_T \left[\frac{P_t(i)}{P_T} \right]^{-\theta_t} - W_T \left(\frac{Y_T}{A_T} \right)^{\phi} \left[\frac{P_t(i)}{P_T} \right]^{-\theta_t \phi} \right\} \tag{1.29}$$

The first-order condition is

$$E_t \sum_{T=t}^{\infty} \alpha^{T-t} M_{t,T} \left\{ Y_T \left[\frac{P_t^*(i)}{P_T} \right]^{-\theta_t} - \theta_t Y_T \left[\frac{P_t^*(i)}{P_T} \right]^{-\theta_t} + \theta_t \phi \frac{W_T}{P_T} \left(\frac{Y_T}{A_T} \right)^{\phi} \left[\frac{P_t^*(i)}{P_T} \right]^{-\theta_t \phi - 1} \right\} = 0 \tag{1.30}$$

By substituting in for the real wage from (1.5), and using the definition of the stochastic discount factor, the first-order condition can be expressed as

$$\begin{aligned} & E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \chi_T^C C_T^{-\delta} Y_T \left(\frac{P_T}{P_t} \right)^{\theta_t - 1} \\ &= \left[\frac{P_t}{P_t^*(i)} \right]^{\theta_t \phi + 1 - \theta_t} E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \frac{\theta_t \phi}{\theta_t - 1} \chi_T^L L_T^{\psi} \left(\frac{Y_T}{A_T} \right)^{\phi} \left(\frac{P_T}{P_t} \right)^{\theta_t \phi} \end{aligned} \tag{1.31}$$

Defining

$$F_t \;\; \equiv \;\; E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \chi_T^C C_T^{-\delta} Y_T \left(\frac{P_T}{P_t} \right)^{\theta_t - 1} \tag{1.32}$$

[16] This demand schedule is standard in models with monopolistically competitive producers, see Woodford (2003) for a derivation.

$$K_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \frac{\theta_t \phi}{\theta_t - 1} \chi_T^L L_T^{\psi} \left(\frac{Y_T}{A_T}\right)^{\phi} \left(\frac{P_T}{P_t}\right)^{\theta_t \phi} \tag{1.33}$$

followed by rearranging yields

$$\frac{P_t^*(i)}{P_t} = \left(\frac{K_t}{F_t}\right)^{\frac{1}{\theta_t(\phi-1)+1}} \tag{1.34}$$

Benigno and Woodford (2005) show in an analogous problem that (1.34) is both necessary and sufficient for an optimum. Note that F_t and K_t can be written recursively:

$$F_t = \chi_t^C C_t^{-\delta} Y_t + \alpha\beta E_t \Pi_{t+1}^{\theta_t - 1} F_{t+1} \tag{1.35}$$

$$K_t = \frac{\theta_t \phi}{\theta_t - 1} \chi_t^L L_t^{\psi} \left(\frac{Y_t}{A_t}\right)^{\phi} + \alpha\beta E_t \Pi_{t+1}^{\theta_t \phi} K_{t+1} \tag{1.36}$$

The above equations are those shown in the paper.

The price level in the economy is

$$P_t = \left[(1-\alpha) P_t^*(i)^{1-\theta_t} + \alpha P_{t-1}^{1-\theta_t}\right]^{\frac{1}{1-\theta_t}} \tag{1.37}$$

Rearranging yields

$$\frac{P_t^*(i)}{P_t} = \left(\frac{1 - \alpha \Pi_t^{\theta_t - 1}}{1-\alpha}\right)^{\frac{1}{1-\theta_t}} \tag{1.38}$$

Combining the above equation with (1.34) gives

$$\frac{1 - \alpha \Pi_t^{\theta_t - 1}}{1-\alpha} = \left(\frac{F_t}{K_t}\right)^{\frac{\theta_t - 1}{\theta_t(\phi-1)+1}} \tag{1.39}$$

which is the same as equation (1.7) in Section 1.2.2.

1.5.2 Asset Demand Equations

In the GTL model, the asset shares $as^B \equiv \frac{P^B B}{P^S S}$ and $as^Q \equiv \frac{P^Q Q}{P^S S} = 1 - as^B$ are

$$as_t^k = \frac{P_t^k g^k}{P_t^S s_t} + \left(1 - \frac{P_t^B g^B + P_t^Q g^Q}{P_t^S s_t}\right) \frac{a_1^k + \sum_l a_2^{kl} \log\left(\frac{P_t^l}{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}\right)}{\sum_l a_1^l + \sum_k \sum_l a_2^{kl} \log\left(\frac{P_t^l}{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}\right)} \tag{1.40}$$

where $k, l \in \{B, Q\}$ and symmetry requires that $a_2^{kl} = a_2^{lk}$.

Demand curves are quasi-homothetic if they have linear Engel curves, i.e. the demand for an asset increases linearly with income. We impose quasi-homotheticity to limit wealth effects, which implies that $a_2^{BB} + a_2^{QB} = 0$ and $a_2^{BQ} + a_2^{QQ} = 0$. Together with symmetry this implies that $a_2^{BB} = a_2^{QQ} = -a_2^{BQ} = -a_2^{QB}$. Making the common normalisation $a_1^B + a_1^Q = 1$ and defining $a_1 \equiv a_1^B$ and $a_2 \equiv a_2^{BB}$, the asset shares are

$$
\begin{aligned}
as_t^B &= \frac{P_t^B g^B}{P_t^S s_t} + \left(1 - \frac{P_t^B g^B + P_t^Q g^Q}{P_t^S s_t}\right) \times \\
&\left[a_1 + a_2 \log\left(\frac{P_t^B}{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}\right) - a_2 \log\left(\frac{P_t^Q}{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}\right)\right] \qquad (1.41)\\
as_t^Q &= \frac{P_t^Q g^Q}{P_t^S s_t} + \left(1 - \frac{P_t^B g^B + P_t^Q g^Q}{P_t^S s_t}\right) \times \\
&\left[1 - a_1 - a_2 \log\left(\frac{P_t^B}{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}\right) + a_2 \log\left(\frac{P_t^Q}{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}\right)\right] \qquad (1.42)
\end{aligned}
$$

which after rearranging is

$$
\frac{B_{t,t+1}}{P_t} = g^B + \frac{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}{P_t^B}\left[a_1 + a_2 \log\left(\frac{P_t^B}{P_t^Q}\right)\right] \qquad (1.43)
$$

$$
\frac{Q_{t,t+\tau}}{P_t} = g^Q + \frac{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}{P_t^Q}\left[1 - a_1 - a_2 \log\left(\frac{P_t^B}{P_t^Q}\right)\right] \qquad (1.44)
$$

Marshallian demands satisfy i) positivity, ii) adding up, iii) homogeneity of degree zero in prices and income and iv) symmetry and negative semi-definiteness of the matrix of substitution effects. Thus, the above demand system is only consistent with the firm's maximisation problem stated in section 1.2.3 if it satisfies these four "integrability conditions" (Barnett and Serletis, 2008). (1.43) and (1.44) satisfy ii) and iii), i) will be satisfied by an adequate calibration and iv) remains to be checked post simulation.

1.5.3 Government Budget Constraint

The budget constraint of the representative household is given by

$$P_t C_t + T_t + P_t^S S_{t,t+1} = S_{t-1,t} + W_t L_t + (1 - t_\pi)(P_t Y_t - W_t L_t) \tag{1.45}$$

Additionally, we have

$$\begin{aligned} Y_t &= C_t + G_t & (1.46) \\ P_t^S S_{t,t+1} &= P_t^B B_{t,t+1} + P_t^Q Q_{t,t+\tau} & (1.47) \\ S_{t-1,t} &= B_{t-1,t} + \frac{1}{\tau}\sum_{j=1}^{\tau} Q_{t-j,t+\tau-j} & (1.48) \\ \bar{Q}_{t,t+\tau} &= Q_{t,t+\tau} + Q_{t,t+\tau}^{CB} & (1.49) \end{aligned}$$

Central bank profits from asset purchases are given by

$$\pi_t^{CB} \equiv \frac{1}{\tau}\sum_{j=1}^{\tau} Q_{t-j,t+\tau-j}^{CB} - P_t^Q Q_{t,t+\tau}^{CB} \tag{1.50}$$

and firm profits are equal to $\int_0^1 [P_t(i)Y_t(i) - W_t L_t(i)] = P_t Y_t - W_t L_t$.

Using the relationships listed above, the household budget constraint can be re-written as follows.

$$\begin{aligned} & P_t C_t + T_t + P_t^S S_{t,t+1} \\ & = S_{t-1,t} + W_t L_t + (1 - t_\pi)(P_t Y_t - W_t L_t) & (1.51) \\ & P_t C_t + T_t + P_t^B B_{t,t+1} + P_t^Q Q_{t,t+\tau} + t_\pi (P_t Y_t - W_t L_t) \\ & = B_{t-1,t} + \frac{1}{\tau}\sum_{j=1}^{\tau} Q_{t-j,t+\tau-j} + P_t Y_t & (1.52) \\ & T_t + P_t^B B_{t,t+1} + P_t^Q (\bar{Q}_{t,t+\tau} - Q_{t,t+\tau}^{CB}) + t_\pi (P_t Y_t - W_t L_t) \\ & = B_{t-1,t} + \frac{1}{\tau}\sum_{j=1}^{\tau} (\bar{Q}_{t-j,t+\tau-j} - Q_{t-j,t+\tau-j}^{CB}) + P_t G_t & (1.53) \\ & T_t + P_t^B B_{t,t+1} + P_t^Q \bar{Q}_{t,t+\tau} + t_\pi (P_t Y_t - W_t L_t) + \pi_t^{CB} \end{aligned}$$

$$= B_{t-1,t} + \frac{1}{\tau}\sum_{j=1}^{\tau} \bar{Q}_{t-j,t+\tau-j} + P_t G_t \tag{1.54}$$

This is the government budget constraint shown in the text.

1.5.4 Price Dispersion

The labour market clearing condition is

$$L_t = \int_0^1 L_t(i)di \tag{1.55}$$

or, using the production function of firm i

$$L_t = \int_0^1 \left[\frac{Y_t(i)}{A_t}\right]^{\phi} di \tag{1.56}$$

Together with (1.28) this implies

$$L_t = \int_0^1 \left\{\frac{Y_t}{A_t}\left[\frac{P_t(i)}{P_t}\right]^{-\theta_t}\right\}^{\phi} di = \left(\frac{Y_t}{A_t}\right)^{\phi} \int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t \phi} di \tag{1.57}$$

Defining $D_t \equiv \int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t \phi} di$ and rearranging yields

$$Y_t = A_t \left(\frac{L_t}{D_t}\right)^{\frac{1}{\phi}} \tag{1.58}$$

The law of motion of price dispersion (1.26) is derived as follows. Since only a share $1-\alpha$ of firms adjusts their price in every period, the dispersion of prices in t can be written

$$\int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t \phi} di = (1-\alpha)\left[\frac{P_t^*(i)}{P_t}\right]^{-\theta_t \phi} + \alpha \int_0^{\alpha} \left[\frac{P_{t-1}(i)}{P_t}\right]^{-\theta_t \phi} di \tag{1.59}$$

where without loss of generality it is assumed that the firms that are not able to reset their price in t are those indexed by $i \in [0, \alpha]$. After simple manipulations one obtains

$$\int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t \phi} di = (1-\alpha)\left[\frac{P_t^*(i)}{P_t}\right]^{-\theta_t \phi} + \alpha \Pi_t^{\theta_t \phi} \int_0^{\alpha} \left[\frac{P_{t-1}(i)}{P_{t-1}}\right]^{-\theta_t \phi} di \tag{1.60}$$

Since the distribution of prices among the firms that are unable to change their price in $t-1$ is the same as the distribution of all prices in $t-1$, the above equation becomes

$$\int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t\phi} di = (1-\alpha)\left[\frac{P_t^*(i)}{P_t}\right]^{-\theta_t\phi} + \alpha\Pi_t^{\theta_t\phi}\int_0^1 \left[\frac{P_{t-1}(i)}{P_{t-1}}\right]^{-\theta_t\phi} di \tag{1.61}$$

and adequate substitutions yield the final result

$$D_t = (1-\alpha)\left(\frac{1-\alpha\Pi_t^{\theta_t-1}}{1-\alpha}\right)^{\frac{\theta_t\phi}{\theta_t-1}} + \alpha\Pi_t^{\theta_t\phi}D_{t-1} \tag{1.62}$$

1.5.5 Loss Conversion

We define the loss from the "baseline" policy as

$$\Omega^b = \omega_\Pi \mathrm{Var}(\Pi_t^b - \Pi^b) + \omega_Y \mathrm{Var}(Y_t^b - Y^b) + \omega_i \mathrm{Var}(i_t^b - i^b) + \omega_Q \mathrm{Var}(i_t^{Q,b} - i^{Q,b}) \tag{1.63}$$

and similarly the loss from an alternative policy that we would like to compare with the baseline policy as

$$\begin{aligned} \Omega^a &= \omega_\Pi \mathrm{Var}(\Pi_t^a - \Pi^a) + \omega_Y \mathrm{Var}(Y_t^a - Y^a) + \omega_i \mathrm{Var}(i_t^a - i^a) + \omega_Q \mathrm{Var}(i_t^{Q,a} - i^{Q,a}) &(1.64)\\ &= \omega_\Pi \mathrm{E}(\Pi_t^a - \Pi^a)^2 + \omega_Y \mathrm{Var}(Y_t^a - Y^a) + \omega_i \mathrm{Var}(i_t^a - i^a) + \omega_Q \mathrm{Var}(i_t^{Q,a} - i^{Q,a}) &(1.65)\end{aligned}$$

The equivalent change in steady state inflation $\Delta\pi^*$ is then implicitly defined as

$$\omega_\Pi \mathrm{E}\left[\Pi_t^a + \Delta\pi^* - \Pi^a\right]^2 + \omega_Y \mathrm{Var}(Y_t^a - Y^a) + \omega_i \mathrm{Var}(i_t^a - i^a) + \omega_Q \mathrm{Var}(i_t^{Q,a} - i^{Q,a}) = \Omega^b \tag{1.66}$$

the constant additional amount of inflation that would raise the loss from the alternative policy to the level of the baseline policy. Solving for $\Delta\pi^*$ gives

$$\begin{aligned} &\omega_\Pi \left[\mathrm{E}(\Pi_t^a - \Pi^a)^2 + (\Delta\pi^*)^2\right] + \omega_Y \mathrm{Var}(Y_t^a - Y^a) + \omega_i \mathrm{Var}(i_t^a - i^a) + \omega_Q \mathrm{Var}(i_t^{Q,a} - i^{Q,a}) \\ &= \Omega^b &(1.67)\\ &\Omega^a + \omega_\Pi(\Delta\pi^*)^2 = \Omega^b &(1.68)\end{aligned}$$

$$\Delta\pi^* = \sqrt{\frac{\Omega^b - \Omega^a}{\omega^\Pi}} \tag{1.69}$$

which when annualised is the measure used in the text.

1.5.6 Stationary Model

Definitions: $w_t \equiv \frac{W_t}{P_t}$, $s_t \equiv \frac{S_{t,t+1}}{P_t}$, $b_t \equiv \frac{B_{t,t+1}}{P_t}$, $q_t \equiv \frac{Q_{t,t+\tau}}{P_t}$, $\bar{q}_t \equiv \frac{\bar{Q}_{t,t+\tau}}{P_t}$, $q_t^{CB} \equiv \frac{Q^{CB}_{t,t+\tau}}{P_t}$

$$C_t + P_t^S s_t + G_t = \frac{s_{t-1}}{\Pi_t} + t_{\mathcal{P}} w_t L_t + (1 - t_{\mathcal{P}})Y_t \tag{1.70}$$

$$1 = \beta E_t \left[\frac{\chi_{t+1}^C}{\chi_t^C} \left(\frac{C_{t+1}}{C_t} \right)^{-\delta} \frac{1}{\Pi_{t+1}} \right] \frac{1}{P_t^S} \tag{1.71}$$

$$w_t = \frac{\chi_t^L}{\chi_t^C} L_t^\psi C_t^\delta \tag{1.72}$$

$$\frac{1 - \alpha \Pi_t^{\theta_t - 1}}{1 - \alpha} = \left(\frac{F_t}{K_t} \right)^{\frac{\theta_t - 1}{\theta_t(\phi - 1) + 1}} \tag{1.73}$$

$$F_t = \chi_t^C C_t^{-\delta} Y_t + \alpha\beta E_t \Pi_{t+1}^{\theta_t - 1} F_{t+1} \tag{1.74}$$

$$K_t = \frac{\theta_t \phi}{\theta_t - 1} \chi_t^L L_t^\psi \left(\frac{Y_t}{A_t} \right)^\phi + \alpha\beta E_t \Pi_{t+1}^{\theta_t \phi} K_{t+1} \tag{1.75}$$

$$s_t = b_t + \frac{1}{\tau} \left(q_t + \sum_{k=1}^{\tau - 1} \frac{q_{t-k}}{\prod_{j=0}^{k-1} \Pi_{t-j}} \right) \tag{1.76}$$

$$1 + i_t = \frac{1}{P_t^B} \tag{1.77}$$

$$P_t^Q = \frac{1}{\tau} \frac{1}{1 + i_t^Q} \frac{1 - \left(\frac{1}{1 + i_t^Q} \right)^\tau}{1 - \frac{1}{1 + i_t^Q}} \tag{1.78}$$

$$b_t = g^B + \frac{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}{P_t^B} \left[a_1 + a_2 \log \left(\frac{P_t^B}{P_t^Q} \right) \right] \tag{1.79}$$

$$q_t = g^Q + \frac{P_t^S s_t - P_t^B g^B - P_t^Q g^Q}{P_t^Q} \left[1 - a_1 - a_2 \log \left(\frac{P_t^B}{P_t^Q} \right) \right] \tag{1.80}$$

$$\bar{q}_t = fY \tag{1.81}$$

$$\frac{1 + i_t}{1 + i} = \left(\frac{\Pi_t}{\Pi} \right)^{\gamma_\Pi} \left(\frac{Y_t}{Y} \right)^{\gamma_Y} \nu_t \tag{1.82}$$

$$\frac{\bar{q}_t - q_t^{CB}}{\bar{q}_t} = \left(\frac{\Pi_t}{\Pi} \right)^{\gamma_\Pi^{QE}} \left(\frac{Y_t}{Y} \right)^{\gamma_Y^{QE}} \xi_t \tag{1.83}$$

$$
\begin{aligned}
\bar{q}_t &= q_t + q_t^{CB} && (1.84)\\
Y_t &= C_t + G_t && (1.85)\\
Y_t &= A_t \left(\frac{L_t}{D_t}\right)^{\frac{1}{\phi}} && (1.86)\\
D_t &= (1-\alpha)\left(\frac{1-\alpha\Pi_t^{\theta_t - 1}}{1-\alpha}\right)^{\frac{\theta_t\phi}{\theta_t - 1}} + \alpha\Pi_t^{\theta_t\phi} D_{t-1} && (1.87)\\
\ln(\chi_t^C) &= \rho_C \ln(\chi_{t-1}^C) + \varepsilon_t^C && (1.88)\\
\ln(\chi_t^L) &= \rho_L \ln(\chi_{t-1}^L) + \varepsilon_t^L && (1.89)\\
\ln(A_t) &= \rho_A \ln(A_{t-1}) + \varepsilon_t^A && (1.90)\\
\ln\left(\frac{\theta_t}{\theta}\right) &= \rho_\theta \ln\left(\frac{\theta_{t-1}}{\theta}\right) + \varepsilon_t^\theta && (1.91)\\
\ln\left(\frac{G_t}{G}\right) &= \rho_G \ln\left(\frac{G_{t-1}}{G}\right) + \varepsilon_t^G && (1.92)\\
\ln(\nu_t) &= \rho_\nu \ln(\nu_{t-1}) + \varepsilon_t^\nu && (1.93)\\
\ln(\xi_t) &= \rho_\xi \ln(\xi_{t-1}) + \varepsilon_t^\xi && (1.94)
\end{aligned}
$$

1.5.7 Steady State

This section describes the steady state of the stationary model summarised in 1.5.6.

$$
\begin{aligned}
A &= \chi^C = \chi^L = \nu = \xi = 1 && (1.95)\\
D &= \frac{1-\alpha}{1-\alpha\Pi^{\theta\phi}}\left(\frac{1-\alpha\Pi^{\theta-1}}{1-\alpha}\right)^{\frac{\theta\phi}{\theta-1}} && (1.96)\\
Y &= \left[\frac{1-\alpha\beta\Pi^{\theta-1}}{1-\alpha\beta\Pi^{\theta\phi}}\frac{\theta\phi}{\theta-1}D^\psi (1-\bar{g})^\delta \left(\frac{1-\alpha\Pi^{\theta-1}}{1-\alpha}\right)^{\frac{\theta(\phi-1)+1}{\theta-1}}\right]^{\frac{1}{1-\delta-\phi(\psi+1)}} && (1.97)\\
G &= \bar{g}Y && (1.98)\\
C &= Y - G && (1.99)\\
F &= \frac{YC^{-\delta}}{1-\alpha\beta\Pi^{\theta-1}} && (1.100)\\
K &= \frac{1}{1-\alpha\beta\Pi^{\theta\phi}}\frac{\theta\phi}{\theta-1}D^\psi Y^{\phi(\psi+1)} && (1.101)\\
L &= DY^\phi && (1.102)\\
w &= L^\psi C^\delta && (1.103)
\end{aligned}
$$

$$
\begin{aligned}
P^S &= \frac{\beta}{\Pi} && (1.104)\\
q^{CB} &= 0 && (1.105)\\
\bar{q} &= fY && (1.106)\\
q &= \bar{q} && (1.107)\\
s &= \frac{\Pi}{1-\beta}\left[C + G - t_{\mathcal{P}} wL - (1-t_{\mathcal{P}})\, Y\right] && (1.108)\\
b &= s - \frac{q}{\tau}\left(1 - \frac{1}{\Pi^{\tau}}\right)\frac{\Pi}{\Pi - 1} && (1.109)
\end{aligned}
$$

The asset demand equations (1.110) and (1.111) jointly determine P^B and P^Q

$$
\begin{aligned}
b &= g^B + \frac{P^S s - P^B g^B - P^Q g^Q}{P^B}\left[a_1 + a_2 \log\left(\frac{P^B}{P^Q}\right)\right] && (1.110)\\
q &= g^Q + \frac{P^S s - P^B g^B - P^Q g^Q}{P^Q}\left[1 - a_1 - a_2 \log\left(\frac{P^B}{P^Q}\right)\right] && (1.111)
\end{aligned}
$$

Finally,

$$
i = \frac{1}{P^B} - 1 \qquad (1.112)
$$

and i^Q is given implicitly by

$$
P^Q = \frac{1}{\tau}\frac{1}{1+i^Q}\frac{1-\left(\frac{1}{1+i^Q}\right)^{\tau}}{1-\frac{1}{1+i^Q}} \qquad (1.113)
$$

Chapter 2

Unconventional Monetary Policy in a Currency Union

Chapter Summary

In a currency union with a common central bank, monetary policy implemented through common short-term interest rate targets is unable to address idiosyncratic shocks to the business cycles of the member states. This chapter investigates whether unconventional monetary policy in the form of central bank purchases of government debt can help to smooth idiosyncratic shocks when fiscal coordination is not attainable. Purchases of government bonds reduce the interest that fiscal policy makers have to pay on outstanding debt and thus loosen the government budget constraint of the country in which they are carried out, which leads to adjustments in spending, increased aggregate demand and higher inflation. Due to home bias in government spending and segmentation in the market for government bonds, this mechanism can be used by the central bank to achieve differential effects on demand and prices across member countries allowing it to address idiosyncratic shocks. A second-order approximation to welfare relates welfare losses to inflation, deviations of consumption and the terms of trade from their respective natural rates and unconventional interventions. A monetary policy authority that is equipped with the unconventional tool is able to address idiosyncratic shocks reflected in volatility of the natural terms of trade more effectively and to achieve higher welfare than one that cannot make use of government bond purchases.

2.1 Introduction

The member countries of a currency union share a common monetary policy institution—the ECB in the case of the Euro Area. As a result, the widely-discussed tension between a monetary policy stance that is common towards all members and less than perfectly correlated business cycles across member countries arises. While, for example, a strong stimulus might be needed in some parts of the union, other parts could require only weakly expansionary policy, leaving the centralised policy maker most likely adopting an intermediate strategy that may in fact be individually suboptimal for most if not all member states. It is often argued that close fiscal policy coordination is required to address this issue. Governments, however, are subject to constraints that make doubtful whether fiscal policy can fulfil the role that it would optimally play in smoothing idiosyncratic shocks, especially since, in the absence of a unified institutional framework, fiscal policy authorities are likely to put higher weight on domestic welfare compared to welfare in the union as a whole. This chapter takes a lack of fiscal coordination as given and investigates how unconventional monetary policy, more precisely—central bank purchases of government bonds—can be used to stabilise idiosyncratic disturbances within a currency union.

A model of a currency union consisting of two countries with a common central bank but separate fiscal authorities is laid out. The local fiscal authorities finance spending partly by debt issuance. The market for government debt is segmented across asset classes and countries, giving rise to the possibility of sustained differences in the yield of government debt issued in the two countries. Central bank purchases of the bond given out by a particular country increase its price and decrease the interest that the respective government has to pay on it. This loosens the government's budget constraint inducing it to increase spending. Given that government spending is subject to home bias, domestic demand and prices increase, allowing the central bank to affect relative prices and thus the allocation of productive activity across countries by taking debt from the two countries on and off its balance sheet.

A second-order approximation of welfare relates welfare losses to inflation in both countries, deviations of consumption and the terms of trade from their respective natural rates as well as usage of unconventional monetary policy. All welfare losses result from nominal price rigidities and costs associated with the unconventional monetary policy tool.[1] The fact that output prices can be adjusted only infrequently causes two types of distortion. First, there is price dispersion within each country despite of all producers of a country producing output with the same technology. As a result, a benevolent social planner could increase welfare by re-distributing productive activity until marginal cost is equated across all firms in a given country. Second, the aggregate price level of all goods produced in one country relative to that of all goods produced in the other country deviates from the optimal relative price level implied by differences in the production technologies of the two countries. The social planner could thus increase welfare further by re-distributing productive activity across countries. Conventional monetary policy is effective at addressing the first type of distortion, but not the second. Welfare gains from using unconventional monetary policy arise from its ability to partly mitigate the second, the cross-country type of distortion.

The model employed here builds on the currency union model developed by Benigno (2004) and more broadly on the open economy model introduced in Obstfeld and Rogoff (1995). A large literature is concerned with the optimal combination of fiscal and monetary policy in a currency union. Beetsma and Jensen (2005) derive the welfare-maximising monetary and fiscal policy mix and compare it to several suboptimal rules when governments have access to lump-sum taxation. Ferrero (2009) introduces distortionary taxation, evaluates the welfare consequences of regulation that requires governments to balance their budgets and derives simple rules that approximate the optimal policy. Leith and Wren-Lewis (2006) also study the implications of deficit constraints on stabilisation policy but obtain their results in an overlapping generations model. The previously mentioned articles make use of two-country models, implying that each individual country is "large" compared to the entire union. Gali and Monacelli (2008) analyse the optimal policy mix when each country is atomistic. This chapter makes use of a two-country model with lump-sum taxation and government debt issuance. It adds to the previous literature by taking the absence of fiscal coordination as given and considering central bank purchases of government debt as an additional policy tool instead.

D'Amico et al. (2012), D'Amico and King (2013), Gagnon et al. (2011) and Krishnamurthy

[1] Additional distortions from market power are eliminated using a steady state subsidy as in Rotemberg and Woodford (1998).

and Vissing-Jorgensen (2011), among others, present empirical evidence for a negative effect of purchases of US Treasuries by the Fed on the yield of the securities purchased. Meier (2009) and Joyce et al. (2011a,b) provide equivalent results for the QE programme of the Bank of England. Early theoretical contributions by Wallace (1981) and Eggertsson and Woodford (2003) contain irrelevance propositions for open market operations in perfect financial markets suggesting that models with market imperfections are required to explain the empirically observed effects. Andrés et al. (2004), Chen et al. (2012), Ellison and Tischbirek (2014) and Vayanos and Vila (2009) obtain a link between the quantity of a particular bond purchased by the central bank and its yield by introducing segmentation in government bond markets and limits to arbitrage. While these papers consider closed economies with bond markets that are segmented along the maturity dimension, in the model presented here, bond markets are also segmented by country of issuance. Evidence for this type of segmentation can be found, for example, in French and Poterba (1991) and Fidora et al. (2006) who study home bias in financial asset holdings and Krishnamurthy et al. (2014) who study the ECB's recent government bond purchase programmes.

The remainder is organised as follows. Section 2.2 introduces the model. Section 2.3 shows a Linear-Quadratic (LQ) approximation to the central bank's optimal policy problem and discusses the calibration for numerical experiments. Section 2.4 is concerned with characteristics of the optimal policy mix. Section 2.5 contains a welfare analysis. The final section concludes. Derivations are relegated to a detailed appendix.

2.2 Model

The model introduces unconventional monetary policy in the form of central bank purchases of government debt to the currency union framework studied by Benigno (2004). Two countries, "Home" (H) and "Foreign" (F), form a currency union with a common central bank but separate fiscal policy institutions. The countries are indexed by i with $i \in \{H, F\}$, while the agents that inhabit the two countries are indexed by j. For simplicity it is assumed that agents are both producers of a single differentiated good and consumers of a basket of all goods produced in the union. Agents are uniformly distributed on the interval $[0, 1]$, where agents with $j \in [0, n)$ reside in H and those indexed by $j \in [n, 1]$ are located in F.

2.2.1 Consumer Problem

The preferences of a generic household j in country i are described by the following utility function

$$U_t^j = \mathrm{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ U(C_T^j) + L\left(\frac{M_T^j}{P_T}\right) - V\left[y_T(j), z_T^i\right] \right\} \tag{2.1}$$

where $i = H$ if $j \in [0, n)$ and $i = F$ if $j \in [n, 1]$. $\beta \in (0, 1)$ is the discount factor. U and L are increasing and concave functions of consumption and liquidity services from real money holdings, respectively. V is an increasing and convex function of agent j's output $y_T(j)$ and a country-specific supply shock z_T^i. A low draw of z_T^i reflects diminished production costs, which is interpreted as a positive technology shock below. Each household consumes all goods produced in the currency union.

$$C_t^j \equiv \frac{(C_{H,t}^j)^n (C_{F,t}^j)^{1-n}}{n^n (1-n)^{1-n}} \tag{2.2}$$

C_t^j is a CES aggregate of a basket of goods from H, $C_{H,t}^j$, and one from F, $C_{F,t}^j$, where the elasticity of substitution approaches one, implying Cobb-Douglas preferences between the basket of goods from H and the basket of goods from F.[2] The weights of domestically and foreign produced consumption goods are equal to the respective country size. The baskets $C_{H,t}^j$ and $C_{F,t}^j$ are composed as follows

$$C_{H,t}^j \equiv \left[\left(\frac{1}{n}\right)^{\frac{1}{\sigma}} \int_0^n c_t^j(h)^{\frac{\sigma-1}{\sigma}} dh \right]^{\frac{\sigma}{\sigma-1}} \tag{2.3}$$

$$C_{F,t}^j \equiv \left[\left(\frac{1}{1-n}\right)^{\frac{1}{\sigma}} \int_n^1 c_t^j(f)^{\frac{\sigma-1}{\sigma}} df \right]^{\frac{\sigma}{\sigma-1}} \tag{2.4}$$

Here $\sigma > 1$ is the elasticity of substitution between goods from the same country. Agent j's demand for generic goods h and f with $h \in [0, n)$ and $f \in [n, 1]$ is

$$c_t^j(h) = \left[\frac{p_t(h)}{P_{H,t}}\right]^{-\sigma} T_t^{1-n} C_t^j \tag{2.5}$$

$$c_t^j(f) = \left[\frac{p_t(f)}{P_{F,t}}\right]^{-\sigma} T_t^{-n} C_t^j \tag{2.6}$$

[2]The assumption of a unitary elasticity of substitution is made to ensure equilibrium determinacy despite of incomplete financial markets. See Section 2.2.3 for more details.

where $T_t \equiv P_{F,t}/P_{H,t}$ are the terms of trade of country F or, alternatively, the competitiveness of H. The law of one price holds, $p_t^H(j) = p_t^F(j)$ for all $j \in [0,1]$, which together with the fact that the composition of the basket of goods consumed in both countries is identical implies that purchasing power parity holds. The aggregate price index is found as the expenditure minimising price of one unit of the basket C_t^j. It is given by

$$P_t = (P_{H,t})^n (P_{F,t})^{1-n} \tag{2.7}$$

The price indices $P_{H,t}$ and $P_{F,t}$ that correspond to the basket of home goods and the basket of foreign goods defined in (2.3) and (2.4), respectively, are given by

$$P_{H,t} = \left[\left(\frac{1}{n}\right)\int_0^n p_t(h)^{1-\sigma} dh\right]^{\frac{1}{1-\sigma}} \tag{2.8}$$

$$P_{F,t} = \left[\left(\frac{1}{1-n}\right)\int_n^1 p_t(f)^{1-\sigma} df\right]^{\frac{1}{1-\sigma}} \tag{2.9}$$

There are four different types of monetary assets in the currency union. A complete set of state-contingent securities issued and traded only within each country, a non-contingent security traded across all agents, a non-contingent bond issued by the fiscal authorities of H and F that is only traded within the two countries and a non-interest-bearing asset that could be labelled “money”. Equation (2.10) below is the budget constraint of an agent j in units of the consumption basket that includes goods from both H and F.

$$\begin{aligned} q_t^i B_t^{i,j} + \frac{B_t^j}{P_t(1+i_t)} + \frac{Q_t^{i,j}}{P_t(1+i_t^Q)(1-\epsilon_t^i Q_{CB,t}^i)} + \frac{M_t^j}{P_t} + C_t^j + \frac{T_t^i}{P_t} = \\ \mathbb{1}_{1\times S} B_{t-1}^{i,j} + \frac{B_{t-1}^j}{P_t} + \frac{Q_{t-1}^{i,j}}{P_t} + \frac{M_{t-1}^j}{P_t} + (1-\tau)\frac{p_t(j)y_t(j)}{P_t} \end{aligned} \tag{2.10}$$

$B_t^{i,j}$ is an $S \times 1$ vector of state-contingent real securities, issued and traded only in the country i in which j resides. There exists a complete set of such state-contingent securities in H and F, where S is the number of possible states in period $t+1$.[3] q_t^i denotes the corresponding vector

[3]The state-contingent securities are assumed to be in zero net supply, which means that $\int_0^n B_t^{H,j} dj$ and $\int_n^1 B_t^{F,j} dj$ are equal to an $S \times 1$ vector of zeros.

of security prices. B_t^j are holdings of a non-contingent nominal bond that is traded within and across the two countries. By investing the amount $B_t^j/(1+i_t)$ in period t, agent j secures a payment of B_t^j in $t+1$, where i_t is the one-period nominal interest rate assumed to be directly controlled by the union's central bank. i_t is safe and known in t but not in advance.

$Q_t^{i,j}$ are j's holdings of a non-contingent nominal bond issued by the government of country i in which j resides. Unconventional monetary policy works through purchases and sales of this asset. To break the irrelevance result given in Wallace (1981) and thus to give non-standard monetary policy "a chance to work" (Chen et al., 2012, p. F299), central bank bond purchases have a direct effect on the yield of the bond purchased and, furthermore, this effect is not immediately undone by arbitrage. More specifically, the gross yield of government bonds is assumed to be given by $(1+i_t^Q)(1-\epsilon_t^i Q_{CB,t}^i)$, where $i_t^Q \equiv i_t + \psi$. It depends on the interest rate set by the central bank, the government bond premium ψ and an additional factor capturing the effects of non-standard central bank purchases of this bond. Government debt can be thought of as a long-term bond whose entire stock is sold in the period following its acquisition as in Faraglia et al. (2013), for example. ψ can then be interpreted as a term premium. $Q_{CB,t}^i$ is the amount of the bond issued by the government of i that is purchased by the central bank in period t and ϵ_t^i determines the size of the negative effect of central bank bond purchases on the yield of this bond.[4] Note that, consistent with the empirical literature described in the introduction, this specification of the interest paid on government debt requires bond markets to be segmented and therefore gaps in the yields of the government bonds issued in H and F not to be immediately closed by arbitrageurs.

The government of country i issues the total amount of debt $\bar{Q}_t^i$. Households have access only to the residual debt not purchased by the central bank $\bar{Q}_t^i - Q_{CB,t}^i$. The premium ψ is assumed to be large enough so that $(1+i_t+\psi)(1-\epsilon_t^i Q_{CB,t}^i) > i_t$ at all times causing the households to

[4]In Chen et al. (2012), the risk premium on long-term bonds is assumed to contain an exogenous component that is made dependent on the quantity of long-term debt held by the public. Comparable to this risk premium component, ϵ_t^i is determined outside of the model. The precise mechanism by which central bank asset purchases affect asset yields is at the heart of current research. D'Amico et al. (2012) and Krishnamurthy et al. (2014) give empirical evidence for channels related to signalling, the re-allocation of duration risk and the preferred-habitat theory of the term structure. The relative importance of these channels is subject to an ongoing debate. This study does not take a stand in that debate and adopts the approach taken by Chen et al. (2012). An overview of the estimated impact of bond purchases by the Fed on 10-year Treasury yields found in the literature is given in Table 1 of Chen et al. (2012).

hold all government debt available to them and to use the non-contingent security for savings beyond this amount. In addition, agent j has real money holdings at the amount of M_t^j/P_t, consumes C_t^j units of the basket defined above, pays lump-sum taxes at the nominal value of T_t^i and receives nominal income from production $p_t(j)y_t(j)$, which is subject to a proportional tax or subsidy τ.

None of the agents is initially endowed with financial assets, i.e. $B_{-1}^{i,j} = B_{-1}^j = Q_{-1}^{i,j} = M_{-1}^j = 0$ for all $j \in [0,1]$. Since all agents have identical initial wealth and identical preferences and financial markets are complete within each country, there is perfect risk sharing in consumption within H and F so that, in all periods $t \geq 0$, $C_t^j = C_t^H$ for all $j \in [0,n)$ and $C_t^j = C_t^F$ for all $j \in [n,1]$. The consumer problem can thus be analysed for representative agents from H and F. The first-order conditions of the agents' optimisation problem, where agents are now indexed by $i \in \{H,F\}$, can be combined to give

$$L_{M/P}\left(\frac{M_t^i}{P_t}\right) = \frac{i_t}{1+i_t}U_C(C_t^i) \tag{2.11}$$

$$Q_t^i = \frac{1}{n^i}\left(\bar{Q}_t - Q_{CB,t}^i\right) \tag{2.12}$$

$$U_C(C_t^i) = (1+i_t)\beta \mathrm{E}_t\left[U_C(C_{t+1}^i)\frac{1}{\Pi_{t+1}}\right] \tag{2.13}$$

where $n^i = n$ if $i = H$ and $n^i = 1-n$ if $i = F$. Equation (2.11) characterises the optimal amount of real money balances. Equation (2.12) describes the corner solution in government bond holdings and Equation (2.13) is the Euler equation associated with the non-contingent security that is traded between all agents of the union. $\Pi_{t+1} \equiv P_{t+1}/P_t$ is the price inflation of the consumption basket that includes all goods produced in the union.

2.2.2 Monetary and Fiscal Policy Institutions

The central bank sets the union-wide nominal interest rate, controls the supply of money and, in addition, is allowed to purchase government debt from both countries. Through debt purchases, it is able to affect the price and yield of government bonds in H and F. Each country has a fiscal authority that issues debt, collects taxes and spends funds on a basket of goods that is entirely made up of domestically produced goods. The fiscal policy makers plan their budgets based on an estimate of central bank bond purchases in their respective country. When the realised values are observed, they revise taxes and spending according to a simple rule. Knowledge of this rule and the expectations formed by the fiscal policy authorities allows the central bank to prompt them to revise spending upwards or downwards by purchasing the appropriate amount of debt, affecting aggregate demand and inflation accordingly.

Central Bank

The central bank's real surplus or deficit in period t is given by

$$\begin{aligned} \frac{\Delta_t}{P_t} &= \frac{M_t^S}{P_t} - \frac{M_{t-1}^S}{P_t} + \frac{Q_{CB,t-1}^H}{P_t} - \frac{Q_{CB,t}^H}{P_t(1+i_t^Q)(1-\epsilon_t^H Q_{CB,t}^H)} \\ &\quad + \frac{Q_{CB,t-1}^F}{P_t} - \frac{Q_{CB,t}^F}{P_t(1+i_t^Q)(1-\epsilon_t^F Q_{CB,t}^F)} \end{aligned} \tag{2.14}$$

M_t^S is the nominal money supply, which exactly covers the demand for money in the union, i.e. $M_t^S = \int_0^1 M_t^j dj$. Central bank profits are made up of increases in money supply and net revenues from asset purchases. The central bank surplus is transferred to the local governments of H and F at the end of each period. The real amounts received by the two countries are

$$\frac{\Delta_t^H}{P_t} = n\left(\frac{M_t^S}{P_t} - \frac{M_{t-1}^S}{P_t}\right) + \frac{Q_{CB,t-1}^H}{P_t} - \frac{Q_{CB,t}^H}{P_t(1+i_t^Q)(1-\epsilon_t^H Q_{CB,t}^H)} \tag{2.15}$$

$$\frac{\Delta_t^F}{P_t} = (1-n)\left(\frac{M_t^S}{P_t} - \frac{M_{t-1}^S}{P_t}\right) + \frac{Q_{CB,t-1}^F}{P_t} - \frac{Q_{CB,t}^F}{P_t(1+i_t^Q)(1-\epsilon_t^F Q_{CB,t}^F)} \tag{2.16}$$

Central bank surpluses and deficits are divided among the member countries of the union according to their size, with the exception of net revenues from unconventional interventions, which are fully transferred to their source country. This splitting rule is intended to capture the institutional framework of the Euro Area. The ECB distributes profits to the national central banks in proportion to their "paid-up shares".[5] The national central banks in turn pay their profits to the local governments. The members' shares are approximated here by country size. Net revenues from unconventional monetary policy are treated as an exception to this rule to

[5] See Article 33 of the Statute of the European System of Central Banks and of the European Central Bank, Protocol No 4 to the Treaty on the Functioning of the European Union.

separate the bond yield and price effects of central bank asset purchases from de facto fiscal transfers. Equations (2.15) and (2.16) imply that no resources are shifted between countries. Unconventional monetary policy works as a result of the effect of central bank bond purchases on local government bond prices. If the revenues from unconventional asset purchases were also split according to country size, the central bank could act essentially as a central fiscal policy maker and re-distribute resources between both countries. It is doubtful however if this role of the ECB were in accordance with its mandate. The splitting rules assumed here leave this problem for future discussion.

Local Governments

At the beginning of each period, the fiscal authorities of H and F plan their budgets for the current period. Central bank purchases of government debt do not take place until a later point in time in the same period. The fiscal policy makers therefore plan taxes and spending based on their expectations about government bond purchases by the central bank, $Q^{H,e}_{CB,t}$ and $Q^{F,e}_{CB,t}$, respectively. This timing assumption reflects the fact that changes to fiscal policy require more time to be implemented than open market operations by the central bank. The projected values of taxes and spending must satisfy the government budget constraints that are given by

$$\int_0^n \frac{T_t^{P,H}}{P_t}dj + \frac{1}{(1+i_t^Q)(1-\epsilon_t^H Q^{H,e}_{CB,t})}\left(\int_0^n \frac{Q_t^{H,j}}{P_t}dj + \frac{Q^H_{CB,t}}{P_t}\right) + \tau\int_0^n \frac{p_t(j)y_t(j)}{P_t}dj + \frac{\Delta_t^H}{P_t}$$
$$= \int_0^n \frac{p_t(j)g_t^P(j)}{P_t}dj + \int_0^n \frac{Q_{t-1}^{H,j}}{P_t}dj + \frac{Q^H_{CB,t-1}}{P_t} \tag{2.17}$$
$$\int_n^1 \frac{T_t^{P,F}}{P_t}dj + \frac{1}{(1+i_t^Q)(1-\epsilon_t^F Q^{F,e}_{CB,t})}\left(\int_n^1 \frac{Q_t^{F,j}}{P_t}dj + \frac{Q^F_{CB,t}}{P_t}\right) + \tau\int_n^1 \frac{p_t(j)y_t(j)}{P_t}dj + \frac{\Delta_t^F}{P_t}$$
$$= \int_n^1 \frac{p_t(j)g_t^P(j)}{P_t}dj + \int_n^1 \frac{Q_{t-1}^{F,j}}{P_t}dj + \frac{Q^F_{CB,t-1}}{P_t} \tag{2.18}$$

$T_t^{P,i}$ and $g_t^P(j)$ are planned per capita lump-sum taxes in country i and planned government consumption of product j, respectively. The central bank is assumed to inform the local governments about the profits that it will transfer to them at the end of the period, allowing the local fiscal authorities to plan with the correct amount of funds.[6] Clearing in the markets for government bonds implies that the constraints on the planned budgets can be written more succinctly as

$$\int_0^n \frac{T_t^{P,H}}{P_t}dj + \frac{\bar{Q}_t^H}{P_t(1+i_t^Q)(1-\epsilon_t^H Q_{CB,t}^{H,e})} + \tau \int_0^n \frac{p_t(j)y_t(j)}{P_t}dj + \frac{\Delta_t^H}{P_t} =$$
$$\int_0^n \frac{p_t(j)g_t^P(j)}{P_t}dj + \frac{\bar{Q}_{t-1}^H}{P_t} \tag{2.19}$$
$$\int_n^1 \frac{T_t^{P,F}}{P_t}dj + \frac{\bar{Q}_t^F}{P_t(1+i_t^Q)(1-\epsilon_t^F Q_{CB,t}^{F,e})} + \tau \int_n^1 \frac{p_t(j)y_t(j)}{P_t}dj + \frac{\Delta_t^F}{P_t} =$$
$$\int_n^1 \frac{p_t(j)g_t^P(j)}{P_t}dj + \frac{\bar{Q}_{t-1}^F}{P_t} \tag{2.20}$$

Planned spending on the good produced by j is given by

$$g_t^P(j) = \left[\frac{p_t(j)}{P_{H,t}}\right]^{-\sigma} G_t^{P,H} \quad \forall j \in [0,n) \tag{2.21}$$

$$g_t^P(j) = \left[\frac{p_t(j)}{P_{F,t}}\right]^{-\sigma} G_t^{P,F} \quad \forall j \in [n,1] \tag{2.22}$$

where $G_t^{P,H}$ and $G_t^{P,F}$ represent planned aggregate real spending in H and F. Note that governments allocate their expenditures only to domestic producers.[7] $G_t^{P,i}$ is determined by an exogenous signal, which can be thought of as resulting from political debate and advice given, for example, by infrastructure or education experts. It is mean zero and should be interpreted as the deviation from a long-run trend that is not modelled to retain the focus on fluctuations at the business cycle frequency.

To account for the fact that preliminary budgets are based on projected values of $Q_{CB,t}^i$, the budget of each country is supplemented with a rule that determines how planned taxes and planned spending shall be adjusted when the true value of $Q_{CB,t}^i$ becomes observable. The rules that the fiscal policy makers of H and F are assumed to follow are described below. If taxes are not adjusted one-to-one with the change in funds available due to higher or lower than projected

[6]This information, however, does not permit the government of country i to infer $Q_{CB,t}^i$. This is the case, for example, because money supply is also not observable at this point in time.

[7]The assumption of complete home bias in government expenditures is made merely for simplicity; all that is required to obtain results of the kind discussed below is some, potentially small, bias towards domestic goods.

government bond prices, spending is raised above or decreased below the level initially planned. As a result, the central bank can influence aggregate demand through the effect of government bond purchases on bond prices.

When the amount of bonds purchased by the central bank becomes observable, the local governments determine T_t^i and $g_t(j)$, the revised values of $T_t^{P,i}$ and $g_t^P(j)$. Revised taxes and spending, which are then implemented, have to satisfy the budget constraints given by

$$\int_0^n \frac{T_t^H}{P_t}dj + \frac{\bar{Q}_t^H}{P_t(1+i_t^Q)(1-\epsilon_t^H Q_{CB,t}^H)} + \tau\int_0^n \frac{p_t(j)y_t(j)}{P_t}dj + \frac{\Delta_t^H}{P_t} = \int_0^n \frac{p_t(j)g_t(j)}{P_t}dj + \frac{\bar{Q}_{t-1}^H}{P_t} \tag{2.23}$$

$$\int_n^1 \frac{T_t^F}{P_t}dj + \frac{\bar{Q}_t^F}{P_t(1+i_t^Q)(1-\epsilon_t^F Q_{CB,t}^F)} + \tau\int_n^1 \frac{p_t(j)y_t(j)}{P_t}dj + \frac{\Delta_t^F}{P_t} = \int_n^1 \frac{p_t(j)g_t(j)}{P_t}dj + \frac{\bar{Q}_{t-1}^F}{P_t} \tag{2.24}$$

In both countries, income consists of revised tax revenues, debt and central bank profits. Expenditures comprise revised spending on consumption goods as well as debt service payments.

In the case of country H, for example, one obtains the equation below by subtracting the constraint employed for preliminary budgeting (2.19) from the constraint that revised spending and taxes have to satisfy (2.23).

$$\frac{\bar{Q}_t^H}{P_t}\left[\frac{1}{(1+i_t^Q)(1-\epsilon_t^H Q_{CB,t}^H)} - \frac{1}{(1+i_t^Q)(1-\epsilon_t^H Q_{CB,t}^{H,e})}\right] = \int_0^n \frac{p_t(j)g_t(j)}{P_t}dj - \int_0^n \frac{p_t(j)g_t^P(j)}{P_t}dj + \int_0^n \frac{T_t^{P,H}}{P_t}dj - \int_0^n \frac{T_t^H}{P_t}dj \tag{2.25}$$

which can be re-written as

$$\frac{\bar{Q}_t^H}{P_t}\left(P_t^{QH} - P_t^{QH,e}\right) = \int_0^n \frac{p_t(j)g_t(j)}{P_t}dj - \int_0^n \frac{p_t(j)g_t^P(j)}{P_t}dj + \int_0^n \frac{T_t^{P,H}}{P_t}dj - \int_0^n \frac{T_t^H}{P_t}dj \tag{2.26}$$

where $P_t^{QH} \equiv \left[(1+i_t^Q)(1-\epsilon_t^H Q_{CB,t}^H)\right]^{-1}$ and $P_t^{QH,e} \equiv \left[(1+i_t^Q)(1-\epsilon_t^H Q_{CB,t}^{H,e})\right]^{-1}$ are the realised price of the government bond issued in H and the fiscal authority's projection thereof, respectively. If central bank bond purchases exceed the amount expected by the government of

H, i.e. if $Q^{H}_{CB,t} > Q^{H,e}_{CB,t}$ and thus $P^{QH}_t > P^{QH,e}_t$, either government spending has to be increased above or taxes have to be decreased below the planned level for the government budget constraint to be satisfied. The budget adjustment rule determines to which extent the adjustments take place at these two margins. The rule that the fiscal authority of H is assumed to follow here is

$$\int_0^n \frac{p_t(j)g_t(j)}{P_t}dj - \int_0^n \frac{p_t(j)g^P_t(j)}{P_t}dj = \theta^H \frac{\bar{Q}^H_t}{P_t}\left(P^{QH}_t - P^{QH,e}_t\right) \tag{2.27}$$

with $\theta^H \in [0,1]$. The budget constraints at both stages then immediately imply

$$\int_0^n \frac{T^{P,H}_t}{P_t}dj - \int_0^n \frac{T^H_t}{P_t}dj = \left(1-\theta^H\right)\frac{\bar{Q}^H_t}{P_t}\left(P^{QH}_t - P^{QH,e}_t\right) \tag{2.28}$$

The equations above state that government expenditures on consumption goods are adjusted proportionally to the projection error of revenue from new debt. In the extreme case $\theta^H = 1$, government spending is revised one-for-one with the change in funds available. If $\theta^H = 0$, lump-sum taxes are adjusted and spending is unaffected by central bank interventions in the government bond market. Note that adding (2.28) to (2.27) gives (2.26), which implies that the two equations jointly lead to the correct amount of budget revision. The government of F uses an analogous rule with parameter $\theta^F \in [0,1]$.

2.2.3 Aggregate Demand

What is the effect that central bank asset purchases have on aggregate demand in each country? Aggregate demand for individual goods from H and F is given by

$$y_t(h) = \int_0^1 c^j_t(h)dj + g_t(h) = \left[\frac{p_t(h)}{P_{H,t}}\right]^{-\sigma}\left(T_t^{1-n}C^U_t + G^H_t\right) \tag{2.29}$$

$$y_t(f) = \int_0^1 c^j_t(f)dj + g_t(f) = \left[\frac{p_t(f)}{P_{F,t}}\right]^{-\sigma}\left(T_t^{-n}C^U_t + G^F_t\right) \tag{2.30}$$

where $C^U_t \equiv \int_0^1 C^j_t dj$ is consumption aggregated over the entire union. The demand for the goods baskets from both countries is

$$Y^H_t = \left[\frac{1}{n}\int_0^n y_t(h)^{\frac{\sigma-1}{\sigma}}dh\right]^{\frac{\sigma}{\sigma-1}} = T_t^{1-n}C^U_t + G^H_t \tag{2.31}$$

$$Y_t^F = \left[\frac{1}{1-n}\int_n^1 y_t(f)^{\frac{\sigma-1}{\sigma}} df\right]^{\frac{\sigma}{\sigma-1}} = T_t^{-n} C_t^U + G_t^F \tag{2.32}$$

Aggregate demand depends on the terms of trade, private union-wide consumption and government spending, which can be decomposed as shown in the equations (2.33) and (2.34).[8]

$$G_t^H = G_t^{P,H} + \theta^H \frac{T_t^{1-n}}{nP_t} \bar{Q}_t^H \left(P_t^{QH} - P_t^{QH,e}\right) \tag{2.33}$$

$$G_t^F = G_t^{P,F} + \theta^F \frac{T_t^{-n}}{(1-n)P_t} \bar{Q}_t^F \left(P_t^{QF} - P_t^{QF,e}\right) \tag{2.34}$$

Final government spending consists of planned spending, the exogenous amount of expenditures that local governments initially considered appropriate, and revisions thereof due to central bank interventions. The final term of the sums in the two equations above is discussed below in detail.

Before examining the effects of central bank bond purchases on aggregate demand, it is convenient to characterise consumption and net foreign bond holdings further. The following lemma is based on analogous results from Benigno (2004), Corsetti and Pesenti (2001) and Obstfeld and Rogoff (1998).

Lemma 1

a) All households select the same quantity of consumption.

$C_t^i = C_t^U \equiv C_t \ \forall t \geq 0, i \in \{H, F\}$

b) The non-contingent security traded between all households of the union is redundant.

$B_t^i = 0 \ \forall t \geq 0, i \in \{H, F\}$

Despite of incomplete financial markets at the union-level, there is perfect consumption insurance and the non-contingent security is not traded. Therefore, monetary policy can induce stationarity of consumption implying that the equilibrium indeterminacy issues of open economy models with incomplete asset markets described in Schmitt-Grohe and Uribe (2003) do not play a role here. Preferences over the baskets from H and F are Cobb-Douglas (CES with unitary elasticity of substitution); each country therefore earns a constant share of real union-wide income. Since

[8] The details can be found in the appendix.

the Cobb-Douglas weighting parameters are chosen to equal the country sizes of H and F, real per capita income is equal across countries resulting in identical per capita consumption.[9]

Aggregate demand in H and F is thus given by

$$Y_t^H = T_t^{1-n} C_t + G_t^{P,H} + \bar{\mathcal{P}}_t^H \tag{2.35}$$

$$Y_t^F = T_t^{-n} C_t + G_t^{P,F} + \bar{\mathcal{P}}_t^F \tag{2.36}$$

where

$$\begin{aligned} \bar{\mathcal{P}}_t^H &\equiv \theta^H \frac{T_t^{1-n}}{nP_t} \bar{Q}_t^H \left(P_t^{QH} - P_t^{QH,e} \right) \\ &= \theta^H \frac{T_t^{1-n}}{nP_t} \bar{Q}_t^H \left[\frac{1}{(1+i_t^Q)(1-\epsilon_t^H Q_{CB,t}^H)} - \frac{1}{(1+i_t^Q)(1-\epsilon_t^H Q_{CB,t}^{H,e})} \right] \end{aligned} \tag{2.37}$$

$$\begin{aligned} \bar{\mathcal{P}}_t^F &\equiv \theta^F \frac{T_t^{-n}}{(1-n)P_t} \bar{Q}_t^F \left(P_t^{QF} - P_t^{QF,e} \right) \\ &= \theta^F \frac{T_t^{-n}}{(1-n)P_t} \bar{Q}_t^F \left[\frac{1}{(1+i_t^Q)(1-\epsilon_t^F Q_{CB,t}^F)} - \frac{1}{(1+i_t^Q)(1-\epsilon_t^F Q_{CB,t}^{F,e})} \right] \end{aligned} \tag{2.38}$$

Aggregate demand can be expressed as the sum of private consumption, planned government spending and additional demand caused by central bank asset purchases, which depends on four factors. For country H these are: the expectational error made by the government with respect to the price at which it can sell its bonds $P_t^{QH} - P_t^{QH,e}$, the total quantity of bonds sold $\bar{Q}_t^H$, a rate of conversion into domestic prices and per capita terms $T_t^{1-n}/(nP_t)$, and the share of funds available due to unconventional measures which is allocated to spending θ^H. In this environment, central bank purchases of government debt only affect aggregate demand if they are not fully anticipated by the local governments. This is the case here, because it is assumed that $Q_{CB,t}^{i,e}$ is formed according to the simple rule

$$Q_{CB,t}^{i,e} = \arg\min\,(1-\gamma)\left(Q_{CB,t}^{i,e} - \mathrm{E}_t Q_{CB,t}^i\right)^2 + \gamma\left(Q_{CB,t}^{i,e} - Q_{CB}^i\right)^2 \tag{2.39}$$

with $\gamma \in (0, 1]$, implying

$$Q_{CB,t}^{i,e} = (1-\gamma)\mathrm{E}_t Q_{CB,t}^i + \gamma Q_{CB}^i \tag{2.40}$$

[9]See Section 2.7.2 of the appendix for more details on the proof.

When forecasting bond purchases by the central bank, the local governments put weight $1-\gamma$ on the rational expectation of $Q^i_{CB,t}$ based on information available in the same period. $\mathrm{E}_t Q^i_{CB,t}$ is equal to the value that is realised later, given the governments' knowledge of the central bank's objective and shocks in the current period. However, they also put a small positive weight γ on the event that the central bank phases out its active involvement in government bond markets and returns to the long-run equilibrium value of purchases Q^i_{CB}. This long-run value is normalised to zero. Fear that the central bank could not honour its commitment to bond purchases therefore leads local governments to under-estimate the size of unconventional open market operations, yielding $Q^{i,e}_{CB,t} = (1-\gamma)Q^i_{CB,t}$.[10]

It follows that

$$\frac{\partial\left(P_t^{Qi} - P_t^{Qi,e}\right)}{\partial Q^i_{CB,t}} > 0 \tag{2.41}$$

allowing the central bank to affect aggregate demand through the unconventional tool, as can be seen from the equations (2.35) to (2.38). Different degrees of expansion or contraction can be achieved in H and F by purchasing or selling the appropriate amounts of government debt. The central bank is able to implement any desired values of $\bar{\mathcal{P}}^H_t$ and $\bar{\mathcal{P}}^F_t$ by choosing $Q^H_{CB,t}$ and $Q^F_{CB,t}$ appropriately. It is therefore assumed below that the central bank chooses $\bar{\mathcal{P}}^H_t$ and $\bar{\mathcal{P}}^F_t$ directly rather than $Q^H_{CB,t}$ and $Q^F_{CB,t}$.

2.2.4 Producer Problem

As in Calvo (1983), only a fraction $1-\alpha^i$ of the firms resident in country $i \in \{H, F\}$ can change the price of their respective good. All firms that reside in the same country and are able to adjust their price in a given period t face the same profit maximisation problem. Let $p^*_t(h)$ be the price that a firm $h \in [0, n)$ chooses if it can re-optimise its price in t and let $p^*_t(f)$ be equivalently defined for a firm $f \in [n, 1]$. Then $P_{H,t}$ and $P_{F,t}$ evolve according to

$$P_{H,t} = \left[\alpha^H \left(P_{H,t-1}\right)^{1-\sigma} + \left(1-\alpha^H\right) p^*_t(h)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \tag{2.42}$$

$$P_{F,t} = \left[\alpha^F \left(P_{F,t-1}\right)^{1-\sigma} + \left(1-\alpha^F\right) p^*_t(f)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \tag{2.43}$$

[10] It is even conceivable that governments do not plan with the uncertain extra funds available due to asset purchases at all, so that $Q^{i,e}_{CB,t} = 0$.

A generic firm j that is able to adjust its price in period t sets $p_t(j)$ to maximise the expected discounted stream of future profits given by

$$\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^i \beta\right)^{T-t} \left\{ \frac{U_C(C_T)}{P_T} (1-\tau) p_t(j) y_{t,T}(j) - V\left[y_{t,T}(j), z_T^i\right] \right\} \tag{2.44}$$

where $i = H$ if $j \in [0, n)$ and $i = F$ if $j \in [n, 1]$. Revenues are weighted by the marginal utility of consumption. $y_{t,T}(j)$ denotes the firm's output in period T when it could last change its price in period t. The optimisation is subject to the relevant demand constraint, that is, (2.45) and (2.46) for a firm from H and F, respectively.

$$y_{t,T}(h) = \left[\frac{p_t(h)}{P_{H,T}}\right]^{-\sigma} Y_T^H \tag{2.45}$$

$$y_{t,T}(f) = \left[\frac{p_t(f)}{P_{F,T}}\right]^{-\sigma} Y_T^F \tag{2.46}$$

The first-order condition of this problem is

$$\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^i \beta\right)^{T-t} \left\{ (1-\sigma)(1-\tau) \frac{U_C(C_T)}{P_T} \left[\frac{p_t^*(j)}{P_{i,T}}\right]^{-\sigma} Y_T^i + \sigma V_y\left[y_{t,T}(j), z_T^i\right] \frac{p_t^*(j)^{-\sigma-1}}{\left(P_{i,T}\right)^{-\sigma}} Y_T^i \right\} = 0 \tag{2.47}$$

Making the appropriate substitutions and solving for $p_t^*(j)$ gives

$$p_t^*(j) = \frac{\sigma}{(\sigma-1)(1-\tau)} \frac{\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^i \beta\right)^{T-t} V_y\left[y_{t,T}(j), z_T^i\right] y_{t,T}(j)}{\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^i \beta\right)^{T-t} \frac{U_C(C_T)}{P_T} y_{t,T}(j)} \tag{2.48}$$

The marginal tax rate τ is assumed to equal $1/(1-\sigma)$, which, in steady state, precisely eliminates the mark-up that firms charge as a result of their market power.

$$p_t^*(j) = \frac{\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^i \beta\right)^{T-t} V_y\left[y_{t,T}(j), z_T^i\right] y_{t,T}(j)}{\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^i \beta\right)^{T-t} \frac{U_C(C_T)}{P_T} y_{t,T}(j)} \tag{2.49}$$

Equation (2.49) then gives the optimal price charged by a firm that can re-optimise its price in period t.

2.3 Model Approximation

This section describes the Linear-Quadratic (LQ) approximation of the optimisation problem faced by the central bank. The resulting first-order conditions together with the approximated model constraints can be solved for the determinate rational expectations equilibrium using the method outlined in Klein (2000).

2.3.1 Steady State

The approximation point is non-stochastic with $G_t^{P,i} = G^{P,i} = 0$ and $z_t^i = z^i = 0$ for all t and symmetric across countries. It exhibits central bank inactivity with respect to the unconventional tool, i.e. $\bar{\mathcal{P}}^H = \bar{\mathcal{P}}^F = \bar{\mathcal{P}} = 0$. The central bank does not intervene in the markets for government debt in the absence of shocks.

The consumption Euler equation (2.13) requires the gross nominal interest rate to equal the inverse of the discount factor, so

$$1 + i = \frac{1}{\beta} \tag{2.50}$$

The prices set by all producers are assumed to be identical, $p(j) = p(j')$ for all $j, j' \in [0, 1]$. From the definitions (2.7) to (2.9) it follows then that

$$P_H = P_F = P = p(j) \tag{2.51}$$

and, furthermore, that inflation and the terms of trade, defined as $T \equiv P_F/P_H$, equal unity. The absence of shocks together with the previous assumptions about the symmetry of prices and central bank inactivity implies $y(j) = C$ for all j and thus

$$Y^H = Y^F = C \tag{2.52}$$

Aggregate production and price indexes are symmetric across countries.

The first-order condition of the producers' profit maximisation problem in steady state becomes

$$U_C(C) = \frac{\sigma}{(\sigma - 1)(1 - \tau)} V_y\left[y(j), 0\right] \tag{2.53}$$

When the proportional tax rate τ is chosen such that the mark-up that producers are able to charge due to market power is just offset, the marginal utility of consuming an additional unit of the consumption basket equals the marginal disutility from producing it.

2.3.2 Dynamics

In order to describe the the dynamics of the approximated model, it is convenient to introduce the following definitions. For each variable X, $\tilde{X}_t \equiv \log X_t - \log X$ denotes log-deviations from steady state under flexible prices. $\hat{X}_t \equiv \log X_t - \log X$ is defined equivalently for the case of sticky output prices outlined in Section 2.2.4. Variables indexed by "U" are union-wide aggregates defined as $X^U \equiv nX^H + (1-n)X^F$ and variables with the superscript "R" are relative in the sense that they are given by $X^R \equiv X^F - X^H$.

Dynamics under Flexible Prices

The flexible price equilibrium is described by the model equations where (2.48), the price setting equation under sticky prices, has to be replaced with its flexible price counterpart. I assume that, in the hypothetical case of flexible prices, the central bank would refrain from interventions in the asset markets of both countries so that $\bar{\mathcal{P}}_t^H = \bar{\mathcal{P}}_t^F = 0$ for all t. The values of all variables in the equilibrium under flexible prices are referred to as their "natural" rates below.

If producers are able to adjust their output price in each period, the first-order condition of the profit maximisation problem becomes

$$p_t(j) = \frac{\sigma}{(\sigma-1)(1-\tau)} V_y\left[y_t(j), z_t^i\right]$$

Prices are set at a mark-up over marginal cost. Since this condition is identical for all producers that are located within the same country, it implies that

$$P_{H,t} = \frac{\sigma}{(\sigma-1)(1-\tau)} V_y\left(Y_t^H, z_t^H\right) \tag{2.54}$$

$$P_{F,t} = \frac{\sigma}{(\sigma-1)(1-\tau)} V_y\left(Y_t^F, z_t^F\right) \tag{2.55}$$

By log-linearising and combining the consumer and fiscal policy blocks of the model with log-linear versions of (2.54) and (2.55), the union-wide aggregates in the flexible price equilibrium can be expressed as functions of the exogenous processes only:[11]

$$\begin{aligned}
\tilde{T}_t &= \frac{\eta}{1+\eta}\left(g_t^{P,R} - \zeta_t^R\right) && (2.56)\\
\tilde{C}_t &= \frac{\eta}{\rho+\eta}\left(\zeta_t^U - g_t^{P,U}\right) && (2.57)\\
\tilde{Y}_t^U &= \frac{\eta}{\rho+\eta}\zeta_t^U + \frac{\rho}{\rho+\eta}g_t^{P,U} && (2.58)\\
\tilde{i}_t - \mathrm{E}_t\tilde{\Pi}_{t+1}^U &= \frac{\rho\eta}{\rho+\eta}\mathrm{E}_t\left[\zeta_{t+1}^U - \zeta_t^U - \left(g_{t+1}^{P,U} - g_t^{P,U}\right)\right] && (2.59)
\end{aligned}$$

where the parameters η and ρ are defined as $\eta \equiv \frac{V_{yy}(C,0)}{V_y(C,0)}C$ and $\rho \equiv -\frac{U_{CC}(C)}{U_C(C)}C$ and the shock processes are normalised as follows:

$$\begin{aligned}
g_t^{P,i} &\equiv \frac{G_t^{P,i} - G^{P,i}}{Y^i} && (2.60)\\
\zeta_t^i &\equiv -\frac{V_{yz}(C,0)}{CV_{yy}(C,0)}z_t^i && (2.61)
\end{aligned}$$

$g_t^{P,i}$ is the exogenous government spending process expressed here as a deviation from its mean, normalised by steady state output. ζ_t^i is defined as the technology disturbance z_t^i multiplied by a negative scaling factor. Below, $g_t^{P,i}$ and ζ_t^i are referred to as demand and supply shocks, respectively. Remember that an increase in z_t^i raises the production costs in country i and thus has adverse effects on supply. Since ζ_t^i correlates negatively with z_t^i, a high value of ζ_t^i has a positive effect on production, affecting supply in i in a positive way.

Equation (2.58) shows that positive values of both shocks increase natural world output. At the same time, from both (2.57) and (2.58) it becomes obvious that a positive government spending shock does not raise output one-for-one but partly crowds out natural consumption. Recalling the definitions of $g_t^{P,R}$ and ζ_t^R, one can see from (2.56) that domestic demand shocks raise the relative price level of domestic to foreign goods and domestic supply shocks have the opposite effect. The natural real rate of interest, depends positively on expected improvements in technology and negatively on expected increases in the exogenous component of demand.

In a currency union, shocks are likely to be correlated; however, we have no reason to believe

[11] The derivations are shown in the appendix.

that $\zeta_t^H = \zeta_t^F$ and $g_t^{P,H} = g_t^{P,F}$ at all times. A central aim of this chapter is to investigate the role that central bank asset purchases can play in stabilising imperfectly correlated shocks. Figure 2.1 shows selected loci of the natural terms of trade and natural consumption in ζ_t^H,ζ_t^F-space in the absence of demand shocks.[12] When discussing the dynamic properties of the model, it is convenient to think of shocks as exogenous shifts in $\tilde{T}_t$ and $\tilde{C}_t$. The figure below shows how these can result from asymmetric supply disturbances. A situation, in which $\tilde{T}_t = 0.1$ and $\tilde{C}_t = 0$, for example, can be viewed as a combination of a positive supply shock in H and a negative supply shock in F of magnitudes fully determined by the intersection of the corresponding loci. An analogous diagram could be constructed for the case of demand shocks and the absence of technology shocks. Both types of disturbances together yield infinitely many possible combinations of shocks underlying each $\left(\tilde{T}_t, \tilde{C}_t\right)$-pair.

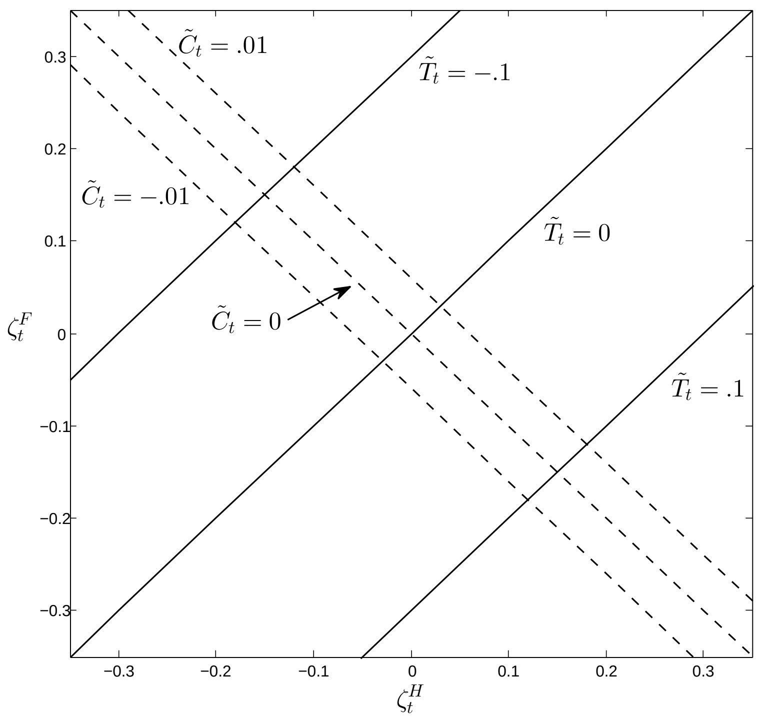

Figure 2.1: Natural consumption and natural terms of trade as a function of supply shocks

[12]Figure 1 is drawn using the calibration described later.

LQ Approximation of the Policy Problem

Aggregate demand is given by the approximated versions of the country-specific demand functions, (2.35) and (2.36).[13]

$$\begin{aligned} \hat{Y}_t^H &= (1-n)\hat{T}_t + \hat{C}_t + g_t^{P,H} + \mathcal{P}_t^H & (2.62) \\ \hat{Y}_t^F &= -n\hat{T}_t + \hat{C}_t + g_t^{P,F} + \mathcal{P}_t^F & (2.63) \end{aligned}$$

$\mathcal{P}_t^H$ and $\mathcal{P}_t^F$ are terms capturing unconventional monetary policy in H and F defined as follows.[14]

$$\mathcal{P}_t^i \equiv \frac{\bar{\mathcal{P}}_t^i - \bar{\mathcal{P}}^i}{Y^i} \tag{2.64}$$

$\mathcal{P}_t^H$ and $\mathcal{P}_t^F$ are assumed to be under direct control of the central bank. Demand in each country depends positively on the competitiveness of domestically produced goods, private consumption, planned government spending and unconventional monetary policy.

Private consumption satisfies the log-linearised Euler equation

$$\hat{C}_t = \mathrm{E}_t\hat{C}_{t+1} - \frac{1}{\rho}\left(\hat{\imath}_t - \mathrm{E}_t\hat{\Pi}_{t+1}^U\right) \tag{2.65}$$

For given expectations about consumption and inflation of the aggregate price level, there is a one-to-one relationship between consumption and the interest rate. Rather than choosing the short-term interest rate, the central bank can therefore be viewed as selecting consumption directly—the level of the interest rate required to implement the desired level of consumption is then determined by the equation above.

Inflation in the price of goods produced in H and F is given by the following two Phillips curves.

$$\begin{aligned} \hat{\Pi}_t^H &= \beta\mathrm{E}_t\hat{\Pi}_{t+1}^H + (1-n)a_T^H\left(\hat{T}_t - \tilde{T}_t\right) + a_C^H\left(\hat{C}_t - \tilde{C}_t\right) + a_{\mathcal{P}}^H\mathcal{P}_t^H & (2.66) \\ \hat{\Pi}_t^F &= \beta\mathrm{E}_t\hat{\Pi}_{t+1}^F - na_T^F\left(\hat{T}_t - \tilde{T}_t\right) + a_C^F\left(\hat{C}_t - \tilde{C}_t\right) + a_{\mathcal{P}}^F\mathcal{P}_t^F & (2.67) \end{aligned}$$

[13]The appendix shows the derivations of the LQ approximation of the optimal policy problem.

[14]Since $\mathcal{P}_t^H$ and $\mathcal{P}_t^F$ are zero in the steady state used as the approximation point, they are expressed in linear-deviation form from steady state rather than log-deviations.

As in Benigno (2004) and Ferrero (2009), inflation depends not only on expected inflation and the consumption gap, but also on deviations of the terms of trade from their natural rate. Higher competitiveness of country H increases output and inflation in H and has the opposite effect in F. Additionally, inflation is increased by the effects of central bank asset purchases. The terms of trade evolve according to the following law of motion

$$\hat{T}_t = \hat{T}_{t-1} + \hat{\Pi}_t^F - \hat{\Pi}_t^H \tag{2.68}$$

Two motivations for employing the unconventional policy tool can be distinguished. The first one is stabilising *relative* disturbances between H and F. To see this, note that $\hat{T}_t$ depends on relative inflation in the two countries. Subtracting (2.66) from (2.67) shows that the inflation differential depends on $a_{\mathcal{P}}^F \mathcal{P}_t^F - a_{\mathcal{P}}^H \mathcal{P}_t^H$. Thus, $\mathcal{P}_t^H$ and $\mathcal{P}_t^F$ can be used to influence the relative price level between the member countries. The second one is stimulating or depressing economic activity in the union *as a whole*. Taking the weighted average of (2.62) and (2.63) gives

$$\hat{Y}_t^U = \hat{C}_t + g_t^{P,U} + \mathcal{P}_t^U \tag{2.69}$$

Union output can be boosted by choosing $\mathcal{P}_t^U > 0$ comparable to a fiscal stimulus package in a closed economy. Results from a number of recent contributions obtained in closed-economy models suggest that this type of intervention may be beneficial; however, the focus of this chapter lies on the question how well unconventional monetary policy is suited to address the challenges specific to a monetary union.[15] To separate the two issues described above, I restrict my attention to policies that give rise to paths of the unconventional monetary policy tool with

$$\mathcal{P}_t^U = 0 \tag{2.70}$$

for all t. In each period, the average intervention across the entire union is required to equal zero, which implies that asset purchases in one country have to be met with sales in the other, eliminating the central bank's ability to affect union-wide output through asset purchases. Ad-

[15] For example, Chen et al. (2012) and Ellison and Tischbirek (2014) consider central bank purchases of government debt in closed economy models.

ditionally, (2.70) ensures that long-run affects on the central bank's balance sheet are small, $\mathcal{P}_t^i$ is centred around the approximation point at which $\mathcal{P}^i = 0$ and the appropriate transversality conditions are not violated.

Welfare is measured as the expected discounted sum of aggregate per-period welfare.

$$W_0 = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t w_t \tag{2.71}$$

where per-period union-wide welfare w_t is defined as $w_t \equiv n w_t^H + (1-n) w_t^F$ and per-period welfare in H and F is given by

$$w_t^H \equiv \frac{1}{n} \int_0^n \left\{ U(C_t) - V\left[y_t(h), z_t^H\right] - W\left(\mathcal{P}_t^H\right) \right\} dh \tag{2.72}$$

$$w_t^F \equiv \frac{1}{1-n} \int_n^1 \left\{ U(C_t) - V\left[y_t(f), z_t^F\right] - W\left(\mathcal{P}_t^F\right) \right\} df \tag{2.73}$$

As in Benigno (2004) and Woodford (2003), the welfare measure abstracts from the utility derived from liquidity services assuming that this type of utility is small. Liquidity consequently does not enter the approximated model, which, similar to the model outlined in Woodford (2003), can be viewed as the cashless limit of an economy with money.

Unconventional monetary policy in the form of central bank asset purchases has been criticised on a number of grounds. Important concerns have to do with the uncertainty surrounding this type of policy with respect to the precise transmission mechanism, its effects on financial stability, its effect on expectation formation etc. This suggests that the true welfare costs of unconventional government bond purchases go beyond the influence that it has on relative prices and the allocation of productive activity. To account for these costs, the strategy adopted here is to add the cost function W to household utility, which is intended to proxy for the costs connected to $\mathcal{P}_t^i$ that are of non-allocative nature. It is assumed that W is continuous and sufficiently differentiable with $W(0) = 0$, $W_{\mathcal{P}}\left(\left|\mathcal{P}_t^i\right|\right) > 0$ and $W_{\mathcal{P}\mathcal{P}}\left(\left|\mathcal{P}_t^i\right|\right) \geq 0$. W enters the approximation of welfare in a clear cut way, which will be discussed in detail below.

A second-order approximation to the welfare criterion is

$$W_0 = -\Omega \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t L_t \tag{2.74}$$

where Ω is a parameter that is defined in the appendix. When the Arrow-Pratt coefficient of relative risk aversion in consumption ρ equals one, per-period loss is given by

$$\begin{aligned} L_t \;=\; & \Lambda_C \left(\hat{C}_t - \tilde{C}_t\right)^2 + \Lambda_C n(1-n)\left(\hat{T}_t - \tilde{T}_t\right)^2 + \gamma\left(\hat{\Pi}_t^H\right)^2 + (1-\gamma)\left(\hat{\Pi}_t^F\right)^2 \\ & + \Lambda_{\mathcal{P}}\left\{\frac{1}{2}\left(1+\frac{\kappa}{\eta}\right)\left[n\left(\mathcal{P}_t^H\right)^2 + (1-n)\left(\mathcal{P}_t^F\right)^2\right] - n(1-n)\mathcal{P}_t^R\left(\hat{T}_t - \delta\tilde{T}_t\right)\right\} \\ & + \mathrm{tip} + O(3) \end{aligned} \tag{2.75}$$

with $\delta \equiv (\eta+1)/\eta$.[16] In the absence of unconventional monetary policy, per-period loss is standard. Loss increases quadratically with deviations of consumption and the terms of trade from their respective levels under flexible prices and with inflation in output prices. When the central bank makes use of the unconventional tool, additional welfare costs arise. These losses depend on the squared size of the intervention in both countries and on the interaction of the relative size of interventions with a modified terms of trade gap.

Sticky prices and costs from unconventional policy are the sole sources of inefficiency in the model. Welfare losses from market power are eliminated by the subsidy that guarantees that long-run output is at its efficient level.[17] Price rigidities cause distortions in relative prices, which lead to an inefficient allocation of productive activity within and across countries. All producers of a given country produce output with the same technology. Price dispersion within H and F is inefficient as it implies that producers operate at different marginal cost. Aggregate cost from production within both countries could therefore be lowered by re-allocating productive activity such that production is equally distributed among all producers located in the same country. Price rigidities furthermore distort the relative price level between H and F away from

[16] The definitions of all other parameters can be found in the appendix. "tip" are terms independent of policy.

[17] Rotemberg and Woodford (1998) distinguish between welfare losses arising from relatively stable sources of inefficiency like market power and those resulting from inefficient short-run fluctuations. They argue that monetary policy should be responsible for minimising the latter type of losses, while other kinds of policy are more suitable for addressing the former. Since the focus is on short-term stabilisation policy here, I make use of their strategy to eliminate the negative long-run effects of market power with an appropriate subsidy.

the optimal value implied by differences in technology. As a result, union-wide production costs could be lowered by re-allocating productive activity between the two countries.

From the Phillips curves (2.66) and (2.67), it can be seen that $\mathcal{P}_t^H$ and $\mathcal{P}_t^F$ have a direct effect on inflation. The strength of this effect is governed by $a_{\mathcal{P}}^H$ and $a_{\mathcal{P}}^F$. The parameter $\Lambda_{\mathcal{P}}$ that determines the size of welfare losses from unconventional monetary policy is zero if $a_{\mathcal{P}}^H$ and $a_{\mathcal{P}}^F$ are zero, which shows that the welfare losses of the unconventional instrument arise from its effect on inflation. When expected inflation, the consumption gap and the terms of trade gap are zero, positive or negative values of $\mathcal{P}_t^H$ and $\mathcal{P}_t^F$ steer inflation away from zero and therefore cause distortions in relative prices, which is reflected in the quadratic welfare costs from $\mathcal{P}_t^H$ and $\mathcal{P}_t^F$. The final term implies that dead-weight losses from the unconventional tool are lowered when it is employed in a country with an inefficiently low level of inflation and vice versa. Welfare losses are shown by contrasting the dynamics under nominal price rigidities with counter-factual dynamics that would arise under fully flexible prices and complete central bank inactivity in government bond markets. This second feature of the flexible price equilibrium used as a reference however implies that it could be improved upon by adequate asset purchases resulting in the small modification of the terms of trade gap in the final term of the loss function.[18] The cost function W enters welfare only through the parameter κ, which is defined as $\kappa \equiv \frac{W_{\mathcal{P}\mathcal{P}}(\mathcal{P})}{U_C(C)C}$. Welfare costs of bond purchases that arise independent of their effects on relative prices therefore depend on the convexity of W. If W is linear, for example, κ equals zero and W plays no role.

The welfare criterion is purely quadratic. This is achieved through a combination of the steady state subsidy described before and appropriate substitutions; the methods used here are described in detail in Benigno and Woodford (2012). As a result, welfare is correctly approximated to the second order despite of the policy derived from the LQ problem necessarily being of first order only. Furthermore, the policy rule obtained from the LQ problem is a correct first-order approximation of the optimal policy in the non-linear problem.

[18]Beetsma and Jensen (2005) analyse the optimal monetary and fiscal policy mix for a currency union in a comparable model. They avoid a similar modification of the terms of trade gap in a cross term with the fiscal instrument by deriving natural government spending levels that permit writing the fiscal instrument in gap form. This route is not followed here to avoid a dependency of loss on the unintuitive concept of "natural unconventional monetary policy".

2.3.3 Calibration

The calibration follows Benigno (2004) and Ferrero (2009). It is summarised in Table 2.1.

Parameter	Value	Description
β	0.99	Households discount factor
σ	7.66	Intratemporal elasticity of substitution between consumption goods
ρ	1	Coefficient of relative risk aversion
η	0.47	Elasticity of production
α_H	0.75	Price stickiness in H
α_F	0.75	Price stickiness in F
κ	$[0, 2]$	Normalised convexity of $W(\cdot)$ at the approximation point
n	$\{0.5, 0.8\}$	Size of country H

Table 2.1: Calibration

A period is assumed to be a quarter. β is set to 0.99, giving rise to an annualised interest rate of about 4 per cent in steady state. As in Benigno (2004), the intratemporal elasticity of substitution between consumption goods of the same country is set to 7.66, implying a steady state mark-up of about 15 per cent. In Benigno (2004), the coefficient of relative risk aversion is 1/6. This is low compared to Ferrero (2009) who uses a value of 3. I choose an intermediate value of 1, which is consistent with a logarithmic sub-utility function for consumption. Assuming that output at firm-level is produced using a simple CRS production function with labour as the only input, η is the inverse Frisch elasticity of labour supply.[19] Ferrero (2009) sets this value to 0.47, which I follow here. The price stickiness parameters are equal in both countries at 0.75. This implies that prices are changed every year on average. κ is defined as $\kappa \equiv \frac{W_{\mathcal{PP}}(\mathcal{P})}{U_C(C)C}$. Assuming CES functional forms for $U(\cdot)$ and $V(\cdot)$, equation (2.53) implies that $C = 1$ and $U_C(C) = 1$. Since $\mathcal{P} = 0$, κ in this case equals $W_{\mathcal{PP}}(0)$, giving the convexity of the cost function $W(\cdot)$ at the steady state used as the approximation point. It appears plausible to assume that each additional unit of intervention in asset markets carries a larger welfare cost, which implies that κ is non-negative. Since it is difficult to find a precise estimate of κ, results are shown for a large

[19] Suppose that the disutility from production $V\left[y_t(j), z_t^i\right]$ results from supplying $l_t(j)$ units of labour where $y_t(j) = \exp(-z_t^i)l_t(j)$ so that V can be expressed as $V\left[l(j)\right]$ in steady state. Since utility is additively separable in consumption and labour supply, the inverse Frisch elasticity is given by $\frac{V_{ll}}{V_l}l(j)$ and can be written as $\frac{V_{yy}}{V_y}y(j)$ which is precisely equal to η given that $y(j) = C$.

range of values. Furthermore, two scenarios will be considered, one in which H and F are of equal size and one in which one country, here H, is significantly larger than the other, containing 80 per cent of all productive capacity and consumers of the union.

2.4 Optimal Policy Coordination

The central bank of the union combines conventional and unconventional monetary policy to stabilise disturbances to the economy. This section describes the optimal mix of the two instruments. Conventional policy takes the form of targets for the short-term nominal interest rate $\hat{\imath}_t$, which the central bank is assumed to control directly. Unconventional monetary policy refers to interventions in the markets for government debt, captured by $\mathcal{P}_t^H$ and $\mathcal{P}_t^F$. The central bank maximises welfare given by (2.74) subject to the constraints posed by the structure of the economy.[20] It is assumed to possess a credible commitment device and to choose policy from the "timeless perspective".[21] The optimisation yields a time-invariant set of first-order conditions, which, together with the constraints, forms a system of linear expectational difference equations that can be solved for the equilibrium laws of motion of the endogenous variables.

To be able to isolate the effects of conventional and unconventional monetary policy, let it be true now that the degree of price stickiness is identical in H and F, that is $\alpha_H = \alpha_F$. Benigno (2004) shows that a central bank that solely has to rely on the conventional instrument is unable to affect the relative price level between the two countries in this case; the same is true here. The terms of trade evolve according to

$$\hat{T}_t = \hat{T}_{t-1} + \hat{\Pi}_t^F - \hat{\Pi}_t^H \tag{2.76}$$

where $\hat{\Pi}_t^F - \hat{\Pi}_t^H$ can be found by subtracting the Phillips curve for H from that for F,

$$\hat{\Pi}_t^F - \hat{\Pi}_t^H \quad = \quad \beta \mathrm{E}_t \left(\hat{\Pi}_{t+1}^F - \hat{\Pi}_{t+1}^H \right) - a \left(\hat{T}_t - \tilde{T}_t \right) + a_{\mathcal{P}} \left(\mathcal{P}_t^F - \mathcal{P}_t^H \right) \tag{2.77}$$

[20] The policy optimisation problem is described in detail in the appendix.

[21] See Woodford (1999) and Woodford (2003) for details on optimal policy from the timeless perspective.

with $a \equiv a_C^H = a_C^F = a_T^H = a_T^T$ and $a_{\mathcal{P}} \equiv a_{\mathcal{P}}^H = a_{\mathcal{P}}^F$.[22] Since the consumption gap has a symmetric effect on inflation in both countries, it does not enter relative inflation. But as changes in the nominal interest rate only feed into the rest of the economy through changes to the households' consumption and savings behaviour, the inflation rate differential and thus the terms of trade are independent of $\hat{\imath}_t$. Using the conventional tool alone, the central bank can therefore not actively close the terms of trade gap in response to an exogenous disturbance that moves the natural rate of the terms of trade away from its steady state level. However, $\hat{\Pi}_t^F - \hat{\Pi}_t^H$ depends on the unconventional tool, allowing the central bank to affect the terms of trade gap in this way. As a result, all policy induced smoothing of terms of trade shocks is entirely due to unconventional policy in the special case considered here.

Combining (2.76) and (2.77) gives

$$\hat{T}_t = \frac{1}{1+\beta+a}\left[\hat{T}_{t-1} + \beta \mathrm{E}_t \hat{T}_{t+1} + a_{\mathcal{P}}\left(\mathcal{P}_t^F - \mathcal{P}_t^H\right) + a\tilde{T}_t\right] \tag{2.78}$$

The terms of trade in period t depend on a lag, their natural rate, expectations formed in t about the terms of trade in $t+1$ and the relative strength of unconventional policy interventions. When $\tilde{T}_t > 0$ and $\hat{T}_{t-1} = 0$, for example, which could be interpreted as a cost-push shock in F and a deflationary shock in H, it is a well-known result that the central bank aims to create deflationary expectations in F and inflationary expectations in H. The fact that $\hat{T}_t$ depends positively on $\mathrm{E}_t\hat{T}_{t+1}$ adds to the incentive of the central bank to create such expectations to steer the terms of trade back to target. Since a quadratic cost is attached to both, the commitment to future inflation or deflation and unconventional interventions, generally a mix of the two minimises welfare losses, giving rise to a persistent unconventional policy response along the terms of trade and inflation adjustment path. The proposition below characterises the optimal policy further.[23]

Proposition 1 *For $\alpha_H = \alpha_F$, in the equilibrium generated by the optimal commitment policy chosen from the timeless perspective, the following is true for all $t \geq 0$.*

a) Average inflation is entirely stabilised.

$n\hat{\Pi}_t^H + (1-n)\hat{\Pi}_t^F = 0$

[22] These relationships hold for $\alpha_H = \alpha_F$ and $\rho = 1$.

[23] The derivations can be found in Section 2.7.6 of the appendix.

b) The short-term interest rate only depends on the natural rate of consumption.

$\hat{i}_t = -\rho \tilde{C}_t$

c) $\mathcal{P}_t^R$ *is made dependent on* $\mathcal{P}_{t-1}^R$, $\mathrm{E}_t \mathcal{P}_{t+1}^R$, *past, present and expected future values of the terms of trade as well as current and lagged values of its natural rate.*

$\mathcal{P}_t^R = \mathrm{E}_t f\left(\mathcal{P}_{t-1}^R, \mathcal{P}_{t+1}^R, \hat{T}_{t-1}, \hat{T}_t, \hat{T}_{t+1}, \tilde{T}_{t-1}, \tilde{T}_t\right)$

A result from Benigno (2004) is that a central bank that is equipped with the conventional tool only should stabilise average union-wide inflation. Part *a*) of the proposition above shows that this is still true when the unconventional tool is at its disposal as well. Note that this implies that small deviations of inflation from target in a large country are accompanied by large deviations in a small country. For example, if inflation is 50 basis points below target in 80 per cent of the union, it should be 200 basis points above target in the rest of the currency union. Large inflation differentials are therefore not necessarily an indicator for suboptimal conventional or unconventional monetary policy. It has been argued before that conventional monetary policy is unable to affect the relative price level between H and F. Since parallel shifts in inflation in response to shocks to the natural terms of trade are not welfare-improving, $\hat{i}_t$ is adjusted to close the consumption gap at all times, which results in the simple rule for $\hat{i}_t$ given in *b*). The final part shows that $\mathcal{P}_t^R$ has an autoregressive component, leading to persistence in the unconventional tool and that the path of the terms of trade only depends on the the paths of the corresponding natural rate and the relative strength of unconventional policy in both countries. The results described in *b*) and *c*) give insights for the question of how to design simple policy rules that approximate the optimal policy as closely as possible.

2.4.1 Countries of Equal Size

Figure 2.2 shows the impulse responses of an unexpected 10% one-off increase in the natural terms of trade in period $t = 1$ for different values of the cost parameter κ. The two countries are assumed to be equal sized.

When confronted with a shock to the natural terms of trade, the monetary policy maker faces a fundamental trade-off with respect to the optimal adjustment of the terms of trade. By letting

$\hat{T}_t$ rise upon impact of the shock, contemporaneous losses from an increase in the terms of trade gap are decreased. To achieve the increase in $\hat{T}_t$, inflation has to be allowed to fall below target in H and to rise above target in F, which leads to contemporaneous welfare costs. Since the terms of trade are elevated entering into the next period in this case, additional welfare costs arise from steering $\hat{T}_t$ back to target. These additional costs result from a succession of positive terms of trade gaps, increased inflation in H, decreased inflation in F and unconventional interventions along the adjustment path. In contrast, when the terms of trade are not permitted to rise in $t = 1$, large contemporaneous costs resulting from the terms of trade gap and unconventional policy necessary to prevent inflation from falling in H and rising in F have to be incurred. However, the shock is fully stabilised upon impact, which means that future losses are avoided.

The degree to which it is optimal to let $\hat{T}_t$ initially increase depends on the costs associated with unconventional policy. If $\kappa = 0$, unconventional policy is sufficiently uncostly to make it optimal to stabilise inflation and thus the terms of trade entirely. Large asset purchases in H limit the deflationary effect of the shock and large sales in F limit the inflationary pressure that it causes there. As κ increases, larger initial reactions of $\hat{T}_t$ are permitted, since stabilising $\hat{T}_t$ becomes more costly. Even for large values of κ, the terms of trade increase in $t = 1$ is moderate, however. This is due to the fact that deviations of inflation from target are significantly more costly than all other components of the welfare loss function.

Unconventional policy plays two roles. It is used in $t = 1$ to dampen the impact of the terms of trade shock on inflation in both countries and, in the subsequent periods, to steer the terms of trade back to target in case $\hat{T}_t$ was initially allowed to increase. Since it is optimal to prevent large movements in $\hat{T}_t$ due to the reasons outlined above, the unconventional tool is more heavily used at the time at which the unexpected shift in the natural terms of trade occurs compared to the following periods. For the parameter constellations considered here, unconventional policy in $t = 1$ is applied at about 16% of steady state output for $\kappa = 0$, 5% for $\kappa = 1$ and 3% for $\kappa = 2$. In $t = 2$ these shares fall to about 0.00%, 0.09% and 0.07% and then gradually decline.

Conventional monetary policy is inactive in response to the terms of trade shock. This can be seen from the fact that consumption remains entirely unaffected by it. The bigger κ is, the more restrictive are the constraints posed by the Phillips curves and the law of motion of the terms of trade resulting in more volatile reactions of the corresponding multipliers. The constraint associated with the requirement that the average unconventional intervention be zero at all times does no bind, since it has been imposed through appropriate substitution into the welfare criterion.

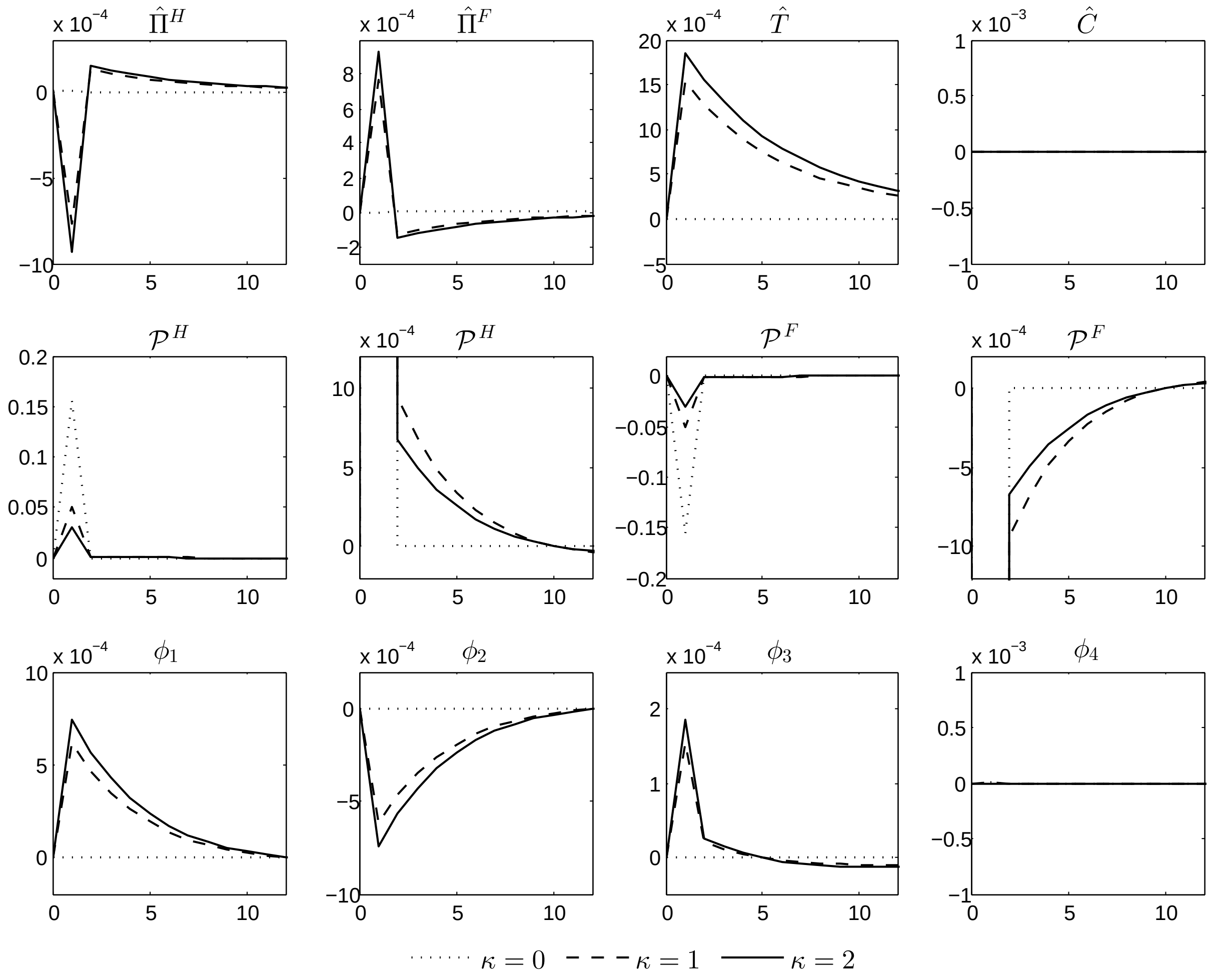

Figure 2.2: Impulse responses to a 10% increase in natural terms of trade for different values of κ

Figure 2.3 compares the impulse responses for the two cases that unconventional monetary policy can be used without restrictions and that central bank purchases of government debt are restricted to be zero at all times. The responses for the case that unconventional policy can be used are depicted for an intermediate value of the cost parameter of $\kappa = 1$. It can be seen that bond market interventions by the central bank stabilise the terms of trade and inflation in both countries significantly. In particular, the unconventional tool causes the initial impact on inflation in both countries and the terms of trade to be significantly less pronounced. The additional monetary policy tool also works to reduce the tightness of the restrictions posed by the Euler equation as can be seen from the response of the corresponding Lagrange multipliers. The comparison between regimes in which unconventional monetary policy can and cannot be used is continued in Section 2.5 which contains quantitative results for the welfare implications of having access to the unconventional tool.

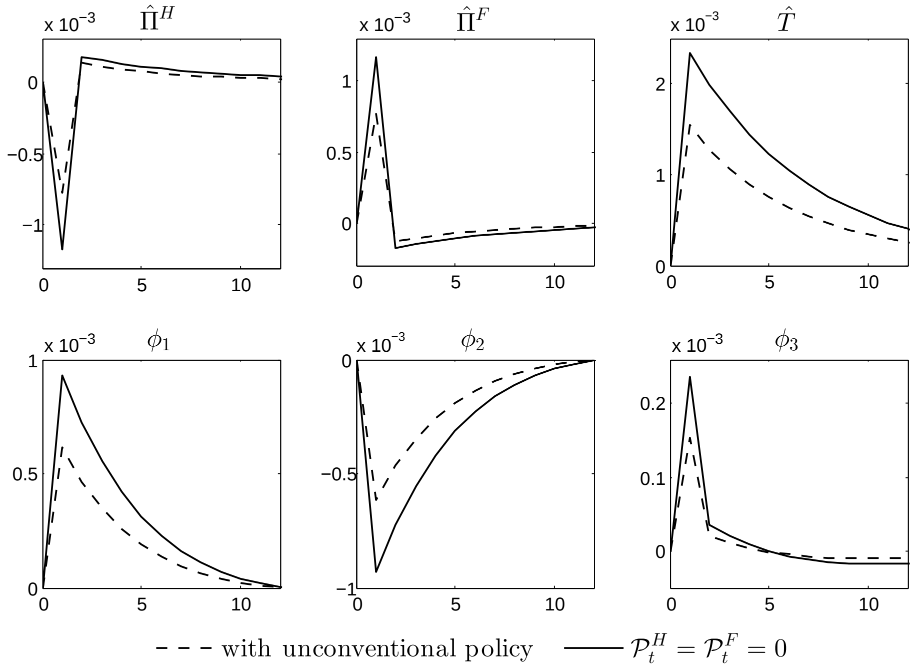

Figure 2.3: Comparison to the case without unconventional policy ($\kappa = 1$)

2.4.2 Countries of Unequal Size

When the member countries of the currency union are of unequal size, optimal unconventional policy is asymmetric. The impulse responses to a 10% increase in the natural terms of trade for the case that country H contains 80 per cent of all consumers and productive capacity are shown in Figure 2.4.

For each κ, the same initial increase in the terms of trade as in the case of identical countries is permitted. However, since the contribution of F to aggregate welfare is smaller than that of

H, higher inflation volatility is allowed there. A large share of the goods consumed in the small country is imported from H while only a small fraction of the goods produced in H are imported from F, which implies that the terms of trade shock increases inflation in F more than in H as can be seen from the two Phillips curves. To dampen the initial impact of the disturbance and to steer inflation in both countries and the terms of trade back to target from period $t = 2$ onwards, the unconventional tool is therefore used more heavily in F than in H. Shocks that affect the relative price level between two countries of a currency union thus result not only in more inflation volatility but also in stronger unconventional interventions in the smaller of the two countries if policy is chosen optimally.

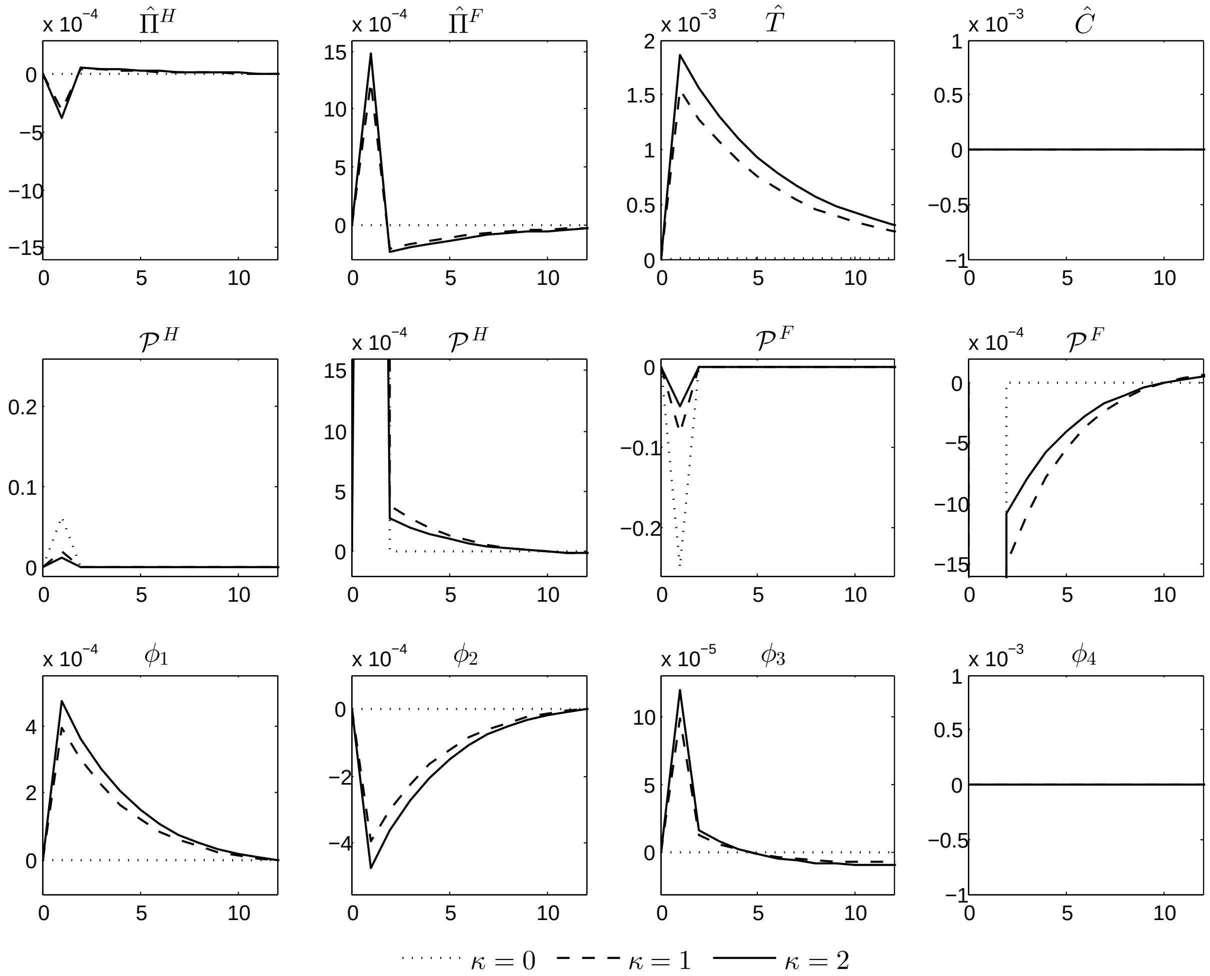

Figure 2.4: Impulse responses to a terms of trade shock in a union of a large and a small country

2.5 Welfare

To perform a welfare analysis, stochastic processes for the exogenous shocks have to be specified. It was shown above that the central bank employs the conventional tool to close the consumption gap at all times when the degrees of price stickiness are equal in both countries. This implies that welfare losses may arise from movements in the natural terms of trade but not from variation in the natural rate of consumption. I continue to consider the case $\alpha^H = \alpha^F$ here to separate the effects of the two monetary tools from each other and therefore only describe the calibration of the process of $\tilde{T}_t$. If the underlying exogenous components of $\tilde{T}_t$, that is—technology and government spending shocks—are assumed to follow AR(1) processes, then, according to Granger and Morris (1976), $\tilde{T}_t$ follows a higher order ARMA process. To be able to derive a closed form expression for welfare, the strategy chosen here is to approximate the micro-founded process of $\tilde{T}_t$ using an AR(1) process and to calculate welfare based on the approximated time series.

The law of motion of the natural terms of trade is approximated by

$$\tilde{T}_t^a = \rho_T \tilde{T}_{t-1}^a + u_{T,t} \tag{2.79}$$

where $u_{T,t} \sim N(0, \sigma_T^2)$. To derive ρ_T and σ_T^2, recall that $\tilde{T}_t$ is defined as

$$\tilde{T}_t = \frac{\eta}{1+\eta}\left(g_t^{P,R} - \zeta_t^R\right) \tag{2.80}$$

with $g_t^{P,i} \equiv \frac{G_t^{P,i} - G^{P,i}}{Y^i}$ and $\zeta_t^i \equiv -\frac{V_{yz}(C,0)}{CV_{yy}(C,0)} z_t^i$. The disutility from production V can be thought of as arising from labour exerted in the production process. Assuming that the production function is $y_t(j) = \exp(-z_t^i) l_t(j)$ where $l_t(j)$ is labour and that $V(\cdot)$ is a standard CES function of $l_t(j)$, given by $V[l_t(j)] = \frac{l_t(j)^{1+\eta}}{1+\eta}$, the supply side shock is given by $\zeta_t^i = -\frac{1+\eta}{\eta} z_t^i$. Both types of "deep" shocks evolve according to AR(1) processes, that is

$$z_t^i = \rho_z z_{t-1}^i + \varepsilon_t^{z,i} \tag{2.81}$$

$$\frac{G_t^{P,i}}{Y^i} = \rho_G \frac{G_{t-1}^{P,i}}{Y^i} + \varepsilon_t^{G,i} \tag{2.82}$$

where $\left(\varepsilon_t^{z,H}, \varepsilon_t^{z,F}\right) \sim N(0, \Sigma^z)$, $\left(\varepsilon_t^{G,H}, \varepsilon_t^{G,F}\right) \sim N(0, \Sigma^G)$ and

$$\Sigma^z \equiv \begin{bmatrix} \sigma_{z,H}^2 & \sigma_{z,HF}^2 \\ \sigma_{z,FH}^2 & \sigma_{z,F}^2 \end{bmatrix}, \quad \Sigma^G \equiv \begin{bmatrix} \sigma_{G,H}^2 & \sigma_{G,HF}^2 \\ \sigma_{G,FH}^2 & \sigma_{G,F}^2 \end{bmatrix} \tag{2.83}$$

This specification allows for correlation between the evolution of technology in both countries and for correlated government spending shocks.[24] Demand and supply side shocks are assumed to be independent. Using equation (2.80), one can derive the variance of $\tilde{T}_t$ as a function of (calibrated) parameters. The variance of $\tilde{T}_t^a$ is chosen to match this variance, more precisely

$$\begin{aligned}
\text{Var}\left(\tilde{T}_t^a\right) = \text{Var}\left(\tilde{T}_t\right) &= \left(\frac{\eta}{1+\eta}\right)^2 \text{Var}\left[g_t^{P,F} - g_t^{P,H} - \left(\zeta_t^F - \zeta_t^H\right)\right] \\
&= \left(\frac{\eta}{1+\eta}\right)^2 \left[\text{Var}\left(g_t^{P,F}\right) + \text{Var}\left(g_t^{P,H}\right) + \text{Var}\left(\zeta_t^F\right) + \text{Var}\left(\zeta_t^H\right)\right. \\
&\quad \left. -2\text{Cov}\left(g_t^{P,H}, g_t^{P,F}\right) - 2\text{Cov}\left(\zeta_t^H, \zeta_t^F\right)\right] \\
&= \left(\frac{\eta}{1+\eta}\right)^2 \left[\frac{\sigma_{G,F}^2}{1-\rho_G^2} + \frac{\sigma_{G,H}^2}{1-\rho_G^2} + \left(\frac{1+\eta}{\eta}\right)^2 \frac{\sigma_{z,F}^2}{1-\rho_z^2} + \left(\frac{1+\eta}{\eta}\right)^2 \frac{\sigma_{z,H}^2}{1-\rho_z^2}\right. \\
&\quad \left. -2\frac{\sigma_{G,HF}^2}{1-\rho_G^2} - 2\left(\frac{1+\eta}{\eta}\right)^2 \frac{\sigma_{z,HF}^2}{1-\rho_z^2}\right]
\end{aligned} \tag{2.84}$$

The persistence parameter is found numerically. To do so, I simulate $\tilde{T}_t$ for one million periods and calculate ρ_T using the OLS/ML estimator, so $\rho_T = \sum \tilde{T}_t \tilde{T}_{t-1} / \sum \tilde{T}_{t-1}^2$.

Table 2.2 summarises the calibration of the demand and supply shock sequences. The variance and persistence of the shock processes are assumed to be equal in H and F. The estimates of the persistence parameters are taken from Smets and Wouters (2005), which includes an estimation of a DSGE model for the Euro Area based on quarterly data from 1983 to mid-2002. The variances are estimates reported in Duarte and Wolman (2008). Duarte and Wolman estimate the laws of motion of technology shocks in the traded goods sector and the government spending share of GDP using French and German quarterly data from the 1990s and early 2000s.[25]

[24] Government spending is normalised by steady state output to facilitate the comparability with the previous literature.

[25] The variance estimates reported in Smets and Wouters (2005) are significantly larger than those from Duarte and Wolman (2008). Larger variances imply higher potential welfare gains from unconventional policy. In order not to over-state possible welfare gains, the smaller set of variances is used here, which is also in line with the calibrations from Gali (2008) and others.

Parameter	Value	Description
ρ_z	0.94	Persistence of technology shocks
ρ_G	0.98	Persistence of government spending shock
$\sigma^2_{z,H}$, $\sigma^2_{z,F}$	$0.16 * 10^{-3}$	Variance of technology shock
$\sigma^2_{G,H}$, $\sigma^2_{G,F}$	$0.21 * 10^{-3}$	Variance of government spending shock

Table 2.2: Additional Calibration

Figure 2.5 shows the welfare gains from employing the unconventional tool as a function of the respective correlation of demand and supply shocks across the two countries. The welfare measure λ is given by

$$\lambda = 1 - e^{(1-\beta)(W^a - W)} \tag{2.85}$$

W and W^a are the unconditional expectation of welfare under the policy that is optimal from the timeless perspective when the central bank is able to credibly commit and under an alternative policy, respectively. Here, the alternative policy is optimal subject to the same qualifications as W if the central bank does not have access to the unconventional monetary tool. λ translates the welfare differential between both policies into units of steady state consumption.[26] If both types of disturbances are perfectly correlated, no welfare gains are obtainable. Perfect correlation of demand and supply shocks implies that $\tilde{T}_t = \tilde{T} = 0$ for all t. To see this, note that for $\sigma^2_{z,H} = \sigma^2_{z,F} = \sigma^2_z$ and $\sigma^2_{G,H} = \sigma^2_{G,F} = \sigma^2_G$, $\operatorname{Var}\left(\tilde{T}_t\right)$ can be written as

$$\begin{aligned} \operatorname{Var}\left(\tilde{T}_t\right) = \left(\frac{\eta}{1+\eta}\right)^2 \Bigg\{ & \frac{2}{1-\rho_G^2}\sigma_G^2 \left[1 - \operatorname{Corr}(\varepsilon_t^{G,H}, \varepsilon_t^{G,F})\right] + \\ & \frac{2}{1-\rho_z^2}\left(\frac{1+\eta}{\eta}\right)^2 \sigma_z^2 \left[1 - \operatorname{Corr}(\varepsilon_t^{z,H}, \varepsilon_t^{z,F})\right] \Bigg\} \end{aligned} \tag{2.86}$$

where $\operatorname{Corr}(\cdot,\cdot) \in [-1,1]$ is Pearson's correlation coefficient. Thus, if the shocks within the currency union are perfectly correlated across countries, the natural terms of trade remain at their steady state level and there are no gains from employing unconventional monetary policy. The lower is the correlation of the disturbances, the larger becomes the variance of $\tilde{T}_t$ and the higher are the welfare gains associated with central bank asset purchases. This link results from

[26]See Section 2.7.7 in the appendix for the details.

the fact that, in contrast to conventional short-term interest rate policy, the unconventional monetary policy tool can be used to close the gap between the terms of trade and their natural rate. In addition, the variance of the individual shocks, their persistence and the inverse elasticity of substitution in production matter for the variance of $\tilde{T}_t$ and thus for the gains obtainable from employing asset purchases.

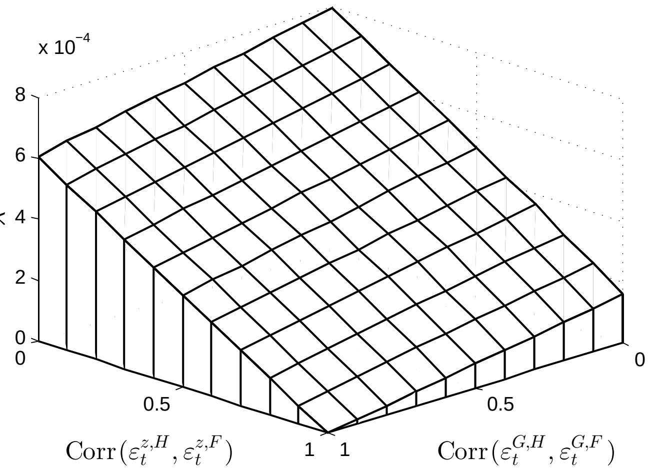

Figure 2.5: Welfare gains as a function of shock correlations (for $\kappa = 1$)

The magnitude of the welfare gains from the unconventional tool depends on κ, the parameter that reflects the costs associated with unconventional interventions. Figure 2.6 contrasts the gains obtainable for different values of κ. Shown are isoquants of λ with a step size of 0.01 percentage points. When starting at the point of perfect correlation $(1, 1)$ and moving into the direction of lower correlation of both disturbances, i.e. to the south-west, welfare increases faster for lower values of κ. For $\kappa = 1$, if government spending shocks are entirely uncorrelated and the correlation coefficient of productivity shocks equals 0.8, the welfare gains from unconventional interventions are at a moderate value of around 0.03% of steady state consumption. The calibration chosen here implies that welfare gains increase faster with declining correlation in supply shocks than in demand shocks, emphasising the importance for the central bank to monitor productivity differences within the currency union.

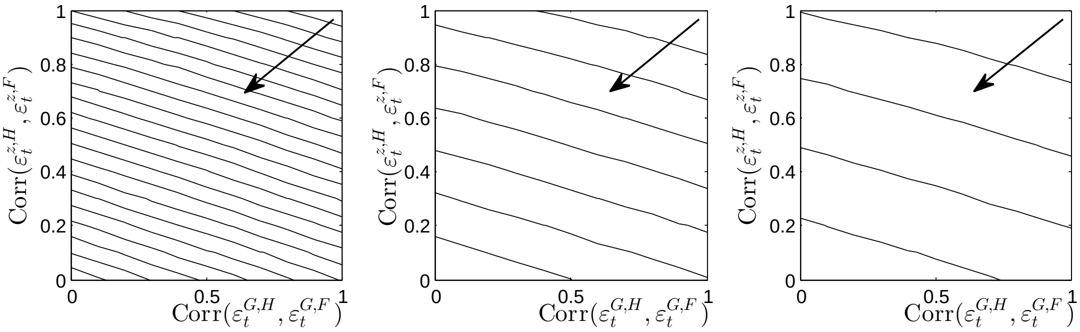

Figure 2.6: Welfare gains for $\kappa = 0$ (left), $\kappa = 1$ (centre), $\kappa = 2$ (right)

2.6 Conclusion

By purchasing debt issued by one of the member countries, the central bank reduces the yield and increases the price of bonds given out by the government of the country in which the operation was carried out. The government's budget constraint is loosened, causing taxes to fall, spending to increase and thus aggregate demand and inflation to rise. The mechanism described here allows the central bank to "correct" fiscal policy decisions made by the member states. A lack of fiscal co-ordination is balanced by the common monetary policy institution that can use the unconventional tool. For countries of similar structure, the optimal commitment policy from the timeless perspective stabilises average union-wide inflation, closes the consumption gap by adjusting the short-term nominal interest rate accordingly and uses asset purchases to respond to movements in the natural terms of trade. In response to less than perfectly correlated demand and supply shocks that cause an unexpected increase in the natural terms of trade, government bonds are bought and sold upon impact of the shock at a large scale in H and F, respectively. The same measures are taken over the following periods but at a lower scale and of gradually declining magnitude to steer the terms of trade and inflation back to target.

The mechanism considered here is only one way in which central bank asset purchases work. A number of empirical contributions emphasise the effects of asset purchases on the savings decision of households through changes in the supply of bonds with particular maturities, the re-allocation of risk between the central bank and the private sector, and the informational content of central bank asset purchases with respect to the path of future short-term rates. As long as unconventional central bank operations possess an asymmetric effect on aggregate demand in the member countries, they may be useful for smoothing idiosyncratic shocks. Previous contributions view central bank asset purchases as a tool that should be used only in crisis times. In my model, the result that unconventional policy may be of particular importance if fiscal co-ordination cannot be reached is independent from the prevailing economic conditions.

2.7 Appendix

2.7.1 Aggregate Demand

The derivations are shown for H only, analogous steps lead to the result for country F. Since $g_t(j) = \left[\frac{p_t(j)}{P_{H,t}}\right]^{-\sigma} G_t^H$ for all home goods, we have

$$g_t(j) = \left[\frac{p_t(j)}{P_{H,t}}\right]^{-\sigma} \left(G_t^{P,H} + X_t\right) \tag{2.87}$$

where X_t is the difference between planned and revised government spending defined by

$$X_t = G_t^H - G_t^{P,H}$$

Multiplying (2.21) and (2.87) with $\frac{p_t(j)}{P_t}$ and integrating over all agents in H gives

$$\int_0^n \frac{p_t(j) g_t^P(j)}{P_t} dj = \int_0^n \frac{p_t(j)}{P_t} \left[\frac{p_t(j)}{P_{H,t}}\right]^{-\sigma} G_t^{P,H} dj \tag{2.88}$$

$$\int_0^n \frac{p_t(j) g_t(j)}{P_t} dj = \int_0^n \frac{p_t(j)}{P_t} \left[\frac{p_t(j)}{P_{H,t}}\right]^{-\sigma} \left(G_t^{P,H} + X_t\right) dj \tag{2.89}$$

Subtracting the top from the bottom equation and simplifying yields

$$\int_0^n \frac{p_t(j) g_t(j)}{P_t} dj - \int_0^n \frac{p_t(j) g_t^P(j)}{P_t} dj = \int_0^n \frac{p_t(j)^{1-\sigma} \left(P_{H,t}\right)^{\sigma}}{P_t} X_t dj \tag{2.90}$$

Using (2.27), this condition implies that

$$\int_0^n \frac{p_t(j)^{1-\sigma} \left(P_{H,t}\right)^{\sigma}}{P_t} X_t dj = \theta^H \frac{\bar{Q}_t^H}{P_t} \left(P_t^{QH} - P_t^{QH,e}\right) \tag{2.91}$$

From the definition of $P_{H,t}$ follows that $\int_0^n p_t(j)^{1-\sigma} dj = n \left(P_{H,t}\right)^{1-\sigma}$, so

$$n \frac{P_{H,t}}{P_t} X_t = \theta^H \frac{\bar{Q}_t^H}{P_t} \left(P_t^{QH} - P_t^{QH,e}\right) \tag{2.92}$$

Noting that $\frac{P_{H,t}}{P_t}$ can be expressed as T_t^{n-1}, one can write the solution for X_t as

$$X_t = \theta^H \frac{T_t^{1-n}}{nP_t} \bar{Q}_t^H \left(P_t^{QH} - P_t^{QH,e} \right) \tag{2.93}$$

which confirms equation (2.33) from the text.

2.7.2 Equality of Private Consumption Across Countries

The real central bank profit share transferred to H is given by

$$\frac{\Delta_t^H}{P_t} = n\left(\frac{M_t^S}{P_t} - \frac{M_{t-1}^S}{P_t}\right) + \frac{Q_{CB,t-1}^H}{P_t} - \frac{P_t^{QH} Q_{CB,t}^H}{P_t} \tag{2.94}$$

where $P_t^{QH} \equiv [(1+i_t^Q)(1-\epsilon_t^H Q_{CB,t}^H)]^{-1}$. Substituting for $\frac{\Delta_t^H}{P_t}$ in the budget constraint of the government of H and solving for aggregate real taxes gives

$$\begin{aligned} \int_0^n \frac{T_t^H}{P_t} dj &= \int_0^n \frac{p_t(j)g_t(j)}{P_t} dj - \tau \int_0^n \frac{p_t(j)y_t(j)}{P_t} dj \\ &- \left[\int_0^n \frac{P_t^{QH} Q_t^{H,j}}{P_t} dj - \int_0^n \frac{Q_{t-1}^{H,j}}{P_t} dj \right] - n\left(\frac{M_t^S}{P_t} - \frac{M_{t-1}^S}{P_t}\right) \end{aligned} \tag{2.95}$$

The budget constraint of an agent $j \in [0, n)$, integrated over all agents of H, is given by

$$\begin{aligned} &\int_0^n q_t^H B_t^{H,j} dj + \int_0^n \frac{B_t^j}{P_t(1+i_t)} dj + \int_0^n \frac{P_t^{QH} Q_t^{H,j}}{P_t} dj + \int_0^n \frac{M_t^j}{P_t} dj + \int_0^n C_t^j dj + \int_0^n \frac{T_t^H}{P_t} dj \\ &= \int_0^n \mathbb{1}_{1\times S} B_{t-1}^{H,j} dj + \int_0^n \frac{B_{t-1}^j}{P_t} dj + \int_0^n \frac{Q_{t-1}^{H,j}}{P_t} dj + \int_0^n \frac{M_{t-1}^j}{P_t} dj + (1-\tau) \int_0^n \frac{p_t(j)y_t(j)}{P_t} \end{aligned} \tag{2.96}$$

Together with the previous equation and using the fact that the state-contingent securities are in zero net supply within each country, this implies that the resource constraint for H is

$$\begin{aligned} &\int_0^n \frac{B_t^j}{P_t(1+i_t)} dj + \int_0^n \frac{M_t^j}{P_t} dj + \int_0^n C_t^j dj + \int_0^n \frac{p_t(j)g_t(j)}{P_t} dj - n\left(\frac{M_t^S}{P_t} - \frac{M_{t-1}^S}{P_t}\right) \\ &= \int_0^n \frac{B_{t-1}^j}{P_t} dj + \int_0^n \frac{M_{t-1}^j}{P_t} dj + \int_0^n \frac{p_t(j)y_t(j)}{P_t} dj \end{aligned} \tag{2.97}$$

We now observe that if $C_t^H = C_t^F = C_t$, then the optimality condition for money demand (2.11) implies that $M_t^H = M_t^F = M_t$. Using these relationships, which will be verified later, together with the fact that the money market clears in equilibrium and expressing integrals in terms of quantities pertaining to the representative agent of H where possible, one can re-write this equation as

$$n\frac{B_t^H}{P_t(1+i_t)} - n\frac{B_{t-1}^H}{P_t} + nC_t^H = \int_0^n \frac{p_t(j)\left[y_t(j) - g_t(j)\right]}{P_t}dj \tag{2.98}$$

Given (2.29), this can be expressed as

$$n\frac{B_t^H}{P_t(1+i_t)} - n\frac{B_{t-1}^H}{P_t} + nC_t^H = \int_0^n \frac{p_t(j)}{P_t}\left(\frac{p_t(j)}{P_{H,t}}\right)^{-\sigma} T_t^{1-n}C_t^U dj \tag{2.99}$$

Since $\int_0^n p_t(j)^{1-\sigma}dj = n\left(P_{H,t}\right)^{1-\sigma}$ and $T_t^{1-n} = \frac{P_t}{P_{H,t}}$, we get

$$\frac{B_t^H}{P_t(1+i_t)} - \frac{B_{t-1}^H}{P_t} + C_t^H = C_t^U \tag{2.100}$$

Analogous steps lead to a symmetric condition for country F. The proof proceeds from here as shown in Benigno (2003).[27] Benigno shows that given the resource constraints for both countries, the optimality conditions from the agents' utility maximisation problem and the assumption of symmetric initial wealth, B_t^i is zero for all $t \geq 0$, which immediately implies that $C_t^i = C_t \;\; \forall t \geq 0$.

2.7.3 Flexible Price Equilibrium

To derive equation (2.56), one can first combine equations (2.54) and (2.55). This yields

$$T_t \equiv \frac{P_{F,t}}{P_{H,t}} = \frac{V_y\left(Y_t^F, z_t^F\right)}{V_y\left(Y_t^H, z_t^H\right)}$$

which after log-linearisation becomes

$$\tilde{T}_t = \frac{V_{yy}\left(C,0\right)}{V_y\left(C,0\right)}C\left(\tilde{Y}_t^F - \tilde{Y}_t^H\right) + \frac{V_{yz}\left(C,0\right)}{V_y\left(C,0\right)}z_t^F - \frac{V_{yz}\left(C,0\right)}{V_y\left(C,0\right)}z_t^H \tag{2.101}$$

[27]See Benigno (2003), pp. i – ii.

Aggregate demand in each country is given by

$$\begin{aligned} Y_t^H &= T_t^{1-n} C_t + G_t^{P,H} \\ Y_t^F &= T_t^{-n} C_t + G_t^{P,F} \end{aligned}$$

Log-linearising these two relationships yields

$$\begin{aligned} \tilde{Y}_t^H &= (1-n)\tilde{T}_t + \tilde{C}_t + g_t^{P,H} \\ \tilde{Y}_t^F &= -n\tilde{T}_t + \tilde{C}_t + g_t^{P,F} \end{aligned}$$

where $g_t^{P,H} \equiv \frac{G_t^{P,H} - G^{P,H}}{Y^H}$ and $g_t^{P,F} \equiv \frac{G_t^{P,F} - G^{P,F}}{Y^F}$. Subtracting the top from the bottom equation gives

$$\tilde{Y}_t^F - \tilde{Y}_t^H = -\tilde{T}_t + g_t^{P,F} - g_t^{P,H} \tag{2.102}$$

By entering (2.102) into (2.101), defining $\eta \equiv \frac{V_{yy}(C,0)}{V_y(C,0)} C$ as well as $\zeta_t^i \equiv \frac{V_{yz}(C,0)}{V_{yy}(C,0)C} z_t^i$ for $i \in \{H, F\}$ and simplifying, one obtains

$$\tilde{T}_t = \frac{\eta}{1+\eta} \left(g_t^{P,R} - \zeta_t^R \right) \tag{2.103}$$

Aggregate demand in H and F can also be used to establish a connection between world output and consumption. Adding the two demand equations weighted by country size yields

$$nY_t^H + (1-n)Y_t^F = nT_t^{1-n} C_t + (1-n)T_t^{-n} C_t + nG_t^{P,H} + (1-n)G_t^{P,F}$$

or

$$Y_t^U = nT_t^{1-n} C_t + (1-n)T_t^{-n} C_t + G_t^{P,U}$$

Log-linearising this equation and making use of the fact that $Y^U = C$ and $T = 1$ gives

$$\tilde{Y}_t^U = \tilde{C}_t + g_t^{P,U} \tag{2.104}$$

Now turn to the Euler equation to derive equation (2.57). Using the definition of the price index P_t, the Euler equation can be written as

$$U_C\left(C_t\right) = \left(1+i_t\right)\beta \mathrm{E}_t \left[U_C\left(C_{t+1}\right) \frac{(P_{H,t})^n (P_{F,t})^{1-n}}{(P_{H,t+1})^n (P_{F,t+1})^{1-n}}\right]$$

Plugging in from (2.54) and (2.55) and re-arranging yields

$$\frac{U_C\left(C_t\right)}{V_y\left(Y_t^H, z_t^H\right)^n V_y\left(Y_t^F, z_t^F\right)^{1-n}} = \left(1+i_t\right)\beta \mathrm{E}_t \frac{U_C\left(C_{t+1}\right)}{V_y\left(Y_{t+1}^H, z_{t+1}^H\right)^n V_y\left(Y_{t+1}^F, z_{t+1}^F\right)^{1-n}}$$

Movements of the nominal interest rate have no real effects under flexible prices. For simplicity, I thus set $1+i_t$ equal to its steady state $1+i$, which implies that

$$\frac{U_C\left(C_t\right)}{V_y\left(Y_t^H, z_t^H\right)^n V_y\left(Y_t^F, z_t^F\right)^{1-n}} = \mathrm{E}_t \frac{U_C\left(C_{t+1}\right)}{V_y\left(Y_{t+1}^H, z_{t+1}^H\right)^n V_y\left(Y_{t+1}^F, z_{t+1}^F\right)^{1-n}} \tag{2.105}$$

Log-linearising (2.105), substituting out $\tilde{Y}_t^U$ using (2.104), defining $\rho \equiv -\frac{U_{CC}(C)}{U_C(C)}C$ and making the appropriate substitutions for the shock processes gives the following condition:

$$-\left(\rho+\eta\right)\tilde{C}_t + \eta\left(\zeta_t^U - g_t^{P,U}\right) = -\left(\rho+\eta\right)\mathrm{E}_t\tilde{C}_{t+1}$$

To solve this equation for $\tilde{C}_t$ as a function of the exogenous disturbances only, I guess that $\tilde{C}_t$ is a linear combination of ζ_t^U and $g_t^{P,U}$, i.e. $\tilde{C}_t = a\zeta_t^U + bg_t^{P,U}$, which implies that

$$\begin{aligned} -\left(\rho+\eta\right)\left(a\zeta_t^U + bg_t^{P,U}\right) + \eta\left(\zeta_t^U - g_t^{P,U}\right) &= 0 \\ \Rightarrow \quad \left(a - \frac{\eta}{\rho+\eta}\right)\zeta_t^U + \left(b + \frac{\eta}{\rho+\eta}\right)g_t^{P,U} &= 0 \end{aligned}$$

It has to be true that $a = \frac{\eta}{\rho+\eta}$ and $b = -\frac{\eta}{\rho+\eta}$ and the initial guess is verified. $\tilde{C}_t$ is therefore given by

$$\tilde{C}_t = \frac{\eta}{\rho+\eta}\left(\zeta_t^U - g_t^{P,U}\right) \tag{2.106}$$

which is equation (2.57) shown in the text.

Together with (2.104), the above expression for $\tilde{C}_t$ immediately implies equation (2.58) from the text stated below for completeness.

$$\tilde{Y}_t^U = \frac{\eta}{\rho + \eta}\zeta_t^U + \frac{\rho}{\rho + \eta}g_t^{P,U} \tag{2.107}$$

The final expression left to derive under flexible prices is that for the natural real rate of interest. The Euler equation can be brought in the following form

$$\frac{U_C\left(C_t\right)}{\left(1+i_t\right)\beta} = \mathrm{E}_t\frac{U_C\left(C_{t+1}\right)}{\Pi_{t+1}} \tag{2.108}$$

Log-linearising this equation, using the definition of ρ and re-arranging yields

$$\tilde{C}_t = \mathrm{E}_t\tilde{C}_{t+1} - \frac{1}{\rho}\left(\tilde{i}_t - \mathrm{E}_t\tilde{\Pi}_{t+1}^U\right) \tag{2.109}$$

Plugging in for $\tilde{C}_t$ and $\tilde{C}_{t+1}$ from equation (2.106) and solving for the real interest rate gives

$$\tilde{i}_t - \mathrm{E}_t\tilde{\Pi}_{t+1}^U = \frac{\rho\eta}{\rho + \eta}\mathrm{E}_t\left[\zeta_{t+1}^U - \zeta_t^U - \left(g_{t+1}^{P,U} - g_t^{P,U}\right)\right] \tag{2.110}$$

which is precisely equation (2.59) from the text.

2.7.4 Sticky Price Equilibrium

Country-specific Demand

The approximations to aggregate demand are derived by log-linearising (2.35) and (2.36). Log-linearising the aggregate demand condition for H and subtracting the same relationship in steady state gives

$$C\hat{Y}_t^H = C(1-n)\hat{T}_t + C\hat{C}_t + G_t^{P,H} + \bar{\mathcal{P}}_t^H \tag{2.111}$$

Dividing by steady-state consumption C yields

$$\hat{Y}_t^H = (1-n)\hat{T}_t + \hat{C}_t + g_t^{P,H} + \mathcal{P}_t^H \tag{2.112}$$

where $g_t^{P,H} \equiv \frac{G_t^{P,H} - G^{P,H}}{Y^H}$ and $\mathcal{P}_t^H \equiv \frac{\bar{\mathcal{P}}_t^H - \bar{\mathcal{P}}^H}{Y^H}$. This is the same as (2.62) stated in Section 2.3.2. Using the analogous procedure and definitions for country F results in equation (2.63).

Euler Equation and Law of Motion of Terms of Trade

The Euler equation under sticky prices is analogous to the one under flexible prices. It is given by

$$\hat{C}_t = \mathrm{E}_t \hat{C}_{t+1} - \frac{1}{\rho}\left(\hat{\imath}_t - \mathrm{E}_t \hat{\Pi}_{t+1}^U\right) \tag{2.113}$$

The terms of trade are defined as $T_t \equiv \frac{P_{F,t}}{P_{H,t}}$, which can be re-written as $T_t = \frac{P_{F,t}}{P_{F,t-1}} \frac{P_{H,t-1}}{P_{H,t}} T_{t-1}$. Using the definition of the inflation rate and log-linearising yields

$$\hat{T}_t = \hat{T}_{t-1} + \hat{\Pi}_t^F - \hat{\Pi}_t^H \tag{2.114}$$

Phillips Curves

In the case of aggregate supply, the derivations are shown again only for country H. The first-order condition for the optimal price of a producer in H can be written as

$$(\sigma - 1)(1-\tau)\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^H \beta\right)^{T-t} \frac{U_C(C_T)}{P_T} p_t^*(h) y_{t,T}(h) = \sigma \mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^H \beta\right)^{T-t} V_y\left[y_{t,T}(h), z_T^H\right] y_{t,T}(h)$$

where the demand for the product produced by h in period T when a price adjustment was last possible in t is given by

$$y_{t,T}(h) = \left[\frac{p_t(h)}{P_{H,T}}\right]^{-\sigma} \left(T_T^{1-n} C_T + G_T^{P,H} + \bar{\mathcal{P}}_T^H\right) \tag{2.115}$$

Note that $\frac{1}{P_t} = \frac{T_t^{n-1}}{P_{H,t}}$, which allows to express the condition above as

$$\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^H \beta\right)^{T-t} \left\{(1-\sigma)(1-\tau) U_C(C_T) \frac{p_t^*(h)}{P_{H,T}} T_T^{n-1} y_{t,T}(h) + \sigma V_y\left[y_{t,T}(h), z_T^H\right] y_{t,T}(h)\right\} = 0$$

Log-linearising this equation yields

$$\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^H \beta\right)^{T-t} \Bigg\{(1-\sigma)(1-\tau)U_C C \left[1 + \frac{U_{CC}}{U_C} C\hat{C}_T + \hat{p}_{t,T} + (n-1)\hat{T}_T + \hat{y}_{t,T}(h)\right] + \sigma V_y C \left[1 + \hat{y}_{t,T}(h) + \frac{V_{yy}}{V_y} C\hat{y}_{t,T}(h) + \frac{V_{yz}}{V_y} z_T^H\right]\Bigg\} = 0$$

Here the arguments of functions evaluated at steady state have been dropped for readability reasons, e.g. $U_C \equiv U_C(C)$, and $\hat{p}_{t,T}$ is defined as $\hat{p}_{t,T} \equiv \log \frac{p_t^*(h)}{P_{H,T}} - \log \frac{p^*(h)}{P_H} = \log p_t^*(h) - \log P_{H,T}$. Since $(1-\sigma)(1-\tau)U_C + \sigma V_y = 0$, this equation can be simplified to

$$\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^H \beta\right)^{T-t} \Bigg\{(1-\sigma)(1-\tau)U_C \left[\frac{U_{CC}}{U_C} C\hat{C}_T + \hat{p}_{t,T} + (n-1)\hat{T}_T\right] + \sigma \left[V_{yy} C\hat{y}_{t,T}(h) + V_{yz} z_T^H\right]\Bigg\} = 0 \tag{2.116}$$

At this point (2.115) has to be log-linearised to find $\hat{y}_{t,T}(h)$:

$$\hat{y}_{t,T}(h) = -\sigma\hat{p}_{t,T} + (1-n)\hat{T}_T + \hat{C}_T + g_T^{P,H} + \mathcal{P}_T^H \tag{2.117}$$

Together with (2.116), this implies

$$\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^H \beta\right)^{T-t} \Bigg\{(1-\sigma)(1-\tau)U_C \left[\frac{U_{CC}}{U_C} C\hat{C}_T + \hat{p}_{t,T} + (n-1)\hat{T}_T\right] + \sigma V_{yy} C \left[-\sigma\hat{p}_{t,T} + (1-n)\hat{T}_T + \hat{C}_T + g_T^{P,H} + \mathcal{P}_T^H\right] + \sigma V_{yz} z_T^H\Bigg\} = 0$$

Dividing by $(1-\sigma)(1-\tau)U_C$ and noticing that $\frac{\sigma V_{yy} C}{(1-\sigma)(1-\tau)U_C} = -\eta$, $\frac{\sigma V_{yz} z_T^H}{(1-\sigma)(1-\tau)U_C} = \eta\zeta_T^H$ and $\frac{U_{CC}C}{U_C} = -\rho$ gives

$$\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^H \beta\right)^{T-t} \Bigg\{\hat{p}_{t,T} - (1-n)\hat{T}_T - \rho\hat{C}_T - \eta \left[-\sigma\hat{p}_{t,T} + (1-n)\hat{T}_T + \hat{C}_T + g_T^{P,H} + \mathcal{P}_T^H - \zeta_T^H\right]\Bigg\} = 0 \tag{2.118}$$

which simplifies to

$$\mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^H \beta\right)^{T-t} \Big[(1+\sigma\eta)\, \hat{p}_{t,T} - (1+\eta)\,(1-n)\hat{T}_T - (\rho+\eta)\, \hat{C}_T - \eta \mathcal{P}_T^H - \eta \left(g_T^{P,H} - \zeta_T^H\right)\Big] = 0$$

$$\begin{aligned} \mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^H \beta\right)^{T-t} \hat{p}_{t,T} &= \mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^H \beta\right)^{T-t} \left[\frac{1+\eta}{1+\sigma\eta}(1-n)\hat{T}_T + \frac{\rho+\eta}{1+\sigma\eta}\hat{C}_T \right. \\ &\quad \left. + \frac{\eta}{1+\sigma\eta}\mathcal{P}_T^H + \frac{\eta}{1+\sigma\eta}\left(g_T^{P,H} - \zeta_T^H\right)\right] \end{aligned} \tag{2.119}$$

Now let log-inflation of home goods be defined as $\hat{\Pi}_t^H \equiv \log \frac{P_{H,t}}{P_{H,t-1}} - \log \frac{P_H}{P_H} = \log P_{H,t} - \log P_{H,t-1}$. From the definitions of $\hat{\Pi}_t^H$ and $\hat{p}_{t,T}$ it follows that

$$\hat{p}_{t,T} = \hat{p}_{t,t} - \sum_{s=1}^{T-t} \hat{\Pi}_{t+s}^H \tag{2.120}$$

By log-linearising the law of motion of the price index for goods from H, it can furthermore be shown that

$$\hat{p}_{t,t} = \frac{\alpha^H}{1-\alpha^H}\hat{\Pi}_t^H$$

which, together with (2.120) implies

$$\hat{p}_{t,T} = \frac{\alpha^H}{1-\alpha^H}\hat{\Pi}_t^H - \sum_{s=1}^{T-t} \hat{\Pi}_{t+s}^H \tag{2.121}$$

Inserting this expression for $\hat{p}_{t,T}$ into equation (2.119) yields

$$\begin{aligned} \frac{1}{1-\alpha^H\beta}\frac{\alpha^H}{1-\alpha^H}\hat{\Pi}_t^H - \mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^H \beta\right)^{T-t} \sum_{s=1}^{T-t} \hat{\Pi}_{t+s}^H &= \mathrm{E}_t \sum_{T=t}^{\infty} \left(\alpha^H \beta\right)^{T-t} \left[\frac{1+\eta}{1+\sigma\eta}(1-n)\hat{T}_T \right. \\ &\quad \left. + \frac{\rho+\eta}{1+\sigma\eta}\hat{C}_T + \frac{\eta}{1+\sigma\eta}\mathcal{P}_T^H + \frac{\eta}{1+\sigma\eta}\left(g_T^{P,H} - \zeta_T^H\right)\right] \end{aligned}$$

or

$$\frac{1}{1-\alpha^H\beta}\frac{\alpha^H}{1-\alpha^H}\hat{\Pi}_t^H - \mathrm{E}_t\sum_{T=t+1}^{\infty}\left(\alpha^H\beta\right)^{T-t}\frac{\hat{\Pi}_T^H}{1-\alpha^H\beta} = \mathrm{E}_t\sum_{T=t}^{\infty}\left(\alpha^H\beta\right)^{T-t}\left[\frac{1+\eta}{1+\sigma\eta}(1-n)\hat{T}_T \right.$$
$$\left. + \frac{\rho+\eta}{1+\sigma\eta}\hat{C}_T + \frac{\eta}{1+\sigma\eta}\mathcal{P}_T^H + \frac{\eta}{1+\sigma\eta}\left(g_T^{P,H} - \zeta_T^H\right)\right]$$

Iterating the equation above forward one period, multiplying by $\alpha^H\beta$, taking the expectation conditional on the information set in t and using the law of iterated expectations gives

$$\frac{\alpha^H\beta}{1-\alpha^H\beta}\frac{\alpha^H}{1-\alpha^H}\mathrm{E}_t\hat{\Pi}_{t+1}^H - \mathrm{E}_t\sum_{T=t+2}^{\infty}\left(\alpha^H\beta\right)^{T-t}\frac{\hat{\Pi}_T^H}{1-\alpha^H\beta} = \mathrm{E}_t\sum_{T=t+1}^{\infty}\left(\alpha^H\beta\right)^{T-t}$$
$$\left[\frac{1+\eta}{1+\sigma\eta}(1-n)\hat{T}_T + \frac{\rho+\eta}{1+\sigma\eta}\hat{C}_T + \frac{\eta}{1+\sigma\eta}\mathcal{P}_T^H + \frac{\eta}{1+\sigma\eta}\left(g_T^{P,H} - \zeta_T^H\right)\right] \tag{2.122}$$

Equation (2.122) can now be subtracted from (2.122) to obtain

$$\hat{\Pi}_t^H - \beta\mathrm{E}_t\hat{\Pi}_{t+1}^H = \frac{(1-\alpha^H\beta)(1-\alpha^H)}{\alpha^H}\left[\frac{1+\eta}{1+\sigma\eta}(1-n)\hat{T}_t + \frac{\rho+\eta}{1+\sigma\eta}\hat{C}_t + \right.$$
$$\left.\frac{\eta}{1+\sigma\eta}\mathcal{P}_t^H + \frac{\eta}{1+\sigma\eta}\left(g_t^{P,H} - \zeta_t^H\right)\right]$$

which, in more succinct form, becomes

$$\hat{\Pi}_t^H - \beta\mathrm{E}_t\hat{\Pi}_{t+1}^H = a_T^H(1-n)\hat{T}_t + a_C^H\hat{C}_t + a_{\mathcal{P}}^H\mathcal{P}_t^H + $$
$$\frac{(1-\alpha^H\beta)(1-\alpha^H)}{\alpha^H}\frac{\eta}{1+\sigma\eta}\left(g_t^{P,H} - \zeta_t^H\right) \tag{2.123}$$

where

$$\begin{aligned}
a_T^H &\equiv \frac{(1-\alpha^H\beta)(1-\alpha^H)}{\alpha^H}\frac{1+\eta}{1+\sigma\eta} \\
a_C^H &\equiv \frac{(1-\alpha^H\beta)(1-\alpha^H)}{\alpha^H}\frac{\rho+\eta}{1+\sigma\eta} \\
a_{\mathcal{P}}^H &\equiv \frac{(1-\alpha^H\beta)(1-\alpha^H)}{\alpha^H}\frac{\eta}{1+\sigma\eta}
\end{aligned}$$

Equation (2.123) can be simplified further by noting that

$$
\begin{aligned}
\frac{(1-\alpha^H\beta)(1-\alpha^H)}{\alpha^H}\frac{\eta}{1+\sigma\eta}\left(g_t^{P,H}-\zeta_t^H\right) &= \\
-\frac{(1-\alpha^H\beta)(1-\alpha^H)}{\alpha^H}\frac{1+\eta}{1+\sigma\eta}(1-n)\frac{\eta}{1+\eta}\left[g_t^{P,F}-g_t^{P,H}-\left(\zeta_t^F-\zeta_t^H\right)\right] & \\
-\frac{(1-\alpha^H\beta)(1-\alpha^H)}{\alpha^H}\frac{\rho+\eta}{1+\sigma\eta}\frac{\eta}{\rho+\eta}\left\{n\zeta_t^H+(1-n)\zeta_t^F-\left[ng_t^{P,H}+(1-n)\zeta_t^F\right]\right\} &= \\
-(1-n)a_T^H\tilde{T}_t-a_C^H\tilde{C}_t &
\end{aligned}
$$

Using this relationship, the aggregate supply function given in the text is obtained:

$$
\hat{\Pi}_t^H=\beta \mathrm{E}_t\hat{\Pi}_{t+1}^H+a_T^H(1-n)\left(\hat{T}_t-\tilde{T}_t\right)+a_C^H\left(\hat{C}_t-\tilde{C}_t\right)+a_{\mathcal{P}}^H\mathcal{P}_t^H \tag{2.124}
$$

Going through the equivalent steps for country F yields

$$
\hat{\Pi}_t^F=\beta \mathrm{E}_t\hat{\Pi}_{t+1}^F+a_T^F(-n)\left(\hat{T}_t-\tilde{T}_t\right)+a_C^F\left(\hat{C}_t-\tilde{C}_t\right)+a_{\mathcal{P}}^F\mathcal{P}_t^F \tag{2.125}
$$

where

$$
\begin{aligned}
a_T^F &\equiv \frac{(1-\alpha^F\beta)(1-\alpha^F)}{\alpha^F}\frac{1+\eta}{1+\sigma\eta} \\
a_C^F &\equiv \frac{(1-\alpha^F\beta)(1-\alpha^F)}{\alpha^F}\frac{\rho+\eta}{1+\sigma\eta} \\
a_{\mathcal{P}}^F &\equiv \frac{(1-\alpha^F\beta)(1-\alpha^F)}{\alpha^F}\frac{\eta}{1+\sigma\eta}
\end{aligned}
$$

Welfare Function

The welfare function that serves as a basis for policy decisions is

$$
W_0=\mathrm{E}_0\sum_{t=0}^{\infty}\beta^t w_t
$$

where per-period union-wide welfare w_t is defined as $w_t \equiv n w_t^H + (1-n) w_t^F$ and per-period welfare in H and F is given by average utility within each respective country

$$
\begin{aligned}
w_t^H &\equiv \frac{1}{n}\int_0^n \left\{U(C_t) - V\left[y_t(h), z_t^H\right]\right\} dh - W\left(\mathcal{P}_t^H\right) \qquad (2.126)\\
w_t^F &\equiv \frac{1}{1-n}\int_n^1 \left\{U(C_t) - V\left[y_t(f), z_t^F\right]\right\} df - W\left(\mathcal{P}_t^F\right) \qquad (2.127)
\end{aligned}
$$

Here, a second-order approximation of W_0 is derived. When analogous steps are carried out for H and F, only the derivations for country H are shown as before.

A second-order Taylor approximation of the utility from consumption is given by

$$U(C_t) = U(C) + U_C(C)(C_t - C) + \frac{1}{2}U_{CC}(C)(C_t - C)^2 + O(3)$$

C_t can be written as

$$C_t = C + C\hat{C}_t + \frac{1}{2}C\hat{C}_t^2 + O(3)$$

Combining the two equations above yields

$$U(C_t) = U_C(C)C\left[\hat{C}_t + \frac{1}{2}\hat{C}_t^2(1-\rho)\right] + \text{tip} + O(3) \qquad (2.128)$$

where terms that are independent of policy, such as expressions that depend on steady state values only, are collected under ‘tip.’

Similarly, expanding the disutility from labour to the second order gives

$$
\begin{aligned}
V\left[y_t(h), z_t^H\right] &= V(C,0) + V_y(C,0)\left[y_t(h) - C\right] + V_z(C,0)z_t^H + \\
&\quad \frac{1}{2}V_{yy}(C,0)\left[y_t(h) - C\right]^2 + \frac{1}{2}V_{zz}(C,0)\left(z_t^H\right)^2 + V_{yz}(C,0)\left[y_t(h) - C\right] z_t^H
\end{aligned}
$$

and $y_t(h)$ can be written as

$$y_t(h) = C + C\hat{y}_t(h) + \frac{1}{2}C\hat{y}_t(h)^2 + O(3)$$

which can be inserted into the approximation of $V\left[y_t(h), z_t^H\right]$ to obtain

$$\begin{aligned} V\left[y_t(h), z_t^H\right] &= V_y(C,0)\left[C\hat{y}_t(h) + \frac{1}{2}C\hat{y}_t(h)^2\right] + \frac{1}{2}V_{yy}(C,0)\left[C\hat{y}_t(h) + \frac{1}{2}C\hat{y}_t(h)^2\right]^2 \\ &\quad + V_{yz}(C,0)\left[C\hat{y}_t(h) + \frac{1}{2}C\hat{y}_t(h)^2\right] z_t^H + \text{tip} + O(3) \\ &= V_y(C,0)C\left[\hat{y}_t(h) + \frac{1}{2}\hat{y}_t(h)^2 + \frac{\eta}{2}\hat{y}_t(h)^2 - \eta\hat{y}_t(h)\zeta_t^H\right] + \text{tip} + O(3) \end{aligned}$$

It is assumed that a tax/subsidy is in place, which precisely offsets the effects of market power in steady state so that $U_C(C) = V_y(C,0)$. The equation above thus becomes

$$V\left[y_t(h), z_t^H\right] = U_C(C)C\left[\hat{y}_t(h) + \frac{1}{2}\hat{y}_t(h)^2 + \frac{\eta}{2}\hat{y}_t(h)^2 - \eta\hat{y}_t(h)\zeta_t^H\right] + \text{tip} + O(3)$$

Integrating this expression over all households that reside in H and dividing by n gives an expression for the average disutility from labour in H.

$$\begin{aligned} \frac{1}{n}\int_0^n V\left[y_t(h), z_t^H\right] dh &= U_C(C)C\left[\frac{1}{n}\int_0^n \hat{y}_t(h)dh + \frac{1}{2}\frac{1}{n}\int_0^n \hat{y}_t(h)^2 dh + \right. \\ &\quad \left. \frac{\eta}{2}\frac{1}{n}\int_0^n \hat{y}_t(h)^2 dh - \eta\frac{1}{n}\int_0^n \hat{y}_t(h)\zeta_t^H dh\right] + \text{tip} + O(3) \end{aligned}$$

Note that $\mathrm{E}_h x(h) = \frac{1}{n}\int_0^n x(h)dh$ and $\mathrm{Var}_h x(h) = \frac{1}{n}\int_0^n x(h)^2 dh - \left[\frac{1}{n}\int_0^n x(h)dh\right]^2$ for any variable $x(h)$. This implies

$$\begin{aligned} \frac{1}{n}\int_0^n V\left[y_t(h), z_t^H\right] dh &= U_C(C)C\left[\mathrm{E}_h\hat{y}_t(h) + \frac{1}{2}\left\{\mathrm{Var}_h\hat{y}_t(h) + \left[\mathrm{E}_h\hat{y}_t(h)\right]^2\right\} + \right. \\ &\quad \left. \frac{\eta}{2}\left\{\mathrm{Var}_h\hat{y}_t(h) + \left[\mathrm{E}_h\hat{y}_t(h)\right]^2\right\} - \eta\mathrm{E}_h\hat{y}_t(h)\zeta_t^H\right] + \text{tip} + O(3) \end{aligned} \tag{2.129}$$

$\mathrm{E}_h\hat{y}_t(h)$ can be expressed in terms of $\hat{Y}_t^H$. The relationship between these two terms is derived now.

Remember that output in H is given by

$$Y_t^H = \left[\frac{1}{n}\int_0^n y_t(h)^{\frac{\sigma-1}{\sigma}} dh\right]^{\frac{\sigma}{\sigma-1}}$$

Approximating this equation to the second order gives

$$\hat{Y}_t^H + \frac{1}{2}\frac{\sigma-1}{\sigma}\left(\hat{Y}_t^H\right)^2 = \mathrm{E}_h\hat{y}_t(h) + \frac{1}{2}\frac{\sigma-1}{\sigma}\mathrm{Var}_h\hat{y}_t(h) + \frac{1}{2}\frac{\sigma-1}{\sigma}\left[\mathrm{E}_h\hat{y}_t(h)\right]^2 + O(3)$$

It can be shown that it is furthermore true that $\left(\hat{Y}_t^H\right)^2 = \left[\mathrm{E}_h\hat{y}_t(h)\right]^2 + O(3)$, which implies

$$\hat{Y}_t^H = \mathrm{E}_h\hat{y}_t(h) + \frac{1}{2}\frac{\sigma-1}{\sigma}\mathrm{Var}_h\hat{y}_t(h) + O(3)$$

Using this equation to substitute for $\mathrm{E}_h\hat{y}_t(h)$ in (2.129) and collecting terms gives

$$\begin{aligned} \frac{1}{n}\int_0^n V\left[y_t(h), z_t^H\right]dh &= U_C(C)C\left[\hat{Y}_t^H + \frac{1+\eta}{2}\left(\hat{Y}_t^H\right)^2 + \right. \\ & \left. \frac{1}{2}\left(\frac{1}{\sigma}+\eta\right)\mathrm{Var}_h\hat{y}_t(h) - \eta\hat{Y}_t^H\zeta_t^H\right] + \mathrm{tip} + O(3) \end{aligned} \tag{2.130}$$

A second-order approximation of the disutility from the unconventional tool is given by

$$W(\mathcal{P}_t^H) = W(\mathcal{P}^H) + W_{\mathcal{P}}(\mathcal{P}^H)\mathcal{P}_t^H + \frac{1}{2}W_{\mathcal{P}\mathcal{P}}(\mathcal{P}^H)\left(\mathcal{P}_t^H\right)^2 + O(3)$$

Since $W(\mathcal{P}^H) = 0$, we have that

$$W(\mathcal{P}_t^H) = W_{\mathcal{P}}(\mathcal{P}^H)\mathcal{P}_t^H + \frac{1}{2}W_{\mathcal{P}\mathcal{P}}(\mathcal{P}^H)\left(\mathcal{P}_t^H\right)^2 + O(3) \tag{2.131}$$

Entering (2.128), (2.130) and (2.131) into the definition of welfare in H (2.126) yields

$$\begin{aligned} w_t^H &= U_C(C)C\left[\hat{C}_t + \frac{1}{2}(1-\rho)\hat{C}_t^2 - \hat{Y}_t^H - \frac{1+\eta}{2}\left(\hat{Y}_t^H\right)^2 - \frac{1}{2}\left(\frac{1}{\sigma}+\eta\right)\mathrm{Var}_h\hat{y}_t(h) + \eta\hat{Y}_t^H\zeta_t^H\right] \\ & -\left[W_{\mathcal{P}}(\mathcal{P}^H)\mathcal{P}_t^H + \frac{1}{2}W_{\mathcal{P}\mathcal{P}}(\mathcal{P}^H)\left(\mathcal{P}_t^H\right)^2\right] + \mathrm{tip} + O(3) \end{aligned}$$

Equivalent steps imply that

$$\begin{aligned} w_t^F &= U_C(C)C\left[\hat{C}_t + \frac{1}{2}(1-\rho)\hat{C}_t^2 - \hat{Y}_t^F - \frac{1+\eta}{2}\left(\hat{Y}_t^F\right)^2 - \frac{1}{2}\left(\frac{1}{\sigma}+\eta\right)\mathrm{Var}_f\hat{y}_t(f) + \eta\hat{Y}_t^F\zeta_t^F\right] \\ & -\left[W_{\mathcal{P}}(\mathcal{P}^F)\mathcal{P}_t^F + \frac{1}{2}W_{\mathcal{P}\mathcal{P}}(\mathcal{P}^F)\left(\mathcal{P}_t^F\right)^2\right] + \mathrm{tip} + O(3) \end{aligned}$$

Since $w_t \equiv nw_t^H + (1-n)w_t^F$, per-period welfare is given by

$$\begin{aligned} w_t &= U_C(C)C\left\{\hat{C}_t + \frac{1}{2}(1-\rho)\hat{C}_t^2 - \left[n\hat{Y}_t^H + (1-n)\hat{Y}_t^F\right]\right. \\ &\quad -\frac{1+\eta}{2}\left[n\left(\hat{Y}_t^H\right)^2 + (1-n)\left(\hat{Y}_t^F\right)^2\right] + \eta\left[n\hat{Y}_t^H\zeta_t^H + (1-n)\hat{Y}_t^F\zeta_t^F\right] \\ &\quad \left.-\frac{1}{2}\left(\frac{1}{\sigma}+\eta\right)\left[n\mathrm{Var}_h\hat{y}_t(h) + (1-n)\mathrm{Var}_f\hat{y}_t(f)\right]\right\} \\ &\quad -W_{\mathcal{P}}(\mathcal{P})\mathcal{P}_t^U - \frac{1}{2}W_{\mathcal{P}\mathcal{P}}(\mathcal{P})\left[n(\mathcal{P}_t^H)^2 + (1-n)(\mathcal{P}_t^F)^2\right] + \text{tip} + O(3) \end{aligned} \tag{2.132}$$

To find expressions for $\hat{Y}_t^H$ and $\hat{Y}_t^F$, output in both countries has to be approximated to the second order.

$$\begin{aligned} \hat{Y}_t^H &= \hat{C}_t + (1-n)\hat{T}_t + \frac{1}{2}\hat{C}_t^2 + \frac{1}{2}(1-n)^2\hat{T}_t^2 + (1-n)\hat{C}_t\hat{T}_t + g_t^{P,H} + \mathcal{P}_t^H - \frac{1}{2}\left(\hat{Y}_t^H\right)^2 + O(3) \\ \hat{Y}_t^F &= \hat{C}_t - n\hat{T}_t + \frac{1}{2}\hat{C}_t^2 + \frac{1}{2}n^2\hat{T}_t^2 - n\hat{C}_t\hat{T}_t + g_t^{P,F} + \mathcal{P}_t^F - \frac{1}{2}\left(\hat{Y}_t^F\right)^2 + O(3) \end{aligned}$$

This immediately implies

$$\begin{aligned} n\left(\hat{Y}_t^H\right)^2 + (1-n)\left(\hat{Y}_t^F\right)^2 &= \hat{C}_t^2 + n(1-n)\hat{T}_t^2 + n\left(g_t^{P,H}\right)^2 + (1-n)\left(g_t^{P,F}\right)^2 + n\left(\mathcal{P}_t^H\right)^2 \\ &\quad +(1-n)\left(\mathcal{P}_t^F\right)^2 + 2\hat{C}_t g_t^{P,U} + 2\hat{C}_t\mathcal{P}_t^U - 2n(1-n)\hat{T}_t g_t^{P,R} \\ &\quad -2n(1-n)\hat{T}_t\mathcal{P}_t^R + 2\left[ng_t^{P,H}\mathcal{P}_t^H + (1-n)g_t^{P,F}\mathcal{P}_t^F\right] + O(3) \end{aligned}$$

and

$$\begin{aligned} n\hat{Y}_t^H + (1-n)\hat{Y}_t^F &= \hat{C}_t + g_t^{P,U} + \mathcal{P}_t^U - \frac{1}{2}\left[n\left(g_t^{P,H}\right)^2 + (1-n)\left(g_t^{P,F}\right)^2\right] \\ &\quad -\frac{1}{2}\left[n\left(\mathcal{P}_t^H\right)^2 + (1-n)\left(\mathcal{P}_t^F\right)^2\right] - \hat{C}_t g_t^{P,U} - \hat{C}_t\mathcal{P}_t^U + n(1-n)\hat{T}_t g_t^{P,R} \\ &\quad +n(1-n)\hat{T}\mathcal{P}_t^R - \left[ng_t^{P,H}\mathcal{P}_t^H + (1-n)g_t^{P,F}\mathcal{P}_t^F\right] + O(3) \end{aligned}$$

and furthermore

$$\begin{aligned} n\hat{Y}_t^H\zeta_t^H + (1-n)\hat{Y}_t^F\zeta_t^F &= \hat{C}_t\zeta_t^U - n(1-n)\hat{T}_t\zeta_t^R + ng_t^{P,H}\zeta_t^H + (1-n)g_t^{P,F}\zeta_t^F \\ &\quad +n\mathcal{P}_t^H\zeta_t^H + (1-n)\mathcal{P}_t^F\zeta_t^F + O(3) \end{aligned}$$

Plugging the equations found above into (2.132) and simplifying yields

$$
\begin{aligned}
w_t \;=\;& U_C(C)C\left\{-\frac{1}{2}(\rho+\eta)\hat{C}_t^2-\eta\hat{C}_t g_t^{P,U}+\eta n(1-n)\hat{T}_t g_t^{P,R}-\frac{1}{2}(1+\eta)\,n(1-n)\hat{T}_t^2\right.\\
&+\eta\hat{C}_t\zeta_t^U-\eta n(1-n)\hat{T}_t\zeta_t^R-\eta\hat{C}_t\mathcal{P}_t^U\\
&-\left[1+\frac{W_{\mathcal{P}}(\mathcal{P})}{U_C(C)C}\right]\mathcal{P}_t^U-\frac{1}{2}\left[\eta+\frac{W_{\mathcal{P}\mathcal{P}}(\mathcal{P})}{U_C(C)C}\right]\left[n\left(\mathcal{P}_t^H\right)^2+(1-n)\left(\mathcal{P}_t^F\right)^2\right]\\
&+\eta n(1-n)\hat{T}_t\mathcal{P}_t^R-\eta\left[n g_t^{P,H}\mathcal{P}_t^H+(1-n)g_t^{P,F}\mathcal{P}_t^F\right]+\eta\left[n\zeta_t^H\mathcal{P}_t^H+(1-n)\zeta_t^F\mathcal{P}_t^F\right]\\
&\left.-\frac{1}{2}\left(\frac{1}{\sigma}+\eta\right)\left[n\mathrm{Var}_h\hat{y}_t(h)+(1-n)\mathrm{Var}_f\hat{y}_t(f)\right]\right\}+\mathrm{tip}+O(3)
\end{aligned}
$$

which can be re-written as

$$
\begin{aligned}
w_t \;=\;& -U_C(C)C\left\{\frac{1}{2}(\rho+\eta)\left(\hat{C}_t-\tilde{C}_t\right)^2+\frac{1}{2}(1+\eta)n(1-n)\left(\hat{T}_t-\tilde{T}_t\right)^2\right.\\
&+\left[1+\frac{W_{\mathcal{P}}(\mathcal{P})}{U_C(C)C}\right]\mathcal{P}_t^U+\frac{1}{2}\eta\left(1+\frac{\kappa}{\eta}\right)\left[n\left(\mathcal{P}_t^H\right)^2+(1-n)\left(\mathcal{P}_t^F\right)^2\right]+\eta\hat{C}_t\mathcal{P}_t^U\\
&-\eta n(1-n)\hat{T}_t\mathcal{P}_t^R+\eta\left[n g_t^{P,H}\mathcal{P}_t^H+(1-n)g_t^{P,F}\mathcal{P}_t^F\right]-\eta\left[n\zeta_t^H\mathcal{P}_t^H+(1-n)\zeta_t^F\mathcal{P}_t^F\right]\\
&\left.+\frac{1}{2}\left(\frac{1}{\sigma}+\eta\right)\left[n\mathrm{Var}_h\hat{y}_t(h)+(1-n)\mathrm{Var}_f\hat{y}_t(f)\right]\right\}+\mathrm{tip}+O(3) \qquad (2.133)
\end{aligned}
$$

where $\kappa\equiv\frac{W_{\mathcal{P}\mathcal{P}}(\mathcal{P})}{U_C(C)C}$. The variance terms in (2.133) can be related to country-specific inflation.

In Benigno (2003), based on Rotemberg and Woodford (1998), it is shown that

$$
\mathrm{Var}_h\hat{y}_t(h)=\sigma^2\left(\alpha^H\right)^{t+1}\mathrm{Var}_h\log p_{-1}(h)+\sigma^2\frac{\alpha^H}{1-\alpha^H}\sum_{s=0}^{t}\left(\alpha^H\right)^{t-s}\left(\hat{\Pi}_s^H\right)^2+O(3)
$$

The first term on the right hand side is independent of policy in all periods $t\geq 0$ and therefore

$$
\sum_{t=0}^{\infty}\beta^t\mathrm{Var}_h\hat{y}_t(h)=\sigma^2\frac{\alpha^H}{1-\alpha^H}\frac{1}{1-\alpha^H\beta}\sum_{t=0}^{\infty}\beta^t\left(\hat{\Pi}_t^H\right)^2+\mathrm{tip}+O(3)
$$

which immediately implies

$$
\begin{aligned}
\sum_{t=0}^{\infty}\beta^t\left[n\mathrm{Var}_h\hat{y}_t(h)+(1-n)\mathrm{Var}_f\hat{y}_t(f)\right] \;=\;& \sigma^2 n\gamma^H\sum_{t=0}^{\infty}\beta^t\left(\hat{\Pi}_t^H\right)^2\\
&+\sigma^2(1-n)\gamma^F\sum_{t=0}^{\infty}\beta^t\left(\hat{\Pi}_t^F\right)^2+\mathrm{tip}+O(3)
\end{aligned}
$$

where

$$\gamma^H \equiv \frac{\alpha^H}{(1-\alpha^H)(1-\alpha^H\beta)}$$
$$\gamma^F \equiv \frac{\alpha^F}{(1-\alpha^F)(1-\alpha^F\beta)}$$

It thus follows that

$$\sum_{t=0}^{\infty}\beta^t\left[n\mathrm{Var}_h\hat{y}_t(h)+(1-n)\mathrm{Var}_f\hat{y}_t(f)\right] = \sigma^2\left[n\gamma^H+(1-n)\gamma^F\right]\sum_{t=0}^{\infty}\beta^t\left[\gamma\left(\hat{\Pi}_t^H\right)^2+ (1-\gamma)\left(\hat{\Pi}_t^F\right)^2\right]+\mathrm{tip}+O(3)$$

with

$$\gamma \equiv \frac{n\gamma^H}{n\gamma^H+(1-n)\gamma^F}$$

Using the fact that only policies with $\mathcal{P}_t^U=0$ are considered and since we also have $a_T^i = \frac{1}{\gamma^i}\frac{1+\eta}{1+\sigma\eta}$, $a_C^i = \frac{1}{\gamma^i}\frac{\rho+\eta}{1+\sigma\eta}$ and $a_{\mathcal{P}}^i = \frac{1}{\gamma^i}\frac{\eta}{1+\sigma\eta}$ for $i\in\{H,F\}$, welfare can be written as

$$W_0 = -\Omega\mathrm{E}_0\sum_{t=0}^{\infty}\beta^t L_t$$

where the loss function L_t is defined as

$$\begin{aligned} L_t &= \Lambda_C\left(\hat{C}_t-\tilde{C}_t\right)^2+\Lambda_T n(1-n)\left(\hat{T}_t-\tilde{T}_t\right)^2+\gamma\left(\hat{\Pi}_t^H\right)^2+(1-\gamma)\left(\hat{\Pi}_t^F\right)^2 \\ &+\Lambda_{\mathcal{P}}\left\{\frac{1}{2}\left(1+\frac{\kappa}{\eta}\right)\left[n\left(\mathcal{P}_t^H\right)^2+(1-n)\left(\mathcal{P}_t^F\right)^2\right]-n(1-n)\hat{T}_t\mathcal{P}_t^R\right. \\ &\left.+n\mathcal{P}_t^H\left(g_t^{P,H}-\zeta_t^H\right)+(1-n)\mathcal{P}_t^F\left(g_t^{P,F}-\zeta_t^F\right)\right\}+\mathrm{tip}+O(3) \end{aligned} \tag{2.134}$$

with

$$\Omega \equiv \frac{1}{2}U_C(C)C\sigma(1+\sigma\eta)\left[n\gamma^H+(1-n)\gamma^F\right]$$
$$\Lambda_C \equiv \frac{1}{\sigma}\frac{a_C^H a_C^F}{na_C^F+(1-n)a_C^H}$$
$$\Lambda_T \equiv \frac{1}{\sigma}\frac{a_T^H a_T^F}{na_T^F+(1-n)a_T^H}$$

$$\Lambda_{\mathcal{P}} \equiv \frac{2}{\sigma} \frac{a_{\mathcal{P}}^{H} a_{\mathcal{P}}^{F}}{n a_{\mathcal{P}}^{F} + (1-n) a_{\mathcal{P}}^{H}}$$

For $\rho = 1$, the per-period loss function can be simplified further. Since in this case

$$\begin{aligned} n\mathcal{P}_t^H \left(g_t^{P,H} - \zeta_t^H\right) + (1-n)\mathcal{P}_t^F \left(g_t^{P,F} - \zeta_t^F\right) &= \frac{\eta+1}{\eta}\left[n(1-n)\tilde{T}_t\mathcal{P}_t^R - \tilde{C}_t\mathcal{P}_t^U\right] \\ &\quad + \frac{\eta+1}{\eta} n(1-n)\tilde{T}_t\mathcal{P}_t^R \end{aligned}$$

and $\Lambda_C = \Lambda_T$, per-period loss is given by

$$\begin{aligned} L_t &= \Lambda_C\left(\hat{C}_t - \tilde{C}_t\right)^2 + \Lambda_C n(1-n)\left(\hat{T}_t - \tilde{T}_t\right)^2 + \gamma\left(\hat{\Pi}_t^H\right)^2 + (1-\gamma)\left(\hat{\Pi}_t^F\right)^2 \\ &\quad + \Lambda_{\mathcal{P}}\left\{\frac{1}{2}\left(1+\frac{\kappa}{\eta}\right)\left[n\left(\mathcal{P}_t^H\right)^2 + (1-n)\left(\mathcal{P}_t^F\right)^2\right] - n(1-n)\mathcal{P}_t^R\left(\hat{T}_t - \delta\tilde{T}_t\right)\right\} \\ &\quad + \text{tip} + O(3) \end{aligned} \quad (2.135)$$

where $\delta \equiv \frac{\eta+1}{\eta}$.

2.7.5 Optimal Policy

The optimal policy problem is to maximise (2.74) subject to (2.66) to (2.68), (2.70), suitable initial conditions that reflect that policy is assumed to be chosen "from the timeless perspective" and transversality conditions. The first-order conditions are

$$2\gamma\hat{\Pi}_t^H + \phi_{1,t} + \phi_{3,t} - \phi_{1,t-1} = 0 \quad (2.136)$$

$$2(1-\gamma)\hat{\Pi}_t^F + \phi_{2,t} - \phi_{3,t} - \phi_{2,t-1} = 0 \quad (2.137)$$

$$\begin{aligned} &2\Lambda_C n(1-n)\left(\hat{T}_t - \tilde{T}_t\right) - \Lambda_{\mathcal{P}} n(1-n)\left(\mathcal{P}_t^F - \mathcal{P}_t^H\right) \\ &-a_T^H(1-n)\phi_{1,t} + a_T^F n\phi_{2,t} + \phi_{3,t} - \beta \mathrm{E}_t\phi_{3,t+1} = 0 \end{aligned} \quad (2.138)$$

$$2\Lambda_C\left(\hat{C}_t - \tilde{C}_t\right) - a_C^H\phi_{1,t} - a_C^F\phi_{2,t} = 0 \quad (2.139)$$

$$\Lambda_{\mathcal{P}}\left(1+\frac{\kappa}{\eta}\right) n\mathcal{P}_t^H + \Lambda_{\mathcal{P}} n(1-n)\left(\hat{T}_t - \delta\tilde{T}_t\right) - a_{\mathcal{P}}^H\phi_{1,t} + n\phi_{4,t} = 0 \quad (2.140)$$

$$\Lambda_{\mathcal{P}}\left(1+\frac{\kappa}{\eta}\right)(1-n)\mathcal{P}_t^F - \Lambda_{\mathcal{P}} n(1-n)\left(\hat{T}_t - \delta\tilde{T}_t\right) - a_{\mathcal{P}}^F\phi_{2,t} + (1-n)\phi_{4,t} = 0 \quad (2.141)$$

for all $t \geq 0$, where $\phi_{1,t}$ to $\phi_{4,t}$ are the multipliers associated with the constraints.

Defining $x_t^n \equiv \left(\hat{\Pi}_t^H, \hat{\Pi}_t^F, \hat{C}_t, \phi_{1,t}, \phi_{2,t}, \phi_{3,t}, \phi_{4,t}, \mathcal{P}_t^H, \mathcal{P}_t^F\right)'$, $x_t^p \equiv \left(\hat{T}_{t-1}, \phi'_{1,t}, \phi'_{2,t}\right)'$ with $\phi'_{1,t} \equiv \phi_{1,t-1}$ and $\phi'_{2,t} \equiv \phi_{2,t-1}$ as well as $\varepsilon_t \equiv \left(\tilde{T}_t, \tilde{C}_t\right)'$, the evolution of all endogenous variables in equilibrium is given by

$$B \begin{bmatrix} x_{t+1}^p \\ \mathrm{E}_t x_{t+1}^n \end{bmatrix} = A \begin{bmatrix} x_t^p \\ x_t^n \end{bmatrix} + F\varepsilon_t$$

The 12×12 matrices B and A and the 12×2 matrix F are defined as

$$B \equiv \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\Lambda_C n(1-n) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1-n)a_T^H & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-na_T^F & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\Lambda_{\mathcal{P}} n(1-n) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_{\mathcal{P}} n(1-n) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$A \equiv \begin{bmatrix}
1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 2\gamma & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 2(1-\gamma) & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_T^H(1-n) & -a_T^F n & -1 & 0 & -\Lambda_{\mathcal{P}} n(1-n) & \Lambda_{\mathcal{P}} n(1-n) \\
0 & 0 & 0 & 0 & 0 & 2\Lambda_C & -a_H^C & -a_F^C & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -a_H^C & 0 & 0 & 0 & 0 & -a_{\mathcal{P}}^H & 0 \\
0 & 0 & 0 & 0 & 1 & -a_F^C & 0 & 0 & 0 & 0 & 0 & -a_{\mathcal{P}}^F \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n & (1-n) \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{\mathcal{P}}^H & 0 & 0 & n & \Lambda_{\mathcal{P}}\left(1+\frac{\kappa}{\eta}\right)n & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{\mathcal{P}}^F & 0 & 1-n & 0 & \Lambda_{\mathcal{P}}\left(1+\frac{\kappa}{\eta}\right)(1-n)
\end{bmatrix}$$

$$F' \equiv \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 2\Lambda_C n(1-n) & 0 & (1-n)a_T^H & -na_T^F & 0 & -\Lambda_{\mathcal{P}} n(1-n)\delta & \Lambda_{\mathcal{P}} n(1-n)\delta \\
0 & 0 & 0 & 0 & 0 & 0 & -2\Lambda_C & a_C^H & a_C^F & 0 & 0 & 0
\end{bmatrix}$$

This is a system of expectational difference equations which can be solved using the method outlined by Klein (2000). To do so, the matrix pair B, A has to be decomposed such that $B = QTZ'$ and $A = QSZ'$ with $QQ' = Q'Q = ZZ' = Z'Z = I_{12}$. After appropriate matrix partitioning and reordering such that the eigenvalues associated with the system, given by the ratio of the diagonal elements of S and T, increase in absolute value, the solution can be written as

$$x_{t+1}^{p} = \left[B_{11} - B_{12}\left(Z_{22}'\right)^{-1} Z_{21}'\right]^{-1}\left[A_{11} - A_{12}\left(Z_{22}'\right)^{-1} Z_{21}'\right] x_t^p + \left[B_{11} - B_{12}\left(Z_{22}'\right)^{-1} Z_{21}'\right]^{-1}\left[F_1 - A_{12}\left(Z_{22}'\right)^{-1} S_{22}^{-1}\left(Q_{21}'F_1 + Q_{22}'F_2\right)\right]\varepsilon_t \tag{2.142}$$

$$x_t^n = -\left(Z_{22}'\right)^{-1} Z_{21}' x_t^p - \left(Z_{22}'\right)^{-1} S_{22}^{-1}\left(Q_{21}'F_1 + Q_{22}'F_2\right)\varepsilon_t \tag{2.143}$$

2.7.6 Proposition 1

For $\alpha_H = \alpha_F$ together with $\rho = 1$, one can define $a \equiv a_C^H = a_C^F = a_T^H = a_T^T$ and $a_{\mathcal{P}} \equiv a_{\mathcal{P}}^H = a_{\mathcal{P}}^F$. It is argued in the text that $\hat{C}_t$ is independent of $\left\{\tilde{T}_t\right\}$. Deviations of $\hat{C}_t$ from $\tilde{C}_t$ are costly, which implies that $\hat{C}_t - \tilde{C}_t = 0$.[28] In this case, (2.139) implies

$$\phi_{1,t} + \phi_{2,t} = 0 \tag{2.144}$$

Since $\gamma = n$ for $\alpha_H = \alpha_F$, (2.136) and (2.137) then immediately give

$$n\hat{\Pi}_t^H + (1-n)\hat{\Pi}_t^F = 0 \tag{2.145}$$

verifying part a) of the proposition.

Given that average inflation is zero and that $\hat{C}_t = \tilde{C}_t$ for all t, the Euler equation immediately implies

$$\hat{i}_t = -\rho\tilde{C}_t \tag{2.146}$$

which is part b).

[28] It is straightforward to show this more formally using a proof by contradiction.

Using (2.136) to eliminate $\phi_{3,t}$ from (2.138) gives

$$\begin{aligned}\beta \mathrm{E}_t \Pi^H_{t+1} &= -\frac{\Lambda_C n(1-n)}{\gamma}(\hat{T}_t - \tilde{T}_t) + \frac{\Lambda_{\mathcal{P}} n(1-n)}{2\gamma}\mathcal{P}^R_t + \Pi^H_t \\ &\quad + \frac{1}{2\gamma}\left[a(1-n)\phi_{1,t} - an\phi_{2,t} + \phi_{1,t} - \phi_{1,t-1} - \beta \mathrm{E}_t(\phi_{1,t+1} - \phi_{1,t})\right]\end{aligned}$$

By solving the Phillips curve for H for $\beta \mathrm{E}_t \Pi^H_{t+1}$ and substituting, one obtains

$$\begin{aligned}a_{\mathcal{P}} \mathcal{P}^H_t &= -\left[(1-n)a - \frac{\Lambda_C n(1-n)}{\gamma}\right](\hat{T}_t - \tilde{T}_t) - \frac{\Lambda_{\mathcal{P}} n(1-n)}{2\gamma}\mathcal{P}^R_t \\ &\quad - \frac{1}{2\gamma}\left[a(1-n)\phi_{1,t} - an\phi_{2,t} + \phi_{1,t} - \phi_{1,t-1} - \beta \mathrm{E}_t(\phi_{1,t+1} - \phi_{1,t})\right]\end{aligned}$$

Analogous steps for F yield

$$\begin{aligned}a_{\mathcal{P}} \mathcal{P}^F_t &= -\left[\frac{\Lambda_C n(1-n)}{1-\gamma} - na\right](\hat{T}_t - \tilde{T}_t) + \frac{\Lambda_{\mathcal{P}} n(1-n)}{2(1-\gamma)}\mathcal{P}^R_t \\ &\quad - \frac{1}{2(1-\gamma)}\left[-a(1-n)\phi_{1,t} + an\phi_{2,t} + \phi_{2,t} - \phi_{2,t-1} - \beta \mathrm{E}_t(\phi_{2,t+1} - \phi_{2,t})\right]\end{aligned}$$

Taking the difference of these two equations and simplifying gives

$$\begin{aligned}a_{\mathcal{P}} \mathcal{P}^R_t &= -\left(\Lambda_C - a\right)(\hat{T}_t - \tilde{T}_t) + \frac{\Lambda_{\mathcal{P}}}{2}\mathcal{P}^R_t \\ &\quad - \frac{1}{2(1-\gamma)}\left[-a(1-n)\phi_{1,t} + an\phi_{2,t} + \phi_{2,t} - \phi_{2,t-1} - \beta \mathrm{E}_t(\phi_{2,t+1} - \phi_{2,t})\right] \\ &\quad + \frac{1}{2\gamma}\left[a(1-n)\phi_{1,t} - an\phi_{2,t} + \phi_{1,t} - \phi_{1,t-1} - \beta \mathrm{E}_t(\phi_{1,t+1} - \phi_{1,t})\right] \qquad (2.147)\end{aligned}$$

Equations (2.76) and (2.77) can be combined to

$$\hat{T}_t - \hat{T}_{t-1} = \beta \mathrm{E}_t(\hat{T}_{t+1} - \hat{T}_t) - a(\hat{T}_t - \tilde{T}_t) + a_{\mathcal{P}} \mathcal{P}^R_t$$

Together with (2.147) this equation can be written as

$$\begin{aligned}\hat{T}_t - \hat{T}_{t-1} &= \beta \mathrm{E}_t(\hat{T}_{t+1} - \hat{T}_t) - a(\hat{T}_t - \tilde{T}_t) - \left(\Lambda_C - a\right)(\hat{T}_t - \tilde{T}_t) + \frac{\Lambda_{\mathcal{P}}}{2}\mathcal{P}^R_t \\ &\quad + \frac{a}{2\gamma(1-\gamma)}\left[(1-n)\phi_{1,t} - n\phi_{2,t}\right] + \frac{1+\beta}{2}\left(\frac{\phi_{1,t}}{n} - \frac{\phi_{2,t}}{1-n}\right)\end{aligned}$$

$$-\frac{1}{2}\left(\frac{\phi_{1,t-1}}{n}-\frac{\phi_{2,t-1}}{1-n}\right)-\frac{\beta}{2}\mathrm{E}_t\left(\frac{\phi_{1,t+1}}{n}-\frac{\phi_{2,t+1}}{1-n}\right) \tag{2.148}$$

To substitute out the multipliers, turn to (2.140) and (2.141). Adding both equations yields

$$\Lambda_{\mathcal{P}}\left(1+\frac{\kappa}{\eta}\right)\left[n\mathcal{P}_t^H+(1-n)\mathcal{P}_t^F\right]-a(\phi_{1,t}+\phi_{2,t})+\phi_{4,t}=0$$

The first two components of the sum are zero due to (2.144) and (2.70). This implies that $\phi_{4,t}=0$. Using this fact, one can combine (2.140) and (2.141) to obtain

$$(1-n)\phi_{1,t}-n\phi_{2,t}=\frac{1}{a_{\mathcal{P}}}\left[-\Lambda_{\mathcal{P}}n(1-n)\left(1+\frac{\kappa}{\eta}\right)\mathcal{P}_t^R+\Lambda_{\mathcal{P}}n(1-n)(\hat{T}_t-\delta\tilde{T}_t)\right]$$

and

$$\frac{\phi_{1,t}}{n}-\frac{\phi_{2,t}}{1-n}=\frac{1}{a_{\mathcal{P}}}\left[-\Lambda_{\mathcal{P}}\left(1+\frac{\kappa}{\eta}\right)\mathcal{P}_t^R+\Lambda_{\mathcal{P}}(\hat{T}_t-\delta\tilde{T}_t)\right]$$

Inserting these two expressions into (2.148) and simplifying yields

$$\begin{aligned}\mathcal{P}_t^R \quad &= \Psi\left\{\left[1+\beta+\Lambda_C-\frac{(1+\beta+a)\Lambda_{\mathcal{P}}}{2a_{\mathcal{P}}}\right]\hat{T}_t-\left(1-\frac{\Lambda_P}{2a_{\mathcal{P}}}\right)\hat{T}_{t-1}-\left(1-\frac{\Lambda_P}{2a_{\mathcal{P}}}\right)\beta\mathrm{E}_t\hat{T}_{t+1}-\right.\\ &\left.\left[\Lambda_C-\frac{\delta(1+\beta+a)\Lambda_{\mathcal{P}}}{2a_{\mathcal{P}}}\right]\tilde{T}_t-\frac{\delta\Lambda_{\mathcal{P}}}{2a_{\mathcal{P}}}\tilde{T}_{t-1}-\frac{\Lambda_P}{2a_{\mathcal{P}}}\left(1+\frac{\kappa}{\eta}\right)\mathcal{P}_{t-1}^R-\frac{\Lambda_P}{2a_{\mathcal{P}}}\left(1+\frac{\kappa}{\eta}\right)\beta\mathrm{E}_t\mathcal{P}_{t+1}^R\right\}\end{aligned}$$

with

$$\Psi\equiv\frac{2a_{\mathcal{P}}}{\Lambda_{\mathcal{P}}a_{\mathcal{P}}-\Lambda_{\mathcal{P}}\left(1+\frac{\kappa}{\eta}\right)(1+\beta+a)}$$

which can be written as in part c) of Proposition 1.

2.7.7 Welfare

Unconditional Expectation of Welfare

To evaluate welfare, the strategy adopted here is to follow a number of recent contributions in considering the unconditional expectation of the welfare criterion in period $t=0$.[29] The

[29]The derivations in this section are based on analogous calculations in the appendix to Ferrero (2009).

second-order approximation to welfare derived above can be written as

$$-\frac{1-\beta}{\Omega}W_0 = (1-\beta)\,\mathrm{E}_0\sum_{t=0}^{\infty}\beta^t L_t$$

Taking the unconditional expectation based on the probability distribution of the exogenous variables in $t=0$ on both sides and using the law of iterated expectations yields

$$-\frac{1-\beta}{\Omega}W = (1-\beta)\,\mathrm{E}\sum_{t=0}^{\infty}\beta^t L_t \tag{2.149}$$

where $W \equiv \mathrm{E}(W_0)$.

In order to find a simple expression for per period loss L_t, let the vector of all endogenous variables x_t be defined as $x_t \equiv \left(\hat{\Pi}_t^H, \hat{\Pi}_t^F, \hat{C}_t, \phi_{3,t}, \phi_{4,t}, \phi_{5,t}, \phi_{6,t}, \mathcal{P}_t^H, \mathcal{P}_t^F, \hat{T}_t, \phi_{1,t}, \phi_{2,t}, \tilde{T}_t^a, \tilde{C}_t\right)'$. The equilibrium laws of motion derived using Klein's method, the analogues of (2.142) and (2.143) for the case of AR(1)-shocks, can then be re-written as

$$x_t = Kx_{t-1} + Nu_t$$

with $u_t \equiv (u_{T,t}, u_{C,t})'$. K is a 14×14 matrix whose first nine elements of each row are zeroes and N is 14×2. Now define $y_t \equiv \left(\hat{\Pi}_t^H, \hat{\Pi}_t^F, \hat{C}_t - \tilde{C}_t, \hat{T}_t - \tilde{T}_t^a, \hat{T}_t - \delta\tilde{T}_t^a, \mathcal{P}_t^H, \mathcal{P}_t^F, \mathcal{P}_t^F - \mathcal{P}_t^H\right)'$ and note that one can find a matrix M such that

$$y_t = Mx_t$$

M is a simple 8×14 matrix that is not shown here for conciseness. y_t is constructed so that it contains the elements of the per period loss function. L_t is therefore given by the quadratic expression

$$L_t = y_t'\Gamma y_t$$

where

$$\Gamma \equiv \begin{bmatrix} \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Lambda_C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n(1-n)\Lambda_C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Lambda_{\mathcal{P}} n(1-n) \\ 0 & 0 & 0 & 0 & 0 & \Lambda_{\mathcal{P}}\frac{1}{2}n\left(1+\frac{\kappa}{\eta}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{\mathcal{P}}\frac{1}{2}(1-n)\left(1+\frac{\kappa}{\eta}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the definitions above, L_t becomes

$$\begin{aligned} L_t &= x_t' M' \Gamma M x_t \\ &= (Kx_{t-1} + Nu_t)' M' \Gamma M (Kx_{t-1} + Nu_t) \\ &= x_{t-1}' K' M' \Gamma M K x_{t-1} + x_{t-1}' K' M' \Gamma M N u_t + u_t' N' M' \Gamma M K x_{t-1} + u_t' N' M' \Gamma M N u_t \\ &= x_{t-1}' K' M' \Gamma M K x_{t-1} + 2 u_t' N' M' \Gamma M K x_{t-1} + u_t' N' M' \Gamma M N u_t \end{aligned}$$

Substituting back into (2.149) yields

$$-\frac{1-\beta}{\Omega} W = (1-\beta)\,\mathrm{E}\sum_{t=0}^{\infty} \beta^t \left(x_{t-1}' K' M' \Gamma M K x_{t-1} + 2u_t' N' M' \Gamma M K x_{t-1} + u_t' N' M' \Gamma M N u_t\right) \tag{2.150}$$

The individual components of the right-hand-side of this equation are now considered in turn starting with the last one.

$$\begin{aligned} (1-\beta)\,\mathrm{E}\sum_{t=0}^{\infty} \beta^t u_t' N' M' \Gamma M N u_t &= (1-\beta)\sum_{t=0}^{\infty} \beta^t \mathrm{E}\left[\mathrm{tr}(u_t' N' M' \Gamma M N u_t)\right] \\ &= (1-\beta)\sum_{t=0}^{\infty} \beta^t \mathrm{tr}\left[\mathrm{E}(u_t u_t') N' M' \Gamma M N\right] \\ &= \mathrm{tr}\left(\Sigma N' M' \Gamma M N\right) \end{aligned} \tag{2.151}$$

where $\Sigma \equiv \mathrm{E}(u_t u_t')$ is the covariance matrix associated with the white noise components of the disturbances.

$$\begin{aligned}(1-\beta)\,\mathrm{E}\sum_{t=0}^{\infty}\beta^t 2u_t'N'M'\Gamma MKx_{t-1} &= (1-\beta)\sum_{t=0}^{\infty}\beta^t 2\mathrm{E}\left[\mathrm{tr}(u_t'N'M'\Gamma MKx_{t-1})\right] \\ &= (1-\beta)\sum_{t=0}^{\infty}\beta^t 2\mathrm{tr}\left[\mathrm{E}(x_{t-1}u_t')N'M'\Gamma MK\right] \\ &= 0 \qquad (2.152)\end{aligned}$$

since $\mathrm{E}(x_{t-1}u_t') = 0$.

$$\begin{aligned}(1-\beta)\,\mathrm{E}\sum_{t=0}^{\infty}\beta^t x_{t-1}'K'M'\Gamma MKx_{t-1} &= (1-\beta)\sum_{t=0}^{\infty}\beta^t \mathrm{E}\left[\mathrm{tr}(x_{t-1}'K'M'\Gamma MKx_{t-1})\right] \\ &= \mathrm{tr}\left[(1-\beta)\sum_{t=0}^{\infty}\beta^t K'M'\Gamma MK\mathrm{E}\left(x_{t-1}x_{t-1}'\right)\right] \\ &= \mathrm{tr}\left(K'M'\Gamma MKJ\right) \qquad (2.153)\end{aligned}$$

where J is defined as $J \equiv (1-\beta)\sum_{t=0}^{\infty}\beta^t\mathrm{E}\left(x_{t-1}x_{t-1}'\right)$. By making use of the starting conditions $x_{-1} = \mathbf{0}$, J can be re-arranged as follows.

$$\begin{aligned}J &\equiv (1-\beta)\sum_{t=0}^{\infty}\beta^t\mathrm{E}\left(x_{t-1}x_{t-1}'\right) \\ &= \beta(1-\beta)\sum_{t=0}^{\infty}\beta^t\mathrm{E}\left(x_t x_t'\right) \\ &= \beta(1-\beta)\sum_{t=0}^{\infty}\beta^t\mathrm{E}\left[(Kx_{t-1}+Nu_t)(Kx_{t-1}+Nu_t)'\right] \\ &= \beta(1-\beta)\sum_{t=0}^{\infty}\beta^t\mathrm{E}\left[(Kx_{t-1}x_{t-1}'K' + Kx_{t-1}u_t'N' + Nu_tx_{t-1}'K' + Nu_tu_t'N')\right] \\ &= \beta\left(KJK' + N\Sigma N'\right)\end{aligned}$$

This expression can be solved for J using the fact that for any matrices A, B, C, $\mathrm{vec}(ABC) = (C'\otimes A)\mathrm{vec}(B)$. Hence,

$$\begin{aligned}\mathrm{vec}(J) &= \beta\mathrm{vec}(KJK') + \beta\mathrm{vec}(N\Sigma N') \\ &= \beta(K\otimes K)\mathrm{vec}(J) + \beta\mathrm{vec}(N\Sigma N')\end{aligned}$$

$$\Rightarrow \quad \mathrm{vec}(J)\,[\mathrm{I}-\beta(K\otimes K)] = \beta \mathrm{vec}(N\Sigma N')$$
$$\Rightarrow \quad \mathrm{vec}(J) = \beta\,[\mathrm{I}-\beta(K\otimes K)]^{-1}\,\mathrm{vec}(N\Sigma N') \tag{2.154}$$

By combining the equations (2.150) through (2.153), one obtains

$$W = -\frac{\Omega}{1-\beta}\left[\mathrm{tr}\left(\Sigma N'M'\Gamma MN\right)+\mathrm{tr}\left(K'M'\Gamma MKJ\right)\right] \tag{2.155}$$

where J is implicitly given by (2.154).

Consumption equivalent of welfare differentials

Based on the unconditional welfare measure derived in the previous section, one can compare suboptimal policies with the Ramsey policy by converting welfare differentials into consumption equivalents. The value associated with the Ramsey policy is given by

$$V^r \equiv \mathrm{E}\sum_{t=0}^{\infty}\beta^t\left\{U(C_t^r)-\int_0^n V\left[y_t^r(h),z_t^H\right]dh-\int_n^1 V\left[y_t^r(f),z_t^F\right]df \right.$$
$$\left. -nW\left(\mathcal{P}_t^{H,r}\right)-(1-n)W\left(\mathcal{P}_t^{F,r}\right)\right\}$$

Similarly, the value associated with an alternative policy is

$$V^a \equiv \mathrm{E}\sum_{t=0}^{\infty}\beta^t\left\{U(C_t^a)-\int_0^n V\left[y_t^a(h),z_t^H\right]dh-\int_n^1 V\left[y_t^a(f),z_t^F\right]df \right.$$
$$\left. -nW\left(\mathcal{P}_t^{H,a}\right)-(1-n)W\left(\mathcal{P}_t^{F,a}\right)\right\}$$

λ is then defined as the fraction of consumption under the Ramsey policy that a household would have to give up to be as well off under the Ramsey policy as under the alternative policy.

$$\mathrm{E}\sum_{t=0}^{\infty}\beta^t\left\{U\left[C_t^r(1-\lambda)\right]-\int_0^n V\left[y_t^r(h),z_t^H\right]dh-\int_n^1 V\left[y_t^r(f),z_t^F\right]df \right.$$
$$\left. -nW\left(\mathcal{P}_t^{H,r}\right)-(1-n)W\left(\mathcal{P}_t^{F,r}\right)\right\} = V^a$$

To be able to solve this condition for λ, consistent with the previous assumption $\rho = 1$ it is assumed that $U(\cdot) = \log(\cdot)$. Therefore,

$$\begin{aligned}
V^a &= \mathrm{E}\sum_{t=0}^{\infty}\beta^t\left\{\log\left(C_t^r\right)+\log(1-\lambda)-\int_0^n V\left[y_t^r(h), z_t^H\right]dh-\int_n^1 V\left[y_t^r(f), z_t^F\right]df\right.\\
&\quad\left.-nW\left(\mathcal{P}_t^{H,r}\right)-(1-n)W\left(\mathcal{P}_t^{F,r}\right)\right\}\\
&= \frac{1}{1-\beta}\log(1-\lambda)+V^r\\
\log(1-\lambda) &= (1-\beta)\left(V^a-V^r\right)\\
\lambda &= 1-e^{(1-\beta)(V^a-V^r)}
\end{aligned}$$

In this equation, V^r is approximated to the second order by W given in (2.155). V^a is approximated by W^a which can be derived in an analogous way. The consumption equivalent is then given by

$$\lambda = 1 - e^{(1-\beta)(W^a - W)} \tag{2.156}$$

which is the measure used in the text.

Chapter 3

Long-Term Government Debt and Household Portfolio Composition over the Life Cycle

Chapter Summary

This chapter studies the participation decision in the markets for equity and long-term government debt and portfolio shares conditional on participation over the course of the life cycle. Empirical evidence is obtained using a quasi-panel constructed from seven consecutive waves of the Survey of Consumer Finances. Three different strategies are employed to identify the effects of observation time, respondent age and cohort. I find that participation in the long-term government debt market, similar to that in the equity market, takes an inverse U-shape. The conditional long-term government debt share is weakly increasing and the conditional equity share weakly decreasing over the working life. The main features of the data can be explained using a portfolio choice model in which investors face idiosyncratic risk through non-financial income and fixed probabilities of retirement and death, as well as aggregate risk through asset returns.

3.1 Introduction

Analyses of portfolio choice over the life cycle generally focus on holdings of stock and a risk-free asset, neglecting holdings of long-term bonds. With short-term interest rates close to the zero lower bound in Europe and the US more or less since the outbreak of a the Financial Crisis, central banks like the Federal Reserve and the Bank of England have purchased large amounts of long-term government debt as a part of their respective unconventional monetary policy programmes. As a result, a considerable interest has arisen in the role of long-term government debt in household portfolios and the motives for portfolio rebalancing. Using data from the Survey of Consumer Finances (SCF) and a portfolio-choice model in which agents can invest in three assets—stock, long-term government debt and a riskless asset—this chapter studies the asset-market participation decision and the composition of household portfolios over the course of the life cycle.

Empirical results are based on a data set constructed from seven consecutive waves of the SCF. The joint existence of birth cohort, time and age influences on the objects of interest, participation in the markets for equity and long-term government debt and the respective portfolio shares conditional on participation, result in a well-known identification problem.[1] Using three different identification strategies, a latent variable model with a participation equation is estimated. Similar to participation in the stock market, participation in the market for long-term government debt takes an inverse U-shape. While the conditional portfolio share of stock is declining with age, the conditional share of long-term government debt is mildly increasing until the age of around sixty and declining somewhat thereafter.[2]

A portfolio choice model with uninsurable labour income risk, stochastic stock and long-term government debt returns as well as random retirement and death ages is used to gain intuition for these results. Following Viceira (2001), retirement in the employment stage and death in the retirement stage occur with constant probabilities, permitting life-cycle effects to be incorporated into a tractable environment with an ex ante infinite planning horizon. Fixed

[1] Browning et al. (2012) give a detailed description of the "age-period-cohort" problem. According to them, the problem can be traced back at least to Ryder (1965).

[2] The results regarding equity holdings are in line with findings by Fagereng et al. (2013) and Gomes and Michaelides (2005).

costs of asset market participation prevent agents from investing in stock and government debt at young age. The participation rate increases as they accumulate wealth and later decreases again due to agents retiring and running down their savings. The average long-term debt and stock share conditional on participation respectively increases and decreases with age. This is the case, since the portfolio income of market participants increases on average, implying that the ratio of portfolio income to labour income increases. As a result, agents with CRRA utility rebalance their portfolios to reduce their risk exposure. Government debt is suitable for this purpose, as its return is less volatile than that of stock but higher in expectation than that of the risk-free asset.

A large literature is concerned with portfolio choice over the course of the life cycle. Early contributions by Merton (1969, 1971) and Samuelson (1969) analyse optimal portfolio choice abstracting from the asset market participation decision. More recent examples include, but are not limited to, Alan (2006), Cocco et al. (2005), Fagereng et al. (2013), Gomes and Michaelides (2005) and Haliassos and Michaelides (2003). In these papers, households are restricted to holdings of stock and a riskless asset. I relax this constraint by adding a long-term government bond to the portfolio choice problem. Campbell and Viceira (2001) and Wachter (2003) also consider asset allocation problems with long-term bonds but do not study life-cycle effects.

The chapter is organised as follows. Section 3.2 presents the empirical results. It starts by describing the data set, then discusses parameter identification and finally shows estimation results. Section 3.3 contains a description of the model, its calibration and the resulting policy functions. Section 3.4 presents simulations and a comparison of the model predictions with the empirical findings. Section 3.5 concludes.

3.2 Empirical Results

This section presents empirical findings on long-term government bond holdings of US households. In addition, results for equity holdings are shown that confirm stylised facts previously described in the literature.

3.2.1 Data

The data is taken from the Survey of Consumer Finances (SCF), which is administered by the Federal Reserve Board. The SCF is a cross-sectional survey that is repeated in three-year intervals. It contains detailed information on assets, liabilities and socio-economic characteristics of a large number of US families. The data set employed here is constructed from the seven survey waves collected between 1989 and 2007.[3] Since the data consists of repeated cross-sections rather than a panel, I am not able to track *individuals* over time. However, due to the large amount of households included in each survey wave, one can track *cohorts* of individuals defined by their birth year over the sample period.[4] I revisit this point below in more detail.

It should be noted that there is an intentional over-sampling of wealthy households in the SCF relative to the entire US population. This is done to allow for more precise estimates of financial asset holdings, which are highly concentrated among households in the upper tail of the wealth distribution, and to correct for the fact that the non-response rate is positively correlated with wealth.[5] The benefit of a much improved estimation precision comes at the cost of being able to make inferences for wealthier households only. Descriptive statistics of the sample are shown in Table 3.5 in the appendix.

The SCF contains information on a large variety of assets held by households, which I divide into three categories. These categories are long-term government debt, stocks and a residual category that mainly includes cash/liquidity, short-term sovereign debt and corporate bonds. Most assets that appear in household balance sheets can be fully attributed to one of these three categories. When this is not the case, a careful partial assignment is done based on additional information about the institutions that issue the asset in question.

According to the "Monthly Statement of the Public Debt of the United States" from December 2007, 22.1% of total marketable debt held by the public took the form of Bills (maturity of one year or less), while the remaining 77.9% were issued in the form of Notes, Bonds and TIPS (maturity of two years or more). Assuming that agents are homogeneous in regards to

[3]Data from 2010 is also available; it is not used here to avoid bias introduced by crisis-specific effects. Data from before 1989 is not used due to changes in the availability of a subset of variables.

[4]This fact is emphasised also in Deaton and Paxson (1994).

[5]See Kennickell (2008).

the maturity composition of government debt in their portfolios, 77.9% of the marketable US government debt held by a household is assigned to long-term government debt and the rest to the residual category.

Funds held in individual retirement accounts (IRAs) are also divided into more than one category. An IRA is a tax-advantaged retirement savings plan. Funds transferred into an IRA can be requested to be allocated to a large variety of financial assets. The Employee Benefit Research Institute (EBRI) collects data on the allocation of assets in IRAs.[6] Based on this data, I attribute 45.8% of the funds held in an IRA to stocks, 18.4% to long-term government debt and the remainder to the last category. The assignment of all assets into the three broad categories is described in detail in Section 3.6.1 in the appendix and summarised in Table 3.3.

Figures 3.1-3.3 illustrate the average shares of the portfolio categories constructed in this way. Each line represents the average portfolio share of a given birth-year cohort at a particular age. Since data points are available only every three years, both respondent age and cohort (i.e. birth year) are divided into three-year intervals. For example, the earliest data available is from 1989. The youngest age group considered includes households with a household head aged 26-28.[7] Individuals that are 26-28 of age in 1989 belong to the birth-year cohort 1961-63. The 1961-63 cohort is sampled seven times between 1989 and 2007. Its members are aged 29-31 in 1992, 32-34 in 1995 and so on; they are last sampled at the age of 44-46 in 2007. In Figures 3.1-3.3, the blue line at the very left represents the average portfolio shares held by the 1961-63 cohort. Similarly, the green line starting with age group 29-31 represents the average portfolio shares of the 1958-60 cohort. Altogether, the sample contains the eleven cohorts born between 1931-33 and 1961-63, each observed at seven consecutive age groups between the ages of 26-28 and 74-76.[8]

[6] See EBRI Note "IRA Asset Allocation and Characteristics of the CDHP Population, 2005-2010" from May 2011, available at www.ebri.org/publications/notes.

[7] In the SCF, the terms "household" or "family" refer to a "primary economic unit", which consists of a core couple or economically dominant single individual and all other individuals in the household that are financially interdependent with that couple or individual. The "head" is defined as the single economically-dominant individual in a household without a core couple, the male in a household with a mixed-sex core couple and the older individual in the case of a same-sex core couple.

[8] Only cohorts that fall inside this age interval at all seven survey waves are considered. Younger and older age groups are not examined due to a lack of sufficient data. Table 3.4 in Section 3.6.2 of the appendix shows the number of observations for each cohort-age pair. Note that the data contains multiple imputations as is explained in more detail in the table notes.

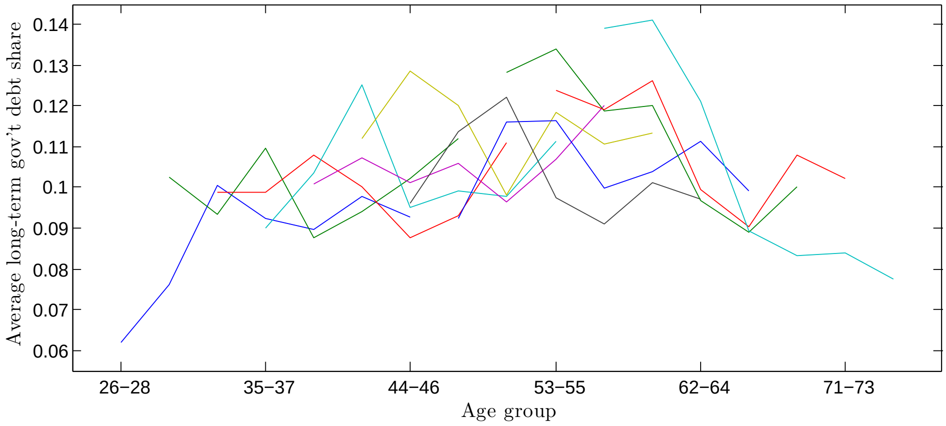

Figure 3.1: Average portfolio share of long-term government debt

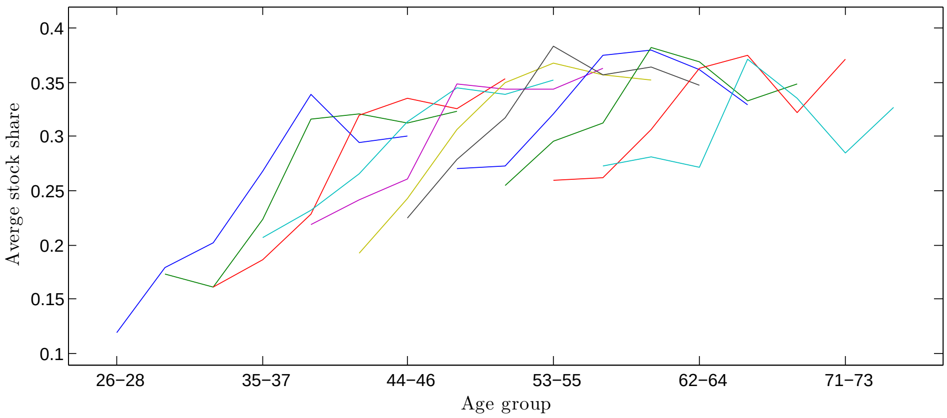

Figure 3.2: Average portfolio share of equity

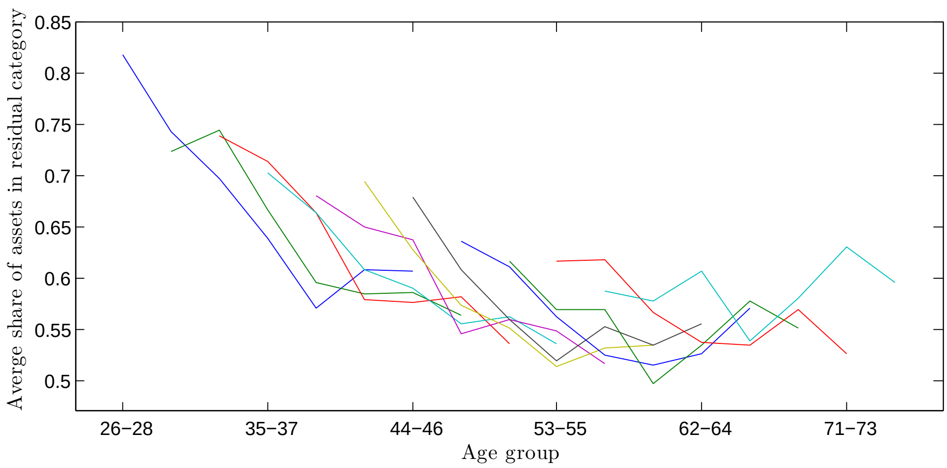

Figure 3.3: Average portfolio share of cash/liquidity, corporate bonds and other financial assets

The average portfolio share of stocks is increasing in household portfolios with a peak in the late fifties or early sixties of the household head. Average long-term government debt holdings behave in a somewhat similar way, although the pattern is less well pronounced. The average share of assets in the residual category resembles the mirror image of the share of equity holdings. However, the unconditional averages computed in the figures 3.1-3.3 do *not* provide a clear picture of life-cycle effects on the portfolio composition of households. The reasons are twofold.

First, the adjustments visible in the figures can be due to changes at the internal or external margin. A number of papers report that the rate of participation in stock markets first increases and later decreases significantly over the life cycle.[9] This suggests that the inverse U-shape in Figure 3.2 largely results from households entering and exiting the stock market rather than adjustments at the internal margin. An important question explored below is whether or not the same is true for holdings of long-term government debt.

Second, it is not possible from the figures to disentangle the effects of age, observation period and cohort. The objects of interest here are age effects, that is, life-cycle related effects like those from education, family formation and retirement. Period effects result from events that occur at the time of data collection. An example for a period effect is the dot-com-bubble and its bursting, which is reflected in the survey waves from 2001 and 2004. Cohort effects include cohort-specific experiences like growing up during war time as was the case for the oldest cohorts in the sample.

Even if, for example, we were to observe a figure of the same kind as Figures 3.1-3.3 in which all lines were perfectly aligned such that they formed one single upward-sloping line, we could not say whether this was due to pure age effects or a combination of time and cohort effects. A pure age effect that led older households to invest a higher share of their portfolio in a given asset would generate such a result. However, even if the age effect were zero, time effects could cause each individual line to be sloped upwards and cohort effects of increasing size could also result in all lines aligning precisely in the way previously mentioned. The econometric concept underlying this problem is perfect multicollinearity (birth year + age = observation period). I lay out the strategy pursued to identify age effects despite of this relationship in the next section.

[9] See Ameriks and Zeldes (2001), Fagereng et al. (2013) and Guiso et al. (2003). Haliassos (2008) contains a more general summary of the literature on limited participation in asset markets.

3.2.2 Identification

Parameter identification is endangered as a result of two features of the data, sample selection and perfect multicollinearity of age, time of observation and birth year. I describe how both of these problems are addressed in turn.

Sample Selection

The self-selection of agents into participants and non-participants in the market for a given financial asset results in a standard sample selection problem. To address this issue, I employ a latent variable model with a Probit selection equation as suggested, for example, by Heckman (1979).[10] Formally, the model is given by

$$s_i = x'_{2,i}\beta_2 + \sigma_{12}\lambda\left(x'_{1,i}\hat{\beta}_1\right) + e_{2,i} \tag{3.1}$$

$\forall i$ with $s_i > 0$, where s_i is the portfolio share of the asset in question, $\lambda \equiv \phi\left(x'_{1,i}\hat{\beta}_1\right)/\Phi\left(x'_{1,i}\hat{\beta}_1\right)$ is the Inverse Mills Ratio and $\hat{\beta}_1$ is obtained from estimating the first-stage Probit model

$$\begin{aligned} \Pr(P_i = 1|x_{1,i}) &= \Pr\left(x'_{1,i}\beta_1 + e_{1,i} > 0\right) \\ &= \Phi\left(x'_{1,i}\beta_1\right) \end{aligned} \tag{3.2}$$

$P_i = 1$ if i is a participant in the market for the asset under consideration and $P_i = 0$ otherwise. The error terms are normal, $e_{1,i} \sim N(0,1)$ and $e_{2,i} \sim N(0,\sigma^2)$, with $\text{Cov}(e_{1,i}, e_{2,i}) = \sigma_{12}$. A naive estimation of (3.1), that is without the inverse Mills ratio as a regressor, may lead to omitted variable bias unless $\sigma_{12} = 0$. Since $\lambda(\cdot)$ is approximately linear over a large part of its domain, the model is only well-identified for $x_{1,i} \neq x_{2,i}$. I discuss in the section on estimation which of the variables in $x_{1,i}$ are excluded in the second stage to ensure that this condition holds.

[10]The model is sometimes referred to as a "type 2 Tobit model" and the corresponding two-step estimation procedure as the "Heckit" estimator.

The "age-period-cohort problem"

Due to perfect multicollinearity of age, sampling period and year of birth, a simple linear model that aims to separate age, period and cohort effects is under-identified. To see this, consider the following example.[11] Let a_i denote the age of respondents, t_i the time period in which they are sampled and c_i their year of birth. Suppose that data is available for two consecutive time periods, $t_i \in \left\{t^1, t^2\right\}$, and that three consecutive cohorts are sampled in both periods, $c_i \in \left\{c^1, c^2, c^3\right\}$. Age can then take on four distinct values, $a_i \in \left\{a^1, a^2, a^3, a^4\right\}$.[12]

A linear regression of some variable of interest y_i on a constant and complete sets of age, period and cohort dummies is given by

$$\begin{aligned} y_i = \beta_0 &+ \alpha_1 a_i^1 + \alpha_2 a_i^2 + \alpha_3 a_i^3 + \alpha_4 a_i^4 \\ &+ \theta_1 t_i^1 + \theta_2 t_i^2 \\ &+ \gamma_1 c_i^1 + \gamma_2 c_i^2 + \gamma_3 c_i^3 + e_i \end{aligned} \tag{3.3}$$

where $x_i^n = 1$ if $x_i = x^n$ and $x_i^n = 0$ otherwise for $x \in \{a, t, c\}$ and $n \in \{1, 2, 3, 4\}$. Of course, the sum of each set of binary variables is equal to the constant. This can be easily avoided by making use of the normalisation $\beta_0 = \theta_1 = \gamma_1 = 0$ and re-interpreting the remaining parameters accordingly. The equation above becomes

$$\begin{aligned} y_i = &\alpha_1 a_i^1 + \alpha_2 a_i^2 + \alpha_3 a_i^3 + \alpha_4 a_i^4 \\ &+ \theta_2 t_i^2 \\ &+ \gamma_2 c_i^2 + \gamma_3 c_i^3 + e_i \end{aligned} \tag{3.4}$$

However, the fact that there exists a linear relationship between the age, observation period and cohort of each respondent implies that the data matrix corresponding to regression equation (3.4) is not invertible and parameter estimates cannot be computed. More precisely, the linear

[11] The example used here builds on the exposition in Browning et al. (2012).
[12] For example, if $t_i \in \{2000, 2001\}$ and $c_i \in \{1950, 1951, 1952\}$ then $a_i \in \{48, 49, 50, 51\}$.

relationship between age, period and cohort implies[13]

$$2a_i^1 + a_i^2 - a_i^4 + t_i^2 = c_i^2 + 2c_i^3 \tag{3.5}$$

Inserting (3.5) into (3.4) yields

$$\begin{aligned} y_i = {} & \tilde{\alpha}_1 a_i^1 + \tilde{\alpha}_2 a_i^2 + \tilde{\alpha}_3 a_i^3 + \tilde{\alpha}_4 a_i^4 \\ & + \tilde{\gamma}_2 c_i^2 + \tilde{\gamma}_3 c_i^3 + e_i \end{aligned} \tag{3.6}$$

where

$$\begin{pmatrix} \tilde{\alpha}_1 \\ \tilde{\alpha}_2 \\ \tilde{\alpha}_3 \\ \tilde{\alpha}_4 \\ \tilde{\gamma}_2 \\ \tilde{\gamma}_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \theta_2 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \tag{3.7}$$

Using equation (3.6), the six reduced form parameters $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \tilde{\gamma}_2, \tilde{\gamma}_3$ can be estimated. From (3.7) it is clear though that it is not possible to back out the seven structural parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \theta_2, \gamma_2, \gamma_3$ from knowledge of the reduced form parameters. The structural parameters are under-identified, unless at least *one* parameter restriction is imposed. It can be easily shown that this result generalises to scenarios with more observation periods and cohorts.

To be able to judge the robustness of estimation results, I pursue three distinct strategies of achieving parameter identification. The first is to impose an equality restriction on neighbouring cohort effects, i.e. I impose

$$\gamma_n = \gamma_{n+1} \tag{3.8}$$

for some n. Although this restriction formally reduces the generality of the model, the bias it

[13] Continuing the previous example, a person that is born say in 1952 and surveyed in 2001 is aged 49 when surveyed; thus $t_i^2 = c_i^3 = a_i^2 = 1$ and $a_i^1 = a_i^4 = c_i^2 = 0$. It is straight forward to verify that (3.5) holds for all six such combinations of binary-variable values for which birth year and age sum to observation period.

introduces is only minimal if two neighbouring cohorts can be identified that have a sufficiently similar history. For a number of cohorts in the sample, this seems to be a plausible assumption.

The second strategy was suggested by Deaton and Paxson (1994).[14] The idea is to attribute cyclical fluctuations to time effects and trends to age and cohort effects. This is achieved by requiring time effects to sum to zero and to be orthogonal to a linear trend, that is

$$g'\theta = 0 \tag{3.9}$$

where $g = (0, 1, 2, ..., T-1)'$ is the trend, θ is the vector of coefficients on the time dummies and T is the number of observation periods. This set of assumptions correctly identifies all effects if indeed only age and cohort effects are trending. In the context here, one cannot be sure however that there is no trend in time effects. In particular, in the time period examined (1989-2007), stocks became a more widely-used mode of saving. Imposing (3.9) when a trend in time effects is present in the data could cause the coefficients on the age and cohort variables to pick up this trend and therefore to be biased. In the second-stage regression, I therefore de-trend the dependent variable, the portfolio share of a given asset, by subtracting its cross-sectional average at each time period. Since this is not feasible in a binary dependent variable model, I add a linear time trend as an explanatory variable at the first stage. Note that this implies that one of the age dummies has to be excluded from the Probit model.

The final identification strategy is to replace all time dummies in (3.4) with the first p principal components of a large set of stationary macroeconomic time series ranging from the first until the last sampling period. This resolves the linear dependence of the independent variables. As before, the asset share is de-trended and a trend is added to the selection equation to control for time trends. To the extent to which the principal components contain the effects otherwise picked up by the time dummies, this modification allows to estimate age, period and cohort effects without parameter restrictions. When the variable to be explained is the portfolio share of a particular asset, it is difficult to find examples of important time effects that are not reflected in macro variables.

[14] See also Deaton (1997).

3.2.3 Estimation and Results

For long-term government debt and stocks, this section shows estimation results from the decomposition of the participation decision and the portfolio choice conditional on participation into age, period and cohort effects, controlling for a number of socio-economic characteristics. Beginning with the second-stage regression, the equations estimated in case of the first identification strategy (parameter restriction on cohort effects) are

$$s_i = a_i'\alpha_2 + t_i'\theta_2 + c_i'\gamma_2 + \varsigma\lambda_i + z_{2,i}'\delta_2 + e_{2,i} \tag{3.10}$$

$$\Pr(P_i = 1|x_{1,i}) = \Phi\left(a_i'\alpha_1 + t_i'\theta_1 + c_i'\gamma_1 + z_{1,i}'\delta_1\right) \tag{3.11}$$

a_i is a complete set of age dummies for seventeen age groups, $a_i = \left(a_i^{26-28}, a_i^{29-31}, \ldots, a_i^{74-76}\right)'$; t_i is a vector of six year dummies, $t_i = \left(t_i^{1992}, t_i^{1995}, \ldots, t_i^{2007}\right)'$; and c_i contains ten cohort dummies, $c_i = \left(c_i^{1934-36}, c_i^{1937-39}, \ldots, c_i^{1961-63}\right)'$. $z_{2,i}$ and $z_{1,i}$ are additional family-specific controls and $x_{1,i} = (a_i, t_i, c_i, z_{1,i})'$. In case of the other two strategies, the equations are modified as explained in the previous section.

An exclusion restriction is required to avoid weak parameter identification. The variable excluded from the second stage of the model for long-term government debt is a binary variable indicating whether the household head is "not doing any work for pay at the present time". As we would expect, it has a highly significant negative effect on the participation decision. The hypothesis is that it does not directly affect the portfolio share of long-term government debt for those who hold the asset. Why should this be the case? The average age of those responding positively is significantly larger than that of individuals responding negatively (57.9 compared to 48.9) suggesting that positive responses come, at a disproportionate amount, from retired individuals. Of course, retired individuals differ systematically from not retired ones; they are older on average. But age effects are controlled for in the second stage regression so that the hypothesis holds true for this group unless there are effects from being retired that the age dummies do not pick up. Positive responses also come from individuals that have not yet reached retirement age. These cases are unproblematic if there are other household members that provide the family with labour income, since the source of family income should not affect portfolio shares. To address cases where labour income is diminished as a result of one or more household members being involuntarily unemployed and the family participates in the market for long-term debt nevertheless, a set of educational variables is used to control for potential common drivers of unemployment and the long-term government debt share. In a regression of the residuals from the second stage on all explanatory variables including the employment indicator, the instrument is insignificant well beyond the 10% level.

The finance literature suggests that stock holdings are sensitive to changes in labour income. In the case of the stock share, I therefore employ a different exclusion restriction. The assumption maintained here is that, at low levels of education, a marginal amount of additional schooling increases the likelihood of stock market participation but not the stock share conditional on participation. Compared to some years of high-school attendance, a completed high-school education may provide individuals with knowledge about investment opportunities and lower the costs of access to the stock market, but should have no direct effect on the stock share conditional on participation when other characteristics such as the occupation of the individual are controlled for. In the data, having earned a high-school diploma and visited college for some time without graduating has a strong positive effect on the participation probability. It is insignificant though at the 5% level in a regression of the second-stage residuals on all explanatory variables.

The estimation results are plotted in Figure 3.4 and summarised in Table 3.1. Panels a) and c) show the marginal values, the average predicted probabilities, of being a stockholder and a long-term government debt holder, respectively, for each age group. In panels b) and d), estimates of the age effects from the corresponding second-stage regressions are plotted.[15] Results are shown for all three identification strategies outlined before, including separate sets of results for two different cohort restrictions. The first cohort restriction equates the effects of the first two cohorts in the sample, 1934-36 and 1937-39, the second one those from the first two post-war cohorts, 1946-48 and 1949-51. The cohort effects of the oldest respondents are restricted, since it seems likely for any differences between them to wash out over the years until the sampling period. The second restriction appears to be reasonable from a historical

[15]Estimation results are graphed only for age groups that are observed in at least two cohorts. That is, coefficients on the first and last age group are not shown, since they are estimated with considerable error.

perspective. The first three principal components of macro variables are used when the time dummies are dropped from the model.

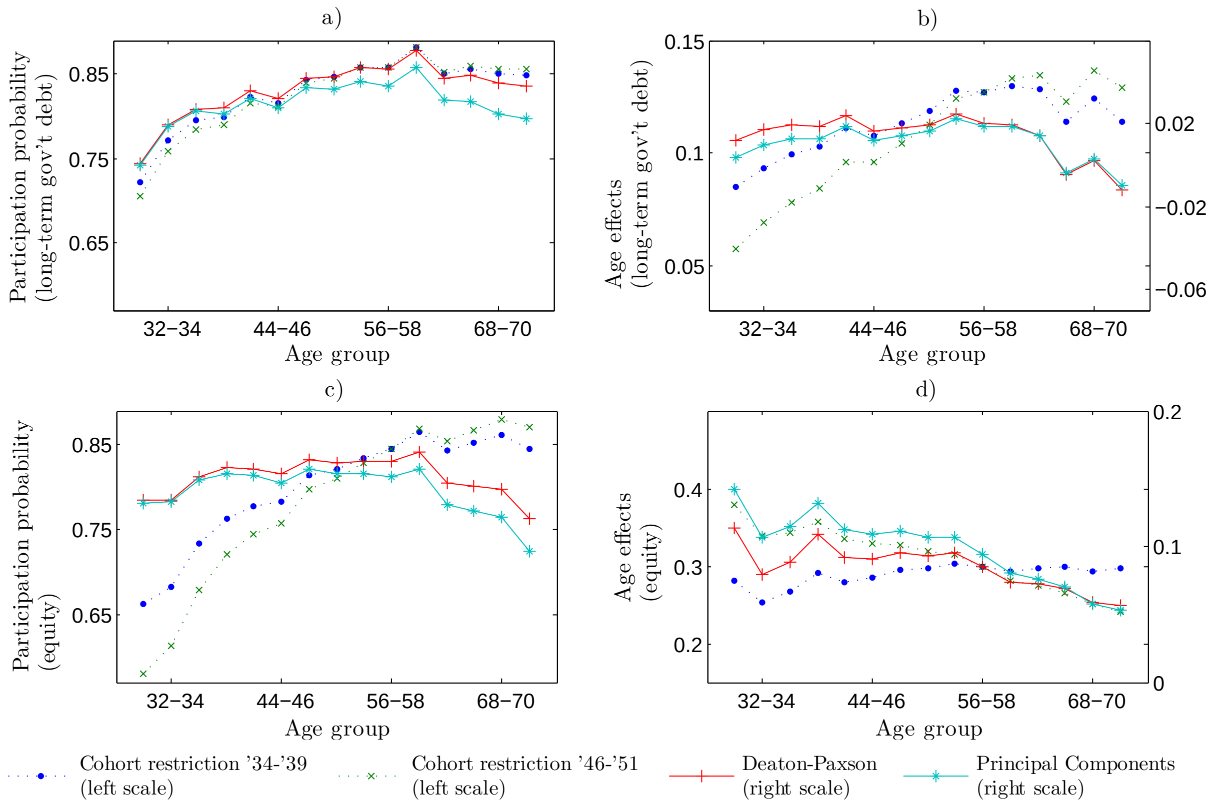

Figure 3.4: Predicted participation probabilities and estimated age effects on conditional asset shares

In all four cases, the profile of participation in the market for long-term government debt shows a clear hump shape. Participation rates rise until the age of 59-61 and then begin to decline. Until 62-64, the age effects on the conditional portfolio share of long-term government debt are slightly increasing under the two cohort restrictions and more or less flat under the Deaton-Paxson restrictions and when principal components are used to capture time effects. Note that the size of the effects is necessarily different as the dependent variable is de-trended in case of the latter two models and all models are estimated without a constant. A decline is visible at age 62-64 coinciding with a strong increase in the number of retirements. Overall, the results suggest that there is a clear inverse U-shape in participation rates and that the conditional portfolio share is non-decreasing until retirement, but potentially declines thereafter.

Stock market participation takes an almost identical shape to participation in the market for long-term government debt when the Deaton-Paxson restrictions are used and when the time dummies are replaced by principal components. Under the cohort restrictions, the participation profile also increases for younger age groups, although at a faster rate, but lacks the decline for older age groups. An explanation for this result could be that there is a time trend in stock market participation that the time dummies do not absorb completely, but that is eliminated by de-trending. The conditional stock share is monotone declining from 39-41 onwards. An exception is the case of the first cohort restriction, for which the stock share is flat.

Stock market participation and the portfolio share of stocks have been a topic of debate in the literature. In summarising the existing empirical evidence, Gomes and Michaelides (2005) state that 1) "participation rates increase during working life", 2) "there is some evidence suggesting that they might decrease during retirement" and 3) "there is no clear pattern of equity holdings over the life cycle". I interpret the results presented here as strong support for 1) and weaker support for 2). In more recent work, Fagereng et al. (2013) find evidence for the conditional stock share to decline over the life cycle using data from Norway. Regarding 3), my estimates seem to be more in line with the findings of Fagereng et al. (2013).

	Cohort Restr. 1		Cohort Restr. 2		Deaton-Paxson		Principal Comp.	
	1st st.	2nd st.	1st st.	2nd st.	1st st.	2nd st.	1st st.	2nd st.
λ		0.065**		0.065**		0.065**		0.066**
		(0.021)		(0.021)		(0.021)		(0.021)
ρ		0.445		0.445		0.445		0.449
min(N_{imp})	17,202	12,673	17,202	12,673	17,202	12,673	17,202	12,673
Total N	86,030	63,437	86,030	63,437	86,030	63,437	86,030	63,437
Sign. tests								
Age Eff's	96.8***	106.1***	99.2***	108.5***	96.4***	27.6**	82.4***	25.9*
(d.o.f.)	(17)	(17)	(17)	(17)	(16)	(17)	(16)	(17)
Time Eff's	12.8**	43.3***	12.8**	44.5***	12.7**	5.9		
(d.o.f.)	(6)	(6)	(6)	(6)	(5)	(5)		
Cohort Eff's	11.5	15.1*	11.5	15.1*	11.6	26.0***	11.7	16.9*
(d.o.f.)	(9)	(9)	(9)	(9)	(10)	(10)	(10)	(10)
Pri. Comp's							9.5**	5.4
(d.o.f.)							(3)	(3)

Notes: Results of first and second stage estimation shown for four models—cohort effects equated for '34-'36 and '37-'39 (Cohort Restr. 1), cohort effects equated for '46-'48 and '49-'51 (Cohort Restr. 2), Deaton-Paxson restrictions, principal components of macro variables replacing time dummies. Models estimated using two-step estimator (Heckit). Data is multiply imputed; for each respondent, there are five observations in the data. Point estimates are averages over five separate estimations. Strd. errors (in parentheses) are adjusted in an appropriate way. N_{imp} is number of obs. for imputation imp $\in \{1, 2, ..., 5\}$. For joint significant tests, avrg. Wald test stat. shown (each $\sim \chi^2$), degrees of freedom in parentheses. *p<0.1, **p<0.05, ***p<0.01.

Table 3.1: Estimation results for long-term government debt

The coefficient on the inverse Mills ratio λ is significant and the correlation of first-stage and second-stage residuals ρ is estimated to be positive in all four models, confirming the need to address the selection problem. SCF data is multiply imputed. The size of the smallest imputation group $\min(N_{\text{imp}})$ therefore gives a more accurate picture of the number of respondents in the sample than the total amount of observations N.[16] Age effects are generally highly significant. Time effects play an important role in the selection equations and the conditional share regressions under cohort restrictions but cease to do so in the conditional share equations when the dependent variable is de-meaned. Conditional asset shares thus appear to be more influenced by time trends than by high-frequency time effects. The participation decision however is strongly influenced by time effects even in the two models that include a time trend as a regressor at the first stage. Cohort effects are of lower importance and significant only at the second stage.

3.3 Model

There is a large number of investors that are faced with an asset market participation decision and, conditional on participation, an asset allocation problem in each period of their lives. Investors are born employed. They retire and subsequently die with fixed probabilities in every period, providing them with a motive to save for retirement and to deplete their asset stock once retired. Asset market participation is costly, but allows an investor to hold stocks and long-term government debt, while a non-participant is able to save only through a riskless and low-interest bearing asset that is comparable to short-term bonds or cash.[17] Investors are subject to idiosyncratic and aggregate risk. Idiosyncratic risk, which is assumed to be uninsurable, arises from non-financial income as well as the stochastic nature of retirement and death; aggregate risk results from the returns on stocks and long-term government debt.

[16]Point estimates and standard errors are corrected for data imputation. See also the notes to Tables 3.1 and 3.4.

[17]I do not distinguish between stock market participation and long-term government debt market participation in the model to reduce the dimensionality of the portfolio choice problem. Participation in both markets is highly correlated in the sample with a coefficient of 0.80.

3.3.1 Life Cycle Stages

Each investor $i \in I$ is born in period $t = 0$ and, over the course of their life, goes through an employment stage and a retirement stage. Both of these stages are of stochastic length. Note that the model describes the decisions of a large number of agents belonging to the *same* generation and that, as a result, there is no interaction between different, potentially overlapping, generations. Investors are born employed and supply labour inelastically as long as they remain employed, which entitles them to an exogenous income stream given by[18]

$$Y_{i,t} = P_{i,t}U_{i,t}, \qquad \ln U_{i,t} \sim N(-0.5\sigma_u^2, \sigma_u^2) \tag{3.12}$$

$$P_{i,t} = GP_{i,t-1}N_{i,t}, \quad \ln N_{i,t} \sim N(-0.5\sigma_n^2, \sigma_n^2) \tag{3.13}$$

Labour income has a transitory component $U_{i,t}$ and a permanent component $P_{i,t}$. The logarithm of $P_{i,t}$ follows a random walk with drift $\ln G > 0$. Note that the expectation of the shock to permanent income $N_{i,t}$ and the expectation of the transitory shock $U_{i,t}$ equal one, so that, in expectation, the income from labour of all working agents grows at the common rate $G - 1$.[19] Retired investors receive a pension $\Omega_{i,t} = \omega P_{i,t}$, which is a fraction of the permanent income that they would obtain if they remained employed. This specification captures the empirical fact that differences in income that develop over the course of the working life persist among retired investors.

As in Viceira (2001), an individual that is employed in the current period retires with probability π^r at the beginning of the next period and, once retired, dies in each period with probability π^d. By assuming stochastic working and retirement spells, life-cycle effects are introduced to the model without relinquishing the computational benefits of an infinite-horizon model (Viceira, 2001).[20] For the numerical calculations below, π^r and π^d will be calibrated to obtain realistic average lengths of the employment and retirement stages.

[18]This specification of the income process is due to Carroll (1992).

[19]In general, if $\ln x \sim N(\mu, \sigma)$, then $\mathrm{E}x = \exp\left(\mu + 0.5\sigma^2\right)$, therefore here $\mathrm{E}U_{i,t} = \mathrm{E}N_{i,t} = 1$.

[20]Gertler (1999) follows a comparable strategy in an OLG framework. Blanchard (1985) works with a constant death hazard and labour income that is a decreasing function of age.

3.3.2 Investment Opportunities

There are three types of assets available to the investors: a one-period bond, stocks and long-term government debt. Long-term government debt has a maturity of δ periods. If long-term government debt were modelled as fully liquid, that is, if securities bought in t could be sold freely in every period $t+1, t+2, ..., t+\delta$, it would be indistinguishable from short-term debt and differences in the expected yield of both assets could not be justified.[21] A strategy frequently adopted in the literature is to assume at the other extreme that long-term bonds are entirely illiquid, or more precisely, that they have to be held until maturity. Aside from understating the liquidity of long-term government debt, this assumption leads to a big inflation of the state space as δ becomes large, causing exact solutions to portfolio choice problems to be computationally burdensome.

A specification is proposed here that allows investors to access some of the funds held in the form of long-term debt in each period and that makes portfolio optimisation computationally straightforward. An investor that has purchased long-term government bonds in $t-1$ at the amount of $Q_{i,t-1}$ receives a Calvo-type signal for each infinitesimal unit of $r_{q,t}Q_{i,t-1}$ in t indicating whether it can be sold or not, where $r_{q,t}$ is the annual gross return on the long-term government bond. A positive signal is received with probability δ^{-1}, implying that each infinitesimal unit has to be held on average for δ periods. Thus, portfolios are chosen in all periods subject to the constraint

$$Q_{i,t} \geq (1-\delta^{-1})r_{q,t}Q_{i,t-1} \tag{3.14}$$

Since the probability of being able to sell a given unit of long-term debt is time-constant, all long-term debt held by i can be summarised by a single state variable.[22]

[21] This were true unless, for example, there were limits to arbitrage between both bonds and investors had a special preference for holding one over the other or there were maturity-specific costs attached to government debt holdings. See, for example, Andrés et al. (2004) and Chen et al. (2012) for more details.

[22] Modelling long-term government bonds as a perpetuity as in Woodford (2001) would equally permit all long-term debt to be represented by one state variable. The specification chosen here emphasises the imperfect liquidity of long-term government debt, which is an important characteristic of assets such as US savings bonds and tax-exempt bonds. The two specifications are similar though. In the case of a perpetuity, the payoff stream from a one-dollar investment in long-term debt is $\rho, \rho^2, \rho^3, \dots$ for some $\rho \in \left[0, \beta^{-1}\right)$. Here, if government debt is run down at the fastest possible rate, this stream is $(1-\delta^{-1})r_{q,t+1}, (1-\delta^{-1})^2 r_{q,t+1}r_{q,t+2}, (1-\delta^{-1})^3 r_{q,t+1}r_{q,t+2}r_{q,t+3}, \dots$ with $(1-\delta^{-1}) \in [0,1)$.

The one-period bond yields a riskless gross return of r_b. Following Bonaparte et al. (2012), the gross stock return $r_{s,t}$ evolves according to a two-state Markov process, $r_{s,t} \in \left\{r_s^l, r_s^h\right\}$, with mean r_s and standard deviation σ_{rs}. Bonaparte et al. (2012) estimate the serial correlation of annual stock returns to be not significantly different from zero using US stock data. $r_{s,t}$ is therefore assumed to be i.i.d. across periods. Long-term government debt also follows a two-state Markov process. The mean, standard deviation and transition matrix are given by r_q, σ_{rq} and Γ_{rq}, respectively. No restrictions on Γ_{rq} are placed allowing for persistence in the government bond return process.

Holdings of the one-period bond $B_{i,t}$ are costless. Investments in stocks $S_{i,t}$ and long-term government debt $Q_{i,t}$ are associated with a cost $\Psi_{i,t} = \psi P_{i,t}$ that has to be paid in each period of active participation in the markets for stocks or long-term government debt. $\Psi_{i,t}$ can be interpreted as opportunity costs associated with the acquisition of information about financial markets. Following Alan (2006), this cost is made dependent on the permanent component of labour income.[23] In addition, equity holdings are subject to a variable cost $\psi_s S_{i,t}$, reflecting the costs of maintaining a stock portfolio.

3.3.3 Optimisation Problem

Each investor $i \in I$ solves the problem described in this section. The index i is suppressed below for expositional clarity. Employed investors take the retirement stage into account. The problem is therefore solved backwards.

Retired Investors

The budget constraint of a retired investor that participates in financial markets is given by

$$C + S(1 + \psi_s) + B + Q + \Psi = r_s S_{-1} + r_b B_{-1} + r_q Q_{-1} + \Omega \tag{3.15}$$

The sum of expenditures on consumption, stocks, short-term bonds, long-term government debt and the costs of asset market participation and stock holdings have to equal income, which is

[23] In Alan (2006), however, the cost is incurred only the first time an agent enters the stock market and not, for example, at a later re-entry.

given by the gross return on last period's investments and pension income. Not all funds that an investor brings into the current period can be used for consumption or investments in short-term bonds or stocks. Defining "cash on hand" as

$$X_{r,p} \equiv r_s S_{-1} + r_b B_{-1} + \delta^{-1} r_q Q_{-1} + \Omega \tag{3.16}$$

and letting illiquid assets be given by

$$Z \equiv (1 - \delta^{-1}) r_q Q_{-1}, \tag{3.17}$$

one can express the budget constraint as

$$C + S(1 + \psi_s) + B + Q + \Psi = X_{r,p} + Z \tag{3.18}$$

In Equation (3.18), income is divided into liquid funds $X_{r,p}$ that can be freely allocated towards all types of expenditures and illiquid funds Z which are tied to a re-investment in long-term government debt. Using this notation, the liquidity constraint on long-term government debt becomes

$$Q \geq Z \tag{3.19}$$

requiring investors to carry an amount of long-term debt forward into the next period that is at least as large as the amount of illiquid assets brought into the period.

A retired investor that does not participate in financial markets has the following budget constraint

$$C + B + Q = r_b B_{-1} + r_q Q_{-1} + \Omega \tag{3.20}$$

which can be re-written as

$$C + B + Q = X_{r,n} + Z \tag{3.21}$$

with cash on hand now defined as

$$X_{r,n} \equiv r_b B_{-1} + \delta^{-1} r_q Q_{-1} + \Omega \tag{3.22}$$

An agent that has accumulated long-term government debt but subsequently left financial markets is assumed to run down their holdings of long-term government debt at the fastest permissible rate. The constraint on Q for a non-participant therefore is binding, so

$$Q = Z \tag{3.23}$$

If the agent wanted to increase their holdings of long-term debt beyond this level, they would have to rejoin asset markets and pay the participation fee. A non-participant that has brought no long-term debt into the period can therefore not invest in long-term debt without becoming an asset market participant. Combining (3.21) and (3.23), the budget constraint can be expressed more succinctly as

$$C + B = X_{r,n} \tag{3.24}$$

The equation above is independent of Q, reflecting the fact that existing holdings of long-term government debt are run down at a rate that is fully determined by its return r_q and its maturity δ. All remaining savings are carried out by means of the riskless savings device.

The value of the problem of a retired investor is given by

$$v_r\left(X_r, Z, r_q, P\right) = \max\left\{v_{r,p}\left(X_{r,p}, Z, r_q, P\right), v_{r,n}\left(X_{r,n}, Z, r_q, P\right)\right\} \tag{3.25}$$

where $v_{r,p}$ and $v_{r,n}$ with $v_{r,k} : \mathbb{R}^2 \times \left(\mathbb{R}^+\right)^2 \to \mathbb{R}$ and $k \in \{p, n\}$ are the respective value functions corresponding to an investor that participates and one that does not participate in financial markets in the current period. For each point in the state space, the investor decides to participate if the value from participating is higher than that from not participating. In the event of participation in the current period, the value of the problem is

$$v_{r,p}\left(X_{r,p}, Z, r_q, P\right) = \max_{C,S,B,Q} \quad u(C) + \beta\left(1 - \pi^d\right) \mathrm{E}_{P', r_s', r_q' | P, r_q} v_r\left(X_r', Z', r_q', P'\right) \tag{3.26}$$

subject to (3.16)–(3.19), (3.22) and borrowing constraints $(S, B, Q) \geq (\underline{S}, \underline{B}, \underline{Q})$. $u(\cdot)$ is the period utility function with $u : \mathbb{R}^+ \to \mathbb{R}$, $u'(C) > 0$ and $u''(C) < 0$ for all $C \in \mathbb{R}^+$. The value from non-participation is

$$v_{r,n}\left(X_{r,n}, Z, r_q, P\right) = \max_{C,B} \quad u(C) + \beta\left(1 - \pi^d\right) \mathrm{E}_{P', r_q' | P, r_q} v_r\left(X_r', Z', r_q', P'\right) \tag{3.27}$$

subject to (3.16), (3.17), (3.22), (3.24) and $B \geq \underline{B}$.

In each period, investors are faced with a participation problem, a consumption-savings problem and, in case of participation, a portfolio choice problem. The probability of dying π^d increases the rate at which future consumption utility is discounted, effectively making retired agents more impatient than employed agents.

Employed Investors

Employed investors differ from retired ones in two ways. They receive labour rather than pension income and have to account for retirement and longevity risk rather than only then second.

The budget constraint of a financial market participant is

$$C + S(1 + \psi_s) + B + Q + \Psi = X_{e,p} + Z \tag{3.28}$$

where cash on hand is now given by

$$X_{e,p} \equiv r_s S_{-1} + r_b B_{-1} + \delta^{-1} r_q Q_{-1} + Y \tag{3.29}$$

Since $Q = Z$ and $S = 0$ if the participation fee Ψ is not paid, the budget constraint of a non-participant is

$$C + B = X_{e,n} \tag{3.30}$$

with

$$X_{e,n} \equiv r_b B_{-1} + \delta^{-1} r_q Q_{-1} + Y \tag{3.31}$$

As above, an investor decides whether or not to participate in each period and for each combination of states by comparing the value from participating with that from not participating.

The value of the problem for an employed investor therefore is

$$v_e\left(X_e, Z, r_q, P\right) = \max\left\{v_{e,p}\left(X_{e,p}, Z, r_q, P\right), v_{e,n}\left(X_{e,n}, Z, r_q, P\right)\right\} \tag{3.32}$$

where $v_{e,k} : \mathbb{R}^2 \times \left(\mathbb{R}^+\right)^2 \rightarrow \mathbb{R}$ for $k \in \{p, n\}$. The optimisation problem of a financial market participant is captured by

$$\begin{aligned} v_{e,p}\left(X_{e,p}, Z, r_q, P\right) = \max_{C,S,B,Q} \quad & u(C) + \beta\left(1-\pi^r\right)\mathrm{E}_{P',r_s',r_q'|P,r_q} v_e\left(X_e', Z', r_q', P'\right) \\ & + \beta\pi^r \mathrm{E}_{P',r_s',r_q'|P,r_q} v_r\left(X_r', Z', r_q', P'\right) \end{aligned} \tag{3.33}$$

together with the definition of $X_{j,k}$ for $j \in \{e, r\}$ and $k \in \{p, n\}$ as well as the restriction on long-term debt (3.19), the budget constraint (3.28) and borrowing constraints. Similarly, the problem of an agent that stays out of financial markets is described by

$$\begin{aligned} v_{e,n}\left(X_{e,n}, Z, r_q, P\right) = \max_{C,B} \quad & u(C) + \beta\left(1-\pi^r\right)\mathrm{E}_{P',r_q'|P,r_q} v_e\left(X_e', Z', r_q', P'\right) \\ & + \beta\pi^r \mathrm{E}_{P',r_q'|P,r_q} v_r\left(X_r', Z', r_q', P'\right) \end{aligned} \tag{3.34}$$

subject to the definition of $X_{j,k}$, the budget constraint (3.30) and $B \geq \underline{B}$. When forming their expectation about the value of the problem in the next period, agents evaluate whether they will participate in financial markets for both possible future employment states.

3.3.4 Normalisation

The model has to be solved numerically. One state variable can be eliminated from the problem by dividing all endogenous variables by the permanent component of labour income P as in Carroll (1992). Below, lower case letters denote normalised variables, e.g. $x_{j,k} \equiv X_{j,k}/P$. Period utility is assumed to be of CRRA form with coefficient of relative risk aversion γ. This implies that all value functions introduced above are homogeneous of degree $1-\gamma$ (Haliassos and Michaelides, 2002) making the problem "scale-independent" (Gomes and Michaelides, 2005).[24] For example, the Bellman equation of an employed investor that participates in financial markets

[24] See also Alan (2006).

in the current period can be expressed as

$$v_{e,p}\left(x_{e,p},z,r_q\right)=\max_{c,s,b,q}\quad u(c)+\beta\left(1-\pi^r\right)\mathrm{E}_{U',N',r_q',r_s'|r_q}\left(GN'\right)^{1-\gamma}v_e\left(x_e',z',r_q'\right)$$
$$+\beta\pi^r\mathrm{E}_{N',r_q',r_s'|r_q}\left(GN'\right)^{1-\gamma}v_r\left(x_r',z',r_q'\right)\tag{3.35}$$

Note that the stochastic growth rate of the permanent component of income $P'/P=GN'$ raised to $1-\gamma$ now pre-multiplies v_e and v_r in the expected values, accounting for uncertainty resulting from permanent income shocks. All normalised model equations together with more details on their derivation are listed in Section 3.6.3 of the appendix.

3.3.5 Calibration

A period in the model corresponds to a year. Since the age group 26-28 is the youngest contained in the sample used to obtain the empirical results discussed in Section 3.2, it is assumed that agents are born at the age of 27. The retirement and death probabilities are calibrated to yield an average retirement age of 62 and a life expectancy of of 82 years, respectively. δ is chosen such that the partially liquid bond approximates 5-year government debt, which is in line with the average maturity of all outstanding US debt of about 60 months.

A number of papers that examine stock holdings over the life cycle contain estimates of the discount factor and the coefficient of relative risk aversion. Although results vary considerably, the picture that emerges is that a relatively high degree of discounting and risk aversion is required to be able to match the data. The estimates reported in Bonaparte et al. (2012) are $\beta=0.69$ and $\gamma=7.24$. Alan (2006) estimates $\beta=0.92$ and $\gamma=1.6$. Cagetti (2003) shows that both variables strongly depend on education with estimates for groups of different education levels ranging between 0.781 and 1.14 for β and between 2.40 and 8.13 for γ. I set $\beta=0.85$ and $\gamma=5$, which is in the interior of the set of estimates.

Following Haliassos and Michaelides (2003) and Viceira (2001), permanent income growth is set to 3%. The standard deviations of the shocks to the transitory and permanent components of income are chosen based on estimations by Carroll (1992). A relatively low value of 0.3 for the replacement ratio captures the fact that only about half of the US population are covered by pension plans. As in Bonaparte et al. (2012), the risk-free rate is set to 1.02. Bonaparte et al. further estimate the average net return of stock in the US to be 6.33% with a a standard deviation of 0.155 and no serial correlation, which I also adopt here. The mean excess return of 5-year government debt over the risk-free rate is set to 0.5%, the standard deviation of 5-year US debt over the sampling period was about 0.01. The 5-year government debt return is modelled using a two-state Markov process, whose transition matrix is found by first fitting and AR(1) process to the data and then finding the transition probabilities that best describe the estimated AR(1) process.[25] r_q remains in state $k \in \{l, h\}$ with probability 0.663 and switches with the converse probability.[26] The cost parameters are set to 2% and 1% of initial wealth, respectively. The role of these parameters will be discussed below in more detail. Following Bonaparte et al. (2012) among others, the possibility of borrowing is excluded in the model.

Parameter	Value	Description
π_r	1/35	Retirement probability for employed investors
π_d	1/20	Death probability for retired investors
δ	5	Maturity of government debt
β	0.85	Discount factor
γ	5	Coefficient of relative risk aversion
G	1.03	Drift in permanent component of income
σ_u	0.16	Standard deviation of log of transitory income shock
σ_n	0.12	Standard deviation of log of permanent income shock
ω	0.3	Pension income parameter
r_b	1.02	Gross return on riskless bond
r_q	1.025	Mean gross return of δ-year government bond
r_s	1.0633	Mean gross stock return
σ_{rq}	0.01	Standard deviation of δ-year government bond return
σ_{rs}	0.155	Standard deviation of stock return
$\Gamma_{rq,kk}$	0.663	Probability for long-term bond return to remain in state $k \in \{l, h\}$
ψ	0.02	Cost of participating in financial markets
ψ_s	0.01	Proportional cost parameter of stock holdings
$\underline{B}$, $\underline{Q}$, $\underline{S}$	0	Borrowing limits

Table 3.2: Calibration

[25]See Adda and Cooper (2003).

[26]Modelling the long-term bond return as i.i.d. while maintaining the same values for its mean and variance raises the long-term debt share and lowers the stock share of asset market participants but yields qualitatively the same results as those shown below.

3.3.6 Policy Functions

The policy functions show how investors rebalance their portfolios as financial wealth increases. In the figures 3.5 and 3.6, the optimal choice of financial assets is plotted against cash on hand for the case of an employed and a retired investor, respectively. All variables are normalised by the trending permanent component of income as described above.

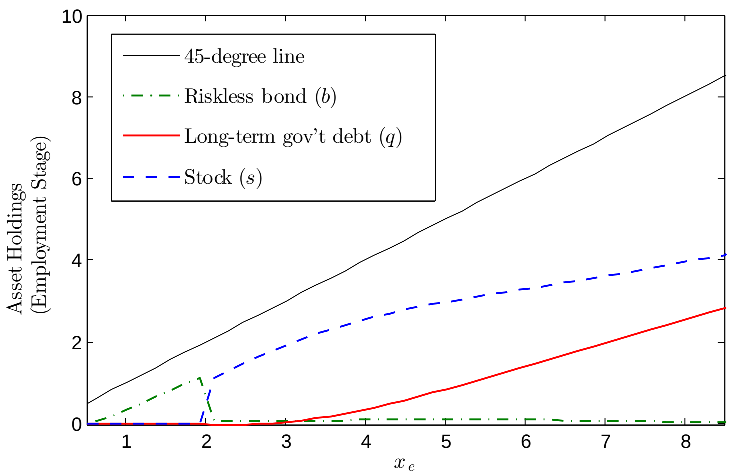

Figure 3.5: Policy functions of employed investors (for $z = 0$, $r_q = r_q^l$)

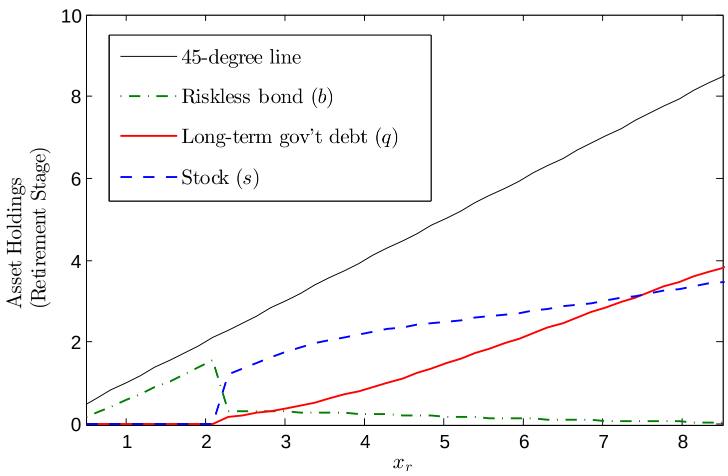

Figure 3.6: Policy functions of retired investors (for $z = 0$, $r_q = r_q^l$)

The solution is highly non-linear. At very low levels of cash on hand, investors in both employment states do not participate in financial markets. All savings are done using the risk-free asset, since the increase in expected utility due to the excess return from investing in stock and government debt does not exceed the participation cost ψ. Note that in the non-participation phase, consumption of an employed investor is higher than that of a retired investor with an equal amount of cash on hand, as can be seen from the higher distance between the function representing holdings of the one-period bond and the 45-degree line. Employed investors have an incentive to consume more, because labour income exceeds pension income. They also have an incentive to consume less, since retired investors die with probability π_d in the next period, effectively making retired agents less patient. For the parameter values chosen here, the first effect dominates the second.

As cash on hand increases, the benefits from participating in financial markets outweigh the costs. Holdings of the riskless asset then play only a very minor role.[27] The threshold value for cash on hand at which asset market participation becomes attractive is slightly smaller for employed investors, reflecting the fact that employed agents invest a higher fraction of cash on hand in stocks and therefore have a higher expected benefit from asset market participation. At the employment stage, investors choose a more risky portfolio than in the retirement stage, since labour income provides them with a stream of income that is higher than non-financial income at retirement, allowing them to better smooth consumption in response to volatile portfolio returns. Additionally, employed investors aim to build up wealth in anticipation of retirement, whereas retired investors run down their wealth.

At larger values of cash on hand, the gap between stock and government-debt holdings narrows. In the case of retired agents, this gap declines faster with government debt holdings eventually overtaking stock holdings. As wealth is accumulated, investors reduce the risk that they are exposed to *through their financial portfolio* by rebalancing their portfolios away from stock and towards government debt. To understand this, suppose first that labour and retirement income is non-stochastic. At low levels of cash on hand, safe labour and retirement income make up a large part of overall income. Thus, investments in financial assets can be comparably risky without being exposed to too much risk on aggregate. At high levels of cash on hand, labour and

[27]There is no intrinsic value of holding the riskless asset in the model. A more significant portfolio share would be obtained for asset market participants, for example, if this this asset were interpreted as "money" and a cash-in-advance constraint were introduced.

retirement income are only a small fraction of overall income. As a result, the investor would be exposed to more risk on aggregate if an equally large share of financial investments were made in the form of stocks as at low levels of cash on hand. But this would not be optimal given that relative risk aversion is assumed to be constant.[28] Viceira (2001) shows that investors with stochastic labour and retirement income behave in a similar way. More precisely, in choosing their financial portfolios, agents with risky non-financial income that is uncorrelated with asset returns behave as if their non-financial income were an investment in a safe asset, although of a value below the discounted expected value of future non-financial income.[29] Thus, they also reduce the stock share in their financial portfolio as wealth is accumulated.

A comparable effect obtains in models with labour income, in which investors choose between stock and a safe asset only.[30] In this class of models, it is the share of the riskless asset that increases with cash on hand as the stock share is reduced. These models therefore give rise to implausibly high holdings of the riskless asset for large values of cash on hand, which highlights the need to add government debt holdings to quantitative models of portfolio choice.

3.4 Simulation Results

By means of simulations, this section shows how agents adjust their holdings of financial assets at the internal and external margin over the course of the life cycle. Simulation results are averaged over I individuals with idiosyncratic shocks $\left\{U_{i,t}, N_{i,t}, \mathbb{1}^e_{i,t}, \mathbb{1}^r_{i,t}\right\}_{t=0}^{\infty}$, where $\mathbb{1}^e_{i,t}$ and $\mathbb{1}^r_{i,t}$ are indicators that equal one respectively if i is employed and retired in t, and over J aggregate shock sequences $\{r_{s,t}, r_{q,t}\}_{t=0}^{\infty}$. I and J are chosen sufficiently large to obtain full convergence and guarantee that the observed effects are not driven by an individual sequence of shocks.

The line most to the top in in Figure 3.7 gives the fraction of the initial population that is alive in period t, the dashed line beneath it the fraction of the initial population that is employed in t. The difference between the two represents the fraction of retired agents. Both

[28]The initial increase in the risk exposure of employed investors that are participating, i.e. at cash on hand levels beyond the "kink" in holdings of the riskless asset, is a result of the lower bound on government-debt and riskless bond holdings. If they were unconstrained, investors would hold negative amounts of at least of one of the two assets initially.

[29]This is shown formally in Proposition 3 of Viceira (2001).

[30]See, for example, Alan (2006) and Fagereng et al. (2013).

lines are fully determined by the exogenous constant hazard rates of retiring in the employment state and dying in the retirement state. Participation in financial markets as a share of the population that is alive in t is given by the solid line. It is zero at first, since initial wealth is too low for it to be profitable to pay the participation cost ψ and then increases sharply as agents on average accumulate wealth. Only agents with a series of low income draws and agents that enter the retirement stage very quickly do not participate in financial markets then. As more and more individuals retire and fall back to income levels at which participation is not profitable, the participation rate declines. This decline is slower than the retirement rate, since a number of individuals die without having run down their wealth to levels at which asset market participation ceases to be optimal.

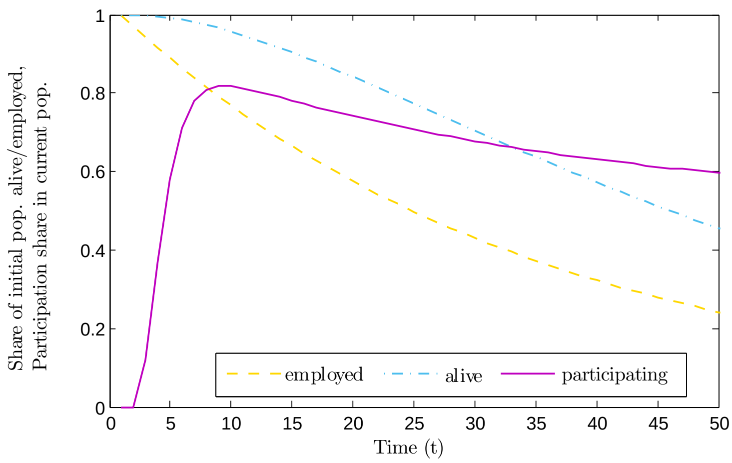

Figure 3.7: Fraction of initial population alive at t, fraction of initial population employed at t and fraction of current population participating in financial markets at t

Figure 3.8 shows average asset holdings for all agents alive in a given period. For the first years, all individuals exclusively invest in the safe asset. Then, average holdings of the safe asset decline and stock holdings increase sharply. Average holdings of government debt grow at a slower rate but eventually make up a significant fraction of all financial assets held. Note that after around twenty years, average asset holdings remain roughly constant with a small decline in the quantity of stock holdings. This is the case as a result of three effects that partly offset each other. Financial market participants accumulate more assets, which means that they aim to hold more stock and government debt. However, since they desire to hold a smaller stock share in their financial portfolio, stock holdings increase at a smaller rate than government-debt holdings for this group. At the same time, a fraction of individuals quit financial markets in each period. Thus, analogous to the selection problem described in Section 3.2.2, changes in financial market participation mask portfolio rebalancing effects that take place over the life cycle.

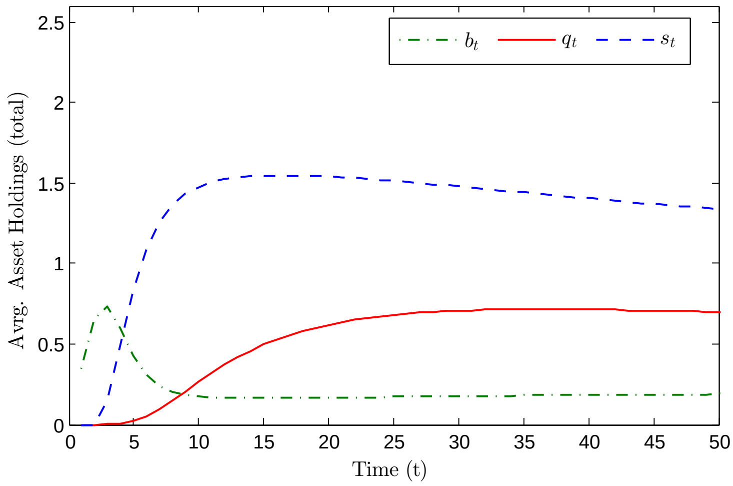

Figure 3.8: Average total holdings of financial assets over time

To isolate life-cycle effects it is therefore instructive to consider average asset holdings of the subgroup of individuals that participate in financial markets in a given period. Figure 3.9 plots the participation rate and asset holdings conditional on participation, now expressed as portfolio shares, implied by the model against the estimates obtained using SCF data. The estimates shown are those that use the Deaton-Paxson identifying assumptions and principal components of macroeconomic aggregates to capture time effects, respectively.

The asset market participation rate in the model is hump-shaped, a pattern that can also be found in the stock market and the market for long-term government debt in the data. The similarity of the participation schedule in both markets suggests that considering participation in both markets jointly, as is done in the model employed here, does not introduce a significant error. While capturing both entry and exit, the model gives rise to a peak in the participation rate that occurs too early compared to the data. The constant retirement hazard implies that agents have an incentive to quickly build up a stock of assets to meet the threat of retiring. A number of contributions to the literature on retirement saving suggest that a significant fraction of households in the US under-save, which would help explain the delay in asset market participation observed in the SCF.[31]

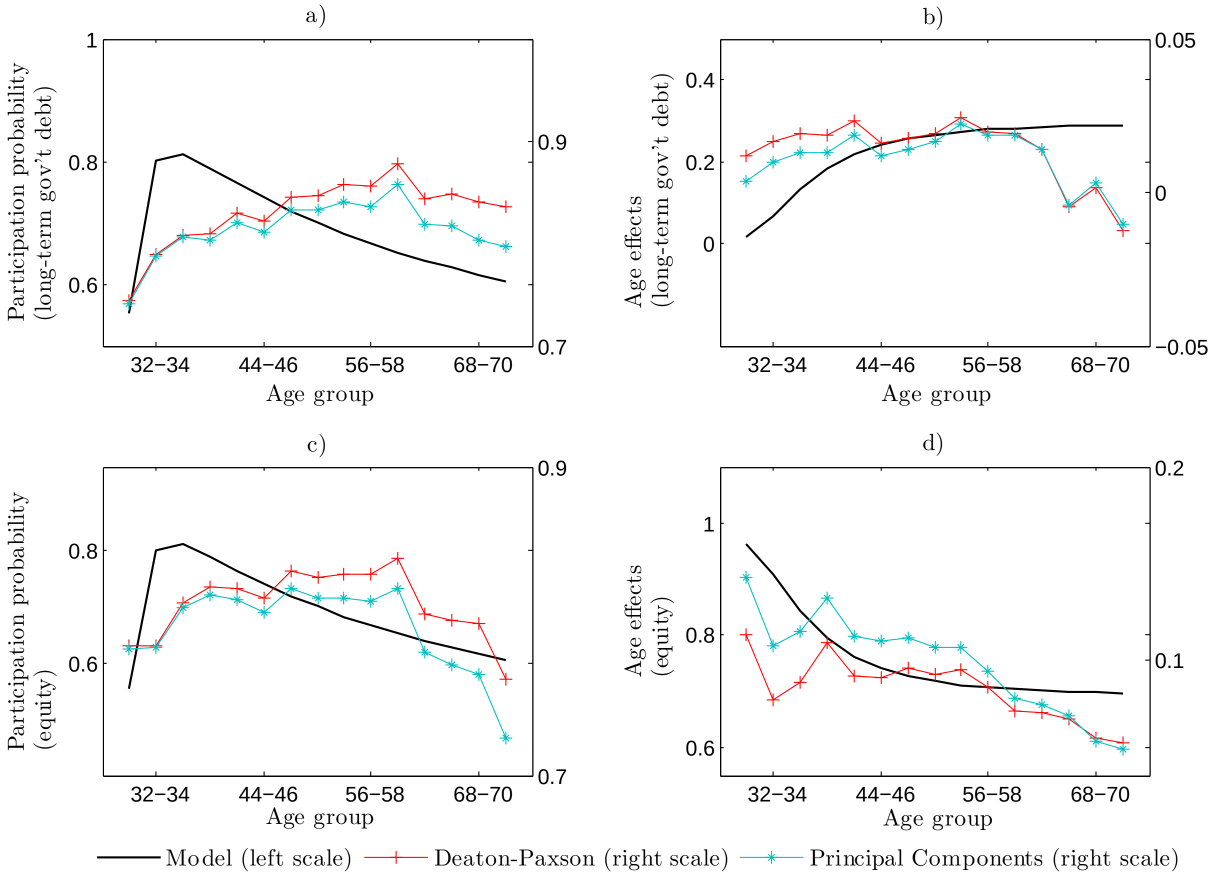

Figure 3.9: Participation and conditional portfolio shares

The conditional government debt share implied by the model is increasing at first and later levels off. Until age 59-61 this model prediction seems to be broadly in line with the data. As explained above, it results from investors rebalancing their portfolios towards safer assets as they accumulate wealth. Accordingly, the model predicts a decline in the conditional stock share, which is also present in the data. The long-term debt share and the stock share found in the data decline further for older age groups, implying that the share of the residual category increases. The government debt share generated by the model remains approximately constant, because the retirement rate at this age is smaller than in the data and thus asset accumulation

[31] Skinner (2007) contains an overview of this literature.

of employed investors dampens the effects of asset de-accumulation by retired agents, which by itself would result in an a declining long-term government debt share. The fact that the residual asset category, not the stock share, increases towards the end of the life cycle points towards an additional mechanism being at work in the data such as perhaps an increase in risk aversion at advanced retirement age.

3.5 Conclusion

The composition of household portfolios changes systematically over the course of the life cycle. At young age, asset market participation is low, implying that households predominantly save through riskless assets such as cash or bank deposits. As households accumulate wealth, participating in the markets for equity and government debt becomes more profitable. With investment income initially being small compared to labour income, agents can afford to invest a large share of their financial portfolio in equity without being exposed to an excessive amount of risk on aggregate. Additional accumulation of wealth implies that investment income as a fraction of total income increases. As a result, the average asset-market participant reduces the equity share and increases the share of safer assets to limit their risk exposure. In my model, there are two safer alternatives to equity holdings, a long-term bond and an entirely risk-free asset. The model is able to explain key features of the data when the long-term bond is interpreted as government debt. During retirement, households run down their savings and partly quit financial markets, giving rise to an inverse U-shaped asset-market participation profile.

The precise way in which households rebalance their portfolios strongly depend on the assumptions made about their risk preferences. While portfolio choice problems are commonly analysed under the assumption that relative risk aversion is constant, we might expect risk preferences to vary with income and age. Incorporating these effects into life-cycle models of portfolio choice may improve their fit to the data significantly.

The distribution of long-term government debt holdings across age groups potentially has important policy implications. For example, unconventional monetary policy tools that include purchases of long-term government debt, such as the Bank of England's QE programme or the LSAP programmes of the Federal Reserve, may have unwanted welfare implications. Long-term government debt plays a significant role in the portfolio of agents that are close to retirement. Lowering the yield of long-term debt through asset purchases by the central bank implies that these agents either have to accept a lower portfolio return or increase stock holdings and be exposed to more risk. Both may lead to under-saving for retirement which may necessitate higher benefit payments in the future.

3.6 Appendix

3.6.1 Asset Categorisation

This section shows how the set of financial assets sampled in the SCF is divided into the three broad categories stocks, long-term government debt and residual assets. I begin by discussing assignments that require explanation. The categorisation of all assets is summarised in Table 3.3.

Savings bonds and tax-exempt bonds are issued by the Department of the Treasury and municipal, county or state governments, respectively. They typically have a maturity of at least five years and are therefore treated as long-term government debt.

US government (agency) debt contains Bills, Notes and Bonds. As explained in the text, Bills are classified as short-term and Notes/Bonds as long-term. According to the "Monthly Statement of the Public Debt of the United States" from December 2007, 22.1% of total marketable debt held by the public were Bills and the remainder Notes/Bonds. Households are assumed to hold long-term and short-term debt at this proportion.

Certificates of deposit are a fixed-term savings product issued by banks, credit unions and thrift institutions; they are allocated to the residual category.

Government bond investment funds are allocated in the same way as direct holdings of US government debt.

Hybrid investment funds are allocated equally to stocks and government debt. Government debt is again split into long-term and short-term debt as described above.

Individual Retirement Accounts (IRAs) and Keogh Plans can include a variety of different financial assets. The Employee Benefit Research Institute (EBRI) estimates that, in 2008, a total of 45.8% of funds in IRAs were held in the form of equity (38.5% in equity mutual funds, directly held individual stocks, and other 100%-equity investment vehicles; they add to this 60% of balanced, lifestyle/lifecycle, target-date funds, and any other funds that have a partial investment in equities and bonds, which make up 12.1% of all funds in IRAs).[32] I assign 18.4% to long-term government debt (the remaining 40% of hybrid equity-bond funds and all

[32]See "IRA Asset Allocation and Characteristics of the CDHP Population, 2005-2010", May 2011, available at www.ebri.org/publications/notes.

bond investments). Money and other assets make up 35.8% and are classified as "residual" for the purposes of this paper. Keogh Plans are similar to IRAs but designed for self-employed individuals. I assume that funds held in Keogh Plans are invested as in IRAs.

Life insurance and annuities are assigned to the residual category, since life insurances and annuities are claims to a safe future payoff (stream).

Other managed assets include trusts and managed investment accounts which are counted towards stocks here.

Asset Type	Asset Sub-Type	Stocks	Long-Term Gov't Debt	Residual
Stocks		1	0	0
Bonds				
	Savings bonds	0	1	0
	Tax-exempt bonds	0	1	0
	US gov't (agency) debt	0	0.779	0.221
	Corporate and foreign bonds	0	0	1
	Mortgage-backed bonds	0	0	1
Transaction accounts				
	Checking accounts	0	0	1
	Saving accounts	0	0	1
	Money market deposit accounts	0	0	1
	Money market mutual funds	0	0	1
	Call accounts at brokerages	0	0	1
Certificates of deposit		0	0	1
Pooled investment funds				
	Stock funds	1	0	0
	Tax-free bond funds	0	1	0
	Government bond funds	0	0.779	0.221
	Other bond funds	0	0	1
	Hybrid funds	0.5	0.389	0.111
	Other funds	0	0	1
Retirement accounts				
	IRAs and Keogh plans	0.458	0.184	0.358
	Thrift plans	0.458	0.184	0.358
	Current pensions	0	0	1
Life insurance, annuities		0	0	1
Other managed assets		1	0	0

Table 3.3: Asset Categorisation

3.6.2 Descriptive Statistics

Age group	Cohort (birth year) '31-'33	'34-'36	'37-'39	'40-'42	'43-'45	'46-'48	'49-'51	'52-'54	'55-'57	'58-'60	'61-'63
26-28	0	0	0	0	0	0	0	0	0	0	495
29-31	0	0	0	0	0	0	0	0	0	510	858
32-34	0	0	0	0	0	0	0	0	822	1,009	1,047
35-37	0	0	0	0	0	0	0	850	1,102	1,200	1,123
38-40	0	0	0	0	0	0	906	1,155	1,227	1,305	1,296
41-43	0	0	0	0	0	1,125	1,125	1,328	1,447	1,601	1,362
44-46	0	0	0	0	890	1,229	1,462	1,386	1,517	1,365	1,230
47-49	0	0	0	810	1,162	1,265	1,411	1,690	1,608	1,511	0
50-52	0	0	834	1,035	1,286	1,519	1,327	1,725	1,535	0	0
53-55	0	916	1,000	1,086	1,206	1,480	1,321	1,566	0	0	0
56-58	825	800	937	1,117	1,180	1,595	1,295	0	0	0	0
59-61	855	945	1,010	1,016	1,292	1,437	0	0	0	0	0
62-64	943	782	905	1,097	1,235	0	0	0	0	0	0
65-67	739	756	867	1,020	0	0	0	0	0	0	0
68-70	733	711	760	0	0	0	0	0	0	0	0
71-73	690	705	0	0	0	0	0	0	0	0	0
74-76	518	0	0	0	0	0	0	0	0	0	0
Σ	5,303	5,615	6,313	7,181	8,251	9,650	8,847	9,700	9,258	8,501	7,411

Notes: N=86,030. SCF data is multiply imputed. For each respondent there are five observations in the data. To approximate the number of respondents in each field in the table, the number shown therefore has to be divided by five. Age groups and cohorts are chosen such that there is a sufficient number of responses in each age-cohort pairing. Figures are not multiples of five if asset shares are missing for some but not all imputations.

Table 3.4: Distribution of observations across age groups and cohorts

Variable	Mean	Variance	Min	Max
Age of household (h.h.) head	50.32	10.04	26	76
Marital Status of h.h. head (=1 if married)	1.26	0.44	1	2
Number of children living in h.h.	0.98	1.20	0	10
Overall wage income of h.h. (in thousands)	210.04	1,143.30	-86.92	88,630.81
Highest education of h.h. head				
None	0.07	0.26	0	1
School diploma	0.24	0.42	0	1
Some college education	0.16	0.36	0	1
College diploma	0.53	0.50	0	1
Ethnic background of h.h. head				
White and non-Hispanic	0.84	0.36	0	1
Black	0.08	0.27	0	1
Hispanic	0.04	0.20	0	1
Other	0.04	0.19	0	1
Occupational category of h.h. head				
Managerial and professional	0.45	0.50	0	1
Technical, sales and services	0.21	0.41	0	1
Other	0.18	0.39	0	1
Not working	0.15	0.36	0	1

Notes: Nominal variables in 2010 US dollar.

Table 3.5: Selected descriptive statistics

3.6.3 Normalisation

First consider the amount of cash on hand brought into the period for $j \in \{e, r\}$ and $k \in \{p, n\}$. In case of participation in the employment state, cash on hand can be normalised as follows

$$X_{e,p} \equiv r_s S_{-1} + r_b B_{-1} + \delta^{-1} r_q Q_{-1} + Y \tag{3.36}$$

$$\frac{X_{e,p}}{P} = r_s \frac{S_{-1}}{P} + r_b \frac{B_{-1}}{P} + \delta^{-1} r_q \frac{Q_{-1}}{P} + \frac{Y}{P} \tag{3.37}$$

$$= r_s \frac{S_{-1}P_{-1}}{P_{-1}P} + r_b \frac{B_{-1}P_{-1}}{P_{-1}P} + \delta^{-1} r_q \frac{Q_{-1}P_{-1}}{P_{-1}P} + \frac{PU}{P} \tag{3.38}$$

$$x_{e,p} = \frac{P_{-1}}{P} \left(r_s s_{-1} + r_b b_{-1} + \delta^{-1} r_q q_{-1} \right) + U \tag{3.39}$$

$$= \frac{r_s s_{-1} + r_b b_{-1} + \delta^{-1} r_q q_{-1}}{GN} + U \tag{3.40}$$

In the other three cases, analogous steps yield

$$x_{e,n} = \frac{r_b b_{-1} + \delta^{-1} r_q q_{-1}}{GN} + U \tag{3.41}$$

$$x_{r,p} = \frac{r_s s_{-1} + r_b b_{-1} + \delta^{-1} r_q q_{-1}}{GN} + \omega \tag{3.42}$$

$$x_{r,n} = \frac{r_b b_{-1} + \delta^{-1} r_q q_{-1}}{GN} + \omega \tag{3.43}$$

and normalised illiquid assets brought forward are

$$z = \frac{(1 - \delta^{-1}) r_q q_{-1}}{GN} \tag{3.44}$$

The four Bellman equations can be transformed in a similar way. Starting again with the case of an employed financial-markets participant, we have

$$\begin{aligned} v_{e,p}\left(X_{e,p}, Z, r_q, P\right) = \max_{C,S,B,Q} \quad & u(C) + \beta\left(1 - \pi^r\right) \mathrm{E}_{P', r_s', r_q' | P, r_q} v_e\left(X_e', Z', r_q', P'\right) \\ & + \beta \pi^r \mathrm{E}_{P', r_s', r_q' | P, r_q} v_r\left(X_r', Z', r_q', P'\right) \end{aligned} \tag{3.45}$$

$$\begin{aligned} \frac{v_{e,p}\left(X_{e,p}, Z, r_q, P\right)}{P^{1-\gamma}} = \max_{C,S,B,Q} \quad & \frac{u(C)}{P^{1-\gamma}} + \beta\left(1 - \pi^r\right) \mathrm{E}_{P', r_s', r_q' | P, r_q} \frac{v_e\left(X_e', Z', r_q', P'\right)}{P^{1-\gamma}} \\ & + \beta \pi^r \mathrm{E}_{P', r_s', r_q' | P, r_q} \frac{v_r\left(X_r', Z', r_q', P'\right)}{P^{1-\gamma}} \end{aligned} \tag{3.46}$$

$$v_{e,p}\left(\frac{X_{e,p}}{P},\frac{Z}{P},r_q,1\right)=\max_{C,S,B,Q}\quad u\left(\frac{C}{P}\right)+\beta\left(1-\pi^r\right)\mathrm{E}_{P',r_s',r_q'|P,r_q}v_e\left(\frac{X_e'}{P},\frac{Z'}{P},r_q',\frac{P'}{P}\right)$$
$$+\beta\pi^r\mathrm{E}_{P',r_s',r_q'|P,r_q}v_r\left(\frac{X_r'}{P},\frac{Z'}{P},r_q',\frac{P'}{P}\right) \tag{3.47}$$
$$v_{e,p}\left(x_{e,p},z,r_q\right)=\max_{c,s,b,q}\quad u\left(c\right)+\beta\left(1-\pi^r\right)\mathrm{E}_{P',r_s',r_q'|P,r_q}v_e\left(x_e'\frac{P'}{P},z'\frac{P'}{P},r_q',\frac{P'}{P}\right)$$
$$+\beta\pi^r\mathrm{E}_{P',r_s',r_q'|P,r_q}v_r\left(x_r'\frac{P'}{P},z'\frac{P'}{P},r_q',\frac{P'}{P}\right) \tag{3.48}$$
$$v_{e,p}\left(x_{e,p},z,r_q\right)=\max_{c,s,b,q}\quad u(c)+\beta\left(1-\pi^r\right)\mathrm{E}_{U',N',r_s',r_q'|r_q}\left(GN'\right)^{1-\gamma}v_e\left(x_e',z',r_q'\right)$$
$$+\beta\pi^r\mathrm{E}_{N',r_s',r_q'|r_q}\left(GN'\right)^{1-\gamma}v_r\left(x_r',z',r_q'\right) \tag{3.49}$$

The remaining equations are

$$v_{e,n}\left(x_{e,n},z,r_q\right)=\max_{c,b}\quad u(c)+\beta\left(1-\pi^r\right)\mathrm{E}_{U',N',r_q'|r_q}\left(GN'\right)^{1-\gamma}v_e\left(x_e',z',r_q'\right)$$
$$+\beta\pi^r\mathrm{E}_{N',r_q'|r_q}\left(GN'\right)^{1-\gamma}v_r\left(x_r',z',r_q'\right) \tag{3.50}$$
$$v_{r,p}\left(x_{r,p},z,r_q\right)=\max_{c,s,b,q}\quad u(c)+\beta\left(1-\pi^d\right)\mathrm{E}_{N',r_s',r_q'|r_q}\left(GN'\right)^{1-\gamma}v_r\left(x_r',z',r_q'\right) \tag{3.51}$$
$$v_{r,n}\left(x_{r,n},z,r_q\right)=\max_{c,b}\quad u(c)+\beta\left(1-\pi^d\right)\mathrm{E}_{N',r_q'|r_q}\left(GN'\right)^{1-\gamma}v_r\left(x_r',z',r_q'\right) \tag{3.52}$$

where

$$v_e\left(x_e,z,r_q\right)=\max\left\{v_{e,p}\left(x_{e,p},z,r_q\right),v_{e,n}\left(x_{e,n},z,r_q\right)\right\} \tag{3.53}$$
$$v_r\left(x_r,z,r_q\right)=\max\left\{v_{r,p}\left(x_{r,p},z,r_q\right),v_{r,n}\left(x_{r,n},z,r_q\right)\right\} \tag{3.54}$$

The corresponding budget constraints in normalised form are given by

$$c+s(1+\psi_s)+b+q+\psi=x_{e,p}+z \tag{3.55}$$
$$c+b=x_{e,n} \tag{3.56}$$
$$c+s(1+\psi_s)+b+q+\psi=x_{r,p}+z \tag{3.57}$$
$$c+b=x_{r,n} \tag{3.58}$$

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