Investigating Necessity: Essays on the Role and Status of Metaphysical Modality

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To Sue and Brian

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Abstract

Metaphysical necessity is one of contemporary philosophy's widely-used tools for theorising. Since the 1970s, it has played central roles in metaphysics, epistemology, philosophy of language, philosophy of science, and beyond. Throughout philosophy, it is used to formulate important theories and hypotheses, and features pivotally in various influential arguments. Due to its wide use, metaphysical necessity has a multifarious theoretical role. To take a few examples, it is expected to be the broadest objective necessity, license paradigm essentialist claims, and obey liberal principles of recombination. Nevertheless, there has never been an attempt to investigate whether the many postulates of this role and applications of metaphysical necessity are *compatible*, in the sense that they can all be satisfied by a single notion of necessity. The aim of this thesis is to provide such an investigation.

The Introduction outlines the themes of the thesis, and provides an overview of each individual essay. Essay One addresses certain foundational issues by criticising a popular form of scepticism towards metaphysical necessity. Essay Two investigates whether a family of natural principles of recombination can be integrated with the thesis that metaphysical necessity is the broadest objective necessity. Similarly, Essay Three studies whether paradigm portions of essentialist discourse can also be integrated with the same thesis. Essay Four draws on material from the previous essays to answer Timothy Williamson's notorious expressive challenge to contingentist theories of modality. Finally, Essay Five provides a formal characterisation of so-called relative necessities which supports various claims and arguments made elsewhere in the thesis. Together, the essays constitute an investigation into the role and status of metaphysical necessity.

TeXShop counted 71,583 words in this thesis.

Introduction

This thesis comprises five independent but thematically related essays. They are unified by their primary subject of interest, metaphysical necessity. Taken together, the essays constitute an investigation into the theoretical *role* and *status* of metaphysical necessity.

The essays may be grouped by their points of emphasis. The first essay of the thesis concerns the status of metaphysical necessity. In particular, it seeks to rebut an increasingly popular, moderate form of scepticism about metaphysical necessity. With this foundation in place, the remaining essays address different features of the theoretical role of metaphysical necessity. Such features include: its status as the broadest objective necessity; the principles of recombination to which metaphysical necessity is expected to conform; the metaphysical necessity of paradigm essentialism claims, and many others. The arguments of these remaining essays are guided by the hypothesis that various postulates of metaphysical necessity's theoretical role are incompatible, in the sense that they cannot all be satisfied by a single notion of necessity. The essays isolate such tensions and draw important morals for our theorising with metaphysical necessity.

The thesis has two further, subsidiary themes. First, the methodology of each essay is to investigate a specific feature of metaphysical necessity by studying how the notion interacts with other modalities, broadly construed. By studying such interactions, one is able to gain insight into the role of metaphysical necessity which may not have been apparent in isolation. In order to implement this methodology carefully, each essay uses a distinctive bi- or multi-modal logic to formalise the interactions of interest. Second, each essay assumes a non-reductive realist, or *modalist*, attitude towards all notions of necessity.[1] As a consequence, the resources of possible worlds model theory which each essay employs are to be understood purely instrumentally, as mathematical tools for modelling certain modal hypotheses or providing a consequence relation for a particular modal language.

Given the historical focus on reductionist accounts of metaphysical necessity, it may be surprising that a thesis on metaphysical necessity omits discussion of reductionism almost entirely. However, the omission is intentional. Reductionist accounts of metaphysical necessity face the difficult demand to recover the exact role and extension of metaphysical necessity. If reductionist accounts are unable to do so, they distort the role of metaphysical necessity in a way that is liable to prejudice the current investigation. It is primarily for this reason that a non-reductive realist attitude is adopted throughout this thesis: the suggestion is to first clarify the theoretical role of metaphysical necessity in order to provide a more refined target for subsequently attempted reductions.

Relatedly, it is tempting to speculate that the understandable preoccupation with the question of reduction has diverted attention away from questions about the theoretical role of metaphysical necessity itself. In my view, although such questions remain largely unanswered by the extant literature, they are no less philosophically pressing than the question of reduction itself. In part, this motivated my felt need for the project of this thesis: an investigation of necessity, which is itself of necessity.

Outline of the Essays

Essay One: From Physical to Metaphysical Modality

From Physical to Metaphysical Modality (**FPTM**) advances an argument against an increasingly popular breed of scepticism about metaphysical necessity. The essay first contends that we ought to believe in more familiar, scientifically respectable modalities such as physical necessity, the notion of necessity largely operative in scientific practice. Whatever one thinks of metaphysical necessity, these less exotic

[1] Arthur Prior (1968) initially gave explicit voice to the modalist position, and it received its first systematic exposition in Fine (1977).

modalities have clear theoretical roles and purposes. However, the essay argues that by examining how these modalities relate to and interact with one another—that is, by examining their *logic*—it can be seen that they must be restrictions of some other, broader modality which satisfies the theoretical role of metaphysical necessity. Thus, the essay concludes, belief in familiar modalities mandates belief in metaphysical modality. In the formal appendix, various claims and arguments made in the main body are formalised in an axiomatic system of higher-order logic.

Essay Two: Modal Expansionism

Modal Expansionism (**ME**) studies a family of recombiniatorial paradoxes which afflict metaphysical necessity. The essay diagnoses the paradoxes as highlighting that certain recombinatorial principles simply cannot be satisfied by a single notion of necessity. However, in response to this diagnosis, the essay develops and investigates a novel view of metaphysical necessity according to which it exhibits a type of indefinite extensibility, similar in kind to the indefinite extensibility which many suggest that set-theoretic notions exhibit. The essay philosophically motivates this idea by highlighting parallels between the various paradoxes of set theory (especially the Burali-Forti paradox) and the recombiniatorial paradoxes in question. In light of these parallels, the essay develops the indefinite extensbility idea within a new bimodal logic, drawing on recent work in modal set theory by Øystein Linnebo (2010), James Studd (2013), and Gabriel Uzquiano (2015). Using this formal framework, the essay establishes the consistency of various bimodal recombinatorial principles by model-theoretic means, thus ensuring the proposed conception of metaphysical necessity avoids the recombiniatorial paradoxes under discussion. In this regard, **ME** specifies a new conception of metaphysical necessity that consistently integrates a body of recombinatorial principles which speak to the various motivations for the initial paradox-prone principles of recombination.

Essay Three: Essence, Tolerance, and Paradox

Essence, Tolerance, and Paradox (**ETP**) argues for the claim that if metaphysical necessity meets various essentialist expectations, it is a source of significant indeterminacy. However, in addition, the essay argues that if metaphysical necessity exhibits such indeterminacy it is a poor candidate for the broadest objective necessity. Thus, it is suggested that we should not be optimistic about metaphysical necessity both being the broadest objective necessity and licensing large portions of essentialist discourse. Indeed, the essay draws the moral that the broadest objective necessity must be disentangled from whichever notion of necessity meets the demands of most essentialist judgements. In light of their disentanglement, the essay provides some tentative metasemantic considerations for identifying which notion of necessity deserves the name 'metaphysical necessity'.

Essay Four: Contingentism and Compossibility

Contingentism and Compossibility (**C&C**) applies the results of **ETP** to a central debate in the metaphysics of modality: the debate been necessitists and contingentists. One of the most forceful arguments against contingentism is that its advocates are unable to meet Williamson's (2010; 2013) expressive challenge to recursively map the entirety of necessitist discourse to a neutral body of discourse. (The challenge is closed connected to Fine's (1977) discussion of translations between actualist and possibilist discourse.) In effect, the problem for contingentists is generated by putative cases of incompossible individuals. However, drawing on the arguments of **ETP**, **C&C** argues that the broadest objective necessity should not be expected to generate the apparent cases of incompossibility. In light of that point, the essay shows that in the presence of such a modality the entirety of necessitist discourse can be mapped to a neutral body of discourse, regardless of whether the modality in question is identified with metaphysical necessity or not. To establish this formal claim, the essay develops a model theory which is based on a notion of a proposition's being true *over* some one or more metaphysically possible worlds, as opposed to being true *at* a particular metaphysically possible world.

Essay Five: Relative Necessity and Propositional Quantification

Relative Necessity and Propositional Quantification (**RNPQ**) is a short technical essay on how to characterise so-called relative necessities in terms of an absolute, broadest necessity. Following Humberstone (1981), the essay argues that the best known account of relative necessity due to Smiley (1960) is beset with substantial technical faults. Moreover, the essay argues that a recent account of relative necessity due to Bob Hale and Jessica Leech exhibits some concerning features too. Thus, **RNPQ** provides a new account of relative necessities in terms of propositional quantification that functions as desired. This new account of relative necessity supplements and substantiates various points and arguments made elsewhere in the thesis, especially in **FPTM**. However, **RNPQ** may be read completely independently of the other essays.

Final Remarks

Although the essays in this thesis belong to an investigation of metaphysical necessity, I wish to emphasise to the reader that the investigation is far from reaching its conclusion. However, I hope that what emerges from the thesis is a conception of metaphysical necessity which is both familiar to philosophical practice yet revisionary in crucial respects. In so far as there is a central moral of this conception, it is that metaphysical necessity is much more stringent than is typically appreciated. Whilst writing the thesis, this stringency moral has become the point of which I have been most firmly convinced about metaphysical necessity. Hopefully, the arguments of what follows convey to the reader how this came to be.

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Contents

1

From Physical to Metaphysical Modality

Abstract

Metaphysical necessity is not without its sceptics, whilst various other objective modalities are typically held in much better standing. Nevertheless, in contrast to what common breeds of scepticism about metaphysical necessity would recommend, this paper argues that one ought to believe in metaphysical necessity provided one believes in much more familiar notions of necessity. More specifically, the paper first argues that we ought to believe in familiar, scientifically respectable modalities such as physical necessity. Whatever one thinks of metaphysical modality, these less contested modalities have clear theoretical roles and purposes. However, the paper argues that by examining how these modalities relate to and interact with one another—that is, by examining their *logic*—it can be seen that they must be restrictions of some other, broader modality which satisfies the theoretical role of metaphysical modality. The central moral will be that it is inherently unstable to believe in more familiar modalities whilst refusing to believe in metaphysical necessity. The discussion will also provide novel insight into the theoretical role and extent of metaphysical necessity. In the formal appendix, various claims and arguments made in the main body are formalised in an axiomatic system of higher-order logic.

1.1 Scepticism about Metaphysical Modality

It is difficult to resist the urge to periodise post-war metaphysics. One popular, albeit oversimplified, account demarcates notable historical periods in terms of the dominant philosophical ideology and concepts of the day.[1] According to this version of recent history, the 1950s and 1960s were the *extensional era*, shaped by W.V. Quine's advocacy of austere first-order logical resources. In reaction to this austerity, the 1970s and 1980s saw an explosion of intensional theorising, whose technical foundations were laid by the earlier work of figures like Ruth Barcan Marcus and Saul Kripke. During this so-called *intensional era*, metaphysical modality took centre stage and was used pervasively in philosophical theorising.

Although it is harder to categorise more recent history, popular philosophical opinion appears increasingly to view the current period as the *hyperintensional era*. The account offers a neat narrative. Just as the tools of the extensional era were replaced by more discriminating modal concepts, which permitted distinctions of finer grain, those modal tools are now being replaced by hyperintensional ideology, which promises further improvement of distinction. In a recent paper, Daniel Nolan (2014, p. 150) writes:

> In this paper I want to predict, and argue for, a shift of the same kind [as that from extensional to intensional concepts], if perhaps not of the same magnitude. I will defend the thesis that metaphysics will, and should, increasingly make use of resources that cannot be adequately captured by modal distinctions, that is, distinctions that do not distinguish between necessary equivalents.

Theodore Sider speaks in a related vein in his 2016 John Locke Lectures:

> Friends of the post-modal [hyperintensional] revolution think that modal conceptual tools need to be supplemented, or perhaps even replaced, by one or more of these post-modal [hyperintensional] concepts. A

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[1] For in-print versions of this popular account, see Sider (2011, pp. 3-4) and Nolan (2014).

> vague theme has been that modal concepts are too crude for many purposes, in that even after modal questions are settled, there remain important questions that can be raised only by using the post-modal [hyperintensional] tools.

Nevertheless, as is common with attempts to categorise ongoing periods of history, the narrative raises various difficult questions. One such question will be the focus of this essay: what becomes of metaphysical modality in the hyperintensional era?

Philosophical notions like metaphysical modality are typically characterised by their theoretical role.[2] Indeed, metaphysical modality had a role which *was* clear. Perhaps most importantly, and as Kripke (1980, p. 99) emphasised, metaphysical necessity was intended to be 'necessity in the highest degree', or, as Stalnaker (2003, p. 202) put it 'necessity in the widest sense'.[3] This idea persisted and remained key to characterising metaphysical modality. As van Inwagen (1997, p. 72) writes of what is metaphysically possible:

> [It is] Possible *tout court.* Possible *simpliciter.* Possible *period.*

In addition to this, metaphysical modality was posited to fulfil various other theoretical needs. Centrally, metaphysical modality was used to characterise the notion of supervenience, which in turn figured in accounts of widely used philosophical notions such as intrinsicality and reduction. Moreover, at points, it was common to see properties, relations and propositions individuated by criteria of necessary equivalence, and essentialist claims analysed as *de re* modal predications.[4] However, advocates of hyperintensional metaphysics argue that almost all aspects of this role are now satisfied by hyperintensional notions. For instance, it is becoming established opinion that reduction is not to be understood via supervenience,

[2]One need not make Lewis's (1970) strong assumption that theoretical terms are analytically equivalent to the description of their role. Indeed, it may be that what a theoretical term denotes does not satisfy every postulate of the term's assigned theoretical role.

[3]Kripke was aware of the imprecision involved in characterising metaphysical necessity as such: his entire quote (with my emphasis) includes the qualification that metaphysical necessity was intended to be 'necessity in the highest degree—*whatever that means*'. Nevertheless, the resources introduced in §1.3 will allow us to sharpen Kripke's characterisation whilst preserving its core idea.

[4]For an influential discussion of supervenience, see Kim (1993). See Langton & Lewis (1998) for a modal analysis of intrinsicality. Lewis (1986, chap. 1) discusses modally individuating properties and propositions.

but instead in terms of Kit Fine's (2001; 2012) notion of grounding, or some other such variant, such as Sider's (2011) notion of structure, or Bennett's (2017) notion of building. Moreover, grounding has been used to analyse the notion of intrinsicality, and has featured as a criterion for hyperintensionally individuating properties, relations, and propositions.[5] Furthermore, since Fine's influential 'Essence and Modality' (1994), essentialist claims have been divorced from *de re* modal predications, with the latter being thought of as too coarse to capture *what lies in the nature* of an object. Yet, typically, if a tool's theoretical role is better satisfied by other notions, that tool is redundant. And although Quine's extensional influence may have waned, his more general maxim still applies: only posit that which best theory requires.

Thus, perhaps inadvertently, advocates of the so-called hyperintensional revolution have breathed new life into scepticism about metaphysical modality. In conversation, one often meets the charge that metaphysical modality is just a philosopher's fiction, a notion not required for any serious or genuine theoretical enterprise. Yet in the hyperintensional era, when metaphysical modality's theoretical work has been almost entirely subsumed by other notions, it has become more difficult to allay this concern. This failure only exacerbates modal scepticism. Without a clear theoretical role, should we not abandon belief in metaphysical modality and, perhaps, modal ideology altogether?[6,7] In this regard, the hyperintensional shift may be more radical than the intensional shift which preceded it.

Although both the above narrative and argument seem compelling, this paper will undermine them. From independently plausible assumptions, it will be shown

[5]See Bader (2013) and Marshall (2013) for grounding-theoretic analyses of intrinsicality. See Fine (2017) and Rosen (2015) amongst others for hyperintensional accounts of propositions and properties.

[6]Divers (2018) carefully articulates a pragmatic scepticism about metaphysical necessity which echoes this concern. The aim of the current article is to show that metaphysical necessity is, as Divers (*ibid.*, p. 7) would put it, 'Field-indispensible'. That is, the aim is to connect metaphysical necessity with notions that are of pragmatic value to science in order to confer similar value onto metaphysical necessity itself.

[7]One way in which this narrative simplifies matters is that it ignores the explicit claims of various hyperintensional research programs (for instance, Fine's (1994; 2000) essentialist program) to have the capacity to define the notion of metaphysical necessity in terms of an item of new hyperintensional ideology. Nevertheless, this paper seeks a more general response to the sceptical worry, not one which is tied to the success of any particular hyperintensional metaphysic.

that metaphysical modality remains in good standing, even within the apparent hyperintensional era. Here is a preview of the argument. First, it will be argued that we ought to believe in more familiar modalities, such as physical modality. Whatever one thinks of metaphysical modality, these less exotic modalities have clear theoretical roles and purposes. However, by examining how these modalities relate to and interact with one another—that is, by examining their *logic*—it will be argued that they must be restrictions of some other, broader modality. Finally, it will be argued that this broader modality satisfies the theoretical role of metaphysical modality. This will vindicate the claim that belief in familiar modalities mandates belief in metaphysical modality.

1.2 Physical Modality

1.2.1 Physical Possibility

The argument's central premise, whose motivation will occupy the main body of this paper, is a conditional:

Bridge If one ought to believe in physical modality, then one ought to believe in metaphysical modality.

In this claim and throughout, modalities are taken to be certain properties of propositions.[8,9] In refusing to believe in a modality, then, one is to be understood as denying that there is a certain property of propositions that can be properly classified as modal. This understanding of belief in modality is compatible with accounts on which belief in modality is characterised in terms of a preparedness to add a proposition as a premise over a certain range of suppositions; see McFetridge

[8]Higher-order logic is a natural tool for regimenting this claim, as explored in recent work by Cian Dorr (2016, sect. 7) and Andrew Bacon (2018). For example, with the assumption that propositions form a Boolean algebra under the usual truth-functional operations, Bacon specifies a highly natural criterion for when a property of propositions is a necessity operator. See also **Definition 3** of the appendix. Moreover, Dorr (2016, pp. 69-70) provides discussion which is relevant to the project of characterising notions of necessity in non-Boolean settings.

[9]Nominalists (i.e. those who deny that there are properties of propositions, or even any propositions at all) may reinterpret this claim in their preferred way of simulating non-nominalist discourse.

(1990) and Divers & Elstein (2012) for discussion of such accounts. On the proposed conception of modalities, 'belief in modality' applies to modal reductionists too, even those of a deflationist variety, such as advocates of Sider's (2011) neo-conventionalist program.

The antecedent of **Bridge** should be plausible, for physical modality can be understood in terms of the notion of a law of nature in various ways, which itself is a notion placed in good standing by scientific practice. According to one popular gloss, a world v is physically possible from a world w iff every one of w's laws is true at v.[10,11] A proposition is then physically possible at a world w iff it is true at a world physically possible from w, and physically necessary at w iff true at all such worlds. Thus, physically necessary propositions are entailed by the laws of nature, and physically possible propositions are consistent with those laws.[12] Importantly, here and throughout 'law' is used to refer to the propositions p such that it is a law that p; 'law' will never be used to refer to the claim that *p is a law*.

According to another compatible gloss, physical possibility simply tracks the models of ultimate physical theory, which can themselves be viewed as solutions of the laws of nature. On this conception, a proposition ϕ is physically possible iff $\ulcorner \phi \urcorner$ is true in some model of the laws of nature, where instances of ϕ are sentences of the mathematical language for ultimate physics. As Tim Maudlin (2007, pp. 14-17; p. 64) describes it:[13]

[10] For a recent and explicit appeal to this gloss, see Hale & Leech (2017). Interestingly, Hale & Leech reject various definitions of physical modality solely on the basis that they are inconsistent with this gloss, which suggests that this gloss is fairly well entrenched. See also Lewis (1986, p. 20), and van Frassen (1989, pp. 44-45). Importantly, in §1.4.1, various *different* glosses of physical possibility are considered which superficially may appear immune to the central argument. It will be argued that upon further scrutiny they still lead to **Bridge**, but by other argumentative means.

[11] Marc Lange (2007) suggests that there may in fact be two notions of physical necessity, one corresponding to consistency with the laws, the other corresponding to consistency with what he dubs the meta-laws, that is laws about laws. According to Lange, the latter notion of necessity is broader than the former (in the sense of broadness specified in §1.3.2). As will be argued in §1.3.4, this paper's central argument is compatible with there being modalities that are broader than our working notion of physical necessity.

[12] See below for exactly how these notions of entailment and consistency ought to be understood.

[13] As a matter of Maudlin exegesis, it is not entirely obvious whether Maudlin intends his gloss to permit variance in the laws from physically possible world to world. As will be argued in §1.2.3, some of Maudlin's remarks elsewhere suggests that such variance is permitted. However, §1.4 will address versions of Maudlin's gloss which forbid the laws to vary from physically possible

> The [physically] possible worlds consistent with a set of laws are described by the models of a theory that formulates those laws. [...] Once we have the set of models, physical necessity is easily defined in the usual way, using the models as mutually accessible possible worlds. [...] The content of the laws can be expressed without modal notions, and suffices to determine a class of models. The models can then be treated as 'possible worlds' in the usual way, and so provide truth conditions for claims about nomic [i.e physical] possibility and necessity. The laws themselves, of course, turn out to be nomically [i.e physically] necessary, since they obtain in all the models.

> Physicists do make assertions about what is physically possible and impossible. Cosmologists, for example, regard both open and closed universes as physically possible, and study the features of each. How they manage this is relatively clear: they have the field equations of General Relativity, and regard the physically possible universes as models of these laws. [...] So once we have an account of physical law, the account of physical possibility is near to hand.

According to Maudlin's gloss, physical modality charts the possibilities and necessities of ultimate physics. It tracks the models of scientific theories, and therefore is constrained by the discovery of laws. It is even employed in scientific practice. Thus, its theoretical role is clear: our best contemporary physical theory and its future extensions warrant belief in physical modality.

1.2.2 Extant Arguments for Bridge

It is important first to consider a direct argument for **Bridge** based on the initial gloss of physical modality. Ultimately, a more general argument will be offered in favour of **Bridge**, but it will be instructive to consider this first attempt.

Many philosophers might be tempted to think that physical possibility *reduces* to mere consistency with the laws of nature. In recent work, however, Timothy Williamson (2016) argues for **Bridge** by exploiting this popular reduction. As Williamson notes, it would be insufficient to understand the reductive claim merely in terms of logical consistency. After all, there are many propositions which

world to world.

are not physically possible but yet logically consistent with the laws of nature.[14] For instance, given the generality and non-parochial character of laws of nature, the propositions that Hesperus is not Phosphorous and that Theresa May is a photocopier will be logically consistent with them. But of course neither of these are physical possibilities. One might attempt to re-gloss physical possibility as logical consistency with the laws of nature, all true identities, distinctions, and essentialist claims. However, this is dangerously close just to explicating the notion of consistency in terms of metaphysical modality, a modality according to which all true identities, distinctions and essentialist claims are necessary. Indeed, as the argument goes, the only obvious way to safeguard against counterexamples similar to the above would be to define physical modality as metaphysical compossibility with the laws of nature. In that case, metaphysical modality would be partly responsible for making physical modality intelligible, hence belief in the latter would require belief in the former.

Nevertheless, Williamson's method of motivating **Bridge** will not convince everyone. For many will equate physical possibility with consistency with the laws of nature only as a mere heuristic, and not as an attempted reduction of physical modality. Indeed, sceptics of metaphysical modality might naturally take this anti-reductionist position; if one rejects the need for metaphysical modality, it is much more plausible to think that physical necessity is a natural modal stopping point in the sense that it is not to be understood as a restriction of some broader modality. Of course, this precludes such anti-reductionists from offering some independent criterion for tracking physical possibility. But as anti-reductionists they might welcome this. Instead, they might offer the following mere *necessary*

[14] Typically, logical consistency is defined as a relation between sentences and not between propositions: some sentences are logically consistent iff there is some interpretation of their non-logical expressions under which those sentences are all true. The literature, however, freely moves between the sentential and propositional notions of logical consistency. See for example, McFetridge (1990) and Hale (1996). Rumfitt (2010), however, is more careful—as is Stalnaker (2003, p. 203) in his related discussion of conceptual necessity. Nonetheless, Bacon's (2018) propositional notion of broad necessity is a good candidate for the propositional notion of logical necessity which various authors have had in mind. When 'logical necessity' and 'consistency' are employed on behalf of different opponents in what follows, it will always be the propositional notions thereof.

condition on physical possibility.

Possibility p is physically possible at a world w only if it is logically consistent with the laws of nature of w.

This would at least provide them with some principled basis for making claims about physical possibility.

In spite of their resistance, the remainder of this paper targets this particular anti-reductionism about physical possibility. It will be argued that such anti-reductionists must still recognise a notion of metaphysical necessity. Since, as suggested above, physical modality could then be understood as a restriction of metaphysical modality, this undercuts any motivation for the anti-reductionist view. Sceptics of metaphysical modality thus face a dilemma: either disavow the theoretically fruitful notion of physical modality or renounce their scepticism.

1.2.3 Conceptions of Laws

Before the main argument is outlined, it is important to first survey some key ideas about laws of nature.

In the literature, there are various theories of laws of nature, each with different and sometimes opposing criteria of lawhood.[15] Broadly speaking, there are two rival conceptions of laws: reductionist and non-reductionist conceptions. In rough terms, reductionist conceptions all endorse the claim that the particular matters of fact of a world completely determine that world's laws, and non-reductionist conceptions reject this claim.[16]

Since David Lewis' (1983, pp. 366-368) Humean conception of laws is the best developed, most popular reductionist theory, it will be used it as a representative of reductionist views throughout this paper.

[15] John Carroll's (2016) *Stanford Encyclopedia* article gives an overview of these criteria and the ways in which they are in opposition.

[16] Historically speaking, this 'reductionist' or 'determination' claim has been understood in terms of supervenience—see for instance Lewis (1973, 1983, 1986, 1994) and Loewer (1996)—but in principle it could be formulated in hyperintensional terms. For a recent example, see Bhogal (forthcoming, p. 10). Throughout this paper, 'reduction' and its converse 'determination' will be used as proxy for these various notions.

Humean A regularity p is a law of nature if and only if p is a theorem of the simplest and strongest axiomatisation of particular matters of fact.

For Lewis, to be a law of nature at a world is to be a certain universal generalisation which belongs to the best systematisation of that world's particular matters of fact.

Of course, a basic constraint on this system is that it must not prove anything false about the particular matters of fact it systematises. However, Lewis also required that the axiomatisation must strike the optimal balance between simplicity and strength. Humeans have proposed to understand this notion of simplicity in syntactic terms: the simplicity of an axiomatisation can be measured by the total complexity or length of formulae which figure as its axioms. Exactly which complexity measure Humeans should use is a vexed question, however, since there are several available candidates.[17] For instance, there is the crude method of simply counting the connectives. However, Williams (2007, p. 377, n.30) and Dorr & Hawthorne (2013, p. 16) suggest that an expression's simplicity could be measured computationally as the length of the shortest computer program that has the expression as an output. Indeed, there might be various appropriate measures which can be taken together to yield a weighted verdict on the expression's complexity. Yet, however simplicity is calibrated, it is crucial that the language in question be *canonical* insofar as all its predicates denote perfectly natural properties and relations, otherwise, as Lewis (1983, p. 367) noted, the simplicity constraint can be trivalised. Similar to simplicity, strength is a also measure-theoretic notion: a system S is at least as strong as another S' when S proves as much as S' does according to some suitable measure. Note that in this sense one system may be at least as strong as another even if it is about a completely different subject matter, and thus does not include the weaker system; proof-theoretic inclusion, however, is of course a special case of measure-theoretic inclusion. Relatedly, Bhogal & Perry (2017, pp. 78-79; p. 82) offer a helpful and compatible gloss of strength

[17]Humeans face difficult questions regarding non-finitely axiomatisable theories in this regard. With even countable theories, the connective counting method will ignore important differences of complexity. Moreover, with seriously infinitary theories, the computational method may have similar drawbacks. Following the extant Humean literature, the discussion of Humeanism in this paper will attempt to circumnavigate such issues.

according to which the strength of a system is calibrated by how many non-actual distributions of particular matters of fact it rules out; see also Dorr & Hawthorne (2013, p. 20). In §1.4.2, the details about how Humeans have formulated the notions simplicity and strength will become relevant.

Three further pieces of commentary about Humeanism are in order. First, in his early discussions, Lewis (1973, p. 73) makes the qualification that the laws are 'contingent'. This qualification does not appear in his subsequent discussions, but we shall assume, as is plausible, that Humeans will not regard trivial universal generalisations as laws. Secondly, Lewis is also explicit that a regularity p is a law of nature if and only if p is a *theorem*, and not necessarily an axiom, of the best system. For instance, he (1983, p. 367; my emphasis) writes "a law is any regularity that earns *inclusion* in the best system". Similarly, elsewhere (1994, p. 478) Lewis states simply that "[a] regularity is a law iff it is a theorem of the best system"; in that same article, Lewis quotes Ramsey's (1990, p. 150) remark that the laws are the "consequences of those propositions which we should take as axioms if we knew everything and organized it as simply as possible in a deductive system". In contrast, some discussions of Humeanism appear only to count the *axioms* as the laws, as in Bhogal & Perry (2017, p. 78); this will not be the working conception of Humeanism adopted in this paper. Finally, how to understand Lewis's notion of a 'regularity' is a matter of exegetical contention. One interpretation simply equates regularities with universal generalisations whereas another requires regularities to be *explanatory* universal generalisations. These two different interpretations will be compatible with the central argument, but will nevertheless be re-emphasised at points of significance.

In contrast to the reductionist market, the landscape of non-reductionist views is much more vast. Non-reductionist views are, however, unified in their denial of lawhood supervening on the non-nomic. For example, Marc Lange (2009a) and John Carroll (2008) reject the Humean's supervenience claim, but still analyse laws in terms of different respective nomic notions. Famously, David Armstrong (1978, 1983), Fred Dretske (1977) and Michael Tooley (1977) also rejected the attempted

Humean reduction, and analysed laws in terms of a 'contingent necessitation' relation holding between universals. More recently, Tim Maudlin (2007) takes lawhood to be a simply primitive status of certain true propositions, and not something which is to be non-trivially analysed in either nomic or non-nomic terms. Although there is a clearly a wealth of non-reductionist views, Maudlin's primitivist conception will be the chosen representative of non-reductionist views throughout this paper. This is primarily because the primitivist makes a far more radical non-reductionist claim than other non-reductionists; given the extremity of the primitivist's claim, it should be clear how the various arguments apply to more moderate non-reductionist theories.

At this point, it is important to highlight that different conceptions of laws of nature will admit different spaces of physical possibilities. To appreciate this, recall the above models-based gloss of physical modality on which physical possibility simply tracks the models of scientific theories. As Maudlin puts it in the above quote, "the models can then be treated as 'possible worlds' in the usual way". However, since these models are mere, austere set-theoretic constructs, they are not informationally rich enough to serve as possible worlds in the sense of maximally consistent classes of propositions. Thus, a more sophisticated alternative is one on which each model is associated with a possible world, or a world proposition, which describes the physical state that set-theoretic construct is intended to model.[18] Yet, crucially, opposing conceptions of laws will disagree on whether there is a unique world associated with each model. Given the Humean claim that the laws of a world supervene on its non-nomic facts, for them each model will be associated with a unique possible

[18]This alternative requires three clarificatory notes. First, Maudlin (2007, p. 38; his emphasis) is clearly aware of this type of alternative, as he writes in the same article quoted above:

> One might maintain that we seek theories that are *metaphysically adequate*, i.e. theories whose models stand in one-to-one correspondence with the physically possible states of affairs, each model being isomorphic to a state.

Second, it may be that a single physically possible world is associated with two different models because the models differ merely representationally, perhaps due to some artefact of the formalism. Third, for simplicity, it may be assumed that each model determines all the non-nomic facts of a world which is associated with it. This simplifying assumption merely reduces the complexity of what follows and is not crucial to any of the following arguments.

world. In contrast, on primitivist conceptions of laws, such as the one Maudlin himself advocates, each model will be associated with a class of worlds which agree on the particular matters of fact but differ as to their laws.[19] Indeed, Maudlin (*ibid.*, p. 67-68) himself is explicit about this fact in his influential criticism of the Humean view of laws. In that criticism, he argues that two different sets of laws can have models with the same physical state—in particular, he points out that Minkowski space-time is a vacuum solution to the field equations of General Relativity and of other theories of gravity. In order to advance this criticism, the primitivist cannot think that models of scientific theories determine what the laws are at all worlds based on those models. Hence, they must think that each model of ultimate physics is associated with a class of worlds with varying laws. This difference between reductionists and non-reductionists will become relevant at certain points.

Finally, it bears emphasis that on the scientific essentialist conception of laws of nature found in Bird (2007), Ellis & Lierse (1994), Shoemaker (1998), amongst others, **Bridge** is immediate. Crudely put, according to scientific essentialists, at least some properties which feature in the fundamental physical laws have their nomico-dispositional role as a matter of metaphysical necessity. That is, it is metaphysically necessary that those properties figure in the laws which they actually do, and have the same dispositional profile as they actually do. Thus, as should be abundantly clear from even this crude formulation, scientific essentialism has been developed in a framework which simply takes the notion of metaphysical necessity for granted; see in particular Bird (2007, p. 3; p. 43; p. 48) who is explicit on this point. Of course, there is scope for a scientific essentialist view developed in terms of a Finean non-modal notion of essence. However, although this view would not simply take metaphysical necessity for granted, presumably its advocates will

[19]Since the models of scientific theories used in practice are set-theoretic, there is a slight complication at this point. After all, it would be both unfortunate and artificial to conclude that physically possible worlds are subject to the size constraints to which sets conform: for example, this would have the consequence that it is physically necessary that there are not more than set-many things. Thus, even Humeans should want to first associate each set-theoretic model with a class of non-set-theoretic 'models' which are not subject to any cardinality constraints. (See Rayo & Uzquiano (1999) for a non-set-theoretic treatment of 'models' using higher-order resources.) Each of these new models can then be associated with a unique world for Humeans, and a class of worlds for primitivists.

want to retain the key scientific essentialist selling point that laws of nature are true as a matter of metaphysical necessity. (On this selling point see Bird (2007, chap. 8, especially pp. 169-170).) Thus, scientific essentialism would remain hospitable to **Bridge**. More generally, since scientific essentialists already accept the central conclusion, hereafter their conception of laws is set aside.

1.3 Bridging Modalities

1.3.1 Objective Modalities

In this section, **Bridge** will be motivated in a more robust and dialectically effective manner. Specifically, unlike Williamson's argument, it will not be assumed that physical possibility is equated with a type of consistency with the laws of nature. As initially advertised, the argument will proceed as follows. First, it will be shown that physical modality *must* be a restriction of some other modality. Second, by exploring the logic of physical modality, it will be argued that the modality of which it is a restriction fits a key postulate of metaphysical modality's theoretical role. To anticipate what is to come, the argument's strategy will be first to examine how physical modality interacts with and relates to other *objective* modalities. This will require some extended set up which nevertheless ought to be of interest in its own right.

Put evocatively, the objective modalities are those modalities which regard variation in *worldly* circumstance. The term comes from Williamson (2016) who contrasts the objective modalities with epistemic, linguistic, and intentional modalities, such as knowability a priori and logical necessity, which instead concern variation along a broadly representational dimension.[20] Moreover, deontic modalities are not to be classified as objective in the intended sense either, for they concern not variation in worldly circumstance but rather requirements on how those circumstances ought to be. Examples of objective modalities include various

[20] See also Rosen (2006, p. 16) on 'real' modalities, which he glosses as 'alethic, non-epistemic, and sometimes substantive or synthetic'.

familiar modalities, such as: technological possibility, what is possible given the current state of technology; biological possibility, what is possible given our current biological makeup; practical possibility, what is possible given our current means, and of course physical possibility.[21] As Williamson (2016) notes, the objective modalities are roughly the modalities expressed in natural language by expressions belonging to Kratzer's (2012, p. 61) class of circumstantial modals. Although no explicit definition of the objective modalities has been offered, this characterisation should suffice to isolate a unified family of modalities that is familiar to philosophical practice.

The family of objective modalities forms a class OBJ, to each member of which corresponds an objective necessity operator $\Box_i$, $\Box_j$, ... (Distinctions between use and mention will be left implicit throughout, and $\Diamond_i$ is to be understood as an abbreviation for $\neg\Box_i\neg$.) In the main body of text, we will work in an object language containing propositional variables $p, q, ...$, and, as logical constants, the truth-functional connectives (including infinite conjunction), all the aforementioned modal operators, and a universal quantifier binding propositional variables. We assume that the quantification theory for this language is classical in the sense that the propositional quantifiers validate universal instantiation and generalisation. This language is used to state various structural principles about the objective modalities and argue that they motivate belief in metaphysical necessity. Toward this end, we say that a formula is *metaphysically necessarily universal* ('MNU') iff the result of prefixing it with arbitrary strings of objective necessity operators and universal quantifiers is true on the intended interpretation of the necessity operators and truth-functional connectives, and the intended unrestricted reading of the quantifiers.[22] The MNU formulas of the object-language will provide structural constraints on the objective modalities. For example, each objective necessity operator $\Box_i$ can be

[21] It is doubtful that there are totally precise, context invariant notions of technological, biological, and practical possibility. A more plausible view is one on which such notions are contextually variable, with the context supplying additional restricting information such as the laws of nature, or a certain timeframe. Indeed, even within a single context, such restricted modals may still exhibit a reasonable amount of vagueness.

[22] See Williamson (2013, pp. 92-96) for the use of MNU in the context of modal propositional logics. As above, Williamson uses it as an object-linguistic means of reflecting metalinguistic claims about validity for some appropriate consequence relation.

said to have a normal modal logic partly due to the following being MNU:[23]

K $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$.

Abbreviated talk of some formula 'holding' (or 'failing') should always be understood as shorthand for the claim that it is (not) MNU.

Although the presentation in the main body will remain at this fairly informal level, the formal appendix uses an axiomatic system of higher-order logic to formalise the various principles about the objective modalities which are used in the main body. For example, in the appendix, the above claim that each objective necessity operator obeys the **K** axiom is formalised by axiom **Kripke**. In the system of higher-order logic, it is then shown that one can prove analogues of all of the major claims made in the main body about the objective modalities and metaphysical necessity. An advantage of formalising the principles in higher-order logic is that one need not state some of them in terms of an infinitary conjunction operator. Instead, in the particular higher-order system, one can define propositional operators which behave like the conjunctions of objective necessity operators which are used in the main body.

There are two final pieces of set up. First, it is assumed throughout that concatenations or sequences of modal operators are finite unless specified otherwise. Second, as a convenient simplification, we shall assume *propositional necessitism*: the view that it is a non-contingent matter (in every objective sense) which propositions there are.[24] Given that modalities were taken to be certain properties of propositions, it is natural to conjecture that propositional necessitism induces non-contingency (in every objective sense) in what modalities there are too. Indeed, this too will be assumed throughout.

[23]For the quantified propositional logic of objective modality to be normal, in addition its theorems must be closed under modus ponens, universal generalisation, necessitation (for every objective modality), and uniform substitution, all of which we assume. Since the logic under consideration uses infinitary conjunction, it must contain typical, additional axioms governing the distribution of modals over infinitary conjunction.

[24]The assumption of propositional necessitism has been questioned by various authors such as Arthur Prior (1967) and Robert Stalnaker (2011). One might expect that it could be dropped for current purposes at the cost of complicating various principles and definitions following the general strategy of Fine (1977). However, whether that strategy would ultimately succeed is far beyond the current scope.

1.3.2 Closure Conditions and Broadness

As Williamson (2016, pp. 456-460) remarks, it is extremely plausible that the family of objective modalities obeys certain closure properties. One example is a type of union operation (in **Un** and **Con**, $\Box_i, \Box_j, ...$ is an at most countably infinite sequence):[25]

Un If $\Box_i, \Box_j, ...$ are objective necessity operators, there is some notion of objective necessity according to which all and only that which is possible according to either $\Box_i$ or $\Box_j$ or ... is possible.

For objective modalities $\Box_i, \Box_j, ...$, the modality given by **Un** is their *union modality*. It is worth noting that Williamson (2016, p. 457) endorses an analogue of **Un** (and **Con** below) without the restriction to countable sequences. However, merely the countable principles will be required in what follows.

Why would the union modality of some objective modalities not itself be an objective modality? Since it only countenances possibilities which are countenanced as possibilities by some other objective modalities, it is difficult to see what would disqualify it from being objective itself. Moreover, **Un** is highly plausible in practice. To take an example, consider technological modality and biological modality. It may be that some technological possibilities are not biologically possible. For instance, we might have the technology to enable Usain to run 100 metres in three seconds provided Usain's skeleton could sustain a level of pressure which, given his actual biological makeup, his skeleton could not. Similarly, it may be that some biological possibilities are not technologically possible. For example, it is trivial that in any scenario which shares our current technology, there are computers: thus it is technologically necessary that there are computers (at some point in time). But there are of course many scenarios in which humans possess the same biological makeup as ourselves yet simply fail to invent computerised machines at all (and nothing else invents them). Nevertheless, we ought to believe in an objective union modality according to which all and only biological and

[25]Interestingly, in the appendix it is shown that the arguments in the main body merely require a weaker closure condition (axiom **Closure**).

technological possibilities are possible. For that union modality only recognises possibilities which are known to be objective.

In the above example, notice that the union necessary propositions are all and only those which are both biologically and technologically necessary. Thus, the relevant union necessity is just expressed by the conjunction of biological necessity and technological necessity. More generally, we can make the following observation about union modalities and object-language formulae, which will become helpful at various points:

Con Let $\Box_k, \Box_m, ...$ be objective necessity operators and $\Box_j$ be the objective necessity operator expressing their union modality given by **Un**. Where $\Box_i$ is any objective necessity operator and I some index set containing only $k, m, ...$, the following is MNU:

$$\forall p \Box_i (\Box_j p \leftrightarrow \bigwedge_{\alpha \in I} \Box_\alpha p)$$

The family of objective modalities obeys other plausible closure properties. One such property is the idea that combining objective modalities should never result in loss of objectivity. After all, if a proposition is objectively possible from some actually objective possibility, surely it ought to be actually objectively possible too: the facts about which possibilities are objective should be necessary in every objective sense. As a result, we require the following condition:[26]

Comp If $\Box_i$ and $\Box_j$ are objective necessity operators, the string $\Box_i\Box_j$ expresses a notion of objective necessity too.

For objective modalities $\Box_i$ and $\Box_j$, the modality given by **Comp** is their *composition modality*. Notice that **Comp** implies that all finite concatenations of (possibly distinct) objective necessity operators are themselves objective necessity operators. Helpfully, **Comp** also implies the following principle for single modal operators:

It If $\Box_i$ is an objective necessity operator, then the string $\Box_i\Box_i$ expresses an objective modality too.

[26]See the **Composition** axiom in the appendix.

In other words, **It** states that iterations of objective modal operators always express species of objective modality. In some cases this may be because $\Box_i\Box_i\phi$ is equivalent to $\Box_i\phi$; in others, concatenations may express genuinely new notions of necessity.

There are further structural conditions to which the objective modalities conform. In particular, there is a natural partial ordering of the objective modalities according to their *broadness*. This notion of modal broadness might be familiar from examples: physical modality is broader than practical modality since there are physically possible acts which are nevertheless beyond what is practically feasible for us to do, such as travelling from London to New York in three hours. To this end, we define the broadness ordering on objective modalities via the holding of certain object-language formulae (where $\Box_i$ and $\Box_j$ are objective necessity operators).[27]

ALAB $\Box_i$ is *at least as broad as* $\Box_j$ in OBJ iff $\forall p\Box_k(\Box_i p \rightarrow \Box_j p)$ is MNU, whenever $\Box_k$ is an objective necessity operator.

The intuitive idea behind this definition is that a notion of objective necessity $\Box_i$ is at least as broad as $\Box_j$ when whatever is necessary according to $\Box_i$ is necessary according to $\Box_j$; or, alternatively, when whatever is possible according to $\Box_j$ is possible according to $\Box_i$. In addition, we say that $\Box_i$ is *broader than* $\Box_j$ in OBJ iff $\Box_i$ is at least as broad as $\Box_j$ in OBJ and not vice versa. Moreover, $\Box_i$ is said to be *independent from* $\Box_j$ in OBJ when neither is at least as broad as the other in OBJ; since independence is symmetric, we shall often talk of 'independent modalities'. A notion of objective necessity $\Box_i$ expresses the *broadest* necessity in OBJ iff it

[27]See Bacon (2018) for the basis of definition, and also Hale (1996) for a related idea. The predicate 'as broad as (with respect to the objective modalities' is defined in **Definition 8** of the appendix. For current purposes, however, two consequences are of note. First, notice that it is a consequence of **ALAB** that broadness holds with objective necessity. To see why, first suppose that $\Box_i$ is *at least as broad as* $\Box_j$ for objective necessity operators $\Box_i$ and $\Box_j$. Were this objectively contingent, the formula $\neg\Box_l\Box_k(\Box_i p \rightarrow \Box_j p)$ would be true for some objective necessity operators $\Box_k, \Box_l$ and for some proposition p. But since $\Box_k$ and $\Box_l$ are objective necessity operators, by **Comp** so is $\Box_l\Box_k$. Thus, **ALAB** implies $\Box_l\Box_k(\Box_i p \rightarrow \Box_j p)$, which contradicts the above. Second, notice that if for a given objective necessity operator $\Box_j$ the formula $\forall p\Box_k(\Box_i p \rightarrow \Box_j p)$ is true whenever $\Box_k$ is an objective necessity operator, then the formula $\forall p\Box_l(\Box_i p \rightarrow \Box_j p)$ is MNU for every objective necessity operator $\Box_l$ given propositional necessitism. For take an arbitrary finite string S of objective necessity operators. By **Comp**, the string $S\Box_l$ is an objective necessity operator, and so the formula $\forall pS\Box_l(\Box_i p \rightarrow \Box_j p)$ is true, which is equivalent to the claim that $S\forall p\Box_l(\Box_i p \rightarrow \Box_j p)$ in a necessitist setting.

is at least as broad as all notions of objective necessity. Informally, $\Box_i$ is the broadest in a family when every possibility recognised by any modality in that family is recognised as possible by $\Box_i$. When the context allows, the qualification 'in OBJ' will be dropped for convenience.

Supplementing the above, we say that notions of objective necessity $\Box_i$ and $\Box_j$ are identical when they apply to all and only the same propositions as a matter of objective necessity (in every sense). More precisely (for objective necessity operators $\Box_i$ and $\Box_j$):[28]

Id $\Box_i = \Box_j$ iff $\forall p \Box_k(\Box_i p \leftrightarrow \Box_j p)$ is MNU, whenever $\Box_k$ is an objective necessity operator.

Id ensures that the above order on OBJ is antisymmetric, a consequence of which is that if there is broadest objective necessity, it is unique.[29]

This conception of the objective modalities motivates further principles, which, though not required by the argument, merit brief comment. First, given **ALAB**, **Comp** and **K**, broadness remains constant as objective modalities are combined in the following sense.[30]

Const If $\Box_i$ and $\Box_j$ are objective necessity operators with $\Box_i$ at least as broad as $\Box_j$, and $\Box_m, \Box_l, ..., \Box_z$ are all objective necessity operators, $\Box_m\Box_l \ldots \Box_z\Box_i$ is at least as broad as $\Box_m\Box_l \ldots \Box_z\Box_j$.

[28]See **Identity** in the appendix. In a hyperintensional setting, mere intensional equivalence might be too coarse a way of individuating modalities. In that case, there would be intensionally equivalent distinct modalities which are still nevertheless distinct. Nevertheless, were that to occur, the central argument would essentially remain unaffected, since (as we shall see) each of these 'maximal' modalities could still be characterised by physical modality up to intensional equivalence.

[29]In recent work, Fritz (2016), Clarke-Doane (2017), Roberts (2019), and Rayo (MS) have either entertained or developed a view on which the objective modalities are indefinitely extensible. According to the various ways of developing that view, the thesis of indefinite extensibility is perfectly compatible with there being a broadest objective necessity. For example, on Roberts's view there can be a broadest objective necessity according to each univocal interpretation of the objective necessity operators. Similarly, on Rayo's view, there can be a broadest objective necessity relative to each set of 'distinctions'. In analogy, on Fine's (2006) view of indefinitely extensible first-order quantification one can recover a notion of *unrestricted* quantification relative to each postulational context.

[30]Note that the analogue of **Const** does not hold for the relation of broader-than, since adding further modalities might induce modal collapse.

A related principle is the claim that broadness remains constant as objective modalities iterate. For one might think that if one modality is at least as broad as another then any n-fold iteration of the former is also at least as broad as any n-fold iteration of the latter:

ItBr If $\Box_i$ and $\Box_j$ are objective necessity operators with $\Box_i$ at least as broad as $\Box_j$, then $\Box_i\Box_i$ is at least as broad as $\Box_j\Box_j$.

Again, this principle is given by **ALAB**, **Comp** and **K**. Moreover, it is suggested by extrapolation from examples. If physical modality is broader than practical modality, then what is practically possibly possible ought to be physically possibly possible too. Informally, worlds reached by chaining together n steps of practical possibility should not be unreachable by n length chains of physical possibility. Of course, it could conceivably be that the realm of physical possibility encompasses the entirety of objective modal space, so that what is physically possibly possible just is what is physically possible. But then n length chains of practical possibility will still be no broader than n length chains of physical possibility, for by **It** those chains of practical possibility will not take one outside of objective modal space.

With these preliminaries in place, we can now turn to the argument for **Bridge**.

1.3.3 Bridge: First Step

Let $\blacksquare$ express physical necessity. The first point to appreciate is that **T** axiom for $\blacksquare$ is MNU:

T $\blacksquare p \to p$

If **T** failed to be MNU, there would be some physically necessary proposition which was false. But this would be absurd: since each physical law is true, everything which follows from those laws must itself be true. In other words, every world will be physically possible relative to itself. Supplementing this, one would expect the actual world to be associated with a model of the actual laws of nature, namely the intended model. Thus, whatever is true in all worlds based on models of the laws will be actually true too. However, from the fact that **T** is MNU, we may conclude that the following is also MNU:

$$\blacksquare\blacksquare p \rightarrow \blacksquare p$$

By the definition of an MNU formula, the following is also MNU whenever $\Box_k$ is an objective necessity operator:

As Broad $\forall p \Box_k(\blacksquare\blacksquare p \rightarrow \blacksquare p)$

Therefore, by the above definitions, the modality expressed by the string $\blacksquare\blacksquare$ is at least as broad as that expressed by $\blacksquare$.

Marc Lange (2000, chap. 6) appears to reject **T** for physical modality. As he (*ibid.*, p. 161) writes: "'It is a law that all Fs are G' can be true, even if 'All Fs are G' is false." However, it seems incorrect to interpret Lange as actually claiming that there are false laws. Instead, his point is that some laws are accompanied by certain provisos. For instance, the above quotation continues as follows: "This formulation, though possessing a certain salutary shock value, is somewhat misleading. I emphasize that when 'All Fs are G' is used to state a law, there may be a great deal implicit in it about how it is to be applied [...] To interpret the law as simply that all Fs are G is to disregard these provisos." Thus, as Lange suggests, to treat the mere universal generalisation as the law may ultimately be too crude; instead, a more qualified, nuanced generalisation may instead be the relevant law. Consequently, one can respect Lange's point about laws being subject to various provisos whilst still accepting **T** for physical necessity.

The second point to appreciate is that $\blacksquare$ is not the broadest modality in OBJ. Part of the reason is that the **4** axiom fails to be MNU for $\blacksquare$:[31]

4 $\blacksquare p \rightarrow \blacksquare\blacksquare p$

[31] As Lloyd Humberstone (1981) shows, if one crudely defines physical necessity as a restriction of logical necessity, one can derive that its logic is **S4**. But this is widely viewed as a shortcoming of the crude definition, and not a substantive conclusion about the logical of physical necessity. See Hale and Leech (2017) for discussion.

For if something is physically necessary, it need not be physically necessary that it is physically necessary. To appreciate this, consider the following case structure.

> *Against 4*
>
> Consider a world w_1 in which it is a physical law that all Fs are G. Now consider a world w_2 in which all Fs are G, but not as a matter of law. w_2 is physically possible from w_1. But, at w_2 it is physically possible that not all Fs are indeed G. Thus, at w_1 it is physically necessary that all Fs are G yet not physically necessarily necessary.

$$\begin{tikzcd} \overset{\text{Law: } \forall x(Fx \to Gx)}{w_1} \arrow[r] & \overset{\forall x(Fx \to Gx)}{w_2} \arrow[r] & \overset{\exists x(Fx \land \neg Gx)}{w_3} \end{tikzcd}$$

If cases with this structure are realisable, **4** fails. The key question thus becomes whether either reductionist or non-reductionist conceptions of laws rule out this type of case. To forestall anticipation, it will be argued that on the above models-based gloss of physical modality both conceptions of laws admit such cases.

First consider the reductionist conception of laws, of which the Humean view was the representative. As argued in §1.2.3, according to Humeans each model of the laws of nature will be associated with a unique physically possible world. This was because Humeans maintain that the particular matters of fact of a world completely determine its laws. However, there will be worlds based on models of the laws in which those laws do not belong to the best systematisation of particular matters of fact at those worlds. This may arise for various reasons. For example, consider the law that all like charged particles repel, and a world based on a model of that law in which all like charged particles *do* repel but there are extremely few pairs of like charged particles due to bizarre initial conditions.[32] It is difficult to see why this merely accidental generalisation—that all like charged particles repel—would be a theorem of the best systematisation of that world's particular matters of fact. Similarly, surely not all generalisations of the form 'all Fs are

[32]Nathan Salmon (1989, pp. 8-9) discusses a similar counterexample to **4** for physical necessity. There, he also presents a counterexample to the **B** axiom schema for physical modality ($\phi \to \blacksquare\blacklozenge\phi$).

G' are laws at worlds where they are *vacuously* true.[33] However, such worlds will be physically possible from worlds where it is a law that all Fs are G provided they do not conflict with any other the laws. Humeans, then, are in no position to reject the above type of counterexample to **4**.[34]

Now consider the non-reductionist conception of laws, of which the primitivist view was the representative. As emphasised in §1.2.1, on the primitivist conception of laws, each model of a scientific theory will not be associated with a unique physically possible world, but rather a class of such worlds. All such worlds will share the same particular matters of fact, but will differ as to their laws. After all, according to the primitivist, the particular matters of fact at a world simply do not determine that world's laws. That is to say, for each true proposition the entirety of the non-nomic facts does not determine whether that proposition is a law. Given this lack of determination, the primitivist must recognise that for each law there will be worlds based on models of the laws in which that law is true but not a law. As a result, the primitivist is in not in a position to reject the above type of counterexample to **4** either.

Of course, primitivists might object that not all of these worlds are *physically* possible. Such primitivists must modify their original claim by selecting a proper subclass of the worlds associated with the models of the laws to be physically possible—perhaps those worlds whose laws include the actual laws. But there would appear to be no independent principled reason, especially one stemming from scientific practice, which warrants that modification. Moreover, ultimately,

[33] The claim here is not that there are *no* vacuously true laws; as Tooley (1977, p. 669) and Loewer (1996, p. 187) both argue, there may be cases of vacuously true laws. Instead, the claim is merely that not all vacuously true generalisations are laws. And this must be admitted even by advocates of vacuously true laws, for if 'all Fs are G' is vacuously true so is 'all Fs are not-G'. But were both of these propositions laws, it would follow that in all physically possible worlds (even where there are Fs) all Fs are G and not-G. More generally, it is widely accepted that one crucial improvement of Lewis's Humean theory of lawhood on the simple regularity theory as it does not classify all vacuously true universal generalisations as laws; see Hall (2015, p. 9).

[34] See also Roberts (1998, p. 433). Based on similar considerations, Marc Lange (2009b, p. 198) observes that Humeans will reject the **4**-like 'axiom' that if it is a law that p, then it is a law that it is a law that p. This 'axiom' would motivate the **4** axiom for physical necessity, although not vice-versa. (The vice-versa direction fails since it is physically possible that there are situations in which the actual laws are still true but not laws, yet in which everything else which follows from the actual laws is a law and entails (with perhaps certain auxiliary premises) all the actual laws.) Thus, the Humean has little motivation to accept the **4** axiom for physical necessity.

whether the relevant modality is indeed physical will be besides the point: all the central argument will require is that there is *some* objective modality which conforms to **T** but not **4** that behaves similar enough to physical modality. Clearly, given the models-based gloss of physical modality, the primitivist can recognise one such particular objective modality whether they call it 'physical modality' or not. In fact, the primitivist *must* make sense of such an objective modality for their own theoretical purposes, for they are obliged to explain what they mean by the failure of the laws to supervene on particular matters of fact. Since supervenience is understood modally, the primitivist must employ some objective modality according to which it is possible for the particular matters of fact to be as they actually are but for some actual law to fail to be a law.[35]

The moral is that on either conception of laws **4** fails for $\blacksquare$. Hence, by **ALAB**, $\blacksquare$ is not at least as broad as $\blacksquare\blacksquare$. So, given **As Broad**, $\blacksquare\blacksquare$ is broader than $\blacksquare$. This establishes that physical modality is not the broadest objective modality.

1.3.4 Second Step: First Route

With this crucial lemma, we can argue directly for **Bridge**. In fact, there are two available argumentative routes. One is less controversial, but more complex. The other is more controversial, yet much simpler. For reasons that will become clear, we shall begin with the former.

The initial point of force is that, given the failure of **4**, it is a neat conjecture that any n-length string of $\blacksquare$ operators expresses a modality broader than any $n{-}1$-length string thereof. The result would be that there are infinitely many distinct modalities expressed by different respective concatenations of $\blacksquare$ operators. This result is crucial, because it allows us to define a new notion of necessity as necessity according to *all* concatenations of $\blacksquare$ operators. We write this notion of necessity as $\Box$. The definition proceeds inductively:

$$\blacksquare_0\phi := \phi$$

[35]In Maudlin's (2007, p. 17) words, "[l]aws are ontological primitives at least in that two worlds could differ in their laws but not in any observable respect".

$$\blacksquare_{n+1}\phi := \blacksquare\blacksquare_n\phi$$

$$\Box\phi := \bigwedge_{n\in\omega} \blacksquare_n\phi$$

As promised, the necessity operator $\Box$ expresses necessity according to all concatenations of $\blacksquare$ operators. Given **Un** and **Comp**, $\Box$ is a notion of objective necessity.[36]

$\Box$ is a natural candidate for the broadest objective modality. To put this hypothesis in the terminology of §1.3.4:

Broadest $\forall p\Box_k(\Box p \to \Box_i p)$ is MNU, whenever $\Box_i$ and $\Box_k$ are necessity operators.[37]

And, indeed, we can establish **Broadest** by first defending the claim that there is no objective possibility which is not reachable by chains of physical possibility:

Anti-Isolation For every notion of objective necessity $\Box_i$ and any proposition p, there is a modality expressed by some concatenation $\blacksquare^n$ of only $\blacksquare$ operators, such that $\Diamond_i p \to \blacklozenge^n p$.

Anti-Isolation is a plausible principle. For suppose there is some $\Box_i$-possible world w_i not reachable from the actual world by chains of physical possibility. Recall that physical possibility was glossed as consistency with the laws of nature. It is consistent with the laws of nature that at least one of them fails to be a law *and* that no other laws replace it, as displayed in the above discussion of counterexamples to **4**. Yet this can be iterated an arbitrarily large finite number of times. Thus it ought to be physically possibly ... possible that all the actual laws fail to be laws and no laws replace them. Yet consider such a lawless world. From that world w_i will be physically possible, since trivially the lawless world's laws will all

[36]In terms of Kripke frames, $\Box$'s accessibility relation will just be the reflexive transitive closure of physical accessibility.

[37]See **Theorem 8** of the appendix. Like **ALAB**, notice that if $\forall p\Box_k(\Box p \to \Box_i p)$ is *true* whenever $\Box_i$ and $\Box_k$ are necessity operators, then $\forall p\Box_l(\Box p \to \Box_j p)$ is *MNU* whenever $\Box_j$ and $\Box_l$ are necessity operators too. C.f. n. 27.

be true at w_i.[38] Thus, contrary to supposition, w_i is reachable from the actual world by chains of physical possibility.

This argument for **Anti-Isolation** requires a number of pieces of further commentary. First, the argument assumes that there are only finitely many laws. This may be uncontroversial, but, in fact, the argument could be run on the weaker assumption that there is a finite axiomatic basis from which all laws can be derived. Moreover, even supposing that there are infinitely many laws, it would still be plausible that it is physically possible that some finite subset of those laws are the only laws.[39] But from this possibility the lawless world could then be reached by a finite number of steps of physical possibility.

Second, the argument also makes the assumption that there are no physically necessary connections between distinct laws. To be exact, the assumption is that for any laws it is physically possible that one fails to be a law whilst the others maintain their status as such. (Note that this is not the claim that one law can be *false* whilst the others are laws.) Whilst this may be assumed in scientific practice, it would fail were the laws closed under various logical operations such as conjunction.[40] However, again, the argument only requires the weaker assumption that there is *some* core basis of laws which exhibit the required independence

[38] If the **B** axiom schema for physical modality fails as suggested in n. 32, it need not be that the lawless world will be physically possible from w_i; w_i could have various physical laws which are false at the lawless world.

[39] Remember that according to the Humean view of laws, any regularity which follows from the axioms of the best system is a law. In light of this, it might be thought that the Humean is committed to there being infinitely many laws, since the result of disjoining any theorem with an arbitrary falsehood will also be a theorem. Nevertheless, there would still be a finite axiomatic basis from which all laws could be derived, so the above argument would remain unaffected. Moreover, it is worth emphasising that matters are different if regularities are not mere universal generalisations (c.f. the discussion in §1.2.3 about how to interpret 'regularity'). For, on that conception, although the *theorems* of the system are closed under arbitrary disjunctions, it would not follow that the *regularities* of that system are too, since the result of disjoining a regularity with an arbitrary falsehood need not be a regularity itself.

[40] If Humeans equate regularities with mere universal generalisations, there will always be a regularity which is equivalent to a given conjunction of laws (i.e. the universally quantified conjunction of the relevant open formulae). Thus, assuming lawhood is shared between logical equivalents, on this conception a conjunction of laws would itself be a law. However, if Humeans adopt a more robust understanding of 'regularity', they can deny that the laws are closed under conjunction and other truth-functional operations. For, on this conception, although a system's theorems will be closed conjunction, its regularities will not be: a conjunction of regularities need not itself be a regularity.

and from which all others can be derived. The argument for lawless worlds could then be run in terms of these core laws.

Third, it is important to recall that our anti-reductionist (about physical modality) interlocutor will not accept the gloss of physical possibility as metaphysical compossibility with the laws of nature. Instead, they will accept **Possibility**, which states that logical consistency with the laws of nature is a necessary condition on physical possibility, and thus at best a defeasible indicator thereof. But this criterion itself suggests that w_i is physically possible from the lawless world: trivially, no proposition true at w_i is logically inconsistent with the laws at the lawless world. Moreover, there is no obvious principled way of denying that w_i is physically possible from the lawless world. For, by **T**, w_i is physically possible from *some* law-governed world: itself. Yet if one of w_i's laws failed and was not replaced, w_i would still be physically possible; removing a barrier to physical possibility would not result in fewer worlds being physically possible. But the lawless world is just the final world of this sequence. If the anti-reductionist wants to deny that w_i is physically possible from the lawless world, they must articulate and motivate the radically different conception of physical possibility on which they must rely.[41]

Fourth, recall Maudlin's plausible claim that "the content of the laws can be expressed without modal notions". Given this non-modal conception of laws, it will not be a law that a given proposition p is a law (or a law that it is a law that p, and so on). This prevents the laws themselves mandating that in all physically accessible worlds a certain proposition is a law, which would block the argument for the accessibility of lawless worlds.

Finally, it may be that some laws are metaphysically necessary. For example, Fine (2005, p. 242) suggests that the generalisation that all electrons have negative charge is both a law and metaphysically necessary. Moreover, Fine's discussion also suggests that he may take it to be metaphysically necessarily that *it is a law* that all electrons have negative charge.[42] Were there to be such laws, it may be thought that

[41] Wilson (MS) articulates and develops a novel, radical view of physical possibility which is immune to this step of the current argument. Wilson's view is discussed in §1.4.1 below.

[42] Fine's (2005, p. 242) discussion of vacuously true generalisations, however, suggests that he thinks that for any law it is metaphysically possible that it fails to be a law although remain

the above argument for **Anti-Isolation** could be resisted, for no metaphysically possible world would be lawless. However, this thought would be mistaken. First, there is the less interesting dialectical point that if one admits that some laws are metaphysically necessary, one is using the exact notion of metaphysical necessity which **Anti-Isolation** is intended to motivate. But imagine that Fine's suggestion was rephrased as the claim that as a matter of any objective notion of necessity, necessarily it is a law (and true) that all electrons have negative charge. In that case, no objectively possible world is lawless. Nonetheless, for those laws which are objectively contingent (in perhaps different senses of objective contingency), the above considerations could be used to argue for a physically possibly ... possibly world which lacked any contingent laws, a contingent-lawless world. Now, given that all remaining laws, if any, at that world, would be objectively necessary in every sense, that world's laws are all true at any objective possibility. Thus, from that contingent-lawless world every objective possibility will be physically possible, and so **Anti-Isolation** would be true. As a consequence, **Anti-Isolation** is well-motivated even if certain laws are objectively necessary in every sense, and, indeed, laws in every objective sense of necessity too.

To return to the central thread, notice that the argument for **Anti-Isolation** did not depend on the initial world being the actual world. Thus, the argument supports all necessitated (in every objective sense) versions of **Anti-Isolation**. But together these necessitated claims imply the following lemma:

Reach For any objective necessity operator $\Box_i$, there is a notion of objective necessity expressed by concatenations of only $\blacksquare$ operators which is at least as broad as $\Box_i$.[43]

For suppose that all objective necessitations of **Anti-Isolation** are true, and let $\Box_i$ be any objective necessity operator. Given **Anti-Isolation**, necessarily (in every objective sense) every possibility according to $\Box_i$ is possible according to some

true.

[43]Note that although, as defined, $\Box$ is not itself a concatenation of $\blacksquare$ operators, it can be expressed by all such concatenations. Hence, for $\Box$ there is some notion of objective necessity expressed by concatenations of only $\blacksquare$ operators which is at least as broad as $\Box$: itself.

objective modality corresponding to a concatenation of only $\blacksquare$ operators. Now, take the necessity operator defined by the conjunction of all such concatenations. By **Comp** and **Un**, this necessity operator corresponds to an objective modality. Hence, there is some objective modality expressed by concatenations of only $\blacksquare$ operators according to which all $\Box_i$ possibilities (from every world) are possible (from each respective world). Consequently, this modality is at least as broad as $\Box_i$.[44]

We can now complete the argument that $\Box$ is the broadest objective necessity. As given by **Reach**, for every notion of objective necessity $\Box_i$ there is a notion of objective necessity expressed by concatenations of only $\blacksquare$ operators which is at least as broad as $\Box_i$. Thus, since $\Box$ was defined as necessity according to all concatenations of $\blacksquare$ operators, $\Box$ is at least as broad as every objective necessity. In other words, **Broadest** is true: $\Box$ is the broadest objective modality.

To return to the argument for **Bridge**, recall from the initial discussion of metaphysical modality that it was intended to be 'necessity in the highest degree' or 'necessity *tout court*'. As should now be clear, one can articulate this postulate more precisely as the claim that metaphysical modality is the broadest objective modality.[45] Importantly, however, the preceding argument demonstrated that even if one believes in physical modality, it cannot be the broadest objective modality. Instead, there is a clear candidate for the broadest objective modality: $\Box$. Given that $\Box$ satisfies the key postulate of metaphysical modality's theoretical role, it is natural to identify them. Thus, one can gain independent traction on the notion of metaphysical modality via physical modality: necessity according to the former just is necessity according to all concatenations of the latter. On this conception, if one believes in physical modality, as one ought to, one should believe in metaphysical modality too.

Finally, as a dialectical aside, notice that on this conception those who believed

[44] This informal version of the argument argument depends on the assumptions, made explicit above, that it is necessary (in every objective sense) which objective modalities and propositions there are. However, in the formal version in the appendix does not invoke any such assumptions.

[45] See Williamson (2016) for this refinement of Kripke's initial characterisation. Similarly, Rosen (2006, p. 16) characterises metaphysical necessity as the absolute real modality; see n. 20 above. The restriction to objective modalities is warranted since it is uncontroversial that there are non-objective modalities, such as epistemic necessity for all we know *a priori*, which metaphysical modality was not intended to be broader than.

that physical modality was indeed the broadest modality were committed to belief in metaphysical modality anyway. For them, **Bridge** was immediate. Indeed, interestingly, Kripke (1981, p. 99) himself observes this in *Naming and Necessity*. Nevertheless, the argument now shows that one ought to believe in metaphysical modality as *distinct* from physical modality: a stronger and more interesting claim than **Bridge**. It is for this reason that the argument for **Bridge** did not just proceed via a straightforward application of a highly infinitary version of **Un**, as in Williamson (2016): it is important to recognise that those who believe in physical necessity are committed to belief in a distinct notion of metaphysical necessity. Moreover, it is also crucial to appreciate that the broadest objective necessity can be characterised without appealing to a highly infinitary version of **Un**.

1.3.5 Interlude: Logic

Since every instance of $\Box\phi$ abbreviates an infinitary conjunction of which the relevant instance of ϕ is a conjunct, the **T** axiom for $\Box$ is automatically MNU.

$\mathbf{T}^{\Box}$ $\Box p \to p$

This observation invites the question of whether the characterisation of $\Box$ supports other standard modal axioms.

Interestingly, with the assumption that infinite conjunction distributes over $\blacksquare$, the characterisation of $\Box$ implies that the **4** axiom for $\Box$ is MNU.

$\mathbf{4}^{\Box}$ $\Box p \to \Box\Box p$

Given that $\blacksquare$ is an objective modality and thus has a normal modal logic, it would be highly natural for it to interact with infinite conjunctions as such.[46] The reasoning for the **4** axiom is then quite intuitive. $\Box$ was defined as necessity according to all concatenations of only $\blacksquare$ operators:

$$\Box p := \bigwedge_{n \in \omega} \blacksquare^n p$$

[46]See Williamson (1994, pp. 158-160) and Bacon (MS) for similar arguments with respect to the 'super-definitely' operator and its corresponding **4** axiom.

The following claim is then implied by infinitary conjunction elimination:

$$\bigwedge_{n\in\omega} \blacksquare^n p \to \bigwedge_{n\in\omega} \blacksquare^{n+m} p, \text{ for each } m \in \omega$$

The fact that infinite conjunction distributes over $\blacksquare$ implies the next claim.

$$\bigwedge_{n\in\omega} \blacksquare^{n+m} p \to \blacksquare^m \bigwedge_{n\in\omega} \blacksquare^n p, \text{ for each } m \in \omega$$

Finally, the above two claims and infinitary conjunction introduction imply the following:

$$\bigwedge_{n\in\omega} \blacksquare^n p \to \bigwedge_{m\in\omega} \blacksquare^m \bigwedge_{n\in\omega} \blacksquare^n p$$

Given the definition of $\Box$, this final claim is simply a restatement of $\mathbf{4}^{\Box}$.

There is a slightly more oblique argument for the claim that the **B** axiom is also MNU for $\Box$, regardless of whether it is MNU for $\blacksquare$, but it depends on an additional closure principle about the objective modalities. Informally, the principle states that the right converse of any objective necessity operator is also an objective necessity operator. The principle may be stated formally as follows:[47]

Conv If $\Box_i$ is an objective necessity operator, there is some objective necessity operator $\Box_j$ such that $\forall p \Box_k (p \to \Box_i \neg \Box_j \neg p)$ is MNU whenever $\Box_k$ is an objective necessity operator.

In the current setting, **Conv** and **Anti-Isolation** imply that the **B** axiom for $\Box$ is MNU.

$\mathbf{B}^{\Box}$ $p \to \Box\Diamond p$

To appreciate this, consider $\mathbf{B}^{\Box}$ in its unabbreviated form:

$$p \to \bigwedge_{n\in\omega} \blacksquare_n \bigvee_{n\in\omega} \blacklozenge_n p$$

[47]See axiom **Converses** in the appendix.

Supposing p to be true, the axiom is true if for any finite n and for any objectively possible world w reachable by n steps of physical possibility, there is some finite m such that from w it is physically possible in m steps that p. However, **Conv** and **Anti-Isolation** guarantee exactly this condition. For consider any finite n and any objectively possible world w reachable in n steps of physical possibility. Given **Conv**, there is some objective notion of necessity $\Box_j$ such that from w it is $\Box_j$-possible that p. Thus, by **Anti-Isolation**, there is some finite m such that at w p is m-fold physically possible. Assuming, as is plausible, each objective necessitation of **Anti-Isolation** and the closure principles, this argument for the truth of the **B** axiom generalises to its necessity in every objective sense. Hence $p \to \Box\Diamond p$ is MNU.

Since physical necessity is an objective notion of necessity and thus a normal modal operator, it follows that **S5** is a lower bound on the logic of $\Box$.[48] Adapting an informal argument of Williamson (2013, p. 111), one may then conclude that **S5** is exactly the logic of $\Box$ due to a theorem of Schiller Joe Scroggs (1951), which states that every proper extension of **S5** is the logic of a single finite frame. Thus, on the above characterisation, one can recover the typical logic for metaphysical necessity.

1.3.6 Second Step: Second Route

As stated at the beginning of §1.3.4, there is a simpler yet more controversial argumentative route to the paper's conclusion. To follow this route, consider again the lawless world. Above, it was argued that this world was reachable from any world by a finite number of steps of physical possibility. For, from any given world, it is physically possible that one law fails and is not replaced by any other law. However, more controversially, one might think that from a given world it is physically possible that *every* law at that world fails to be a law and no one of them is replaced. In that case, the lawless world would be simply physically possible from every world. Thus, since every world is trivially physically possible from the lawless world, every world would be physically possibly possible from every world. In other words, $\blacksquare\blacksquare$ would be the broadest objective necessity.

[48] **Theorem 9** of the appendix establishes this claim more rigorously in the setting of higher-order logic.

On this conception, although **4** would fail to be MNU for $\blacksquare$, the following weaker condition would be MNU:

$$\mathbf{4}^{-} \quad \blacksquare\blacksquare p \rightarrow \blacksquare\blacksquare\blacksquare p$$

Given that **T** would still hold, we can conclude that the following is MNU whenever $\Box_k$ is an objective necessity operator:

$$\forall p \Box_k(\blacksquare\blacksquare p \leftrightarrow \blacksquare\blacksquare\blacksquare p)$$

In other words, by **Id** $\blacksquare\blacksquare = \blacksquare\blacksquare\blacksquare$. In terms of possible worlds, this corresponds to the thought that worlds which are reachable by three (or more) steps of physical possibility are in fact reachable in two. Since $\mathbf{4}^{-}$ is weaker than **4**, it does not follow that worlds reachable in two steps of physical possibility are reachable in one. Nevertheless, if $\mathbf{4}^{-}$ holds, the structure of the objective modalities is much simpler than one might have expected. The two-fold iteration of physical necessity is the broadest necessity. Consequently, there is an extremely straightforward route to metaphysical necessity from physical necessity: to be metaphysically possible just is to be physically possibly possible.

This route is more controversial than the initial route. To consider why, recall that on its models-based gloss physical modality is meant to track the models of scientific theories. On this conception, physical modal space was exhaustively characterised by associating a physically possible world (or worlds, for primitivists) with each model of the ultimate physical laws. Thus, in order for the second route to succeed, there must always be a world based on a model of a theory in which all its laws fail to be laws, despite of course being true. This claim is simply hostage to risky empirical fortune, and therefore is a more contentious working assumption. (Interestingly, the idea is less contentious on a primitivist view of laws, since, on that view, the laws do not supervene on the particular matters of fact. For Humeans, however, the second route does appear genuinely more speculative.) In contrast, with the initial route there is far less risk in claiming that there is a world based on a model of a scientific theory in which at least one of the actual laws fails to be a law. Advocates of the second route opt for the more speculative path; whilst this may be efficient, it is important that there is a more circuitous, less risky way to establish **Bridge**.

1.4 Other Opponents

1.4.1 Alternative Conceptions of Physical Necessity

Initially, physical possibility was glossed in terms of laws. On the proposed conception, a world v was physically possible from a world w iff every law of w is true at v. However, there are two salient rival conceptions of physical possibility.

$\blacklozenge^*$ A world v is physically possible* from a world w iff every law of w is a law at v.

$\blacklozenge^\dagger$ A world v is physically possible† from a world w iff v has the exact same laws as w.

John Carroll (1994, p. 18, n.8) assumes the $\blacklozenge^\dagger$ conception without argument, and Tim Maudlin's (2007, pp. 67-68) argument against Humeanism strongly suggests that he endorses the $\blacklozenge^*$ gloss at the very least. Schneider (2007, p. 314) presumes without argument that $\blacklozenge^\dagger$ must be the account of physical modality endorsed by non-reductionists about laws, as suggests Hall (2015, pp. 31-32), who, however, argues that the Humean should endorse our original gloss. Moreover, in effect, Fine (2005, p. 242) assumes the $\blacklozenge^*$ conception too, without mention of any background theory of laws.[49] However, the arguments given above against **4** for $\blacksquare$ from the models-based conception of physical modality should make one appreciate that neither reductionism nor non-reductionism guarantees the **4** axiom merely by themselves. Finally, Alastair Wilson (MS) defends a view, quantum modal realism, according to which, assuming the Everett or 'many worlds' interpretation of quantum mechanics, all and only the objective possibilities are given by the

[49]Fine's (*ibid.*, p. 244) subsequent arguments essentially show that the $\blacklozenge^*$ is incompatible with reductionist views of laws, as he himself is aware; see Fine (*ibid.*, p. 245).

Everett multiverse. On this view, physical necessity obeys the $\blacklozenge^{\dagger}$ gloss and is also the broadest objective modality.

On either conception, **4** is MNU for physical necessity. On the first conception, since any physical possibility v for a world w must have all of w's laws, everything which is physically possible from v must be physically possible from w. On the second conception, any physical possibility v for a world w must have the exact same laws as w, in which case v and w will just share the same space of physical possibilities. However, as was made explicit, the failure of **4** for physical necessity was key to the above argument. Thus, adopting either the $\blacklozenge^{*}$ or $\blacklozenge^{\dagger}$ conception of physical modality may provide those sceptical of metaphysical modality with a potentially live avenue of resistance. Nevertheless, in this section it will be argued that despite initial appearances the opponent of metaphysical modality should find little solace in either gloss.

The first point to appreciate about these rival conceptions is that they depart from the familiar gloss of physical necessity in terms of entailment by the laws of nature. For example, according to both physical possibility* and physical possibility†, it will be physically necessary that each law is physically necessary. However, the laws of nature are entirely non-modal, so how could they entail the physical necessitation of each law?[50] This point holds even in an **S5** logic for physical necessity, in which at most it follows from ψ that it is physically necessarily possible that ψ, from which one cannot derive that ψ is physically necessary.

Moreover, as we saw in §1.3.3, on the models-based understanding of physical modality both the reductionist and non-reductionist conceptions of laws admit counterexamples to **4**. Thus, adopting the $\blacklozenge^{*}$ and $\blacklozenge^{\dagger}$ gloss of physical modality would require both reductionists and non-reductionists to reject this models-based understanding. Hence, to their own detriment, advocates of these rival conceptions of physical modality, $\blacklozenge^{*}$ and $\blacklozenge^{\dagger}$, cannot maintain that physical modality satisfies its typical theoretical role.

In relation to the previous points, notice that every physical possibility* and

[50] Again, recall the Maudlin quote about the non-modal content of laws.

physical possibility† is physically possible. For if the laws of world w are all laws at, or are just the laws of, world v, then clearly the laws of w are true at v. As a result, the initial modality is broader than the latter two modalities. Moreover, with the predicate of propositions 'is a law', one can define the two rival conceptions in terms of the initial modality, yet not vice-versa.[51] This at least suggests that the initial notion of physical necessity is the more theoretically interesting modality of the three, similar to the way in which unrestricted quantifiers are typically more theoretically interesting than their restrictions. The initial notion, then, is perhaps more deserving of the status of physical necessity. Moreover, as was highlighted towards the end of §1.3.3, the central argument merely requires that there is *some* objective modality which conforms to **T** but not **4** that behaves similar enough to physical modality. Yet even if one maintains that either physical possibility* or physical possibility† is *the* semantic value of 'physical modality', given the characterisation of the objective modalities it is still highly natural to think that the current notion of physical modality corresponds to an objective modality.

Nevertheless, despite the above, some might willingly reject the traditional theoretical role of physical modality, and the default conception of the objective modalities, in order to fortify their scepticism towards metaphysical modality. Such opponents will be able to resist the central argument above. However, they face a dilemma either horn of which commits them to **Bridge**. This dilemma is given by a simple question: is physical modality the broadest objective modality?

Suppose they make this equation. On this view, **Bridge** is immediate: for them metaphysical modality just is physical modality. Thus, given this equation, they should not be classified as a thoroughgoing sceptic about metaphysical modality; instead, the opponent who equates physical modality and the broadest objective modality is someone with whom there is disagreement about the role and extension of metaphysical necessity, which effectively generates a more straightforward first-

[51]For the basis of this idea, see Hale & Leech (2017), although Hale & Leech are themselves interested in defining restrictions of absolute necessity (which conforms to its version of **4**). Essay Five of this thesis criticises Hale & Leech's particular proposal and suggests a different way of defining relative necessities.

order dispute.[52] Indeed, Wilson's (MS) quantum modal realist view explicitly makes this equation, which Wilson describes as the equation of physical with metaphysical necessity. However, the dispute with this opponent does not just concern the extension of metaphysical necessity, but rather the structure and extent of the objective modalities more generally. For, after all, on the opponent's view there is no objective sense of possibility in which, for example, superluminal travel is possible. Given the characterisation of the objective modalities as concerning variation in worldly circumstance, this is both striking and radical. On this opponent's view, it is not just the esoteric philosophical claims of metaphysical possibility that fail to be objectively possible, but even more familiar modal claims are relegated to being non-objective too. Although this opponent still endorses **Bridge**, they face a formidable challenge of explaining why we appear to have been systematically mistaken about the realm of objective possibility.[53]

On the other hand, the current opponent might reject the equation of physical modality with the broadest objective modality. Of course, this opponent owes some account of the structure of the objective modalities, in the absence of which it is difficult to assess their view. However, one consideration is particularly salient. Whatever their conception of the objective modalities, as argued in §1.3.2, they still ought to conform to **Un**. And indeed, as Williamson (2016, p. 457) remarks, the version of **Un** which is not restricted to countably infinite sequences of objective modalities is also a plausible closure condition on the objective modalities. Thus, however the objective modalities are structured, the objective modalities will have a maximal member: their union. And, of course, this maximal member is naturally identified with metaphysical modality. As a consequence, this opponent should anticipate to find themselves in a similar position to the opponent who equated physical modality with the broadest objective modality. Neither is a thoroughgoing

[52] Indeed, in this regard, their position is similar to David Lewis's (1986) concrete modal realist view, which has the consequence that it is metaphysically impossible for no individuals to stand in spatiotemporal relations, or for there to be two spatiotemporally disconnected spacetimes. Another salient comparison is Bird (2007).

[53] Wilson (MS, chap. 1, sect. 8) recognises this revisionary character of his quantum modal realist view, but argues that such revisions permit a better modal epistemology than would be otherwise available.

sceptic about metaphysical modality, and so both opponents must engage in a first-order dispute about whether their notion of metaphysical modality—and their conception of the objective modalities—is as plausible as that of rival accounts.

1.4.2 Humean Lawless Worlds

One particular objection to the argument itself merits discussion. The objection relates to David Lewis' (1983) proposed analysis of lawhood, which was considered in §1.2.3:

Humean A regularity p is a law of nature if and only if p is a theorem of the simplest and strongest axiomatisation of particular matters of fact.

In his discussion of Lewis' Humean view, Alexander Bird (2007, p.vii) suggests that, given **Humean**, there cannot be lawless worlds. The reasoning is simple: no matter what the particular matters of fact, there will always be some best systematisation of them, even if by absolute standards the systematisation is utterly inelegant. Thus, for the Humean, every world will possess some laws. As a consequence, the appeal to lawless worlds in motivating **Bridge** trades on the falsity of a well-established theory of laws, which severely limits the argument's scope.

However, although *prima facie* compelling, this line of reasoning is too quick. Upon reflection, it is not at all clear that whatever the particular matters of fact, there will be some *best* systematisation of them. For instance, suppose that there was some world which was nomically chaotic in that there are no highly general nomic regularities there. More specifically, where '$F, G, H, I...$' are predicates of the canonical language and '$a, b, ...$' canonical terms, none of which are co-referential, consider a world w with the following pattern of facts.

$$w \begin{cases} Fa \wedge Ga \\ Hb \wedge Ib \\ . \\ . \\ . \end{cases}$$

There are different axiomatisations of how things are at w which are tied with one another with respect to simplicity and strength:

S1 $Fa \wedge Ga$, $Hb \wedge Ib$, ...

S2 $\forall x((Fx \wedge Gx) \vee (Hx \wedge Ix) \vee ...)$

First, note that neither system appears at least as strong as the other. For starters, neither proof-theoretically includes the other. **S1** will not prove **S2**'s single axiom, since the conjunction of all its axioms does not prove **S2**'s axiom. In order to secure that implication, one would need the universal generalisation that $a, b, ...$ are all the things there are. But, for the **Humean**, totality facts like these will not typically be categorised as laws: there ought to be physically possible worlds with different individuals. Indeed, as a consequence, the only regularities **S1** contains will be trivial. Moreover, **S2** will not prove any one of **S1**'s axioms; from the generalisation that everything is either thus-and-so or such-and-such, one cannot derive *which* things are thus-and-so and *which* things are such-and-such. More generally, neither system looks to prove much more than the other in the above measure-theoretic sense. Furthermore, on any reasonable measure, both systems will be of roughly the same complexity. After all, **S1** is non-finitely axiomatisable and **S2**'s single axiom contains infinitely many expressions. As a result, there will be no best systematisation of particular matters of fact at w.

Of course, the two systems may nevertheless differ in their strength in certain respects. For instance, Humeans may think that the quantificational form of **S2** in some sense imbues it with extra strength, perhaps in light of its more ambitious generality. But, the best systematisation at a given world need not just contain universal generalisations; as Lewis (1983, p. 41) himself writes, “particular facts may gain entry if they contribute enough to collective simplicity and strength”. Thus, the quantificational form of **S2** would not automatically break the tie between the systems. Moreover, recall Bhogal & Perry’s (2017, pp. 78-79; p. 82) gloss of a system’s strength as a measure of the non-actual distributions it rules out (this gloss is mentioned in §1.2.3 above). If there are haecceitistic differences between distributions of particular matters of fact, **S1** will rule out all non-actual distributions given merely by non-trivial permutations of individuals, unlike **S2**. However, given the Humean’s criteria for simplicity and strength, these differences would appear still to be compatible with the systems being of equal overall strength.

Even Lewis himself recognises that it is optimistic to think that there will always be a best systematisation of a given world. He (1983, p. 41; my emphasis) thus makes the following proviso.

> [A law] is any regularity that earns inclusion in the ideal system. (*Or, in the case of ties, in every ideal system.*).

But, in the above case, the intersection of both systems will contain only tautologies and disjunctions of **S1** and **S2** theorems. However, as mentioned above, the only regularities of **S1** are trivial generalisations, so the regularities in the intersection of both systems will be trivial, which simply deprives them of the explanatory capacity and relevance which are required of laws.[54] Thus, there will be no laws at this chaotic world.

It is also worth mentioning that Lewis himself later came to the opinion that there would be no laws in the absence of a clearly best system. In his (1994, p. 479) own words:

> Likewise for the threat that two very systems are tied for best. [...] I used to say that the laws are then the theorems common to both systems, which could leave us with next to no laws. Now I’ll admit that in this unfortunate case there would be no very good deservers of the name of laws.

[54]Indeed, as mentioned in §1.2.3 above, the Humean might not even want to classify trivial universal generalisations as regularities to begin with.

Indeed, in his later work, Lewis seemed not just to require that laws belong to the best system, but rather that the laws need to be sufficiently explanatory in addition.[55] As one way of developing this idea, Lewis (*ibid.*, p. 479) seemed to require that the best system would be robustly such:

> [I]f nature were unkind, [...] in that case the theorems of the barely-best system would not very well deserve the name of laws.

Interestingly, on this conception, the Humean can easily recognise that there are lawless worlds: even if a chaotic world's particular matters of fact admit some best systematisation, if there are extremely close competing systematisations, the regularities of the best system will not qualify as laws. Consequently, on the various ways in which one might develop the Humean view, chaos can induce lawlessness.

1.5 Concluding Remarks

To summarise, in theorising about the role of physical modality within the objective modalities, one is able to rebut a moderate breed of scepticism about metaphysical modality. Accordingly, there is significant pressure to admit belief in metaphysical modality provided one makes use of physical modality, be it in theorising or in everyday conversation.

Of course, it may turn out that this conception of metaphysical modality appears revisionary in its role and extension. More concretely, it may be that propositions we thought were metaphysically (im)possible are not indeed so. After all, there may be little reason to think that essentialist propositions about an object's origins or membership of kinds should be necessary according to the broadest objective modality defined as such. Similarly, there may be little reason to think that certain 'laws' of metaphysics such as the truth of nominalism or platonism, or various mereological truths, should be necessary according to the proposed definition of metaphysical modality.[56] If such claims fail to be necessary according to the broadest

[55] Thanks to Ted Sider for highlighting this point. In conversation, Sider also suggested that this conception of laws was motivated independently, and could serve as a response to Bird's argument.

[56] See Rosen (2006) and, more recently, Schaffer (2017) for talk of laws of metaphysics.

objective necessity, one might ask whether this notion *really is* metaphysical necessity. But arguably this is not a novel concern. For it may not have been antecedently clear that we have any reason to expect the broadest objective necessity to license such necessities anyway. As Cian Dorr (2016, p. 69) writes:

> 'Metaphysically necessary' was introduced into the philosophical vernacular partly through general formulas—for example, the equation of metaphysical necessity with "unrestricted" or "absolute" necessity, or 'necessity in the highest degree—whatever that means' (Kripke 1972, p. 99)—but partly also through philosophers exchanging opinions about which truths are, in fact, metaphysically necessary—e.g. that nothing is green and red all over, that Nixon is not an inanimate object, that a certain lectern is not made of ice, etc. [...] Given the character of 'metaphysically necessary' as a term of art, charity to the explicit explanatory remarks made by those who introduced the term should weigh especially heavily with us in deciding how to interpret it, while charity to claims that they made where they thought of themselves as taking a stand on philosophically controversial questions should count for relatively little. If we can make sense of a notion of necessity with a good claim to labels like 'necessity in the highest degree', it would be perverse to interpret 'metaphysically necessary' as expressing something more restricted just for the sake of being more charitable to the case-by-case modal pronouncements of, say, Saul Kripke.

As Dorr notes, there may already be internal tensions within the theoretical role of metaphysical modality.[57] On the one hand, we require metaphysical modality to be the broadest objective necessity, yet on the other we speak as if it licenses what appear to be non-maximal objective necessities, such as that Nixon is not an inanimate object. However, from a metasemantic perspective, the initial stipulative requirement that metaphysical modality be the broadest objective necessity ought to take precedent. If, as is plausible, the modality expressed by $\Box$ is an eligible semantic target, we may have just been wrong about the nature and role of metaphysical necessity all along.

Moreover, even if the broadest objective necessity does not classify as necessary, for example, propositions about objects' origins and material constitution, it can be used to characterise a narrower notion of objective necessity which does classify such propositions as necessary. Indeed, the project of characterising narrower

[57]See also Rosen (2006) for a very similar point.

necessities in terms of a broadest notion of necessity was first explored by Smiley (1963), and has recently been developed by Hale & Leech (2017) and in Essay Five of this thesis. This narrower notion of necessity would be a notion of objective necessity that fits the profile of the 'case-by-case modal pronouncements' of various philosophers, and would be placed in good standing by its characterisation in terms of the broadest objective necessity. Thus, the central argument of this paper would motivate belief in the narrower notion of necessity too, regardless of whether it is the semantic value of 'metaphysical necessity' or not.

Ultimately, however, distinguishing between the broadest objective necessity and such narrower notions may turn out to be a conciliatory proposal. For advocates of metaphysical necessity, it ought to be exciting since there will be new modal terrain for them to theorise about. Moreover, for previous sceptics of metaphysical necessity, it ought to be a welcome conclusion: in believing in metaphysical necessity, they need not endorse a body of esoteric essentialist claims and speculative necessary truths which may have initially driven their scepticism.

Formal Appendix

This appendix uses an axiomatic system of higher-order logic to formalise various principles about objective modalities from the main body of text, such as **Anti-Isolation**. In this higher-order setting, an objective operator necessity operator is defined which corresponds to the countable conjunction of all finite iterations of physical necessity, the *closure* of physical necessity.[58] In the presence of principles like **Anti-Isolation**, it is then proven axiomatically that the closure of physical necessity is indeed the broadest objective necessity and includes an **S5** logic. In a natural extension of the system, it is shown that this broadest objective necessity is unique.

Throughout the appendix, informal glosses are provided of the definitions and theorems in order to connect the technical material with the less formal presentation in the main body. The appendix uses single spacing to improve readability.

Syntax

The language is that of the simply typed λ-calculus with the single base type, t. *Typ* is the least set containing t such that if $\sigma, \tau \in Typ$, then $\sigma \to \tau \in Typ$ too. For a more rigorous outline of the syntax of the language, see Dorr (2016, p. 66); in his terms, our working language is a λK-language for a functionally typed signature.

The signature on which our language is based is $\Sigma = \{\to\} \cup \{\forall_\sigma : \sigma \in Typ\} \cup \{\blacksquare\} \cup \{\mathcal{OB}\}$, where $\to$ has type $t \to (t \to t)$, $\forall_\sigma$ has type $(\sigma \to t) \to t$, $\blacksquare$ has type $t \to t$, and $\mathcal{OB}$ has type $(t \to t) \to t$. On their intended interpretation, $\blacksquare$ expresses physical necessity and $\mathcal{OB}$ the property (of propositional operators) of being an objective necessity operator.

The other connectives such as $\neg$ and $\wedge$ are introduced in the usual ways. Moreover, the standard conventions for abbreviations are adopted, for instance by writing $\forall_\sigma F$ as $\forall x F x$ where x has type σ, $\exists x \phi$ for $\neg \forall x \neg \phi$, and using lowercase p and q for variables of type t. For an expression α and type σ, $\alpha : \sigma$ is used as shorthand for the claim that α is of type σ.

Definitions

We introduce the following abbreviations, the first seven of which are from Bacon (2018):

Definition 1. $\top := \forall p(p \to p)$

Definition 2. $Nec^{-} := \lambda X.X\top$, where $X : t \to t$

[58]A qualification: the assumption that there is a *unique* closure of physical necessity (or any propositional operator) holds only in systems which include the **Functionality** axiom below.

Note: Nec^{-} formalises Bacon's (2018) predicate 'is a weak necessity operator'.

Definition 3. $Nec := \lambda X \forall Y(Nec^{-}(Y) \rightarrow YX\top)$, where $X, Y : t \rightarrow t$

Note: Nec formalises Bacon's (2018) predicate 'is a necessity operator'.

Definition 4. $L := \lambda p \forall X(X\top \rightarrow Xp)$, where $X : t \rightarrow t$

Note: L is Bacon's (2018) broad necessity operator.

Definition 5. $\phi \Rightarrow \psi := L(\phi \rightarrow \psi)$

Note: $\Rightarrow$ is just strict implication with respect to L necessity (i.e. broad necessity).

Definition 6. $=_\sigma := \lambda x \lambda y \forall Z(Zx \rightarrow Zy)$, where $x, y : \sigma$ and $Z : \sigma \rightarrow t$

Note: When the type of variables is clear, the type subscript in $=_\sigma$ will be omitted.

Definition 7. $Br := \lambda Y \lambda Z \forall X \forall p(Nec(X) \rightarrow X(Yp \rightarrow Zp))$, where $X, Y, Z : t \rightarrow t$

Note: Br formalises Bacon's (2018) predicate 'as broad as'.

Definition 8. $Br^{\mathcal{OB}} := \lambda Y \lambda Z \forall X \forall p(\mathcal{OB}(X) \rightarrow X(Yp \rightarrow Zp))$, where $X, Y, Z : t \rightarrow t$

Note: $Br^{\mathcal{OB}}$ formalises the predicate 'as broad as (with respect to the objective modalities)' which on its intended interpretation expresses the notion of broadness used in the main body of text. Thus, on its intended interpretation, $Br^{\mathcal{OB}}(Y)Z$ states that Y is as broad as Z with respect to the objective modalities. To ease readability, we will write $Br^{\mathcal{OB}}(Y)Z$ as $Br^{\mathcal{OB}}(Y, Z)$.

Definition 9. $K := \lambda X \forall p \forall q(X(p \rightarrow q) \rightarrow Xp \rightarrow Xq)$, where $X : t \rightarrow t$

Note: K formalises the predicate 'obeys the **K** axiom'. Thus, on its intended interpretation, $K(X)$ states that X obeys the **K** axiom.

Definition 10. $It := \lambda O_1 \lambda O_2 \forall X(X\lambda p.p \rightarrow (\forall O_3(XO_3 \rightarrow X\lambda p.O_1O_3p) \rightarrow XO_2))$, where $O_1, O_2, O_3 : t \rightarrow t$ and $X : (t \rightarrow t) \rightarrow t$

Note: It formalises the predicate 'is an iteration of'. For $Y : t \rightarrow t$, $It(Y) : (t \rightarrow t) \rightarrow t$. Thus for $Y, Z : t \rightarrow t$, $It(Y)Z$ states that the operator Z is an iteration of Y (this includes the case of zero iterations: truth).

Definition 11. $Cl := \lambda Y \lambda p.\forall X(It(Y)X \rightarrow Xp)$, where $Y, X : t \rightarrow t$

Note: Cl formalises the predicate which applies to a propositional operator Y and returns a closure of that operator, an operator which applies every iteration of Y to a proposition. Thus, for $X : t \to t$, $Cl(X) : t \to t$. In the appendix, $Cl(\blacksquare)$ will play the role of the countable conjunction of all finite iterations of physical necessity which is used in the main body of text.

Definition 12. $Conv := \lambda O_1 \lambda O_2 \forall p(p \Rightarrow O_1 \neg O_2 \neg p)$, where $O_1, O_2 : t \to t$

Note: $Conv$ formalises the predicate 'is a right converse of'. For $Y : t \to t$, $Conv(Y) : (t \to t) \to t$. When $Y, Z : t \to t$, on its intended interpretation $Conv(Y)Z$ states that operator Z is a right converse of operator Y.

HE & HFE

The system **HE** is axiomatised by Bacon (2018) as follows.

PC All instances of propositional tautologies.

MP From A and $A \to B$ infer B

Gen From $A \to B$ infer $A \to \forall_\sigma x B$ when x does not occur free in A

UI $\forall_\sigma x \phi \to \phi[t/x]$ (where $t : \sigma$ and is substitutable for x in ϕ)

$\beta\eta$ $A \leftrightarrow B$ whenever $A : t$ and $B : t$ are $\beta\eta$ equivalent[59]

RE If $\vdash A \leftrightarrow B$ then $\vdash A =_t B$

Bacon shows that **HE** proves (a generalised version of) Booleanism, the view that propositions form a Boolean algebra under the usual truth-functional operations.

As is typical, **HE** proves all instances of the standard distribution schema for quantifiers of any type, which we state explicitly for convenience.

Distribution $\forall_\sigma x(\phi \to \psi) \to (\forall_\sigma x \phi \to \forall_\sigma x \psi)$

The system **HFE** extends **HE** by the following axiom.

Functionality $\forall_\sigma x(Xx =_\tau Yx) \to X =_{\sigma\to\tau} Y$

[59]Terms are $\beta\eta$ equivalent just in case one is obtained from the other by a sequence of substitutions of sub-terms which are either:

Immediately β-equivalent: of the form $(\lambda x \phi)a$ and $\phi[a/x]$, where a is substitutable for x in a and $\phi[a/x]$ is the result of uniformly substituting a for x in ϕ

Immediately η-equivalent: of the form $\lambda x.Fx$ and F, where x is not free in F

HE & HFE Theorems

The following theorems are proved in Bacon (2018).

Theorem 1. (L-Necessity) $\vdash_{\mathbf{HE}} \forall X(X\top \to XL\top)$

On its intended interpretation, Theorem 1 states that Bacon's broad necessity operator L is a necessity operator in the precise sense defined above.

Theorem 2. ($N{>}N^{-}$) $\vdash_{\mathbf{HE}} \forall X(Nec(X) \to Nec^{-}(X))$

On its intended interpretation, Theorem 2 states that every necessity operator is a weak necessity operator in the senses defined above.

Theorem 3. (L-Broadest) $\vdash_{\mathbf{HE}} \forall X \forall Y(Nec(X) \to Nec(Y) \to Y\forall p(Lp \to Xp))$

On its intended interpretation, Theorem 3 states that L is a broadest necessity. Note that this is *not* the claim that L is the broadest objective necessity.

Theorem 4. (L-S4) ***HE*** *includes every substitution instance of the axioms of* ***S4*** *for L.*

Theorem 5. (L-Necessitation) *If* $\vdash_{\boldsymbol{HE}} \phi$*, then* $\vdash_{\boldsymbol{HE}} L\phi$.

Theorem 6. (L-Uniqueness)
$\vdash_{\mathbf{HFE}} \forall X(Nec(X) \to (Br(Y,X) \wedge Br(Z,X) \to Y = Z))$

On its intended interpretation, Theorem 6 states that broadest necessity is unique. The proof of the theorem requires **Functionality**, and hence the claim is not a theorem merely of **HE**.

Theorem 7. (L-Barcan) $\vdash_{\mathbf{HFE}} \forall xLA \to L\forall xA$

Axioms for $\mathcal{OB}$

H(F)E is extended to **H(F)EO** by the following axioms. (For comparison, see the related but different axiomatisation of the objective modalities in Williamson (2016).)

Inclusion $\forall X(\mathcal{OB}(X) \to Nec(X))$

On its intended interpretation, **Inclusion** states that every objective necessity operator is a necessity operator in Bacon's (2018) sense.

Kripke $\forall X(\mathcal{OB}(X) \to K(X))$

On its intended interpretation, **Kripke** states that every objective necessity operator obeys the **K** axiom. Compare Williamson's (2016) requirement that objective necessity operators commute with conjunction.

Alethic $\mathcal{OB}(\lambda p.p)$

On its intended interpretation, **Alethic** states that truth is an objective necessity operator. Compare Williamson's (2016) Identity postulate.

Composition $\forall X \forall Y(\mathcal{OB}(X) \wedge \mathcal{OB}(Y) \rightarrow \mathcal{OB}(\lambda p.XYp))$

On its intended interpretation, **Composition** states that the objective necessities are closed under composition. Compare Williamson's (2016) Composition postulate.

Basis $\mathcal{OB}(\blacksquare)$

On its intended interpretation, **Basis** states that physical necessity is an objective necessity.

Closure $\forall X(\mathcal{OB}(X) \rightarrow \mathcal{OB}(Cl(X)))$

On its intended interpretation, **Closure** states that a closure of any objective necessity operator is an objective necessity operator. To connect **Closure** with the main body of text, informally it can be thought of as stating that the countable conjunction of all finite iterations of an objective necessity operator expresses an objective necessity itself. Compare Williamson's (2016) Conjunction postulate, which has no restriction to *countable* sets of objective necessity operators.

Converses $\forall X(\mathcal{OB}(X) \rightarrow \exists Y(\mathcal{OB}(Y) \wedge Conv(X)Y))$

On its intended interpretation, **Converses** states that every right converse of an objective necessity operator is an objective necessity operator. Compare Williamson's (2016) Reversal postulate.

Anti-Isolation $\forall X \forall Y(\mathcal{OB}(X) \wedge \mathcal{OB}(Y) \rightarrow Y \exists Z(It(\blacksquare)Z \wedge (Zp \rightarrow Xp)))$

On its intended interpretation, **Anti-Isolation** states that for any objective notions of necessity X, Y, it's Y-necessary that some iteration of physical necessity materially implies X. The formalises the Anti-Isolation principle in the main body of text.

HEO and HFEO Theorems

In the following proofs, the right-hand side of each line states the theorems and rules which are crucial to deriving the formula on the line in question. As is typical, elementary items of justification are omitted. In order to improve readability brackets are omitted at various points when scope is clear from the context.

Theorem 8. ($\mathcal{OB}$-Broadest) $\vdash_{\mathbf{H(F)EO}} \forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Cl(\blacksquare), X))$

On its intended interpretation, Theorem 8 states that a closure of physical necessity is as broad as every notion of objective necessity with respect to the family of objective necessities. In the presence of **Functionality**, the closure of physical necessity is guaranteed to be unique.

Proof. We first establish the following lemma:

Lemma. $\vdash_{\mathbf{HE}}$ $L(\forall p \exists Y(It(\blacksquare)Y \wedge Yp \to Xp) \to \forall p(Cl(\blacksquare)p \to Xp))$

(i) $(It(\blacksquare)Y \to Yp) \to ((It(\blacksquare)Y \wedge (Yp \to Xp)) \to Xp)$ PC

(ii) $\forall Y(It(\blacksquare)Y \to Yp) \to \forall Y((It(\blacksquare)Y \wedge (Yp \to Xp)) \to Xp)$ i, Distribution

(iii) $\forall Y(It(\blacksquare)Y \to Yp) \to ((It(\blacksquare)Y \wedge (Yp \to Xp)) \to Xp)$ ii, UI

(iv) $\forall Y(It(\blacksquare)Y \to Yp) \to (\neg Xp \to \neg(It(\blacksquare)Y \wedge (Yp \to Xp)))$ iii, PC, MP

(v) $\forall Y(It(\blacksquare)Y \to Yp) \to \forall Y(\neg Xp \to \neg(It(\blacksquare)Y \wedge (Yp \to Xp)))$ iv, Gen

(vi) $\forall Y(It(\blacksquare)Y \to Yp) \to (\neg Xp \to \forall Y \neg(It(\blacksquare)Y \wedge (Yp \to Xp)))$ v, Gen

(vii) $\forall Y(It(\blacksquare)Y \to Yp) \to (\neg\forall Y \neg(It(\blacksquare)Y \wedge (Yp \to Xp)) \to \neg\neg Xp)$
vi, PC, MP

(viii) $\forall Y(It(\blacksquare)Y \to Yp) \to (\neg\forall Y \neg(It(\blacksquare)Y \wedge (Yp \to Xp)) \to Xp)$
vii, PC, MP

(ix) $\forall Y(It(\blacksquare)Y \to Yp) \to (\exists Y(It(\blacksquare)Y \wedge (Yp \to Xp)) \to Xp)$ viii, Def. $\exists$

(x) $\exists Y(It(\blacksquare)Y \wedge (Yp \to Xp)) \to (\forall Y(It(\blacksquare)Y \to Yp) \to Xp)$ ix, PC, MP

(xi) $\forall p \exists Y(It(\blacksquare)Y \wedge (Yp \to Xp)) \to \forall p(\forall Y(It(\blacksquare)Y \to Yp) \to Xp)$
x, Distribution

(xii) $L(\forall p \exists Y(It(\blacksquare)Y \wedge (Yp \to Xp)) \to \forall p(\forall Y(It(\blacksquare)Y \to Yp) \to Xp))$
xi, L-Necessitation

(xiii) $L(\forall p \exists Y(It(\blacksquare)Y \wedge (Yp \to Xp)) \to \forall p(Cl(\blacksquare)p \to Xp))$ Def. Cl

Notice that the above reasoning took place in **HE**. **Theorem 8** may then be established by the following reasoning in **H(F)EO**.

(1) $L(\forall p \exists Y(It(\blacksquare)Y \wedge Yp \to Xp) \to \forall p(Cl(\blacksquare)p \to Xp))$ Lemma

(2) $\forall Z(Z\top \to [Z(\forall p \exists Y(It(\blacksquare)Y \wedge Yp \to Xp) \to \forall p(Cl(\blacksquare)p \to Xp))])$
1, Def. L

(3) $\forall Z(Nec^{-}(Z) \to [Z(\forall p \exists Y(It(\blacksquare)Y \wedge Yp \to Xp) \to \forall p(Cl(\blacksquare)p \to Xp))])$
2, Def. Nec^{-}

(4) $\forall Z(Nec(Z) \to [Z(\forall p \exists Y(It(\blacksquare)Y \wedge Yp \to Xp) \to \forall p(Cl(\blacksquare)p \to Xp))])$
3, $N{>}N^{-}$

(5) $\forall Z(\mathcal{OB}(Z) \to [Z(\forall p \exists Y(It(\blacksquare)Y \wedge Yp \to Xp) \to \forall p(Cl(\blacksquare)p \to Xp))])$
4, Inclusion

(6) $\forall Z(\mathcal{OB}(Z) \to [Z\forall p \exists Y(It(\blacksquare)Y \wedge Yp \to Xp) \to Z\forall p(Cl(\blacksquare)p \to Xp)])$
5, Kripke

(7) $\forall X(\mathcal{OB}(X) \to [\forall Z(\mathcal{OB}(Z) \to [Z\forall p \exists Y(It(\blacksquare)Y \wedge Yp \to Xp) \to Z\forall p(Cl(\blacksquare)p \to Xp)])])$
6, PC, Gen

(8) $\forall X(\mathcal{OB}(X) \to [\forall Z(\mathcal{OB}(Z) \to Z\exists Y(It(\blacksquare)Y \wedge Yp \to Xp))])$ Anti-Isolation

(9) $\forall X(\mathcal{OB}(X) \to [\forall Z(\mathcal{OB}(Z) \to Z\forall p(Cl(\blacksquare)p \to Xp))])$
7, 8, PC, UI, MP, Gen

(10) $\forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Cl(\blacksquare), X))$ 9, Def. $Br^{\mathcal{OB}}$

Theorem 9. (S5) ***H(F)EO*** *includes every substitution instance of the theorems of* ***S5*** *for* $Cl(\blacksquare)$.

Proof. We show that $Cl(\blacksquare)$ obeys the **K**, **T**, **B**, and **4** axioms in **H(F)EO**. Given the rules for **H(F)EO**, it then suffices to show that $Cl(\blacksquare)$ obeys a rule of necessitation in **H(F)EO**.

K for $Cl(\blacksquare)$ is established as follows.

(1) $\mathcal{OB}(\blacksquare)$ Basis

(2) $\mathcal{OB}(Cl(\blacksquare))$ 1, Closure, UI, MP

(3) $K(Cl(\blacksquare))$ 2, Kripke, UI, MP

T for $Cl(\blacksquare)$ is established as follows.

(1) $It(\blacksquare)(\lambda q.q)$ Def. It

(2) $\forall X(It(\blacksquare)X \to Xp) \to (It(\blacksquare)(\lambda q.q) \to (\lambda q.q)p)$ UI

(3) $\forall X(It(\blacksquare)X \to Xp) \to (\lambda q.q)p$ 1, 2, PC, MP

(4) $(\lambda q.q)p \to p$ $\beta\eta$

(5) $\forall X(It(\blacksquare)X \to Xp) \to p$ 3, 4, PC, MP

(6) $Cl(\blacksquare)p \to p$ 5, Def. Cl

4 for $Cl(\blacksquare)$ is established as follows.

(1) $\forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Cl(\blacksquare), X))$ Theorem 8 ($\mathcal{OB}$-Broadest)

(2) $\forall X(\mathcal{OB}(X) \to \forall Y \forall p(\mathcal{OB}(Y) \to Y(Cl(\blacksquare)p \to Xp)))$ 1, Def. $Br^{\mathcal{OB}}$

(3) $\mathcal{OB}(Cl(\blacksquare))$ Basis, Closure

(4) $\forall X \forall Y(\mathcal{OB}(X) \wedge \mathcal{OB}(Y) \to \mathcal{OB}(\lambda p.XYp))$ Composition

(5) $\mathcal{OB}(\lambda p.Cl(\blacksquare)Cl(\blacksquare)p)$ 4, UI, 3, MP

(6) $\mathcal{OB}(Cl(\blacksquare)Cl(\blacksquare))$ 5, $\beta\eta$

(7) $\forall Y \forall p(\mathcal{OB}(Y) \to Y(Cl(\blacksquare)p \to Cl(\blacksquare)Cl(\blacksquare)p))$ 6, UI, 2, MP

(8) $Cl(\blacksquare)p \to Cl(\blacksquare)Cl(\blacksquare)p$ 7, UI, Alethic, MP, $\beta\eta$, UI

B for $Cl(\blacksquare)$ is established by first proving the following standard schema as a lemma.

Lemma $\vdash_{\mathbf{H(F)EO}} \exists v(\phi \wedge (\psi \to \chi)) \to [\forall v(\phi \to (\chi \to \xi)) \to \exists v(\phi \wedge (\psi \to \xi))]$

(i) $(\phi \wedge (\psi \to \chi)) \to [(\phi \to (\chi \to \xi)) \to (\phi \wedge (\psi \to \xi))]$ PC

(ii) $(\phi \to (\chi \to \xi)) \to [(\phi \wedge (\psi \to \chi)) \to (\phi \wedge (\psi \to \xi))]$ i, PC, MP

(iii) $(\phi \to (\chi \to \xi)) \to [\neg(\phi \wedge (\psi \to \xi)) \to \neg(\phi \wedge (\psi \to \chi))]$ ii, PC, MP

(iv) $\forall v(\phi \to (\chi \to \xi)) \to \forall v[\neg(\phi \wedge (\psi \to \xi)) \to \neg(\phi \wedge (\psi \to \chi))]$
iii, Distribution

(v) $\forall v[\neg(\phi \wedge (\psi \to \xi)) \to \neg(\phi \wedge (\psi \to \chi))] \to [\forall v \neg(\phi \wedge (\psi \to \xi)) \to \forall v \neg(\phi \wedge (\psi \to \chi))]$ Distribution

(vi) $\forall v(\phi \to (\chi \to \xi)) \to [\forall v \neg(\phi \wedge (\psi \to \xi)) \to \forall v \neg(\phi \wedge (\psi \to \chi))]$
iv, v, PC, MP

(vii) $\forall v(\phi \to (\chi \to \xi)) \to [\neg \forall v \neg(\phi \wedge (\psi \to \chi)) \to \neg \forall v \neg(\phi \wedge (\psi \to \xi))]$
vi, PC, MP

(viii) $\neg\forall v\neg(\phi \wedge (\psi \to \chi)) \to [\forall v(\phi \to (\chi \to \xi)) \to \neg\forall v\neg(\phi \wedge (\psi \to \xi))]$ vii, PC, MP

(ix) $\exists v(\phi \wedge (\psi \to \chi)) \to [\forall v(\phi \to (\chi \to \xi)) \to \exists v(\phi \wedge (\psi \to \xi))]$ viii, Def. $\exists$

B for $Cl(\blacksquare)$ is then established as follows.

(1) $\forall X(\mathcal{OB}(X) \to \exists Y(\mathcal{OB}(Y) \wedge \forall pL(p \to X\neg Y\neg p)))$ Converses

(2) $\exists Y(\mathcal{OB}(Y) \wedge \forall pL(p \to Cl(\blacksquare)\neg Y\neg p))$ Basis, Closure, 1, UI, MP

(3) $\forall pL(p \to Cl(\blacksquare)\neg Y\neg p) \to (p \to Cl(\blacksquare)\neg Y\neg p)$ UI, Theorem 4

(4) $\exists Y(\mathcal{OB}(Y) \wedge (p \to Cl(\blacksquare)\neg Y\neg p))$ PC, 2, 3, MP

(5) $\forall Y(\mathcal{OB}(Y) \to Br^{\mathcal{OB}}(Cl(\blacksquare), Y))$ Theorem 8

(6) $\forall Y(\mathcal{OB}(Y) \to Cl(\blacksquare)(\neg Y\neg p \to \neg Cl(\blacksquare)\neg p))$ 5, Basis, Closure, UI, MP, PC

(7) $Cl(\blacksquare)(\neg Y\neg p \to \neg Cl(\blacksquare)\neg p) \to (Cl(\blacksquare)\neg Y\neg p \to Cl(\blacksquare)\neg Cl(\blacksquare)\neg p)$ Basis, Closure, Kripke

(8) $\forall Y(\mathcal{OB}(Y) \to (Cl(\blacksquare)\neg Y\neg p \to Cl(\blacksquare)\neg Cl(\blacksquare)\neg p))$ 7, 6, PC, Gen, MP

(9) $\exists Y(\mathcal{OB}(Y) \wedge (p \to Cl(\blacksquare)\neg Cl(\blacksquare)\neg p))$ 4, 8, Lemma

(10) $\exists Y\mathcal{OB}(Y) \wedge (p \to Cl(\blacksquare)\neg Cl(\blacksquare)\neg p)$ 9, PC, Distribution, UI

(11) $p \to Cl(\blacksquare)\neg Cl(\blacksquare)\neg p$ 10, PC, MP

Necessitation for $Cl(\blacksquare)$ in **H(F)EO** is established as follows.

Assume that $\vdash_{\mathbf{H(F)EO}} \phi$. Clearly, $\vdash_{\mathbf{H(F)EO}} \top$ too. Thus, $\vdash_{\mathbf{H(F)EO}} \phi \leftrightarrow \top$, and so by **RE** $\vdash_{\mathbf{H(F)EO}} \phi = \top$. In unabbreviated form, this amounts to: $\vdash_{\mathbf{H(F)EO}} \forall X(X\top \to X\phi)$. However, of course, $\vdash_{\mathbf{H(F)EO}} L\top$. Thus, $\vdash_{\mathbf{H(F)EO}} L\phi$. Given this, it follows by **Basis**, **Closure**, **Inclusion**, and **Theorem 2** that $\vdash_{\mathbf{H(F)EO}} Cl(\blacksquare)\phi$. Thus, if $\vdash_{\mathbf{H(F)EO}} \phi$, then $\vdash_{\mathbf{H(F)EO}} Cl(\blacksquare)\phi$.

Remark. Note that **Converses** was only needed for the proof of **B**. The system which results from removing **Converses** from **H(F)E** still includes all substitution instances of the theorems of **S4** for $Cl(\blacksquare)$.

Extensions

H(F)EO is extended to **H(F)EO**$^=$ by the following axiom.

Identity $\forall Y \forall X(\mathcal{OB}(X) \to (\mathcal{OB}(Y) \to [(Br^{\mathcal{OB}}(Y,X) \wedge Br^{\mathcal{OB}}(X,Y)) \to \forall p(Xp = Yp)]))$

On its intended interpretation, **Identity** states that when objective modalities are broadest objective modalities, the result of applying one to a proposition just is the result of applying the other to that proposition. In the presence of **Functionality**, **Identity** is equivalent to the following.

Identity′ $\forall Y \forall X(\mathcal{OB}(X) \to (\mathcal{OB}(Y) \to [(Br^{\mathcal{OB}}(Y,X) \wedge Br^{\mathcal{OB}}(X,Y)) \to X = Y]))$

On its intended interpretation, **Identity′** states that when objective modalities are broadest objective modalities, they are identical. Philosophically, **Identity′** might be motivated by the thought that objective modalities are pinned down exactly by their relations to other objective modalities. In other words, there are no two objective modalities that share exactly the same profile with respect to all objective modalities.

Theorem 10. ($\mathcal{OB}$ Uniqueness)
$\vdash_{\mathbf{H(F)EO^=}} \forall Y(\mathcal{OB}(Y) \to [\forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Y,X)) \leftrightarrow Cl(\blacksquare) = Y])$

On its intended interpretation, Theorem 10 states that the closure of physical necessity is the only broadest objective necessity.

Proof.

(1) $\forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Cl(\blacksquare), X))$ — Theorem 8 ($\mathcal{OB}$-Broadest)

(2) $\lambda Z \forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Z,X))Cl(\blacksquare)$ — 1, $\beta\eta$

(3) $(Cl(\blacksquare) = Y) \to (\lambda Z \forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Z,X))Cl(\blacksquare) \to \lambda Z \forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Z,X))Y)$ — Def. =

(4) $(Cl(\blacksquare) = Y) \to \lambda Z \forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Z,X))Y$ — 2, 3, PC, MP

(5) $(Cl(\blacksquare) = Y) \to \forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Y,X))$ — 4, $\beta\eta$

(6) $\mathcal{OB}(Y) \to ((Cl(\blacksquare) = Y) \to \forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Y,X)))$ — 5, PC, MP

(7) $\forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Y,X)) \to Br^{\mathcal{OB}}(Y, Cl(\blacksquare))$ — UI, Basis, Closure

(8) $\mathcal{OB}(Y) \to (\forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Y,X)) \to Br^{\mathcal{OB}}(Y, Cl(\blacksquare)))$ — 7, PC, MP

(9) $\mathcal{OB}(Y) \to Br^{\mathcal{OB}}(Cl(\blacksquare), Y)$ — 1, UI, MP

(10) $\mathcal{OB}(Y) \to (\forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Y,X)) \to Br^{\mathcal{OB}}(Cl(\blacksquare), Y))$ — 9, PC, MP

(11) $\mathcal{OB}(Y) \to (\forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Y, X)) \to (Br^{\mathcal{OB}}(Y, Cl(\blacksquare)) \wedge Br^{\mathcal{OB}}(Cl(\blacksquare), Y)))$ 8, 10, PC, MP

(12) $\mathcal{OB}(Y) \to (\forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Y, X)) \to Cl(\blacksquare) = Y)$
11, Identity′, Basis, Closure, PC, MP

(13) $\forall Y(\mathcal{OB}(Y) \to (\forall X(\mathcal{OB}(X) \to Br^{\mathcal{OB}}(Y, X)) \to Cl(\blacksquare) = Y))$
12, PC, Gen, MP

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2
Modal Expansionism

Abstract

There are various well-known paradoxes of modal recombination. This paper offers a solution to a variety of such paradoxes in the form of a new conception of metaphysical modality. On the proposed conception, metaphysical modality exhibits a type of indefinite extensibility. Indeed, for any objective modality there will always be some further, broader objective modality; in other terms, modal space will always be open to *expansion*.

Call a modality *objective* when it is non-epistemic, non-deontic, non-linguistic, and non-intentional. The objective modalities are roughly the modalities expressed by circumstantial modal terms.[1] We routinely use such terms in natural language to express various practical modalities, and, on occasion, nomological modality. Yet, although uncommon in natural language, circumstantial modal auxiliary verbs like 'must' can be used unrestrictedly, that is without restriction to any set of circumstances; on this reading, 'it must be that ϕ' is true when ϕ is the case whatever the circumstances. When read as such, the objective modality expressed by the modal is *metaphysical modality*. Metaphysical modality is accordingly the maximal objective modality: possibility according to any objective modality entails possibility according to metaphysical modality.[2]

The idea that metaphysical modality is the maximal objective modality is widespread. It is frequently assumed in philosophical discussion and regularly appealed to in explanations of the nature of metaphysical necessity. It reflects the popular thought that metaphysical modality in some sense concerns *all* possible worlds. Nevertheless, this paper argues for a new conception of metaphysical modality according to which there is one sense in which this maximality thesis is false. According to the proposed conception, for any objective modality there will always be some further, broader objective modality; or, in other terms, metaphysical modal space will always be open to *expansion*.

The structure of the paper is as follows. §2.1 and §2.2 present a modal recombinatorial paradox due to Kit Fine (2002, pp. 223-224), later reformulated by Peter Fritz (2016).[3] §2.3 outlines the parallels which this recombinatorial paradox

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[1]The term 'objective modality' was introduced in the relevant sense by Williamson (2016), who connects it to Angelika Kratzer's (2012, chap. 2) notion of a circumstantial modal expression. A rough test for whether a modal term is circumstantial is whether it is sensitive to the mode under which a proposition in its scope is presented.

[2]Some philosophers might be tempted by the idea that metaphysical modality is in some sense a restriction of logical modality. This is compatible with the claim that metaphysical modality is the maximal objective modality, for logical modality is a linguistic modality and thus not objective in the relevant sense.

[3]Strictly speaking, Fine and Fritz's paradoxes are closely related but not one and the same; Fine's

bears to the paradoxes of set theory, and suggests a treatment of the former similar to extant treatments of the latter. Using the resources of a novel bimodal logic, §2.4 develops the suggested treatment and its revisionary modal metaphysic. §2.5 and §2.6 respectively discuss objections to and further applications of the newly developed metaphysics of modality.

2.1 Recombination and Paradox

Consider the property of being an elementary particle. This property is intrinsic and therefore it can be possessed independently of whether it is possessed by any other individuals. As a result, the following modal recombinatorial principle will be true (where the operative modality is metaphysical modality):

(R1) For any possible world w, there is a possible world u where all elementary particles at w are elementary particles and there is an elementary particle distinct from every elementary particle at w.

(R1) is plausible because in denying it one objectionably places arbitrary limits on the extent of modal space. Given the purported utterly liberal character of metaphysical modality, for every world w there should be some possible world u where all elementary particles at w are all elementary particles, and there is some elementary particle distinct from every elementary particle at w. After all, could not all elementary particles at w not be elementary particles with some distinct elementary particle? If w contains no elementary particles, surely it is metaphysically possible that there be elementary particles. If w contains one or more elementary particles, surely it is metaphysically possible that there be all of them plus a duplicate of one of them.

Now consider another plausible recombinatorial principle:

(R2) There is some possible world w such that for all possible worlds u all elementary particles at u are elementary particles at w.

puzzle involves classes of possibilia, whereas Fritz's puzzle involves no such resources.

(R2) articulates the idea that there is a possibility in which all elementary particles across modal space are elementary particles. It too is motivated by an aversion to imposing arbitrariness onto modal space. Since something's being an elementary particle is compossible with some other thing's also being an elementary particle, there appears to be no limit as to which possible elementary particles are compossible. Again, given the purported utterly liberal character of metaphysical modality, it should not be the case that, for example, it is possible that all elementary particles across modal space except one are elementary particles, yet not possible that all elementary particles across modal space are elementary particles. To admit that type of division would be to admit a type of objectionable arbitrariness which a theory of metaphysical modality theory ought to eschew.[4]

However, despite (R1) and (R2) both being plausible modal recombinatorial principles, Kit Fine (2002, pp. 223-224) presents an argument that similar such principles together lead to paradox. To adapt Fine's argument, consider w in (R2). From (R1) we can infer that:

(C) There is a possible world u where all elementary particles at w are elementary particles and there is also a elementary particle which is not identical with any of them.

But all the elementary particles at w are all the elementary particles at all worlds. Contradiction. As a result, (R1) and (R2) cannot be satisfied by a single modality, for no one modality can possesses this extreme degree of recombinatorial liberality, including metaphysical modality.

[4]Following David Lewis's (1986, pp. 101-104) discussion, advocates of concrete modal realism might reject (R2) on the basis that some, perhaps large infinite cardinal is an upper bound on the possible size of spacetime. (In his discussion, Lewis is aware that this would still involve some degree of arbitrariness; see Lewis (*ibid.*, p. 103) in particular.) But this response does not seem to address the core concern, for it is surely metaphysically possible that elementary particles can be colocated, just as some philosophers, such as Hawthorne & Uzquiano (2011, p. 55), have claimed bosons can actually be.

2.2 Reformulation

The above formulation of the paradox employed first-order quantification over possible worlds and possibilia. But one should not assume that the paradox is generated by an extensional reduction of modal notions. Indeed, Peter Fritz (2016) has recently shown how to formalise (R1), (R2), and (C) in the language of pure quantified modal logic, which avoids first-order quantification over worlds and possibilia. This should dispel any concern that the paradox is an artefact of reductionism about modal notions.

For a quantified modal language to have the expressive capacity required to formulate the principles, it must be capable of simulating first-order quantification over possible worlds and possibilia, and simulating cross-reference to worlds over which one has previously 'considered'. This expressive requirement is met by enriching the vocabulary of a conventional quantified modal language with infinitely many distinct pairs of indexed Vlach-operators ($\uparrow^n, \downarrow^n$, for every natural number n)—operators that allow one to 'store' a world at a place on a list, and at a later point 'retrieve' that world as the world of evaluation. Specifically, $\uparrow^n$ stores the current world of evaluation at the n^{th} place on a list, and $\downarrow^n$ places the world stored at the n^{th} place on the list as the world of evaluation. To give an example of how the operators work, consider the following formula:

(1) $\Diamond \uparrow^1 \Box \forall x(F(x) \rightarrow \downarrow^1 G(x))$

This can be read in extensional parlance as the claim that there is some possibility at which every F in any possible world is a G.

To be more precise, the exact vocabulary of the quantified modal language (=$\mathcal{L}_1$) in which we shall work consists of countably many individual variables ($x_1, x_2, ...$), countably many predicate letters ($E, F, G...$), $\neg$ (negation) and $\rightarrow$ (material conditional) as the only truth-functional connectives, $\forall$ (universal quantification) as the sole quantifier, and the modal operators $\Box$ (necessity), $\uparrow^n$ and $\downarrow^n$ for every natural number n. The existential quantifier $\exists$ and possibility operator $\Diamond$ are used as abbreviations for $\neg\forall\neg$ and $\neg\Box\neg$, respectively.

Following Kit Fine's (1977, pp. 135-136) early work, Fabrice Correia (2007) has shown how to provide a model-theoretic semantics for languages like $\mathcal{L}_1$. We follow his lead. Thus a *frame* $\mathfrak{F}$ is an ordered pair $\langle \mathcal{W}, \mathcal{D} \rangle$, where $\mathcal{W}$ is a non-empty set understood as a non-empty set of worlds, and $\mathcal{D}$ is a function taking each $w \in \mathcal{W}$ into a non-empty set $\mathcal{D}_w$, understood as that world's domain.[5] A *model* $\mathfrak{M}$ of $\mathcal{L}_1$ based on $\mathfrak{F} = \langle \mathcal{W}, \mathcal{D} \rangle$ is a triple $\langle \mathcal{W}, \mathcal{D}, \mathcal{V} \rangle$, where $\mathcal{V}$ is a function taking any individual constant of the language into a member of $\cup_{w\in\mathcal{W}}\mathcal{D}_w$, and any n-place predicate letter of the language and world into a set of n-tuples of $\cup_{w\in\mathcal{W}}\mathcal{D}_w$, that predicate's extension at that world. Given a model, an *assignment* is a function g mapping each individual variable of the language to a member of $\cup_{w\in\mathcal{W}}\mathcal{D}_w$. Where x is an individual variable, two assignments g and g' are x-alternatives iff g and g' assign the same values for all variables, except possibly x.

For a set S, let $\mathcal{S}^{<\omega}$ be the set of ω-sequences of members of $\mathcal{S}$. For $seq \in \mathcal{S}^{<\omega}$ and $s \in \mathcal{S}$, $seq^{n\to s}$ is the sequence obtained from replacing the n^{th} item in seq with s, and $seq(n)$ is the n^{th} item in seq.

We define '$(\mathfrak{M}, seq, w) \models_g A$' for $\mathfrak{M} = \langle \mathcal{W}, \mathcal{D}, \mathcal{V} \rangle$, $seq \in \mathcal{W}^{<\omega}$, $w \in \mathcal{W}$, g an assignment to the variables in $\mathcal{M}$, and A an formula of the language, recursively as follows ($\mathcal{V}_g(a)$ is used for $g(a)$ if a is an individual variable and for $\mathcal{V}(a)$ if a is an individual constant):

$(\mathfrak{M}, seq, w) \models_g F(a_1, ..., a_n)$ iff $\langle \mathcal{V}_g(a_1), ..., \mathcal{V}_g(a_n) \rangle \in \mathcal{V}(F, w)$;

$(\mathfrak{M}, seq, w) \models_g \neg A$ iff $(\mathfrak{M}, seq, w) \not\models_g A$;

$(\mathfrak{M}, seq, w) \models_g A \to B$ iff either $(\mathfrak{M}, seq, w) \not\models_g A$ or $(\mathfrak{M}, seq, w) \models_g B$;

$(\mathfrak{M}, seq, w) \models_g \forall x A$ iff $(\mathfrak{M}, seq, w) \models_{g'} A$ for every x-alternative g' of g such that $g'(x) \in \mathcal{D}_w$;

$(\mathfrak{M}, seq, w) \models_g \Box A$ iff for every $w' \in \mathcal{W}$, $(\mathfrak{M}, seq, w') \models_g A$;

$(\mathfrak{M}, seq, w) \models_g \uparrow^n A$ iff $(\mathfrak{M}, seq^{n\to w}, w) \models_g A$;

[5]Since frames contain no accessibility relations, the model theory validates an **S5** logic for metaphysical necessity, i.e. $\Box$. This is a harmless simplifying assumption, and could be dropped for current purposes. (For example, as Fritz (2016, Appendix C) notes, the inconsistency remains even if the 4 axiom for metaphysical necessity is rejected.)

$(\mathfrak{M}, seq, w) \models_g \downarrow^n A$ iff $(\mathfrak{M}, seq, seq(n)) \models_g A$.

A formula A is true at $\langle seq, w\rangle$ for $seq \in \mathcal{W}^{<\omega}$ and $w \in \mathcal{W}$ iff for all variable assignments g, $(\mathfrak{M}, seq, w) \models_g A$. A formula A of $\mathcal{L}_1$ is true in an $\mathcal{L}_1$ model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{D}, \mathcal{V}\rangle$ iff it is true at $\langle seq, w\rangle$, for every $seq \in \mathcal{W}^{<\omega}$, and every $w \in \mathcal{W}$. A formula A of $\mathcal{L}_1$ is valid iff it is true in every $\mathcal{L}_1$ model. A set Γ of $\mathcal{L}_1$ formulas entails an $\mathcal{L}_1$ formula A iff for every $\mathcal{L}_1$ model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{D}, \mathcal{V}\rangle$, every $seq \in \mathcal{W}^{<\omega}$, $w \in \mathcal{W}$, and every variable assignment g, if $(\mathfrak{M}, seq, w) \models_g \gamma$ for each $\gamma \in \Gamma$, then $(\mathfrak{M}, seq, w) \models_g A$. For convenience, we shall talk of some $\mathcal{L}_1$ formulas (as opposed to their set) entailing another $\mathcal{L}_1$ formula.

As Fritz (2016) shows, we can use the above apparatus to formalise (R1), (R2), and (C) in the enriched modal language $\mathcal{L}_1$, thus avoiding first-order quantification over worlds. Letting 'E' denote the property of being an elementary particle, we write:

(R1*) $\Box \uparrow^1 \Diamond \uparrow^2 (\downarrow^1 \forall x(E(x) \rightarrow \downarrow^2 E(x)) \wedge \exists x(E(x) \wedge \downarrow^1 \neg E(x)))$

(R2*) $\Diamond \uparrow^1 \Box \forall x(E(x) \rightarrow \downarrow^1 E(x))$

Yet (R1*) entails that $\Box \uparrow^1 \Diamond \uparrow^2 \exists x(E(x) \wedge \downarrow^1 \neg E(x))$—informally, this is because both conjuncts of a necessarily possibly true conjunction are necessarily possible. Since $\uparrow^2$ is redundant in this formula it simplifies as follows:

(C*) $\Box \uparrow^1 \Diamond \exists x(E(x) \wedge \downarrow^1 \neg E(x))$

Yet this is not satisfiable with (R2*), in the sense that it cannot be true in any $\mathcal{L}_1$ model in which (R2*) is.

Accordingly, the paradox can be formulated in an appropriately expressive quantified modal language. Thus (R1)/(R1*) and (R2)/(R2*) cannot both be licensed by a single notion of metaphysical necessity. But the grounds for accepting either principle are essentially the same: both principles are motivated by the fact that rejecting them would be to admit extreme arbitrariness into modal space, hence there is no reason why we should reject one over the other. Moreover, flat-footedly rejecting both seems gratuitous.

What, then, should one conclude from the paradox? Fine (2002, p. 224) entertains the idea of rejecting (R2)/(R2*) by imposing a set-theoretic limitation of size constraint on the domains of possible worlds. But since we need not understand the domains of worlds set-theoretically, this constraint is difficult to motivate once again for reasons of arbitrariness. For why would it not be possible that there are more than set many things? Moreover, Fine's proposal is not obviously well-motivated for modal primitivists who reject an extensional reduction of modal discourse to possible worlds talk. In contrast to Fine, Fritz is more pessimistic; he (2016, p. 8) takes the "puzzle [to] at least chip away at the support for the assumption that the notion of metaphysical modality is in good standing", for the assumption that "we have succeeded in singling out a particular notion of metaphysical necessity" is mostly "a dogma of current metaphysics which is supported by the hope that some core theoretical roles of metaphysical necessity [e.g. recombinatorial principles] suffice pick out a particular distinguished modality."[6]

In what follows, however, it is argued that the prospects are not so bleak for advocates of metaphysical modality. Instead, a novel conception of metaphysical modality is offered which does justice to the theoretical reasons motivating (R1)/(R1*) and (R2)/(R2*) without lapsing into paradox.

2.3 Modal Expansionism

2.3.1 Indefinite Extensibility

As a crude gloss, a linguistic expression is *indefinitely extensible* when given all of its instances one can define, by sole reference to those instances, a new instance of

[6]Fritz (2016, pp. 10-11) later considers but does not endorse a solution similar to the one proposed in this paper. Rayo (MS) also develops this idea in an interestingly different way.

the expression that is not included in the collection of the expression's instances from which it was defined.[7] Roughly speaking, the idea is that such expressions exhibit a distinctive kind of open-endedness insofar as one can indefinitely expand their domain. As Russell (1906, p. 36) explains an early form of the idea:

> [T]here are what we may call *self-reproductive* processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect *all* the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property.

The notion of indefinite extensibility is familiar to philosophical discussions of iterative set theory. On the iterative conception of set, from a given universe of sets one can define, by sole reference to the sets in that universe, a new set which is not amongst the sets from which it was defined. For instance, take any universe V of sets. One can define a set which does not belong to V—namely, the set of of all non-self membered sets in V, for on pain of Russell's paradox it cannot belong to V.

Following Linnebo (2010, pp. 146-147), we can state this idea more precisely. Thus, consider the following comprehension schema of plural logic:[8]

Plural Comprehension $\exists xx \forall u(u \prec xx \leftrightarrow \phi(u))$

The schema seems uncontroversial: for a condition ϕ, of course there are the things that ϕ. Yet despite its seeming innocence, from **Plural Comprehension** one can derive a contradiction with the following unrestricted principle of set formation:

Collapse $\forall xx \exists y \forall u(u \prec xx \leftrightarrow u \in y)$

[7]See Gabriel Uzquiano (2015, p. 147). Note that non-predicate expressions, such as quantifiers, which strictly speaking lack 'instances', are often taken to be indefinitely extensible. See Fine (2006), and the other papers in Rayo & Uzquiano (2006).

[8]We assume that the plural quantifier 'there are some things' is to be read as 'there are zero-or-more things'.

The derivation is straightforward. Informally, instantiating ϕ in **Plural Comprehension** with '$\notin u$' gives us the rr, all and only those sets which are not elements of themselves:

(1) $\forall u(u \prec rr \leftrightarrow u \notin u)$

Yet by **Collapse**, we can infer that there is an r such that:

(2) $\forall u(u \prec rr \leftrightarrow u \in r)$

And we can use (1) and (2) to conclude:

(3) $\forall u(u \in r \leftrightarrow u \notin u)$

Yet this generates a contradiction when the universal quantifier is instantiated with r itself.

Given the elegance and power of the comprehension schema, the natural reaction is to reject **Collapse**. However, it is difficult to provide a principled basis for doing so. For instance, one might be tempted to impose a strict constraint on when pluralities form a set, perhaps pertaining to set-theoretic limitation of size. But as Øystein Linnebo (2010, pp. 152-154) and others have emphasised, one can appreciate the arbitrariness of such constraints. In an attempt to avoid such arbitrariness, an alternative is to replace **Collapse** with a slightly weaker but still well-motivated principle. For whereas **Collapse** states that for any things *there is* a set containing all those things, a weaker principle states that for any things *there could be* a set containing all those things. Yet that there merely could be but is not such a set blocks the above derivation.

This approach is characteristic of advocates of indefinite extensibility. They claim that upon introducing the newly defined set, the set-theoretic universe *expands*; when such an expansion occurs, the membership predicate comes to range over a strictly larger set-theoretic universe. And there is something natural about this picture. For although we saw above that the set of all sets in V which do not belong to themselves cannot itself belong to V, it does of course belong to the power set of V—the next rank in the hierarchy. Thus, in introducing this 'Russell set' and forcing an expansion of the set-theoretic universe, the membership predicate comes to range over the more inclusive power set of V. Moreover, one need not think of this picture in constructivist terms. The idea is that the various set-theoretic universes are mind-independent and pre-given. Expansion is not then to be understood in terms of universe construction, but rather in terms of inter-linguistic semantic shift.

For another example, consider the notion of an ordinal number. Due to the Burali-Forti paradox, it is tempting to think that 'ordinal' is indefinitely extensible.[9] A version of this paradox is generated by two principles:

(BF1) The ordinal corresponding to any well-ordering of ordinals is strictly larger than any ordinal in the series.

(BF2) Since the ordinals form a well-ordering, there is an ordinal corresponding to it.

To appreciate the paradox, consider the ordinal specified in (BF2). By (BF1) this ordinal, Ω, is strictly larger than any ordinal in the series mentioned in (BF2). However, Ω is then distinct from all ordinals in that series, which was meant to include all ordinals. Nevertheless, those who claim that 'ordinal' is indefinitely extensible can avoid this contradiction. On their view, in considering the series of ordinals in (BF2), the interpretation of 'ordinal' expands so as to range over the ordinal corresponding to their well-ordering itself. Thus, post-expansion we are no longer considering the series of *all* ordinals after all. Similar to the set-theoretic example, this expansion is not to be understood in terms of our constructing some new ordinal, but again in terms of inter-linguistic semantic shift.

[9]For a recent endorsement of this idea, see Fine (2006, p. 22). Thanks to an anonymous reviewer for emphasising the pertinence of this example, who also notes that the Russell quote cited above follows Russell's discussion of the Burali-Forti paradox.

Strikingly, the principles (BF1) and (BF2) bear a noticeable resemblance to the two recombinatorial principles in the modal paradox, (R1) and (R2). This suggests that positing indefinite extensibility could be key to avoiding the recombinatorial paradox. For example, consider again (R2), the principle which states that for all possible worlds and the elementary particles at those worlds, there is a further possibility at which all those possible elementary particles are elementary particles. It is natural to view this further possibility as introduced by all antecedent possibilities, in a similar sense to which Ω was introduced above, and to which the Russell set above was introduced by all sets in V. Yet just as—on pain of contradiction—the Russell set could not belong to the universe from which it was defined, this newly introduced possibility cannot be amongst those possibilities from which it was introduced. For by (R1) there is no possibility that contains all possible elementary particles.

The claim is that the utterance of (R2) induces an expansion of modal space; in introducing this further possibility, one reinterprets the modal vocabulary in the language to range over a more inclusive modal space. Therefore, this utterance of (R2) is not true under a univocal interpretation of its modal vocabulary. Although according to some interpretation of the metaphysical modal vocabulary all possible elementary particles are all such that according to some more inclusive interpretation possibly they are all elementary particles, it is not the case that according to a given interpretation possibly all possible elementary particles are all elementary particles according to *that* very interpretation. Thus no contradiction results and the paradox dissolves: the moral is that we must appreciate the disastrous consequences of attempting to interpret the modal vocabulary in one of (R1) or (R2) according to a single modal space.[10]

It is worth emphasising that if the order of the premises were changed so that we read (R2) before (R1), we might read the modal vocabulary in (R2) univocally and the modal vocabulary in (R1) non-univocally. On that presentation of the paradox, the utterance of (R1) would induce an expansion of modal space. For

[10]There are interesting questions about the metasemantic mechanisms by which utterances cause shifts in interpretation which are beyond the scope of this paper. For discussion, see Studd (forthcoming, chap. 8) and Warren (2017).

the utterance of (R2) would introduce a world w such that for all possible worlds u all elementary particles at u are elementary particles at w. Yet the utterance of (R1) would then introduce a further possibility, a world where all elementary particles at w are elementary particles and there is something distinct from every elementary particle at w which is also a elementary particle.

Call the view under consideration *modal expansionism*. According to the expansionist, 'metaphysical necessity' and its interdefinable notions are indefinitely extensible. They maintain that the background modal space over which we modally generalise *can* always be expanded, under specific contextual circumstances. Moreover, given the indefinite extensibility of 'metaphysical necessity', there will never be an ultimate interpretation of metaphysical modal operators. For any modal space over which such operators range could always be expanded via the introduction of a new possibility. Accordingly, there is one sense in which metaphysical modality is not the maximal member of the family of objective modalities: the objective modalities can always be expanded to include broader notions of necessity.[11] Nevertheless, it is crucial to emphasise that there are multiple versions of this maximality thesis in the expansionist setting. For instance, the expansionist can still recognise that, holding fixed the interpretation of metaphysical modal vocabulary, metaphysical modality is the maximal objective modality according to *that* particular interpretation. Indeed, the model theory given below could be generalised to handle other objective modalities in a manner that would completely respect this claim. The upshot is that there is still an important sense in which metaphysical necessity is to be equated with the maximal objective modality.

Finally, to close this section, we note that reductionists about metaphysical modality will have to locate the indefinite extensibility in the non-modal locutions to which they reduce metaphysical modal notions. For instance, on Lewis' (1986) concrete modal realist reduction of metaphysical modal notions, since metaphysical modal operators are purged from total theory in favour of first-order quantifiers over worlds, modal expansionism collapses into indefinite extensibility about first-order

[11] This modal claim is articulated with more care in §2.4.

quantification.[12] Nevertheless, outside of the Lewisian setting, modal expansionism and expansionism about first-order quantification are independent proposals.

2.3.2 Variations

In the next section, modal expansionism is developed in further detail. Before we turn to those details, however, it will help to consider a slight variant of the recombinatorial paradox. The benefits of doing so are twofold. First, it will allow the reader to see another application of the expansionist solution, which should aid their understanding of how it functions. Second, it will help to clarify a potential confusion about the expansionist solution.

The relevant variant of the recombinatorial paradox employs irreducible plural quantification.[13] In this paradox, the background plural logic is assumed to be fairly standard, with plural quantifiers admitting an 'empty' plurality so as to validate the **Plural Comprehension** schema discussed above. (Given that the logic is a *modal* plural logic, it validates the schema in the sense that each of the schema's instances are true in each 'world' in each model.) In many respects, this variant is closer to Fine's original version which employs quantification over proper classes, for, as is known from the work of Uzquiano (2003) and others, such quantification can be understood in terms of irreducible plural quantification.

Like the initial paradox, the variant paradox is based on two principles:

(Pl1) Necessarily, the elementary particles are such that possibly they are elementary particles and there is an elementary particle that is not one of them.

(Pl2) Possibly, the elementary particles are such that necessarily, every elementary particle is one of them.

[12] Fine (2006), amongst others, defends the view that first-order quantification is indefinitely extensible.

[13] See Fritz (2016, Appendix B) for discussion of this version of the paradox. He notes that in the presence of certain background model-theoretic conditions, the claims involved in the plurally formulated paradox are model-theoretically equivalent to their corresponding claims in the initial formulation.

The line of reasoning which generates the contradiction should now be familiar. Suppose, as (Pl2) states, that it is possible that the elementary particles comprise every possible elementary particle. Then, in contrast to (Pl1), it is not the case that in every possibility the elementary particles are possibly elementary particles and accompanied by another elementary particle.[14]

To avoid this contradiction, the expansionist claims that the utterance of (Pl2) is not true on a univocal reading of its modal vocabulary. Similar to before, the expansionist will view the possibility in which the elementary particles comprise all possible elementary particles as introduced (in the above sense) from some initial possibilities. Thus, in order to isolate this thought, the modal expansionist can specify their non-univocal reading of this particular utterance of (Pl2):

(Pl2′) According to a given interpretation of the metaphysical modal vocabulary, there is some more inclusive interpretation according to which it is possible that the elementary particles comprise everything which is a possible elementary particle according to the initial interpretation.

Moreover, as with the initial recombinatorial paradox, there is an available non-univocal reading of (Pl1) utterances too. On this reading, an utterance of (Pl1) can be understood as follows:

(Pl1′) According to a given interpretation of the metaphysical modal vocabulary, it is necessary that the elementary particles are such that there is some more inclusive interpretation according to which it is possible that they are all elementary particles and there is an elementary particle that is not one of them.

Thus, although this paradox differs slightly from the initial one, the expansionist offers a similar solution to both: each paradox is to be avoided by recognising the non-univocal readings of the relevant principles.

[14] As an aside, notice that this version of the paradox makes no use of the Vlach devices. Some may take this to be an advantage, since it is at least contentious whether Vlach operators are legitimate resources for modal primitivists to employ; see, for instance, the exchange between Melia (1992) and Forbes (1992). Timothy Williamson (2013, pp. 333-334) also questions whether Vlach operators can be legitimately used by contingentists.

An interesting question concerning this variant paradox is whether it could be avoided by rejecting **Plural Comprehension**. If this schema is rejected, at some worlds there may simply be no things, the elementary particles. And, given that the plural logic admits an 'empty' plurality, this is different from there being a world at which there are the zero-many elementary particles, to which nothing belongs. Instead, the claim would involve the radical idea that there could be an individual elementary particle e_1, an individual elementary particle e_2, and so on, but no *things* the elementary particles. On a natural way of treating plural definite descriptions, this will allow (Pl2) to be trivially true, which could provide a means of avoiding the paradox.[15]

Stephen Yablo (2006, pp. 151-152) has advocated rejecting **Plural Comprehension** as a way of capturing the indefinitely extensibility of 'set'. On his view, this indefinite extensibility results from a lack of determinacy in which sets there are. In turn, this lack of determinacy falsifies the relevant instance of the comprehension schema: that there these things, the sets. Nevertheless, regardless of whether this is viable in the set-theoretic case, it should be uncontroversial that it is not indefinitely extensible what elementary particles there are. As Fritz (2016, p. 551-552) stresses, entities like elementary particles simply do not exhibit the type of iterative behaviour required for indefinite extensibility. Thus, there is no motivation to reject the instances of the comprehension schema that are relevant to the above recombinatorial paradox.

As a minor aside, notice that Yablo's proposal is different to the above view of the indefinite extensibility of 'set' on which **Collapse** was rejected over **Plural Comprehension**. The rejection of **Collapse** instead of **Plural Comprehension** is advocated by Linnebo (2010, p. 156-158). However, Linnebo accepts the previously mentioned modalised version of **Collapse** and *rejects* a suitably modalised version of **Plural Comprehension**. Due to this, Yablo and Linnebo's approaches to the indefinite extensibility of 'set' are very similar. It is worth noting that modal

[15] See Fritz (2016, Appendix B) for the semantic details of this point. Roughly put, his gloss of 'the elementary particles are ψ' is 'for all things, if anything is one of them iff it is an elementary particle, then ψ'.

expansionism is naturally coupled with Linnebo's rejection of the modalised version of **Plural Comprehension**. The following section will specify an interpretation of the modality used in these modalised principles.

This overall point highlights one crucial feature of the expansionist solution. Specifically, it shows that the expansionist is *not* claiming that at each world the domain of elementary particles is open to expansion. Again, that idea must be accompanied with the implausible thought that the elementary particles exhibit indefinite extensibility. Instead, the expansionist claim is that due to the indefinite extensibility of 'metaphysically necessary', it is indefinitely extensible what metaphysically possible elementary particles there are. This latter idea will be developed carefully and implemented in the following section.

2.4 Model Theory

The expansionist conception of metaphysical modality offers an elegant solution to the recombinatorial paradox. But how exactly should we understand the expansionist's modal claim that we *can* define some new possibility? Recall that such modal talk was used to explicate the indefinite extensibility of 'set' too. But in the set-theoretic case, it would have been inappropriate to read the modal claims as expressing metaphysical modality, for it is metaphysically necessary which sets there are. Similarly, the modal expansionist should not explicate the claim that we *can* define some new metaphysical possibility in terms of metaphysical modality either. For if, as is plausible, **S4** is a lower bound on the logic of metaphysical necessity, it is not metaphysically possible that there is some metaphysical possibility which is not metaphysically possible. So how should this new modality be understood?

Given that expansions occur when the metaphysical modal vocabulary is reinterpreted, a natural understanding of the new modality is as an interpretational one.[16] On this view, the modality concerns possibilities for admissibly reinterpreting

[16]Uzquiano (2015) adopts this understanding of the modality used to explicate indefinite extensibility in set theory, and Studd (2013, p. 706) at least entertains it. With this being said, advocates of modal approaches to indefinite extensibility disagree on how to interpret the modality; see Fine (2006) and Linnebo (2010), for alternative readings of the modal.

the metaphysical modal vocabulary. To be possible is to be true according some such interpretation, to be necessary is to be true according to all of them. And by an *admissible* reinterpretation of the metaphysical modal vocabulary we mean one which respects the postulates of metaphysical modality's theoretical role and conforms to the axioms of its logic.[17]

In order to study how interpretational and metaphysical modality interact, we can enrich the vocabulary of our quantified modal language with a new modal operator $\blacksquare_{\geq}$ (letting $\blacklozenge_{\geq}$ abbreviate $\neg\blacksquare_{\geq}\neg$) expressing this particular species of interpretational necessity. We also introduce analogues $(\uparrow^*, \downarrow^*)$ of the infinitely many distinct pairs of Vlach-operators we saw previously to achieve cross-reference across reinterpretations. To be precise, let $\mathcal{L}_2$ be the language which has the exact vocabulary of $\mathcal{L}_1$ plus $\blacksquare_{\geq}$ and infinitely many distinct pairs of Vlach-operators $\uparrow^*$ and $\downarrow^*$.

To study the bimodal logic of interpretational necessity and metaphysical necessity, it will help to proceed model-theoretically; this model theory need only be treated instrumentally, since it serves the merely mathematical purpose of providing a consequence relation for the logic. To define an appropriate class of models, we first define a *frame* $\mathfrak{F}$ as a quadruple $\langle \mathcal{I}, \mathcal{R}, \mathcal{W}^*, \mathcal{D} \rangle$, where $\mathcal{I}$ is a non-empty set understood as a set of admissible interpretations of the metaphysical modal operators of a language, and $\mathcal{R}$ is a binary relation on $\mathcal{I}$, understood as an accessibility relation over interpretations of the metaphysical modal vocabulary. Informally, when $i\mathcal{R}j$ one can think of j as an interpretation of the metaphysical modal vocabulary at least as inclusive as i, in the sense that all metaphysical possibilities recognised by i, and perhaps more, are recognised as metaphysical possibilities by j. Model-theoretically, this is encoded by $\mathcal{W}^*$, which is a total function from $i \in \mathcal{I}$ to a non-empty set $\mathcal{W}^*_i$, thought of as the set of metaphysically possible worlds according to i. We then define:

[17]For simplicity, we shall assume that the logic of metaphysical necessity is **S5**. In the model theory, the definition of the accessibility relations for metaphysical necessity could be altered to validate a weaker logic. See Salmon (1989) for an argument that the logic of metaphysical necessity is weaker than **S4**.

$$\mathcal{W} = \bigcup_{i \in \mathcal{I}} \mathcal{W}_i^*$$

Accordingly, $\mathcal{D}$ is a function taking each $w \in \mathcal{W}$, into a non-empty set $\mathcal{D}_w$, understood as that metaphysically possible world's domain. We impose the following conditions on frames, the reasons for which will become clear in what is to come:

Transitivity $(\forall i \in \mathcal{I})\,(\forall j \in \mathcal{I})\,(\forall l \in \mathcal{I})\,(i\mathcal{R}j \wedge j\mathcal{R}l \rightarrow i\mathcal{R}l)$

Reflexivity $(\forall i \in \mathcal{I})\,(i\mathcal{R}i)$

Directedness $(\forall i \in \mathcal{I})\,(\forall j \in \mathcal{I})\,(\forall l \in \mathcal{I})\,(i\mathcal{R}j \wedge i\mathcal{R}l \rightarrow (\exists m \in \mathcal{I})(j\mathcal{R}m \wedge l\mathcal{R}m))$

Monotonicity $(\forall i \in \mathcal{I})\,(\forall j \in \mathcal{I})\,(i\mathcal{R}j \rightarrow \mathcal{W}_i^* \subseteq \mathcal{W}_{j,}^*)$

Finally, with each $i \in \mathcal{I}$ we associate an equivalence relation:

$$E_i = \{\langle u, v\rangle : u, v \in \mathcal{W}_i^*\} \cup \{\langle u, u\rangle : u \in \mathcal{W}\}$$

Each of these relations can be thought of as the accessibility relation over metaphysically possible worlds associated with that interpretation. They are each reflexive over $\mathcal{W}$ in order to ensure that the logic for metaphysical necessity is **S5**. Note that given **Monotonicity**, if $i\mathcal{R}j$ then $E_i \subseteq E_j$.

An $\mathcal{L}_2$ *model* $\mathfrak{M}$ based on $\mathfrak{F} = \langle \mathcal{I}, \mathcal{R}, \mathcal{W}^*, \mathcal{D}\rangle$ is a quintuple $\langle \mathcal{I}, \mathcal{R}, \mathcal{W}^*, \mathcal{D}, \mathcal{V}\rangle$, where $\mathcal{V}$ is a function taking any individual constant of the language into a member of $\cup_{w \in \mathcal{W}} \mathcal{D}_w$, and any n-place predicate letter of the language, $i \in \mathcal{I}$, and $u \in \mathcal{W}$, into a set of n-tuples of $\cup_{w \in \mathcal{W}} \mathcal{D}_w$, that predicate's extension at that world at that interpretation. Notice that $\mathcal{V}$ takes any 0-place predicate letter, i.e. an atomic sentence, $i \in \mathcal{I}$, and $u \in \mathcal{W}$ into a set of 0-tuples; in this particular case, we think of $\{\langle\rangle\}$ ('T') as truth and $\{\}$ ('F') as falsity. $\mathcal{V}$ is also subject to the following constraint (where ϕ is an $\mathcal{L}_2$ atomic formula):

Stability $(\forall i \in \mathcal{I})(\forall w \in \mathcal{W})(\forall j \in \mathcal{I})(i\mathcal{R}j \rightarrow \mathcal{V}(\phi, i, w){=}\mathcal{V}(\phi, j, w))$

Given the below clauses for the connectives and quantifier, this condition entails that any $\mathcal{L}_2$ formula which contains no occurrences of any modal operators exhibits the same kind of stability.

An assignment g to the variables in $\mathfrak{M}$ is a function which assigns to each individual variable a member of $\cup_{w \in \mathcal{W}} \mathcal{D}_w$. Where x is a variable, two assignments g and g' are x-alternatives iff g and g' assign the same values for all variables, except possibly x.

We define '$(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g A$' for $\mathfrak{M} = \langle \mathcal{I}, \mathcal{R}, \mathcal{W}^*, \mathcal{D}, \mathcal{V} \rangle$, $\overline{seq} \in \mathcal{I}^{<\omega}$, $i \in \mathcal{I}$, $seq \in \mathcal{W}^{<\omega}$, $w \in \mathcal{W}$, g an assignment to the variables in $\mathfrak{M}$, and A a formula of the language, recursively as follows:

$(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g F(a_1, ..., a_n)$ iff $\langle \mathcal{V}_g(a_1), ..., \mathcal{V}_g(a_n) \rangle \in \mathcal{V}(F, i, w)$;
$(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g \neg A$ iff $(\mathfrak{M}, \overline{seq}, i, seq, w) \not\models_g A$;
$(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g A \rightarrow B$ iff either $(\mathfrak{M}, \overline{seq}, i, seq, w) \not\models_g A$
or $(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g B$;
$(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g \forall x A$ iff $(\mathfrak{M}, \overline{seq}, i, seq, w) \models_{g'} A$ for every x-alternative g'
of g, such that $g'(x) \in \mathcal{D}_w$;
$(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g \Box A$ iff for every u such that wE_iu,
$(\mathfrak{M}, \overline{seq}, i, seq, u) \models_g A$
$(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g \uparrow^n A$ iff $(\mathfrak{M}, \overline{seq}, i, seq^{n \rightarrow w}, w) \models_g A$;
$(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g \downarrow^n A$ iff $(\mathfrak{M}, \overline{seq}, i, seq, seq(n)) \models_g A$;
$(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g \blacksquare_{\geq} A$ iff for every $j \in \mathcal{I}$ such that $i\mathcal{R}j$,
$(\mathfrak{M}, \overline{seq}, j, seq, w) \models_g A$;
$(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g \uparrow^{*n} A$ iff $(\mathfrak{M}, \overline{seq}^{n \rightarrow i}, i, seq, w) \models_g A$;
$(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g \downarrow^{*n} A$ iff $(\mathfrak{M}, \overline{seq}, \overline{seq}(n), seq, w) \models_g A$.

A formula A is true at $\langle \overline{seq}, i, seq, w \rangle$, for $\overline{seq} \in \mathcal{I}^{<\omega}$, $i \in \mathcal{I}$, $seq \in \mathcal{W}^{<\omega}$, $w \in \mathcal{W}$

iff for all variable assignments g, $(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g A$. A formula A of $\mathcal{L}_2$ is true in an $\mathcal{L}_2$ model $\mathfrak{M} = \langle \mathcal{I}, \mathcal{R}, \mathcal{W}^*, \mathcal{D}, \mathcal{V} \rangle$ iff it is true at $\langle \overline{seq}, i, seq, w \rangle$, for every $\overline{seq} \in \mathcal{I}^{<\omega}$, for every $i \in \mathcal{I}$, every $seq \in \mathcal{W}^{<\omega}$, and every $w \in \mathcal{W}$. A formula A of $\mathcal{L}_2$ is valid iff it is true in every $\mathcal{L}_2$ model. A set Γ of $\mathcal{L}_2$ formulas entails an $\mathcal{L}_2$ formula A iff for every $\mathcal{L}_2$ model $\mathfrak{M} = \langle \mathcal{I}, \mathcal{R}, \mathcal{W}^*, \mathcal{D}, \mathcal{V} \rangle$, for every $\overline{seq} \in \mathcal{I}^{<\omega}$, every $i \in \mathcal{I}$, every $seq \in \mathcal{W}^{<\omega}$, every $w \in \mathcal{W}$ and every variable assignment g, if $(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g \gamma$ for each $\gamma \in \Gamma$, then $(\mathfrak{M}, \overline{seq}, i, seq, w) \models_g A$. Again, for convenience, we shall talk of some $\mathcal{L}_2$ formulas (as opposed to their set) entailing another $\mathcal{L}_2$ formula.

An **S4.2** axiom system is sound and complete for the class of models in which the accessibility relation is a directed partial order. Thus **Transitivity**, **Reflexivity**, and **Directedness** ensure that $\blacksquare_\geq$ obeys the following axiom schemas, the addition of all instances of which to the minimal normal modal logic yields system **S4.2**:

(T) $\blacksquare_\geq \phi \to \phi$

(4) $\blacksquare_\geq \phi \to \blacksquare_\geq \blacksquare_\geq \phi$

(G) $\blacklozenge_\geq \blacksquare_\geq \phi \to \blacksquare_\geq \blacklozenge_\geq \phi$

It is helpful to notice that the subset relation on the power set of a set is also a directed partial order, since for any members of the power set s_1, s_2, s_3, if $s_1 \subseteq s_2$ and $s_1 \subseteq s_3$ then there is some member s_4 of the power set such that $s_2 \subseteq s_4$ and $s_3 \subseteq s_4$, namely the union of s_2 and s_3. This neatly parallels the idea that the set of metaphysically possible worlds according to an interpretation is a subset of the set of metaphysically possible worlds according to any at least as inclusive interpretation, and thus lends credibility to the above conditions on $\mathcal{R}$. Note however that the analogy is imperfect insofar as the subset relation on the power set of a set has a maximal element.

Axiom schema (4) is equivalent to $\blacklozenge_\geq \blacklozenge_\geq \phi \to \blacklozenge_\geq \phi$. The **Transitivity** condition which validates all instances of the axiom schema thus corresponds to the thought that in moving through interpretations of the metaphysical modal operators, no previously inadmissible meaning assignment becomes admissible. Of course, previously admissible ones will no longer be in the range of $\blacksquare_{\geq}$ as one moves through this process, since they will not be at least as inclusive as the interpretation to which one has moved, hence why the accessibility relation is not symmetric.

James Studd (2013, p. 703), one advocate of a modal approach to the indefinite extensibility of set-theoretic expressions, requires the following frame condition which entails **Directedness**.

Connectedness $(\forall i \in \mathcal{I})(\forall j \in \mathcal{I})(\forall l \in \mathcal{I})(i\mathcal{R}j \wedge i\mathcal{R}l \to j\mathcal{R}l \vee l\mathcal{R}j)$

In the current setting, this has the effect of validating all instances of the following schema:

(D1) $\blacksquare_{\geq}(\blacksquare_{\geq}\phi \to \psi) \vee \blacksquare_{\geq}(\blacksquare_{\geq}\psi \to \phi)$

However, it would be too restrictive to impose **Connectedness** for current purposes. To see why informally, suppose that at interpretation i there are k metaphysically possible elementary particles and k metaphysically possible spacetime points. One could run through the initial recombinatorial puzzle with which we began to shift to an interpretation, j, at which there are m (for $m > k$) metaphysically possible elementary particles and k metaphysically possible spacetime points.[18] But, at i, one could also run through a version of the initial recombinatorial puzzle in which all instances of 'elementary particle' are replaced with 'spacetime point' to shift to an interpretation, l, at which there are m metaphysically possible spacetime points and k metaphysically possible elementary particles. But it is not the case that j is at least as inclusive as l, or vice versa. The pair thus serves as a counterexample to **Connectedness**.

[18]Of course, this makes the highly plausible assumption that the broad recombinatorial reasoning employed previously guarantees there will be a more inclusive interpretation according to which there are additional metaphysically possible elementary particles despite there being no additional metaphysically possible spacetime points (and vice-versa).

Imposing the **Stability** condition validates the following schema, where ϕ is a formula of the language containing no occurrences of any modal operator:

(ST) $\blacksquare_{\geq}\phi \vee \blacksquare_{\geq}\neg\phi$

This is desirable, since what is non-modally the case at a world should not be affected by the expansion of metaphysical modal space. Additionally, **Monotonicity** has the effect of validating a bimodal converse Barcan schema:

(BCBF) $\blacksquare_{\geq}\Box\forall v\phi \rightarrow \forall v\blacksquare_{\geq}\Box\phi$

In other terms, it has the effect of ensuring that if according to all at least as inclusive interpretations of the metaphysical modal vocabulary it is metaphysically necessary that everything ϕs, then everything is such that according to all at least as inclusive interpretations it is metaphysically necessary that it ϕs. In slogan form: the possibilia may increase, but none are left behind.

It is natural to enquire about the status of other bimodal principles. One might first wonder whether the modalities commute. The following principle is $\mathcal{L}_2$-valid:

(C1) $\blacksquare_{\geq}\Box\phi \;\rightarrow\; \Box\blacksquare_{\geq}\phi$

Proof.

Suppose that $(\mathfrak{M}, \overline{seq}, i, seq, w) \models \blacksquare_{\geq}\Box\phi$, and $(\mathfrak{M}, \overline{seq}, i, seq, w) \not\models \Box\blacksquare_{\geq}\phi$. From the second conjunct, it follows that $(\mathfrak{M}, \overline{seq}, i, seq, u_2) \not\models \blacksquare_{\geq}\phi$ for some u_2 such that wE_iu_2. Therefore $(\mathfrak{M}, \overline{seq}, j, seq, u_2) \not\models \phi$ for some $j \in \mathcal{I}$ such that $i\mathcal{R}j$. But from the first conjunct, it follows that $(\mathfrak{M}, \overline{seq}, l, seq, w) \models \Box\phi$ for all $l \in \mathcal{I}$ such that $i\mathcal{R}l$. This includes j, so $(\mathfrak{M}, \overline{seq}, j, seq, w) \models \Box\phi$. It then follows that $(\mathfrak{M}, \overline{seq}, j, seq, u_1) \models \phi$ for all u_1 such that wE_ju_1. Yet wE_ju_2 because $E_i \subseteq E_j$ and wE_iu_2. Contradiction.

Plausibly, this is the right result: if for any way of more liberally interpreting metaphysical modal operators it is metaphysically necessary that ϕ, then it is metaphysically necessary that for any way of more liberally interpreting metaphysical modal operators, ϕ. In part, this reflects the fact that the admissible interpretations do not change from metaphysically possible world to world when the interpretation is held fixed. However, (C1)'s converse is not $\mathcal{L}_2$-valid:

(C2) $\Box\blacksquare_{\geq}\phi \ \rightarrow\ \blacksquare_{\geq}\Box\phi$

Proof.
Let $\mathfrak{M}$ be an $\mathcal{L}_2$ model such that:

$$\mathcal{I} = \{0, 1\}$$
$$\mathcal{R} = \{\langle 0, 1\rangle, \langle 0, 0\rangle, \langle 1, 1\rangle\}$$
$$\mathcal{W}_0^* = \{2\}$$
$$\mathcal{W}_1^* = \{2, 3\}$$
$$\mathcal{V}(\phi, i, w) = \begin{cases} T, & \forall i \in \mathcal{I}, \text{if}\, \phi = A, w = 2 \\ F, & \text{otherwise} \end{cases}$$
for ϕ an atomic sentence of $\mathcal{L}_2$

Thus for atomic 'A', $(\mathfrak{M}, \overline{seq}, 0, seq, 2) \models A$, $(\mathfrak{M}, \overline{seq}, 1, seq, 2) \models A$ and $(\mathfrak{M}, \overline{seq}, 1, seq, 3) \not\models A$, for arbitrary $\overline{seq}$ and seq. It is easily checked that $(\mathfrak{M}, \overline{seq}, 0, seq, 2) \not\models \Box\blacksquare_{\geq}A \rightarrow \blacksquare_{\geq}\Box A$.

For an informal counterexample to (C2), suppose that at i there is no metaphysically possible world at which there are at least k elementary particles. Let 'A' be the non-modal sentence that there are fewer than k elementary particles. At i, $\Box\blacksquare_{\geq}A$, for since 'A' contains no modal vocabulary the $\blacksquare_{\geq}$ operator is redundant, yet $\Box A$ since at i it is metaphysically necessary that there are fewer than k elementary particles. However, at i it is not the case that $\blacksquare_{\geq}\Box A$. For according to some more inclusive interpretation of the language's metaphysical modal terms, there is a metaphysically possible world that contains more than k elementary particles. So at i it is false that according to every at least as inclusive interpretation of the language's metaphysical modal terms there is a metaphysical modal maximum of k elementary particles.

It is straightforward to check that the principles $\Box\phi \rightarrow \blacksquare_{\geq}\phi$ and $\blacksquare_{\geq}\phi \rightarrow \Box\phi$ fail too. Yet despite neither modality implying the other, certain bimodal formulae of $\mathcal{L}_2$ are of distinctive theoretical interest, such as those concerning which metaphysical modal facts hold according to all interpretations. For example, as noted above, since each E_i is an equivalence relation, at every interpretation the theorems of **S5** hold for $\Box$. Thus suitably interpretationally-modalized versions of the theorems of **S5** for $\Box$ will also hold. Such formulae will concern the metaphysical modal facts whatever the interpretation of the metaphysical modal operators, not merely the metaphysical modal facts according to some particular interpretation.[19] But note that merely prefixing a $\blacksquare_{\geq}$ operator to each theorem of **S5** will not suffice to express such facts. For the resulting formula would concern only all at least as inclusive admissible interpretations of the metaphysical modal operators, and not *all* such interpretations. Thus we can introduce into the language an operator $\blacksquare$ as follows, where R^+ is the transitive closure of $R \cup R^{-1}$:

$$(\mathcal{M}, \overline{seq}, i, seq, w) \models \blacksquare A \text{ iff for each } j \in \mathcal{I} \text{ such that } iR^+j,$$
$$(\mathcal{M}, \overline{seq}, j, seq, w) \models A$$

R^+ is an equivalence relation, so $\blacksquare$ obeys an **S5** modal logic.

The $\blacksquare$ operator allows one to state absolutely general truths about metaphysical modal facts whatever the interpretation of the metaphysical modal operators. For example, take an instance of the **S5** axiom schema:

(S5_1) $\Diamond A \rightarrow \Box\Diamond A$

This is $\blacksquare$-modalized as follows:

(S5_2) $\blacksquare(\Diamond A \rightarrow \Box\Diamond A)$

[19]Compare Uzquiano (2015, p. 155) on modalisations of the axioms of set theory.

(S5$_2$) is valid, since $\blacksquare$ conforms to the rule of necessitation (if $\models \phi$ then $\models \blacksquare\phi$), and as previously mentioned (S5$_1$) is valid since each E_i is an equivalence relation. Thus, at any interpretation, object-language principles prefixed with $\blacksquare$ allow us to state absolutely general truths about the structure of metaphysical necessity at every interpretation.

A simple, but crucial, point to notice is that the model theory allows for the extension of a predicate such as 'metaphysically possible elementary particle' at a particular metaphysically possible world to vary from interpretation to interpretation; specifically, it may expand as one shifts to a more inclusive interpretation.[20] Indeed, we can utilise this feature to formalise an analogue of (R2*) which is true, speaks to the concerns of not imposing arbitrariness which motivate (R2), and from which no paradox results:[21]

$$(\blacksquare\text{R2})\ \uparrow^{*1}\uparrow^{1} \blacklozenge_{\geq} \uparrow^{*2} \Diamond \uparrow^{2}\downarrow^{*1}\downarrow^{1} \Box\forall x(E(x) \rightarrow\downarrow^{*2}\downarrow^{2} E(x))$$

($\blacksquare$R2) says that according to an interpretation, there is some at least as inclusive interpretation of the language such that according to the latter interpretation it is metaphysically possible that everything which is a elementary particle at any metaphysically possible world at the initial interpretation is a elementary particle.

Similarly, with this formal apparatus, one can now formalise an analogue of (R1*) which is true, speaks to the concerns of not imposing arbitrariness with motivate (R1), and from which no paradox results:

$$(\blacksquare\text{R1})\ \uparrow^{*1} \Box \uparrow^{1} \blacklozenge_{\geq} \uparrow^{*2} \Diamond \uparrow^{2} (\downarrow^{*1}\downarrow^{1} \forall x(E(x) \rightarrow\downarrow^{*2}\downarrow^{2} E(x)) \wedge \exists x(E(x) \wedge \downarrow^{*1}\downarrow^{1} \neg E(x)))$$

[20] Strictly speaking, the formal language does not contain complex predicates like 'metaphysically possible elementary particle', but one can think of this point in terms of the satisfaction of open formulae.

[21] The advocate of indefinitely extensible metaphysical modality can maintain that (R2)—a sentence of English—is implicitly interpretational modal and thus true. But since the formal quantified modal language of which (R2*) was a formula did not contain interpretational modal operators (R2*) cannot be implicitly interpretationally modal, and is therefore false.

(■R1) says that according to an interpretation there is some at least as inclusive interpretation at which it is metaphysically possible that everything which is a elementary particle at any metaphysically possible world at the initial interpretation is a elementary particle, and there is some elementary particle which is not a elementary particle at any metaphysically possible world at the initial interpretation.

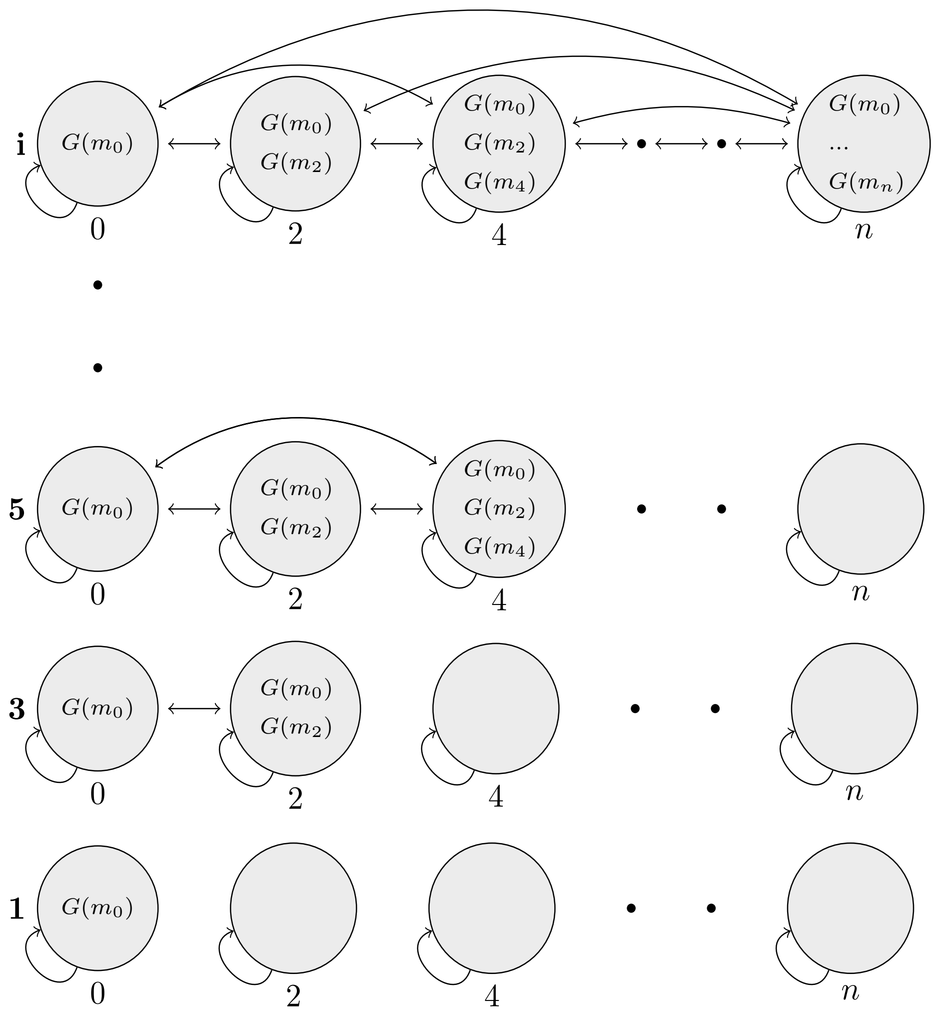
Moreover, unlike (R1*) and (R2*), the replacement principles (■R1) and (■R2) are satisfiable, in the sense that we can construct $\mathcal{L}_2$ models in which they are both true. One class of such models are those which reflect the informal idea that at each interpretation there is a world w which contains all possible elementary particles at that interpretation, but at a more inclusive interpretation there is an additional world u which contains all elementary particles at w and a elementary particle distinct from all those at w:

Let $\mathfrak{M}$ be an $\mathcal{L}_2$-model such that:

$\mathcal{I} = \{x : x \text{ is an odd natural number}\}$
$\mathcal{R} = \{\langle x, y\rangle : x \text{ is less than or equal to } y\}$
$\mathcal{W}_i^* = \{x < i : x \text{ is an even natural number}\}$
$\mathcal{D}_w = \{1/n : n \text{ a non-zero even natural number}\}$ (for all $w \in \mathcal{W}$)

$$\mathcal{V}(\phi, i, w) = \begin{cases} \{1/2, ..., 1/w+2\}, & \text{when}\, \phi = G \,\text{and}\, i > w \\ \{\}, & \text{otherwise} \end{cases}$$

For ease of presentation, let us label 1/n 'm_{n-2}', so that 1/2 is 'm_0', 1/4 is 'm_2' and so on. We can then depict the model as follows:



One can check that (■R1) and (■R2) are both true in $\mathfrak{M}$.

2.5 Revenge

One might wonder whether the modality expressed via the string of modal operators ■□ can supplant metaphysical necessity in various respects. However, this issue is not at all straightforward: if the modality ■□ is too similar to (our previous conception of) metaphysical necessity, revenge paradoxes threaten. In this section, we explore this issue and the various ways in which ■□ might differ from (our previous conception of) metaphysical necessity.

It will help first to consider a similar recombinatorial paradox against the modality expressed by ■□. Call this 'necessity*', and ⧫◇ 'possibility*', and consider the following argument:

(R1′) For any possible* world w, there is a possible* world u where all elementary particles at w are elementary particles and there is something distinct from every elementary particle at w which is also a elementary particle.

(R2′) There is some possible* world w such that for all possible* worlds u all elementary particles at u are elementary particles at w.

Yet considering w in (R2′), from (R1′) we can infer that:

(C′) There is a possible* world u where all elementary particles at w are elementary particles and there is also a elementary particle who is not identical with any of them.

But all the elementary particles at w are all the elementary particles at all possible* worlds. Contradiction. Is this argument sound?

(R1′) is licensed by the modal expansionist solution to the initial recombinatorial paradox. For according to any interpretation there is some more inclusive interpretation according to which for every metaphysically possible world w at the previous interpretation, there is a metaphysically possible world at which every elementary particle at w is a elementary particle and there is a elementary particle distinct from any elementary particle at w.

However, why believe (R2′)? That is, why believe that at an interpretation there is some metaphysically possible world at which all possible* elementary particles—all elementary particles at any metaphysically possible world at any interpretation—are elementary particles? This claim seems antithetical to modal expansionism. For modal expansionists maintain that given any metaphysical modal space one can shift the interpretation to a more inclusive space which contains additional metaphysical possibilities. Such possibilities will include the possibility of there being all possible elementary particles in the initial modal space accompanied by an extra elementary particle. Yet if (R2′) were true, then at some interpretation i, it would be metaphysically possible that all possible* elementary particles—that is, all metaphysically possible elementary particles according to all interpretations—are elementary particles. But the modal expansionist's point throughout has been that at an interpretation i not everything which is a metaphysically possible ϕ according to some more inclusive interpretation is a metaphysically possible ϕ at i.

The crucial point is that one should not expect the recombinatorial principles which govern metaphysical modality to have $\blacksquare\Box$ analogues. In particular, and in contrast to metaphysical modality, the expansionist has introduced nothing to the theoretical role of $\blacksquare\Box$ which licenses the thought that one should be able to collect all $\blacksquare\Box$-possibilities of a certain kind into a single $\blacksquare\Box$-possibility. Compare indefinite extensibility in the set-theoretic case: there is no interpretation of set-theoretic vocabulary according to which every interpretationally possible set is a set.

Nevertheless, this is a negotiable decision point. If one is concerned that the modal expansionist is guilty of imposing arbitrariness onto $\blacksquare\Box$-modal space as result of rejecting (R2′), one can respond more concessively. For one might be tempted to claim that the modality expressed by $\blacksquare\Box$, necessity*, is itself indefinitely extensible. Expansionists of this disposition could extend their treatment of the initial paradox to the revenge version. Of course, this thesis of indefinite extensibility would too have to be explicated modally in terms of a different interpretational modality $\blacksquare^2$ (we relabel the original $\blacksquare$ as $\blacksquare^1$). And similar revenge paradoxes would threaten the modality expressed by $\blacksquare^2\blacksquare^1\Box$. Thus the expansionist would have to iterate the idea that interpretational modalities are indefinitely extensible. Any paradox involving the modality expressed by $\blacksquare^n, \blacksquare^{n-1}, ..., \blacksquare^1\Box$ would be handled by a introducing a new interpretational modality $\blacksquare^{n+1}$. Moreover, on natural ways of developing this proposal, further modalities would have to be introduced at limit states in a typical way: $\blacksquare^\omega\phi \leftrightarrow \bigwedge_{n<\omega} \blacksquare^n\phi$. In turn, these limit modalities would generate further iterations and other limit modalities. However, since it is plausible that 'ordinal' is itself indefinitely extensible, there will be no modality expressed by $\blacksquare^\alpha\phi$, for every interpretationally possible ordinal α.[22]

[22] This final use of 'interpretational possibility' should be understood as invoking the Studd-Linnebo interpretational modality used to explicate the indefinite extensibility of 'set'.

On the resultant view, for any interpretation i of $\blacksquare\Box$ there will be some more inclusive interpretation j of $\blacksquare\Box$ such that the modality expressed by $\blacksquare\Box$ at i is strictly less inclusive than the modality expressed by $\blacksquare\Box$ at j. It is important to emphasise the costs of this response. If the indefinite extensibility of modality is iterated as such, one must forfeit absolute metaphysical modal generality. But if the choice is between imposing arbitrariness on metaphysical modal space and recognising our own expressive limitations, perhaps humility seems preferable to imposition.

2.6 Further Application

Modal expansionism enjoys further benefits. The literature on metaphysical modality contains a wealth of other recombinatorial puzzles and paradoxes which are typically presented as attacks on the notion of metaphysical necessity, or some particular theory thereof. Unlike the recombinatorial paradox considered above, these arguments employ auxiliary machinery such as certain set-theoretic principles or certain conceptions of propositions. In response to these other puzzles and paradoxes, typically philosophers have suggested modifying the auxiliary machinery which they employ. However, these responses are parochial and fail to provide a solution which generalises to other modal recombinatorial paradoxes. In contrast, a benefit of the modal expansionist solution is that it can offer a unified solution to all such paradoxes. This greater generality further motivates the modal expansionist view.

In this section, we shall consider two well-known recombinatorial paradoxes: David Kaplan's (1995) 'Kaplan's paradox', and argument due to Daniel Nolan (1996) which was later developed by Ted Sider (2009).[23] We shall see how modal expansionist treats each paradox, and, along the way, consider how the expansionist approach compares to some other extant solutions. The discussion will highlight aspects of the expansionist approach which may have gone unnoticed.

[23]Kaplan (1995, p. 41, f.n.1) remarks that he first presented the paradox during the 1970s.

2.6.1 Kaplan's Paradox

Kaplan's paradox runs as follows. Suppose the set of metaphysically possible worlds is of cardinality k. According to coarse-grained views of propositions that individuate propositions by a criterion of necessary equivalence, each subset of this set is the proposition true at all and only the metaphysically possible worlds belonging to that subset. Hence by the power set operation, there are 2^k propositions. But consider a person and a time. For any proposition, it is metaphysically possible that it is entertained as the only content that person entertains at that time. Call this a proposition's being *queried*. There are therefore distinct metaphysically possible worlds corresponding to the occurrence of each query. It follows that there are at least 2^k worlds. Contradiction.

Several solutions to the paradox have been offered. Kaplan (1995) himself takes the paradox to demonstrate that we ought to adopt a ramified conception of propositions (and spaces of worlds), and David Lewis (1986, pp. 107-108) draws the moral that some propositions are not in principle entertainable as mental contents. But, as emphasised previously, whatever the merits of such solutions, they lack generality. Clearly, Kaplan's paradox invokes a recombinatorial principle: the premise that for any proposition determined by a metaphysical modal space there is a metaphysically possible world in that space where that proposition is uniquely entertained. Yet given the alleged paradoxes to which similar recombinatorial premises give rise, it would be sensible to expect this particular principle to do so too.

Indeed, the modal expansionist blames the recombinatorial principle and not the particular theory of propositions or mental content. After all, there would appear to be little reason to complicate one's theory of propositions or mental content if one already recognises the tendency of recombinatorial principles to induce expansion. Thus, the expansionist will offer their familiar diagnosis of the paradox. On their view, the recombinatorial principle about the querying of the propositions which a metaphysical modal space determines is only true when its modal vocabulary is read non-univocally. The true reading in question is as follows:

(KP) According to some interpretation, for any proposition determined by the

> metaphysical modal space at that interpretation, there is some more inclusive interpretation according to which there is a metaphysically possible world where that proposition is uniquely entertained (i.e. queried).

Thus, given some metaphysical modal space of cardinality k, there is some more inclusive metaphysical modal space of cardinality 2^k at which all propositions determined by the previous metaphysical modal space are metaphysically possibly queried. Yet no contradiction results from this.

At this point, will be fruitful to compare the expansionist approach to Kaplan's ramificational response to the paradox. Although, as mentioned above, Kaplan's approach is less general than expansionism, it raises some interesting aspects of expansionism which previously may not have been salient.

The basic idea behind Kaplan's ramificational approach is that the space of possibilities is partitioned into *levels* indexed by the natural numbers, and this partition induces a further partition on the space of propositions. As Kaplan (1995, p. 45) evocatively describes it:

> At level 0, there [are] only the possibilities of, say, different distributions of earth, air, fire, and water. Now [at level 1] there are different distributions of earth, air, fire, and water with it being queried whether all is earth, and there are the same distributions with it not being queried whether all is earth. These new possibilities provide new propositions to query—for example, whether it is queried whether it is queried whether all is earth.

On Kaplan's view, then, a level n space of possibilities generates a corresponding level n space of propositions, propositions about level n possibilities. Then, those level n propositions can be queried at level $n+1$ possibilities. Thus, no level n proposition is ever queried at a level n possibility, which allows one to resist the reasoning in Kaplan's paradox.

Clearly, the ramificational approach is similar to expansionist approach. Advocates of the former partition the space of possibilities into successively more inclusive levels, and advocates of the latter claim that any space of possibilities can be expanded to a more inclusive space. More generally, although the exact relation between ramification and indefinite extensibility is vexed, it is natural to view the former as something of a precursor to the latter. (This point is not confined to their applications to metaphysical modality.) However, one must appreciate that—as described by Kaplan—the ramificational approach does not have the capacity to solve the initial recombinatorial puzzle of the elementary particles. For, presumably, matters concerning elementary particles are matters of earth, air, fire, and water, so to speak. And, as Kaplan himself emphasises, no possibilities only about *new* distributions of earth, air, fire, and water are introduced at level 1. Instead, at level 1, there are the same distributions of earth, air, fire, and water along with various level 0 propositions being either queried or not queried.

With this being said, Kaplan's discussion of the ramificational approach raises an interesting issue for expansionists. For given the similarity of both approaches, one might expect that modal expansionism is naturally coupled with an expansionist conception of propositions, similar to how ramifying the space of possibilities induced a ramification of propositions.[24] Indeed, if propositions are identified by a criterion of metaphysically necessary equivalence, an expansionist conception of propositions is simply a corollary of modal expansionism. Whether this is borne out on alternative conceptions of propositions deserves further discussion, although here it would take us too far afield.

2.6.2 Lewis & Williamson

Building on Daniel Nolan's (1996) work, Ted Sider (2009) has argued that both David Lewis' concrete modal realism and Timothy Williamson's first-order necessitism are incompatible with the Urelement Set Axiom. Once again, a recombinatorial principle features centrally in each respective argument:

Moderate Plenitude For each cardinal number v, it is metaphysically possible that there exist v non-sets.

Assuming concrete modal realism, **Moderate Plenitude** entails that there cannot be a set of all non-sets, thus violating the Urelement Set Axiom. For suppose

[24]Yu (2017) discusses what is naturally classified as an expansionist conception of propositions.

that there were a set S of all non-sets. Let its cardinality be k. Let v be some cardinal number larger than k. By **Moderate Plenitude** it is metaphysically possible that there exist v non-sets. Thus by Lewis' concrete modal realism there is a metaphysically possible world containing v non-sets, all of which are members of S. So S's cardinality is not k. Contradiction.

Similarly, first-order necessitism entails that there cannot be a set of all non-sets, again in violation of the Urelement Set Axiom. For suppose that there were a set S of all non-sets. Let its cardinality be k. Let v be some cardinal number larger than k. By Moderate Plenitude it is metaphysically possible that there exist v non-sets. Since the first-order necessitist holds that everything which metaphysically possibly is something is actually something, by **Moderate Plenitude** there are actually v non-sets, all of which are members of S. (This of course assumes that metaphysically possible non-sets are necessarily as such, if they exist.) So S's cardinality is not k. Contradiction.

It may be helpful to notice that Lewisian concrete modal realists are naturally categorised as first-order necessitists.[25] For when the quantifiers are read unrestrictedly, the Lewisian concrete modal realist will endorse the necessitist claim that necessarily everything is necessarily something. Moreover, the above recombinatorial arguments exploit the fact that both Lewisians and Williamson subscribe to this necessitist principle. Seen from this perspective, the argument is a direct objection to first-order necessitism.

Is is tempting to conclude that the objection shows that first-order necessitists must reject the Urelement Set Axiom, and perhaps the iterative conception of set which some take to motivate the axiom.[26] But in an expansionist setting, necessitists need not reject the set-theoretic principle. As one would anticipate, the expansionist rejects **Moderate Plenitude** but endorses a nearby, qualified principle:

Moderate Plenitude* For each cardinal number v, according to some at least as inclusive interpretation it is metaphysically possible that there exist v non-sets.

[25]See, for instance, Williamson (2013, p. 17) and Divers (2014) on this point.
[26]See Sider (2009, p. 256) in particular for this latter point.

This replacement principle speaks to whatever theoretical considerations initially led one to endorse **Moderate Plenitude**, and is compatible with the Urelement Set Axiom on both Lewis' concrete modal realism and Williamson's first-order necessitism. The moral is that when coupled with expansionism, the first-order necessitist is not committed to rejecting any principles of set theory by the recombinatorial argument.

To close, this final discussion suggests a broader project of integrating modal expansionism with a view on which 'set' is itself indefinitely extensible. Such an integrated view may have the potential to handle the modal set-theoretic recombinatorial paradoxes discussed in Hawthorne & Uzquiano (2011), although this investigation must be left for future work.

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3

Essence, Tolerance, and Paradox

Abstract

Metaphysical necessity is typically assumed to have a determinate extension. However, if the extension of metaphysical necessity is fixed by its theoretical role this assumption is far from obvious, for in certain respects the theoretical role of metaphysical necessity is noticeably thin. This paper lifts the determinacy assumption and explores the resultant theoretical terrain. The methodology will be to proceed via a case study of the so-called paradoxes of essentialism. However, in view of that study, the paper will also explore the broader ramifications of the view that metaphysical necessity is a source of indeterminacy.

3.1 Introduction

Metaphysical necessity is typically assumed to have a determinate extension.[1] Kripke's (1980, p. 99) characterisation of metaphysical necessity as 'necessity in the highest degree' may be partly responsible for this assumption, for the characterisation suggests a unique modality. Moreover, Kripke's characterisation has remained central to our conception of metaphysical necessity. For example, in line with Kripke, Stalnaker (2003, p. 202) speaks of metaphysical necessity as 'necessity in the widest sense', and Williamson (2016) of it as the 'maximal objective modality'. In contrast, the litany of ideas philosophers associate with metaphysical necessity suggests a less determinate notion. As Ted Sider (2011, p. 266) writes:[2]

> You may have noticed that the philosopher's conception of metaphysical necessity is frustratingly thin. We are told that logic and mathematics are metaphysically necessary. We are usually told that laws of nature are not. We are told that it is metaphysically necessary that 'nothing can be in two places at once', and so on. This conception falls far short of a full criterion.

Consonant with this, Bacon (2018, chap. 15), Clarke-Doane (2017), Litland & Yli-Vakurri (2016), Nolan (2011, p. 323), Rosen (2006, p. 17), and Torza (2015) have all either argued for or seriously entertained the hypothesis that metaphysical necessity is a source of indeterminacy.[3] Indeed, Kripke himself (1980, p. 115, n. 57; his emphasis) appears to recognise the hypothesis too:

> Just as the question whether an object *actually* has a certain property (e.g. baldness) can be vague, so the question whether the object essentially has a certain property can be vague, even when the question whether it actually has the property is decided.

Given these endorsements, the hypothesis is surely live.

[1]To improve readability, use/mention distinctions will be left implicit throughout.

[2]Sider himself does not state whether he thinks that metaphysical necessity is a source of indeterminacy, although his neo-conventionalist view of modality is naturally coupled with the idea. Sider does, however, note that a notion should not be deemed as 'non-fundamental' purely due to its having a thin theoretical role.

[3]Bacon (2018, chap. 15) entertains the hypothesis in response to one of the paradoxes considered below; the hypothesis is also naturally suggested by Williamson's (1990, chap. 8) discussion of the paradoxes. Rosen (2006, p. 17) compares the notion to Field's (1974) example of the referentially indeterminate Newtonian notion of 'mass'.

Motivated by the above, this paper investigates the hypothesis that metaphysical necessity and its cognate notions are sources of indeterminacy. The aim will be to lift the assumption that metaphysical necessity has a determinate extension and explore the resultant theoretical terrain. To constrain the investigation, it will proceed via a case study of the so-called paradoxes of essentialism. However, in view of that study, the paper will also explore the broader ramifications of the view that metaphysical necessity is a source of indeterminacy.

In further detail, the structure of the paper is as follows. §3.2 considers the so-called paradoxes of essentialism, and charts the surrounding logical geography. Following suggestions from Graham Forbes (1984), Timothy Williamson (1990, chap. 8) and Andrew Bacon (2018, chap. 15), §3.3 then explores the paradoxes under the hypothesis that metaphysical necessity is a source of indeterminacy. It will be argued that the indeterminacy hypothesis permits an elegant treatment of the paradoxes. §3.4 then critically discusses Nathan Salmon's (1986; 1989) famous solution to the paradoxes, and a recent solution based on two-dimensionalist semantics initially suggested by Adam Murray and Jessica Wilson (2012).[4] The point of this critical discussion is to show that advocates of these prominent solutions ought still to take the indeterminacy hypothesis seriously, for varying reasons. The final section, §3.5, draws lessons from §3.2-3 about the theoretical role of metaphysical necessity.

3.1.1 Motivation

Before the main discussion, it is worth clarifying why it is far from obvious that the theoretical role of metaphysical necessity secures the notion a determinate extension.

In most areas of inquiry theoretical posits are introduced and characterised by their role within broader theory. Typically, a posit's role consists of various *postulates* offering constraints on its intended profile, and a series of *connections*

[4]Sarah-Jane Leslie (2011) and John Hawthorne & Juhani Yli-Vakkuri (MS) have developed a promising treatment of the essentialist paradoxes based on an antecedent commitment to a plenitudinous ontology. For reasons of space, this treatment will not be discussed; what follows can be viewed as a treatment of the essentialist paradoxes for those who do not endorse a plenitudinous ontology.

it is expected to bear to other, perhaps better received notions. The history of science abounds with examples. Dmitri Mendeleev's 1869 periodic arrangement of the elements according to their atomic mass predicted an element in the carbon group, between silicon and tin, whose atomic weight was approximately 72. That element was later discovered to be germanium. In this case, the atomic weight of 72 served as a constraint on germanium's intended profile, and germanium's position in the periodic table charted a series of connections which it was expected to bear to other, already discovered elements. Prior to the empirical discovery of germanium, Mendeleev introduced 'ekasilicon' as a provisional name for the element.[5] Plausibly, that name enjoyed determinate reference at the time. Mendeleev's role provided a semantic constraint that sufficed for the expression to select the unique chemical element which fitted the expression's intended profile.

Sometimes, however, clear theoretical roles, with elegant postulates and robust connections, fail to secure determinate reference. As Hartry Field (1973) reminds us, despite Newton's careful attempt to specify a theoretical role for 'mass', there is no one magnitude which best fits the role of Newtonian mass. From Special Relativity, we know that there are two different magnitudes with an equal claim to fit the role of Newtonian mass: relativistic mass (total energy/c^2), and proper mass (non-kinetic energy/c^2). Although neither of these magnitudes perfectly fits the role of Newtonian mass, both do to a convincing extent. Thus, Field argued influentially, Newton's use of 'mass' was referentially indeterminate between relativistic and proper mass.

What separates the referential success of Mendeleev's 'ekasilicon' from the misfortune of Newton's 'mass'? As indicated above, the difference is a type of *metasemantic luck*. Both notions had clear, carefully articulated roles which were embedded in a broader theory. However, unfortunately for Newton, the world was unkind to him in response: unlike germanium, there was just no unique magnitude that satisfied his intended role for mass.

In metasemantic regards, philosophy is not radically different to the empirical sciences. Both disciplines characterise and introduce posits as above, and both are

[5]Derived from the Sanskrit name of 1, 'ekam', 'ekasilicon' connotes being one place down from silicon in the periodic table.

thus hostage to the same vicissitudes of metasemantic luck.[6] Even if philosophical posits have clear, detailed theoretical roles, the danger of indeterminacy threatens. Absent a local means of ensuring that there is no indeterminacy, it ought to be a live hypothesis that certain philosophical notions exhibit it to some degree. Of course, the theoretical role of metaphysical necessity is no clearer than Newton's role for mass. Moreover, unlike the case of germanium, there is no obvious local means of ensuring that metaphysical necessity is determinate. Thus, it ought to be a live hypothesis that metaphysical necessity is indeterminate. However, such indeterminacy does not render the modality unworthy of study. As with Newtonian mass, appreciating its indeterminacy may lead to new insight; indeed, such is the aim of this paper.

3.2 A Family of Paradoxes?

Essence is not an utterly liberal notion: typically, an object's modal profile does not permit total variation in how that object may be.[7] For example, one notable essentialist constraint is the idea that no material object could have originated from material totally different to that from which it actually does. As Kripke (1980, p. 113; final emphasis mine) wrote in his now influential discussion:

> How could a person originating from different parents, from a totally different sperm and egg, be *this very woman*? One can imagine, *given* the woman [Queen Elizabeth II], that various things in her life could have changed: that she should have become a pauper; that her royal blood should have been unknown, and so on. [...] This seems to be possible. And so it's possible that even though she were born of these parents she never became queen. Even though she were born of these parents, like Mark Twain's character [in The Prince and The Pauper] she was switched off with another girl. But what is harder to imagine is her being born of different parents. *It seems to me that anything coming from a different origin would not be this object.*

Let us call this idea **Kripke's Thesis**:

[6] This view of the metasemantics of theoretical terms is given in Lewis (1977), who follows Ramsey (1931).

[7] As a simplification, we shall regiment essentialist claims as *de re* metaphysical necessities. Since even proponents of non-modal conceptions of essence, such as Fine (1994), take essence to imply *de re* necessity, this simplification is harmless for current purposes.

Kripke's Thesis Material objects cannot originate from matter totally different to that from which they do.

There appears to be reasonable textual evidence for attributing Kripke the view that *no* material object can originate from matter totally different to that from which it does; see in particular Kripke (1980, p. 114, n.56). For this reason, **Kripke's Thesis** should be understood in terms of a universal restricted quantifier over material objects (as is **Tolerance** below).

Although **Kripke's Thesis** serves as one constraint on certain objects' modal profiles, ordinary objects are not utterly intolerant to modal variation. Indeed, as Nathan Salmon (1986; 1989) famously noted, essence is reasonably tolerant. For example, although Queen Elizabeth II could not have had a totally different origin, surely she could have originated from a sperm with only one cell difference to the one from which she actually originated. As Salmon (1989, p. 5; his emphasis) describes a similar, now also famous case:

> [A] particular material artifact—say, a particular wooden table which we may call 'Woody'—could have originated from matter slightly different from its actual original matter m^* (while retaining its numerical identity, or its *haecceity*) but not from entirely different matter.

To be explicit again, let us call this thesis **Tolerance**:

Tolerance Material objects can originate from matter slightly different to that from which they do.

Note that **Tolerance** is not itself to be read as implicitly necessitated, although its explicit necessitation will be considered at various points in what follows.

As one might already anticipate, taken together **Kripke's Thesis** and **Tolerance** lead to paradox. This was observed by Chandler (1976) and Nathan Salmon (1986; 1989), the latter of whom credits Chisholm (1967) with the initial basis of the idea.[8] In fact, there are various closely related paradoxes that can be extracted from the combination of **Kripke's Thesis** and **Tolerance** to a single given case. What

[8]Chisholm's article focuses on a version of the paradox which does not invoke the necessity of origin; Chisholm's article was published three years before Kripke's *Naming and Necessity* lectures were delivered at Princeton.

follows is a rough presentation of the type of case in the literature which generates such paradoxes. The point of presenting the case in this rough, incomplete form is to show how different paradoxes can later be extracted from the proposed example.

> Following Salmon's example, let 'Woody' refer to some actual wooden table which originates from six, equally sized and shaped pieces of matter, mm_6. (Here, 'mm_6' is a plural name.) For simplicity, assume that it necessary, and necessarily necessary ... that Woody originates from six parts. Now consider mm_5, the actual pieces of matter just like mm_6 except whereas mm_6 include m_1, mm_5 include n_1—an object distinct from but qualitatively similar to m_1. By **Tolerance**, Woody could have originated from mm_5. But **Tolerance** suggests also that Woody could then originate from mm_4, the pieces of matter just like mm_5 except whereas mm_5 include m_2, mm_4 include n_2—an object distinct from but qualitatively similar to m_2. However, iterating this thought leads to the claim that Woody could then have originated from matter completely different from that which it actually does, in apparent violation of **Kripke's Thesis**.

To extract several different lines of reasoning from this case, it will help to introduce some formal resources in the form of a simple modal propositional logic. The official regimentation will proceed in a language $\mathcal{L}$, which consists of countably many proposition letters ($p_0, p_1, ...$), $\neg$ (negation) and $\rightarrow$ (the material conditional) as the only truth-functional connectives, and $\Box$ (metaphysical necessity), with $\Diamond$ and the other connectives introduced metalinguistically as usual. Informally, think of 'p_6' as the sentence 'Woody originates from six parts, m_1,..., m_6', 'p_5' as the sentence 'Woody originates from six parts, m_1,..., m_5, n_6', 'p_4' as the sentence 'Woody originates from six parts, m_1,..., m_4, n_5, n_6', and so on. Under this informal understanding, in $\mathcal{L}$ the sentence $\neg\Diamond p_0$ formalises the relevant instance of **Kripke's Thesis** for Woody, and the sentence $p_6 \rightarrow \Diamond p_5$ formalises the instance of **Tolerance** concerning Woody. Similarly, sentences of the form $p_i \rightarrow \Diamond p_{i-1}$ (for $1 \leq i \leq 6$) formalise **Tolerance** motivated claims for Woody.

One version of the argument, due to Nathan Salmon (1986; 1989), leads with the idea that **Tolerance** is no contingent accident, and that it is a matter of necessity that material objects can originate from matter slightly different to that from which they do. Working in $\mathcal{L}$, one version of Salmon's argument runs as follows:

(1) p_6

(2) $\Box(p_6 \rightarrow \Diamond p_5)$

.

.

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(7) $\Box(p_1 \rightarrow \Diamond p_0)$

(8) $\Diamond p_0$

Premise (1) is the assumption that Woody originates from the six mm_6. Premises (2)-(7) are all necessitated versions of **Tolerance** motivated claims for Woody. Yet (8) contradicts the claim that it is not possible for Woody to originate from material totally different to that from which it actually does ($\neg\Diamond p_0$), the formalisation of the relevant instance of **Kripke's Thesis**. Following Salmon (1986, p. 41) let us call this argument *Chisholm's Paradox*.

Chisholm's Paradox is valid in any normal system of modal logic that contains **S4** in the sense that it has all instances of the following schemas as theorems:

K $\Box(\phi \rightarrow \psi) \rightarrow \Box\phi \rightarrow \Box\psi$

T $\Box\phi \rightarrow \phi$

4 $\Diamond\Diamond\phi \rightarrow \Diamond\phi$

The **T** schema is required to derive $\Diamond p_5$ from (1) and (2). And **K** and **4** are required to prove that $\Diamond p_{i-1}$ is implied by $\Box(p_i \rightarrow \Diamond p_{i-1})$ and $\Diamond p_i$, for in **K** one can derive the following:

$\mathbf{K}^{\Diamond}$ $\Box(\phi \to \psi) \to \Diamond\phi \to \Diamond\psi$

Yet with $\mathbf{K}^{\Diamond}$ one can show that $\Box(p_i \to \Diamond p_{i-1})$ implies $\Diamond p_i \to \Diamond\Diamond p_{i-1}$. So given that $\Diamond p_i$, it follows that $\Diamond\Diamond p_{i-1}$. Thus from **4**, one may conclude $\Diamond p_{i-1}$.

Another essentialist paradox, however, can be extracted from the above Woody case.[9] This paradox takes the form of a straightforward modal sorites. It begins with the idea of **Tolerance**, that Woody could have originated from matter mm_5 slightly different to that from which it actually does (mm_6). But, the soritical thought goes, if mm_i is slightly different from mm_6, so is mm_{i-1}. Thus, the argument runs, if Woody could have originated from mm_5 it could have originated from mm_4. However, by iterating this thought, Woody could have originated from mm_0. Formally:

(1′) p_6

(2′) $\Diamond p_6 \to \Diamond p_5$

(3′) $\Diamond p_5 \to \Diamond p_4$

.

.

.

(7′) $\Diamond p_1 \to \Diamond p_0$

(8′) $\Diamond p_0$

[9]Salmon (1986, p. 87) was aware of this version of the paradox; it was discussed by Forbes (1984, pp. 172-173) and later by Williamson (1990, chap. 8). As discussed below, Salmon thought that this paradox differed importantly from Chisholm's Paradox. Note also that Williamson focuses on a version of the paradox that is formulated in terms of identities between mere possibilia.

Call this paradox the *Modal Sorites*. The striking feature of this argument is that the only background modal axiom schema it requires is **T**. This is to ensure that $p_6 \rightarrow \Diamond p_6$ is true, which permits one to derive (8′) from the remaining premises via repeated application of modus ponens.[10] Otherwise, the argument does not involve any modal rules of inference or applications of modal axiom schemata. Indeed, (2′) is motivated by the merely unnecessitated version of **Tolerance**, and the other premises are instances of a typical soritical principle. Thus, the argument is valid even in system **T**.

Although the Modal Sorites and Chisholm's Paradox are based on the same principles and case, superficially they might appear quite dissimilar. After all, the Modal Sorites is based on very typical soritical reasoning, whereas the necessitated version of **Tolerance** behind Chisholm's Paradox may not appear at all soritical in nature. Naturally, this has lead to a discussion in the literature of whether they are one and the 'same' paradox. For example, Forbes (1984, pp. 172-173) stresses the similarity of Chisholm's Paradox with the Modal Sorites, and recommends a treatment of the former which follows that of the latter. Salmon (1986, p. 89; his emphasis), however, explicitly argues that Chisholm's Paradox "comes out differently in a very important respect from a standard sorites argument", and emphasises that Chisholm's Paradox is a "paradox of *modality*", presumably in contrast to the Modal Sorites, a paradox of vagueness. In later work, Salmon (1993) repeats this point in response to Williamson (1990, p. 127), who observes the similarity of the Modal Sorites to Chisholm's Paradox and suggests that it would be insufficiently general to weaken the logic of metaphysical necessity in order to handle the latter exclusively.

Clearly, this is a contested issue. However, allusive talk of what is at the 'heart' of each paradox, or what they are genuinely 'about', does little to advance the debate beyond gut instinct. What is required is an agreed upon working sufficient condition for treating the paradoxes uniformly. And, indeed, one naturally suggests itself: logical equivalence. As will be shown, each premise (m) (for $1 \leq m \leq 8$) of

[10] The alternative premise $p_6 \rightarrow \Diamond p_5$, that Woody's originating from the mm_6 implies that he could originate from the mm_5, is plausible anyway. Thus, even the **T** schema is not crucial to the argument.

Chisholm's Paradox is logically equivalent to premise (m') of the Modal Sorites in logics which contain **K45**, which is a normal modal system with all instances of **K**, **4**, and the following schema as axioms:

5 $\Diamond\phi \rightarrow \Box\Diamond\phi$

In this sense, Chisholm's Paradox and the Modal Sorites may be said to be logically equivalent in logics which contain **K45**. Hence, the question of whether the paradoxes are the 'same' is a highly post-theoretic matter.[11]

Pursuing this thought, to show that Chisholm's Paradox and the Modal Sorites are logically equivalent in the above sense, the only non-trivial equivalences that require establishing are between (2)-(7) and (2′)-(7′) all of which have the following respective forms (for $1 \leq n \leq 6$):[12]

(m) $\Box(p_n \rightarrow \Diamond p_{n-1})$

(m') $\Diamond p_n \rightarrow \Diamond p_{n-1}$

Yet the following axiomatic proof establishes that (m) implies (m') in **K4**.

i. $\Box(p_n \rightarrow \Diamond p_{n-1})$ — Premise (m)

ii. $\Diamond p_n \rightarrow \Diamond\Diamond p_{n-1}$ — $\mathbf{K}^{\Diamond}$, MP, (i)

iii. $\Diamond\Diamond p_{n-1} \rightarrow \Diamond p_{n-1}$ — **4**

iv. $\Diamond p_n \rightarrow \Diamond p_{n-1}$ — PL, (ii), (iii)

[11] Of course, if one adopts a modal system weaker than **K45**, one is not *forced* into distinguishing the paradoxes, since logical equivalence was not necessary for a uniform treatment of the paradoxes. However, as will become clearer in what follows, it appears quite ill-motivated to treat the paradoxes uniformly in settings weaker than **K45**.

[12] Forbes (1984, p. 172) notes the following equivalences hold in **S5**. Salmon (1986, p. 87) claims that these equivalences hold in **S4**, but below it will be shown that there are **S4** models in which the equivalences fail.

Moreover, the following axiomatic proof establishes that (m') implies (m) in **K5**, the least normal modal system which extends system **K** with all instances of **5**.[13]

i.	$\Diamond p_{n-1} \rightarrow (p_n \rightarrow \Diamond p_{n-1})$	PL
ii.	$\Box\Diamond p_{n-1} \rightarrow \Box(p_n \rightarrow \Diamond p_{n-1})$	**K**, **Nec**, MP, (i)
iii.	$\Diamond p_{n-1} \rightarrow \Box\Diamond p_{n-1}$	**5**
iv.	$\Diamond p_n \rightarrow \Diamond p_{n-1}$	Premise (m')
v.	$\Diamond p_n \rightarrow \Box(p_n \rightarrow \Diamond p_{n-1})$	PL, (iv), (iii), (ii)
vi.	$\neg\Diamond p_n \rightarrow \Box\neg p_n$	DF$\Diamond$
vii.	$\Box\neg p_n \rightarrow \Box(p_n \rightarrow \Diamond p_{n-1})$	PL, **K**, **Nec**, MP
viii	$\neg\Diamond p_n \rightarrow \Box(p_n \rightarrow \Diamond p_{n-1})$	PL, (vi), (vii)
ix.	$\Box(p_n \rightarrow \Diamond p_{n-1})$	PL, (v), (viii)

Thus, given a logic for metaphysical necessity which includes **K45**, one must treat the paradoxes uniformly.

Interestingly, one can also show that (m) and (m') are *not* equivalent when the logic of necessity does not include **K45**.[14] This highlights that those who take the logic of metaphysical necessity not to include **K45** lose some warrant for treating the paradoxes uniformly, since the two paradoxes are no longer logically equivalent in the sense above. To demonstrate the technical claim, it suffices to establish the following fact.

Fact (EQ-45) Let **KEQ** be the normal modal system which extends **K** with the following axiom: **EQ** $(\Diamond p \rightarrow \Diamond q) \leftrightarrow \Box(p \rightarrow \Diamond q)$. **KEQ** includes all instances of **4** and **5**.

The following axiomatic proof shows that **KEQ** includes the **5** axiom.

[13]In this proof, **Nec** is the standard necessitation rule that if ϕ is a theorem so is $\Box\phi$; note that **Nec** is not applied to the non-theorem premise (m') or anything which is shown to follow from (m').

[14]Thanks to Peter Fritz for helpful discussion here.

i. $(\Diamond\top \to \Diamond q) \to \Box(\top \to \Diamond q)$ **EQ** (LTR)

ii. $\Box(\top \to \Diamond q) \to (\Box\top \to \Box\Diamond q)$ **K**, **Nec**, MP, (i)

iii. $(\Diamond\top \to \Diamond q) \to (\Box\top \to \Box\Diamond q)$ PL, (i), (ii)

iv. $\Diamond q \to (\Diamond\top \to \Diamond q)$ PL

v. $\Diamond q \to (\Box\top \to \Box\Diamond q)$ PL, (iii), (iv)

vi. $\Box\top \to ((\Box\top \to \Box\Diamond q) \to \Box\Diamond q)$ PL

vii. $(\Box\top \to \Box\Diamond q) \to \Box\Diamond q$ PL, **Nec**, MP, (vi)

viii $\Diamond q \to \Box\Diamond q$ PL, (v), (vii)

The next axiomatic proof then shows that **KEQ** includes the **4** axiom.

i. $\Box(\Diamond q \to \Diamond q) \to (\Diamond\Diamond q \to \Diamond q)$ **EQ** (RTL)

ii. $\Box(\Diamond q \to \Diamond q)$ PL, **Nec**

i. $\Diamond\Diamond q \to \Diamond q$ MP, (i), (ii)

Indeed, the non-equivalence of (m) and (m') in **K45** can be brought out vividly by certain non-**K45** models. To construct such models, let a *frame* $\mathfrak{F}$ be a pair $\langle \mathcal{W}, \mathcal{R} \rangle$, where $\mathcal{W}$ is a non-empty set ('the worlds'), and $\mathcal{R}$ is a binary relation on $\mathcal{W}$ ('accessibility'). A *model* $\mathfrak{M}$ of $\mathcal{L}$ based on $\mathfrak{F}$ is a triple $\langle W, \mathcal{R}, \mathcal{V} \rangle$ where $\mathcal{V}$ is a total function from pairs of the form $\langle \phi, w \rangle$, for some proposition letter ϕ and $w \in W$, to either 0 (falsity) or 1 (truth). We define '$(\mathfrak{M}, w) \models \phi$' for $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$, $w \in \mathcal{W}$, and ϕ a formula of the language, recursively as follows (where 'A' is an atomic formula, and ϕ and ψ formulae of $\mathcal{L}$):

$(\mathfrak{M}, w) \models A$ iff $\mathcal{V}(A, w) = 1$;

$(\mathfrak{M}, w) \models \neg\phi$ iff $(\mathfrak{M}, w) \not\models \phi$;

$$(\mathfrak{M}, w) \models \phi \to \psi \text{ iff either } (\mathfrak{M}, w) \not\models \phi \text{ or } (\mathfrak{M}, w) \models \psi;$$

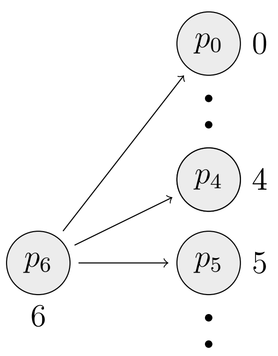
$$(\mathfrak{M}, w) \models \Box\phi \text{ iff } (\mathfrak{M}, u) \models \phi, \text{ for all } u \in W \text{ such that } w\mathcal{R}u.$$

For an $\mathcal{L}$ model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$, a formula ϕ of $\mathcal{L}$ is true at $w \in \mathcal{W}$ iff $(\mathfrak{M}, w) \models \phi$. A formula ϕ of $\mathcal{L}$ is true in an $\mathcal{L}$ model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ iff it is true at all $w \in \mathcal{W}$. A formula ϕ of $\mathcal{L}$ is valid iff it is true in every $\mathcal{L}$ model. A set Γ of $\mathcal{L}$ formulas entails an $\mathcal{L}$ formula ϕ iff for every $\mathcal{L}$ model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ and every $w \in \mathcal{W}$, if $(\mathfrak{M}, w) \models \gamma$ for every $\gamma \in \Gamma$, then $(\mathfrak{M}, w) \models \phi$. We shall say that an argument with premises Γ and conclusion ϕ is valid when Γ entails ϕ.

With this formal apparatus, one can see that the following **S4** model $\mathfrak{M}_4 = \langle \mathcal{W}_4, \mathcal{R}_4, \mathcal{V}_4 \rangle$ shows that (m') does not even imply (m) in **S4**:

$$\mathcal{W}_4 = \mathbb{N}$$
$$\mathcal{R}_4 = \{\langle w, u \rangle : w = 6 \text{ or } w = u\}$$
$$\mathcal{V}_4(p_n, w) = 1 \text{ iff } w = n \leq 6$$

This model can be represented in diagram form. In these diagrams, 'worlds' are represented as labelled circles with the name of the 'world' occurring outside the circle. Inside the circle are the proposition letters which are 'true' at that 'world' in the given model. Moreover, 'accessibility' amongst 'worlds' is represented by arrows, although some instances of 'accessibility' will be left implicit for presentational reasons. For example, if 'accessibility' is reflexive, arrows from circles to themselves are omitted; similarly, arrows are dropped altogether when 'accessibility' is universal.

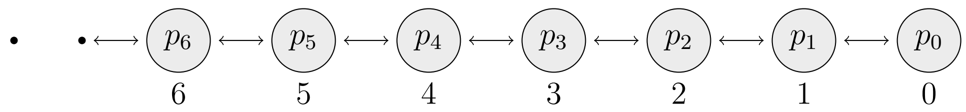


As can be checked, $\mathfrak{M}_4, 6 \models \Diamond p_n \rightarrow \Diamond p_{n-1}$ for positive n. However, $\mathfrak{M}_4, 6 \not\models \Box(p_n \rightarrow \Diamond p_{n-1})$, for positive $n < 6$.

Similarly, we can specify a **T** model $\mathfrak{M}_{\mathrm{T}} = \langle \mathcal{W}_{\mathrm{T}}, \mathcal{R}_{\mathrm{T}}, \mathcal{V}_{\mathrm{T}} \rangle$ to establish that (m) does not imply (m') in **T**:[15]

$$\mathcal{W}_{\mathrm{T}} = \mathbb{N}$$
$$\mathcal{R}_{\mathrm{T}} = \{\langle w, u \rangle : |w - u| = 1\}$$
$$\mathcal{V}_{\mathrm{T}}(p_n, w) = 1 \text{ iff } w = n \leq 6$$

Again in diagram form:



In this model, it can be checked that $\mathfrak{M}_{\mathrm{T}}, 6 \models \Box(p_5 \rightarrow \Diamond p_4)$ despite $\mathfrak{M}_{\mathrm{T}}, 6 \not\models \Diamond p_5 \rightarrow \Diamond p_4$. Thus, weakening the logic of metaphysical necessity in the appropriate way renders (m') and (m) non-equivalent. These technical facts vindicate the above claim that whether the paradoxes should receive a uniform treatment is a post-theoretic matter. Those who endorse a **K45** logic for metaphysical necessity must offer such a treatment, whereas those who weaken the logic in an appropriate way can distinguish the paradoxes. This invites two salient questions:

- Does handling the Modal Sorites with ordinary vagueness-based resources require a weakening of **K45** for metaphysical necessity and thus a separate treatment of each paradox?

- If not, does that vagueness-based apparatus afford a plausible response to Chisholm's Paradox too?

In what follows, these two questions are answered in the negative and affirmative respectively. The latter answer will further contribute to the debate between Salmon,

[15] Below, the function $|.|$ takes an integer to its absolute value. It is also worth emphasising that $\mathfrak{M}_{\mathrm{T}}$ is a **B** model too, since $\mathcal{R}_{\mathrm{T}}$ is symmetric and reflexive over $\mathcal{W}_{\mathrm{T}}$, which emphasises the importance of the **4** schema to securing the implication of (m') from (m).

Forbes, and Williamson about whether a uniform treatment of the paradoxes rides roughshod over their subtleties.

3.3 Necessity as a Source of Indeterminacy

3.3.1 The Modal Sorites

The first aim is to establish that a vagueness-based treatment of the Modal Sorites does not require a weakening of **K45** for metaphysical necessity. In fact, since metaphysical necessity is clearly alethic, we shall establish the stronger claim that a vagueness-based treatment of the Modal Sorites is compatible with an **S5** logic for metaphysical necessity.

It will be instructive to compare the Modal Sorites with a paradigm non-modal sorites argument. Examples of such reasoning are familiar. Imagine 100 audio recordings of one and the same sound ordered gradually by their volume, the first of which is clearly loud and the last of which is clearly not loud. A soritical principle is nearby:

Sorites For every $1 \leq n \leq 100$, if audio n is loud, then $n + 1$ is loud too.

Sorites appears plausible because, the intuitive thought goes, surely no minuscule decrease in volume divides the loud from the not loud. But repeated application of modus ponens on the instances of **Sorites** delivers the result that audio 100 is loud, when clearly it was not. Put alternatively, classical logic forces the result that there *is* some minuscule division between the loud and not loud, if 1 is loud and 100 is not. Thus, if classical logic is to be retained, at least one instance of **Sorites** is not true.

The innovative idea of standard supervaluationist treatments of vagueness was to maintain that although at least one instance of **Sorites** is not true, none of them is false. A further innovation was to show that it might itself be vague which instance of **Sorites** is false. Taken together, these two innovations were intended to ameliorate the overall rejection of **Sorites**. Ideally, the hope is that if metaphysical necessity is a source of indeterminacy, a vagueness-based solution to the Modal Sorites would reflect this standard treatment of paradigm non-modal sorites arguments.[16]

To this end, the aim will be to offer a vagueness-based solution to the Modal Sorites which is extremely similar to standard treatments of paradigm non-modal sorites arguments. To make the details both familiar and clear, we shall assume supervaluationism as the working theory of vagueness. However, since the proposed treatment of the Modal Sorites is intended to be available to both supervaluationists and epistemicists, we shall assume that the background notion of logical consequence is classical. This marks a departure from some versions of supervaluationism which opt for a 'global' consequence relation that validates all classical theorems but invalidates certain classically valid arguments with non-empty premise sets. (See, for example, Fine (1975).) Nevertheless, this will not affect the treatment of the Modal Sorites in any important way. Moreover, it will place advocates of epistemicism in a position to reinterpret most of the following formal machinery straightforwardly in their own terms.

In the literature, there are several suggested vagueness-based treatments of the Modal Sorites. For example, Bacon (2018, pp. 280-281) suggests a standard treatment of the Modal Sorites on which at least one of (2′)-(7′) is indeterminate (although perhaps it is not determinate which one is). According to this treatment, one can handle the Modal Sorites by positing the following type of vagueness in Woody's modal profile:[17]

1 It is determinate that for some $0 \leq n < 6$ Woody could have originated from six parts n or fewer of which it does not actually originate from;

2 It is determinate that for some m (where $0, n < m \leq 6$) Woody could not have originated from six parts m or more of which it does not actually originate from, and

3 It is indeterminate whether for $n < k < m$ above, Woody could have

[16]Bacon (2018, p. 281) points out that the other terms which feature in the premises of the informal Modal Sorites do not appear to be the source of the premises' indeterminacy. See also §3.3.2 for a version of Chisholm's Paradox which uses only precise non-modal vocabulary.

[17]The intended reading of **1-3** is as a nested conjunction ($\mathbf{1} \wedge (\mathbf{2} \wedge \mathbf{3})$). Thus, the numerical variable k as it appears in **3** is 'bound' by the respective occurrences of n in **1** and m in **2**.

originated from six parts k of which it does not actually originate from.

Interestingly, however, Williamson (1990, p. 133) suggests a similar but slightly more radical proposal on which *all* of (2′)-(7′) are indeterminate. On this conception, Woody's modal profile exhibits a much more uniform pattern of vagueness: for $0 < n < 6$, it is indeterminate whether Woody could have originated from six parts n or fewer of which it does not actually originate from.

Let us call these two respective approaches the *standard* and *non-standard* vagueness-based solutions to the Modal Sorites.

Standard At least one of (2′)-(7′) is indeterminate.

Non-Standard All of (2′)-(7′) are indeterminate.

Although **Non-Standard** implies **Standard**, in what follows the focus will be on natural ways of developing **Standard** which are not merely cases of **Non-Standard**. More generally, the aim of this subsection will be to show that natural ways of developing both the **Standard** and **Non-Standard** treatments of the Modal Sorites can retain an **S5** logic for metaphysical necessity.

We require a formal framework for carrying out this investigation. In the literature, there are several model-theoretic frameworks for studying the interaction of vagueness and modality, such as Torza (2015) and Litland & Yli-Vakurri (2016). However, the former uses counterpart-theoretic resources which would unnecessarily complicate the current investigation. Moreover, the latter uses standard relativised product models for bimodal logics with additional contextual parameters, which though elegant in many respects would complicate some of the models used below presentationally. Thus, we shall define a slightly more intuitive class of models for a bimodal language containing operators for determinacy and necessity. The informal idea will be to associate different spaces of metaphysically possible worlds with different precisifications of a language in order to model vague metaphysical necessity.

First, let us expand the language $\mathcal{L}$ to $\mathcal{L}^+$ by adding an operator Δ (determinacy), using the string of symbols $\nabla\phi$ ('it is indeterminate whether ϕ') as a metalinguistic abbreviation for $\neg\Delta\phi \wedge \neg\Delta\neg\phi$. Informally, $\Delta\phi$ can be read as the claim that

ϕ is true according to all admissible ways of sharpening, or making precise, the language.[18] Then, a *frame* is a triple $\langle \mathcal{P}, \mathcal{R}^{\Delta}, \mathcal{W}^{*} \rangle$, where $\mathcal{P}$ is a non-empty set ('the precisifications'), over which the reflexive binary relation $\mathcal{R}^{\Delta}$ ('Δ-accessibility') is defined, and $\mathcal{W}^{*}$ is a total function taking $p \in \mathcal{P}$ to a non-empty set $\mathcal{W}^{*}_{p}$ ('the metaphysically possible worlds according to p').

For a frame $\mathfrak{F} = \langle \mathcal{P}, \mathcal{R}^{\Delta}, \mathcal{W}^{*} \rangle$, we then define:

$$\mathcal{W} = \bigcup_{p \in \mathcal{P}} \mathcal{W}^{*}_{p}$$

We can think of $\mathcal{W}$ as the set of all metaphysically possible worlds according to any precisification. Then, with each $p \in \mathcal{P}$ we associate an equivalence relation:

$$\mathcal{R}^{\Box}_{p} = \{\langle u, v \rangle : u, v \in \mathcal{W}^{*}_{p}\} \cup \{\langle u, u \rangle : u \in \mathcal{W}\}$$

Each of these relations can be thought of as the accessibility relation over metaphysically possible worlds associated with that precisification. They are used in the semantic clause for metaphysical necessity to ensure that its logic is **S5**.

A model $\mathfrak{M}$ of $\mathcal{L}^{+}$ based on a frame $\mathfrak{F} = \langle \mathcal{P}, \mathcal{R}^{\Delta}, \mathcal{W}^{*} \rangle$ is an quadruple $\langle \mathcal{P}, \mathcal{R}^{\Delta}, \mathcal{W}^{*}, \mathcal{V} \rangle$, where $\mathcal{V}$ is a total function from triples of the form $\langle \phi, p, w \rangle$, for some proposition letter ϕ, $p \in \mathcal{P}$, and $w \in \mathcal{W}$, to either 0 (falsity) or 1 (truth).

We now define '$(\mathfrak{M}, p, w) \models \phi$' for $\mathfrak{M} = \langle \mathcal{P}, \mathcal{R}^{\Delta}, \mathcal{W}^{*}, \mathcal{V} \rangle$, $p \in \mathcal{P}$, $w \in \mathcal{W}$, and ϕ a formula of the language, recursively as follows (where 'A' is an atomic formula, and ϕ and ψ formulae of $\mathcal{L}^{+}$):

$(\mathfrak{M}, p, w) \models A$ iff $\mathcal{V}(A, p, w) = 1$;

$(\mathfrak{M}, p, w) \models \neg\phi$ iff $(\mathfrak{M}, p, w) \not\models \phi$;

$(\mathfrak{M}, p, w) \models \phi \rightarrow \psi$ iff either $(\mathfrak{M}, p, w) \not\models \phi$ or $(\mathfrak{M}, p, w) \models \psi$;

[18]Epistemicists can read Δ in their preferred sense, as expressing variation across interpretations of the language not knowably incorrect.

$(\mathfrak{M}, p, w) \models \Box\phi$ iff for every u such that $w\mathcal{R}_p^{\Box}u$, $(\mathfrak{M}, p, u) \models \phi$;

$(\mathfrak{M}, p, w) \models \Delta\phi$ iff for every q such that $p\mathcal{R}^{\Delta}q$, $(\mathfrak{M}, q, w) \models \phi$.

For an $\mathcal{L}^+$ model $\mathfrak{M} = \langle \mathcal{P}, \mathcal{R}^{\Delta}, \mathcal{W}^*, \mathcal{V}\rangle$, a formula ϕ of $\mathcal{L}^+$ is true at $\langle p, w\rangle$ for $p \in \mathcal{P}$ and $w \in \mathcal{W}$ iff $(\mathfrak{M}, p, w) \models \phi$. A formula ϕ of $\mathcal{L}^+$ is true in an $\mathcal{L}^+$ model $\mathfrak{M} = \langle \mathcal{P}, \mathcal{R}^{\Delta}, \mathcal{W}^*, \mathcal{V}\rangle$ iff it is true at $\langle p, w\rangle$ for all $p \in P$ and $w \in \mathcal{W}$; a formula ϕ of $\mathcal{L}^+$ is false in an $\mathcal{L}^+$ model $\langle \mathcal{P}, \mathcal{R}^{\Delta}, \mathcal{W}^*, \mathcal{V}\rangle$ iff it is false at $\langle p, w\rangle$ for all $p \in P$ and $w \in \mathcal{W}$, and a formula ϕ of $\mathcal{L}^+$ is otherwise neither true nor false in $\mathfrak{M}$.[19] A formula ϕ of $\mathcal{L}^+$ is valid iff it is true in every $\mathcal{L}^+$ model.

The model theory validates a number of formulae:

PL All tautologies of propositional logic.

$\mathbf{K}^{\Box}$ $\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$

$\mathbf{T}^{\Box}$ $\Box\phi \to \phi$

$\mathbf{5}^{\Box}$ $\Diamond\phi \to \Box\Diamond\phi$

$\mathbf{K}^{\Delta}$ $\Delta(\phi \to \psi) \to \Delta\phi \to \Delta\psi$

$\mathbf{T}^{\Delta}$ $\Delta\phi \to \phi$

Interestingly, however, the model theory does not validate:

ND $\Box\Delta\phi \to \Delta\Box\phi$

[19] Typically, falsity in a model is understood as non-truth; the slightly different current definition will merely aid the expression of certain points in what follows.

DN $\Delta\Box\phi \to \Box\Delta\phi$

CR $\Diamond\Delta\phi \to \Delta\Diamond\phi$

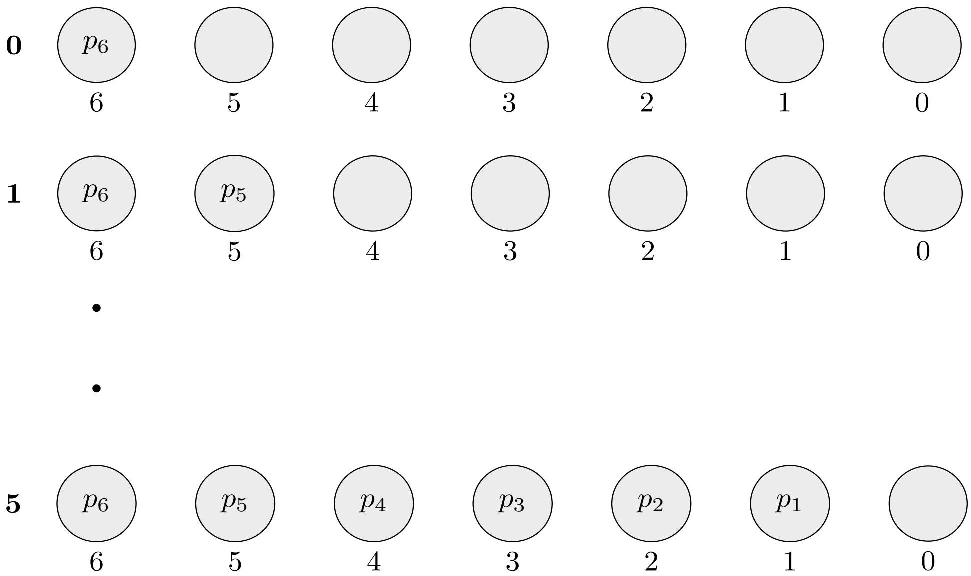
This is because the model theory allows the metaphysically possible worlds to change from precisification to precisification. However, in the above model theory, if one required the precisifications to remain constant from world to world, all instances of **ND**, **DN**, and **CR** would be validated.[20]

We are now in a position to provide both **Standard** and **Non-Standard** models of the Modal Sorites. Due to the simplicity of the latter, it is best to proceed in reverse. Recall that **Non-Standard** was the claim that all of (2′)-(7′) are indeterminate. This, of course, is coupled with the claim that (1′) and the negation of (8′) are determinate. A natural model $\mathfrak{M}_{ns} = \langle \mathcal{P}, \mathcal{R}^{\Delta}, \mathcal{W}^{*}, \mathcal{V} \rangle$ of this proposal can be specified as follows.

$$\mathcal{P} \;=\; \{n \;\in\; \mathbb{N} \;:\; n \;\leq\; 5\}$$
$$\mathcal{R}^{\Delta} \;=\; \{\langle p, q\rangle \;:\; p, q \;\in\; \mathcal{P}\}$$
$$\mathcal{W}^{*}_{n} \;=\; \{m \;\in\; \mathbb{N} \;:\; m \;\leq\; 6\}$$
$$\mathcal{V}_{ns}(p_n, q, m) \;=\; 1 \text{ iff} \begin{cases} 1, & \text{if } m = n \text{ and } m \geq 6 - q \\ 0, & \text{otherwise} \end{cases}$$

This model can be represented in diagram form. In these diagrams, precisification-world pairs are represented as labelled circles with the world name below the circle, and the precisification name to the extreme left of the circle. As before, inside the circle are the proposition letters which are true at that precisification-world pair in the given model. Since $\Box$ has an **S5** logic, '$\Box$-accessibility' is omitted from the diagrams altogether. Similarly, since 'Δ-accessibility' is straightforward in each of the following models, it is omitted too.

[20] Compare Litland and Yli-Vakkuri (2016) for a related point within the setting of their relativised product models.



As can be checked, (2′)-(7′) are neither true nor false in this model. For example, (2′) is the sentence $\Diamond p_6 \to \Diamond p_5$. However, for all $w \in \mathcal{W}$, $(\mathfrak{M}_{ns}, 0, w) \not\models \Diamond p_6 \to \Diamond p_5$ whereas $(\mathfrak{M}_{ns}, q, w) \models \Diamond p_6 \to \Diamond p_5$ when $q > 0$. Likewise, for all $w \in \mathcal{W}$, $(\mathfrak{M}_{ns}, 5, w) \not\models \Diamond p_1 \to \Diamond p_0$ whereas $(\mathfrak{M}_{ns}, q, w) \models \Diamond p_1 \to \Diamond p_0$ when $5 > q$. Indeed, in this model, it is true that ∇(n′) when (n′) is one of (2′)-(7′), which reflects the fact that **Non-Standard** was an *assertion* of their indeterminacy. Moreover, in the model it is both true that Δ(1′) and $\Delta\neg$(8′), as desired. Since the model belongs to the class of defined models for $\mathcal{L}^+$, it also validates an **S5** logic for necessity.

It remains to show that a solution to the Modal Sorites based on **Standard** is compatible with an **S5** logic for necessity too. Recall that **Standard** was the claim that at least one of (2′)-(7′) is indeterminate. We can use a similar model to the one above to capture this idea. The central difference is with the accessibility relation for Δ. Thus, let $\mathfrak{M}_s = \langle \mathcal{P}, \mathcal{R}_s^{\Delta}, \mathcal{W}^*, \mathcal{V} \rangle$, where:

$$\mathcal{R}_s^{\Delta} = \{\langle x, y \rangle : |x - y| \leq 1\}$$

It can be checked that (2′)-(7′) are neither true nor false in the model. However, in the model it is true that $\nabla(2') \vee ... \vee \nabla(7')$. In other words, at least one of (2′)-(7′) is indeterminate, as desired. Moreover, neatly, no member (m') of (2′)-(7′) is such

$\nabla(m')$ is true in the model. Thus, although at least one of (2′)-(7′) is indeterminate, it is not determinate which one is, which reflects typical supervaluationist treatments of paradigm non-modal sorites series. As before, the model belongs to the class of defined models for $\mathcal{L}^+$, so it validates an **S5** logic for necessity.

As a consequence, we may resolve the first question of the investigation: vagueness-based treatments of the Modal Sorites do not require a weakening of **S5** for necessity.

3.3.2 Chisholm's Paradox

We now turn our attention to the second question of the investigation: does the above vagueness-based apparatus afford a plausible response to Chisholm's Paradox? If advocates of the vagueness-based approach retain **S5** for necessity, as they are in a position to do, they must treat Chisholm's Paradox and the Modal Sorites uniformly.[21] This section argues that such a uniform treatment constitutes a plausible response to Chisholm's Paradox. Moreover, working through a vagueness-based solution to Chisholm's Paradox will illuminate further features of the solution.

Recall that Chisholm's Paradox took the following form.

(1) p_6

(2) $\Box(p_6 \to \Diamond p_5)$

.

.

.

[21] It is important to stress the point that the above **S5** equivalences between (m) and (m') would be preserved even if the supervaluationist opted for a 'global' non-classical consequence relation. Typically, global supervaluationist consequence validates all classical theorems but invalidates certain classically valid arguments with non-empty premise sets; see Williamson (1994, chap. 5). Nevertheless, the proof of (m) from (m') does not proceed via argument by cases (which is globally invalid in certain supervaluationist settings); instead one can use the PL theorem schema $((\phi \to \chi) \wedge (\psi \to \chi)) \to ((\phi \vee \psi) \to \chi)$.

(7) $\Box(p_1 \rightarrow \Diamond p_0)$

(8) $\Diamond p_0$

Given the **S5**-equivalence of (2)-(7) with (2′)-(7′) respectively, models $\mathfrak{M}_{ns}$ and $\mathfrak{M}_s$ deliver the same verdicts as before. Thus, to reiterate, in $\mathfrak{M}_{ns}$ it is true that (2)-(7) are all indeterminate, and in $\mathfrak{M}_s$ it is true that one of (2)-(7) is indeterminate despite it not being determinate that any particular one of them is. Moreover, each model endorses the determinate truth of both (1) and the negation of (8). These facts have several interesting upshots.

The first point to appreciate is that both the **Standard** and **Non-Standard** solutions commit to the determinate truth of **Kripke's Thesis**, since in both $\mathfrak{M}_{ns}$ and $\mathfrak{M}_s$ the negation of (8)/(8′) is determinate (and, indeed, determinately determinate and so on). However, neither solution commits to the necessitation of **Tolerance**, whose relevant instance regarding Woody can be viewed as informally implying the conjunction of (2)-(7).[22] In fact, both solutions commit to the determinate falsity of the conjunction of (2)-(7), since there is no precisification in either $\mathfrak{M}_{ns}$ or $\mathfrak{M}_s$ on which all of (2)-(7) are true. In this regard, the conjunction of (2)-(7) is similar to **Sorites** from the non-modal sorites argument above; although it might not be determinate which of (2)-(7) is false, it is determinate that one of them is.

Nevertheless, there is one minor respect in which the two solutions come apart. To appreciate this, let us say that advocates of the **Non-Standard** (**Standard**) solution will *endorse* a sentence in $\mathfrak{M}_{ns}$ ($\mathfrak{M}_s$) if it is true at all precisification-world pairs with 6 as their world component. This is a natural notion of endorsement for the given models, since, in effect, world 6 in the above models served as the actual world. Now consider the instance of **Tolerance** relevant to the Woody case: that Woody can originate from matter slightly different to that from which it does.

[22]Although Williamson (1990, chap. 8) does not discuss the premises of Chisholm's Paradox directly, Nathan Salmon (1993, f.n. 10) notes that Williamson stated in personal correspondence that he rejects the necessitation of **Tolerance** and the conjunction of (2)-(7).

Recall that the official modal propositional language, this instance of **Tolerance** was formalised by the sentence $p_6 \to \Diamond p_5$. However, notice that advocates of the **Non-Standard** solution will not endorse the sentence $p_6 \to \Diamond p_5$. After all, in $\mathfrak{M}_{ns}$ it is *true* that $\nabla(p_6 \to \Diamond p_5)$. Thus, advocates of the **Non-Standard** solution will not endorse $\neg(p_6 \to \Diamond p_5)$ either. In other words, advocates of the **Non-Standard** solution will not endorse the relevant instance of **Tolerance** or its negation; they will simply endorse that it is indeterminate whether Woody's essence is tolerant as such. For comparison, advocates of the **Standard** solution will not endorse the relevant instance of **Tolerance** either, as witnessed by the fact that $(\mathfrak{M}_s, 0, 6) \not\models p_6 \to \Diamond p_5$. However, in contrast to advocates of the **Non-Standard** solution, advocates of the **Standard** solution will *not* endorse the sentence $\nabla(p_6 \to \Diamond p_5)$. Informally, this is because advocates of the **Standard** solution only commit to one of (2)-(7) being indeterminate, whilst maintaining that it is not determinate which one is. Thus, they do not endorse that $p_6 \to \Diamond p_5$ is indeterminate, and so by extension they will refrain from endorsing that it is indeterminate whether Woody can originate from matter slightly different to that from which it does, i.e. the instance of **Tolerance** which concerns Woody. Formally, this is reflected in the **Standard** model by, for example, the fact that $(\mathfrak{M}_s, 5, 6) \not\models \nabla(p_6 \to \Diamond p_5)$.

This small difference aside, the interesting point is that both solutions' outright rejection of the conjunction of (2)-(7) amounts to an assertion of the possibility that Woody's modal profile becomes utterly intolerant at some point. Although this not strict essentialism proper, it is within its vicinity.[23] Unlike typical strict essentialism, the claim is not that there is some actual thing whose essence is utterly intolerant. Instead, the claim is that there are possible cases of such intolerance, despite it not being determinate in which possibility they occur. Nonetheless, it is both an interesting and surprising consequence of the vagueness-based solution; those who hoped to deny any form of strict essentialism will find no solace in the vagueness-based approach.

The above provides an answer to the second question of our investigation, of

[23]Thanks to Timothy Williamson for highlighting this point.

whether the vagueness-based approach affords a plausible solution to Chisholm's Paradox. For although the vagueness-based treatment of the paradox has some surprising features, certainly those features are not implausible. Indeed, these features seem a reasonable price for the retention of **S5** for necessity, which forces them upon those who think that vagueness in necessity is responsible for the Modal Sorites.

This concludes the case study of vague modality. The moral has been that the hypothesis of vague modality can be fruitfully applied to handle the Modal Sorites. Moreover, if one endorses an **S5** logic for necessity, that hypothesis permits a plausible treatment of Chisholm's Paradox too. These two observations should increase one's credence in the hypothesis that metaphysical necessity is vague: it permits a highly natural, integrated treatment of two seemingly related paradoxes.

The next major task is to explore how the indeterminacy hypothesis and conceptual terrain explored during the case study impact the theoretical role of metaphysical necessity. However, before that, we shall consider two alternative solutions to Chisholm's Paradox: Nathan Salmon's (1986; 1989) rejection of **4** for necessity, and a recent proposal based on two-dimensionalist semantics initially suggested by Adam Murrary and Jessica Wilson (2012). The point of this will be to show that proponents of either solution should still treat the hypothesis that necessity is indeterminate as live, for varying reasons.

3.4 Alternative Solutions

3.4.1 Salmon Against S4

Nathan Salmon (1986; 1989) famously argued that the problematic feature of Chisholm's Paradox is the application of axiom schema **4**. He thus denies that **S4** is a lower bound on the logic of necessity.[24] In his view, there are possibly . . . possible worlds which are nevertheless impossible. Salmon exploits this to

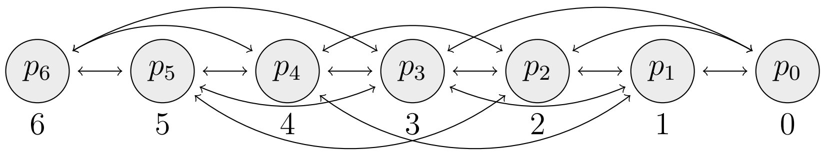
[24] An idiosyncracy of Salmon is that he also rejects the **B** schema for necessity too; see in particular Salmon (1989, p. 4), where he states that although the **B** schema may have no counterexamples, it does not deserve the status of 'logical truth'.

maintain that although it is possibly ... possible that Woody originates from matter totally different to that from which it actually does, it is not possible that Woody does. As a consequence, he endorses **Tolerance** and its necessitation whilst retaining **Kripke's Thesis**.

To appreciate the structure of Salmon's proposal, we can consider a Salmon-inspired $\mathcal{L}$ model of Chisholm's Paradox. Thus let $\mathfrak{M}_{sa} = \langle \mathcal{W}_{sa}, \mathcal{R}_{sa}, \mathcal{V}_{sa} \rangle$ be as follows:

$$\mathcal{W}_{sa} = \{n \in \mathbb{N} : n \leq 6\}$$
$$\mathcal{R}_{sa} = \{\langle u, w \rangle : |w - u| \leq 3\}$$
$$\mathcal{V}_{sa}(P_n, w) = 1 \text{ iff } w = n$$

In diagram form:



Using $\mathcal{R}_{sa}$ model-theoretically encodes the idea that at a given world Woody must originate from at least half the matter it does at that world.[25] For instance, since $|6 - 3| = 3$, at world 6 Woody could originate from six parts, only three of which it does at world 6. Yet, since $|6 - 2| > 3$, at world 6 Woody could not have originated from six parts only two of which it does at world 6. Thus, at world 6, Woody could not have originated from matter more than half of which is different to mm_6.

Crucially, $\mathcal{R}_{sa}$ is not transitive. For example, $|6 - 3| = 3$ and $|3 - 0| = 3$ but $|6 - 0| = 6$, so $6\mathcal{R}_{sa}3$ and $3\mathcal{R}_{sa}0$ but it is not the case that $6\mathcal{R}_{sa}0$. This would be natural to expect in a Salmon-inspired model of the case, since axiom schema **4** is validated by the class of frames in which the accessibility relation

[25] Notice that there are more ways to originate from at least half of one's matter from world 3 than there are from world 0, or 6; this has the odd consequence that the extent of Woody's modal profile will change from world to world. Nevertheless, this would be finessed by a using more realistic description of modal space in the model, as opposed to the simplified version given here which contains only seven worlds.

is transitive. Indeed, the model exhibits a clear failure of the **4** schema since $\mathfrak{M}_{sa}, 6 \models \Diamond\Diamond p_0$ despite $\mathfrak{M}_{sa}, 6 \not\models \Diamond p_0$.

Moreover, as can be checked, each of premises (1)-(7) of Chisholm's Paradox are all true at world 6 in this model, effectively because each world accesses worlds which are at most thrice removed from it. Nevertheless, (8), i.e. $\Diamond p_0$, is not true at world 6 in the model. Thus, the paradoxical conclusion of Chisholm's Paradox is blocked.

In spite of this, since Salmon rejects **S4** for necessity, he cannot guarantee the equivalence of Chisholm's Paradox and the Modal Sorites; indeed, Salmon (1986, p. 87-89) himself embraces this and distinguishes them. It follows that Salmon's rejection of **4** will be of no help in solving the Modal Sorites, for, as we now know, the Modal Sorites is valid in **T**. As a consequence, Salmon must offer a different solution to the Modal Sorites. Sensitive to this concern, Salmon (1986, p. 88) hints at various points that he suspects the Modal Sorites arises due to metaphysical necessity being a source of indeterminacy.[26] The salient point for our current discussion, therefore, is that *even if* one distinguishes between the Modal Sorites and Chisholm's Paradox, as Salmon does, the vagueness hypothesis is still an extremely live solution to the former. In that case, necessity would still exhibit the vagueness-laden structure discussed above.

3.4.2 Two-Dimensionalism

Another alternative solution to Chisholm's Paradox retains the orthodox logic of necessity yet situates the notion in a two-dimensional semantic setting. This view can be articulated by first considering David Chalmers' (2006; 2012) epistemic two-dimensionalist framework.

According to Chalmers' version of two-dimensionalism, with every expression of a language—not just each predicate letter—is associated a two-dimensional intension. For the time being, we can think of a two-dimensional intension as complex semantic value from which two subsidiary intensions can be recovered, a primary intension and a secondary intension. In the simple case of monadic predicates, primary

[26] At other points (1986, f.n.3; p. 113) he suggests that there may well be a 'sharp cutoff point' which renders one of the Modal Sorites' premises simply false.

intensions are functions from metaphysically possible worlds considered as actual to sets of individuals, and secondary intensions are functions from metaphysically possible worlds considered as counterfactual to sets of individuals.[27] Consider some world w_1 where the clear, transparent, drinkable liquid which runs from taps, makes up roughly 55% of the human body, etc., has the chemical composition XYZ. The primary intension of 'water' takes w_1 to XYZ. But the secondary intension we actually assign 'water' takes w_1 to H_20. Roughly speaking, the thought is that when we consider a world as actual, we consider what the expression 'water' would pick out at that world had its usage been the same as it actually is. When we consider a world as counterfactual, we consider what water (i.e. H_20) would have been like at that world.

Clearly, two-dimensionalism and the twin-earth thought experiments used to motivate semantic externalism are closely connected. The lesson of semantic externalism is that there are aspects of an expression's meaning which in some sense depend on how the non-linguistic world turns out. If the clear, transparent, drinkable liquid which runs from taps, makes up roughly 55% of the human body, etc., has the chemical composition XYZ, then 'water' refers to XYZ; if the clear, transparent, drinkable liquid which runs from taps, makes up roughly 55% of the human body, etc., has the chemical composition H_20, as it actually does, then 'water' refers to H_20. Two-dimensionalism incorporates the fact that the semantic values of expressions like 'water' exhibit non-trivial dependence on the non-linguistic world into its semantic theory by assigning them a primary intension whose extension varies with which world is considered as actual.

An upshot to the two-dimensional picture is that truth-value assignments to sentences governed by metaphysical modal operators will be sensitive to which world is considered as actual. For instance, considering the actual world as actual it is metaphysically necessary that water is H_20. But considering w_1 as actual, it is metaphysically impossible that water is H_20, for under that supposition necessarily

[27]This description of the two-dimensional framework makes the assumption of modal monism, the thesis that there is a single modal space over which both primary and secondary intensions are defined. Although this thesis couples elegantly with aspects of the two-dimensionalist program, it is dispensable; see Chalmers (2002).

it is XYZ. It can help to see a tabular representation of the two-dimensional intension the epistemic two-dimensionalist assigns to a given expression. Take the sentence 'water is H_20'. In the following table, the labels of rows correspond to which world is considered as actual, and the labels of the columns correspond to which world is considered as counterfactual:

'water is H_20'	H_20 world	XYZ world
H_20 world	T	T
XYZ world	F	F

The diagonal of Ts and Fs reaching from the top left T to the bottom right F corresponds to the sentence's primary intension—hence its aptronym 'the diagonal intension'. The rows correspond to the secondary intension that the sentence is assigned when the H_20-world and XYZ-world are respectively considered as actual, which clearly represent two different functions. Thus an expression's secondary intension along with its extension varies with which world is considered as actual, hence why metaphysical necessities are sensitive to which world we consider as actual.

Adam Murray and Jessica Wilson (2012) argue for a two-dimensionalist treatment of Chisholm's paradox. However, they offer a 'metaphysical' interpretation of the Chalmersian framework, despite their interpretation being "formally analogous" (p. 198) to it. To be specific, Murray and Wilson depart from epistemic two-dimensionalism in their interpretation of secondary intensions. The traditional line is to treat all secondary intensions, except the one assigned when the actual world is considered as actual, as mere epistemic necessities for all we know a priori. However, Murray and Wilson (2012, p. 211) regard them as genuine 'relativised' metaphysical necessities. Nevertheless, whatever departure Murray and Wilson make, they still endorse (p. 203) the claim that not all truths vary with which worlds are considered as actual, since there must be some 'basic' ways of individuating or specifying worlds which are invariant as such. Indeed, like Chalmers, they (p. 202) hold that "the basic individuation of worlds [...] might proceed by way of, e.g., 'semantically stable' (Bealer 2000) or 'canonical' (Chalmers 2006) descriptions." Effectively, this guarantees that Murray and Wilson's 'metaphysical' interpretation of the Chalmersian framework does not depart foundationally from Chalmers' preferred epistemic construal. At points in the following discussion, this will become significant.

The central moral that two-dimensionalists want to draw from Chisholm's Paradox is that truth-value assignments to essentialist claims about Woody's material makeup are highly sensitive to which world is considered as actual. For on their view, proper names like 'Woody' function similarly to natural kind terms.[28] Thus, just as the denotation of 'water' varies with which world is considered as actual, so does the denotation of 'Woody'. As a consequence, the various denotations of 'Woody' differ in their modal profiles, specifically regarding the matter from which they can originate. The two-dimensionalist claims that acknowledging this dissolves the paradox.

To provide more detail, in Chisholm's Paradox the two-dimensionalist would claim that when we consider world 6 as actual it is necessary that amongst all six of Woody's original parts are, say, at least three of those which it has at 6. However, when we consider world 4 as actual it is not necessary that amongst all six of Woody's original parts are at least three of those which it has at 6, for when world 4 is considered as actual 'Woody' changes its denotation. Thus, when world 4 is considered as actual, it *is* necessary that amongst all six of Woody's parts are at least three of those it has at *4* (and so one of those which it has at 6).

We can consider a simple model of the above case within the two-dimensionalist framework. We work in the previously used propositional language $\mathcal{L}$, and a frame $\mathfrak{F}$ is defined as above too. However, a *model* $\mathfrak{M}$ of $\mathcal{L}$ based on $\mathfrak{F}$ is defined as a triple $\langle W, \mathcal{R}, \mathcal{V}\rangle$, where $\mathcal{V}$ is now a total function from triples of the form $\langle \phi, w, u\rangle$, for some proposition letter ϕ and $w, u \in W$, to either 0 (falsity) or 1 (truth). The idea will be that truth at $\langle w, u\rangle$ corresponds to truth at world u when w is considered as actual. We thus define '$(\mathfrak{M}, w, u) \models \phi$' for $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V}\rangle$, $w, u \in \mathcal{W}$, and ϕ a formula of the language, recursively as follows (where 'A' is an atomic formula, and ϕ and ψ formulae of $\mathcal{L}_p$):

[28] Murray and Wilson (*ibid.*, p. 206) suggest this as one natural way to develop the view.

$(\mathfrak{M}, w, u) \models A$ iff $\mathcal{V}(A, w, u) = 1$;

$(\mathfrak{M}, w, u) \models \neg\phi$ iff $(\mathfrak{M}, w, u) \not\models \phi$;

$(\mathfrak{M}, w, u) \models \phi \to \psi$ iff either $(\mathfrak{M}, w, u) \not\models \phi$ or $(\mathfrak{M}, w, u) \models \psi$;

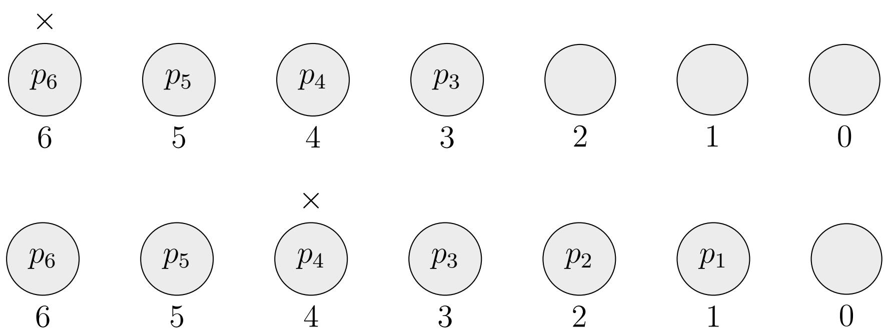
$(\mathfrak{M}, w, u) \models \Box\phi$ iff $(\mathfrak{M}, w, u') \models \phi$, for all $u' \in W$ such that $u\mathcal{R}u'$.

For an $\mathcal{L}$ model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$, a formula ϕ of $\mathcal{L}$ is true at $\langle w, u \rangle$ for $w, u \in \mathcal{W}$ iff $(\mathfrak{M}, w, u) \models \phi$. A formula ϕ of $\mathcal{L}$ is true in an $\mathcal{L}$ model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ iff it is true at $\langle w, u \rangle$ for all $w, u \in \mathcal{W}$. A formula ϕ of $\mathcal{L}$ is valid iff it is true in every $\mathcal{L}$ model.

Consider a particular model $\mathfrak{M}_2 = \langle W_2, \mathcal{R}_2, \mathcal{V}_2 \rangle$ of $\mathcal{L}$, such that:

$$\mathcal{W}_2 = \{n \in \mathbb{N} : n \leq 6\}$$
$$\mathcal{R}_2 = \{\langle u, w \rangle : u, w \in \mathbb{N}\}$$
$$\mathcal{V}_2(p_n, w, m) = \begin{cases} 1, & \text{if } m = n \ \wedge \ |w - n| \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

In diagram form, where the marked node corresponds to which world is considered as actual:



This model is very similar to the Salmon model we considered above. In effect, the modal profile Salmon stipulated Woody to have at a given world is now the modal profile Woody has in that world *when that world is considered as actual.* For instance, considering world 6 as actual, Woody must originate from at least half of the matter it does at 6. Yet considering world 4 as actual, Woody can originate from just less than half of the matter it does at 6. One can see this formally by noticing the following facts about the model:

$$\mathcal{V}_2(p_n, n, n) = 1, \text{ for all } n \in \mathcal{W}$$
$$\mathcal{V}_2(p_2, 6, n) = 0, \text{ for all } n \in \mathcal{W}$$
$$\mathcal{V}_2(p_2, 4, 2) = 1$$

As a result $(\mathfrak{M}_2, 6, 6) \not\models \Diamond p_2$, yet $(\mathfrak{M}_2, 4, 6) \models \Diamond p_2$.

However, despite the similarities with the Salmon model, $\mathfrak{M}_2$ does not serve as a countermodel to axiom schema **4**. For $\Diamond\Diamond\phi \to \Diamond\phi$ is true in $\mathfrak{M}_2$, given the definition of $\mathcal{R}_2$. Indeed, all the axioms of **S5** are true in $\mathfrak{M}_2$.[29] Nevertheless, as mentioned before, this generates the consequence that (2)-(7) are equivalent with (2′)-(7′) respectively. But in this setting that consequence is bizarre. The crucial point of emphasis is that the Modal Sorites involved no nested modal claims, so it is extremely difficult to see why it would result from slipping between different perspectives on which world is considered as actual. Indeed, as stressed previously, the distinctive feature of the Modal Sorites is that it can be run solely from the perspective of the actual world since its premises are all claimed to be actually true. The concern can be put alternatively in terms of the familiar reduction theorem of **S5**:[30]

S5 Reduction Theorem Any nested modal formula is **S5** equivalent to a non-nested formula.

According to the **S5 Reduction Theorem**, nested modal claims are always **S5** equivalent to non-nested claims. Thus, **S5** is simply not hospitable to blaming modal paradox on a shift of perspective induced by nested modal claims; to do

[29] Murray and Wilson (2012, p. 208) view this as a central advantage of the two-dimensionalist solution.

[30] See Hughes & Cresswell (1996, pp. 97-100) for a proof of a more precisely stated version of this theorem, which they also dub the 'S5 Reduction Theorem'. The theorem is used to show that S5 has a modal conjunctive normal form theorem, which weaker modal logics typically lack.

so would be to ignore the characteristic feature of **S5** that both contingency and non-contingency are non-contingent matters. This casts doubt on whether the two-dimensionalist offers a satisfactory solution to the paradoxes of essentialism.

A further worry is that the two-dimensionalist strategy is susceptible to *revenge* versions of Chisholm's Paradox. To appreciate this point, recall the connection between semantic externalism and two-dimensionalism. As was previously emphasised, expressions susceptible to twin-earth style thought experiments—twin-earthable expressions—are assigned primary intensions which assign different secondary intensions to the expression at different worlds considered as actual. But not all expressions are twin-earthable. As Chalmers (2012, p. 374) himself recognises, many expressions are non-twin-earthable, or semantically neutral, in that their primary intension assigns the expression the same secondary intension across all worlds considered as actual.[31] Such expressions include items of purely logical vocabulary like quantifiers and the truth-functional connectives, the predicates which feature in developed mathematical theories, foundational physics, and phenomenal terms like 'consciousness'. Indeed, as Chalmers (2012, pp. 233-243) emphasises, that there are such semantically neutral expressions is crucial to one's being in principle capable of articulating a semantically neutral description of the world—something which is integral to the two-dimensionalist program.

Consequently, if there are variants of Chisholm's Paradox formulated solely in terms of semantically neutral expressions, the two-dimensionalist proposal lacks the resources to handle them. For the modal status of semantically neutral sentences is not sensitive to which world is considered as actual. But there are variants of exactly this kind: in recent work, Andrew Bacon (2018, pp. 280-281) formalises a quantified version of the Modal Sorites in which all the non-modal terms involved in the argument are precise, and Bacon's idea can be adapted to articulate a quantified version of Chisholm's Paradox using only semantically neutral vocabulary.

Our official formulation of the argument was given in $\mathcal{L}$, a purely propositional

[31] Chalmers (2012, pp. 370-375) suggests that non-twin-earthability is only an approximation of semantic neutrality, but, as will become clear, nothing crucial to the following argument depends on this point.

language. But to demonstrate the above point, we can momentarily move to a more complex language. Thus let $\mathcal{L}_c$ contain individual variables $(x_1, x_2, ...)$, plural variables $(xx_1, xx_2, ...)$, the primitive two-place predicates Σ (originates from), $\prec$ (is one of), and $<$ (is an original proper part of), $\neg$ (negation) and $\rightarrow$ (the material conditional) as the only truth-functional connectives, $\forall$ (universal quantification), and $\Box$ (necessity), all of which are semantically neutral. The predicates Σ and $\prec$ take individual variables in their first argument and plural variables in their second argument; the predicate $<$ takes individual variables in both arguments. We also define numerical quantifiers $\exists_n$ read as 'there are at least n' in the typical recursive fashion:

$$\exists_1 xP(x) =_{df} \exists xP(x)$$
$$\exists_{n+1} xP(x) =_{df} \exists x(P(x) \wedge \exists_n y(y \neq x \wedge P(y)))$$

A semantically neutral variant of the paradox then runs as follows.[32] Consider first the premise that there are some things from which some other thing originates:

(1) $\exists xx \exists x \Sigma(x, xx)$

For simplicity, let us assume that actually there is only one complex object and nothing else besides its proper parts, and that this complex object is necessarily composed of six things.[33] Given **Tolerance**, it is possible that one of its actual original proper parts is not one of its original proper parts, even if the complex object exists.

(2) $\exists xx \exists x \Box(\Sigma(x, xx) \rightarrow \Diamond(\exists z(z = x) \wedge \exists_1 y(y \prec xx \wedge \neg y < x)))$

[32]The argument makes the assumption that $\prec$ is rigid, in the sense that if this is one of those, then necessarily this is one of those (provided there are those). See Williamson (2013, chap. 6) and Linnebo (2016) for discussion and motivation.

[33]One could drop the assumption that actually there is only one complex object and nothing else besides its proper parts by just imposing restrictions on the quantifiers in the following claims, provided the restrictions were stated in semantically neutral terms. However, it is preferable to reduce notational complexity.

Yet, as in the initial argument, such **Tolerance** iterates:

(3) $\exists xx \exists x \Box((\exists z(z = x) \wedge \exists_1 y(y \prec xx \wedge \neg y < x)) \rightarrow \Diamond(\exists z(z = x) \wedge \exists_2 y(y \prec xx \wedge \neg y < x)))$

In plainer terms: there are some things xx and some thing x such that, necessarily, if x can exist without at least one of its actual original proper parts as an original proper part, then, in that scenario it is possible for x to exist without at least two of its actual original proper parts as original proper parts. And so on:

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(7) $\exists xx \exists x \Box((\exists z(z = x) \wedge \exists_5 y(y \prec xx \wedge \neg y < x)) \rightarrow \Diamond(\exists z(z = x) \wedge \exists_6 y(y \prec xx \wedge \neg y < x)))$

In any plurally quantified modal system which contains **S4**, one can then derive:

(8) $\exists xx \exists x \Diamond(\exists z(z = x) \wedge \exists_6 y(y \prec xx \wedge \neg y < x))$

Given our assumption that the only actual complex object is necessarily composed of six things, we can thus infer that there is some complex object that could have existed whilst originating from completely different matter:

(9) $\exists xx \exists x(\Sigma(x, xx) \wedge \Diamond(\exists z(z = x) \wedge \exists_6 y(y \prec xx \wedge \neg y < x)))$

But from **Kripke's Thesis** and the assumption that there is only one complex object and nothing else besides its proper parts, the following claim may be inferred:

(10) $\forall xx \forall x(\Sigma(x, xx) \rightarrow \neg\Diamond(\exists z(z = x) \wedge \exists_6 y(y \prec xx \wedge \neg y < x)))$

Yet (10) contradicts (9).

The above is merely a quantified variant of Chisholm's Paradox. The two-dimensional approach, however, does not have the capacity to handle this particular argument, for it is formulated entirely in terms of semantically neutral vocabulary. Specifically, logical expressions like quantifiers do not exhibit the same variability in extension across worlds considered as actual as proper names like 'Woody' do. Since we ought to afford all variations of a paradox the same treatment, the two-dimensionalist's solution lacks the required scope.

One can anticipate the complaint that Σ (originates from) and $<$ (is an original proper part of) are not semantically neutral, perhaps due to it being contested that they are susceptible to twin-earth style thought experiments. However, regardless of whether they are, remember that it is key to the two-dimensionalist program that entire worlds can be described in semantically neutral terms. This includes their facts about mereology and origins. Accordingly, there must be *some* semantically neutral conditions in terms of which Σ- and $<$-facts can be redescribed. A variant of the above argument could then be run in which those neutral conditions are uniformly substituted for Σ and $<$ in the appropriate manner. More generally, given the two-dimensionalist reliance on basic semantically neutral descriptions of worlds, there will always be revenge versions of Chisholm Paradox that they are unable to handle.

3.4.3 Remarks

In this section, we have considered two alternative solutions to Chisholm's Paradox. We first saw that Salmon's anti-**S4** solution failed to generalise to the Modal Sorites. Thus, advocates of Salmon's proposal should still entertain the hypothesis that metaphysical necessity is a source of indeterminacy, a claim strongly suggested by the Modal Sorites. On the other hand, the two-dimensionalist offered a solution to Chisholm's Paradox which retained **S5** for necessity. However, this was a poisoned chalice, for it forced the two-dimensionalist to make unwelcome claims about the Modal Sorites. Moreover, the two-dimensionalist proposal was susceptible to revenge forms of Chisholm's Paradox. Together, these two concerns suggest that the two-dimensionalist proposal is difficult to maintain, which ought to increase one's confidence in the hypothesis that metaphysical necessity is a source of indeterminacy.

3.5 Morals

The hypothesis that metaphysical necessity is a source of indeterminacy reflects an earlier remark of Richard Cartwright (1968, p. 626):

> I see no reason, then, for thinking essentialism unintelligible. At the same time, I do not mean to suggest that it is without its complexities. Chief among these is the obscurity of the grounds on which ratings of attributes as essential or accidental are to be made. Apparently, in any particular case, one is simply to reflect on the question whether the object in question could or could not have lacked the attribute in question. [...] But the criteria to which one appeals in such reflection are sufficiently obscure to leave me, at least, with an embarrassingly large number of undecided cases. [...] The existence of such cases, even in such large number, does not show that there simply is no distinction between essential and accidental attributes of an object. But it does show that the distinction is a good deal less clear than essentialists are wont to suppose.

As Cartwright points out, essentialist discourse is not clearly constrained by any systematic body of principles. Instead, essentialist claims are often made on the basis of relatively *ad hoc* reflection, a pertinent example of which, one might recall, occurs in Kripke's (1980, p. 113) initial discussion of **Kripke's Thesis**:

> But what is harder to imagine is her [Elizabeth II] being born of different parents. It seems to me that anything coming from a different origin would not be this object.

Given this lack of systematic constraint, the theoretical role of essence is far less detailed than, say, the role of germanium or Newtonian mass. Since even the latter was a source of indeterminacy, one should not be surprised if essentialist attributions are loci of indeterminacy too. If we incorporate essentialist claims as constraints on metaphysical necessity *de re*, we may simply fail to isolate a unique modality.

Is this compatible with Kripke's introduction of metaphysical necessity as 'necessity in the highest degree', or Williamson's characterisation of it as 'the maximal objective modality'? We saw above that both glosses carry the presumption of determinacy via uniqueness. But if 'metaphysical necessity' fails to isolate a unique modality, how can this be?

The aim of this section will be to consider two natural ways of developing the indeterminacy hypothesis which can be reconciled with the idea that metaphysical necessity is the maximal objective modality. The guiding thought will be that the maximality postulate is one of, if not *the*, central postulate of metaphysical necessity's role, and, indeed, one of the most clearly articulated. As a consequence, any account of metaphysical necessity must find some means of preserving it.

Before we consider each way of integrating the indeterminacy and maximality claims, it will help to get clear on the latter. The division between objective and non-objective modalities is analogous to the distinction between objective and subjective notions of probability.[34] In the case of probability, objective probabilities are degrees of genuine worldly chance, whereas subjective probabilities are degrees of personal expectation. Similarly, the objective modalities concern variation in worldly circumstance, whereas the non-objective modalities concern variation along another dimension. For instance, physical necessity, practical necessity, and the various historical necessities are all objective in the intended sense: respectively, they concern what must happen given the nomic facts, our current practical means, or how history unfolded to a certain point. In contrast, the merely epistemic or linguistic modalities concern variation along a broadly representational dimension, with the former concerning what is epistemically open given what we know, and the latter concerning what remains true under various reinterpretations of a language. Along with these, the deontic and teleological modalities concern not variation in circumstance but constraints on how actual circumstance ought to be. Although no non-trivial necessary and sufficient conditions are presumed to divide the objective and non-objective modalities, the distinction is clear and useful enough to be of study.

[34]See Williamson (2016) for this characterisation. See also Rosen (2006, p. 16) on 'real' modalities, which he glosses as 'alethic, non-epistemic, and sometimes substantive or synthetic'.

Given this division, one may ask how the objective modalities relate to and interact with one another. A class of interesting questions in this vicinity concerns the comparative *broadness* of objective modalities. Is physical necessity broader than practical possibility? Than all the historical modalities? Or somewhere in between? Answering such questions has the potential to illuminate the connections between the objective modalities. Indeed, one particularly salient question is whether any modality is the broadest, and, if so, which is. As we have already seen, there is a famous candidate: according to Kripke and Williamson's characterisation of metaphysical necessity, metaphysical necessity is the broadest objective modality. On this conception, whatever is metaphysically necessary is necessary in every objective sense.

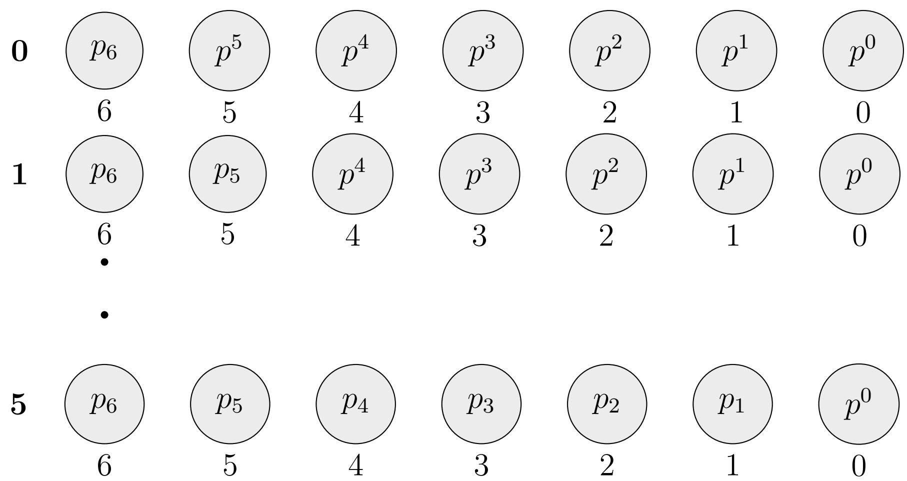
To return to the main thread, it is natural to wonder whether this conception is compatible with the indeterminacy hypothesis. Certainly, at an initial glance they would appear to be in tension. The maximality thesis carries the presumption of uniqueness, whereas, as suggested above, if 'metaphysical necessity' is indeterminate it simply fails to isolate a unique modality.

We can appreciate this by considering a more realistic, less simple model of the Woody case. The needed tweak of realism might have been transparent in the initial presentation. For the initial set up stipulated a series of seven worlds 6-0, in which there were the six mm_6 at world 6, the six mm_5 which differ only from mm_6 by one tiny piece at world 5, and more generally the six mm_i which differ only from mm_6 by $6 - i$ tiny pieces at world i. Yet at the final world 0, Woody does not originate from mm_0. But surely something does. After all, the only stipulated difference between worlds i and j is the difference in make up of mm_i and mm_j. So if something, namely Woody, originates from mm_6 at world 6, something originates from mm_0 at world 0. Call that thing 'Couldy'.

If one assumes the extensionality of composition, the view that no material distinct things share all and only the same proper parts, there will be a symmetry between Couldy and Woody's modal profiles. Wherever Woody is something, Couldy is not, and vice versa. To see this, let us expand $\mathcal{L}$ to $\mathcal{L}^*$ by introducing into the lexicon seven new atomic proposition letters ($p^0, p^1, ..., p^6$). Informally, we think of 'p^i' as the sentence 'Couldy originates from six parts, $6 - i$ of which it does at world 0'. We now want to modify our model so that 'p^i' is true at a world iff 'p_i' is false at that world. Indeed, doing so is straightforward. We can reuse the frame on which the model $\mathfrak{M}_{ns}$ was based, but define a new function $\mathcal{V}^*$ as follows:[35]

$$\mathcal{V}^*_{ns}(p_n, p, m) = \mathcal{V}_{ns}(p_n, p, m)$$

$$\mathcal{V}^*_{ns}(p^n, p, m) \neq \mathcal{V}^*_{ns}(p_n, p, m)$$



The interesting difference between this model and the simplified one concerns the presence of maximal modalities. Since it is natural to think of precisifications of metaphysical modality as corresponding to modalities themselves, we can ask whether one such precisification is the broadest. Indeed, in the initial model there was one maximal precisification: 5. For possibility according to any precisification entailed possibility according to 5. But in the more realistic model there is no longer any precisification that is broader than all others, for there is a precisification where p_n (for $1 \leq n \leq 5$) is possible and a precisification where p^n is possible, but no one precisification according to which both are. Thus, at first glance, the maximality and indeterminacy hypotheses look incompatible.

[35]The situation is parallel on the **Standard** vagueness-based model $\mathfrak{M}_s$.

Despite these appearances, there are at least two ways in which the maximality and indeterminacy hypotheses can be compatibly integrated. The following outlines two such different ways, draws some important contrasts between the approaches, and suggests that one of the approaches fares much better than the other.

On the first *conciliatory* view, paradigm origin essentialism claims such as **Kripke's Thesis** are still incorporated into the role of metaphysical necessity. On the second *revisionary* view, they are not.

Conciliatory Paradigm origin essentialism claims are to be incorporated into the role of metaphysical necessity.

Revisionary Metaphysical necessity does not license various paradigm origin essentialist claims.

Effectively, the conciliatory view is the conception of metaphysical necessity under which we have been operating. At its heart is the idea that essentialist judgements constrain metaphysical necessity *de re*. However, as Cartwright observes, for certain of these judgements it will be indeterminate whether they are necessary. Thus, metaphysical necessity admits of indeterminacy and thus several admissible precisifications.

In order to render this compatible with the maximality thesis it must be determinate that metaphysical necessity is the broadest objective modality. That is, on any precisification thereof, it must be broader than all other objective modalities according to *that* precisification. So, for example, according to every precisification metaphysical necessity must be broader than physical necessity, practical necessity and so on. Yet this is perfectly compatible with the above model. For given any precisification of metaphysical necessity, the model recognises different precisifications which it is not broader than. However, nothing forces the given precisification to recognise those *other* precisifications as corresponding to objective modalities. Thus, a natural picture is one on which although it is determinate that one (and only one) of the above precisifications corresponds to an objective modality—indeed, the broadest one—it is not determinate which one does. On this conception, our use of 'metaphysical necessity' fails to isolate a unique modality, but supervaluating over the various precisifications forces it to be supertrue that whichever precisification is selected corresponds to the broadest objective modality according to *that* very precisification.

Certainly, this appears to be a stable way of integrating the maximality and indeterminacy hypotheses. Nevertheless, it invites certain residual questions, perhaps the most immediate of which is why each precisification would deem the others as corresponding to non-objective modalities. After all, in both the **Standard** and **Non-Standard** vagueness-based models, each precisification deemed at least some others as admissible. Indeed, were they not to, the vagueness-based solution to the paradoxes would be fatally hindered. Moreover, although no independent criterion divides the objective and non-objective modalities, it is difficult to see why one precisification would fall on the objective side but another not. For example, broadly speaking, all would seem to concern variation along a worldly dimension, as opposed to a merely epistemic or representational one. Moreover, none bear any semblance to paradigm deontic or teleological modalities either. Thus there would appear to be little independent basis for any precisification simply to exclude the others from the family of objective modalities. What is needed is some separate means of substantiating this claim, as opposed to leaving it to mere speculation.

The revisionary view of metaphysical modality faces no such challenge. A version of this view is suggested recently by Cian Dorr (2016, p. 69):

> 'Metaphysically necessary' was introduced into the philosophical vernacular partly through general formulas—for example, the equation of metaphysical necessity with "unrestricted" or "absolute" necessity, or 'necessity in the highest degree—whatever that means' (Kripke 1972, p. 99)—but partly also through philosophers exchanging opinions about which truths are, in fact, metaphysically necessary—e.g. that nothing is green and red all over, that Nixon is not an inanimate object, that a certain lectern is not made of ice, etc. [...] Given the character of 'metaphysically necessary' as a term of art, charity to the explicit explanatory remarks made by those who introduced the term should weigh especially heavily with us in deciding how to interpret it, while charity to claims that they made where they thought of themselves as taking a stand on philosophically controversial questions should count for relatively little. If we can make sense of a notion of necessity with a good

> claim to labels like 'necessity in the highest degree', it would be perverse to interpret 'metaphysically necessary' as expressing something more restricted just for the sake of being more charitable to the case-by-case modal pronouncements of, say, Saul Kripke.

Dorr's point reflects our discussion so far: it appears difficult to integrate the maximality thesis with the body of claims about metaphysical necessity made by philosophers on a case-by-case basis. Moreover, in line with the current methodology, Dorr also recognises the importance of respecting the maximality thesis, it being a primary postulate of metaphysical necessity's theoretical role. Thus, on this view, those case-by-case judgements which fail to integrate with the maximality thesis ought to be rejected, and the broadest objective modality—metaphysical necessity—may possess an extension different from that which has been assumed in practice.

On this conception, all of the above precisifications correspond to objective modalities, but none to the broadest objective modality. However if, as would be highly natural, the objective modalities are closed under variations operations such as combination and union, there would be a broadest objective modality.[36] This modality would be broader than each of these precisifications, and, as Dorr would recommend, more deserving of the name 'metaphysical necessity'.

One might think that a natural candidate for this broadest objective modality is determinate necessity or, put alternatively, non-determinate impossibility. For, as can be checked in the above model, if a proposition is possible on any precisification then it is not determinately impossible. Thus, non-determinate impossibility is at least as broad as each of the above precisifications. Nevertheless, there is reason not to think that non-determinate impossibility is the *broadest* objective modality: if the **4** schema for determinate necessity is valid, then the **4** schema for determinacy is valid too.[37] Yet, typically, the **4** schema for determinacy is eschewed in order to recognise borderline clear cases.[38] However if determinate necessity fails to conform

[36] See Williamson (2016) and also Essay One of this thesis.

[37] To appreciate this, it may helpful to note that for a frame $\mathcal{F} = \langle \mathcal{P}, \mathcal{R}^{\Delta}, \mathcal{W}^{*} \rangle$ in which $\mathcal{R}^{\Delta}$ is not transitive, there are models based on that frame in which instances of the schema $\Delta\Box\phi \rightarrow \Delta\Box\Delta\Box\phi$ are not true.

[38] However, Susanne Bobzien (2015) takes exception to the view that determinacy does not conform to the **4** schema.

to its version of the **4** schema, it is not even broader than its twofold iteration, thus disqualifying it as the broadest objective modality.

Moreover, given the above treatment of the Modal Sorites, non-determinate impossibility has the odd feature of licensing all but the last of the following (as can be checked, (D2′)-(D6′) are true in each of the above $\mathcal{L}^+$ models although (D7′) is not):

(D2′) $\neg\Delta\neg\Diamond p_6 \rightarrow \neg\Delta\neg\Diamond p_5$

(D3′) $\neg\Delta\neg\Diamond p_5 \rightarrow \neg\Delta\neg\Diamond p_4$

.

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.

(D7′) $\neg\Delta\neg\Diamond p_1 \rightarrow \neg\Delta\neg\Diamond p_0$

Yet, one might ask, why would p_6-p_1 all be possible in the broadest objective sense yet p_0 (that Woody could originate from totally different matter) fail to be? As before, the possibility of p_0 would not appear to lapse into the realm of the non-objective. Indeed, following Dorr's remark, refusing to acknowledge the objective possibility of p_0 seems worryingly close to misidentifying a restricted modality with metaphysical necessity, the broadest objective modality.

A more plausible view is one on which p_0 is possible in the broadest objective sense. On this conception, Woody could (in the broadest objective sense) originate from entirely different matter, and the necessity of origin is false for the broadest objective sense of necessity. More generally, metaphysical necessity—the broadest objective necessity—does not license various paradigm essentialism claims, especially those over which there is dispute. The moral of the above, then, is not that metaphysical necessity is vague, but rather that there are several indeterminacy-generating restrictions thereof that we have simply mistaken for metaphysical necessity. According to this view, the so-called paradoxes of essentialism are artefacts of underspecified modal restrictions, and not of any deep feature of metaphysical necessity. Indeed, strictly speaking, this is not necessarily a view according to which metaphysical necessity is vague; it is a view on which the role we *mistook* metaphysical necessity to have is a source of significant indeterminacy. Thus, on this revisionary approach, metaphysical necessity has a different extension from that which it is assumed to have in current practice.

3.6 Concluding Remarks

To summarise, the paper has considered the so-called paradoxes of essentialism and showed that they are equivalent in a certain sense if the logic of metaphysical necessity is **S5**. It then established that a vagueness-based solution could preserve **S5** for metaphysical necessity. Given the **S5** equivalence of the paradoxes, the vagueness-based solution to the Modal Sorites immediately extended to Chisholm's Paradox with some unexpected but not implausible consequences. The paper then argued that two rival solutions lead in different ways to the hypothesis that indeterminacy was responsible for at least some of paradoxes of essentialism. In the case of Salmon's solution, this was because his rejection of the **4** schema for metaphysical necessity was of no help in handling the Modal Sorites. In the case of the two-dimensionalist's solution, this was because their proposal mishandled the Modal Sorites and could not handle all versions of Chisholm's Paradox.

Finally, the paper explored two conceptions of metaphysical necessity which integrate the indeterminacy and maximality theses. One conception, the conciliatory view, was able to exploit typical supervaluationist resources to maintain both that metaphysical necessity was the broadest objective modality and a source of indeterminacy. However, motivated by a desire for uniformity amongst the objective modalities, the revisionary conception posited indeterminacy not in metaphysical necessity, but in less broad, restricted modalities. On the resultant view, the paradoxes of essentialism arise partly due to our having misidentified these restricted modalities with metaphysical necessity. Although both conceptions remain live, the revisionary approach offered a less arbitrarily circumscribed conception of objective modality, and was thus suggested to be the more promising view.

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4
Contingentism and Compossibility

Abstract

In recent work, Timothy Williamson (2010; 2013) has argued that quantified contingentist **S5** with identity and second-order quantification is not the correct logic of metaphysical necessity. Specifically, he has argued that contingentists are unable to meet a formal 'expressive challenge', which indicates that they are incapable of expressing modal claims that ought to be intelligible by their own lights. In contrast, necessitists possess the required expressive capacity. In opposition to Williamson, this paper employs the resources of a well-motivated metaphysics of necessity to argue that contingentists can meet the expressive challenge, and thus in effect simulate the entirety of necessitist discourse. In an appendix, it is proven that the metaphysics of necessity in question allows contingentists to meet Williamson's formal expressive challenge.

Over the last two decades, Timothy Williamson has offered a sustained defence of necessitism, the view that necessarily everything necessarily exists.[1] As a key part of his defence, Williamson (2010; 2013) presents an 'expressive challenge' to advocates of necessitism's rival view, contingentism. Crudely stated, the issue is that contingentists are incapable of expressing modal claims which ought to be intelligible by their own lights and are required for crucial parts of modal theorising. In contrast, necessitists face no such problem. If this disparity is genuine, it is a disadvantage for contingentists.

In opposition to Williamson, this paper argues that contingentists can meet the expressive challenge. The strategy will be to meet Williamson's (2010; 2013, chap. 7) demand to provide a recursive mapping of all necessitist discourse to a 'neutral' body of claims which contingentists can treat as the 'cash value' of necessitist talk. Ultimately, this recursive mapping will proceed via the introduction of new expressive resources which both the necessitist and contingentist ought to recognise.

The structure of the paper is as follows. §4.1 outlines some working assumptions and sets out the expressive challenge to contingentism. §4.2 discusses the apparent shortcomings of a natural extant response due to Fine (1977). Drawing on Essay Three of this thesis, §4.3 then motivates a new metaphysics of necessity and §4.4 employs its expressive resources to show informally how the contingentist can meet the expressive challenge. §4.5 discusses a salient objection and, finally, in an appendix it is proven than this extended contingentist theory can map the entirety of necessitist discourse to a neutral body of discourse. The main philosophical payoff is that necessitists can no longer claim any expressive advantage over contingentists, which undermines one of the strongest arguments in favour of necessitism.

[1] There has been an understandable tendency not to use 'exists' when formulating contingentism and necessitism; see Williamson (2013; chap. 1). The risk is that 'exists' is often conflated with 'has spatiotemporal location', whereas here it ought to be understood in the utmost unrestricted sense of 'there is', a sense in which there may be things outside the spatiotemporal realm. For ease of expression, hereafter we simply stipulate that 'exists' is to be understood in the latter sense.

4.1 The Expressive Challenge

4.1.1 Preliminaries

First-order necessitism is the claim that necessarily every individual is necessarily something, where the necessity is interpreted as metaphysical and the quantifiers as unrestricted. More formally, it may be expressed by the following formula of a first-order quantified modal language under the aforementioned interpretation:

NNE $\Box\forall x\Box\exists y(y = x)$

In slogan form, **NNE** states that being is non-contingent. *First-order contingentism* is the negation of **NNE**; for first-order contingentists, possibly some individual possibly fails to be.

Analogues of the first-order necessitism-contingentism debate recur at the higher-orders. The particular interest of this paper is in the debate at the second-order. To regiment this debate, it is natural to state the central positions in a second-order quantified modal language with predicate variables of any finite adicity and second-order quantifiers which bind such variables. It is stipulated that on the intended interpretation of this language object-language predicate variables and second-order quantifiers are to be read in plural terms. That is, predicate variables (under a given assignment) are to be read as denoting some individuals, and predications of the form Xx (under a given assignment) are to be understood as stating that what x denotes (under that assignment) is one of the things denoted by X (under that assignment). One may think of n-place (for $n > 1$) predicate variables as denoting some n-tuples (under a given assignment). In line with this interpretation, second-order quantifiers are to be read as plurally quantifying over some individuals so that, for example, $\forall x\exists XXx$ expresses the claim that everything is one of some things. Moreover, on the intended interpretation, second-order variables are permitted to range over the 'empty' plurality. Although this requires a departure from the English idiom of plural quantification, its advantage is that a permits a proper interpretation of second-order logic, for which there is precedent in Boolos (1984, 1985). This plural regimentation is a harmlessly simplified construal of the second-order debate; its advantage is that it makes parts of the following discussion more tractable. Furthermore, it is expected that the crucial points of the discussion can be transposed without loss of plausibility into a setting in which second-order quantification is treated as *sui generis*.

Two crucial pieces of commentary about the plural regimentation are in order. First, throughout the paper singular quantification over 'pluralities' is to be understood as shorthand for irreducibly plural quantification. Similarly, talk of 'properties' or 'relations' should be understood as shorthand for the above plural idiom too. On occasion, both abbreviations will simplify the mode of expression although generally they will be avoided. Second, 'is one of' is to be understood as *rigid* in the sense that if this is (not) one of those then necessarily this is (not) one of those, provided those exist.[2] This rigid reading will be implemented model-theoretically by not requiring the assignment of 'extensions' to predicate variables to be relativised to a 'world'.[3]

Given this regimentation, *second-order necessitism* is the view that necessarily for all things X necessarily there are some things Y which are exactly the X. Put formally, where X and Y are n-adic predicate variables, second-order necessitists will accept the following principle:

$$\mathbf{NNE}^2_n \quad \Box\forall X\Box\exists Y\forall x_1 \ldots \forall x_n(Xx_1 \ldots x_n \leftrightarrow Yx_1 \ldots x_n)$$

The expression '$\mathbf{NNE}^2$' is used for the set of sentences comprising exactly $\mathbf{NNE}^2_n$ for every n. As in the first-order debate, *second-order contingentism* is the set of sentences comprising exactly the negation of $\mathbf{NNE}^2_n$ for each n. Saliently, on the current plural understanding of second-order quantification under which 'is one of' is rigid, if predicate variables are coextensive under a given assignment then they are necessarily coextensive under that assignment. This is pertinent because in effect it permits coextensionality between predicate variables to serve as a type of identity condition between them. Moreover, the rigid understanding of 'is one of'

[2]See Linnebo (2016, p. 654) a formalisation of the rigidity thesis in plural modal logic.
[3]See **Definition 2.5** of the Appendix Part B.

has the effect of making **NNE** and **NNE**^{2} mutually entailing in the background second-order logic.[4] Nevertheless, to be explicit, we call those who are both first-order and second-order necessitists *necessitists*, and those who are both first-order and second-order contingentists *contingentists*. The question addressed in this paper is whether necessitists have a theoretically worthwhile expressive advantage over contingentists. As mentioned previously, the answer will be in the negative.[5]

Within the contingentist framework, there are various decision points which separate its advocates. Most notably, there is the question of whether an atomic predicate can truly apply to an individual at a world when that individual does not even exist there. The debate centres around whether the following principle ought to be a theorem of contingentist quantified modal logic (where F is an n-place atomic predicate):[6]

Being $\Box\forall x_1 \ldots \Box\forall x_n \Box(Fx_1 \ldots x_n \to (\exists y\, x_1 = y \wedge \cdots \wedge \exists y\, x_n = y))$

The decision matrix concerning **Being** is not a binary one. *Hardline contingentists* accept **Being**. *Moderate contingentists* accept **Being** except for broadly logical predicates like identity. And *mild contingentists* reject **Being** less systematically for predicates which are not 'existence entailing'. For the latter camp, it is true, for example, that a painting could depict Wittgenstein's merely possible son despite his son not being something, for 'depicts' is a non-existence entailing predicate. (We may assume that a particular possible son is fixed by the author's intentional state.) But even mild contingentists will accept **Being** for clearly existence entailing

[4]**NNE** and **NNE**^{2} are mutually entailing over the class of models defined in Appendix Part B.

[5]There are a variety of other positions that will not be considered, such as the combination of first-order contingentism and second-order necessitism. Indeed, since **NNE** and **NNE**^{2} are mutually entailing on the current simplified plural regimentation, this position is not currently available. Nevertheless, such a position is not susceptible to the expressive challenge which contingentists face, but arguably lacks philosophical motivation. See Williamson (2013, chap. 6), and Fine (2002).

[6]Some contingentists have formulated their view in a language with a device for forming complex predicates from open sentences. To be specific, they use the variable binding device λ to combine n distinct variables $v_1, ..., v_n$ with an open sentence ψ to form a one-place predicate $\lambda v_1 \ldots v_n.\psi$. Since such contingentist logics are closed under uniform substitution, the theoremhood of **Being** ensures that complex predicates conform to a version of **Being** too. Thus, in such logics, when a does not exist even the complex predication $\lambda v.\neg\exists y(y = v)a$ will be false. In order to ameliorate this consequence, contingentists who accept **Being** have rejected the equivalence of $\lambda v_1 \ldots v_n.\psi(a_1 \ldots a_n)$ and $\psi[a_1/v_1 \ldots a_n/v_n]$. See Stalnaker (1977) and Williamson (2013, chap. 4) for discussion.

predicates like 'being spatiotemporal'. To draw an analogy, mild contingentists resemble free positive logicians in their attribution of properties to non-existents, whereas hardline contingentists resemble negative free logicians in simply taking all predications of properties to non-existents to be false, which is why Fine (1977a) dubs **Being** 'the Falsehood Convention'.

The details and formal implementation of these different varieties of contingentism differ quite drastically. As a consequence, an answer to Williamson's expressive challenge cannot be offered 'schematically' between different contingentist views. Instead, one simply must select a form of contingentism and determine whether its proponents can meet the challenge. Thus, in what follows, we focus on hardline contingentist theories. An advantage of doing so is that hardline contingentism has been thought to fare worst with respect to such challenges. Therefore, if hardline contingentism meets the expressive challenge in the proposed way, that constitutes good evidence that more moderate contingentist theories may do so too.[7]

4.1.2 Recursive Mappings

As defined above, necessitism is the set of modal generalisations **NNE** and **NNE**^{2}. Given the strength of these generalisations, the necessitist offers a rich modal theory. There has thus been an initial concern that contingentist theories will be insufficiently rich in comparison. More specifically, the worry is that contingentist theories may be unable to mark genuine modal distinctions and acknowledge genuine modal hypotheses which necessitists can articulate with ease. In order to assuage this concern rigorously, contingentists can attempt, and have attempted, to offer a systematic method for generating a contingentist-acceptable analogue of any given necessitist claim. Put grandiosely, the upshot would be that a necessitist based description of modal space could be re-articulated in contingentist terms.

There is a wealth of ways in which one might execute this project. Following Fine's (1977a) related proposal, on might execute the project by constructing

[7]See also Fritz & Goodman (2017) for evidence that hardline contingentism fares particularly poorly with respect to a related 'paraphrase' based expressive challenge.

'analyses' of necessitist discourse in terms of contingentist discourse.[8] Moreover, Goodman & Fritz (2017) suggest that the project should proceed via a sophisticated method of paraphrases between necessitist and contingentist discourse. And, in significant contrast to both of the above, Williamson (2010; 2013) proposes a method based on intra-linguistic mappings from necessitist and contingentist discourse to 'neutral' discourse. Since the recent body of technical results attached to Williamson's (2010) approach lend themselves neatly to the proposal advocated below, his approach will be adopted. It will be clear how to transpose the technical details into the other settings.

Williamson's method requires a substantial amount of setup. To begin with, it is assumed that the necessitist and contingentist speak a common language under the same interpretation, a language where the modal operators are read as expressing metaphysical necessity, and the quantifiers as achieving unrestricted generality. It is also assumed that the language contains predicate variables and a universal quantifier which binds them. As above, this quantifier is to be read plurally, again with unrestricted generality. For a more detailed presentation of the language's syntax, the reader should consult the appendices.

The shared language must also contain devices of modal anaphora, $\uparrow$ and $\downarrow$. The function of these devices is effectively to serve as a type of generalised actuality operator: they are designed to allow the language to simulate reference to worlds which one has previously 'considered'. In particular, the $\uparrow$ operator allows one to 'store' the current world of evaluation, and at a later point the $\downarrow$ operator allows one to 'retrieve' that world and place it as the world of evaluation. To understand how the operators work, consider the following formula.

[8] Fine himself is interested in the actualism-possibilism debate, which bears strong parallels to the necessitism-contingentism debate in this regard. Although Fine discusses 'translations' between actualists and possibilists (see in particular the technical appendix of Fine (1977a)), it is clear that he does not require perfect synonymy between what is translated and its translation. Indeed, elsewhere Fine (2016, p. 554) has suggested "weaker standards" need only be met for a successful reduction. Relatedly, according to Williamson (2013, p. 307, n. 2), in personal communication Fine has stated that he does not require the mappings to preserve meaning. Instead, he would accept a grounding constraint: the mapping must take one from a given necessitist proposition to its contingentist grounds. See Fine (2001) for the notion of grounding, and its use in similar disputes.

$$\Diamond\uparrow\Box\forall x(F(x) \rightarrow \downarrow G(x))$$

In extensional parlance, this is read as the claim that there is some possibility at which every F in any possible world is a G. Without recourse to talk of possible worlds, we might read the sentence as saying: possibly it was once necessary that every F was then G. The devices of modal anaphora permit one to express such claims in the object-language. Ultimately, even more sophisticated versions of these devices might have to be introduced to increase expressive power further, as in Correia (2007), but these more basic devices suffice for current purposes.

Saddled with an apparent opposition to common opinion, typically necessitists attempt to ameliorate their position by highlighting claims close to ¬**NNE** which they can endorse. Although the necessitist accepts that everything necessarily exists, they still accept that some concrete (i.e. spatiotemporal) thing could have failed to be concrete. For, quite generally, the necessitist is happy to recognise that there are many concrete individuals which might have been merely possibly concrete, and thus non-concrete. Thus, we assume that the necessitist is capable of expressing this within the shared language. Accordingly, the language will contain the monadic predicate constant 'C', read as 'is grounded in the concrete' or 'is chunky' for short. We choose this particular interpretation of 'C', as opposed to 'is concrete', since the necessitist might believe in abstracta such as sets, which, though not concrete, are not typically the entities in dispute between necessitists and contingentists. According to Williamson (2010, p. 673), in the operative sense of 'grounding', sets are grounded in the concrete whenever their members are all grounded in the concrete, thus ensuring that any pure set is chunky. Williamson (2010, p. 673) also claims that "numbers may count as grounded in the concrete, perhaps through one or more stages of logicist abstraction".

As one would expect, the necessitist and contingentist agree on all matters chunky. The realm of the chunky, then, constitutes a neutral domain between the two theories; any difference between either position can be traced back to a dispute over the non-chunky. Indeed, to make this thought precise, we can articulate a syntactic criterion for neutral sentences of the language.[9] (In what follows, $X{\leq}C$ is an abbreviation of $\forall x_1 ... \forall x_n(Xx_1...x_n \rightarrow Cx_1 \wedge ... \wedge Cx_n)$, for n-adic X.)

A formula ϕ is *neutral* iff

a. Each n-place predicate constant F of the language which occurs in ϕ is such that every occurrence of it appears within a chunky context $Fx_1 ... x_n \wedge Cx_1 \wedge ... \wedge Cx_n$, and

Each n-place predicate variable V of the language which occurs in ϕ is such that every occurrence of it appears within a chunky context $Vx_1 ... x_n \wedge V{\leq}C$, and

b. Each first-order quantifier $\exists$ of the language which occurs in ϕ is such that every occurrence of it appears within a chunky context $\exists x(Cx \wedge ...)$, and

Each second-order quantifier $\exists$ of the language which occurs in ϕ is such that every occurrence of it appears within a chunky context $\exists X(X \leq C \wedge ...)$.

To appreciate the definition, notice that the claim $\exists xFx \wedge Ga$ is not neutral. But if its quantifier and 'F' are forced to occur within respective chunky contexts, the result is the neutral $\exists x(Cx \wedge Fx \wedge Cx) \wedge Ga \wedge Ca$.[10] To repeat, these neutral formulae are not in dispute between the necessitist and contingentist, at least in virtue of their respective commitments to necessitism or contingentism.

Williamson exploits this neutral domain of agreement to compare necessitism and contingentism via a recursive procedure. First, the necessitist provides a recursive intra-linguistic mapping of non-neutral sentences to neutral sentences which are equivalent by the contingentist's lights. Second, the contingentist provides a recursive intra-linguistic mapping of non-neutral sentences to neutral sentences which are equivalent by the necessitist's lights. If both mappings succeed, each theorist can recognise and acknowledge each other's modal distinctions and hypotheses, for each theory can be seen as comprising a body of claims which, by its advocates' own

[9] Compare Fine's (1981 p.296) notion of a restricted formula.

[10] We shall often replace neutral formulas with simplified formulas that are equivalent to them in the background logic that is specified below.

lights, is equivalent to a body of neutral claims. Yet if one side is able to map the other side's claims to a neutral body but not vice-versa, the former may enjoy an advantage over the latter: in effect, the latter may not be able to recognise genuine modal claims which the other can. Of course, Williamson himself argues that the contingentist is in this position. But before studying the contingentist's shortcoming, it will be instructive to consider the necessitist's success.

4.1.3 Mapping Contingentist to Necessitist Discourse

This section specifies a recursive mapping of each contingentist claim A to a neutral claim $[A]^{\text{CON}}$ which is equivalent to A by the contingentist's lights. According to Williamson (2013, p. 305), since the necessitist and contingentist agree on all neutral formulae, the necessitist can view the neutral claim as the 'kernel of truth' or 'cash-value' of the contingentist's initial claim A. Of course, the necessitist will reject the equivalence of A and $[A]^{\text{CON}}$, but this does not effect the significance of the mapping. In contrast, it further isolates what is at stake in the dispute between necessitists and contingentists.

Strictly speaking, Williamson's method of mapping contingentist to neutral discourse requires one to make some auxiliary assumptions on behalf of the contingentist. The reason is that as stated contingentism is just the negation of **NNE** and each $\mathbf{NNE}^2_n$. Yet, by itself, the theory generated by those axioms in the background logic specified below is logically too weak to license the desired equivalences between contingentist claims and their neutral analogues. Thus, Williamson's discussion focuses on a natural version of contingentism, which is axiomatised in the below natural background logic by the single claim that necessarily everything is chunky.

AuxCon $\Box\forall xCx$

As Williamson (2013, p. 678) recognises, a peculiarity of this auxiliary contingentist theory is that it does not even include $\neg$**NNE** or $\neg\mathbf{NNE}^2_n$ for any finite n, which has the effect of making **AuxCon** consistent with necessitism. Nevertheless, this auxiliary claim ought to be appealing to contingentists: for them being chunky is just equivalent to existing, and of course necessarily everything exists. Moreover, due to this peculiarity, this auxiliary contingentist is not a contingentist in the initially defined sense. Thus, it should be noted that 'contingentism' will be left ambiguous between auxiliary contingentism and contingentism in what follows, although the ambiguity will often be harmless.

The next step is to provide a mapping $[.]^{\text{CON}}$ of contingentist discourse to neutral discourse, where **AuxCon** entails $A \leftrightarrow [A]^{\text{CON}}$, for all formulae A. To reiterate the above, this mapping is one which provides a neutral analogue for each contingentist claim that by **AuxCon**'s lights is equivalent to the initial contingentist claim. Given that each claim $[A]^{\text{CON}}$ must be neutral, they must all conform to the above syntactic requirement on neutral formulae. Specifically, the contingentist's quantified claims must be mapped to claims which are restricted to the chunky. But this suggests a natural way for the necessitist to map the contingentist's claims to a neutral body of discourse. From the necessitist's perspective, the contingentist's ontology is properly included in what there is, so contingentists can be viewed as restricting their attention solely to the realm of the chunky. Indeed, given the current reading of second-order quantification, this applies to contingentist second-order quantification too. Such observations generate an appropriate recursive mapping:

$[Fv_1 \dots v_n]^{\text{CON}} = Fv_1 \dots v_n \wedge Cv_1 \wedge ... \wedge Cv_n$, for n-place non-logical atomic F

$[Vv_1 \dots v_n]^{\text{CON}} = Vv_1 \dots v_n \wedge V{\leq}C$, for n-place variable V

$[v_1 = v_2]^{\text{CON}} = v_1 = v_2 \wedge Cv_1 \wedge Cv_2$

$[\neg A]^{\text{CON}} = \neg[A]^{\text{CON}}$

$[A \wedge B]^{\text{CON}} = [A]^{\text{CON}} \wedge [B]^{\text{CON}}$

$[\Diamond A]^{\text{CON}} = \Diamond[A]^{\text{CON}}$

$$[\uparrow A]^{\text{CON}} = \uparrow[A]^{\text{CON}}$$

$$[\downarrow A]^{\text{CON}} = \downarrow[A]^{\text{CON}}$$

$$[\exists x A]^{\text{CON}} = \exists x(Cx \wedge [A]^{\text{CON}})$$

$$[\exists X A]^{\text{CON}} = \exists X(X \leq C \wedge [A]^{\text{CON}})$$

As may be seen, the function commutes with truth-functional connectives and modal operators, and atomic predications are mapped to formulas in which they occur within chunky contexts. Moreover, crucially, both first and second-order quantifications are mapped to formulas in which they occur in chunky contexts too, thus ensuring that each value of the mapping is a neutral formula.

The mapping generates some welcome results. Suppose the contingentist claims that some mountain could have failed to exist.

1 $\exists x(Mx \wedge \Diamond\forall y \neg y = x)$

The necessitist can use $[.]^{\text{CON}}$ to map the contingentist's claim to a neutral equivalent:[11]

$[\mathbf{1}]^{\text{CON}}$ $\exists x(Cx \wedge Mx \wedge \Diamond\forall y(Cy \rightarrow \neg y = x))$

$[\mathbf{1}]^{\text{CON}}$ states that some chunky mountain is possibly not chunky, a claim which the contingentist will recognise as equivalent to **1**. Similarly, suppose the contingentist claims that possibly there are some one or more things no one of which actually exists.

[11] In what follows, the values of the mapping will often be represented by formulas which are equivalent to them in the background logic specified below from the perspective of the relevant auxiliary theory.

2 $\uparrow\Diamond\exists X\exists x(Xx \wedge \downarrow\forall y\neg Xy)$

The necessitist can again use $[.]^{\text{CON}}$ to map this claim to the neutral claim that possibly there are some one or more chunky things which actually are not chunky:

$[\mathbf{2}]^{\text{CON}}$ $\uparrow\Diamond\exists X(X{\leq}C \wedge \exists x(Cx \wedge Xx \wedge \downarrow\forall x(Cx \rightarrow \neg Xx)))$

These examples indicate that the mapping is on the right track, but the interest is in the more general result of whether **AuxCon** entails all biconditionals of the form $A \leftrightarrow [A]^{\text{CON}}$. In order to ask this question, a background logic must first be specified to host the class of entailments. So as not to beg any questions, this logic must be neutral between necessitism and contingentism. In the appendices, essentially this neutral background logic is specified model-theoretically, but several points are worth mentioning in the main body of text.

Models for the background logic are standard variable domain models for second-order quantified modal logics. The model theory uses the typical definition of consequence ($\models$) on which $\Gamma \models \phi$ iff in every model ϕ is true at every 'world' on a particular assignment at which every $\gamma \in \Gamma$ is. The choice of variable domains has the effect of making **NNE** true in some models, false in others, which constitutes a mild form of neutrality between necessitism and contingentism. In these models, variable assignments assign first-order variables individuals from a non-empty 'outer domain' of which all world domains are subsets. As is typical in variable domain quantified modal logics, this permits variables to be assigned individuals which do not belong to the current world of evaluation. However, unlike typical variable domain quantified logics, in this background logic world domains can be empty. Moreover, individual variables can be assigned items which do not belong to the domain of any world of evaluation. Hence even extremely weak claims such as $\exists x(x = x)$ or $\Diamond\exists y(x = y)$ will not be valid, which displays just how weak the background logic is. This weakness is a virtue, for it ensures that no untoward assumptions are required to show that **AuxCon** entails the relevant biconditionals.

In order to validate **Being**, the model theory requires that the extension of an n-place non-logical predicate at a world includes only n-tuples taken from that world's domain. The model theory thus conforms with our assumption of hardline contingentism. Note also that **Being** should be welcomed by the necessitist, since they will recognise that every individual is included in every world's domain anyway. Finally, we assume that the logic for metaphysical necessity is **S5**. The model theory guarantees this by not using accessibility relations.

In this background logic, Williamson (2010) proves the following result.[12]

AuxCon $\models A \leftrightarrow [A]^{\mathrm{CON}}$, for all formulae A

In less formal terms, in this background logic one can show that for every formula A the proposed recursive mapping generates a neutral formula $[A]^{\mathrm{CON}}$ which by the contingentist's lights is equivalent to A. This vindicates the claim that necessitists can recognise all the modal distinctions and hypotheses which contingentists can. The salient question then becomes whether the contingentist is in a parallel position, to which we now turn.

4.1.4 Mapping Necessitist to Contingentist Discourse

Can the contingentist provide a required mapping $[.]^{\mathrm{NEC}}$ from necessitist discourse to a neutral body of discourse? Again, any such mapping must have the consequence that A is equivalent to $[A]^{\mathrm{NEC}}$ by the necessitist's lights. Similar to the above case, this requires a necessitist theory other than merely **NNE** and **NNE**^{2}. Just as the necessitist concentrated their attention on a certain auxiliary contingentist theory, the contingentist must do the same.

The choice of auxiliary theory is less obvious in this case. However, Williamson focuses on necessitist theories which maintain the natural idea that everything is possibly grounded in the concrete. In addition, Williamson assumes that the necessitist theory includes the claim that necessarily if some non-logical atomic predicate applies to any individuals, those individuals are chunky. This of course

[12]See **Proposition 2.26** of Appendix Part B for essentially the same result.

precludes mere possibilia from belonging to the extension of non-logical predicates at a world, but Williamson (2013, p.687) is explicit that this assumption is required only to simplify matters. Indeed, as an idealisation, one might imagine that the necessitist is using a restricted language with a choice of non-logical atomic predicates that only apply to that which is chunky.

To be explicit, the auxiliary necessitist theory in question may be axiomatised as follows (where F is a n-place non-logical atomic predicate).

AuxNec **NNE** $\wedge \forall x \Diamond Cx \wedge$
$$\ldots \Box \forall z_1 \ldots \forall z_n (Fz_1 \ldots z_n \rightarrow (Cz_1 \wedge \ldots \wedge Cz_n)) \ldots$$

One salient difference to the auxiliary contingentist theory is that although ¬**NNE** was not included in **AuxCon**, **NNE** is included in **AuxNec** since it is required for the mapping to achieve even moderate success.[13] Another salient simplification is that Williamson (2010, p.688) stipulates that there are only finitely many non-logical predicates in the language in order to make **AuxNec** a finite conjunction. This simplification is adopted in what follows and in the appendices.

The task is now to articulate a mapping from any claim A to a neutral claim $[A]^{\text{NEC}}$, such that **AuxNec** $\models A \leftrightarrow [A]^{\text{NEC}}$. The basic idea behind the mapping is to simulate necessitist quantification over possible Fs by modalised neutral quantification over Fs. The thought is that when necessitists claim that there is some possible F which is thus-and-so, contingentists can map this claim to the neutral claim that possibly there is something chunky which is F and thus-and-so. For instance, this idea can be applied as follows:

[there is some possible son of Wittgenstein]$^{\text{NEC}}$=

possibly there is something chunky which is a son of Wittgenstein

This crude strategy is not ultimately successful, however, since it maps necessitist claims to neutral claims which they will view as inequivalent. For example, consider the following result:

[13] See Williamson (2010, p. 688) and **Proposition 1.23** of Appendix Part A.

> [there is some possible son of Wittgenstein which is not a son of Wittgenstein]$^{\text{NEC}}$=
>
> possibly there is something chunky which is a son of Wittgenstein and is not a son of Wittgenstein

Clearly, the initial claim and its neutral putative analogue will not be equivalent according to **AuxNec**.

Nevertheless, a more sophisticated mapping based on essentially the same idea fares better. The problem with the crude strategy is that the 'possibly' operator shifts the perspective to a world at which one then must evaluate the part of the necessitist claim within the scope of the quantifier. But often this will deliver the wrong result. The required tweak is that one first needs to shift attention to a different possible world at which the quantifier 'collects' a chunky individual, after which attention is shifted *back* to the initial world where the remainder of the necessitist's claim is assessed by assigning the 'collected' individual to the variable. The devices of modal anaphora are key to this strategy.

The proposed recursive clauses are stated as follows.

$[Fv_1 \ldots v_n]^{\text{NEC}} = Fv_1 \ldots v_n \wedge Cv_1 \wedge ... \wedge Cv_n$, n-place non-logical atomic F

$[Vv_1 \ldots v_n]^{\text{NEC}} = \Diamond(Vv_1 \ldots v_n \wedge V{\leq}C)$, for n-place variable V

$[v_1 = v_2]^{\text{NEC}} = \Diamond(v_1 = v_2 \wedge Cv_1 \wedge Cv_2)$

$[\neg A]^{\text{NEC}} = \neg[A]^{\text{NEC}}$

$[A \wedge B]^{\text{NEC}} = [A]^{\text{NEC}} \wedge [B]^{\text{NEC}}$

$[\Diamond A]^{\text{NEC}} = \Diamond[A]^{\text{NEC}}$

$[\uparrow A]^{\text{NEC}} = \uparrow[A]^{\text{NEC}}$

$$[\downarrow A]^{\text{NEC}} = \downarrow[A]^{\text{NEC}}$$

$$[\exists xA]^{\text{NEC}} = \uparrow\Diamond\exists x(Cx \wedge \downarrow[A]^{\text{NEC}})$$

$$[\exists XA]^{\text{NEC}} = \uparrow\Diamond\exists X(X{\leq}C \wedge \downarrow[A]^{\text{NEC}})$$

Again, the function commutes with truth-functional connectives and modal operators, and atomic predications involving non-logical predicate constants are mapped to formulas in which they occur within chunky contexts. In addition, identity claims and predications involving second-order variables are mapped to modalisations of formulas in which they occur in chunky contexts, which are themselves neutral. In the case of identity, this is because the auxiliary necessitist may assert identity claims about individuals which are not chunky. Similarly, in the case of predications involving second-order variables, the auxiliary necessitist may recognise that some non-chunky things belong to the extension of a predicate variable under certain assignments. As a result, the proposed mapping attempts to find a neutral analogue of, for example, $Xx_1...x_n$ (under an assignment) by considering a world at which $Xx_1...x_n$ is true (under that assignment) and the individual assigned to each x_i $(1 \leq i \leq n)$ is chunky. Ultimately, this clause will be partly responsible for the deficiencies of the mapping which are discussed below.

A crucial point is that necessitist quantificational claims are mapped to sophisticated modalised quantified statements. A feature of these sophisticated modalisations is that the perspective is returned to the initial world in order to evaluate the part of the necessitist claim within the scope of the quantifier. To take an example, by this more sophisticated mapping we have the following correct result:

> [there is some possible son of Wittgenstein which is not son of Wittgenstein]$^{\text{NEC}}$=
>
> $\uparrow$ possibly there is something chunky $\downarrow$ which is a possible son of Wittgenstein but not a son of Wittgenstein

Put informally, the sophisticated modalised quantifier 'stores' a given world w with the initial $\uparrow$, then moves one to another world u via $\Diamond$, at which one picks up some chunky a in the domain of u and assigns it to a given variable, after which one returns to the stored world w via $\downarrow$ to evaluate $[A]^{\text{NEC}}$, with a assigned to the given variable. The result is a neutral analogue of the initial claim which the auxiliary necessitist will view as equivalent to it.

Given that the proposed recursive mapping enjoys some initial success, one might anticipate a more general result. Indeed, Williamson (2010) proves the following proposition (here, an FO formula is one which contains no occurrence of any second-order quantifier or second-order predicate variable):[14]

AuxNec $\models A \leftrightarrow [A]^{\text{NEC}}$, for all FO formulae A

Whilst this may seem a cause for optimism, unfortunately the result does not hold with total generality; the problem concerns the contingentist's proposed mapping of formulas involving second-order quantifiers and second-order predicate variables.

To anticipate the issue in question, it helps to study the mapping's treatment of second-order quantified claims. According to the proposal above, second-order quantified claims of the form $\exists XA$ are mapped by $[.]^{\text{NEC}}$ to claims of the form $\uparrow\Diamond\exists X(X{\leq}C \wedge \downarrow[A]^{\text{NEC}})$. Less formally, $[.]^{\text{NEC}}$ maps second-order quantified claims of the form 'there are some things such that A' to neutral claims of the form '$\uparrow$possibly there are some chunky things such that $\downarrow[A]^{\text{NEC}}$'. However, if such claims are to be deemed equivalent by the auxiliary necessitist, the necessitist must maintain that arbitrary pluralities of things are such that it is possible for all of their members to be chunky together. Yet there would appear to be a plethora of counterexamples to that expectation. For example, consider the following paradigm cases from Williamson (2013, p. 337):[15]

> *Humans*
>
> A human h could grow from a sperm s and an egg e. A human h^* could grow from the sperm s and an egg e^* distinct from e. But there could not be both a human h and a human h^*. For given the essentiality to humans of their origins (conditional on their being chunky), there

[14]See also **Proposition 1.27** of Appendix Part A for essentially the same result.

[15]Williamson cites Salmon (1987, pp. 47-48) for the first example. Williamson also notes that a suitable metaphysics of events for the second example may be obtained by adding a modal dimension to the account in Davidson (1970).

> could be a human h only by growing from s and e, and there could be a human h^* only by growing from s and e^*. Given the nature of the entities, h cannot grow from from s and e while h^* grows from s and e^*. Thus there could not be both a human h and a human h^*. For h and h^*, to be chunky is to be human. Therefore, it is impossible for h and h^* to be chunky together. Although the chunkiness of h and the chunkiness of h^* are separately possible, they are not compossible.
>
> *Events*
>
> Let *fww* name the concrete, unrepeatable token event of the First World War, in all its terrible detail. Instead of *fww*, a concrete, unrepeatable token event *gep* of a golden era of world peace from 1914 to 1918 could just about have happened, if Princip had missed at Sarajevo. For possible concrete, unrepeatable token events such as *fww* and *gep*, to be chunky is to happen. But *fww* and *gep* could not have both happened. For *fww* could not have happened without a major war in Europe in 1914-1918, while *gep* could not have happened with a major war there then. If an event similar to *fww* had happened somewhere else or at another time, it would not have been *fww*, and if an even similar to *gep* had happened somewhere else or at another time, it would not have been *gep*. Therefore, it is impossible for *fww* and *gep* to be chunky together. Although the chunkiness of *fww* and the chunkiness of *gep* are separately possible, they are not compossible.

In light of such cases, it is highly doubtful that $[.]^{\text{NEC}}$ maps each claim to a neutral claim which the auxiliary necessitist will regard as equivalent to it. Indeed, to confirm this doubt, one can specify counterexamples to the success of the proposed mapping. For example, consider the claim that for any individually possible humans there are some things which are exactly them:

$\Diamond\mathbf{H}$ $\quad \forall x\forall y(\Diamond Hx \wedge \Diamond Hy \rightarrow \exists X\forall z(Xz \leftrightarrow z = x \vee z = y))$

According to $[.]^{\text{NEC}}$, **3** is mapped to claim that, roughly, for any possible individuals which are individually possible humans, possibly there are some chunky things which comprise exactly those possible individuals.

$[\Diamond\mathbf{H}]^{\text{NEC}}$ $\quad \Box\forall x(Cx \rightarrow \Box\forall y(Cy \rightarrow (\Diamond(Hx \wedge Cx) \wedge \Diamond(Hy \wedge Cy) \rightarrow \Diamond\exists X(X{\leq}C \wedge \Box\forall z(Cz \rightarrow (\Diamond(Xz \wedge X{\leq}C) \leftrightarrow \Diamond(z = x \wedge Cz \wedge Cx) \vee \Diamond(z = y \wedge Cz \wedge Cy)))))))$

However, the auxiliary necessitist will not regard **3** as equivalent to $[\mathbf{3}]^{\text{NEC}}$, for it is not the case that, for example, possible human h and possible human h^* are possibly jointly chunky. Thus, it will not be possible that there are some chunky things which comprise exactly h and h^*.

Once one has appreciated the manner of the counterexample, one may notice that $[.]^{\text{NEC}}$ also maps atomic predications involving second-order variables to claims which the auxiliary necessitist will not regard as equivalent to them. To take an example, suppose the necessitist endorses the truth of $Xx_1...x_n$ (for $n > 1$) under an assignment that assigns individuals which are not possibly chunky together to at least two of $x_1...x_n$ respectively. According to $[.]^{\text{NEC}}$, $Xx_1...x_n$ is then mapped to the neutral $\Diamond(Xx_1 \dots x_n \wedge X{\leq}C)$. However, $\Diamond(Xx_1 \dots x_n \wedge X{\leq}C)$ will be false under the same assignment, since at least two of $x_1...x_n$ are assigned individuals which cannot be chunky together. Thus, the mapping fails even in its basic treatment of certain atomic formulae.[16]

The problem for the contingentist is quite general. As Williamson (2010, pp. 738-774) shows, there are formulas of the modal language which are not equivalent in the background logic to *any* neutral formula even in the presence of **AuxNec**. Thus, the problem is not local to the proposed mapping $[.]^{\text{NEC}}$. In this regard, there are deep obstacles facing any contingentist attempt to simulate necessitist discourse. Yet if the contingentist is simply unable to simulate significant portions of necessitist discourse, necessitism would have a clear expressive advantage over contingentism. Moreover, this advantage would be no mere capacity to articulate the arcane: contingentists should want to recognise and articulate the modal claims in question, for some of these particular claims are crucial to formulating the contingentist view itself. Indeed, there are two notable examples.

First, as noticed by Peacocke (1978, pp. 480-481), this particular expressive shortcoming renders the contingentist unable to provide a homophonic semantics

[16]Williamson (2010, p. 705) highlights this fact but acknowledges that it is not as philosophically significant as the previously mentioned failure due to open formulas not being the vehicles of speech acts.

for their object-language.[17] The reason is that in providing such a semantics one needs to quantify over sequences comprising *all* possible individuals for even the basic purpose of providing a semantic clause for the first-order quantifier. Thus, attempting to simulate necessitist generality over such sequences via modalised contingentist second-order quantification will result in failure to construct the semantic theory. This should trouble contingentists, for as Williamson (2013, chap. 3) argues they must already take an instrumentalist attitude towards the apparatus of the Kripke model-theoretic semantics. As a result, contingentists appear to lack any semantic apparatus in terms of which they can theorise about their object-language in a realist manner.

The second example applies to contingentism when the higher-order quantifiers are understood in *sui generis* terms. Typically, contingentists have tried to articulate the exact way in which higher-order 'entities' like relations and propositions vary existentially with which first-order entities there are. One influential idea has been that this variation can be explained by appeal to permutations of modal space.[18,19] To see how this works informally, let f be an *automorphism of modal space* iff it is a permutation on the space of possible individuals (including the possible worlds) such that for any world w and any individual x, x exists in w iff $f(x)$ exists in $f(w)$. One can then *fix* an automorphism on w iff $f(w) = w$ and $f(x) = x$, for all individuals x which exist at w. Then say that an automorphism *preserves* an n-place relation F iff if $f(w) = v$ and $f(x_i) = v_i$ for $1 \leq i \leq n$, then $Fx_1 \ldots x_n$ at w iff $Fv_1 \ldots v_n$ at v. Given these definitions, contingentists may claim that a relation exists at a world w when it is preserved by all automorphisms fixed on w which

[17] Peacocke himself would argue that the contingentist *can* articulate these claims via the infinitary method considered at the beginning of §4.2.

[18] See Fine (1977b) for the precise way of developing this idea; for similar ideas, see Stalnaker (2012, Appendix A) and Williamson (2013, chap. 6). Different ways of developing the broad idea are explored in Goodman & Fritz (2016). As Goodman and Fritz show, the contingentist can implement one way of developing the idea provided they reject **Being** for non-logical atomic predicates. On this view, permutations (which must exist at possible worlds) can hold between incompossible individuals at worlds where at least one of those individuals does not exist. However, this may be extreme for even mild contingentists, who may be inclined to think that such permutations are existence entailing.

[19] Note that the following explanation is provided at the level of the model theory, but the idea has been articulated in a purely modal language in Goodman & Fritz (2016).

preserve qualitative properties and relations. However, certain automorphisms will relate incompossibly chunky individuals, thus precluding the contingentist from simulating necessitist second-order quantification over them. Consequently, contingentists could not appeal to automorphisms of modal space in order to explain the pattern of instantiation in higher-order being; unfortunately for them, no other explanation is obviously available. This marks another crucial respect in which the contingentist would be unable to develop their theory.

4.2 Extant Responses

In order to remedy their failure, contingentists have typically reacted by expanding their theory's expressive resources. One influential strategy, first suggested by Fine (1977a), has been to adopt an infinitary language which permits formulas to be of proper class size. In particular, the language must admit: proper class many individual variables; proper class sized conjunctions, and proper class sized modalised quantifier prefixes. Within this seriously infinitary setting, contingentists can then attempt to simulate necessitist second-order quantification with infinitely long strings of modalised first-order quantifiers. To be exact, Fine's suggestion would be to update the mapping with the following clause for second-order quantifiers.[20]

$$[\exists XA]^{\text{NEC}} = \uparrow\Diamond\exists x_0(Cx_0 \wedge \downarrow\Diamond\exists x_1(Cx_1 \wedge \downarrow\Diamond\exists x_2(Cx_2 \wedge \ldots\downarrow[A]^{\text{NEC}})\ldots)$$

Given the background **S5** logic, the right side of this identity is equivalent to the following:

$$\uparrow\Diamond\exists x_0(Cx_0 \wedge \Diamond\exists x_1(Cx_1 \wedge \Diamond\exists x_2(Cx_2 \wedge \ldots\downarrow[A]^{\text{NEC}})\ldots)$$

The strategy is to simulate necessitist plural quantification over some things X with infinitely long strings of modalised first-order quantifiers which 'select' all the

[20] In this clause, the 'new' first-order variables must be chosen to avoid clash with the variables which occur free in A.

possibilia that are amongst the X. To accompany this clause, Fine's suggestion would also require an update to the clause for open formulae (where x_i are new variables):[21]

$$[Vv]^{\text{NEC}} = \Diamond(Cv \wedge (v = x_0 \vee v = x_1 \vee v = x_2 \vee ...))$$

On this approach, the claim that there are some things to which everything belongs is mapped as follows:

$$[\exists X \forall v\, Xv]^{\text{NEC}} =$$

$$\uparrow\Diamond\exists x_0(Cx_0 \wedge \Diamond\exists x_1(Cx_1 \wedge \Diamond\exists x_2(Cx_2 \wedge ...\Box\forall v\,(Cv \rightarrow \downarrow\Diamond(Cv \wedge (v = x_0 \vee v = x_1 \vee v = x_2 \vee ...))$$

In other words, the initial claim is mapped to the sentence that possibly there is some chunky x_0, and possibly there is some chunky x_1, and ..., such that every possible chunky individual is possibly chunky and either x_0, or x_1, or so on. Whereas the initial necessitist claim states that there are some things to which everything belongs, the mapping generates a claim which uses an infinitely long string of modalised quantifiers to 'select' all possible individuals and then states that necessarily everything is one of them. This claim is neutral and, in a model-theoretic treatment of the infinitary modal language, one would expect the claim to be equivalent (over the appropriate class of models) to the initial claim by the necessitist's lights. Following this suggested procedure, one would also expect that $\Diamond\mathbf{H}$ would get mapped to a neutral sentence that the necessitist will view as its equivalent (over the appropriate class of models), which of course is a welcome result.

However, despite the expected success of the infinitary proposal, Williamson (2016b, pp. 352-361) advances two philosophical considerations which cast doubt

[21]See Williamson (2013, p. 353, n. 40) for how to generalise this to predicate variables of arbitrary finite adicity. Williamson also notes that the above clauses treat V as non-empty, as Fine (2016, p. 555, n. 9) is aware. However, the clauses can be slightly amended to permit the empty case; since empty disjunctions are typically treated as equivalent to contradictions, Williamson suggests that "to allow for the empty case, replace $[A]^{\text{NEC}}$ in the definition of $[\exists XA]^{\text{NEC}}$ by $[A^*]^{\text{NEC}} \vee [A]^{\text{NEC}}$, where $[A^*]^{\text{NEC}}$ results from replacing each atomic formula in which X occurs free in A with a contradiction". In the main body, this minor amendment will be suppressed.

on this particular use of infinitely long strings of modalised quantifiers. The first concerns whether such infinitary strings are even intelligible to contingentists, and the second whether there could in principle be a language large enough to host the required infinitary formulas.

To proceed in order, Williamson's first criticism about intelligibility regards even strings which contain only countably many modalised quantifiers. As Williamson highlights, not all ω-sequences of meaningful sentence operators themselves form meaningful sentence operators. For example, an ω-sequence of negation operators is not itself a meaningful sentence operator, since it is its own negation. Thus, Williamson presses, there is some onus on the contingentist to specify an intended meaning for even countably infinite strings of modalised quantifiers in their own terms. Of course, necessitists can specify an intended meaning quite straightforwardly by appeal to the possible worlds model theory. But given that contingentists must treat the possible worlds model theory merely instrumentally, no such specification is available to them.[22]

In recent work, Kit Fine (2016, pp. 554-557) has suggested a way in which contingentists can specify the required intended meaning without recourse to the possible worlds model theory. Fine's suggestion can be seen as comprising two separate elements. First, Fine suggests a revision to the syntax of the infinitary language. Whereas in typical languages, (modalised) quantifier prefixes are linearly ordered due to the usual formula formation rules, in Fine's suggested language, (modalised) quantifier prefixes need only be partially ordered.[23] Indeed, since the identity relation on a given class induces a partial order on that set, in Fine's proposed infinitary language there can be (modalised) quantifier prefixes in which each (modalised) quantifier occurs only within its own scope. (For convenience, such classes are called *unordered*.) In effect, as Williamson notes, this permits one to treat the sentences used to map necessitist second-order quantified claims as

[22] See Williamson (2013, chap. 3) for an argument that contingentists must take this instrumentalist attitude towards the model theory.

[23] Williamson (2013, pp. 359) himself is aware of such proposals in his discussion. Indeed, this notion of 'branching quantification' has been studied extensively in the non-modal case having been first proposed by Leon Henkin (1961).

having the following structure (where I is an index class used merely to distinguish different occurrences of variables, and the class of modalised quantifier prefixes is partially ordered by the identity relation over that class):

$$[\exists XA]^{\text{NEC}} = \uparrow\{\Diamond\exists x_i(Cx_i\wedge\}_{i\in I}\downarrow[A]^{\text{NEC}})...)$$

To indicate its syntax more explicitly, the sentence on the right side of the equality would have the following form:

$$\begin{pmatrix}\uparrow\Diamond\exists x_0\,(Cx_0\wedge\\ \uparrow\Diamond\exists x_1\,(Cx_1\wedge\\ .\\ .\\ .\end{pmatrix}\downarrow[A]^{\text{NEC}})...)$$

Given this alteration to the syntax, Fine proposes to specify an intended meaning for the infinitary modalised formulas without recourse to the possible worlds model theory. On this point, he writes (2016, p. 555):

> [T]here is something odd in the thought that the contingentist must somehow have recourse to the possible world semantics to explain the modalized quantifiers when no such recourse is required for the unmodalized quantifiers.

Thus, Fine attempts to justify the fact that unordered sets of modalised quantifier prefixes constitute meaningful sentence operators more directly. In short, his claim is that such unordered sets do constitute meaningful sentence operators due to the fact that each individual modalised quantifier prefix is a semantic operation *independent* of the other individual modalised quantifier prefixes. Fine offers no general definition of the notion of operational independence with respect to certain other operations, but suggests that it is intelligible and can be appreciated by its application to various examples. For instance, Fine (2016, pp. 555-556; his emphasis) claims that there is a close analogy with the notion of simultaneous syntactic substitution:

> [C]onsider the case of ordinary syntactic substitution. Given an expression E and some other expression F that it contains, we may then substitute some other expression G for (all occurrences of) F in E, thereby obtaining the expression $E[G/F]$. One substitution may be performed *successively*, after another. Thus, we may first substitute G_1 for F_1 in E and then substitute G_2 for F_2 in the result $E[G_1/F_1]$ to obtain $E[G_1/F_1|G_2/F_2]$ (as long as F_1 and F_2 are non-overlapping expressions in E). But we may also *simultaneously* substitute G_1 for F_1 and G_2 for F_2 to obtain $E[G_1/F_1|G_2/F_2]$ (as long as F_1 and F_2 are non-overlapping expressions in E). The result of a successive substitution will not in general be the same as the result of the corresponding simultaneous substitution. Suppose E is the formula $p \vee q$. Then $E[q/p][p/q]$ will be $p \vee p$ while $E[q/p|p/q]$ will be $q \vee p$.

Fine notes that infinite successive substitutions have no clear meaning. For instance, take a given sentence letter p and consider the infinite successive substitution $p[p/p_1][p_1/p_2][p_2/p_3]$.... Since there is no clear resultant formula, infinite successive substitution does not appear well defined. However, the result of the infinite simultaneous substitution $p[p/p_1|p_1/p_2|p_2/p_3...]$ is just p_1 since the initial formula consisted only of the sentence letter p.

Fine claims that simultaneous substitution differs from successive substitution insofar as simultaneous substitution involves only *independent* syntactic operations on the initial formula. As Fine (2016, p.556) puts it evocatively, "if we were to assign a subordinate to execute to each of the operations, then each of the subordinates could proceed with his task without any interference from the others." It is for this reason, Fine claims, that infinite simultaneous substitution is in good standing whilst successive substitution is not. Similarly, Fine claims that unordered sets of modalised quantifier prefixes constitute meaningful sentence operators due to the fact that each individual modalised quantifier prefix expresses a semantic operation *independent* of those expressed by the other individual modalised quantifier prefixes. Indeed, as Williamson remarks (2013, p. 360), 'very special conditions' are needed for unordered sets of sentence operators to constitute a further sentence operator, and independence of the constituent operators from one another may well be one such condition.

One can appreciate the independence of the modalised quantifier prefixes by noticing that linearly ordered *finite* strings of the type of modalised quantifiers above exhibit logical equivalence up to permutation in ordinary first-order finitary modal logic with Vlach operators. For example, where n is finite, the following two linearly ordered formulas are equivalent over the class of models defined in Appendix Part A:

$$\uparrow\Diamond\exists x_0(Cx_0 \wedge \downarrow\Diamond\exists x_1(Cx_1 \wedge \downarrow \wedge \ ... \ \wedge \Diamond\exists x_n(Cx_n \wedge \downarrow[A]^{\text{NEC}})...)$$

$$\uparrow\Diamond\exists x_1(Cx_1 \wedge \downarrow\Diamond\exists x_0(Cx_0 \wedge \downarrow \wedge \ ... \ \wedge \Diamond\exists x_n(Cx_n \wedge \downarrow[A]^{\text{NEC}})...)$$

This is due the presence of the Vlach operators which ensures that the world of evaluation has been returned to the initial world before evaluating the next embedded quantifier. Indeed, either one of the formulas immediately above is equivalent (over the class of models defined in Appendix Part A) with any formula which results from either of them by merely permuting the modalised quantifier prefix. The fact that such permutations are logically equivalent over that natural class of models constitutes some evidence that each modalised quantifier prefix expresses an operation independent of those expressed by the other prefixes in finite embeddings. Given this putative independence of the individual prefixes in finite settings, it is difficult to see what would generate mutual dependence in infinite settings. By way of comparison, notice that multiple successive substitutions do not exhibit independence of one another in the finite case. Indeed, it is natural to suspect that whether an operation is independent of certain operations is an absolute matter, and not a matter which varies between the operation's occurrence in finite or infinite embeddings of the relevant operations. Thus, since the individual modalised quantifier prefixes appear to exhibit independence from one another in finite embeddings, it seems plausible that they exhibit independence in infinite embeddings too.[24]

[24]As Williamson (2013, p. 358) rightly emphasises, equivalences between permutations of sentence operators does not guarantee that an infinite string of meaningful sentence operators expresses a unique sentence operator which is itself meaningful. However, the current point is that the equivalences between permutation of finite embeddings provide evidence of independent semantic operations.

Williamson (2016b, p. 580) objects to Fine's proposal on the grounds that simultaneous substitution is too dissimilar to the application of an unordered set of modalised quantifier prefixes to a given formula. According to Williamson, simultaneous substitution is a *local* operation, since, as Fine is explicit about in the above quotation, the substitutions must be on non-overlapping expressions in the relevant formula. Nevertheless, Williamson contends that the application of an infinite unordered set of modalised quantifiers to a given formula is importantly non-local, for each of the modalised quantifiers "operates on a whole formula, just as other sentential operators do". However, the salient point is whether the non-locality of an operation suffices to render it not independent of other operations. Yet, granting Williamson's suggestion that sentential operators express non-local operations, there would appear to be cases of non-local operations which are independent of one another. For example, consider the sentential operator $\mathcal{O}^T$ which applies truly to any sentence iff actually Theresa May was Prime Minister on January 1st 2019, and the sentential operator $\mathcal{O}^C$ which applies truly to any sentence iff Jeremy Corbyn actually was not Prime Minister on January 1st 2019. If propositions are not individuated by mere metaphysically necessary equivalence, it is plausible that the propositional functions expressed by $\mathcal{O}^T$ and $\mathcal{O}^C$ are not individuated by co-intensiveness either and are thus distinct.[25] Yet since each operation expresses a necessarily constant propositional function, surely they are independent of one another. This is suggested by the fact that in a language whose vocabulary includes such sentential operators, one would not expect there to be any difference in truth value between sentences which differ merely on the order in which $\mathcal{O}^T$ and $\mathcal{O}^C$ apply to a formula.

Williamson (2016b, p. 580) also objects to Fine's proposal on the basis that the individual expressions in each modalised quantifier prefix are not independent of the operators involved in the other prefix. In Williamson's own words (his emphasis):

> Consider the two modal operators $\Diamond$ and $\Box$. What are we supposed to get when we apply them 'simultaneously' to a formula A? [...] Since there is

[25]For example, on well-motivated of theories of granularity such as Dorr's (2016, p. 47) 'Only Logical Circles' theory, there are co-intensive propositional functions which are nevertheless distinct due to their 'non-logical' contents.

> no natural answer, $\Diamond$ and $\Box$ presumably count as not independent of each other. [...] Thus the operators composing $\Diamond\exists x_1\Box$ are *not* independent of the operators composing $\Diamond\exists x_2\Box$. Why then should we believe that $\Diamond\exists x_1\Box$ is independent of $\Diamond\exists x_2\Box$? That would require some 'cancelling out' of the numerous dependencies between their constituents. That such cancelling out occurs in this particular combination (but not most others) is not obvious; it would require a proof, for which we have no hint.

As Williamson emphasises, the operators composing $\Diamond\exists x_1\Box$ are not independent of those composing $\Diamond\exists x_2\Box$. In light of this, in order for the modalised quantifier prefixes to be considered independent of one another, either, as Williamson suggests, some 'cancelling out' must occur between them, or independent operators may admit dependencies between constituent operators. However, for all that has been said to introduce the notion of independence, it is unclear whether either disjunct is the case. Thus, as the debate stands, there would appear not to be conclusive evidence as to whether or not infinitary strings of modalised quantifiers are intelligible to contingentists.

Nonetheless, Williamson raises a more conclusive concern against Fine's proposal: that there could not even in principle be a language large enough to host the required infinitary formulas. To appreciate this concern, recall that the infinitary proposal must be developed in a language in which there are as many individual variables as there are possible individuals. This is the only way to simulate necessitist quantification over arbitrary pluralities of possible individuals, such as *all* possible individuals as in one of the examples above. However, to begin with, it is not clear that the contingentist is even capable of articulating the claim that there is a one-one correspondence between the individual variables of a language and all possible individuals. The ability to articulate such claims requires exactly the expressive capacity under contention. Moreover, even if the contingentist is granted the required expressive capacity, as Fritz & Goodman (2017) highlight, given highly natural assumptions about the syntax of infinitary languages, there are Cantorian arguments to the effect that no language could satisfy the correspondence requirement. To be exact, according to typical formation rules for complex formulae, for any individual variables VAR there is the formula whose variables are all and only VAR. For example, for any individual variables VAR and monadic predicate X, there is the collection of atomic formulae Xv_i for each v_i in VAR, and hence the infinitary conjunction whose conjuncts are exactly the Xv_i. Yet by a plural version of Cantor's Theorem it follows that there are more formulae than individual variables.[26] But then the individual variables were not equinumerous with the possible individuals, for formulae are individuals too.

In recent work, Fine (2016) himself has acknowledged this Cantorian concern and has thus explored a quasi-fictionalist alternative to the infinitary strategy using a novel 'suppositional calculus'.[27] According to this approach, the contingentist first supposes that necessarily everything is actually something, and then describes the necessitist's view of the modal pluriverse under this supposition. In Fine's suppositional calculus, this supposition can be formalised and a description of the necessitist's pluriverse can be articulated within the scope of the supposition. The contingentist may then use this description within the scope of the supposition to simulate necessitist discourse. However, although Fine's suppositional-based alternative merits further investigation, it would be desirable if the contingentist need not resort to this quasi-fictionalist gambit.[28] Part of the issue is that it is not totally clear how Fine's suppositional calculus could be used to meet Williamson's exact expressive challenge. First of all, within the language of Fine's suppositional calculus it is not clear how exactly to characterise which formulas qualify as neutral. Moreover, even if such a syntactic criterion were offered, it is also unclear whether either the necessitist or the contingentist would recognise suppositional claims as the 'cash-value' of any non-suppositional claim. Such considerations motivate the search for alternative responses to Williamson's expressive challenge.

In light of the above, the following section makes the case for a different approach.

[26] In the context of Bernays-Gödel set theory, Paul Bernays (1942) proved that no binary relational class on a class simulates a class-valued function from that class's members onto its subclasses. Understood in terms of irreducible plural quantification, this result is the plural version of Cantor's Theorem. See Hawthorne & Uzquiano (2011, pp. 60-62) and Uzquiano (2015) for further discussion.

[27] Fine (2016) suggests that the type of class theory developed in Fine (2005) may provide a stable setting in which the infinitary proposal could be carried out, however at the time of writing it has not been explored in print.

[28] Williamson (2016b, pp. 580-583) critically discusses Fine's suppositional-based reduction.

The strategy will be to draw on the novel metaphysics of necessity defended in Essay Three of this thesis, and employ its resources to specify a recursive mapping of necessitist discourse to neutral discourse by a finitary method. Unlike other extant responses, this approach is guided by a suspicion towards the putative cases of incompossible chunkiness, as opposed to an appetite to explore highly infinitary or quasi-fictionalist expressive resources. Indeed, the approach makes a crucial appeal to the claim that there is some notion of objective necessity which fails to license the various paradigm essentialist constraints that motivate the putative cases of incompossible chunkiness. Due to the failure of such constraints, there is some objective notion of possibility according to which arbitrary individually metaphysically possible chunky individuals are possibly all chunky together.

4.3 Compossibility in the Broadest Sense

4.3.1 Broadness and Indeterminacy

The failure of arbitrary individually possibly chunky individuals to be possibly chunky together was the primary hindrance to mapping necessitist discourse to neutral discourse. As evidence of this failure, Williamson cited the two paradigm examples, *Humans* and *Events*. The former example was motivated by the essentiality to material objects of their origins, and the latter by the essentiality to events of their distinctive features, such as involving a major war or occurring during a certain time period. Both examples thus rely on typical, yet fairly noteworthy essentialist assumptions.

A central moral of Essay Three of this thesis was that a notion of objective necessity cannot license such strong essentialist assumptions if it is to be the broadest objective necessity. This claim was motivated by the fact that such assumptions are naturally coupled with a mild *tolerance* towards other claims of *de re* necessity. For example, although origin essentialists maintain that material objects cannot originate from matter totally different to that from which they do, they recognise that material objects may originate from ever so slightly different matter. Similarly, although it may be standard to maintain that *fww* must have occurred during the time period 1914-1918, it is highly plausible that *fww* could have ended one millisecond earlier. Yet, as argued in Essay Three, when combined with such tolerance, these essentialist constraints generate modal sorites series, which strongly indicates that the operative notion of necessity is a source of significant indeterminacy. However, Essay Three also stressed that when a notion of objective necessity exhibits such indeterminacy it becomes an unstable candidate for the broadest objective necessity. Thus, on the conception of the objective modalities articulated in Essay Three, the broadest notion of necessity fails to license essentialist constraints to which metaphysical necessity is often expected to conform.

Of course, the treatment of modal sorites series advocated considered in Essay Three is not the only available approach. As mentioned in Essay Three of this thesis, Sarah-Jane Leslie (2011) and John Hawthorne & Juhani Yli-Vakkuri (MS) have explored a treatment of the modal sorites that uses a plenitudinous ontology, and such treatments need not obviously posit indeterminacy in the operative notion of objective necessity. However, Essay Three presents a live conception of the objective modalities that ought to be investigated in the current setting.

The crucial feature of the resultant view of the objective modalities is that the broadest objective necessity is more stringent than is typically assumed. A corollary of this stringency is that material objects and events exhibit a greater flexibility in their modal profiles. For example, it need not be necessary in the broadest objective sense that material objects possess their actual origins, or that events involve all of their distinctive features, although such facts may be necessary in some narrower sense. Thus, in the broadest objective sense of possibility, humans may originate from different pre-embryonic organisms to those from which they actually do, and *fww* may occur within a totally different time period, in far less or even greater terrible detail. As a result, humans with possibly different origins can be possibly humans together when one originates from totally different matter to the other. Similarly, events which individually possibly occur in the same period can possibly both occur when one happens in a totally different time period to the other.

In addition to arguing that the broadest notion of necessity fails to license the strong essentialist assumptions in question, Essay Three also suggested that metaphysical necessity *is* that very notion of necessity. This identification was motivated by the metasemantic thesis—now familiar to the reader of this thesis—that the central postulate of metaphysical necessity's theoretical role is that it is the broadest objective necessity. On the preferred metasemantic view, this postulate is integral to, indeed almost stipulative of, metaphysical necessity. Consequently, if other purported aspects of metaphysical necessity's theoretical role are incompatible with it, those aspects ought to be revised, not its status as the broadest objective necessity. Thus, according to the proposed view, metaphysical necessity itself fails to license the strong essentialist assumptions behind *Humans* and *Events*. Given this view of metaphysical necessity, it is a reasonable conjecture that the contingentist can recursively map each necessitist claim to a neutral analogue which the necessitist will view as its equivalent.

Nevertheless, the additional metasemantic thesis is more speculative than the arguments for the stringency of the broadest necessity. Whether 'metaphysical necessity' expresses the broadest necessity or some more restricted modality is a delicate metasemantic question. Of course, Kripke's (1980, p. 99) initial, near stipulative characterisation of metaphysical necessity as 'necessity in the highest degree' should hold considerable metasemantic weight. Yet the litany of other ideas and postulates which philosophers (including Kripke) have associated with metaphysical necessity cannot be totally ignored. However, the complementary metasemantic thesis is not even required for the contingentist to construct the required mapping. Interestingly, the weaker claim—merely that there is some objective necessity, perhaps not metaphysical necessity, which fails to license the strong essentialist assumptions in question—suffices entirely for the contingentist's aims. Consequently, the following section outlines a natural way of modelling this broader modality and its interaction with 'metaphysical necessity', or at least a notion of necessity that obeys the essentialist constraints which metaphysical necessity has been expected to obey. (Hereafter, to reflect the reliance on only the weaker claim, we shall just use 'metaphysical necessity' as the name of the restricted modality in question.) In Appendix B this modelling technique is implemented formally to prove that Williamson's exact expressive challenge can be met. However, before the next main section, it will help to connect the working conception of the broadest objective necessity with a complementary view.

4.3.2 Pinning Down Necessity

A highly natural view of modalities is that they are properties of propositions. One promising feature of this view is that it permits a rigorous study of modalities with the use of higher-order logic, a fruitful setting in which to theorise about higher-order entities like properties of propositions. Building on the work of Cresswell (1965) and Suszko (1975), Dorr (2016) and Bacon (2018), amongst others, have begun to explore this conception of modality. In doing so, one important result has emerged: higher-order logic can be used to pin down an exact role for the broadest necessity. Pertinently, this role convergences with the considerations from Essay Three summarised above.

Let *Booleanism* be the theory that propositions form a Boolean algebra with respect to the truth-functional operations. This theory can be axiomatised in classical propositional logic with identity by the familiar Boolean and identity laws, and the rule that logical equivalents within the theory are identical.[29] According to the resultant theory, sentences of the theory such as A and $\neg\neg A$ express the same proposition, as do sentences like $A = \neg\neg A$ and $A \vee \neg A$.

Relatedly, there is only one tautologous proposition $\top$, which is expressed by all theorems of the system. If the language is then extended with a device for binding propositional variables, the property of *being top* can be expressed as $\lambda p.p = \top$. Interestingly, there is an intimate connection between *being top* and propositional identity in the generalised Boolean theory, for that theory proves the following theorem $(\top = (A \leftrightarrow B)) \leftrightarrow (A = B)$. More plainly, the generalised Boolean theory proves that all true propositional identifications are top. As Dorr (2016, p. 68) also

[29] See Dorr (2016, p. 124, n. 59) and Bacon (2018, p. 13).

points out, another interesting feature of *being top* is that Booleanism proves the **K** and **T** axioms for it. With the addition of certain natural axioms, Dorr (2016, p. 124, n. 59) shows that *being top* has an **S5** logic. In slight contrast, Bacon (2018) motivates only an **S4** logic for *being top*, and provides considerations against the axioms which motivate **S5**. Nevertheless, if *being top* has at least an **S4** logic, it is an extremely natural candidate for the broadest objective necessity in the sense that *being top* implies objective necessity in all other senses. More carefully, $p = \top$ implies $\mathcal{O}p$ whenever $\mathcal{O}$ is a propositional operator such that $\mathcal{O}\top$. The implication appeals to a version of Leibniz's Law in the extended propositional calculus described above:

Substitution: $A = B \rightarrow (\phi \rightarrow \phi[A/B])$

For given $p = \top$ and $\mathcal{O}\top$, $\mathcal{O}p$ is implied by the following instance of **Substitution**: $p = \top \rightarrow (\mathcal{O}\top \rightarrow \mathcal{O}p)$. Thus, in the generalised Boolean setting, the exact extension of the broadest objective necessity is pinned down by propositional identity: to be necessary in the broadest sense is to be identical with the tautologous proposition.

Dorr (2016, p. 70) also conjectures that non-Booleans are capable of characterising the broadest objective necessity in terms of propositional identity by slightly alternative means. Indeed, Jeremy Goodman (MS) has developed a non-Boolean theory of propositional granularity based on the notion of aboutness that characterises the broad objective necessity of a proposition as that proposition's being identical to its own self-identity. This indicates that the capacity to pin down the broadest objective necessity by propositional identity is not unique to Booleanism, and appears to be a robust property of different theories of granularity. Thus, although the following discussion takes place in the context of Booleanism, its points should be expected to generalise beyond that setting.

Importantly, according to the generalised Boolean theory distinct propositions cannot be necessarily equivalent in the broadest objective sense. For Booleanism proves that $A \neq B \rightarrow (\top \neq (A \leftrightarrow B))$, as should be clear from the above. Crucially, it follows that if there is independent reason to think that certain propositions are distinct, there is as much reason to think that they are not necessarily equivalent in the broadest objective sense. Saliently, then, if there is independent reason to think that, for example, the proposition that human h exists is distinct from the proposition that human h originates from sperm s and egg e, there is as much reason to think that those propositions are not necessarily equivalent in the broadest objective sense. Similarly, whatever reason there is for thinking that the proposition that *fww* exists is distinct from the proposition that *fww* occurs between 1914-1918 is reason to think that those propositions are not equivalent in the broadest objective sense either. Yet perhaps the central moral of hyperintensional metaphysics in the image of Fine (1994; 2001) is that there is a wealth of reason to think that such propositions are distinct. In particular, neither proposition is logically equivalent to the other according to classical propositional logic with identity, and both propositions seemingly disagree on their subject matter. However, rather than interpret this as evidence that broad necessity is unable to distinguish certain non-identical propositions, it may been taken as evidence of just how stringent necessary equivalence is in the broadest sense.[30] On this view, the so-called hyperintensional revolution in metaphysics involves a reimagining of the intensional, not its supersession.

Put more directly, the program of investigating modalities with the use of higher-order logic appears to concur with the conclusion of Essay Three, that the broadest objective necessity fails to license the essentialist constraints which generate the cases of putative incompossible chunky individuals. For example, it is not broadly necessary that whenever human h exists they originate from sperm s and egg e. Yet such origin essentialism was the primary motivation for the *Humans* case of incompossible chunkiness. Without that reason, there is no immediate reason to think that human h and human h^* are not compossible in the broadest objective sense. More generally, one should expect cases like *Humans* and *Events* to fail with respect to the broadest objective necessity, although they may succeed with some more restricted modality. Given this further motivation, the following section describes how one can model the interaction between broad necessity as just described and what has been dubbed 'metaphysical necessity'.

[30]See Fritz (MS, p. 20) for exactly the same sentiment.

4.4 Modelling Compossibility

4.4.1 Truth Over Some Worlds

On the above conception, broad necessity has an extension radically different from that which metaphysical necessity has been thought to possess. Although the exact extension of broad necessity is an open question, the considerations above suggest one clear feature of it: arbitrary metaphysically possible individually chunky individuals are such that it is broadly possible for them to be chunky together.[31] The aim of this section is to present a way of modelling this specific aspect of broad necessity in relation to metaphysical necessity. As mentioned previously, this modelling technique is then implemented formally in Appendix B to show that Williamson's formal expressive challenge can be met.

Modelling possibility and necessity with a Kripke semantics is evocative. Claims of necessity and possibility are modelled in terms of truth at a 'world' in model, and 'worlds' are associated with 'domains' in order to model individuals' modal profiles. More generally, the model-theoretic construction serves as a tractable template of metaphysical modal space, conceptualised in extensional as opposed to intensional terms. As one would expect, Kripke models are also a natural setting in which to model the interplay between broad and metaphysical necessity. Indeed, there is a simple operation on the class of Kripke models used to study metaphysical necessity which generates models for studying the interaction between metaphysical necessity and broad necessity thus envisaged. Described informally, the operation takes a space of metaphysical possibilities and returns all *pluralities* of those metaphysical possibilities. In each new plural modal space, the metaphysical possibilities can then be recovered as single membered pluralities of metaphysically possible worlds. Moreover, the operation preserves the extension of predicates across metaphysically possible worlds and single membered pluralities thereof. Broad necessity may then be thought of as truth *over* all pluralities of worlds, thus subsuming metaphysical necessity as desired.

[31] Bacon (MS) draws a similar moral.

Although the idea of truth *over* a plurality of worlds is evocative, the notion is merely used to model the interaction between broad necessity and metaphysical necessity, and not as an intended conception of broad possibility. By way of comparison, it may be helpful to compare the typical way of modelling free logic in which an 'outer domain' is used to model the behaviour of non-existents. Of course, the free logician does not genuinely assert that *there are* non-existents belonging to some mysterious outer domain; rather, the outer domain is a mere mathematical instrument used to model the free logician's conceptions of existence and reference. Likewise, truth over a plurality of worlds can be used merely as a model-theoretic instrument for studying a fairly specific feature of how broad necessity interacts with metaphysical necessity, and not a notion which needs to be embedded in the ultimate theory of broad necessity.

By way of contrast, Phillip Bricker (2001) recommends that the Lewisian Modal Realist opt for a reduction of metaphysical modality in terms of plural quantification over worlds and truth over pluralities thereof. Bricker's suggestion is designed to help the Lewisian Modal Realist with the so-called problem of island universes which threatens to undermine Lewisian Modal Realism's comprehensive reduction of modality. However, whereas Bricker proposed to introduce 'truth over some worlds' as a new locution of modal theory, the current proposal is merely to use the notion for an evocative modelling strategy.[32] Perhaps the main benefit of the latter is that, unlike Bricker's development, there is no requirement to think that the extension of any predicate over some worlds is merely the union of its extensions at the metaphysically possible worlds therein. Indeed, for its intended purpose, the model requires almost no systematic connection between how the extensions of predicates over some worlds relate to their extensions at metaphysically possible worlds therein.

As mentioned previously, one exemption to this rule is the behaviour of predicates at single membered pluralities of worlds, which are exactly their extensions at the metaphysically possible worlds therein. Clearly, this is a highly natural requirement given the modelling exercise. The only other such requirement is that the the

[32]The so-called 'span operators' of tense logic are another salient example of a similar notion used as a genuine locution of a theory; see Lewis (2004), Sider (2001, pp.26-28), Brogaard (2006).

extension of 'is chunky' over some worlds is the union of its extensions at the metaphysically possible worlds therein. This requirement is made not due to some metaphysical privileged feature of being chunky, but because the model is designed to capture the key thesis that arbitrary individually metaphysically possible chunky individuals are broadly possibly chunky together. Given this aim, the requirement of 'is chunky' is as motivated as the requirement on predicate extensions at single membered pluralities.

Complementing this, which individuals exist over some worlds is determined exactly by which individuals exist at the metaphysically possible worlds therein. In terms of the model, the 'domain' of individuals for a given plurality of worlds is therefore simply the union of the 'domains' of the metaphysically possible worlds which belong to the plurality in question. In this sense, the 'domain' of a plurality of worlds simply *accumulates* the 'domains' of it members. Thus, the broadly possible individuals are exactly the metaphysically possible individuals, although broad possibility issues less stringent demands on the modal profiles of such individuals. In tandem with the above, this allows the model precisely to capture the compossible chunky existence of arbitrary individually metaphysically possible chunky individuals. Indeed, one of the most notable features of the model theory is that it validates the following formula (where $\Diamond^+$ expresses broad possibility):

Compossibility $\Diamond^+ {\uparrow} \Box \forall x {\downarrow} Cx$

In effect, the fact that the new background logic validates **Compossibility** will ensure the success of the mapping.

To close this section, it bears further emphasis that the modelling technique is an intentionally limited conception of broad necessity. For instance, by design, the model is not a natural setting in which to investigate the connections between broad necessity and fineness of grain. Moreover, the model will simply assume an **S5** logic for broad necessity for convenience, although, as mentioned previously, whether the logic of broad necessity is stronger than **S4** is a contested issue. Nevertheless, even this intentionally limited and simplified conception of broad necessity suffices for the current purpose: mapping the entirety of necessitist discourse to a body of neutral discourse.

4.4.2 Mappings Revisited

The contingentist is now in a position to specify a recursive mapping of necessitist discourse to a body of neutral claims. This section explains informally how the contingentist can provide such a mapping. The formal results which accompany this section are found in Appendix B.

Before specifying the required mapping, the background logic must be updated. As was done previously, the background logic will be specified in terms of its underlying model theory. To this end, recall that the background logic was given by standard variable domain models for second-order quantified modal logics. Following the standard regimentation, these models were quadruples $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$. In these models, $\mathbb{W}$ was a non-empty set (the 'worlds'), $\mathcal{D}$ a non-empty set (the 'individuals'), *dom* a total function (the 'domain' function) from members of $\mathbb{W}$ to possibly empty subsets of $\mathcal{D}$, and $\mathcal{I}$ (the 'interpretation') a total function from constants to members of $\mathcal{D}$ and from n-place predicates and members of $\mathbb{W}$ to sets of n-tuples taken from $\mathcal{D}$. The class of models for our current background logic is given by an operation on the standard variable domain models to classes of new models. In order to specify this operation, one can define a one-many relation $\rightharpoonup$ between standard variable domain models in which either **AuxNec** or **AuxCon** are true and a new model. The $\rightharpoonup$ relation conforms to the following conditions (hereafter, uppercase letters W, U, V are used to denote members of $\mathbb{W}^+$; lowercase letters w, u, v are used to denote singleton sets in $\mathbb{W}^+$):

$\langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle \rightharpoonup \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$ iff

PL $\quad \mathbb{W}^+ = \{A \subseteq \mathbb{W} : |A| \geq 1\}$

Comp 1 $\quad dom^+(W) = \bigcup\limits_{w \in W} dom(w)$

$\mathcal{I}^+$ is a function mapping each non-logical n-place atomic predicate constant F and $W \in \mathbb{W}^+$ to $\mathcal{I}^+(F)(W) \subseteq dom^+(W)^n$ such that:

$$\textbf{Stability} \quad \mathcal{I}^{+}(F)(\{w\}) = \mathcal{I}(F)(w)$$

$$\textbf{FC}^{+} \quad \mathcal{I}^{+}(F)(W) \subseteq \mathcal{I}^{+}(C)(W)^{n}$$

$$\textbf{Comp 2} \quad \mathcal{I}^{+}(C)(W) = \bigcup_{w \in W} \mathcal{I}^{+}(C)(\{w\})$$

Informally, the idea is each standard variable domain model generates a class of models in which: $\mathbb{W}^{+}$ (the 'pluralities of worlds') is all non-empty subsets of $\mathbb{W}$; $\mathcal{D}$ (the 'individuals') stays the same; dom^{+} (the 'plurality domain' function) aggregates the values of the old $dom(w)$ function, and $\mathcal{I}^{+}$ (the 'interpretation') ensures that the old extensions of atomic predicates at 'worlds' are their new extensions at the corresponding single-membered 'pluralities of worlds' and chunkiness is inherited over pluralities of worlds. Variable assignments work as they did previously with the obvious updates.

In more detail, the use of $\mathbb{W}^{+}$ reflects the idea of modelling broad possibility as truth over some metaphysically possible worlds. In line with this, from the perspective of the model theory single membered pluralities of worlds (i.e. singleton subsets of $\mathbb{W}$) are treated as metaphysically possible worlds by the fact that **Stability** requires the extension of predicates at singleton subsets of $\mathbb{W}$ to be exactly the same as its previous extension at the member of the relevant singleton. Members of $\mathbb{W}^{+}$ are required to be non-empty for harmless technical convenience. In addition, **Comp 1** models the thought that arbitrary metaphysically possible individuals are compossible in the broadest objective sense, and **Comp 2** models the thought that arbitrary metaphysically possibly chunky individuals are possibly chunky together in the broadest objective sense. Both of these conditions are natural, well-motivated ways of modelling the interaction between broad necessity and metaphysical necessity. Finally, **FC**$^{+}$ encodes the condition that it is broadly necessary that the extensions of the predicate constants of the shared language must belong to the realm of the chunky. Given that the auxiliary necessitist and auxiliary contingentist agreed on this in the case of metaphysical necessity, **FC**$^{+}$ is a natural condition to require. It may also help to highlight that **Stability** is always satisfiable with **FC**$^{+}$ since new models are generated from previous models in which either **AuxNec** or **AuxCon** are true.

In relation to the final point, the fact that new models are generated from previous models in which either **AuxNec** or **AuxCon** are true implies that *all* new models are ones in which either **AuxNec** or **AuxCon** is true.[33] This marks a mild departure from Williamson's (2010) formal treatment of the expressive challenge, but it is a technical convenience. Moreover, it is philosophically harmless: as long as both the auxiliary necessitist and auxiliary contingentist agree that either side can meet the expressive challenge, neither theorist can claim an expressive advantage over the other.

The notion of consequence ($\models^{+}$) for the new background logic is defined as usual. Given the condition to which $\rightharpoonup$ conforms, **NNE** is true in some new models and false in other new models, thus preserving the neutrality of the background logic. Following the previous precedent, it is required that $\mathcal{I}^{+}(F)(W)$ belongs to $dom^{+}(W)$ when F is a non-logical atomic predicate. Finally, no accessibility relations are used so the logic for broad necessity is **S5**. Due to use of 'pluralities' of worlds as indices, the logic of metaphysical necessity is a sub-system of **S5**, which is merely a harmless effect of the chosen modelling technique; see **Proposition 2.8**.

In this new setting, it must of course be checked that the necessitist can still provide a mapping of contingentist discourse to neutral discourse. However, it can be shown that a mapping $[.]^{\text{CON}+}$ which simply extends $[.]^{\text{CON}}$ by a new, obvious clause for the broad necessity operator achieves this aim.

Proposition 2.26 **AuxCon** $\models^{+} A \leftrightarrow [A]^{\text{CON}+}$, for all formulae A

The reader may consult the appendix for details.

The more interesting question is whether the contingentist is capable of mapping necessitist discourse to a neutral body of discourse. The problem with the contingentist's previously proposed mapping $[.]^{\text{NEC}}$ was that it attempted to map second-order necessitist quantification by prefixing the second-order quantifiers with metaphysical necessity operators. However, that proposal was undermined by the fact that arbitrary individually metaphysically possible chunky individuals are not always jointly metaphysically possibly chunky. This fact produced incorrect

[33]See **Proposition 2.16** of Appendix B.

results for sentences like $\Diamond\mathbf{H}$.

In the current setting, however, the more natural proposal is to map second-order necessitist quantification by prefixing the second-order quantifiers with broad necessity operators. In tandem with this, the clause for atomic predications involving predicate variables must also be updated to reflect that fact that the necessitist may assign the predicate variable an extension which includes individuals for which it is metaphysically impossible that they are chunky together. Since the exact syntactic details are crucial, it will pay to display them in full:

$[Fv_1 \ldots v_n]^{\text{NEC+}} = Fv_1 \ldots v_n \wedge Cv_1 \wedge ... \wedge Cv_n$, n-place non-logical atomic F

$[Vv_1 \ldots v_n]^{\text{NEC+}} = \Diamond^+(Vv_1 \ldots v_n \wedge V{\leq}C)$, for n-place variable V

$[v_1 = v_2]^{\text{NEC+}} = \Diamond(v_1 = v_2 \wedge Cv_1 \wedge Cv_2)$

$[\neg A]^{\text{NEC+}} = \neg[A]^{\text{NEC+}}$

$[A \wedge B]^{\text{NEC+}} = [A]^{\text{NEC+}} \wedge [B]^{\text{NEC+}}$

$[\Diamond^+ A]^{\text{NEC+}} = \Diamond^+[A]^{\text{NEC+}}$

$[\Diamond A]^{\text{NEC+}} = \Diamond[A]^{\text{NEC+}}$

$[\uparrow A]^{\text{NEC+}} = \uparrow[A]^{\text{NEC+}}$

$[\downarrow A]^{\text{NEC+}} = \downarrow[A]^{\text{NEC+}}$

$[\exists xA]^{\text{NEC+}} = \uparrow\Diamond\exists x(Cx \wedge \downarrow[A]^{\text{NEC+}})$

$[\exists XA]^{\text{NEC+}} = \uparrow\Diamond^+\exists X(X{\leq}C \wedge \downarrow[A]^{\text{NEC+}})$

As advertised, second-order quantified formulas are mapped to formulas in which they occur in a chunky context and are prefixed by broad possibility operators. Relatedly, atomic predications involving second-order variables are mapped to formulas in which they occur within a similar environment.

This mapping appears to rectify the issue encountered by sentences such as $\Diamond\mathbf{H}$, which is now mapped to the following sentence by $[.]^{\text{NEC+}}$:

$$[\Diamond\mathbf{H}]^{\text{NEC+}} \;\; \Box\forall x(Cx \rightarrow \Box\forall y(Cy \rightarrow (\Diamond(Hx \wedge Cx) \wedge \Diamond(Hy \wedge Cy) \rightarrow \Diamond^{+}\exists X(X{\leq}C \wedge \Box\forall z(Cz \rightarrow (\Diamond^{+}(Xz \wedge X{\leq}C) \leftrightarrow \Diamond(z = x \wedge Cz \wedge Cx) \vee \Diamond(z = y \wedge Cz \wedge Cy)))))))$$

Informally, $[\Diamond\mathbf{H}]^{\text{NEC+}}$ states that arbitrary metaphysically possible individuals which are individually metaphysically possibly human are such that there is some broad possibility in which there are some chunky things that comprise exactly them. Put evocatively in terms of the modelling metaphor, for arbitrary metaphysically possible humans there are some worlds over which they are all chunky together. Given that arbitrary metaphysically possible chunky individuals are broadly possibly jointly chunky, the auxiliary necessitist ought to recognise that $[\Diamond\mathbf{H}]^{\text{NEC+}}$ is equivalent to $\Diamond\mathbf{H}$.

Indeed, these informal suggestions are confirmed by a technical result. One of the main results of Appendix Part B is that every claim A is mapped by $[.]^{\text{NEC+}}$ to a neutral claim $[A]^{\text{NEC+}}$, which the auxiliary necessitist will view as equivalent to A.

Proposition 2.38 **AuxNec** $\models^{+} [A]^{\text{NEC+}} \leftrightarrow A$

Consequently, if arbitrary metaphysically possible chunky individuals are broadly possibly chunky together, contingentists can recursively specify a neutral equivalent of any necessitist claim, and Williamson's expressive challenge can be met by both parties. Thus, in effect, the contingentist may recognise and articulate analogues of any modal hypothesis which necessitists can express and vice-versa. This constitutes a sense in which the theories stand in expressive parity.

4.5 Revenge

In their recent article, Jeremy Goodman and Peter Fritz (2017) have argued against a related contingentist proposal. They focus on contingentists who introduce a new modality, hyper-possibility, in order to construct the desired mapping. Although Fritz & Goodman (2017, pp. 1091-1092) consider interpretations of hyper-possibility as a type of fictionalist or counterfactual modality, like broad possibility the notion of hyper-possibility they have in mind is intended to be broader than metaphysical possibility.[34] Thus their criticisms are relevant and ought to be considered in the context of the current proposal.

One of their central concerns is that no notion of hyper-possibility can admit the possible joint chunkiness of *arbitrary* individually metaphysically possibly chunky individuals even though it may admit the possible joint chunkiness of arbitrary pairs of individually possibly chunky individuals. They base this concern on issues surrounding the recombinatorial paradoxes discussed Essay Two of this thesis. They write (2017, p. 1092):[35]

> Even if some such modality [hyper-possibility] can be used to evade essentialism-induced incompossibility, there is no reason to think it will evade the second sort of incompossibility discussed in §2.3 [i.e. the sort of incompossibility discussed in Essay Two of this thesis; reference to their own article], namely, the incompossibility that results from the fact that, [metaphysically] necessarily, whatever possible elementary particles there are, it is possible that there be all of them, and one more possible elementary particle.
>
> [...]
>
> Presumably, it is hyper-necessary that whatever hyper-possible elementary particles there are, it is hyper-possible that there be all of them and one more. It follows that the hyper-possible elementary particles are hyper-incompossible. [...] The proponent of the hyper-possibility paraphrase now faces a paraphrase challenge exactly analogous to the one with which we began.

Fritz & Goodman's point is that there is a type of non-essentialism induced incompossibility, a sort of *logical* incompossibility which precludes any notion

[34] They may have in mind Fine's (2016) quasi-fictionalist proposal which was discussed in §4.2.

[35] As mentioned in §4.1.2, Goodman & Fritz are concerned with contingentist *paraphrases* of necessitist discourse; however, their point still applies with equal force.

of hyper-possibility from admitting the joint possible chunkiness of arbitrary metaphysically possibly (or hyper-possibly) chunky individuals.

To appreciate Fritz & Goodman's argument, it will help to consider the exact recombinatorial reasoning to which they are appealing. As outlined in Fine (2002, pp.223-224), Fritz (2016), and Essay Two of this thesis, the recombinatorial reasoning in the case of metaphysical possibility relies on the following two premises:

(R1) For any metaphysically possible world w, there is a metaphysically possible world u where all elementary particles at w are elementary particles and there is an elementary particle distinct from every elementary particle at w.

(R2) There is some metaphysically possible world w such that for all metaphysically possible worlds u all elementary particles at u are elementary particles at w.

(R1) and **(R2)** lead to contradiction. Considering w in **(R2)**, **(R1)** then requires that there be some metaphysically possible world u where every elementary particle at w is an elementary particle and there is some elementary particle distinct from all elementary particles at w. However, this contradicts the claim of **(R2)** that the elementary particles at w are exactly the metaphysically possible elementary particles.

Fritz & Goodman's discussion suggests that the correct response to this recombinatorial paradox is simply to reject **(R2)**, in which case there is no metaphysically possible world at which all metaphysically possible elementary particles are elementary particles. However, **(R1)** and **(R2)** are motivated by almost parallel considerations.[36] Indeed, Essay Two provided a treatment of the paradox and a model thereof in which **(R2)** is true at each metaphysical modal space, and a qualified alternative to **(R1)** was true at each metaphysical modal space too. Thus, it is not uncontentious that the anticipated form of logical incompossibility arises with respect to even metaphysical possibility.

Nevertheless, even granting the denial of **(R2)** to Fritz & Goodman, it is not immediate that arbitrary metaphysically possibly chunky individuals are not such

[36]See Essay Two of this thesis and Fritz (2016, pp. 548-549).

that it is broadly possible that they are chunky together. To appreciate this, notice first that even if **(R2)** fails, the necessitist still recognises that arbitrary metaphysically possible elementary particles are compossible, since actually they all exist. The necessitist is able to recognise this due to their rejection of the essentialist thesis that any metaphysically possible elementary particle is metaphysically necessarily an elementary particle. However, given the characterisation of broad possibility above, one should expect the failure of a similar thesis: that any metaphysically possible elementary particle is broadly necessarily an elementary particle. For if the necessity of origin fails for material objects with respect to broad necessity, nothing would appear to preclude metaphysically possible elementary particles from being such that it is broadly possible that they are otherwise. Supplementing this point, for a given electron e plausibly the proposition that e exists is distinct from the proposition that e has the exact profile of an electron (i.e. has negative charge one and so on). Thus, if broad necessity is pinned down by propositional identifications, it should be expected to eschew the essentiality to elementary particles of being an elementary particle. As a result, it may well be broadly impossible for there to be all metaphysically possible elementary particles as elementary particles. Yet it would not follow that it is broadly impossible for there to be all metaphysically possible elementary particles, for those metaphysically possible elementary particles need not be elementary particles at the broad possibility in question.

To appreciate this point more formally, consider the following formalisation of the claim that it is broadly possible for there to be all metaphysically possible elementary particles as elementary particles (where 'E' expresses the property of being an elementary particle).

(EP) $\Diamond^{+} \uparrow \Box \forall x(E(x) \rightarrow \downarrow E(x))$

The class of models defined in Appendix B includes models in which **(EP)** is false yet the following formula is true.

(EX) $\Diamond^{+} \uparrow \Box \forall x(E(x) \rightarrow \downarrow \exists y(y = x))$

However, **(EX)** simply formalises the claim that there is some broad possibility at which all metaphysically possible elementary particles exist. Thus, on the proposed formal treatment of broad and metaphysical possibility, it may be broadly impossible for there to be all metaphysically possible elementary particles *as elementary particles* without it being broadly impossible for there to be all metaphysically possible elementary particles.[37]

Consequently, there are several reasons to think that Goodman & Fritz's revenge argument does not jeopardise the success of the mapping. Most notably, the features of hyper-possibility which Goodman & Fritz appeal to in their revenge problem are not well motivated with respect to broad possibility.

4.6 Concluding Remarks

One of the most forceful arguments against contingentism has been that its advocates are unable to meet Williamson's (2010; 2013) expressive challenge to recursively map the entirety of necessitist discourse to a neutral body of discourse. This problem is generated by putative cases of individually, but not jointly, possibly chunky individuals. However, by drawing on the arguments of Essay Three of this thesis and on the work of Dorr (2016) and Bacon (2018), this essay has argued that the broadest objective necessity does not generate the apparent cases of incompossibly chunky individuals. In light of this point, the essay has shown that in the presence of such a modality the entirety of necessitist discourse can be mapped to a neutral body of discourse, regardless of whether the modality in question is identified with metaphysical necessity or not. Finally, it was argued that there is no 'revenge' problem facing the proposed solution. The central upshot is that Williamson's

[37] In addition, the class of models defined in Appendix B includes models in which **(EP)** is false and it is *true* that it is broadly possible that all metaphysically possible elementary particles are chunky.

expressive challenge may be met by contingentists, at least given a highly natural conception of broad necessity. In other words, a well-motivated conception of broad necessity undermines one of the strongest arguments in favour of necessitism.

Appendix

The aim of this appendix is to prove that $[.]^{\text{CON}+}$ and $[.]^{\text{NEC}+}$ succeed in mapping each claim to a neutral claim that the respective auxiliary contingentist and necessitist theories will deem as equivalent in the new background logic. Part A primarily adapts the results of Williamson (2010) about first-order languages to a slightly different, but more convenient setting. Part B then extends these results to second-order languages.

Part A: First-order Language

Definition 1.1 The language $\mathcal{L}$ consists of the following items of vocabulary: countably many individual variables $x, y, z, ...$; finitely many atomic non-logical predicate constants of finite arity, including the 1-place constant C; the atomic binary logical predicate $=$; the truth-functional connectives $\neg$, and $\rightarrow$; the unary modal operators $\Box$, $\uparrow$, and $\downarrow$, and the first-order quantifier $\forall$.

The other truth-functional connectives, modal operators and quantifiers are introduced as abbreviations in the usual way.

Definition 1.2 The *well-formed formulas* of $\mathcal{L}$ are given by the following rule. (Here F is an n-place predicate constant and $v, v_1, ..., v_n$ are individual variables.)

$$Fv_1...v_n \mid \neg\psi \mid (\psi \rightarrow \chi) \mid \forall v(\psi) \mid \Box\psi \mid \uparrow\psi \mid \downarrow\psi$$

Definition 1.3 A *model* $\mathfrak{M}$ *of* $\mathcal{L}$ is a quadruple $\langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$. In these models, $\mathbb{W}$ and $\mathcal{D}$ are non-empty sets, dom is a function mapping each $w \in \mathbb{W}$ to $dom(w) \subseteq \mathcal{D}$, and $\mathcal{I}$ is a function mapping each non-logical n-place atomic constant F and $w \in \mathbb{W}$ to $\mathcal{I}(F)(w) \subseteq dom(w)^n$.

Definition 1.4 For a model $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$, a *variable assignment* over $\mathfrak{M}$ is a function g mapping each individual variable v to a member $g(v) \in \mathcal{D}$.

We write $g[d/\alpha]$ for the variable assignment just like g except that it assigns d to α. $\mathcal{I}_g(\beta)$ is used for $g(\beta)$.

Definition 1.5 For a set S, its *sequence set* $S_{<\omega}$ is the set of finite sequences of members of S. We write $\langle\rangle$ for the empty sequence and, for $s \in S_{<\omega}$, s^v is sequence that results from appending $v \in S$ to s.

Definition 1.6 For a model $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I}\rangle$, $w \in \mathbb{W}$, $s \in \mathbb{W}_{<\omega}$, and g a variable assignment over $\mathfrak{M}$, we define the relation $\models$ recursively as follows.

(In this definition, F is a non-logical n-place predicate constant, and $v, v_1, ..., v_n$ are individual variables. For a set S, $diag(S) = \{\langle o, o\rangle : o \in S\}$.)

(At) $(\mathfrak{M}, w, s) \models_g Fv_1...v_n$ iff $\langle \mathcal{I}_g(v_1), \ldots, \mathcal{I}_g(v_n)\rangle \in \mathcal{I}_g(F)(w)$;

(Id) $(\mathfrak{M}, w, s) \models_g v_1 = v_2$ iff $\langle \mathcal{I}_g(v_1), \mathcal{I}_g(v_2)\rangle \in diag(dom(w))$;

($\neg$) $(\mathfrak{M}, w, s) \models_g \neg\phi$ iff $(\mathfrak{M}, w, s) \not\models_g \phi$;

($\wedge$) $(\mathfrak{M}, w, s) \models_g \phi \wedge \psi$ iff $(\mathfrak{M}, w, s) \models_g \phi$ and $(\mathfrak{M}, w, s) \models_g \psi$;

($\Box$) $(\mathfrak{M}, w, s) \models_g \Box\phi$ iff for all $u \in \mathbb{W}$, $(\mathfrak{M}, u, s) \models_g \phi$;

($\forall$) $(\mathfrak{M}, w, s) \models_g \forall v\phi$ iff for all $d \in dom(w)$, $(\mathfrak{M}, w, s) \models_{g[d/v]} \phi$;

($\uparrow$) $(\mathfrak{M}, w, s) \models_g \uparrow\phi$ iff $(\mathfrak{M}, w, s^w) \models_g \phi$;

($\downarrow$) $(\mathfrak{M}, w, s^u) \models_g \downarrow\phi$ iff $(\mathfrak{M}, u, s) \models_g \phi$;

($\downarrow$) $(\mathfrak{M}, w, \langle\rangle) \models_g \downarrow\phi$ iff $(\mathfrak{M}, w, \langle\rangle) \models_g \phi$.

Definition 1.7 A formula ϕ is *valid in a model* $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I}\rangle$ iff $(\mathfrak{M}, w, s) \models_g$

ϕ for all $w \in \mathbb{W}$, $s \in \mathbb{W}_{<\omega}$, and every assignment g over $\mathfrak{M}$. ϕ is *valid* iff ϕ is valid in every model.

For a set of formulas Γ, $(\mathfrak{M}, w, s) \models_g \Gamma$ iff $(\mathfrak{M}, w, s) \models_g \gamma$, for all $\gamma \in \Gamma$. If Γ and Δ are single formulas or sets thereof, $\Gamma \models \Delta$ iff if $(\mathfrak{M}, w, s) \models_g \Gamma$ then $(\mathfrak{M}, w, s) \models_g \Delta$. We say that $\mathfrak{M} \models \Gamma$ iff $(\mathfrak{M}, w, s) \models_g \gamma$ for all $\gamma \in \Gamma$, $w \in \mathbb{W}$, $s \in \mathbb{W}_{<\omega}$, and every assignment g. Moreover, $\models \Gamma$ iff $\mathfrak{M} \models \Gamma$ for all $\mathfrak{M}$.

Definition 1.8 The *auxiliary necessitist* theory **AuxNec** consists of the following axioms:

NNE $\Box\forall x\Box\exists y(x = y)$

$\Diamond$**C** $\forall x\Diamond Cx$

FC $\Box\forall x_1...\forall x_n(Fx_1...x_n \to Cx_1 \wedge ... \wedge Cx_n)$, for each non-logical predicate F

The *auxiliary contingentist* theory **AuxCon** consists of the following axiom:

$\forall$**C** $\Box\forall xCx$

Note that there are models $\mathfrak{M}$ such that $\mathfrak{M} \models$ **AuxNec** and $\mathfrak{M} \models$ **AuxCon**.

Definition 1.9 A model $\mathfrak{M}$ is *admissible* iff either $\mathfrak{M} \models$ **AuxNec** or $\mathfrak{M} \models$ **AuxCon**.

Hereafter in Parts A and B, attention is restricted to admissible $\mathcal{L}$ models, so '$\mathcal{L}$ model' should always be understood as abbreviating 'admissible $\mathcal{L}$ model'. Moreover, A is any formula of $\mathcal{L}$, $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$ is any $\mathcal{L}$ model, $w \in \mathbb{W}$, $s \in \mathbb{W}_{<\omega}$, and g is a variable assignment over $\mathfrak{M}$.

Proposition 1.10 For any model $\mathfrak{M}$, $\mathfrak{M} \models$ **FC** and $\mathfrak{M} \models \Diamond$**C**.

Proof. The proposition is immediate if $\mathfrak{M} \models$ **AuxNec**, so suppose $\mathfrak{M} \models$ **AuxCon**. Given **Definition 1.3**, $\mathfrak{M} \models \forall x_1...\forall x_n(Fx_1...x_n \to \exists y(y =$

$x_1) \wedge ... \wedge \exists y(y = x_n))$. Since $\mathfrak{M} \models \Box\forall x Cx$, it is routine to show as a consequence that $\mathfrak{M} \models$ **FC**. It also routine to show that $\mathfrak{M} \models \Diamond$**C** given that $\mathfrak{M} \models \forall$**C**.

Definition 1.11 For $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$, its *restriction* $\mathfrak{M}^{\mathrm{C}} = \langle \mathbb{W}, \mathcal{D}, dom^{\mathrm{C}}, \mathcal{I}^{\mathrm{C}} \rangle$, where:

$dom^{\mathrm{C}}(w) = \mathcal{I}(C)(w)$

$\mathcal{I}^{\mathrm{C}}(F)(w) = \mathcal{I}(F)(w) \cap \mathcal{I}(C)(w)^n$, for n-place non-logical predicate F

Proposition 1.12 For a model $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$, $\mathfrak{M}^{\mathrm{C}} \models$ **AuxCon**.

Proof. Notice that for arbitrary $w \in \mathbb{W}$, $dom^{\mathrm{C}}(w) = \mathcal{I}(C)(w) = \mathcal{I}^{\mathrm{C}}(C)(w)$. Thus for arbitrary $s \in \mathbb{W}_{<\omega}$, $(\mathfrak{M}^{\mathrm{C}}, w, s) \models_g \forall x Cx$. As a result, $\mathfrak{M}^{\mathrm{C}} \models \Box\forall x Cx$.

Corollary 1.13 For a model $\mathfrak{M}$, $\mathfrak{M}^{\mathrm{C}}$ is admissible.

Proof. Immediate from **Proposition 1.12**.

Definition 1.14 The function $[.]^{\mathrm{CON}}$ is defined over formulae of $\mathcal{L}$ as follows.

$[Fv_1 \dots v_n]^{\mathrm{CON}} = Fv_1 \dots v_n \wedge Cv_1 \wedge ... \wedge Cv_n$, for n-place non-logical atomic F

$[v_1 = v_2]^{\mathrm{CON}} = v_1 = v_2 \wedge Cv_1 \wedge Cv_2$

$[\neg A]^{\mathrm{CON}} = \neg[A]^{\mathrm{CON}}$

$[A \wedge B]^{\mathrm{CON}} = [A]^{\mathrm{CON}} \wedge [B]^{\mathrm{CON}}$

$[\Diamond A]^{\mathrm{CON}} = \Diamond[A]^{\mathrm{CON}}$

$[\uparrow A]^{\mathrm{CON}} = \uparrow[A]^{\mathrm{CON}}$

$[\downarrow A]^{\text{CON}} = \downarrow[A]^{\text{CON}}$

$[\exists x A]^{\text{CON}} \; = \; \exists x(Cx \wedge [A]^{\text{CON}})$

Note: in the following proofs, the item enclosed in the rightmost square brackets states part of the justification for the corresponding inference; elementary items of justification are omitted.

Definition 1.15 A formula A is *neutral* iff for some B, $\models A \leftrightarrow [B]^{\text{CON}}$.

Proposition 1.16 $(\mathfrak{M}, w, s) \models_g [A]^{\text{CON}}$ iff $(\mathfrak{M}^{\text{C}}, w, s) \models_g A$

Proof. By induction on the complexity of A.

Basis 1: $(\mathfrak{M}, w, s) \models_g [Fv_1...v_n]^{\text{CON}}$

iff $(\mathfrak{M}, w, s) \models_g Fv_1 \ldots v_n \wedge Cv_1 \wedge ... \wedge Cv_n$ [Def. $[.]^{\text{CON}}$]

iff $\langle \mathcal{I}_g(v_1), \ldots, \mathcal{I}_g(v_n)\rangle \in \mathcal{I}_g(F)(w) \cap \mathcal{I}_g(C)(w)^n$ [Def. 1.6 **At**; Def. $\cap$]

iff $\langle \mathcal{I}_g^{\text{C}}(v_1), \ldots, \mathcal{I}_g^{\text{C}}(v_n)\rangle \in \mathcal{I}_g^{\text{C}}(F)(w)$ [Def. $\mathcal{I}^{\text{C}}$; Def. dom^{C}]

iff $(\mathfrak{M}^{\text{C}}, w, s) \models_g Fv_1...v_n$ [Def. $\mathfrak{M}^{\text{C}}$; Def. 1.6 **At**]

Basis 2: $(\mathfrak{M}, w, s) \models_g [v_1 = v_2]^{\text{CON}}$

iff $(\mathfrak{M}, w, s) \models_g v_1 = v_2 \wedge Cv_1 \wedge Cv_2$ [Def. $[.]^{\text{CON}}$]

iff $\langle \mathcal{I}_g(v_1), \mathcal{I}_g(v_2)\rangle \in diag(dom(w)) \cap \mathcal{I}_g(C)(w)^2$ [Def. 1.6 **Id**, **At**]

iff $\langle \mathcal{I}_g^{\text{C}}(v_1), \mathcal{I}_g^{\text{C}}(v_2)\rangle \in diag(dom^{\text{C}}(w))$ [Def. $\mathcal{I}^{\text{C}}$; Def. dom^{C}]

iff $(\mathfrak{M}^{\text{C}}, w, s) \models_g v_1 = v_2$ [Def. $\mathfrak{M}^{\text{C}}$; Def. 1.6 **Id**]

Inductive hypothesis: assume that for all g, $(\mathfrak{M}, w, s) \models_g [A]^{\text{CON}}$ iff $(\mathfrak{M}^{\text{C}}, w, s) \models_g A$.

Induction step 1: $(\mathfrak{M}, w, s) \models_g [\exists x A]^{\text{CON}}$

iff $(\mathfrak{M}, w, s) \models_g \exists x(Cx \wedge [A]^{\text{CON}})$ [Def. $[.]^{\text{CON}}$]

iff $(\mathfrak{M}, w, s) \models_{g[d/x]} Cx \wedge [A]^{\text{CON}}$, for some $d \in dom(w)$ [Def. 1.6 $\forall$]

iff $(\mathfrak{M}, w, s) \models_{g[d/x]} [A]^{\text{CON}}$, for some $d \in dom(w) \cap \mathcal{I}_g(C)(w)$ [Def. 1.6 **At**]

iff $(\mathfrak{M}, w, s) \models_{g[d/x]} [A]^{\text{CON}}$, for some $d \in dom^{\text{C}}(w)$ [Def. dom^{C}]

iff $(\mathfrak{M}^{\text{C}}, w, s) \models_{g[d/x]} A$, for some $d \in dom^{\text{C}}(w)$ [IH]

iff $(\mathfrak{M}^{\text{C}}, w, s) \models_g \exists x A$ [Def. 1.6 $\forall$]

The remaining proofs for $\neg, \wedge, \Diamond, \uparrow$, and $\downarrow$ are routine.

Proposition 1.17 If $(\mathfrak{M}, w, s) \models_g$ **AuxCon**, then $\mathfrak{M} = \mathfrak{M}^{\text{C}}$.

Proof. It suffices to show that for arbitrary $u \in \mathbb{W}$, (a) $dom(u) = dom^{\text{C}}(u)$, and (b) $\mathcal{I}(F)(u) = \mathcal{I}^{\text{C}}(F)(u)$, for any n-place atomic predicate F.

(a) $dom(u) = \mathcal{I}(C)(u)$ [$(\mathfrak{M}, w, s) \models_g$ **AuxCon**]

$= dom^{\text{C}}(u)$ [Def. dom^{C}]

(b) $\mathcal{I}(F)(u) = \mathcal{I}(F)(u) \cap dom(u)^n$ [Def. $\mathcal{I}$]

$= \mathcal{I}(F)(u) \cap \mathcal{I}(C)(u)^n$ [$(\mathfrak{M}, w, s) \models_g$ **AuxCon**]

$= \mathcal{I}^{\text{C}}(F)(u)$ [Def. $\mathcal{I}^{\text{C}}$]

Proposition 1.18 **AuxCon** $\models [A]^{\text{CON}} \leftrightarrow A$

Proof. It suffices to show that for arbitrary $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$, arbitrary $w \in \mathbb{W}$, arbitrary $s \in \mathbb{W}_{<\omega}$, and arbitrary assignment g, if $(\mathfrak{M}, w, s) \models_g$ **AuxCon** then $(\mathfrak{M}, w, s) \models_g [A]^{\text{CON}} \leftrightarrow A$. First, suppose that $(\mathfrak{M}, w, s) \models_g$

AuxCon and notice that $\mathfrak{M} = \mathfrak{M}^{\text{C}}$ by **Proposition 1.17**. Now assume that $(\mathfrak{M}, w, s) \models_g [A]^{\text{CON}}$. By **Proposition 1.16**, $(\mathfrak{M}^{\text{C}}, w, s) \models_g A$. Since $\mathfrak{M} = \mathfrak{M}^{\text{C}}$, $(\mathfrak{M}, w, s) \models_g A$. Similarly, assume that $(\mathfrak{M}, w, s) \models_g A$. Since $\mathfrak{M} = \mathfrak{M}^{\text{C}}$, $(\mathfrak{M}^{\text{C}}, w, s) \models_g A$. By **Proposition 1.16**, $(\mathfrak{M}, w, s) \models_g [A]^{\text{CON}}$. Consequently, $(\mathfrak{M}, w, s) \models_g [A]^{\text{CON}} \leftrightarrow A$.

Definition 1.19 For a model $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$, its *possibilification* $\mathfrak{M}^{\text{N}} = \langle \mathbb{W}, \mathcal{D}, dom^{\text{N}}, \mathcal{I}^{\text{C}} \rangle$, where for every $w \in \mathbb{W}$:

$$dom^{\text{N}}(w) \ = \ \bigcup_{u \in \mathbb{W}} \mathcal{I}_g(C)(u)$$

Proposition 1.20 For a model $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$, $\mathfrak{M}^{\text{N}} \models$ **AuxNec**.

Proof. Consider arbitrary $w, u \in \mathbb{W}$:

(a) **NNE**

$$\begin{aligned} dom^{\text{N}}(w) &= \bigcup_{w \in \mathbb{W}} \mathcal{I}(C)(w) && [\text{Def. } dom^{\text{N}}] \\ &= dom^{\text{N}}(u) && [\text{Def. } dom^{\text{N}}] \end{aligned}$$

(b) $\Diamond$**C**

$$\begin{aligned} dom^{\text{N}}(w) &= \bigcup_{w \in \mathbb{W}} \mathcal{I}(C)(w) && [\text{Def. } dom^{\text{N}}] \\ &= \bigcup_{w \in \mathbb{W}} \mathcal{I}(C)(w) \cap \mathcal{I}(C)(w) && [\text{Def. } \cap] \\ &= \bigcup_{w \in \mathbb{W}} \mathcal{I}^{\text{C}}(C)(w) && [\text{Def. } \mathcal{I}^{\text{C}}] \end{aligned}$$

(c) **FC**

Immediate from the definition of $\mathcal{I}^{\text{C}}$.

Corollary 1.21 For a model $\mathfrak{M}$, $\mathfrak{M}^{\text{N}}$ is admissible.

Proof. Immediate from **Proposition 1.20**.

Definition 1.22 The function $[.]^{\text{CON}}$ is defined over formulae of $\mathcal{L}$ as follows.

$[Fv_1 \ldots v_n]^{\text{CON}} = Fv_1 \ldots v_n \wedge Cv_1 \wedge ... \wedge Cv_n$, for n-place non-logical atomic F

$[v_1 = v_2]^{\text{NEC}} = \Diamond(v_1 = v_2 \wedge Cv_1 \wedge Cv_2)$

$[\neg A]^{\text{NEC}} = \neg[A]^{\text{NEC}}$

$[A \wedge B]^{\text{NEC}} = [A]^{\text{NEC}} \wedge [B]^{\text{NEC}}$

$[\Diamond A]^{\text{NEC}} = \Diamond[A]^{\text{NEC}}$

$[\uparrow A]^{\text{NEC}} = \uparrow[A]^{\text{NEC}}$

$[\downarrow A]^{\text{NEC}} = \downarrow[A]^{\text{NEC}}$

$[\exists x A]^{\text{NEC}} = \uparrow\Diamond\exists x(Cx \wedge \downarrow[A]^{\text{NEC}})$

Proposition 1.23 For any model $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$, one may derive the following semantic clause (where $w \in \mathbb{W}$, $s \in \mathbb{W}_{<\omega}$, and g is an assignment over $\mathfrak{M}$):

$(\mathfrak{M}, w, s) \models_g \uparrow\Diamond\exists x(Cx \wedge \downarrow\phi)$ iff $(\mathfrak{M}, w, s) \models_{g[d/x]} \phi$ for some $d \in \bigcup_{u \in \mathbb{W}} \mathcal{I}_g(C)(u)$

Proof. $(\mathfrak{M}, w, s) \models_g \uparrow\Diamond\exists x(Cx \wedge \downarrow\phi)$

iff $(\mathfrak{M}, w, s^w) \models_g \Diamond\exists x(Cx \wedge \downarrow\phi)$

iff $(\mathfrak{M}, u, s^w) \models_g \exists x(Cx \wedge \downarrow\phi)$, for some $u \in \mathbb{W}$

iff $(\mathfrak{M}, u, s^w) \models_{g[d/x]} Cx \wedge \downarrow\phi$, for some $u \in \mathbb{W}$ and some $d \in dom(u)$

iff $(\mathfrak{M}, u, s^w) \models_{g[d/x]} \downarrow\phi$, for some $u \in \mathbb{W}$ and some $d \in \mathcal{I}(C)(u)$

iff $(\mathfrak{M}, w, s) \models_{g[d/x]} \phi$, for some $u \in \mathbb{W}$ and some $d \in \mathcal{I}(C)(u)$

iff $(\mathfrak{M}, w, s) \models_{g[d/x]} \phi$, for some $d \in \bigcup_{u \in \mathbb{W}} \mathcal{I}_g(C)(u)$

All steps follow from the semantics (i.e. Defs. 1.3-1.6).

Proposition 1.24 $(\mathfrak{M}, w, s) \models_g [A]^{\text{NEC}}$ iff $(\mathfrak{M}^{\text{N}}, w, s) \models_g A$

Proof. By induction on the complexity of A.

The base case for non-logical predicates is similar to before.

Basis 2: $(\mathfrak{M}, w, s) \models_g [v_1 = v_2]^{\text{NEC}}$

iff $(\mathfrak{M}, w, s) \models_g \Diamond(v_1 = v_2 \wedge Cv_1 \wedge Cv_2)$ [Def. $[.]^{\text{NEC}}$]

iff $(\mathfrak{M}, u, s) \models_g v_1 = v_2 \wedge Cv_1 \wedge Cv_2$, for some $u \in \mathbb{W}$ [Def. 1.6 $\Box$]

iff $\langle \mathcal{I}_g(v_1), \mathcal{I}_g(v_2) \rangle \in diag(dom(u)) \cap \mathcal{I}(C)(u)^2$, for some $u \in \mathbb{W}$

[Def. 1.6 **Id** & **At**]

iff $\langle \mathcal{I}_g(v_1), \mathcal{I}_g(v_2) \rangle \in diag(\bigcup_{u \in \mathbb{W}} \mathcal{I}_g(C)(u))$ [Def. $\bigcup$]

iff $\langle \mathcal{I}_g(v_1), \mathcal{I}_g(v_2) \rangle \in diag(dom^{\text{N}}(w))$ [Def. dom^{N}]

Inductive hypothesis: assume that for all g, $(\mathfrak{M}, w, s) \models_g [A]^{\text{NEC}}$ iff $(\mathfrak{M}^{\text{N}}, w, s) \models_g A$

Induction step 1: $(\mathfrak{M}, w, s) \models_g [\exists x A]^{\text{NEC}}$

iff $(\mathfrak{M}, w, s) \models_g \uparrow\Diamond\exists x(Cx \wedge \downarrow[A]^{\text{NEC}})$ [Def. $[.]^{\text{NEC}}$]

iff $(\mathfrak{M}, w, s) \models_{g[d/x]} [A]^{\text{NEC}}$, for some $d \in \bigcup_{u \in \mathbb{W}} \mathcal{I}_g(C)(u)$ [Pr. 1.23]

iff $(\mathfrak{M}, w, s) \models_{g[d/x]} [A]^{\text{NEC}}$, for some $d \in dom^{\text{N}}(w)$ [Def. dom^{N}]

iff $(\mathfrak{M}^{\text{N}}, w, s) \models_{g[d/x]} A$, for some $d \in dom^{\text{N}}(w)$ [IH] iff $(\mathfrak{M}^{\text{N}}, W, s) \models_g \exists x A$ [Def. 1.6 $\forall$]

The remaining proofs for $\neg, \wedge, \Diamond, \uparrow$, and $\downarrow$ are routine.

Proposition 1.25 $[A]^{\text{NEC}}$ is neutral.

Proof. By induction on the complexity of A.

Base case: Notice immediately that $[Fv_1 \ldots v_n]^{\text{NEC}} = [Fv_1 \ldots v_n]^{\text{CON}}$. Moreover, $[v_1 = v_2]^{\text{NEC}} = \Diamond(v_1 = v_2 \wedge Cv_1 \wedge Cv_2) = \Diamond[v_1 = v_2]^{\text{CON}} = [\Diamond\, v_1 = v_2]^{\text{CON}}$.

Inductive hypothesis: $[A]^{\text{NEC}}$ is neutral.

Induction step 1: Consider $\uparrow\Diamond\exists x(Cx \wedge \downarrow[A]^{\text{NEC}})$. By hypothesis, for some B, $\models\ \uparrow\Diamond\exists x(Cx \wedge \downarrow[A]^{\text{NEC}}) \leftrightarrow \uparrow\Diamond\exists x(Cx \wedge \downarrow[B]^{\text{CON}})$. Thus, $\models [\exists x A]^{\text{NEC}} \leftrightarrow [\uparrow\Diamond\exists x\downarrow B]^{\text{CON}}$.

The remaining proofs for $\neg, \wedge, \Diamond, \uparrow$, and $\downarrow$ are routine.

Proposition 1.26 If $(\mathfrak{M}, w, s) \models_g$ **AuxNec**, then $\mathfrak{M} = \mathfrak{M}^{\text{N}}$.

Proof. It suffices to show that for arbitrary $u \in \mathbb{W}$, (a) $dom(u) = dom^{\text{N}}(u)$, and (b) $\mathcal{I}(F)(u) = \mathcal{I}^{\text{C}}(F)(u)$, for any n-place atomic predicate F.

(a) $dom(u) = \bigcup_{v \in \mathbb{W}} \mathcal{I}_g(C)(v)$ [$(\mathfrak{M}, w, s) \models_g \Diamond$**C**, **NNE** & Def. 1.3]

$= dom^{\text{N}}(u)$ [Def. dom^{N}]

(b) $\mathcal{I}(F)(u) = \mathcal{I}(F)(u) \cap dom(u)^n$ [Def. 1.3]

$= \mathcal{I}(F)(u) \cap \mathcal{I}(C)(u)^n$ [$(\mathfrak{M}, w, s) \models_g$ **FC**]

$= \mathcal{I}^{\text{C}}(F)(u)$ [Def. $\mathcal{I}^{\text{C}}$]

Proposition 1.27 AuxNec $\models [A]^{\text{NEC}} \leftrightarrow A$

Proof. Similar to **Proposition 1.18**, using **Propositions 1.24** & **1.26** instead of **Propositions 1.16** & **1.17**.

Part B: Second-order Language

Attention is still restricted to admissible $\mathcal{L}$ models throughout Part B.

Definition 2.1 The language $\mathcal{L}^+$ extends $\mathcal{L}$ by the addition of: countably many n-place second-order variables, $X^n, Y^n, Z^n, ...$, for each natural number n; the unary modal operator $\Box^+$, and the second-order quantifier $\forall$ (allowing context to disambiguate different uses of $\forall$).

The other truth-functional connectives, modal operators and quantifiers are introduced as abbreviations in the usual way.

Definition 2.2 The *well-formed formulas* of $\mathcal{L}^+$ are given by the following rule. (Here X is either an n-place predicate constant or variable, V is an n-place predicate variable, and $v, v_1, ..., v_n$ are individual variables.)

$$Xv_1...v_n \mid \neg\psi \mid (\psi \to \chi) \mid \forall v(\psi) \mid \forall V(\psi) \mid \Box\psi \mid \Box^+\psi \mid \uparrow\psi \mid \downarrow\psi$$

Definition 2.3 For a model $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$ of $\mathcal{L}$, $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$ is an *extension* of $\mathfrak{M}$ iff:

PL $\quad \mathbb{W}^+ = \{A \subseteq \mathbb{W} : |A| \geq 1\}$

Comp 1 $\quad dom^+(W) = \bigcup_{w \in W} dom(w)$

$\mathcal{I}^+$ is a function mapping each non-logical n-place atomic predicate constant F and $W \in \mathbb{W}^+$ to $\mathcal{I}^+(F)(W) \subseteq dom^+(W)^n$ such that:

Stability $\mathcal{I}^+(F)(\{w\}) = \mathcal{I}(F)(w)$

FC$^+$ $\mathcal{I}^+(F)(W) \subseteq \mathcal{I}^+(C)(W)^n$

Comp 2 $\mathcal{I}^+(C)(W) = \bigcup_{w \in W} \mathcal{I}^+(C)(\{w\})$

Reflecting the discussion in the main body of text, **Comp 1** models the thought that arbitrary metaphysically possible individuals are compossible in some broader objective sense, and **Comp 2** models the thought that arbitrary metaphysically possibly chunky individuals are possibly chunky together in some broader sense.

Definition 2.4 A *model* $\mathfrak{M}^+$ of $\mathcal{L}^+$ is an extension of some model $\mathfrak{M}$ of $\mathcal{L}$.

Hereafter in Part B, A is any formula of $\mathcal{L}^+$, $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$ is any $\mathcal{L}^+$ model (where $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$ is the $\mathcal{L}$ model of which $\mathfrak{M}^+$ is the extension), $W \in \mathbb{W}^+$, $s \in \mathbb{W}^+_{<\omega}$, and g is a variable assignment over $\mathfrak{M}$. When sufficiently clear, context will be left to determine whether $\mathfrak{M}^+$ is a particular extension of $\mathfrak{M}$ or any extension thereof.

Definition 2.5 For a model $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$, a *variable assignment* over $\mathfrak{M}^+$ is a function g mapping each individual variable v to a member $g(v) \in \mathcal{D}$, and each n-place predicate variable V to $g(V) \subseteq \mathcal{D}^n$.

We write $g[d/\alpha]$ for the variable assignment just like g except that it assigns d to α. $\mathcal{I}^+_g(\beta)$ is used for $g(\beta)$ when β is either an individual or predicate variable; $\mathcal{I}^+_g(\beta)$ is used for $\mathcal{I}^+(\beta)$ if β is a predicate constant.

Definition 2.6 For an $\mathcal{L}^+$ model $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$, $w \in \mathbb{W}^+$, $s \in \mathbb{W}^+_{<\omega}$, and g a variable assignment over $\mathfrak{M}^+$, we define the relation $\models^+$ recursively as follows.

(In this definition, F is a non-logical n-place predicate constant, $v, v_1, ..., v_n$ are individual variables, and V is an n-place predicate variable.)

(At1) $(\mathfrak{M}^+, W, s) \models^+_g Fv_1...v_n$ iff $\langle \mathcal{I}^+_g(v_1), \ldots, \mathcal{I}^+_g(v_n)\rangle \in \mathcal{I}^+_g(F)(W)$;

(At2) $(\mathfrak{M}^+, W, s) \models^+_g Vv_1...v_n$ iff $\langle \mathcal{I}^+_g(v_1), \ldots, \mathcal{I}^+_g(v_n)\rangle \in \mathcal{I}^+_g(V) \subseteq dom^+(W)^n$;

(Id) $(\mathfrak{M}^+, W, s) \models^+_g v_1 = v_2$ iff $\langle \mathcal{I}^+_g(v_1), \mathcal{I}^+_g(v_2)\rangle \in diag(dom^+(W))$;

$(\neg)$ $(\mathfrak{M}^+, W, s) \models^+_g \neg\phi$ iff $(\mathfrak{M}^+, W, s) \not\models^+_g \phi$;

$(\wedge)$ $(\mathfrak{M}^+, W, s) \models^+_g \phi \wedge \psi$ iff $(\mathfrak{M}^+, W, s) \models^+_g \phi$ and $(\mathfrak{M}^+, W, s) \models^+_g \psi$;

$(\Box^+)$ $(\mathfrak{M}^+, W, s) \models^+_g \Box^+\phi$ iff for all $U \in \mathbb{W}^+$, $(\mathfrak{M}^+, U, s) \models^+_g \phi$;

$(\Box)$ $(\mathfrak{M}^+, W, s) \models^+_g \Box\phi$ iff for all singleton $u \in \mathbb{W}^+$, $(\mathfrak{M}^+, u, s) \models^+_g \phi$;

$(\forall v)$ $(\mathfrak{M}^+, W, s) \models^+_g \forall v\phi$ iff for all $d \in dom^+(W)$, $(\mathfrak{M}^+, W, s) \models^+_{g[d/v]} \phi$;

$(\forall V)$ $(\mathfrak{M}^+, W, s) \models^+_g \forall V\phi$ iff for all $S \subseteq dom^+(W)^n$, $(\mathfrak{M}^+, W, s) \models^+_{g[S/V]} \phi$;

$(\uparrow)$ $(\mathfrak{M}^+, W, s) \models^+_g \uparrow\phi$ iff $(\mathfrak{M}^+, W, s^W) \models^+_g \phi$;

$(\downarrow)$ $(\mathfrak{M}^+, W, s^U) \models^+_g \downarrow\phi$ iff $(\mathfrak{M}^+, U, s) \models^+_g \phi$;

$(\downarrow)$ $(\mathfrak{M}^+, W, \langle\rangle) \models^+_g \downarrow\phi$ iff $(\mathfrak{M}^+, W, \langle\rangle) \models^+_g \phi$.

Definition 2.7 A formula ϕ of $\mathcal{L}^+$ is *valid in a model* $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+\rangle$ iff $(\mathfrak{M}^+, W, s) \models^+_g \phi$ for all $W \in \mathbb{W}^+$, $s \in \mathbb{W}^+_{<\omega}$, and every assignment g. ϕ is $\mathcal{L}^+$-*valid* iff ϕ is valid in every $\mathcal{L}^+$ model.

For a set of formulas Γ, $(\mathfrak{M}^+, W, s) \models^+_g \Gamma$ iff $(\mathfrak{M}^+, W, s) \models^+_g \gamma$, for all $\gamma \in \Gamma$. If Γ and Δ are single formulas or sets thereof, $\Gamma \models^+ \Delta$ iff if $(\mathfrak{M}^+, W, s) \models^+_g \Gamma$ then $(\mathfrak{M}^+, W, s) \models^+_g \Delta$. We say that $\mathfrak{M}^+ \models^+ \Gamma$ iff $(\mathfrak{M}^+, W, s) \models^+_g \gamma$ for all $\gamma \in \Gamma$, $W \in$

$\mathbb{W}^+$, $s \in \mathbb{W}^+_{+<\omega}$, and every assignment g. Moreover, $\models^+ \Gamma$ iff $\mathfrak{M}^+ \models^+ \Gamma$ for all $\mathfrak{M}^+$.

Proposition 2.8 The model theory validates all tautologies of propositional logic, and an **S5** logic for $\Box^+$. It validates all instances of the following **S5**-like schemas for $\Box$:

5 $\Diamond\phi \to \Box\Diamond\phi$

4 $\Box\phi \to \Box\Box\phi$

D $\Box\phi \to \Diamond\phi$

K $\Box\phi \to \Box(\phi \to \psi) \to \Box\psi$

$\Box$**T** $\Box(\Box\phi \to \phi)$

$\Box$**B** $\Box(\phi \to \Box\Diamond\phi)$

The modified versions of the usual T and B axioms for $\Box$ are due to the unusual semantics for $\Box$ (since indices are now to be thought of as 'pluralities' of worlds with single worlds as a special case). Nevertheless, in effect, the modified T and B schemas furnish $\Box$ with an **S5** logic as far as single worlds are concerned, which suffices for all purposes in what follows.

In addition, the model theory validates all instances of the following schema:

IMP $\Box^+\phi \to \Box\phi$

Put informally, this reflects that $\Box^+$ is broader than $\Box$.

Proof. The proofs for all but **IMP**, $\Box$**T**, and $\Box$**B** are routine.

For **IMP**, take arbitrary $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$, arbitrary $W \in \mathbb{W}^+$, arbitrary $s \in \mathbb{W}^+_{<\omega}$, and arbitrary assignment g. If $(\mathfrak{M}^+, W, s) \models^+_g \Box^+\phi$, it would follow that $(\mathfrak{M}^+, U, s) \models^+_g \phi$ for all $U \in \mathbb{W}^+$. This includes all singleton $u \in \mathbb{W}^+$, hence $(\mathfrak{M}^+, W, s) \models^+_g \Box\phi$. Since g, s, W and $\mathfrak{M}^+$ were arbitrary, this suffices to establish the claim.

For $\Box\mathbf{T}$, take arbitrary $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$, arbitrary $W \in \mathbb{W}^+$, arbitrary $s \in \mathbb{W}^+_{<\omega}$, and arbitrary assignment g. Clearly, for any singleton $w \in \mathbb{W}^+$, $(\mathfrak{M}^+, w, s) \models^+_g \Box\phi \to \phi$. Thus, $(\mathfrak{M}^+, W, s) \models^+_g \Box(\Box\phi \to \phi)$. Since g, s, W and $\mathfrak{M}^+$ were arbitrary, this suffices to establish the claim.

For $\Box\mathbf{B}$, the argument is similar to $\Box\mathbf{T}$.

Proposition 2.9 For $\mathcal{O} \in \{\Box, \Diamond\}$ and $\mathcal{O}^+ \in \{\Box^+, \Diamond^+\}$, all instances of the following schemas are valid.

(i) $\mathcal{O}\mathcal{O}^+\phi \leftrightarrow \mathcal{O}^+\phi$

(ii) $\mathcal{O}^+\mathcal{O}\phi \leftrightarrow \mathcal{O}\phi$

Proof of (i). Suppose $\mathcal{O}^+ = \Box^+$, and consider $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$, $W \in \mathbb{W}^+$, $s \in \mathbb{W}^+_{<\omega}$, and an assignment g. [$\Leftarrow$] If $(\mathfrak{M}^+, W, s) \models^+_g \Box^+\phi$ then for all $U \in \mathbb{W}^+, (\mathfrak{M}^+, U, s) \models^+_g \phi$. So take any singleton $u \in \mathbb{W}^+$, clearly $(\mathfrak{M}^+, u, s) \models^+_g \Box^+\phi$ also. Thus, $(\mathfrak{M}^+, W, s) \models^+ \Box\Box^+\phi$. [$\Rightarrow$] By a similar argument.

The proofs for the other cases are similar.

Proof of (ii). Similar to (i).

Proposition 2.10 Let $\Delta = \{\Box, \Diamond, \Box^+, \Diamond^+\}$, and S be any non-empty finite sequence of operators from Δ which terminates in $\mathcal{O}$. For any formula $\psi = S\phi$, $\models^+ \psi \leftrightarrow \mathcal{O}\phi$.

Proof. By induction on the length of S. The base case is immediate, so for arbitrary $n \geq 1$ assume that for each n-length sequence S' of operators from Δ the claim holds. We have the following cases (where S' is an arbitrary n-length sequence of operators from Δ).

(i) $\psi = \Box S'\phi$

(ii) $\psi = \Diamond S'\phi$

(iii) $\psi = \Box^+ S'\phi$

(iv) $\psi = \Diamond^+ S'\phi$

Consider (i). By hypothesis, $\models^+ S'\phi \leftrightarrow \mathcal{O}\phi$ for some $\mathcal{O} \in \Delta$. Thus, depending on the choice of $\mathcal{O}$, one of the following is the case: $\models^+ \psi \leftrightarrow \Box\Box\phi$; $\models^+ \psi \leftrightarrow \Box\Diamond\phi$; $\models^+ \psi \leftrightarrow \Box\Box^+\phi$, or $\models^+ \psi \leftrightarrow \Box\Diamond^+\phi$. However, $\models^+ \Box\Box\phi \leftrightarrow \Box\phi$ and $\models^+ \Box\Diamond\phi \leftrightarrow \Diamond\phi$ by the **S5**-like logic for $\Box$ (see **Proposition 2.8**). Moreover, $\models^+ \Box\Box^+\phi \leftrightarrow \Box^+\phi$ and $\models^+ \Box\Diamond^+\phi \leftrightarrow \Diamond^+\phi$ by **Proposition 2.9**. Cases (ii)-(iv) can be established in a similar fashion. By induction on the length of S, this establishes **Proposition 2.10**.

Proposition 2.11 All instances of the following two schemas are valid.

$\Diamond\Box^+\phi \ \rightarrow \ \Box^+\Diamond\phi$

Proof. As noted in **Proposition 2.8**, all instances of $\Box^+\phi \rightarrow \Box\phi$ are valid, as are all instances of the schema $\Box\phi \rightarrow \Diamond\phi$. Thus, all instances of $\Box^+\phi \rightarrow \Diamond\phi$ are valid. All instances of $\Diamond\Box^+\phi \rightarrow \Box^+\Diamond\phi$ are then valid by **Proposition 2.9** and familiar propositional logic.

$\Diamond^+\Box\phi \ \rightarrow \ \Box\Diamond^+\phi$

Proof. Given **Proposition 2.8**, all instances of $\Diamond\phi \rightarrow \Diamond^+\phi$ (the contrapositive of **IMP**) are valid, as are all instances of $\Box\phi \rightarrow \Diamond\phi$. All instances of $\Diamond^+\Box\phi \rightarrow \Box\Diamond^+\phi$ are then valid by **Proposition 2.9** and familiar propositional logic.

On this basis, the logic may be said to be *Church-Rosser*.

Definition 2.12 A model $\mathfrak{M}^+$ of $\mathcal{L}^+$ is *admissible*$^+$ iff either $\mathfrak{M}^+ \models^+$ **AuxNec** or $\mathfrak{M}^+ \models^+$ **AuxCon**.

Proposition 2.13 For any any $\mathcal{L}^+$ model $\mathfrak{M}^+$, $\mathfrak{M}^+ \models^+ \mathbf{FC}$.

Proof. Immediate from **Definition 2.3 FC$^+$**.

Proposition 2.14 $\mathfrak{M} \models$ **AuxNec** iff $\mathfrak{M}^+ \models^+$ **AuxNec**, for any extension $\mathfrak{M}^+$ of $\mathfrak{M}$.

Proof. $\Rightarrow$

The proof for **NNE** is routine.

For $\Diamond$**C**, suppose that $\mathfrak{M} \models$ **AuxNec** and consider any extension $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$ of $\mathfrak{M}$, and arbitrary $W \in \mathbb{W}^+$. For arbitrary $d \in dom^+(W)$, one may reason as follows:

$d \in dom(w)$ for some $w \in W$ [Def. 2.3 **Comp 1**]

$d \in \mathcal{I}(C)(w)$ for some $w \in \mathbb{W}$ [$\mathfrak{M} \models \Diamond\mathbf{C}$]

$d \in \mathcal{I}^+(C)(\{w\})$ for some $w \in \mathbb{W}$ [Def. 2.3 **Stability**]

Since d was arbitrary, it follows that $(\mathfrak{M}^+, W, s) \models^+_g \forall x \Diamond Cx$, for all s and g.

The proof for **FC** is a simple corollary of **Proposition 2.13**.

$\Leftarrow$

Again, the proof for **NNE** is routine.

For $\Diamond$**C**, suppose that an arbitrary extension $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$ of $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$ is such that $\mathfrak{M}^+ \models^+$ **AuxNec**. Consider arbitrary $w \in \mathbb{W}$. For arbitrary $d \in dom(w)$, one may reason as follows:

$d \in dom^+(\{w\})$ [Def. 2.3 **Comp 1**]

$d \in \mathcal{I}^+(C)(\{u\})$ for some $u \in \mathbb{W}$ [$\mathfrak{M}^+ \models^+ \Diamond\mathbf{C}$]

$d \in \mathcal{I}(C)(u)$ for some $u \in \mathbb{W}$ [Def. 2.3 **Stabilty**]

Since d was arbitrary, it follows that $(\mathfrak{M}, w, s) \models_g \forall x \Diamond Cx$, for all s and g.

The proof for **FC** is a simple corollary of **Proposition 2.13** and **Definition 2.3 Stability**.

Proposition 2.15 $\mathfrak{M} \models$ **AuxCon** iff $\mathfrak{M}^+ \models^+$ **AuxCon**, for any extension $\mathfrak{M}^+$ of $\mathfrak{M}$.

Proof. It is routine to show that $\mathfrak{M} \models \Box\forall x Cx$ iff $\mathfrak{M}^+ \models^+ \Box\forall x Cx$.

Proposition 2.16 For every $\mathcal{L}$ model $\mathfrak{M}$, $\mathfrak{M}^+$ is admissible$^+$.

Proof. Since attention is still restricted to admissible $\mathcal{L}$ models, the proposition is immediate from **Propositions 2.14** & **2.15**.

Proposition 2.17 For any $\mathcal{L}^+$ model $\mathfrak{M}^+$, $\mathfrak{M}^+ \models^+ \Diamond$**C**.

Proof. The proof is similar to **Proposition 1.10** using the fact that each $\mathcal{L}^+$ model is admissible$^+$.

Definition 2.18 For a model $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$, is *restriction*$^+$ $\mathfrak{M}^{+\mathrm{C}} = \langle \mathbb{W}^+, \mathcal{D}, dom^{+\mathrm{C}}, \mathcal{I}^{+\mathrm{C}} \rangle$ where:

$$dom^{+\mathrm{C}}(W) = \mathcal{I}^+(C)(W)$$

$$\mathcal{I}^{+\mathrm{C}}(F)(W) = \mathcal{I}^+(F)(W) \cap \mathcal{I}^+(C)(W)^n, \text{ for } n\text{-place non-logical predicate } F$$

Corollary 2.19 For a model $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$ such that $\mathfrak{M}^{+\mathrm{C}} = \langle \mathbb{W}^+, \mathcal{D}, dom^{+\mathrm{C}}, \mathcal{I}^{+\mathrm{C}} \rangle$ the following identity holds:

$$dom^{+\mathrm{C}}(W) = \bigcup_{w \in W} dom^{+\mathrm{C}}(\{w\})$$

Proof.
$$\begin{aligned} dom^{+\mathrm{C}}(W) &= \mathcal{I}^+(C)(W) && [\text{Def. } dom^{+\mathrm{C}}] \\ &= \bigcup_{w \in W} \mathcal{I}^+(C)(\{w\}) && [\text{Def. 2.3 } \textbf{Comp 2}] \\ &= \bigcup_{w \in W} dom^{+\mathrm{C}}(\{w\}) && [\text{Def. } dom^{+\mathrm{C}}] \end{aligned}$$

Proposition 2.20 For any $\mathcal{L}$ model $\mathfrak{M}$ and any $\mathcal{L}^+$ model $\mathfrak{M}^+$ which is an extension of $\mathfrak{M}$, $\mathfrak{M}^{+\mathrm{C}}$ is an extension of $\mathfrak{M}^{\mathrm{C}}$.

Proof. Consider an arbitrary $\mathcal{L}$ model $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$, and an arbitrary extension $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$ of $\mathfrak{M}$. We first establish the following lemmas.

Lemma 1 $\bigcup_{w \in W} dom^{\mathrm{C}}(w) = dom^{+\mathrm{C}}(W)$

Proof.

$$\begin{aligned} \bigcup_{w \in W} dom^{\mathrm{C}}(w) &= \bigcup_{w \in W} \mathcal{I}(C)(w) && [\text{Def. } dom^{\mathrm{C}}] \\ &= \bigcup_{w \in W} \mathcal{I}^+(C)(\{w\}) && [\text{Def. 2.3 } \textbf{Stability}] \\ &= \bigcup_{w \in W} dom^{+\mathrm{C}}(\{w\}) && [\text{Def. } dom^{+\mathrm{C}}] \\ &= dom^{+\mathrm{C}}(W) && [\text{Corr. 2.19}] \end{aligned}$$

Lemma 2 $\mathcal{I} = \mathcal{I}^{\mathrm{C}}$

Proof.

$$\begin{aligned} \mathcal{I}(F)(W) &= \mathcal{I}(F)(W) \cap \mathcal{I}(C)(W) && [\mathfrak{M} \models \textbf{FC}] \\ &= \mathcal{I}^{\mathrm{C}}(F)(W) && [\text{Def. } \mathcal{I}^{\mathrm{C}}] \end{aligned}$$

Lemma 3 $\mathcal{I}^+ = \mathcal{I}^{+\mathrm{C}}$

Proof.

$$\begin{aligned} \mathcal{I}^+(F)(W) &= \mathcal{I}(F)^+(W) \cap \mathcal{I}^+(C)(W) && [\text{Def. 2.3 } \textbf{FC}^+] \\ &= \mathcal{I}^{+\mathrm{C}}(F)(W) && [\text{Def. } \mathcal{I}^{+\mathrm{C}}] \end{aligned}$$

Given Lemmas 1-3 and that $\mathfrak{M}^+$ is an extension of $\mathfrak{M}$, $\mathfrak{M}^{+\mathrm{C}}$ meets the conditions in **Definition 2.3** for being an extension of $\mathfrak{M}^{\mathrm{C}}$.

Corollary 2.21.1 For any $\mathcal{L}$ model $\mathfrak{M}$, $\mathfrak{M}^{+\mathrm{C}}$ is a model of $\mathcal{L}^+$.

Proof. Immediate from **Proposition 2.20** and **Definition 2.4**.

Corollary 2.21.2 For any $\mathcal{L}$ model $\mathfrak{M}$, $\mathfrak{M}^{+\mathrm{C}} \models^+$ **AuxCon**.

Proof. $\mathfrak{M}^{\mathrm{C}} \models$ **AuxCon** by **Proposition 1.12**. By **Proposition 2.20**, $\mathfrak{M}^{+\mathrm{C}}$ is an extension of $\mathfrak{M}^{\mathrm{C}}$. Thus, by **Proposition 2.15**, $\mathfrak{M}^{+\mathrm{C}} \models^{+}$ **AuxCon**.

Corollary 2.21.3 For any $\mathcal{L}$ model $\mathfrak{M}$, $\mathfrak{M}^{+\mathrm{C}}$ is admissible^{+}.

Proof. Immediate from **Corollary 2.21.2**.

Definition 2.22 The function $[.]^{\text{CON+}}$ is defined over formulae of $\mathcal{L}$ as follows. ($X{\leq}C$ is an abbreviation of $\forall x(Xx \to Cx)$)

$[Fv_1 \ldots v_n]^{\text{CON+}} = Fv_1 \ldots v_n \wedge Cv_1 \wedge ... \wedge Cv_n$, for n-place non-logical atomic F

$[Vv_1 \ldots v_n]^{\text{CON+}} = Vv_1 \ldots v_n \wedge V{\leq}C$, for n-place variable V

$[v_1 = v_2]^{\text{CON+}} = v_1 = v_2 \wedge Cv_1 \wedge Cv_2$

$[\neg A]^{\text{CON+}} = \neg[A]^{\text{CON+}}$

$[A \wedge B]^{\text{CON+}} = [A]^{\text{CON+}} \wedge [B]^{\text{CON+}}$

$[\Diamond A]^{\text{CON+}} = \Diamond[A]^{\text{CON+}}$

$[\Diamond^{+} A]^{\text{CON+}} = \Diamond^{+}[A]^{\text{CON+}}$

$[\uparrow A]^{\text{CON+}} = \uparrow[A]^{\text{CON+}}$

$[\downarrow A]^{\text{CON+}} = \downarrow[A]^{\text{CON+}}$

$[\exists x A]^{\text{CON+}} = \exists x(Cx \wedge [A]^{\text{CON+}})$

$[\exists X A]^{\text{CON+}} = \exists X(X{\leq}C \wedge [A]^{\text{CON+}})$

Definition 2.23 A formula A is *neutral*$^+$ iff for some B, $\models^+ A \leftrightarrow [B]^{\text{CON}+}$.

Proposition 2.24 $(\mathfrak{M}^+, W, s) \models^+_g [A]^{\text{CON}+}$ iff $(\mathfrak{M}^{+\text{C}}, W, s) \models^+_g A$

Proof. By induction on the complexity of A.

Basis 1: $(\mathfrak{M}^+, w, s) \models^+_g [Fv_1...v_n]^{\text{CON}+}$

iff $(\mathfrak{M}^+, w, s) \models^+_g Fv_1 \ldots v_n \wedge Cv_1 \wedge ... \wedge Cv_n$ [Def. $[.]^{\text{CON}+}$]

iff $\langle \mathcal{I}^+_g(v_1), \ldots, \mathcal{I}^+_g(v_n)\rangle \in \mathcal{I}^+_g(F)(w) \cap \mathcal{I}^+_g(C)(w)^n$ [Def. 1.6 **At1**]

iff $\langle \mathcal{I}^{+\text{C}}_g(v_1), \ldots, \mathcal{I}^{+\text{C}}_g(v_n)\rangle \in \mathcal{I}^{+\text{C}}_g(F)(w)$ [Def. 2.3 $\mathcal{I}^{+\text{C}}$]

iff $(\mathfrak{M}^{+\text{C}}, w, s) \models^+_g Fv_1...v_n$ [Def. $\mathfrak{M}^{+\text{C}}$; Def. 1.6 **At1**]

Basis 2: $(\mathfrak{M}^+, W, s) \models^+_g [v_1 = v_2]^{\text{CON}+}$

iff $(\mathfrak{M}^+, W, s) \models^+_g v_1 = v_2 \wedge Cv_1 \wedge Cv_2$ [Def. $[.]^{\text{CON}+}$]

iff $\langle \mathcal{I}^+_g(v_1), \mathcal{I}^+_g(v_2)\rangle \in diag(dom^+(W)) \cap \mathcal{I}^+_g(C)(W)^2$ [Def. 1.6 **Id**, **At1**]

iff $\langle \mathcal{I}^{+\text{C}}_g(v_1), \mathcal{I}^{+\text{C}}_g(v_2)\rangle \in diag(dom^{\text{C}}(W))$ [Def. $\mathcal{I}^{+\text{C}}$; Def. dom^{C}]

iff $(\mathfrak{M}^{+\text{C}}, W, s) \models^+_g v_1 = v_2$ [Def. $\mathfrak{M}^{+\text{C}}$; Def. 1.6 **Id**]

Basis 3: $(\mathfrak{M}^+, W, s) \models^+_g [Vv_1...v_n]^{\text{CON}+}$

iff $(\mathfrak{M}^+, W, s) \models_g Vv_1...v_n \wedge V{\le}C$ [Def. $[.]^{\text{CON}+}$]

iff $\langle \mathcal{I}^+_g(v_1), \ldots, \mathcal{I}^+_g(v_n)\rangle \in \mathcal{I}^+_g(V) \subseteq dom^+(W)^n \cap \mathcal{I}^+_g(C)(W)^n$ [Def. 1.6 **At2**; Def. $V{\le}C$]

iff $\langle \mathcal{I}^{+\text{C}}_g(v_1), \ldots, \mathcal{I}^{+\text{C}}_g(v_n)\rangle \in \mathcal{I}^{+\text{C}}_g(V) \subseteq dom^{+\text{C}}(W)^n$ [Def. $\mathcal{I}^{+\text{C}}$; Def. $dom^{+\text{C}}$]

iff $(\mathfrak{M}^{+\text{C}}, W, s) \models^+_g Vv_1...v_n$ [Def. **At2**]

Inductive hypothesis: assume that for all g, $(\mathfrak{M}^+, W, s) \models_g^+ [A]^{\text{CON}+}$ iff $(\mathfrak{M}^{\text{C}+}, W, s) \models_g^+ A$

Induction step 1: $(\mathfrak{M}^+, W, s) \models_g^+ [\exists X A]^{\text{CON}+}$

iff $(\mathfrak{M}^+, W, s) \models_g^+ \exists X(X{\leq}C \wedge [A]^{\text{CON}+})$ [Def. $[.]^{\text{CON}+}$]

iff $(\mathfrak{M}^+, W, s) \models_{g[S/X]}^+ X{\leq}C \wedge [A]^{\text{CON}+}$, for some $S \subseteq dom^+(W)^n$ [Def. 2.6 $\forall V$]

iff $(\mathfrak{M}^+, W, s) \models_{g[S/X]}^+ [A]^{\text{CON}+}$, for some $S \subseteq \mathcal{I}_g^+(C)(W)^n$ [Def. $X{\leq}C$]

iff $(\mathfrak{M}^{+\text{C}}, W, s) \models_{g[S/X]}^+ A$, for some $S \subseteq \mathcal{I}_g^+(C)(W)^n$ [IH]

iff $(\mathfrak{M}^{+\text{C}}, W, s) \models_{g[S/X]}^+ A$, for some $S \subseteq dom^{+\text{C}}(W)^n$ [Def. $dom^{+\text{C}}$]

iff $(\mathfrak{M}^{+\text{C}}, W, s) \models_{g[S/X]}^+ \exists X A$ [Def. 2.6 $\forall V$]

The proof for $\exists x A$ is similar to that in **Proposition 1.16**. The remaining proofs for $\neg, \wedge, \Diamond, \uparrow$, and $\downarrow$ are routine.

Proposition 2.25 If $(\mathfrak{M}^+, W, s) \models_g^+$ **AuxCon**, then $\mathfrak{M}^+ = \mathfrak{M}^{+\text{C}}$.

Proof. It suffices to show that for arbitrary $U \in \mathbb{W}^+$, (a) $dom^+(U) = dom^{+\text{C}}(U)$, and (b) $\mathcal{I}^+(F)(U) = \mathcal{I}^{+\text{C}}(F)(U)$

(a) $dom^+(U) = \mathcal{I}^+(C)(U)$ [$(\mathfrak{M}^+, W, s) \models_g^+$ **AuxCon**; Def. 2.3 **Comp 1 & 2**]

$= dom^{+\text{C}}(U)$ [Def. $dom^{+\text{C}}$]

(b) $\mathcal{I}^+(F)(U) = \mathcal{I}^+(F)(U) \cap \mathcal{I}^+(C)(U)$ [Def. 2.3 **FC**$^+$]

$= \mathcal{I}^{+\text{C}}(F)(U)$ [Def. $\mathcal{I}^{+\text{C}}$]

Proposition 2.26 **AuxCon** $\models^+ [A]^{\text{CON}+} \leftrightarrow A$

Proof. Similar to **Proposition 1.18**, using **Propositions 2.24** & **2.25** instead of **Propositions 1.16** & **1.17**.

Definition 2.27 For a model $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$, its *possibilification*$^+$ $\mathfrak{M}^{+\text{N}} = \langle \mathbb{W}^+, \mathcal{D}, dom^{+\text{N}}, \mathcal{I}^{+\text{C}} \rangle$ where:

$$dom^{+\text{N}}(W) = \bigcup_{w \in \mathbb{W}} \mathcal{I}^+(C)(\{w\})$$

Corollary 2.28 For a model $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$, and any extension $\mathfrak{M}^{+\text{N}} = \langle \mathbb{W}^+, \mathcal{D}, dom^{+\text{N}}, \mathcal{I}^{+\text{C}} \rangle$ the following equation holds:

$$dom^{+\text{N}}(W) = dom^{+\text{N}}(U)$$

Proof. Immediate from **Definition 2.27**.

Proposition 2.29 For any $\mathcal{L}$ model $\mathfrak{M}$ and any $\mathcal{L}^+$ model $\mathfrak{M}^+$ which is an extension of $\mathfrak{M}$, $\mathfrak{M}^{+\text{N}}$ is an extension of $\mathfrak{M}^{\text{N}}$.

Proof. Consider an arbitrary $\mathcal{L}$ model $\mathfrak{M} = \langle \mathbb{W}, \mathcal{D}, dom, \mathcal{I} \rangle$, and an arbitrary extension $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+ \rangle$ of $\mathfrak{M}$. We first establish the following lemmas.

Lemma 1 $dom^{+\text{N}}(W) = \bigcup_{w \in \mathbb{W}} dom^{\text{N}}(w)$

Proof.
$$\begin{aligned}
dom^{+\text{N}}(W) &= \bigcup_{w \in \mathbb{W}} \mathcal{I}^+(C)(\{w\}) && [\text{Def. } dom^{+\text{N}}] \\
&= \bigcup_{w \in \mathbb{W}} \mathcal{I}(C)(w) && [\text{Def. 2.3 } \textbf{Stability}] \\
&= dom^{\text{N}}(w) && [\text{Def. } dom^{\text{N}}] \\
&= \bigcup_{w \in \mathbb{W}} dom^{\text{N}}(w) && [\mathfrak{M}^{\text{N}} \models \textbf{NNE}]
\end{aligned}$$

Given Lemma 1, Lemmas 2-3 from **Proposition 2.20**, and that $\mathfrak{M}^+$ is an extension of $\mathfrak{M}$, $\mathfrak{M}^{+\text{N}}$ meets the conditions in **Definition 2.3** for being an extension of $\mathfrak{M}^{\text{N}}$.

Corollary 2.30.1 For any $\mathcal{L}$ model $\mathfrak{M}$, $\mathfrak{M}^{+\text{N}}$ is a model of $\mathcal{L}^+$.

Proof. Immediate from **Proposition 2.29** and **Definition 2.4**.

Corollary 2.30.2 For any $\mathcal{L}$ model $\mathfrak{M}$, $\mathfrak{M}^{+\text{N}} \models^+$ **AuxNec**.

Proof. $\mathfrak{M}^{\text{N}} \models$ **AuxNec** by **Proposition 1.20**. By **Proposition 2.29**, $\mathfrak{M}^{+\text{N}}$ is an extension of $\mathfrak{M}^{\text{N}}$. Thus, by **Proposition 2.14**, $\mathfrak{M}^{+\text{N}} \models^+$ **AuxNec**.

Corollary 2.30.3 For any $\mathcal{L}$ model $\mathfrak{M}$, $\mathfrak{M}^{+\text{N}}$ is admissible$^+$.

Proof. Immediate from **Corollary 2.30.2**.

Definition 2.31 The function $[.]^{\text{NEC+}}$ is defined over formulae of $\mathcal{L}^+$ as follows. (Again, $X{\leq}C$ is an abbreviation of $\forall x(Xx \rightarrow Cx)$).

$[Fv_1 \ldots v_n]^{\text{NEC+}} = Fv_1 \ldots v_n \wedge Cv_1 \wedge ... \wedge Cv_n$, for n-place non-logical atomic F

$[Vv_1 \ldots v_n]^{\text{NEC+}} = \Diamond^+(Vv_1 \ldots v_n \wedge V{\leq}C)$, for n-place variable V

$[v_1 = v_2]^{\text{NEC+}} = \Diamond(v_1 = v_2 \wedge Cv_1 \wedge Cv_2)$

$[\neg A]^{\text{NEC+}} = \neg[A]^{\text{NEC+}}$

$[A \wedge B]^{\text{NEC+}} = [A]^{\text{NEC+}} \wedge [B]^{\text{NEC+}}$

$[\Diamond^+ A]^{\text{NEC+}} = \Diamond^+[A]^{\text{NEC+}}$

$[\Diamond A]^{\text{NEC+}} = \Diamond[A]^{\text{NEC+}}$

$[\uparrow A]^{\text{NEC+}} = \uparrow[A]^{\text{NEC+}}$

$[\downarrow A]^{\text{NEC+}} = \downarrow[A]^{\text{NEC+}}$

$[\exists x A]^{\text{NEC+}} = \uparrow\Diamond\exists x(Cx \wedge \downarrow[A]^{\text{NEC+}})$

$[\exists X A]^{\text{NEC+}} = \uparrow\Diamond^{+}\exists X(X{\leq}C \wedge \downarrow[A]^{\text{NEC+}})$

Proposition 2.32 For all $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+\rangle$ and all $W \in \mathbb{W}^+$:

$$S \subseteq \bigcup_{W\in\mathbb{W}^+} \mathcal{I}^+(C)(W) \text{ iff } S \subseteq \mathcal{I}^+(C)(U), \text{ for some } U \in \mathbb{W}^+$$

Proof. $\Leftarrow$

Immediate.

$\Rightarrow$

Consider arbitrary $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+\rangle$ and notice the following identities:

$$\begin{aligned} \bigcup_{W\in\mathbb{W}^+} \mathcal{I}^+(C)(W) &= \bigcup_{w\in\mathbb{W}} \mathcal{I}^+(C)(\{w\}) && [\textbf{Stability}, \textbf{Comp 2}] \\ &= \mathcal{I}^+(C)(\mathbb{W}) && [\textbf{Comp 2}] \end{aligned}$$

However, clearly $\mathbb{W} \in \mathbb{W}^+$ since $\mathbb{W}$ is non-empty. Thus, there is some $U \in \mathbb{W}^+$ such that $S \subseteq \mathcal{I}^+(C)(U)$.

Proposition 2.33 For any model $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+\rangle$, one may derive the following semantic clause (where $W \in \mathbb{W}^+$, $s \in \mathbb{W}^+_{<\omega}$, and g is an assignment over $\mathfrak{M}^+$):

$$(\mathfrak{M}^+, W, s) \models^+_g \uparrow\Diamond\exists x(Cx{\wedge}{\downarrow}\phi) \text{ iff } (\mathfrak{M}^+, W, s) \models^+_{g[d/x]} \phi \text{ for some } d \in \bigcup_{u\in\mathbb{W}} \mathcal{I}^+_g(C)(\{u\})$$

Proof. Similar to **Proposition 1.23**.

Proposition 2.34 For any model $\mathfrak{M}^+ = \langle \mathbb{W}^+, \mathcal{D}, dom^+, \mathcal{I}^+\rangle$, one may derive the following semantic clause (where $W \in \mathbb{W}^+$, $s \in \mathbb{W}^+_{<\omega}$, and g is an assignment over $\mathfrak{M}^+$):

$(\mathfrak{M}^+, W, s) \models^+_g \uparrow\Diamond^+\exists X(X{\leq}C{\wedge}{\downarrow}\phi)$ iff $(\mathfrak{M}^+, W, s) \models^+_{g[S/X]} \phi$ for some $S \subseteq \mathcal{I}^+(C)(U)^n$ for some $U \in \mathbb{W}^+$

Proof. Similar to that of **Proposition 1.23**.

Proposition 2.35 $(\mathfrak{M}^+, W, s) \models^+_g [A]^{\text{NEC}+}$ iff $(\mathfrak{M}^{+\text{N}}, W, s) \models^+_g A$

Proof. By induction on the complexity of A.

Basis 1: $(\mathfrak{M}^+, W, s) \models^+_g [Vv_1...v_n]^{\text{NEC}+}$

iff $(\mathfrak{M}^+, W, s) \models_g \Diamond^+(Vv_1...v_n \wedge V{\leq}C)$ [Def. $[.]^{\text{NEC}+}$]

iff $(\mathfrak{M}^+, U, s) \models_g Vv_1...v_n \wedge V{\leq}C$, for some $U \in \mathbb{W}^+$ [Def. 2.6 $\Box$]

iff $\langle \mathcal{I}^+_g(v_1), \ldots, \mathcal{I}^+_g(v_n)\rangle \in \mathcal{I}^+_g(V) \subseteq \mathcal{I}^+_g(C)(U)^n$, for some $U \in \mathbb{W}^+$ [Def. 1.6 **At2**; Def. $V{\leq}C$]

iff $\langle \mathcal{I}^+_g(v_1), \ldots, \mathcal{I}^+_g(v_n)\rangle \in \mathcal{I}^+_g(V) \subseteq \bigcup\limits_{W\in\mathbb{W}^+} \mathcal{I}^+(C)(W)^n$ [Pr. 2.32]

iff $\langle \mathcal{I}^+_g(v_1), \ldots, \mathcal{I}^+_g(v_n)\rangle \in \mathcal{I}^+_g(V) \subseteq \bigcup\limits_{w\in\mathbb{W}} \mathcal{I}^+(C)(\{w\})^n$ [Def. 2.3 **Comp 2**]

iff $\langle \mathcal{I}^{+\text{C}}_g(v_1), \ldots, \mathcal{I}^{+\text{C}}_g(v_n)\rangle \in \mathcal{I}^{+\text{C}}_g(V) \subseteq dom^{+\text{N}}(W)^n$ [Def. 2.3 $\mathcal{I}^{+\text{C}}$, $dom^{+\text{N}}$]

iff $(\mathfrak{M}^{+\text{N}}, W, s) \models^+_g Vv_1...v_n$ [Def. 2.6 **At2**]

Basis 2: $(\mathfrak{M}^+, W, s) \models^+_g [v_1 = v_2]^{\text{NEC}+}$

iff $(\mathfrak{M}^+, W, s) \models^+_g \Diamond(v_1 = v_2 \wedge Cv_1 \wedge Cv_2)$ [Def. $[.]^{\text{NEC}+}$]

iff $(\mathfrak{M}^+, u, s) \models^+_g v_1 = v_2 \wedge Cv_1 \wedge Cv_2$, for some singleton $u \in \mathbb{W}^+$ [Def. 2.6 $\Box$]

iff $\langle \mathcal{I}^+_g(v_1), \mathcal{I}^+_g(v_n)\rangle \in diag(dom^+(u)) \cap \mathcal{I}^+_g(C)(u)^2$, for some singleton $u \in \mathbb{W}^+$ [Def. 2.6 **Id**, **At1**]

iff $\langle \mathcal{I}^+_g(v_1), \mathcal{I}^+_g(v_n)\rangle \in diag(\bigcup\limits_{u\in\mathbb{W}} \mathcal{I}^+(C)(\{u\}))$ [Def. $\bigcup$]

iff $\langle \mathcal{I}^+_g(v_1), \mathcal{I}^+_g(v_n)\rangle \in diag(dom^{+\text{N}}(W))$ [Def. $dom^{+\text{N}}$]

$(\mathfrak{M}^{+\text{N}}, W, s) \models^+_g v_1 = v_2$ [Def. 2.6 **Id**]

The proof for $Fv_1...v_n$ is similar to that in **Proposition 2.24**.

Inductive hypothesis: assume that for all g, $(\mathfrak{M}^+, W, s) \models^+_g [A]^{\text{NEC}+}$ iff $(\mathfrak{M}^{\text{N}+}, W, s) \models^+_g A$

Induction step 1: $(\mathfrak{M}^+, W, s) \models^+_g [\exists X A]^{\text{NEC}+}$

iff $(\mathfrak{M}^+, W, s) \models^+_g \uparrow\Diamond^+\exists X(X{\leq}C \wedge \downarrow[A]^{\text{NEC}+})$ [Def. $[.]^{\text{NEC}+}$]

iff $(\mathfrak{M}^+, W, s) \models^+_{g[S/X]} [A]^{\text{NEC}+}$ for some $S \subseteq \mathcal{I}^+(C)(U)^n$ for some $U \in \mathbb{W}^+$ [Pr. 2.34]

iff $(\mathfrak{M}^+, W, s) \models^+_{g[S/X]} [A]^{\text{NEC}+}$ for some $S \subseteq \bigcup_{W \in \mathbb{W}^+} \mathcal{I}^+(C)(W)^n$ [Pr. 2.32]

iff $(\mathfrak{M}^+, W, s) \models^+_{g[S/X]} [A]^{\text{NEC}+}$ for some $S \subseteq \bigcup_{w \in \mathbb{W}} \mathcal{I}^+(C)(\{w\})^n$ [Def. 2.3 **Comp 2**]

iff $(\mathfrak{M}^+, W, s) \models^+_{g[S/X]} [A]^{\text{NEC}+}$ for some $S \subseteq dom^{+\text{N}}(W)^n$ [Def. 2.3 $dom^{+\text{N}}$]

iff $(M^{+\text{N}}, W, s) \models^+_{g[S/X]} A$ for some $S \subseteq dom^{+\text{N}}(W)^n$ [IH]

iff $(M^{+\text{N}}, W, s) \models^+_g \exists X A$ [Def. 2.6 $\forall V$]

The proof for $\exists x A$ is similar to that in **Proposition 1.24**. The remaining proofs for $\neg, \wedge, \Diamond, \uparrow$, and $\downarrow$ are routine.

Proposition 2.36 If $(\mathfrak{M}^+, W, s) \models^+_g$ **AuxNec**, then $\mathfrak{M}^+ = \mathfrak{M}^{+\text{N}}$.

Proof. Similar to **Proposition 1.26**.

Proposition 2.37 $[A]^{\text{NEC}+}$ is neutral.

Proof. By induction on the complexity of A.

Base case: Notice immediately that $[Fv_1 \ldots v_n]^{\text{NEC+}} = [Fv_1 \ldots v_n]^{\text{CON+}}$. Moreover, $[Vv_1 \ldots v_n]^{\text{NEC+}} = \Diamond^+(v_1 \ldots v_n \wedge V{\leq}C) = \Diamond^+[Vv_1 \ldots v_n]^{\text{CON+}} = [\Diamond^+ Vv_1 \ldots v_n]^{\text{CON+}}$. Finally, $[v_1 = v_2]^{\text{NEC+}} = \Diamond(v_1 = v_2 \wedge V{\leq}C) = \Diamond[v_1 = v_2]^{\text{CON+}} = [\Diamond\, v_1 = v_2]^{\text{CON+}}$.

Inductive hypothesis: $[A]^{\text{NEC+}}$ is neutral.

Induction step 1: Consider $\uparrow\Diamond^+\exists X(X{\leq}C \wedge \downarrow[A]^{\text{NEC+}})$. By hypothesis, for some B, $\models \uparrow\Diamond^+\exists X(X{\leq}C \wedge \downarrow[A]^{\text{NEC+}}) \leftrightarrow \uparrow\Diamond^+\exists X(X{\leq}C \wedge \downarrow[B]^{\text{CON+}})$. Thus, $\models [\exists XA]^{\text{NEC+}} \leftrightarrow [\uparrow\Diamond\exists X\downarrow B]^{\text{CON+}}$.

The proof for $\exists x$ is similar to that in **Proposition 1.25**. The remaining proofs for $\neg, \wedge, \Diamond, \uparrow$, and $\downarrow$ are routine.

Proposition 2.38 AuxNec $\models^+ [A]^{\text{NEC+}} \leftrightarrow A$

Proof. Similar to **Proposition 1.18**, using **Propositions 2.35** & **2.36** instead of **Propositions 1.16** & **1.17**.

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5

Relative Necessity and Propositional Quantification

Abstract

Following Timothy Smiley's (1963) influential proposal, it has become standard practice to characterise notions of relative necessity in terms of simple strict conditionals. However, Lloyd Humberstone (1981) and others have highlighted various flaws with Smiley's now standard account of relative necessity. In their recent article, Bob Hale and Jessica Leech (2017) propose a novel account of relative necessity designed to overcome the problems facing the standard account. Nevertheless, the current article argues that Hale & Leech's account suffers from its own defects, some of which Hale & Leech are aware of but underplay. To supplement this criticism, the article offers an alternative account of relative necessity which overcomes these defects. This alternative account is developed in a quantified modal propositional logic and is shown model-theoretically to meet several desiderata of an account of relative necessity.

There are many different notions of necessity. For example, it may well be *practically* impossible for someone to run from London to Winchester in under three hours, but surely it is *physically* possible for that person to do so. Similarly, it is *physically* necessary that nothing goes faster than the speed of light, but many would claim that it is *metaphysically* possible for something to do so.

Although there are various different kinds of necessity, it is commonly assumed that there is an *absolute* notion of necessity which is broader than all other notions of necessity. Moreover, given this notion of absolute necessity, it is also widely assumed that all other kinds of necessity can be characterised in terms of it. Indeed, since Timothy Smiley's (1963) influential proposal, it has become standard practice to characterise notions of relative necessity in terms of certain absolutely strict conditionals. According to this proposal, the relative necessity of some proposition p can be characterised in terms of the absolute strict implication of p by some suitable sentential constant. For instance, on this view, the physical necessity of a proposition p is equated with the absolute strict implication of p by a sentential constant expressing the laws of physics.

Nevertheless, Lloyd Humberstone (1981) and others have highlighted various flaws with Smiley's now standard account of relative necessity. Aware of these flaws, in their recent article Bob Hale and Jessica Leech (2017) propose a novel account of relative necessity designed to overcome the problems facing the standard account. However, although Hale & Leech's account is an improvement on the standard account in certain respects, it suffers from its own defects. The current article outlines these defects, some of which Hale & Leech are aware of but underplay, and emphasises their seriousness. The article then defends an account of relative necessity without these defects, to which the use of propositional quantification is key. It is then shown model-theoretically that this new account meets several desiderata of an account of relative necessity. The article concludes with a brief discussion of

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certain complexities which arise if various background assumptions are lifted.

5.1 Relative Necessity

5.1.1 Absolute Necessity

On the prevailing conception, for a notion of necessity to be absolute is for it to be the *broadest* necessity: that is, it is a matter of necessity that whatever is absolutely necessary is necessary in any other sense. This conception is found in both Ian McFetridge (1990) and Bob Hale's (1996) classic discussions of absolute necessity. Moreover, in recent work, Andrew Bacon (2018) has shown how to formalise this conception in a typed quantified modal language with quantification into operator position.

A slightly less formal version of Bacon's official characterisation can be stated with the aid of the following notion:[1]

ALAB $\Box_i$ is *at least as broad as* $\Box_j$ iff $\forall p \Box_k(\Box_i p \rightarrow \Box_j p)$ is true, whenever $\Box_k$ is a necessity operator

ALAB encodes the popular idea that one notion of necessity is at least as broad as another just in case, as a matter of necessity (in every sense), the latter deems as necessary every proposition which the former deems necessary; the broader a notion of necessity, the more stringent it is. Put alternatively, a notion of necessity

[1] Bacon's official working language is a functional type theory in which each type has a corresponding universal quantifier used to generalise universally over things of that type. In Bacon's (2018) terms, **ALAB** can be written as follows:

$$\Box_i \geq \Box_j \leftrightarrow \forall \Box_k (N\Box_k \rightarrow \forall p \Box_k(\Box_i p \rightarrow \Box_j p))$$

Here, the subscripted boxes are expressions which take sentences as arguments and result in another sentence, '$\geq$' is an expression which takes expressions of the same type as subscripted boxes as arguments and results in a sentence, and 'N' is a predicate which takes as arguments expressions of the same type as subscripted boxes to produce a sentence. On their intended interpretation, '$\geq$' denotes the relation of *being-at-least-as-broad-as* and 'N' denotes the property (of properties of propositions) of *being-a-necessity-operator*, which is defined in Bacon (2018, p. 2). Although **ALAB** in the main body of text mixes object- and meta-language, it should be understood as an informal statement of the above formalisation which does not. Indeed, similar principles in the main body of text may always be understood as informal statements of their formalisations in this typed formal language.

is at least as broad as another just in case, as a matter of necessity (in every sense), any proposition which is possible according to the latter is possible according to the former; the broader a notion of necessity, the more inclusive its corresponding notion of possibility. To take an example, it is plausible that physical necessity is at least as broad as practical necessity since, as a matter of necessity (in every sense), anything which is practically possible is physically possible too. Given **ALAB**, the claim that absolute necessity ($\Box$) is the broadest necessity can be expressed as follows:[2]

Absolute $\forall p \Box_k(\Box p \rightarrow \Box_j p)$ is true, whenever $\Box_k$ and $\Box_j$ are necessity operators

To reiterate, according to this characterisation, absolute necessity is at least as broad as any notion of necessity: for any proposition, it is a necessary matter (in every sense) that if that proposition is absolutely necessary it is necessary in every sense.

It bears emphasis that **ALAB** and **Absolute** function as intended only if *propositional necessitism* is assumed, here understood as the view that, in any sense of necessity, it is necessary which propositions there are. Were propositional necessitism rejected, the propositional quantifiers in **ALAB** and **Absolute** would not range over all absolutely possible propositions, depriving the principles of their intended generality. As is typical in the literature, propositional necessitism will be a working assumption in what follows.

According to one popular view, a natural candidate for absolute necessity is logical necessity. However, it is often unclear how logical necessity—thought of as a propositional operator, not a predicate of expressions—is intended to be characterised. Nevertheless, under the assumption that propositions form a Boolean algebra under the usual truth-functional operations, logical necessity can be characterised quite naturally. In the Boolean setting, there is a unique tautologous proposition—the top element of the algebra—which may be thought of as *the* logical truth. Thus, in that setting, it is highly natural to characterise logically necessity as simply being identical with the tautologous proposition.[3] Given such

[2]The assumption that there is a unique broadest necessity will be discussed shortly below.

[3]Bacon (2018) investigates this characterisation of logical necessity (although Bacon uses the term ‘broad necessity’ instead). See also Cresswell (1965) and Suszko (1971). Dorr (2016, p. 70) discusses how one might define a similar notion of necessity in a non-Boolean setting.

elegance, the article will simply adopt the hypothesis of Booleanism to provide a stable background to the discussion. To be more precise, the article will adopt the assumption that propositions form a *complete* Boolean algebra under the usual truth-functional operations, an assumption which since Fine (1970) is familiar to the study of propositionally quantified modal logics. As will be seen, Booleanism is an elegant working setting in which to adjudicate competing accounts of relative necessity. Moreover, the final section discusses how the arguments of the article may be adapted to non-Boolean settings.

Interestingly, given only the austere characterisation of absolute necessity in **Absolute**, one should expect its logic to meet various conditions. To take a clear example, notice that **Absolute** strongly suggests that $\Box$ will conform to the **4** axiom:

4 $\Box p \rightarrow \Box\Box p$

After all, it is highly plausible that concatenations of modal operators express notions of necessity, and so $\Box\Box$ will itself be a species of necessity. Thus, given **ALAB**, $\Box$ must be at least as broad as $\Box\Box$. Similarly, it is also plausible that $\Box$ conforms to the **T** axiom, since surely $\Box$ is at least as broad as some *alethic* notion of necessity:

T $\Box p \rightarrow p$

Finally, it is equally plausible that $\Box$ conforms to the **K** axiom which is partly characteristic of normal modal operators:

K $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$

Given that it is natural to expect that the ultimate logic of $\Box$ should be closed under modus ponens, necessitation, and uniform substitution, **S4** will be treated as a lower bound on the logic of absolute necessity.[4]

As a final piece of set up, we make the working assumption that absolute necessity is unique. This assumption is usually taken to be implicit, however we

[4]Indeed, in Bacon's (2018) axiomatic treatment of broad necessity one can derive the principles of **S4** for broad necessity.

shall enforce it explicitly by requiring notions of necessity to be individuated by a criterion of intensional equivalence.[5]

Identity $\Box_i = \Box_j$ iff $\forall p \Box_k(\Box_i p \leftrightarrow \Box_j p)$ is true, whenever $\Box_k$ is a necessity operator

Informally, **Identity** states that notions of necessity are the same just in case it is necessary (in all senses) that they apply to exactly the same propositions. This has the consequence that absolute necessity is unique, for if $\Box$ and $\Box'$ are absolute notions of necessity, then by **Absolute** they are at least as broad as one another. However, by **ALAB** and **Identity** it must be that $\Box$ and $\Box'$ are one and the same.

5.1.2 Characterising Relative Necessity

Given this notion of absolute necessity, one may hope to characterise non-absolute, relative notions of necessity in terms of it. Broadly speaking, relative necessities are types of parameterised modals. For example, one might think of physical necessity as necessity *given* the laws of physics. Similarly, one might think of metaphysical necessity as necessity *given* some particular body of metaphysical truths. Historically, however, these parameter conditions have been formalised via absolutely strict conditionals, as in Smiley's (1963) influential account:

> If we define $\mathbf{O}A$ [$\Box_i A$] as $\mathbf{L}(T \to A)$ [$\Box(T \to A)$] then to assert $\mathbf{O}A$ [$\Box_i A$] is to assert that T strictly implies A or that A is necessary relative to T. Since the pattern of the definition is independent of the particular interpretation that may be put on T we can say that to the extent that the standard alethic modal systems embody the idea of absolute or logical necessity, the corresponding **O**-systems embody the idea of relative necessity—necessity relative to an arbitrary proposition or body of propositions. They should therefore be appropriate for the formalisation of any modal notion that can be analysed in terms of relative necessity.

[5] **Identity** will be rejected by those who think that intensional equivalence (according to every notion of necessity) is too coarse-grained a criterion for individuating notions of necessity. According to that conception, there may be several absolute notions of necessity, all of which are at least as broad as all other notions of necessity. However, those who reject **Identity** may still wish to engage in the project of characterising non-absolute notions of necessity in terms of the absolute necessities. See both Dorr (2016, sect. 5) and Bacon (2018) for relevant discussion.

According to Smiley's account, for instance, a proposition p is physically necessary iff $\Box(L \rightarrow p)$ is true, where L is a sentential constant expressing the laws of physics.[6] Similarly, according to his account a proposition p is metaphysically necessary iff $\Box(M \rightarrow p)$ is true, where M is some sentential constant expressing the 'laws' of metaphysics (such as claims of identity and distinctness, and various essentialist claims). Call this the *standard account*:

SA $\Box_i\phi =_{df} \Box(\Psi \rightarrow \phi)$, where Ψ is a sentential constant which expresses the 'i-laws'

At this point, two notes are of importance. First, since **SA** and related definitions below introduce relative necessity 'operators' as metalinguistic abbreviations for complex object language formulae, the use of 'necessity operator' throughout should be understood as covering the object language formulae corresponding to such metalinguistic abbreviations. Second, the use of 'i-laws' should be understood very liberally. The 'i-laws' for a given notion of relative i-necessity are just some body of propositions which serve to relativise the notion of necessity, and need not have the status of scientific laws. For example, in the case of *necessity given Theresa May's current goals*, the relevant body of propositions will comprise just those propositions which she wishes to realise (for instance, that Parliament passes her Withdrawal Agreement).

Despite the influence of Smiley's standard account, Lloyd Humberstone (1981) and others have highlighted various flaws with it.[7] Moreover, in their recent article, Bob Hale and Jessica Leech have carefully articulated and developed the problems facing the standard account. They raise two central issues. First, given that **S4** is a lower bound on the logic of $\Box$, every notion of relative necessity will conform to its own version of the **4** axiom. Second, for the same reason, any relative necessities $\Box_i$ and $\Box_j$ will conform to a bizarre 'mixed' **4** axiom $\Box_i p \rightarrow \Box_j\Box_i p$. Although the

[6] Williamson (2016, pp. 462-463) notes that various truths of metaphysics must either be expressed by L or conjoined to it for reasons of adequacy.

[7] Humberstone (1981) attributes the problems for Smiley's account to an observation made by Kit Fine in conversation.

former is a special case of the latter, each problem shall be considered separately in order to appreciate the nature of the issue.

To begin with the former, consider the following **S4** valid schematic line of reasoning:

(1)	$\Box\phi \to (\psi \to \Box\phi)$	PL Tautology
(2)	$\Box(\Box\phi \to (\psi \to \Box\phi))$	Necessitation, (1)
(3)	$\Box\Box\phi \to \Box(\psi \to \Box\phi)$	**K**, MP, (2)
(4)	$\Box\phi \to \Box\Box\phi$	**4**
(5)	$\Box\phi \to \Box(\psi \to \Box\phi)$	PL, (3), (4)

Now consider any proposition p and sentential constant L expressing the laws of physics. The following is an instance of (5):

$$\Box(L \to p) \to \Box(L \to \Box(L \to p))$$

Yet, by **SA**, this just abbreviates an instance of the **4** axiom for physical necessity. And the argument generalises to every notion of relative necessity. But surely not all notions of relative necessity conform to **4**. To take an example, consider the notion of practical necessity, of what is necessary given our current practical means. At some point in recent history, it was practically impossible for us to engineer smartphones. However, at that point, surely it was within our practical means to engineer technology which would provide us with the practical means to engineer smartphones. Thus, although it was practically impossible for us to engineer smartphones, it was still practically possibly possible for us to do so, which constitutes a failure of **4**.

To see the second problem, consider the sentential constant L expressing the actual laws of physics and the sentential constant M expressing the 'laws' of metaphysics. The following is another instance of (5):

$$\Box(L \to p) \to \Box(M \to \Box(L \to p))$$

Yet, by **SA**, this just abbreviates the claim that if a given proposition p is physically necessary then it is metaphysically necessary that p is physically necessary. But for some propositions this is false: even though superluminal travel is physically impossible, there are metaphysically possible worlds with different laws of physics at which it is physically possible that bodies go faster than the actual speed of light.

5.2 Hale & Leech's Proposal

5.2.1 The Proposal

Hale & Leech clearly appreciate the force of the above concerns. After highlighting the shortcomings of Humberstone's (2004) two-dimensional solution, they offer their own definition of relative necessity. According to Hale & Leech's proposal, each relative necessity operator $\Box_i$ is to be defined in terms of a corresponding propositional operator π_i. In the case of, say, physical necessity, here denoted as $\Box_p$, the claim $\pi_p(p)$ expresses that p is a law of physics.

For their first attempt, Hale & Leech propose that i-necessity can be defined as follows (where q is the first variable not free in ϕ under some fixed ordering):[8]

HL1 $\Box_i\phi =_{df} \exists q(\pi_i(q) \wedge \Box(q \rightarrow \phi))$

In other words, p is i-necessary iff there is some i-law which strictly implies p. An instance of **HL1** is the claim that p is physically necessary iff there is some law of physics which strictly implies it.

One might immediately notice that this is somewhat weak. After all, surely the physical necessities ought to be closed under conjunction. Indeed, all instances of the following schema are valid in any normal modal logic:

K$\wedge$ $(\Box_i\phi \wedge \Box_i\psi) \rightarrow \Box_i(\phi \wedge \psi)$

But **HL1** does not guarantee closure under conjunction, and so precludes any relative necessity operator from being a normal modal operator. To see this informally,

[8]Hale & Leech (2017, p. 13) do not state their account schematically, but instead state it in terms of propositional variables. When stated in schematic form, one needs the condition that q is the first variable not free in ϕ in order to avoid clash of variables.

imagine there were only two laws r and r' such that r strictly implies p and r' strictly implies q. Since neither r nor r' strictly implies $p \wedge q$, **K**$\wedge$ fails for physical necessity.

Hale & Leech are aware of this issue, and offer a more sophisticated definition of relative necessity to handle the problem. In this definition, Γ is a finite set of propositional variables q_1, ..., q_n, the first n variables (given some fixed order) not free in ϕ, and $\pi_i(q_j)$ abbreviates the finitary conjunction $\pi_i(q_1) \wedge ... \wedge \pi_i(q_n)$:

$$\textbf{HL2} \;\; \Box_i\phi =_{df} \exists q_1 \ldots \exists q_n(\pi_i(q_j) \wedge \Box(\bigwedge_{q_j \in \Gamma} q_j \rightarrow \phi))$$

According to this account, p is i-necessary iff there are some propositions, the i-laws, the conjunction of which strictly implies p. An instance of **HL2** is the claim that p is physically necessary iff there are some propositions, the laws of physics, the conjunction of which strictly implies p. As can be easily verified, **HL2** validates **K**$\wedge$ for each relative necessity.

In **HL2**, Hale & Leech (p. 18, f.n.23) ultimately want to treat n as variable. They recognise that this requires a departure from the resources of ordinary quantified modal propositional logic. For instance, they entertain the idea of introducing explicit quantification over the natural numbers into the object-language in order to permit variability. An alternative would be to state their ultimate account of relative necessity in terms of a countably infinite disjunction of finite **HL2** instances. Relatedly, Hale & Leech assume that n is finite. In principle, this finitary simplification may ultimately have to be lifted in order to accommodate relative necessities based on infinitely many 'laws'. After all, one may expect there to be such notions of relative necessity. Thus, the ultimate regimentation of **HL2** may have to be stated in a language with infinitary conjunction and whose quantifiers could bind infinite sets of variables. Although this creates additional technical complexity, there should be no technical obstacle to introducing such infinitary resources to modal languages. Indeed, some key invariance results in modal logic only hold in modal languages with infinite conjunction binding arbitrary sets of formulae.[9]

[9]See Goranko and Otto (2006, p. 24); see also van Benthem (2010, pp. 30-31) for a simple explanation of the invariance results.

Nevertheless, the focus of what follows will be on the simple **HL2** account, since it shares many features of interest with these more sophisticated refinements.

Neatly, **HL2** overcomes the two above problems with **S4**. To see this, consider the **4** axiom for $\Box_i$ according to **HL2**:

4′ $\exists q_1 \ldots \exists q_n(\pi_i(q_j) \wedge \Box(\bigwedge_{q_j \in \Gamma} q_j \rightarrow p)) \rightarrow \exists q_1 \ldots \exists q_n(\pi_i(q_j) \wedge \Box(\bigwedge_{q_j \in \Gamma} q_j \rightarrow \exists q_1 \ldots \exists q_n(\pi_i(q_j) \wedge \Box(\bigwedge_{q_j \in \Gamma} q_j \rightarrow p))))$

This formula is not a theorem of **S4** (or even **S5**): it is invalidated in models similar to the one given in §5.3.2 below.[10] Thus, the definition of relative necessity does not force all notions of relative necessity to conform to the **4** axiom. A similar argument shows that **HL2** overcomes the 'mixed' **4** problem too.

To close the discussion of **HL2**, it may be assumed on behalf of Hale & Leech that a given notion of relative necessity $\Box_i$ has *dual* notion of possibility $\Diamond_i$ to be understood as follows (where again $q_1, ..., q_n$ are the first n variables, given some fixed order, not free in ϕ):

HL2◊ $\Diamond_i \phi =_{df} \neg\exists q_1 \ldots \exists q_n(\pi_i(q_j) \wedge \Box(\bigwedge_{q_j \in \Gamma} q_j \rightarrow \neg\phi))$

Thus, in logically equivalent form, the formula $\Diamond_i p$ abbreviates the following formula:

$$\forall q_1 \ldots \forall q_n(\pi_i(q_j) \rightarrow \Diamond(\bigwedge_{q_j \in \Gamma} q_j \wedge \phi))$$

That is to say, p is i-possible according to **HL2** iff for all i-laws it is absolutely possible that they are true along with p. If one endorses **HL2**, **HL2◊** is an extremely natural way to define relative notions of possibility; in what follows, talk of relative possibility on Hale & Leech's view should be understood in these terms.

[10] Officially, Hale & Leech do not provide a model-theoretic semantics for their language, although they (p. 15) note that "the most obvious method" would be to follow Fine's classic (1970) semantics. In Fine's framework, propositions are interpreted as sets of worlds and propositional variables and quantifiers range over the power set of worlds in a model. Hale & Leech (p. 16) emphasise that they are "reasonably confident that with further work, [their] informal [proof] sketches can turned into rigorous model-theoretic [proofs]". At points, their informal presentation will be followed when discussing their view; however, the current paper's positive proposal will be developed explicitly in the setting of a model-theoretic framework similiar to that of Fine (1970).

5.2.2 Against HL2

Despite the advantages of **HL2** over the standard account, it suffers from several serious problems.

The first is perhaps the most serious: **HL2** fails to have the consequence that logical necessity is at least as broad as any relative necessity. Since logical necessity was considered as absolute *because* it was at least as broad as every notion of necessity, this is a seriously worrying defect. Indeed, put alternatively, Hale and Leech cannot even recognise a notion of necessity which is absolute in the required sense, which is arguably a fatal consequence of an account of relative necessity.

To see how this issue arises, fix on a given merely relative necessity, say physical necessity ($\Box_p$), and consider the following consequence of **Absolute**:

[**1**] $\Box(\Box p \rightarrow \Box_p p)$

Since $\Box$ conforms to **T**, we may conclude the following:

[**2**] $\Box p \rightarrow \Box_p p$

But one can see informally why Hale & Leech's proposal does not validate [**2**]. Let $\top$ be the tautology $\forall p(p \rightarrow p)$. Given the usual definition of validity as truth at all worlds in all models, $\top$ will be true at all worlds in every model and therefore valid. Since $\Box$ has a normal modal logic, $\Box\top$ will be valid too. But consider some world w in some model where there are no physical laws. Straightforwardly, at w the following would be false:

[**3**] $\exists q_1 \ldots \exists q_n(\pi_i(q_j) \wedge \Box(\bigwedge_{q_j \in \Gamma} q_j \rightarrow \top))$

The reason is that there are no physical laws at w, and so no proposition will be in the extension of π_i there. But, by **HL2**, [**3**] is equivalent to $\Box_p\top$. Thus, [**2**] is not valid according to **HL2**, and so **Absolute** will not be valid either. Put alternatively, if there are i-lawless worlds for a given notion of i-necessity, absolute necessity will not be as broad as that notion of necessity.

Overall, it is worrying even if there is a *single* notion of relative necessity which absolute necessity is not as broad as. However, it is worth emphasising just how widespread this phenomenon is according to **HL2**. After all, for many notions of relative i-necessity it will be absolutely possible that there are i-lawless worlds. For example, consider again *necessity given Theresa May's current goals.* Although May has various actual goals, it is absolutely possible that her goals could have been different. For example, instead of aiming for Parliament to pass her Withdrawal Agreement she might have desired for the United Kingdom to retain full membership of the European Single Market. Likewise, there is an absolute possibility in which May has no goals whatsoever, which would be a lawless world in the relevant sense. Since it is plausible that the phenomenon of absolutely possible i-lawless worlds arises for many notions of i-necessity, **HL2** predicts that absolute necessity is not as broad as myriad relative necessities.

Hale & Leech are aware of a version of the problem with **Absolute** but they attempt to underplay it.[11] They write (pp. 18-19; their emphasis):

> That [**Absolute**] should be validated is, no doubt, just what one would think, if one thinks about relative necessity in essentially world-terms: that is, so that what is *physically* necessary, for instance, is essentially just what is true throughout a *restricted* range of all *logically* possible worlds. For then, since anything logically necessary (i.e. true throughout the whole unrestricted range) must be true throughout any restriction of it, it must also be physically necessary. Our answer to this is that it is just a mistake to think of forms of relative necessity as fundamentally to be understood in world terms. If we drop that prejudice, then it can seem entirely natural and correct to characterise or define a form of relative necessity in such a way that logical necessity does not automatically ensure relative necessity.

Nonetheless, despite Hale & Leech's insistence that **Absolute** should only be appealing to those in the grip of a worlds-based reduction of modality, it ought to be clear from the initial set up that this is just not true. The reason why **Absolute** is so plausible is because absolute necessity is characterised by the fact that it is at least as broad as all other notions of necessity. Indeed, how else

[11] They attribute the version of the problem they discuss to David Makinson; in their discussion, the problem is presented under a guise which is much different to the current one.

would it be characterised? Its absoluteness consists in this feature. Moreover, claims of comparative broadness can be used to provide insight into the relations between different notions of necessity, which is surely an ambition of our modal theorising.[12] The capacity to express such claims does not seem like an artefact of the worlds-based conception, but rather a requirement of modal theory.

At the heart of the above problem is Hale & Leech's treatment of relative i-necessities at worlds at which there are no i-laws. According to **HL2**, at worlds with no i-laws *no* proposition is i-necessary since claims of i-necessity involve existential quantification over i-laws. But this is difficult to square with familiar glosses of relative necessities. For example, consider the following typical necessary condition on physical possibility:

PP If a proposition is physically possible at world, it is logically consistent with the world's laws of physics.

Indeed, Hale & Leech (p. 2) appear to endorse this very condition:

> Thus on one common view, physical necessity is a matter of following from the laws of physics, and physical possibility is compatibility [i.e. logical consistency] with them.

However, according to **HL2**/**HL2**$\Diamond$, at worlds with no laws of physics it is vacuously true that every proposition whatsoever (including contradictory propositions) is physically possible. Yet this violates the highly plausible condition that, whatever the circumstances, only logically possible propositions are physically possible. Moreover, as stressed above, since no proposition is physically necessary at lawless worlds according to **HL2**, not even arbitrary tautologies will be physically necessary there either. But on the typical definition of validity as truth at all worlds in all models, it would follow that notions of relative necessity do not even have a normal modal logic, since they would fail to conform to the rule of necessitation.

There are thus two related concerns facing **HL2**: the problem with **Absolute**, and the mishandling of lawless worlds. In effect, the latter is responsible for the

[12]For recent examples of such insight, see Williamson (2016) and Bacon (2018). See also Hale's (1996) classic and influential discussion of absolute necessity.

former. Moreover, there is the subsidiary problem of the rule of necessitation failing for notions of relative necessity, which again is an artefact of the mishandling of lawless worlds. Together, these problems motivate the need for a new definition of relative necessity, preferably in terms of absolute necessity. In the next section, we consider a novel definition which overcomes each of them as well as the initial **S4** problems. This proposal will be developed model-theoretically in a quantified modal propositional logic.

5.3 A New Proposal

5.3.1 Regimentation

The intuitive idea of the new proposal is that relative i-necessity is a matter of being strictly implied by the weakest proposition that strictly implies all i-laws. On the following way of developing this proposal, **Absolute** will be validated and the issue with lawless worlds will be rectified. Moreover, the proposal will ensure a normal modal logic for each notion of relative necessity. Finally, the similarity of the new proposal with **HL2** will allow it to overcome the **S4**-related problems that afflicted the standard account. We shall postpone the exact statement of the new proposal until some technical resources have been introduced.

The technical resources are to be introduced model-theoretically, so we must first specify a language.

Def 1. The *language* $\mathcal{L}$ is defined using: countably many propositional variables $(p, q, ...)$; countably many monadic predicate constants $(\pi_1, \pi_2, ...)$; conjunction $(\wedge)$ and negation $(\neg)$ as the only truth-functional connectives; the unary modal operator $\Box$ (absolute necessity), and a universal quantifier binding propositional variables $(\forall)$.

The *well-formed formulas* of the language are given by the following rule (where α is a propositional variable and ξ is a monadic predicate constant):

$$\phi := \alpha \mid \xi(\alpha) \mid \neg\psi \mid (\psi \wedge \chi) \mid \forall\alpha(\psi) \mid \Box\psi$$

We write $\exists$ as an abbreviation for the string $\neg\forall\neg$ and $\Diamond$ as an abbreviation for the string $\neg\Box\neg$. We introduce $\rightarrow, \vee$, and $\leftrightarrow$ by the usual abbreviations.

The class of models for this language shall not permit contingency in what propositions there are. Thus, to this end, we shall use constant domain models for quantified propositional logics; the class of models is similar to that defined in the classic Fine (1970) which, in effect, employs a constant domain model theory.

Def 2. A *model* $\mathfrak{M}$ of the language is a pair $\langle \mathcal{W}, \mathcal{V} \rangle$, where:

$\mathcal{W}$ is a non-empty set ('the worlds'), and

$\mathcal{V}$ is a total function ('the interpretation function') from pairs $\langle \xi, w \rangle$ (for ξ a predicate constant, and $w \in \mathcal{W}$) to subsets of $\mathcal{P}(\mathcal{W})$.

Def 3. A *variable assignment* g for the language is a total function from propositional variables to subsets of $\mathcal{W}$. We write $g(\alpha/d)$ for a variable assignment which differs from g at most in that it assigns d to α.

The next step is to define a truth at a world on an assignment.

Def 4. We define '$(\mathfrak{M}, w) \models_g \phi$' ('true at w on g') for ϕ a well-formed formula, $\mathfrak{M} = \langle \mathcal{W}, \mathcal{V} \rangle$, and $w \in \mathcal{W}$ recursively in the following manner. In this definition, α is propositional variable, ξ is a monadic predicate constant, and ϕ and ψ are both well-formed formulae.

$(\mathfrak{M}, w) \models_g \alpha$ iff $w \in g(\alpha)$

$(\mathfrak{M}, w) \models_g \xi(\alpha)$ iff $g(\alpha) \in \mathcal{V}(\xi, w)$

$(\mathfrak{M}, w) \models_g \neg\phi$ iff $(\mathfrak{M}, w) \not\models_g \phi$

$(\mathfrak{M}, w) \models_g \phi \wedge \psi$ iff $(\mathfrak{M}, w) \models_g \phi$ and $(\mathfrak{M}, w) \models_g \psi$

$$(\mathfrak{M}, w) \models_g \Box\phi \text{ iff } (\mathfrak{M}, u) \models_g \phi, \text{ for all } u \in W$$

$$(\mathfrak{M}, w) \models_g \forall\alpha(\phi) \text{ iff } (\mathfrak{M}, w) \models_{g(\alpha/p)} \phi, \text{ for all } p \in \mathcal{P}(W)$$

Note that this semantics validates the Barcan formula and its converse for the propositional quantifiers, as is standard in necessitist logics which validate **S5**.

Def 5. A formula ϕ is *valid in* $\mathfrak{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ iff $(\mathfrak{M}, w) \models_g \phi$ for all $w \in \mathcal{W}$, and all variable assignments g. A formula ϕ is *valid* iff it is valid in all models.

With this technical apparatus, we are now in a position to state the new proposed definition of relative necessity. Intuitively, the idea is that p is i-necessary at a world iff it is strictly implied by the weakest proposition that strictly implies every i-law at that world. For example, p is physically necessary at a world iff it is strictly implied by the weakest proposition that strictly implies every physical law at that world. To appreciate this idea, it may help to emphasise that in cases where there are non-zero finitely many i-laws at a given world, the weakest proposition that strictly implies every i-law at that world will just be the proposition expressed by the finite conjunction of the i-laws. In this regard, the new account of relative necessity is not a huge departure from **HL2**. To state this definition, we introduce two metalinguistic abbreviations:

$$\mathcal{O}^{\pi_i} p =_{df} \forall q(\pi_i(q) \to \Box(p \to q))$$

$$\mathcal{L}^{\pi_i} p =_{df} \mathcal{O}^{\pi_i}(p) \wedge \forall q(\mathcal{O}^{\pi_i}(q) \to \Box(q \to p))$$

Put informally, $\mathcal{O}^{\pi}_i p$ means that p strictly implies all propositions that are i-laws, and $\mathcal{L}^{\pi_i} p$ means that p is the weakest such proposition with respect to strict implication. To aid one's understanding of these notions, it may help to notice that since propositions are modelled as sets of absolutely possible worlds, one proposition's strictly implying another is modelled as the former being a subset of the latter. Given this equivalence, $\mathcal{O}^{\pi_i} p$ can be understood as the claim that

p is a subset of all i-laws, and $\mathcal{L}^{\pi_i}p$ can be understood as the claim that p is the greatest lower bound on the i-laws with respect to subset inclusion. When $\mathcal{L}^{\pi_i}p$ is true at a world on a given assignment, whatever p is assigned will be referred to as the *i-lawful proposition* of that world.

It is important to stress that since propositions are treated as sets of worlds in a given model, the domain of propositions (i.e. the power set of the worlds in the model) will form a *complete* Boolean algebra under the usual set-theoretic operations in the sense that every subset of propositions has a least upper bound and a greatest lower bound with respect to subset inclusion. Thus, for any set of propositions there is guaranteed to be a weakest proposition that strictly implies all its members. Helpfully, this means that at each world exactly one proposition will satisfy $\mathcal{L}^{\pi_i}$ for a given predicate π_i.

The new definition of relative necessity can now be stated as follows (where q is the first variable not free in ϕ):

RN $\Box_i\phi =_{df} \exists q(\mathcal{L}^{\pi_i}(q) \wedge \Box(q \to \phi))$

To restate the intuitive gloss: i-necessity at a world is a matter of being strictly implied by the weakest proposition that strictly implies all i-laws at that world.

5.3.2 Application

In certain respects, **RN** is similar to both **HL2** and **SA**. This section will highlight these similarities, but also emphasise its crucial differences. These differences will ensure that **RN** overcomes each of the problems which **HL2** and **SA** faced.

4 Axiom Schema

Recall that one problem with **SA** is that it validated all instances of the **4** axiom schema for every notion of relative necessity. It may be shown that **RN** does not have this feature.

Consider a relative necessity operator $\Box_i$. According to **RN**, the **4** axiom schema for $\Box_i$ is stated as follows:

4 $\exists q(\mathcal{L}^{\pi_i}(q) \wedge \Box(q \rightarrow \phi)) \rightarrow [\exists q(\mathcal{L}^{\pi_i}(q) \wedge \Box(q \rightarrow \exists q(\mathcal{L}^{\pi_i}(q) \wedge \Box(q \rightarrow \phi))))]$

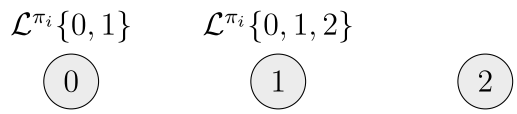
The following is a countermodel to an instance of **4**. Let $\mathfrak{M} = \langle \mathcal{W}, \mathcal{V} \rangle$, such that:

$\mathcal{W} = \{0, 1, 2\}$

$\mathcal{V}(\pi_i, 0) = \{\{0, 1\}, \{0, 1, 2\}\}$

$\mathcal{V}(\pi_i, 1) = \{\{0, 1, 2\}\}$

$\mathcal{V}(\xi, w) = \emptyset$, for all $w \in \mathcal{W}$ and all ξ otherwise



Let g be a variable assignment such that $g(p) = \{0, 1\}$. Say that $\mathbf{4}^p$ is the instance of **4** with p substituted for ϕ. With g as above, it can be checked that $(\mathfrak{M}, 0) \not\models_g \mathbf{4}^p$.

The **4** axiom schema according to **RN** is a special case of the 'mixed' **4ij** schema according to **RN**, which is stated as follows:

4ij $\exists q(\mathcal{L}^{\pi_i}(q) \wedge \Box(q \rightarrow \phi)) \rightarrow [\exists q(\mathcal{L}^{\pi_i}(q) \wedge \Box(q \rightarrow \exists q(\mathcal{L}^{\pi_j}(q) \wedge \Box(q \rightarrow \phi))))]$

As a consequence, instances of **4ij** are invalidated by the model theory too.

Absolute

Next, it can be shown that $\forall p \Box_k(\Box p \rightarrow \Box_j p)$ is valid whenever $\Box_k$ and $\Box_j$ are necessity operators. To this end, the following helpful lemma may be established.

First, for a set S and X a set of sets, the *intersection of X with respect to S* ($\cap_S X$) is defined as follows.

$\cap_S X = \{y \in S : y \in x \text{ for all } x \in X\}$

For a predicate constant π_i and model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{V} \rangle$, $\mathcal{R}_i$ is a binary relation over $\mathcal{W}$ defined as follows.

$$\mathcal{R}_i = \{\langle x, y \rangle : \ y \in \cap_{\mathcal{W}} \mathcal{V}(\pi_i, x)\}$$

We may first notice that for a model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ and a variable assignment g the following is true:[13]

$$\dagger \ \ (\mathfrak{M}, w) \models_g \mathcal{L}^{\pi_i}(p) \text{ iff } \cap_{\mathcal{W}} \mathcal{V}(\pi_i, w) = g(p)$$

In intuitive terms, † and the definition of $\mathcal{R}_i$ establish that, from a given world, the i-accessible worlds are exactly those in which that world's i-lawful proposition is true. With this in mind, one can use $\mathcal{R}_i$ to derive a semantic clause for a given notion of i-necessity.

Theorem 1. *For $\Box_i$ a notion of relative necessity, one may derive the following semantic clause:*

$$(\mathfrak{M}, w) \models_g \Box_i \phi \textit{ iff for all } u \textit{ such that } w\mathcal{R}_i u,\ (\mathfrak{M}, u) \models_g \phi$$

Proof. All steps are given by either the semantic clauses, †, or the definition of $\mathcal{R}_i$.

$$(\mathfrak{M}, w) \models_g \exists q(\mathcal{L}^{\pi_i}(q) \wedge \Box(q \to \phi)) \qquad \text{iff}$$

$$(\mathfrak{M}, w) \models_{g(q/p)} \mathcal{L}^{\pi_i}(q) \wedge \Box(q \to \phi) \text{ for some } p \in \mathcal{P}(\mathcal{W}) \qquad \text{iff}$$

$$\cap_{\mathcal{W}} \mathcal{V}(\pi_i, w) = g_{(q/p)}(q) \text{ and } (\mathfrak{M}, w) \models_{g(q/p)} \Box(q \to \phi) \qquad \text{iff}$$

[13] *Proof sketch.* Since, as mentioned previously, $(\mathfrak{M}, w) \models_g \mathcal{L}^{\pi_i}(p)$ equates to $g(p)$ being the greatest lower bound on $\mathcal{V}(\pi_i, w)$ w.r.t. $\subseteq$ in $\mathcal{P}(\mathcal{W})$ (hereafter these qualifications are left implicit), it suffices to show that $\cap_{\mathcal{W}} \mathcal{V}(\pi_i, w)$ is also the greatest lower bound on $\mathcal{V}(\pi_i, w)$. It helps to separate the cases, so first assume that $\mathcal{V}(\pi_i, w) \neq \emptyset$. Clearly, for any $s \in \mathcal{V}(\pi_i, w)$, $\cap_{\mathcal{W}} \mathcal{V}(\pi_i, w) \subseteq s$. Moreover, $\cap_{\mathcal{W}} \mathcal{V}(\pi_i, w)$ contains *every* u such that $u \in s$ for all $s \in \mathcal{V}(\pi_i, w)$, so there is no lower bound Δ on $\mathcal{V}(\pi_i, w)$ such that $\cap_{\mathcal{W}} \mathcal{V}(\pi_i, w) \subset \Delta$. However, if $\mathcal{V}(\pi_i, w) = \emptyset$ then it is vacuously true that $\cap_{\mathcal{W}} \mathcal{V}(\pi_i, w)$ is a lower bound on $\mathcal{V}(\pi_i, w)$. Moreover, given that $\mathcal{V}(\pi_i, w) = \emptyset$, it follows that $\cap_{\mathcal{W}} \mathcal{V}(\pi_i, w) = \mathcal{W}$ and is thus the greatest lower bound on $\mathcal{V}(\pi_i, w)$.

$$(\mathfrak{M}, u) \models_g \phi \text{ for all } u \in \cap_{\mathcal{W}} \mathcal{V}(\pi_i, w) = g_{(q/p)}(q) \qquad \text{iff}$$

$$(\mathfrak{M}, u) \models_g \phi \text{ for all } u \text{ such that } v\mathcal{R}_i u$$

To return to the main thread, notice that for each π_i and model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{V} \rangle$, $\mathcal{R}_i \subseteq \mathcal{W} \times \mathcal{W}$. Thus, **Theorem 1** can be used to establish that $\Box p \to \Box_i p$ is valid for $\Box_i$ whenever $\Box_i$ is a relative necessity operator. However, as can easily be checked by using the semantics and definition of validity, $\Box$ obeys the rule of necessitation.

Necessitation If ϕ is valid, then $\Box \phi$ is valid.

Given the above, we may derive a generalised rule of necessitation for any given notion of relative necessity $\Box_i$.

Generalised Necessitation If ϕ is valid, then $\Box_i \phi$ is valid.

As a consequence, $\Box_k(\Box p \to \Box_i p)$ is valid whenever $\Box_k$ and $\Box_i$ are necessity operators. Using the semantics, one can easily verify that **Absolute** is valid as a consequence.

Relatedly, an advantage of **RN** is that it admits the following fact.

Theorem 2. *Let $\mathfrak{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ be a model such that $\mathcal{R} \subseteq \mathcal{W}^2$. There is a model $\mathfrak{M}' = \langle \mathcal{W}, \mathcal{V}' \rangle$ such that $(\mathfrak{M}', w) \models_g \Box_i \phi$ iff for all u such that $w\mathcal{R}u$ $(\mathfrak{M}', u) \models_g \phi$.*

Proof. For any $w \in \mathcal{W}$, let $\mathcal{R}(w) = \{v : w\mathcal{R}v\}$. Let $\mathcal{V}'$ be any interpretation function such that $\mathcal{R}(w) = \cap_{\mathcal{W}} \mathcal{V}'(\pi_i, w)$. Observe that $w\mathcal{R}u$ iff $u \in \mathcal{R}(w)$ iff $u \in \cap_{\mathcal{W}} \mathcal{V}'(\pi_i, w)$. Hence $(\mathfrak{M}', w) \models_g \Box_i \phi$ iff for all u such that $w\mathcal{R}u$ $(\mathfrak{M}', u) \models_g \phi$.

Both **Theorem 2** and **Theorem 1** are reasonable desiderata on an account of relative necessity. If met, respectively they ensure that to each accessibility relation in a model corresponds an in principle definable notion of relative necessity, and vice-versa. However, notice that **SA** fails to meet an analogue of **Theorem 2**, since its notions of relative necessity correspond only to transitive accessibility relations. Similarly, notice that **HL2** fails to meet an analogue of **Theorem 1**: according to **HL2**, for a given notion of relative i-necessity at i-lawless worlds it is i-possible that contradictions are true, but on standard model-theoretic treatments of normal modal logics, such notions of relative necessity will not correspond to definable accessibility relations in models.

Lawless Worlds

Recall that according to **HL2** at worlds with no i-laws no proposition is i-necessary and every proposition is i-possible. This was in tension with familiar glosses of i-necessity. For example, if physical necessity is simply absolute necessity given the laws of physics, if there are no laws of physics, then there are no propositions to restrict what is physically necessary. Thus, one would expect that physical necessity would collapse into absolute necessity. However, it can be shown that **RN** meets this expectation and thus conforms with this familiar gloss of i-necessity.

Consider a model $\mathfrak{M}^1 = \langle \mathcal{W}, \mathcal{V}^1 \rangle$ just like $\mathfrak{M}$ before, except:

$\mathcal{V}^1(\pi_i, w) = \emptyset$, for all $w \in \mathcal{W}$

We want to show that $(\mathfrak{M}^1, w) \models_g \Box p$ iff $(\mathfrak{M}^1, w) \models_g \Box_i p$ for an arbitrary propositional variable p. Of course, the left-to-right direction is immediate from **Absolute**. But, more helpfully, notice that since $\mathcal{V}^1(\pi_i, w) = \emptyset$, $\cap_{\mathcal{W}} \mathcal{V}^1(\pi_i, w) = \mathcal{W}$. Thus for all $u \in \mathcal{W}$, $w\mathcal{R}_i u$. As intended, when there are no i-laws at a given world the i-possible worlds from that world are all and only the absolutely possible worlds, which demonstrates the collapse of i-necessity and absolute necessity at such i-lawless worlds. This is preferable to the verdict of **HL2** that even absolutely impossible propositions are i-possible at i-lawless worlds.

Relatedly, we can note that if the i-laws are inconsistent at some world in some model, in the sense that their intersection with respect to the set of all worlds in that model is empty, then at that world everything is i-necessary but nothing is i-possible. This is another welcome result, for such inconsistent sets of laws and their corresponding relative necessities exhibit a certain degeneracy. To appreciate the result, consider a model $\mathfrak{M}' = \langle \mathcal{W}, \mathcal{D}, \mathcal{V}' \rangle$ just like $\mathfrak{M}^1$ directly above, except:

$$\mathcal{V}'(\pi_i, w) = \{S, \mathcal{W}{-}S\}, \text{ for some } S \subseteq \mathcal{W}$$

Speaking informally, it follows that the only proposition (i.e. subset of $\mathcal{W}$) which strictly implies all the i-laws at w is the impossible proposition, that is $\emptyset$. As a consequence, $w\mathcal{R}_i u$ iff $u \in \emptyset$.[14] In other words, there is no $u \in \mathcal{W}$ such that $w\mathcal{R}_i u$. Thus, given the derived semantic clause above, vacuously every ϕ is i-necessary at w and no ϕ is i-possible at w.

Comparison with HL2 and RN

As is clear from the above, **RN** has some crucially beneficial advantages over both **HL2** and **SA**. However, it also bears similarities to both accounts.

To appreciate one helpful similarity it bears to **HL2**, first recall that **RN** relies on a conception of propositions according to which they form a complete Boolean algebra under the usual truth-functional operations. Thus, assuming a notion of conjunction which binds arbitrary sets of formula, it would be natural to view whichever proposition satisfies $\mathcal{L}^{\pi_i}$ (for a given predicate π_i) as the conjunction of the i-laws. This view would require that the conjunction of the empty set of propositions is simply the tautologous proposition, which is plausible (under the assumption of Booleanism) since the conjunction a set of propositions is true iff each proposition in the set is.

In light of this, **HL2** is clearly on the right track. To see this, imagine that **HL2** is amended to: (i) allow for propositional quantifiers to bind arbitrary sets of propositional variables; (ii) permit conjunction to bind arbitrary sets of formula, and (iii) equate empty conjunctions with the tautologous proposition. Given such amendments, one would expect **HL2** to be equivalent to **RN** in an appropriate extension of $\mathcal{L}$ to this quite extreme infinitary setting. In fact, Strohminger &

[14]In effect, this condition would force i-necessity to exhibit system **VER**-like features on the relevant model, a system which is axiomatised by adding all instances of the schema $\Box_i\phi$ to system **K**.

Yli-Vakkuri (2019) sketch an analysis of relative necessity similar to **HL2** in terms of infinite conjunction of exactly this kind, and suggest that they expect it to overcome the problems facing **HL2**. Nevertheless, rather than complicate the propositional language by resort to such infinitary measures, the advantage of **RN** is that it provides a less circuitous account of relative necessity simply in terms of already available propositional quantification. Indeed, Strohminger & Yli-Vakkuri (2019, p. 6, n.11; p. 7, n.12) themselves suggest that a 'higher-order analysis' of relative necessity is needed.

Relatedly, notice that **RN** is interestingly similar to **SA** too. Clearly, the standard account attempts to express the natural idea that, for example, physical necessity is absolute necessity relative to the laws of physics. However, the standard account formalises this idea problematically with the use of sentential *constants*, which would, for example, express the laws of physics of some specific world. In effect, it is this use of constants which generates the **4** axiom and mixed **4** axiom issues for the standard account. What is needed is some means of formalising the variability of the laws of physics from absolutely possible world to world in the language of propositional modal logic. Yet **RN** provides exactly this means: as stressed above, the i-lawful proposition will typically vary from world to world, but the i-lawful proposition of a world w will be true wherever the i-laws of w all are. Thus, **RN** may be viewed as a better expression of the natural idea which **SA** was intended to formalise.

5.4 Concluding Remarks

This article has criticised Hale & Leech's (2017) account of relative necessity, which itself resulted from their criticisms of the standard account of relative necessity. The paper then proposed a new account of relative necessity which met the criticisms of both Hale & Leech's account and the standard account.

To close the discussion, it is interesting to note that certain complications arise once various working assumptions are lifted. To begin with, from the outset it was made explicit that propositional necessitism was a working assumption of the discussion. If this assumption is lifted, **RN** may be an inadequate definition of relative necessity; in such a contingentist setting, the propositional quantifier in **RN** will not range over all absolutely possible propositions, depriving **RN** of its intended generality. This may come as no surprise: as is typical, non-modal quantificational claims lose their intended generality when simply copied from a necessitist setting and pasted into a contingentist one. When faced with such cases, a natural contingentist reaction has been to offer certain modalised contingentist analogue principles following the lead of Fine (1977). However, as Fine himself and recent commentators have recognised, this modalising strategy is difficult to execute in the case of propositional quantification.[15] Unfortunately, at the time of writing, such difficulties make it unclear whether an adequate analogue of **RN** can be articulated in a contingentist setting. Indeed, clarity on this issue is likely to be gained only if a more general contingentist method of simulating higher order necessitist discourse can be established.

Finally, it was made explicit that the account relied on a conception of propositions according to which it is absolutely necessary that there is a weakest proposition which strictly implies all i-laws. On alternative conceptions of propositions, however, there may be be no such unique proposition. In particular, on some conceptions of propositions they may not form a complete Boolean algebra under the usual truth-functional operations. For example, in certain cases, there may be various minimally weak propositions which strictly imply all i-laws, but no unique candidate. Questions regarding how to define relative necessity in such settings are left for future work, but one may expect that **RN** could still serve as the basis of such definitions.

[15]See in particular Williamson (2013, chap. 7) and Fritz & Goodman (2017).

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