The Circle Method and Diagonal Cubic Forms

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1 Introduction

Let

$$F(\mathbf{x}) = \sum_{i=0}^{n} F_i x_i^3 \qquad (F_i \in \mathbb{Z} - \{0\}) \tag{1}$$

be an integral cubic form in n variables. This paper is motivated by the problem of describing the distribution of the integral zeros of F. We shall assume $F(\mathbf{x})$ to be fixed throughout this work, so that all order constants, for example, may depend on the coefficients F_i. For values of n which are not too small the Hardy-Littlewood circle method may be used to tackle the distribution problem successfully. For example, let real numbers $\alpha_i < \beta_i$ be given for $1 \le i \le n$, and set

$$\mathcal{B} = \{\mathbf{x} \in \mathbb{R}^n : \ \alpha_i \le x_i \le \beta_i, \ (1 \le i \le n)\}.$$

One then seeks an asymptotic formula for the number $N(P, \mathcal{B})$ of integral zeros of $F(\mathbf{x})$ in the box $P\mathcal{B}$, as $P \to \infty$. This is expected to take the form

$$N(P, \mathcal{B}) = cP^{n-3} + o(P^{n-3}), \tag{2}$$

where c is a constant depending only on F and $\mathcal{B}$. For $n \ge 9$, the methods of Hardy and Littlewood [3] yield an asymptotic formula

$$N(P, \mathcal{B}) = cP^{n-3} + O(P^{n-3-\delta}) \tag{3}$$

where δ is a positive constant depending only on n. When $n = 8$, a weaker asymptotic formula

$$N(P, \mathcal{B}) = cP^{n-3} + O(P^{n-3}(\log P)^{-\delta})$$

follows from techniques of Vaughan [12]. Finally, for $n = 7$ one may obtain a lower bound of the correct order of magnitude P^{n-3}, (in those cases when one expects c to be positive), by the methods of Vaughan [13]. When $n \le 6$ the circle method has yielded no unconditional results of interest.

Hooley [8] has made some significant progress, again via the circle method, subject to an unproved hypothesis concerning certain Hasse-Weil L-functions associated to the form F. (This hypothesis is Hypothesis 'HW$_6$' which we shall introduce in §4.) Under this assumption Hooley's method enables one to establish the formula (3) for $n = 7$ or 8.

We expect (3) to remain true when $n \geq 5$, but for $n = 4$ rational lines in the surface $F(\mathbf{x}) = 0$ may contribute $\gg P^2$ to $N(P, \mathcal{B})$. Thus if $F_1 = F_2$ and $F_3 = F_4$, for example, then $F(\mathbf{x}) = 0$ will have $\gg P^2$ integer solutions $|x_i| \leq P$, arising from the lines $x_1 = -x_2$, $x_3 = -x_4$. Thus for $n = 4$ we expect that

$$N(P, \mathcal{B}) = cP^2 + o(P^2), \tag{4}$$

with the constant c reflecting the contribution from rational lines. Until now it has appeared to be inherent in the circle method that, if the analysis is to succeed at all, the resulting asymptotic formula must necessarily take the form (2). It is not at all clear from the usual formulation of the method where a main term of the type one sees in (4) can originate. There is a general consensus among those working on the circle method that the main term must arise from the minor arcs, but this viewpoint says nothing more than that the major arc contribution, which can always be calculated, is $o(P^2)$.

The goal of this paper is to introduce the ideas of Hooley [8] into the author's recent analysis [5] of the circle method. This latter work showed how an identity of Duke, Friedlander and Iwaniec [2] produces a very convenient form of Hooley's 'double Kloosterman refinement'. Many of the technical difficulties in Hooley's approach are avoided, and this enables one to push the analysis further. We shall confine our attention to the cases $n = 6$ and $n = 4$.

Our first result improves that of Hooley [8].

Theorem 1 *Let $F = \sum_1^6 x_i^3$ and assume Hypothesis HW$_6$. Then if ε is any positive constant, the equation $F(\mathbf{x}) = 0$ has $O_\varepsilon(P^{3+\varepsilon})$ integral solutions in the region $|\mathbf{x}| \leq P$. Thus if $r_3(n)$ denotes the number of representations of n as a sum of 3 non-negative cubes, then*

$$\sum_{n \leq X} r_3(n)^2 \ll_\varepsilon X^{1+\varepsilon}. \tag{5}$$

The bound (5) is best possible, apart from the exponent ε, and leads to many corollaries, of the type considered by Hooley [8; Chapter II]. Hooley obtains only the exponent $\frac{20}{19} + \varepsilon$, although more recently [9], he has refined his approach to give another proof of (5). The reader should recall, for comparison, that the best unconditional estimate of the type given by (5) has exponent $\frac{7}{6} + \varepsilon$. This is a straighforward deduction from Hua's inequality.

It is an easy corollary of Theorem 1 that an arbitrary diagonal form $F(\mathbf{x})$ in 6 variables has $O_\varepsilon(P^{3+\varepsilon})$ integral zeros in the region $|\mathbf{x}| \leq P$, subject to Hypothesis HW$_6$ for the form $F(\mathbf{x}) = \sum_{i=1}^6 x_i^3$.

For $n = 4$ we have the following result.

Theorem 2 *Let $F(\mathbf{x})$ be given by (1) with $n=4$, and assume Hypothesis HW_4. Then if ε is any positive constant, the equation $F(\mathbf{x})=0$ has $O_\varepsilon(P^{3/2+\varepsilon})$ integral solutions in the region $|\mathbf{x}|\le P$, excluding those which lie on rational lines in the surface $F=0$. Such lines take the form $b_ix_i+b_jx_j=0$, $b_kx_k+b_lx_l=0$, where i,j,k,l are distinct indices, and $F_ib_i^{-3}=F_jb_j^{-3}$, $F_kb_k^{-3}=F_lb_l^{-3}$.*

Thus we have, for the first time, an asymptotic formula of the shape (4), proven via the circle method. Unfortunately it is not easy to describe succinctly just how the main term cP^2 arises in our analysis. The reader is encouraged to study the relevent material in §8.

Theorem 2 may be compared with the corresponding bounds for the equation

$$x^3+y^3=z^3+w^3,$$

due to Hooley [7], Wooley [15] and, recently, Heath-Brown [6]. The first two of these show that there are $O_\varepsilon(P^{5/3+\varepsilon})$ solutions not on rational lines, and the third improves the result to $O_\varepsilon(P^{4/3+\varepsilon})$. Thus Theorem 2 is sharper than the results of Hooley and Wooley, but weaker than the author's recent bound. Theorem 2 is, of course, conditional, whereas the other estimates are not, but it has the all important advantage of applying to any diagonal form, while the other methods are only capable of partial generalization. Indeed we take this opportunity to point out that the methods of this paper appear in principle to be capable of extension to non-diagonal forms. It is only difficulties of a purely technical nature that currently prevent such a generalization.

2 Preliminaries

The author's paper [5] is set up using weighted counting functions. Rather than using a box $P\mathcal{B}$ we will employ a weight $w(P^{-1}\mathbf{x})$, where we may think of w as being an approximation to the the characteristic function of $\mathcal{B}$. Instead of investigating $N(P,\mathcal{B})$ we shall consider

$$N(F,w)=N(F,w,P)=\sum w(P^{-1}\mathbf{x}),$$

the sum being taken over all $\mathbf{x}\in\mathbb{Z}^n$ for which $F(\mathbf{x})=0$. This approach allows us, in principle, to handle regions other than boxes, but in practice this is not of much interest, given the form of the results we shall obtain. The main advantage in introducing the weight w is that many of the estimates in the argument become sharper if one allows w to be many times differentiable. Since the results contained in Theorems 1 and 2 are upper bounds only, rather than asymptotic formulae, it will suffice to consider the weight

$$w(\mathbf{x})=w_0(|\mathbf{x}|-2),$$

where

$$w_0(x) = \begin{cases} \exp(-(1-x^2)^{-1}), & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Thus $w(\mathbf{x})$ is infinitely differentiable, and supported on the multi-dimensional annulus $1 \leq |\mathbf{x}| \leq 3$. It is then clear that it will be enough, for Theorem 1, to show that

$$N(F, w, P) \ll_\varepsilon P^{3+\varepsilon},$$

since the result as stated will follow on summing for $P, P/2, P/4, \ldots$. Similarly for Theorem 2 it will be enough to show that

$$N^*(F, w, P) \ll_\varepsilon P^{3/2+\varepsilon},$$

where $N^*(F, w, P)$ counts only those integer zeros of F which do not lie on rational lines.

Having given an appropriate formulation of the problem, we proceed to apply the circle method, in the form given in the author's work [5; Theorem 2]. This immediately yields an expression for $N(F, w)$ of the form

$$N(F, w) = c_Q P^{-3} \sum_{\mathbf{c} \in \mathbb{Z}^n} \sum_{q=1}^{\infty} q^{-n} S_q(\mathbf{c}) I_q(\mathbf{c}), \tag{6}$$

where

$$S_q(\mathbf{c}) = \sum_{a \pmod{q}}{}^{*} \sum_{\mathbf{b} \pmod{q}} e_q(aF(\mathbf{b}) + \mathbf{c}.\mathbf{b}),$$

and

$$I_q(\mathbf{c}) = P^n \int_{\mathbb{R}^n} w(\mathbf{x}) h(\frac{q}{Q}, F(\mathbf{x})) e_q(-P\mathbf{c}.\mathbf{x}) d\mathbf{x}.$$

Here we introduce some notation that will be standard throughout this paper; a sum $\Sigma^*_{a \pmod{q}}$ will be subject to $(a, q) = 1$; a sum for $\mathbf{x} \pmod{q}$ will mean that each component of $\mathbf{x}$ runs over a complete set of residues modulo q; and an integral $\int f(\mathbf{x}) d\mathbf{x}$ will be the n-fold repeated integral over $\mathbb{R}^n$.

Throughout the paper the parameter Q will be taken to be $P^{3/2}$. To define the constant c_Q and the function $h(x, y)$ we write

$$\omega(x) = 4c_0^{-1} w_0(4x - 3),$$

where

$$c_0 = \int_{-\infty}^{\infty} w_0(x) dx,$$

and we set

$$\sum_{q=1}^{\infty} \omega(q/Q) = c_Q^{-1} Q \tag{7}$$

and

$$h(x,y)=\sum_{j=1}^{\infty}\frac{1}{xj}\{\omega(xj)-\omega(|y|/xj)\}. \tag{8}$$

As is shown in [5;§3] we have

$$c_Q=1+O_N(Q^{-N})$$

for any fixed $N>0$. Moreover $h(x,y)$ vanishes unless $0<x\le\max(1,2|y|)$, and outside this range we have $h(x,y)\ll x^{-1}$. It follows that $I_q(\mathbf{c})=0$ for $q\gg Q$, so that the sum over q in (6) is finite.

The strategy for the proof of our theorems is merely to estimate the sum over q and $\mathbf{c}$ in (6). We shall use our Hypotheses HW_n to demonstrate some cancellation in the summation with respect to q, but we shall not be able to use any cancellation in the sum over $\mathbf{c}$. For most values of $\mathbf{c}$ we shall obtain a satisfactory conclusion. One would normaly expect the value $\mathbf{c}=\mathbf{0}$ to provide the main term of an asymptotic formula, but in our case this main term would be of order $O(P^{n-3})$, which is negligible. However when $n=4$ certain other values of $\mathbf{c}$ produce contributions that cannot be bounded satisfactorily, and we have then to show that these contributions account for points of the surface $F=0$ which lie on rational lines.

3 The Integral $I_q(\mathbf{c})$

In this section we shall consider the integral

$$I_q(\mathbf{c})=P^n\int_{\mathbb{R}^n}w(\mathbf{x})h(Q^{-1}q,F(\mathbf{x}))e_q(-P\mathbf{c}.\mathbf{x})d\mathbf{x}.$$

We shall begin by recording some results from the author's work [5]. For the case $\mathbf{c}=\mathbf{0}$ we have

$$I_q(\mathbf{0})=P^n\{\sigma_\infty(F,w)+O_N((q/Q)^N)\} \tag{9}$$

for any $N\ge1$ and all $q\ll Q$, by [5; Lemma 13]. The constant $\sigma_\infty(F,w)$ is, in fact, the 'singular integral'. However we shall only need to know that it is independent of P. From [5; Lemma 16] we also have

$$\frac{\partial^j I_q(\mathbf{0})}{\partial q^j}\ll P^n q^{-j},\ \ (j=0,1). \tag{10}$$

For general values of $\mathbf{c}$ we see that [5; Lemma 14] yields

$$I_q(\mathbf{c})\ll P^n r^{-1}|I(r;\mathbf{u})| \tag{11}$$

and

$$\frac{\partial I_q(\mathbf{c})}{\partial q} \ll P^n q^{-1} r^{-1} |I(r;\mathbf{u})|, \tag{12}$$

where $r = Q^{-1}q$ and $\mathbf{u} = q^{-1}P\mathbf{c}$. Here according to [5; Lemma 17], the integral $I(r;\mathbf{u}) = I(\mathbf{u})$ takes the form

$$I(r;\mathbf{u}) = \int_{-\infty}^{\infty} p(t) \int_{\mathbb{R}^n} w_3(\mathbf{x}) e(tF(\mathbf{x}) - \mathbf{u}.\mathbf{x}) d\mathbf{x} dt,$$

where w_3 is a certain continuous function of compact support, and

$$p(t) \ll_N r(r|t|)^{-N}$$

for any $N \geq 0$. Moreover we have the estimates

$$I(r;\mathbf{u}) \ll r \tag{13}$$

and

$$I(r;\mathbf{u}) \ll_N r^{-N} |\mathbf{u}|^{-N} \tag{14}$$

for any $N \geq 0$, by [5; Lemmas 15 & 18]. In particular it follows that

$$I_q(\mathbf{c}) \ll_{\varepsilon,N} |\mathbf{c}|^{-N} \tag{15}$$

when $|\mathbf{c}| > P^{1/2+\varepsilon}$, for any $N > 0$, and any $\varepsilon > 0$.

Our starting point for a more sophisticated bound for $I(\mathbf{u})$ is Lemma 20 of [5], which we state here as follows.

Lemma 1 *Let $R \geq 1$. If $|\mathbf{u}| \geq R^3$ then there exist positive constants A, B and C, and a value of t in the range*

$$A|\mathbf{u}| \leq |t| \leq B|\mathbf{u}|,$$

such that

$$I(\mathbf{u}) \ll_N R^{-N} + r|\mathbf{u}|\text{meas}(\mathcal{S}_t),$$

with

$$\mathcal{S}_t = \{\mathbf{x} \in \text{supp}(w) : \ |t\nabla F(\mathbf{x}) - \mathbf{u}| \leq CR|\mathbf{u}|^{1/2}\}.$$

The second condition in the definition of $\mathcal{S}_t$ yields

$$3tF_i x_i^2 - u_i \ll R|\mathbf{u}|^{1/2}, \quad (1 \leq i \leq n).$$

It follows that x_i is restricted to an interval of length $O(R|u_i|^{-1/2})$ in case $|u_i| \gg R|\mathbf{u}|^{1/2}$, and of length $O(R^{1/2}|\mathbf{u}|^{-1/4})$ otherwise. We therefore see that

$$I(\mathbf{u}) \ll_N R^{-N} + R^n r|\mathbf{u}| \prod_{i=1}^{n} \min\{|u_i|^{-1/2}, \, |\mathbf{u}|^{-1/4}\}.$$

We shall take $R = P^{\varepsilon/n}$. Then if $|\mathbf{u}| \le Q^2$ we will have

$$r|\mathbf{u}|\prod_{i=1}^{n}\min\{|u_i|^{-1/2}\,,\,|\mathbf{u}|^{-1/4}\} \ge r|\mathbf{u}|^{1-n/2} \ge Q^{-1}.Q^{2-n} \ge P^{-\varepsilon N/n},$$

providing that we choose N big enough. It therefore follows that

$$I(\mathbf{u}) \ll P^{\varepsilon} r|\mathbf{u}|\prod_{i=1}^{n}\min\{|u_i|^{-1/2}\,,\,|\mathbf{u}|^{-1/4}\} \tag{16}$$

for $P^{3\varepsilon/n} \le |\mathbf{u}| \le Q^2$. When $|\mathbf{u}| \ge Q^2$ we have

$$r|\mathbf{u}|\prod_{i=1}^{n}\min\{|u_i|^{-1/2}\,,\,|\mathbf{u}|^{-1/4}\} \ge r|\mathbf{u}|^{1-n/2} \ge (r|\mathbf{u}|)^{-n+1},$$

where the final inequality is a consequence of the bound $r^2|\mathbf{u}| \ge Q^{-2}|\mathbf{u}| \ge 1$. We therefore see that (16) is a consequence of (14) when $|\mathbf{u}| \ge Q^2$.

If $|\mathbf{u}| \le P^{3\varepsilon/n}$ we have

$$r|\mathbf{u}|\prod_{i=1}^{n}\min\{|u_i|^{-1/2}\,,\,|\mathbf{u}|^{-1/4}\} \ge r|\mathbf{u}|^{1-n/2} \ge rP^{3(1-n/2)\varepsilon/n},$$

and since $3(n/2-1)/n \le 2$, we find from (13) that (16) again holds, with ε replaced by 2ε. We may now deduce as follows.

Lemma 2 *If* $|\mathbf{c}| > P^{1/2+\varepsilon}$ *we have*

$$I_q(\mathbf{c}) \ll_{\varepsilon,N} |\mathbf{c}|^{-N}, \tag{17}$$

for any $N > 0$. *Moreover, for any* $\mathbf{c} \ne \mathbf{0}$ *we have*

$$I_q(\mathbf{c}) \ll_{\varepsilon} \frac{P|\mathbf{c}|}{q} P^{n+\varepsilon}\prod_{i=1}^{n}\min\{(\frac{q}{P|c_i|})^{1/2}\,,\,(\frac{q}{P|\mathbf{c}|})^{1/4}\}$$

and

$$\frac{\partial}{\partial q} I_q(\mathbf{c}) \ll_{\varepsilon} \frac{P|\mathbf{c}|}{q^2} P^{n+\varepsilon}\prod_{i=1}^{n}\min\{(\frac{q}{P|c_i|})^{1/2}\,,\,(\frac{q}{P|\mathbf{c}|})^{1/4}\}.$$

We also have

$$I_q(\mathbf{c}) \ll P^n \tag{18}$$

and

$$\frac{\partial}{\partial q} I_q(\mathbf{c}) \ll q^{-1}P^n.$$

4 The sum $S_q(\mathbf{c})$

In this section we shall give some of the fundamental properties of the sums $S_q(\mathbf{c})$, and examine their behaviour in the case in which q is square-free.

We begin with the following results.

Lemma 3 *We have*

$$S_{uv}(\mathbf{c}) = S_u(\mathbf{c})S_v(\mathbf{c})$$

for any coprime positive integers u and v.

Lemma 4 *If $(k,u)=1$ then $S_u(k\mathbf{c}) = S_u(\mathbf{c})$.*

The proofs are trivial, and we omit them.

Lemma 3 shows that it suffices to estimate $S_q(\mathbf{c})$ when q is a prime power, and we begin by examining the case in which q is prime. The sums in question have already been investigated by the author [4; Lemmas 11 & 12]. We state the results as follows.

Lemma 5 *If $p \nmid G(\mathbf{c})$ then*

$$S_p(\mathbf{c}) \ll p^{(n+1)/2}.$$

In general we have

$$S_p(\mathbf{c}) \ll p^{(n+2)/2}.$$

The polynomial $G(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$ which occurs here was taken to be an irreducible form for which

$$F(\mathbf{x}) | G(\nabla F(\mathbf{x})). \tag{19}$$

However, since F is diagonal in our case, given by (1), it is clear that we may take

$$G(\mathbf{x}) = (\prod_i F_i)^{2^{n-2}} \prod \{(F_1^{-1}X_1^3)^{1/2} \pm (F_2^{-1}X_2^3)^{1/2} \pm \ldots \pm (F_n^{-1}X_n^3)^{1/2}\}, \tag{20}$$

where the $\pm$ signs run over all 2^{n-1} possible combinations. In particular we see that $G(\mathbf{x})$ has degree $2^{n-1} \times 3$, and is irreducible providing that $n \geq 3$.

The estimates of Lemma 5 are in fact a simple consequence of Deligne's bounds for exponential sums [1]. Note that the results were initially established under the assumption that $F(\mathbf{x})$ is non-singular modulo p. However, this latter condition holds for all but finitely many primes p, depending only on the coefficients F_i. These finitely many primes may be catered for by adjusting the order constant appropriately.

To handle prime power moduli one of our basic tools is the following.

Lemma 6 *If $t \geq 2$ then $S_{p^t}(\mathbf{c}) = 0$ unless $p | G(\mathbf{c})$.*

This is an immediate consequence of Heath-Brown [5; Lemma 24] which states that

$$S_{p^t}(\mathbf{c}) = p^{s(n+1)} \sum_{d(\text{mod } p^{t-s})}^{*} \quad \sum_{\mathbf{x}(\text{mod } p^{t-s})}^{(1)} e_{p^t}(dF(\mathbf{x}) + \mathbf{x}.\mathbf{c}), \tag{21}$$

where $t \geq 2,\ s = [t/2]$ and $\Sigma^{(1)}$ indicates the conditions

$$p^s | F(\mathbf{x}) \quad \text{and} \quad p^s | d\nabla F(\mathbf{x}) + \mathbf{c}.$$

The sum $\Sigma^{(1)}$ will therefore be empty unless $p|G(\mathbf{c})$, in view of (19).

Lemma 5 provides good bounds for individual values of $S_q(\mathbf{c})$ when q is square-free. However these alone are insufficient for our purposes, and we therefore examine the possibility of cancellations occuring in sums of the form $\sum_q S_q(\mathbf{c})$. Such sums are intimately connected to the Hasse-Weil L-functions, as we proceed to show. The theory here is somewhat simpler when $G(\mathbf{c}) \neq 0$, as we henceforth suppose.

Let $\mathcal{V}$ and $\mathcal{V}(\mathbf{c})$ denote the projective varieties defined over $\mathbb{C}$ by the equations $F(\mathbf{x}) = 0$ and $F(\mathbf{x}) = \mathbf{c}.\mathbf{x} = 0$ respectively, and let $\mathcal{V}(p)$ and $\mathcal{V}(\mathbf{c}; p)$ denote the correponding varieties over $\mathbf{F}_p$. We now define $\rho(p^r)$ and $\rho(\mathbf{c}; p^r)$ to be the number of points of $\mathcal{V}(p)$ and $\mathcal{V}(\mathbf{c}; p)$ that have coordinates in $\mathbf{F}_{p^r}$. Then, as in (47) of Hooley's paper [8] we have

$$S_p(\mathbf{c}) = p\{pE(\mathbf{c}; p) - E(p)\},$$

for $p \nmid G(\mathbf{c})$, where

$$E(\mathbf{c}; p^r) = \rho(\mathbf{c}; p^r) - \frac{p^{(n-2)r} - 1}{p - 1}, \quad E(p^r) = \rho(p^r) - \frac{p^{(n-1)r} - 1}{p - 1}.$$

It is an easy exercise involving Gauss sums to show that

$$E(p) \ll p^{(n-2)/2}. \tag{22}$$

Since this is sharp enough for our applications we now focus our attention on the term $E(\mathbf{c}; p)$. We note at once that a trivial bound yields $\rho(\mathbf{c}; p^r) \ll p^{(n-1)r}$, whence also

$$E(\mathbf{c}; p^r) \ll p^{(n-1)r}. \tag{23}$$

We begin by defining the local L-function for $p \nmid G(\mathbf{c})$ by

$$L_p(\mathbf{c}; s) = \exp\{-\sum_{r=1}^{\infty} r^{-1} E(\mathbf{c}; p^r) p^{-rs}\},$$

This is the quotient of the zeta-function of projective $(n-2)$-space by that of $\mathcal{V}(\mathbf{c}; p)$. When $p|G(\mathbf{c})$ the corresponding local L-function is more difficult to define, but according to Serre [11] it takes the form

$$L_p(\mathbf{c}; s) = \prod_j (1 - \lambda_{j,p} p^{-s})^{-1},$$

where the coefficients satisfy

$$1 \le |\lambda_{j,p}| \le p^{(n-3)/2}.$$

Moroever the number of factors is bounded in terms of n, there being at most 2 for $n=4$ and at most 10 for $n=6$. We now set

$$L(\mathbf{c};s) = \prod_p L_p(\mathbf{c};s),$$

this being the Hasse-Weil L-function for the variety $\mathcal{V}(\mathbf{c})$. It is immediate from (23) that the product over primes is convergent, and hence that the function $L(\mathbf{c};s)$ defines a holomorphic function, in the region $\sigma > n$. When $n=4$ the function $L(\mathbf{c};s)$ is the usual L-function of the Jacobian of $\mathcal{V}(\mathbf{c})$. Thus $L(\mathbf{c};s)$ is the L-function of an elliptic curve. It should be noted that the above definitions need some slight modification when n is odd, but this case does not concern us.

Associated to $\mathcal{V}(\mathbf{c})$ there is a conductor

$$B(\mathbf{c}) = \prod_{p|G(\mathbf{c})} p^{a_p},$$

in which the exponents a_p are non-negative integers, bounded in terms of n. We now define

$$\xi(\mathbf{c};s) = (2\pi)^{-s}\Gamma(s)B(\mathbf{c})^{s/2}L(\mathbf{c};s)$$

for $n=4$, and

$$\xi(\mathbf{c};s) = (2\pi)^{-5s}\Gamma(s-1)^5B(\mathbf{c})^{s/2}L(\mathbf{c};s)$$

for $n=6$.

We are at last in a position to state the hypothesis HW_n, this being the particular case relevent to us of the general conjecture given by Serre [11].

Hypothesis $\mathbf{HW}_n$ *Assume that $G(\mathbf{c}) \ne 0$, and that $n=4$ or 6. Then*

1. *$\xi(\mathbf{c};s)$ has a meromorphic extension to $\mathbb{C}$, with its only possible poles being at $s=5/2$ or $3/2$ in case $n=6$. Moreover $\xi(\mathbf{c};s)$ has finite order.*

2. *There is a functional equation*

$$\xi(\mathbf{c};s) = W(\mathbf{c})\xi(\mathbf{c};n-2-s)$$

with $W(\mathbf{c}) = \pm 1$.

3. *All zeros of $\xi(\mathbf{c};s)$ lie on $\sigma = (n-2)/2$.*

It should be remarked that parts 1 and 2 of Hypothesis HW_4 have been proved by Wiles [14] in some important cases. It would be interesting to know whether Theorem 2 can be established subject only to part 3 of the hypothesis.

We may now proceed exactly as in Hooley [8; pp 73-75] to deduce the following estimate, which corresponds precisely to [8; Lemma 10].

Lemma 7 *Assume Hypothesis HW_n, and let $\varepsilon > 0$. Then*

$$\sum_{\substack{q\le y\\ (q,G(\mathbf{c}))=1}} q^{-(n+1)/2}S_q(\mathbf{c}) \ll_\varepsilon |\mathbf{c}|^\varepsilon y^{1/2+\varepsilon}.$$

Note that the lemma is trivial when $G(\mathbf{c}) = 0$, and that only square-free values of q will be counted, by virtue of Lemma 6.

5 Averages of $S_q(\mathbf{c})$ for square-full q

In order to describe the average of $S_q(\mathbf{c})$ which we shall consider in this section we must introduce a little notation. We select a non-empty set $\mathcal{T}$ of indices $i \in \{1, \ldots, n\}$ and we put $t = \#\mathcal{T}$. For each $i \in \mathcal{T}$ we choose a positive number C_i. We then consider the set $\mathcal{R}$ of vectors $\mathbf{c}$ for which $C_i < |c_i| \le 2C_i$, for $i \in \mathcal{T}$, and $c_i = 0$ for all other i. Our aim is to estimate

$$A = \sum_{\mathbf{c}\in\mathcal{R},G(\mathbf{c})\ne 0}\ \sum_{X<q\le 2X} |S_q(\mathbf{c})|,$$

the sum over q being restricted to square-full moduli.

We begin by recording the estimates for $S_q(\mathbf{c})$ which we shall use. In the first place, we have the multiplicative property given by Lemma 3. We also have

$$|S_{p^k}(\mathbf{c})| \le \sideset{}{^*}\sum_{a(\text{mod }p^k)} \prod_{i=1}^n \Big|\sum_{b(\text{mod }p^k)} e_{p^k}(aF_ib^3 + c_ib)\Big|.$$

The inner sum here has been thoroughly investigated by Hooley [8; pages 67 & 68]. To summarize his conclusions we define $\{p^k, c\} = 1$ for $k = 2$, and if $k \ge 3$ we set $\{p^k, c\} = 0$ if $p||c$, and $\{p^k, c\} = (p^k, c)$ otherwise. We extend the definition to square-full q by setting

$$\{q, c\} = \prod_{p^k||q} \{p^k, c\}.$$

Hooley's results then show that

$$\sum_{b(\text{mod }p^k)} e_{p^k}(aF_ib^3 + cb) \ll p^{k/2}\{p^k, c\}^{1/4} \tag{24}$$

for $k \ge 2$. We shall also use the estimate

$$\sum_{b(\text{mod }p^k)} e_{p^k}(aF_ib^3 + cb) \ll p^{2k/3}, \tag{25}$$

due to Hua [10]. In fact both (24) and (25) were originally established under the assumption that $p \nmid F_i$. To handle the alternative case we suppose that $p^j || F_i$, and we assume that $k > j$, since (24) and (25) are trivial otherwise. We then set $b = b_1 + p^{k-j} b_2$, with b_1 and b_2 running modulo p^{k-j} and p^j respectively. It follows that

$$\sum_{b (\bmod p^k)} e_{p^k}(aF_i b^3 + cb) = \sum_{b_1 (\bmod p^{k-j})} e_{p^k}(aF_i b_1^3 + cb_1) \sum_{b_2 (\bmod p^j)} e_{p^j}(cb_2).$$

The sum therefore vanishes unless $p^j | c$. In this latter case we get

$$p^j \sum_{b_1 (\bmod p^{k-j})} e_{p^{k-j}}(aF_i p^{-j} b_1^3 + cp^{-j} b_1)$$

and the required estimates then follow from the case $p \nmid F_i$.

As a corollary of (24) and (25) we obtain the following bound.

Lemma 8 *We have*

$$S_{p^2}(\mathbf{c}) \ll p^{2+n}.$$

Moreover, if the highest common factor of p^k and $c_1, \ldots, c_n$ is H_p, and there are at least m indices i for which $(p^k, c_i) = H_p$, then

$$S_{p^k}(\mathbf{c}) \ll p^{k+2(n-m)k/3+mk/2} H_p^{m/4}.$$

For a general square-full q we shall write

$$q = q_* \prod_{i \in \mathcal{T}} q_i$$

with the various factors defined as follows. We take q_* to be the product of those prime powers $p^k || q$ for which either $k = 2$ or $p \nmid c_i$ whenever $i \in \mathcal{T}$. The remaining factors q_i are defined as the products of those prime powers $p^k || q$ for which $p | c_i$ but $p \nmid c_j$ for any $j \in \mathcal{T}$ with $j < i$. The factors q_i are thus cube-full. The way that a particular value of q is split up may, of course, depend on the value of $\mathbf{c}$ under consideration. The bounds (24) and (25) then show that

$$S_q(\mathbf{c}) \ll q^{1+n/2+(n-t)/6+\varepsilon} \prod_{i,j \in \mathcal{T}} \{q_i, c_j\}^{1/4}.$$

However, when $k \geq 2$, we see from Lemma 6 that $S_{p^k}(\mathbf{c}) = 0$ unless $p | G(\mathbf{c})$. We therefore have

$$S_q(\mathbf{c}) \ll \eta(q, \mathbf{c}) q^{1+n/2+(n-t)/6+\varepsilon} \prod_{i,j \in \mathcal{T}} \{q_i, c_j\}^{1/4},$$

where $\eta(q, \mathbf{c}) = 1$ if $p | G(\mathbf{c})$ for each prime $p | q_*$, and $\eta(q, \mathbf{c}) = 0$ otherwise.

We now split the available ranges for q_* and the q_i into intervals

$$q_* \in (X_* \, , \, 2X_*], \qquad q_i \in (X_i \, , \, 2X_i].$$

If we allow X_* and the X_i to run over powers of 2 there will be $O((\log X)^{t+1})$ sets of intervals. Thus

$$A \ll X^{1+n/2+(n-t)/6+2\varepsilon} \sum_{\mathbf{c}\in\mathcal{R}, G(\mathbf{c})\neq 0} \sum_{X_i<q_i\le 2X_i} \prod_{i,j\in\mathcal{T}} \{q_i, c_j\}^{1/4} S_{\mathbf{c}},$$

where

$$S_{\mathbf{c}} = \sum_{X_*<q_*\le 2X_*} \eta(q, \mathbf{c}).$$

Now for a given value of $G \neq 0$, there are $O((N|G|)^{\varepsilon})$ possible $n \le N$ for which $p|G$ for every prime factor of n. To see this one merely notes that the number of such n is at most

$$\sum_{p|n\Rightarrow p|G} (N/n)^{\varepsilon} = N^{\varepsilon} \prod_{p|G} (1-p^{-\varepsilon})^{-1} \le N^{\varepsilon} C_{\varepsilon}^{\omega(|G|)} \ll_{\varepsilon} (N|G|)^{\varepsilon},$$

where $C_{\varepsilon} = (1-2^{-\varepsilon})^{-1}$. It follows that

$$\sum_{X_*<q_*\le 2X_*} \eta(q, \mathbf{c}) \ll (XC^D)^{\varepsilon},$$

where D is the degree of the form G and $C = \max C_i$. We now deduce, with a new value of ε, that

$$A \ll X^{1+n/2+(n-t)/6} (XC)^{\varepsilon} \sum_{\mathbf{c}\in\mathcal{R}} \sum_{X_i<q_i\le 2X_i} \prod_{i,j\in\mathcal{T}} \{q_i, c_j\}^{1/4}.$$

At this point we remove the restrictions imposed on the q_i by their original definition, and suppose merely that they are coprime and cube-full and that $p|c_i$ for all primes $p|q_i$. We may then factorize the expression on the right above, to yield

$$A \ll X^{1+n/2+(n-t)/6} (XC)^{\varepsilon} \sum_{X_i<q_i\le 2X_i} \prod_{j\in\mathcal{T}} S(j), \tag{26}$$

with

$$S(j) = \sum_{C_j<c_j\le 2C_j} \prod_{i\in\mathcal{T}} \{q_i, c_j\}^{1/4}.$$

We put $r = \prod p^2$, the product being over primes $p|q_j$. Then c_j only contributes when $r|c_j$, in which case

$$\{q_j, c_j\} = r(q_j/r, c_j/r).$$

It follows that

$$S(j) \le r^{1/4} \sum_{C_j/r<d\le 2C_j/r} (q_0, d)^{1/4},$$

where $q_0 = r^{-1}\prod q_i$. However

$$\sum_{d\le D} (\kappa, d) \le \kappa^{\varepsilon} D$$

in general, whence $S(j) \ll r^{-3/4} C_j X^{\varepsilon}$. We therefore see from (26) that

$$A \ll X^{1+n/2+(n-t)/6}(XC)^{\varepsilon} \sum_{X_i<q_i\le 2X_i} R^{-3/4} X^{n\varepsilon} \#\mathcal{R},$$

where $R = \prod p^2$, the product being over primes $p | \prod q_i$. However if, for a general q, we take r to be the product of p^2 for each $p|r$, then

$$\sum_{X<q\le 2X} r^{-3/4} \le \sum_{q=1}^{\infty} r^{-3/4} = \prod_p (1+p^{-3/2}) \ll 1.$$

We conclude that

$$A \ll X^{1+n/2+(n-t)/6}(XC)^{3\varepsilon} X^{n\varepsilon} \#\mathcal{R}.$$

On re-defining ε we may therefore summarize our analysis as follows.

Lemma 9 *Let a set $\mathcal{T}$ of $t \ge 1$ indices $i \in \{1, \ldots, n\}$ and positive numbers C_i for each $i \in \mathcal{T}$ be given. Define $\mathcal{R}$ to be the set of vectors $\mathbf{c}$ for which $C_i < |c_i| \le 2C_i$, for $i \in \mathcal{T}$, and $c_i = 0$ for all other i. Set $C = \max C_i$. Then*

$$A = \sum_{\mathbf{c}\in\mathcal{R}, G(\mathbf{c})\ne 0} \sum_{X<q\le 2X} |S_q(\mathbf{c})| \ll X^{1+n/2+(n-t)/6}(XC)^{\varepsilon} \#\mathcal{R},$$

the sum over q being restricted to square-full moduli.

6 Terms With $G(\mathbf{c}) \ne 0$

In this section we shall consider the contribution to (6) arising from terms with $G(\mathbf{c}) \ne 0$. Our goal will be to estimate the sum

$$A = \sum_{X<q\le 2X} \sum_{G(\mathbf{c})\ne 0} q^{-n} S_q(\mathbf{c}) I_q(\mathbf{c}).$$

Since $I_q(\mathbf{c}) = 0$ for $q \gg Q$, we may take $X \ll Q \ll P^{3/2}$. In view of the bound (17) the terms with $|\mathbf{c}| > P^{1/2+\varepsilon}$ make a contribution $O(1)$ which will be negligible. For the remaining terms we break up the range into sets $\mathcal{R}$, as in the previous section. Thus for each i we have either $c_i = 0$ or $C_i < |c_i| \le 2C_i$. There will be $O((\log P)^n)$ such subsets $\mathcal{R}$. As before we write $\mathcal{T}$ for the set of indices for which $C_i \neq 0$, and set $t = \#\mathcal{T}$, so that $t \ge 1$. Moreover we set $C = \max C_i$.

We proceed by factoring q into two coprime factors as $q = q_1 q_2$, with q_1 square-free and q_2 square-full. Thus Lemma 3 yields

$$A \ll (\log P)^n |\sum_{q_2} \sum_{\mathbf{c}\in\mathcal{R},\ G(\mathbf{c})\neq 0} q_2^{-n} S_{q_2}(\mathbf{c}) \sum_{q_1} q_1^{-n} S_{q_1}(\mathbf{c}) I_q(\mathbf{c})|.$$

Since $G(\mathbf{c}) \neq 0$, we can estimate the inner sum using partial summation based on Lemmas 2 and 7. This gives

$$\sum_{y<q_1\le 2y} q_1^{-n} S_{q_1}(\mathbf{c}) I_q(\mathbf{c})$$

$$\ll |\mathbf{c}|^{\varepsilon} y^{1-n/2+\varepsilon} \frac{P|\mathbf{c}|}{X} P^{n+\varepsilon} \prod_{i=1}^{n} \min\{(\frac{X}{P|c_i|})^{1/2}, (\frac{X}{P|\mathbf{c}|})^{1/4}\}.$$

Taking $y = X/q_2$ and redefining ε, this leads to

$$A \ll P^{1+n+\varepsilon} X^{-n/2} C(\frac{X}{PC})^{(n-t)/4} \prod_{i\in\mathcal{T}} \min\{(\frac{X}{PC_i})^{1/2}, (\frac{X}{PC})^{1/4}\} B(\mathcal{R}),$$

with

$$B(\mathcal{R}) = \sum_{q_2\le 2X} q_2^{-1-n/2} \sum_{\mathbf{c}\in\mathcal{R},\ G(\mathbf{c})\neq 0} |S_{q_2}(\mathbf{c})|.$$

To estimate $B(\mathcal{R})$ we divide the range $q_2 \le 2X$ into $O(\log X)$ subintervals $Y < q_2 \le 2Y$. Thus, for some such $\mathcal{R}$ and Y we have

$$B(\mathcal{R}) \ll P^{\varepsilon} Y^{-1-n/2} S(\mathcal{R}, Y),$$

where

$$S(\mathcal{R}, Y) = \sum_{\mathbf{c}\in\mathcal{R},\ G(\mathbf{c})\neq 0} \sum_{Y<q_2\le 2Y} |S_{q_2}(\mathbf{c})|.$$

The sum $S(\mathcal{R}, Y)$ is in precisely the correct form for Lemma 9 to be applied, and we deduce that

$$B(\mathcal{R}) \ll P^{4\varepsilon} Y^{(n-t)/6} (\#\mathcal{R}).$$

Since $\#\mathcal{R} \ll \prod_{i\in\mathcal{T}} C_i$, it follows that

$$A \ll P^{1+n+5\varepsilon} X^{-n/2} Y^{(n-t)/6} C(\frac{X}{PC})^{(n-t)/4} \Pi,$$

with

$$\Pi = \prod_{i\in\mathcal{T}} \min\{(\frac{XC_i}{P})^{1/2}, C_i(\frac{X}{PC})^{1/4}\}.$$

However $C_i \le C$, so that

$$\Pi \le C^t (\frac{X}{PC})^{t/4} \min\{\frac{X}{PC}, 1\}^{t/4},$$

which leads to the bound

$$A \ll P^{1+n+5\varepsilon} X^{-n/2+(n-t)/6} C^{1+t} (\frac{X}{PC})^{n/4} \min\{\frac{X}{PC}, 1\}^{t/4}$$

on observing that $Y \ll X$. The expression on the right must be maximal either at $t = 0$ or at $t = n$. In the former case we have

$$A \ll P^{1+3n/4+5\varepsilon} X^{-n/12} C^{1-n/4} \ll P^{1+3n/4+5\varepsilon},$$

while in the latter we conclude that

$$\begin{aligned} A &\ll P^{1+3n/4+5\varepsilon} X^{-n/4} C^{1+3n/4} \min\{\frac{X}{PC}, 1\}^{n/4} \\ &\ll P^{1+3n/4+5\varepsilon} X^{-n/4} C^{1+3n/4} (\frac{X}{PC})^{n/4} \\ &= P^{1+n/2+5\varepsilon} C^{1+n/2}. \end{aligned}$$

Since $C \ll P^{1/2+\varepsilon}$, we conclude, on redefining ε, that

$$A \ll P^{3/2+3n/4+\varepsilon},$$

in either case.

On combining the possible ranges of q, which number $O(\log P)$, we can now summarize the results of this section as follows.

Lemma 10 *Suppose that Hypothesis HW_n holds. Then for any $\varepsilon > 0$ we have*

$$\sum_{q=1}^{\infty} \sum_{\mathbf{c} \in \mathbb{Z}^n,\ G(\mathbf{c}) \ne 0} q^{-n} S_q(\mathbf{c}) I_q(\mathbf{c}) \ll P^{3/2+3n/4+\varepsilon}.$$

7 Terms With $G(\mathbf{c}) = 0$

In this section we shall consider the contribution to (6) arising from terms with $G(\mathbf{c}) = 0$. From (20) we see that if $G(\mathbf{c}) = 0$ for an integer vector $\mathbf{c}$, then

$$(F_1^{-1} c_1^3)^{1/2} + \ldots + (F_n^{-1} c_n^3)^{1/2} = 0,$$

with a suitable choice of signs for the square-roots. We partition the indices $1, \ldots, n$ into subsets $\mathcal{I}(k)$ according to the square-free part, m_k say, of $F_i c_i^3$.

(If $c_i = 0$ we take $m_k = 1$.) It follows that there are integers d_i such that $F_i c_i^3 = m_k d_i^2$ for $i \in \mathcal{I}(k)$, and

$$\sum_{i \in \mathcal{I}(k)} F_i^{-1} d_i = 0$$

for each set $\mathcal{I}(k)$. Since $c_i^2 | m_k d_i^2$ it follows that $c_i | d_i$. Thus on writing $d_i = c_i e_i$ we see that

$$c_i = m_k F_i^{-1} e_i^2 \quad (i \in \mathcal{I}(k)),$$

and

$$\sum_{i \in \mathcal{I}(k)} F_i (\frac{e_i}{F_i})^3 = 0. \tag{27}$$

We proceed to count how many solutions of $G(\mathbf{c}) = 0$ can lie in the region $|\mathbf{c}| \le C$. This will entail estimating the number of solutions of (27) for which $|e_i| \le E$, say, where $E \ll \sqrt{C/|m_k|}$. When $\#\mathcal{I}(k) = 1$ the equation (27) implies that $e_i = 0$. For $2 \le \#\mathcal{I}(k) \le 4$ we write the number of solutions of (27) as

$$\int_0^1 \prod_{i \in \mathcal{I}(k)} \{ \sum_{m \le E} e(\alpha F_i' m^3) \} d\alpha,$$

where $F_i' = F_i^{-1} \prod_{j \in \mathcal{I}(k)} F_j$. On applying Hölder's inequality, together with the bound

$$\int_0^1 | \sum_{m \le E} e(\alpha F' m^3)|^4 d\alpha \ll E^2,$$

we deduce that (27) has $O(E^2)$ solutions with $|e_i| \le E$. When $\#\mathcal{I}(k) = 5$ or 6 we set $\mathcal{I}(k) = r$ and use a similar argument, based on the bound

$$\int_0^1 | \sum_{m \le E} e(\alpha F' m^3)|^r d\alpha \ll E^{r-2},$$

to show that there are $O(E^{r-2})$ solutions. These bounds are, of course, very weak, but they suffice for our purposes.

We now see that the number N, say, of solutions of $G(\mathbf{c}) = 0$ with $|\mathbf{c}| \le C$, corresponding to a given partition of the indices $1, \ldots, n$ into sets $\mathcal{I}(k)$, is

$$\ll \sum_{m_k} \prod_k (\frac{C}{|m_k|})^{e_k/2},$$

where we take

$$e_k = \begin{cases} 0, & \#\mathcal{I}(k) = 1, \\ 2, & 2 \le \#\mathcal{I}(k) \le 4, \\ 3, & \#\mathcal{I}(k) = 5, \\ 4, & \#\mathcal{I}(k) = 6. \end{cases}$$

Since $m_k = 1$ whenever $\#\mathcal{I}(k) = 1$ we deduce, on summing over admissable values of $m_k \ll C$, that

$$N \ll \prod_k C^{e_k/2+\varepsilon}.$$

Moreover, on considering the possible partitions of the indices $1, \ldots, n$ we find that $N \ll_\varepsilon C^{3+\varepsilon}$ for $n = 6$, with a new value of ε.

The case $n = 4$ requires slightly more care. Here the above argument shows that $N \ll_\varepsilon C^{1+\varepsilon}$ except when there are exactly two sets $\mathcal{I}(k)$, each of cardinality 2. We call a solution of $G(\mathbf{c}) = 0$, 'special', if none of the c_i are zero, and there are exactly two pairs of indices (i, j) with $i < j$, for which

$$(F_i^{-1}c_i^3)^{1/2} + (F_j^{-1}c_j^3)^{1/2} = 0, \tag{28}$$

with a suitable choice of signs for the square-roots. Otherwise we shall call the solution 'ordinary.' Suppose now that $\mathbf{c}$ is an ordinary solution of $G(\mathbf{c})$ for which (28) holds. Then

$$(F_k^{-1}c_k^3)^{1/2} + (F_l^{-1}c_l^3)^{1/2} = 0$$

for the other two indices k, l. If c_i, say, is zero, then c_j is also zero, and c_k determines c_l. If (28) holds for more than two pairs of indices, the numbers $F_i^{-1}c_i^3$ must all be the same, so that c_1, say determines the remaining c_i. It follows that there are $O(C)$ ordinary solutions of $G(\mathbf{c})$ for which (28) holds.

We can now summarize our conclusions as follows.

Lemma 11 *When $n = 6$ the equation $G(\mathbf{c}) = 0$ has $O_\varepsilon(C^{3+\varepsilon})$ solutions with $|\mathbf{c}| \le C$. For $n = 4$ there are $O_\varepsilon(C^{1+\varepsilon})$ ordinary solutions.*

We proceed to estimate the sum

$$A = \sum_{X<q\le 2X} \sum_{C<|\mathbf{c}|\le 2C,\ G(\mathbf{c})=0} q^{-n} S_q(\mathbf{c}) I_q(\mathbf{c}),$$

where the sum over $\mathbf{c}$ is restricted to ordinary solutions of $G(\mathbf{c}) = 0$ when $n = 4$. In view of the bound (17) the sum is negligible if $C > P^{1/2+\varepsilon}$. We therefore suppose henceforth that $C \le P^{1/2+\varepsilon}$. If we write D for the degree of the form $G(\mathbf{x})$ we see that $G(\mathbf{x})$ contains monomials $G_i x_i^D$ for every i. It follows that if $G(\mathbf{c}) = 0$ then there must be at least two indices i for which $|c_i| \gg C$. Lemma 2 therefore yields

$$I_q(\mathbf{c}) \ll_\varepsilon \frac{P|\mathbf{c}|}{q} P^{n+\varepsilon} \prod_{i=1}^n \min\{(\frac{q}{P|c_i|})^{1/2}, (\frac{q}{P|\mathbf{c}|})^{1/4}\} \ll P^{n+\varepsilon}(\frac{X}{PC})^{(n-2)/4}. \tag{29}$$

To handle the sum $S_q(\mathbf{c})$ we factor q into coprime factors as $q = q_1 q_2 q_3$ where q_1 is cube-free, q_2 is cube-full, and q_3 is the product of prime powers

$p^e||q$, for which $p|3\prod F_i$. We split the available ranges for the factors q_i into ranges $X_i < q_i \le 2X_i$, and deduce from (29) that

$$A \ll_\varepsilon P^{n+2\varepsilon} X^{-n} (\frac{X}{PC})^{(n-2)/4} \sum_{X_i < q_i \le 2X_i} \sum_{C<|\mathbf{c}|\le 2C,\ G(\mathbf{c})=0} |S_q(\mathbf{c})|,$$

for a suitable set of ranges with $X \ll \prod X_i \ll X$.

Now if $p^k||q_2$ and we write H_p for the highest common factor of p^k and $c_1, \ldots, c_n$, as in Lemma 8, then we will have $\mathbf{c} = H_p\mathbf{c}'$, say for an appropriate integer vector $\mathbf{c}'$. However, if D is the degree of the form $G(\mathbf{x})$, then one sees that $G(\mathbf{x})$ contains monomials $G_i X_i^D$, where any prime factor of the coefficient G_i must divide $3\prod F_i$. If $p|q_2$ and $H_p \ne p^k$ it therefore follows from the fact that $G(\mathbf{c}') = 0$ that at least two of the c_i' must be coprime to p. On the other hand, if $H_p = p^k$ then $(p^k, c_i) = H_p$ for every value of i. In the notation of Lemma 8 we may then take $m = 2$, whatever the value of H_p, giving

$$S_{p^k}(\mathbf{c}) \ll p^{2k/3+2nk/3} H_p^{1/2}.$$

For $p^k||q_1$ or q_3 we find, from Lemmas 5 and 8, that

$$S_{p^k}(\mathbf{c}) \ll p^{k+nk/2}$$

and

$$S_{p^k}(\mathbf{c}) \ll p^{k+2nk/3}$$

respectively. These bounds may now be combined, in view of Lemma 3, to produce

$$S_q(\mathbf{c}) \ll P^\varepsilon q_1^{1+n/2} q_2^{2/3+2n/3} q_3^{1+2n/3} H^{1/2},$$

where $H = \prod H_p$.

We now observe that H takes $O_\varepsilon(P^\varepsilon)$ values for each possible q_2, each of which divides $\mathbf{c}$. It follows that there is some such H for which

$$A \ll_\varepsilon P^{n+4\varepsilon} X^{-n} X_1^{1+n/2} X_2^{2/3+2n/3} X_3^{1+2n/3} H^{1/2} (\frac{X}{PC})^{(n-2)/4} \mathcal{N}_1 \mathcal{N}_2(H).$$

where

$$\mathcal{N}_1 = \#\{(q_1, q_2, q_3) : X_i < q_i \le 2X_i\}$$

and

$$\mathcal{N}_2(H) = \#\{\mathbf{c} : C < |\mathbf{c}| \le 2C,\ H|\mathbf{c}\ ,\ G(\mathbf{c}) = 0\}.$$

Here we should recall that for $n = 4$ only ordinary solutions of $G(\mathbf{c}) = 0$ are considered. It is an easy exercise to show that

$$\mathcal{N}_1 \ll_\varepsilon X_1 X_2^{1/3} X_3^\varepsilon,$$

and Lemma 11 shows that

$$\mathcal{N}_2(H) \ll_\varepsilon (\frac{C}{H})^{n-3+\varepsilon}.$$

On combining our estimates we now see that

$$A \ll_\varepsilon P^{n+7\varepsilon} X^{-n} X_1^{2+n/2} X_2^{1+2n/3} X_3^{1+2n/3} H^{1/2} (\frac{X}{PC})^{(n-2)/4} (\frac{C}{H})^{n-3}.$$

Since $2+n/2 \geq 1+2n/3$ for $n=4$ or 6, and $n-3 \geq 1/2$, this simplifies to give

$$A \ll_\varepsilon P^{n+7\varepsilon} X^{2-n/2} (\frac{X}{PC})^{(n-2)/4} C^{n-3}.$$

The variable C is effectively restricted to the range $1 \ll C \ll P^{1/2+\varepsilon}$, and the above bound for A is clearly increasing with respect to C, since $n-3 \geq (n-2)/4$. We therefore find that

$$\begin{aligned} A &\ll_\varepsilon P^{n+10\varepsilon} X^{2-n/2} (\frac{X}{P^{3/2}})^{(n-2)/4} P^{(n-3)/2} \\ &= P^{(9n-6)/8+10\varepsilon} X^{(6-n)/4} \\ &\ll P^{3/2+3n/4+10\varepsilon}, \end{aligned}$$

since $X \ll P^{3/2}$.

We may now summarize the results of this section by combining the possible ranges of q and $\mathbf{c}$, which number $O_\varepsilon(P^\varepsilon)$, to produce the following lemma.

Lemma 12 *For any $\varepsilon > 0$ we have*

$$\sum_{q=1}^{\infty} \sum_{\mathbf{c} \in \mathbb{Z}^n,\ G(\mathbf{c})=0} q^{-n} S_q(\mathbf{c}) I_q(\mathbf{c}) \ll_\varepsilon P^{3/2+3n/4+\varepsilon},$$

where the sum over $\mathbf{c}$ *is for non-zero vectors, and is restricted to ordinary solutions of $G(\mathbf{c}) = 0$ for $n = 4$.*

We conclude with a simple treatment of the case $\mathbf{c} = \mathbf{0}$. We have $I_q(\mathbf{0}) \ll P^n$, as in (18). Moreover if we take $m = 0$ in Lemma 8 we obtain

$$S_q(\mathbf{0}) \ll P^\varepsilon q_1^{1+n/2} q_2^{1+2n/3},$$

where $q = q_1 q_2$ with coprime factors q_1, q_2 which are cube-free and cube-full respectively. Now

$$\begin{aligned} \sum_{q=1}^{\infty} q^{-n} S_q(\mathbf{0}) I_q(\mathbf{0}) &\ll P^{n+\varepsilon} \sum_{q \ll Q} q^{-n} q_1^{1+n/2} q_2^{1+2n/3} \\ &\ll P^{n+\varepsilon} \sum_{q_1 \ll Q} q_1^{1-n/2} \sum_{q_2 \ll Q/q_1} q_2^{1-n/3} \\ &\ll P^{n+2\varepsilon} \sum_{q_1 \ll Q} q_1^{1-n/2} \\ &\ll P^{n+3\varepsilon}, \end{aligned}$$

This contribution is thus $O_\varepsilon(P^{3/2+3n/4+\varepsilon})$ as in Lemma 12.

We now see that Theorem 1 follows from Lemmas 10 and 12, by virtue of (6).

8 The case $n = 4$: Points on Rational Lines

In order to eliminate points that lie on rational lines we shall show how they correspond to special solutions of $G(\mathbf{c}) = 0$. We shall in fact prove the following result.

Lemma 13 *For any $\varepsilon > 0$ we have*

$$c_Q^{-1}P^{-3}\sum_{q=1}^{\infty}\sum_{\mathbf{c}}{}^{\text{spec}}\, q^{-4}S_q(\mathbf{c})I_q(\mathbf{c}) = \sum_{\mathbf{x}}{}^{\text{line}}\, w(P^{-1}\mathbf{x}) + O_\varepsilon(P^{3/2+\varepsilon}), \qquad (30)$$

where the sum over $\mathbf{c}$ is for special solutions of $G(\mathbf{c}) = 0$ for which

$$(F_1^{-1}c_1^3)^{1/2} \pm (F_2^{-1}c_2^3)^{1/2} = (F_3^{-1}c_3^3)^{1/2} \pm (F_4^{-1}c_4^3)^{1/2} = 0, \qquad (31)$$

and the sum over $\mathbf{x}$ is for integral solutions of

$$F_1x_1^3 + F_2x_2^3 = F_3x_3^3 + F_4x_4^3 = 0.$$

This result shows that the contribution to (6) arising from special solutions $\mathbf{c}$ does indeed correspond to the contribution to $N(F, w)$ from points on rational lines. Thus Lemma 13, in conjunction with Lemmas 10 and 12, completes the proof of Theorem 2.

We begin the proof of Lemma 13 by showing that solutions $\mathbf{c}$ of (31) for which one or more of the c_i are zero may be included on the left hand side of (30). To do this we note that there are $O(P^{1/2+\varepsilon})$ such vectors $\mathbf{c}$ in the region $|\mathbf{c}| \le P^{1/2+\varepsilon}$, and any larger values of $\mathbf{c}$ make a negligible contribution, by (17). In view of (18) it therefore remains to show that

$$\sum_{q \ll Q} q^{-4}|S_q(\mathbf{c})| \ll P^{\varepsilon}$$

uniformly in $\mathbf{c}$. However Lemmas 5 and 8, with $m = 0$, yield

$$S_q(\mathbf{c}) \ll q^{\varepsilon}q_1^3q_2^{11/3}$$

for $n = 4$, where q_1 is the cube-free part of q and q_2 is the cube-full part. We conclude that

$$\sum_{q \ll Q} q^{-4}|S_q(\mathbf{c})| \ll P^{2\varepsilon}\{\sum_{q_1 \ll Q} q_1^{-1}\}\{\sum_{q_2 \ll Q} q_2^{-1/3}\},$$

which is sufficient, since each of the sums on the right is $O(P^{\varepsilon})$.

We now observe that if one or other of F_1/F_2 or F_3/F_4 is not a rational cube, there are no special solutions $\mathbf{c}$, and $O(P)$ points $\mathbf{x}$, so that the lemma is trivial. We may therefore take

$$F_1 = \lambda\rho_1^3, \;\; F_2 = \lambda\rho_2^3, \;\; F_3 = \mu\rho_3^3, \;\; F_4 = \mu\rho_4^3,$$

with $(\rho_1, \rho_2) = (\rho_3, \rho_4) = 1$, and set

$$c_1 = \rho_1 r_1, \;\; c_2 = \rho_2 r_1, \;\; c_3 = \rho_3 r_2, \;\; c_4 = \rho_4 r_2.$$

It is also natural to make a unimodular integer change of variables

$$y_1 = \rho_1 x_1 + \rho_2 x_2, \quad y_2 = \rho_3 x_3 + \rho_4 x_4 \tag{32}$$

and

$$z_1 = \rho_1' x_1 + \rho_2' x_2, \quad z_2 = \rho_3' x_3 + \rho_4' x_4, \tag{33}$$

so that

$$F(\mathbf{x}) = y_1 Q_1(y_1, z_1) + y_2 Q_2(y_2, z_2) = F^{(*)}(\mathbf{y}, \mathbf{z}),$$

say. A simple calculation shows that

$$Q_1(y, z) = \frac{1}{4}\lambda(y^2 + 3\{2\rho_1\rho_2 z - (\rho_1\rho_2' + \rho_1'\rho_2)y\}^2), \tag{34}$$

and similarly for Q_2.

We proceed to examine

$$\sum_{\mathbf{c}}{}^{\text{spec}} S_q(\mathbf{c}) I_q(\mathbf{c}).$$

In the above notation we find that

$$S_q(\mathbf{c}) = \sum_{a(\text{mod } q)}{}^{*} \sum_{\mathbf{g},\mathbf{h}(\text{mod } q)} e_q(aF^{(*)}(\mathbf{g}, \mathbf{h}) + \mathbf{r}.\mathbf{g}).$$

We substitute for $\mathbf{x}$ in terms of $\mathbf{y}$ and $\mathbf{z}$ in the integral $I_q(\mathbf{c})$, the Jacobian of the transformation being identically 1. We then put $\mathbf{y} = P^{-1}(\mathbf{g} + q\mathbf{v})$, whence

$$\sum_{\mathbf{c}}{}^{\text{spec}} S_q(\mathbf{c}) I_q(\mathbf{c}) = P^2 q^2 \sum_{\mathbf{g}(\text{mod } q)} \int_{\mathbb{R}^2} \{ \sum_{\mathbf{r} \in \mathbb{Z}^2} \int_{\mathbb{R}^2} f_{\mathbf{g},\mathbf{z}}(\mathbf{v}) e(-\mathbf{r}.\mathbf{v}) d\mathbf{v} \} d\mathbf{z},$$

where

$$f_{\mathbf{g},\mathbf{z}}(\mathbf{v}) = \sum_{a(\text{mod } q)}{}^{*} \sum_{\mathbf{h}(\text{mod } q)} e_q(aF^{(*)}(\mathbf{g}, \mathbf{h})) w(\mathbf{x}) h\left(\frac{q}{Q}, F^{(*)}(P^{-1}\{\mathbf{g} + q\mathbf{v}\}, \mathbf{z})\right).$$

According to the Poisson summation formula we have

$$\sum_{\mathbf{r}\in\mathbb{Z}^2}\int_{\mathbb{R}^2} f_{\mathbf{g},\mathbf{z}}(\mathbf{v})e(-\mathbf{r}.\mathbf{v})d\mathbf{v} = \sum_{\mathbf{s}\in\mathbb{Z}^2} f_{\mathbf{g},\mathbf{z}}(\mathbf{s}).$$

However, if we write $\mathbf{j} = \mathbf{g} + q\mathbf{s}$, we find that

$$f_{\mathbf{g},\mathbf{z}}(\mathbf{s}) = \sum_{a(\text{mod } q)}{}^{*} \sum_{\mathbf{h}(\text{mod } q)} e_q(aF^{(*)}(\mathbf{j},\mathbf{h}))w(\mathbf{x})h\left(\frac{q}{Q}, F^{(*)}(P^{-1}\mathbf{j},\mathbf{z})\right).$$

It then follows on substituting $\mathbf{z} = P^{-1}\mathbf{t}$ that

$$\sum_{\mathbf{c}}{}^{\text{spec}} S_q(\mathbf{c})I_q(\mathbf{c}) = q^2 \sum_{\mathbf{j}\in\mathbb{Z}^2} T_q(\mathbf{j})J_q(\mathbf{j}),$$

with

$$T_q(\mathbf{j}) = \sum_{a(\text{mod } q)}{}^{*} \sum_{\mathbf{h}(\text{mod } q)} e_q(aF^{(*)}(\mathbf{j},\mathbf{h}))$$

and

$$J_q(\mathbf{j}) = \int_{\mathbb{R}^2} w(P^{-1}\mathbf{x}(\mathbf{j},\mathbf{t}))h(\frac{q}{Q}, \frac{F^{(*)}(\mathbf{j},\mathbf{t})}{Q^2})d\mathbf{t}.$$

Here the vector $\mathbf{x}(\mathbf{y},\mathbf{z})$ is given as the inverse of the linear transformation specified in (32) and (33). We may now conclude as follows.

Lemma 14 *We have*

$$c_Q^{-1}P^{-3}\sum_{q=1}^{\infty}\sum_{\mathbf{c}}{}^{\text{spec}} q^{-4}S_q(\mathbf{c})I_q(\mathbf{c})$$

$$= c_Q^{-1}P^{-3}\sum_{q=1}^{\infty}q^{-2}\sum_{\mathbf{j}\in\mathbb{Z}^2} T_q(\mathbf{j})J_q(\mathbf{j}) + O_\varepsilon(P^{3/2+\varepsilon}).$$

We end this section by showing how the terms with $\mathbf{j} = \mathbf{0}$ count points of the surface $F^{(*)}(\mathbf{y},\mathbf{z}) = 0$ on the line $\mathbf{y} = 0$. It will then remain to estimate the contribution from other values of $\mathbf{j}$.

It is clear from the definitions, and in particular from (8), that $T_q(\mathbf{0}) = q^2\phi(q)$ and

$$h(Q^{-1}q, 0) = \sum_{j=1}^{\infty}\frac{Q}{qj}\omega(\frac{qj}{Q}).$$

Thus

$$\sum_{q=1}^{\infty}q^{-2}T_q(\mathbf{0})h(Q^{-1}q,0) \quad = \quad Q\sum_{q,j=1}^{\infty}\frac{\phi(q)}{qj}\omega(\frac{qj}{Q})$$

$$
\begin{aligned}
&= Q\sum_{n=1}^{\infty}\sum_{q|n}\frac{\phi(q)}{n}\omega(\frac{n}{Q}) \\
&= Q\sum_{n=1}^{\infty}\omega(\frac{n}{Q}) \\
&= c_Q^{-1}Q^2,
\end{aligned}
$$

by (7). We therefore deduce that

$$c_Q^{-1}P^{-3}\sum_{q=1}^{\infty}q^{-2}T_q(\mathbf{0})J_q(\mathbf{0}) = \int_{\mathbb{R}^2} w(P^{-1}\mathbf{x}(\mathbf{0},\mathbf{t}))d\mathbf{t}. \tag{35}$$

On the other hand,

$$\sum_{\mathbf{x}}^{\text{line}} w(P^{-1}\mathbf{x}) = \sum_{\mathbf{z}\in\mathbb{Z}^2} w(P^{-1}\mathbf{x}(\mathbf{0},\mathbf{z})),$$

which is

$$\sum_{\mathbf{m}\in\mathbb{Z}^2}\int_{\mathbb{R}^2} w(P^{-1}\mathbf{x}(\mathbf{0},\mathbf{t}))e(-\mathbf{m}.\mathbf{t})d\mathbf{t},$$

by the Poisson summation formula. The integral for $\mathbf{m}=\mathbf{0}$ is exactly that occuring in (35), and the remaining ones will be dealt with by the following estimate, for which see the author's work [5; Lemma 10].

Lemma 15 *Let $W(\mathbf{t})$ be an infinitely differentiable function of compact support, and let $f(\mathbf{t})$ be an infinitely differentiable real valued function defined on* supp(W)*. Suppose that there is a positive real number λ, and a set $A = \{A_2, A_3, A_4, \ldots\}$ of positive real numbers such that, for all $\mathbf{t}\in$ supp(W) we have*

$$|\nabla f| \geq \lambda$$

and

$$\left|\frac{\partial^{j_1+\ldots+j_n}f(\mathbf{x})}{\partial^{j_1}t_1\ldots\partial^{j_n}t_n}\right| \leq A_j\lambda, \quad (j=j_1+\ldots+j_n\geq 2). \tag{36}$$

Then for any $N>0$ we have

$$\int W(\mathbf{t})e(f(\mathbf{t}))d\mathbf{x} \ll_{N,W,A} \lambda^{-N}.$$

In our application we take $W(\mathbf{t}) = w(\mathbf{x}(\mathbf{0},\mathbf{t}))$, $f(\mathbf{t}) = -P\mathbf{m}.\mathbf{t}$, and $N=4$. This allows us to choose $\lambda = P|\mathbf{m}|$, and $A_i = 0$ for all $i\geq 2$, so that the lemma yields

$$\int_{\mathbb{R}^2} w(P^{-1}\mathbf{x}(\mathbf{0},\mathbf{t}))e(-\mathbf{m}.\mathbf{t})d\mathbf{t} \ll P^{-2}|\mathbf{m}|^{-4}.$$

We therefore have

$$\sum_{\mathbf{m}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}}\int_{\mathbb{R}^2} w(P^{-1}\mathbf{x}(\mathbf{0},\mathbf{t}))e(-\mathbf{m}.\mathbf{t})d\mathbf{t} \ll P^{-2},$$

whence

$$\sum_{\mathbf{x}}^{\text{line}} w(P^{-1}\mathbf{x}) = \int_{\mathbb{R}^2} w(P^{-1}\mathbf{x}(\mathbf{0},\mathbf{t}))d\mathbf{t} + O(P^{-2}).$$

The sum for $\mathbf{j}=\mathbf{0}$ in Lemma 14 therefore produces the main term on the right of (30), as claimed.

9 Completion of the Argument for $n = 4$

It remains to estimate the terms in Lemma 14 for which $\mathbf{j} \neq \mathbf{0}$. We begin by examining the integral $J_q(\mathbf{j})$. Here the weight $w(P^{-1}\mathbf{x}(\mathbf{j},\mathbf{t}))$ vanishes unless $\mathbf{j},\mathbf{t} \ll P$, in which case it is $O(1)$. We proceed to estimate the function

$$m(\mathbf{j},\eta) = \text{meas}\{\mathbf{t} \ll P : \ |F^{(*)}(\mathbf{j},\mathbf{t})| \ll \eta\}.$$

In order to do this we begin by making a linear change of variables from $\mathbf{t}$ to $\mathbf{u}$ say, replacing

$$2\rho_1\rho_2 t_1 - (\rho_1\rho_2' + \rho_1'\rho_2)j_1$$

by u_1, and similarly for t_2. According to (34) the condition $|F^{(*)}(\mathbf{j},\mathbf{t})| \ll \eta$ becomes

$$\lambda j_1\{3u_1^2 + j_1^2\} + \mu j_2\{3u_2^2 + j_2^2\} \ll \eta,$$

and since the Jacobian of our transformation is constant, of order 1, it now suffices to examine the set of $\mathbf{u} \ll P$ for which the above inequality holds. We decompose the available region for $\mathbf{u}$ into subsets of the form $U_i \le |u_i| \le 2U_i$, and consider the measure corresponding to such a subset. Since $u_1^2 = A + O(\eta/|j_1|)$ for some $A = A(j_1, j_2, u_2)$, the variable u_1 is restricted to a set of measure $O(\eta/(|j_1|U_1))$ for each u_2. This yields a bound $O(\eta U_2/(|j_1|U_1))$ for the measure corresponding to a subset $U_i \le |u_i| \le 2U_i$. We may obtain an estimate $O(\eta U_1/(|j_2|U_2))$ similarly, and on compaing the two we see that we also have a bound $O(\eta|j_1j_2|^{-1/2})$. If U_1 or U_2 is less than P^{-1} we can use the trivial bound $O(U_1U_2)$, which contributes $O(1)$ in total for all such U_1, U_2. There are $O(\log^2 P)$ pairs $U_1, U_2 \ge P^{-1}$, and hence we find that

$$m(\mathbf{j},\eta) \ll 1 + (\log P)^2\eta|j_1j_2|^{-1/2}.$$

When $j_2 = 0$, say, a similar but simpler argument shows that

$$m(\mathbf{j},\eta) \ll P\eta^{1/2}|j_1|^{-1/2}.$$

We are now ready to bound the integral $J_q(\mathbf{j})$. This will be accomplished using the estimate $h(x,y) \ll \min\{x^{-1}, x|y|^{-2}\}$ which follows from Lemma 5 of the author's work [5], on taking $m = n = 0$ and $N = 2$. We now have

$$J_q(\mathbf{j}) \ll (\log P) \max_{qQ \ll \eta \ll P^3} \frac{q}{Q} (\frac{\eta}{P^3})^{-2} m(\mathbf{j}, \eta),$$

whence

$$J_q(\mathbf{j}) \ll (\log P)^3 \frac{P^3}{\sqrt{|j_1 j_2|}} \tag{37}$$

for $j_1 j_2 \neq 0$, and

$$J_q(\mathbf{j}) \ll (\log P)^3 \frac{P^{13/4}}{\sqrt{q|j_1|}} \tag{38}$$

for $j_2 = 0$, say.

We turn now to the problem of estimating $T_q(\mathbf{j})$. It is an elementary exercise to verify that

$$T_{uv}(\mathbf{j}) = T_u(\mathbf{j}) T_v(\mathbf{j}), \quad (u, v) = 1,$$

so it suffices to consider prime power values of q. We have

$$|\sum_{\mathbf{h} (\mathrm{mod}\ q)} e_q(aF^{(*)}(\mathbf{j}, \mathbf{h}))| \leq |\sum_{h_1} e_q(aj_1 Q_1(j_1, h_1))| \times |\sum_{h_2} e_q(aj_2 Q_2(j_2, h_2))|.$$

Moreover

$$\begin{aligned} |\sum_{h_1} e_q(aj_1 Q_1(j_1, h_1))|^2 &= \sum_{h_1, h (\mathrm{mod}\ q)} e_q(aj_1\{Q_1(j_1, h + h_1) - Q_1(j_1, h_1)\}) \\ &\leq \sum_{h (\mathrm{mod}\ q)} |\sum_{h_1 (\mathrm{mod}\ q)} e_q(aj_1 h \frac{\partial}{\partial h_1} Q_1(j_1, h_1))| \\ &= \sum_{h (\mathrm{mod}\ q)} |\sum_{h_1 (\mathrm{mod}\ q)} e_q(3a\lambda \rho_1^2 \rho_2^2 j_1 h h_1)| \\ &= q\#\{h \,(\mathrm{mod}\ q) :\ q | 3a\lambda \rho_1^2 \rho_2^2 j_1 h\} \\ &\ll q(q, j_1), \end{aligned}$$

on using (34). We obtain a similar bound for the sum involving Q_2, and we deduce that

$$T_q(\mathbf{j}) \ll q^2 (q, j_1)^{1/2} (q, j_2)^{1/2}.$$

This estimate is inadequate when q is cube-free, so we investigate more carefully the cases in which q is a prime or the square of a prime. It will be enough to examine the cases $q = p$ or p^2 when $p \nmid (j_1, j_2)$. On performing the summation over a we have

$$T_p(\mathbf{j}) = p\#\{\mathbf{h} \,(\mathrm{mod}\ p) :\ p | F^{(*)}(\mathbf{j}, \mathbf{h})\} - p^2.$$

If Q is a non-singular ternary quadratic form modulo p, then $p|Q(h_1,h_2,1)$ has $p+O(1)$ solutions modulo p. It follows, in view of (34), that $T_p(\mathbf{j}) \ll p$, providing that $p\nmid j_1j_2F_0(\mathbf{j})$, where $F_0(\mathbf{j}) = \lambda j_1^3+\mu j_2^3$. Since we are assuming that $p\nmid (j_1,j_2)$ we will clearly have $T_p(\mathbf{j}) \ll p^2$ if $p|j_1j_2F_0(\mathbf{j})$.

To analyse $T_{p^2}(\mathbf{j})$ we assume that $p\nmid 6\lambda\mu\prod\rho_i$, and make the obvious change of variable to obtain

$$T_{p^2}(\mathbf{j}) = \sum_{a(\bmod\ p^2)}{}^{*}\ e_{p^2}(\frac{a}{4}F_0(\mathbf{j}))\sum_{\mathbf{k}(\bmod\ p^2)} e_{p^2}(3a\{\lambda j_1k_1^2+\mu j_2k_2^2\}).$$

We now set $\mathbf{k} = \mathbf{u}+p\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ both run modulo p. Then

$$\begin{aligned}\sum_{k_1(\bmod\ p^2)} e_{p^2}(3a\lambda j_1k_1^2) &= \sum_{u_1(\bmod\ p)} e_{p^2}(3a\lambda j_1u_1^2)\sum_{v_1(\bmod\ p)} e_p(6a\lambda j_1u_1v_1)\\ &= p\sum_{u_1(\bmod\ p):\ p|j_1u_1} e_{p^2}(3a\lambda j_1u_1^2),\end{aligned}$$

and similarly for the other factor. If $p\nmid j_1j_2$ it follows that

$$T_{p^2}(\mathbf{j}) = p^2\sum_{a(\bmod\ p^2)}{}^{*}\ e_{p^2}(\frac{a}{4}F_0(\mathbf{j})) = \left\{\begin{array}{cc} 0, & p\nmid F_0(\mathbf{j}),\\ -p^3, & p||F_0(\mathbf{j}),\\ p^4-p^3, & p^2|F_0(\mathbf{j}),\end{array}\right.$$

whence

$$T_{p^2}(\mathbf{j}) \ll p^2(p^2,F_0(\mathbf{j})).$$

On the other hand, if $p|j_1$, say, then

$$T_{p^2}(\mathbf{j}) = p^2\sum_{u_1(\bmod\ p)}\ \sum_{a(\bmod\ p^2)}{}^{*}\ e_{p^2}(a\{\frac{1}{4}F_0(\mathbf{j})+3\lambda j_1u_1^2\}).$$

The inner sum vanishes unless $p|F_0(\mathbf{j})+12\lambda j_1u_1^2$, but as $p|j_1$ this implies $p|\mu j_2^3$. It follows that $T_{p^2}(\mathbf{j})$ vanishes if p divides exactly one of j_1 and j_2. We may therefore conclude that

$$T_{p^2}(\mathbf{j}) \ll p^2(p^2,F_0(\mathbf{j}))$$

whenever $p\nmid (j_1,j_2)$.

We shall summarize our bounds for $T_q(\mathbf{j})$ as follows.

Lemma 16 *We have*

$$T_q(\mathbf{j}) \ll q^2(q,j_1)^{1/2}(q,j_2)^{1/2}$$

for any q. When $p\nmid (j_1,j_2)$ and $q=p$ or p^2 we also have

$$T_q(\mathbf{j}) \ll q(q,j_1j_2F_0(\mathbf{j})).$$

We are now ready to estimate

$$S=\sum_{q\ll Q}\sum_{\mathbf{j}\ll P}q^{-2}T_q(\mathbf{j})J_q(\mathbf{j})$$

for those integer vectors with $j_1j_2F_0(\mathbf{j})\neq 0$. In view of (37) we have

$$S\ll P^{3+\varepsilon}Q^{\sigma-2}\sum_{\mathbf{j}}\sum_{q=1}^{\infty}q^{-\sigma}|j_1j_2|^{-1/2}|T_q(\mathbf{j})|,$$

for any $\sigma>2$. We shall see in due course that the infinite sum converges for suitable σ.

For each value of $\mathbf{j}$ we define a set $S(\mathbf{j})$ by taking $q\in S(\mathbf{j})$ if $p|(j_1,j_2)$ whenever $p||q$ or $p^2||q$. Similarly we define $T(\mathbf{j})$ by taking $q\in T(\mathbf{j})$ if q is cube-free and $(q,j_1j_2)=1$. Thus every integer can be factored uniquely into coprime components as q_1q_2 with $q_1\in S(\mathbf{j})$ and $q_2\in T(\mathbf{j})$. This decomposition allows us to write

$$S\ll P^{3+\varepsilon}Q^{\sigma-2}\sum_{\mathbf{j}}|j_1j_2|^{-1/2}\{\sum_{q\in S(\mathbf{j})}q^{-\sigma}|T_q(\mathbf{j})|\}\Sigma(\mathbf{j}),$$

where

$$\Sigma(\mathbf{j})=\sum_{q\in T(\mathbf{j})}q^{-\sigma}|T_q(\mathbf{j})|.$$

We may factorize further to get

$$\Sigma(\mathbf{j})=\prod_{p\nmid(j_1,j_2)}\{1+p^{-\sigma}|T_p(\mathbf{j})|+p^{-2\sigma}|T_{p^2}(\mathbf{j})|\}.$$

For those primes $p\nmid j_1j_2F_0(\mathbf{j})$, Lemma 16 shows that the corresponding factor in the above product is $1+O(p^{1-\sigma})$. These produce a product which is $O(1)$. Primes p dividing $j_1j_2F_0(\mathbf{j})$ similarly produce factors $1+O(p^{2-\sigma})$, for $\sigma>2$. The product of these is $O(|\mathbf{j}|^{\varepsilon})$. It therefore follows that $\Sigma(\mathbf{j})\ll P^{\varepsilon}$ for fixed $\sigma>2$, so that

$$S\ll P^{3+2\varepsilon}Q^{\sigma-2}\sum_{\mathbf{j}}|j_1j_2|^{-1/2}\sum_{q\in S(\mathbf{j})}q^{-\sigma}|T_q(\mathbf{j})|.$$

To handle $q\in S(\mathbf{j})$ we write $n(q)=\prod p$ for those primes p for which $p||q$ or $p^2||q$. Thus $n(q)|\mathbf{j}$, so that $\mathbf{j}=n(q)\mathbf{k}$, say, with $\mathbf{k}\ll P/n(q)$. Using Lemma 16 we now have

$$\begin{aligned}&\sum_{\mathbf{j}}|j_1j_2|^{-1/2}\sum_{q\in S(\mathbf{j})}q^{-\sigma}|T_q(\mathbf{j})|\\ &\ll\ \sum_{q=1}^{\infty}q^{-\sigma}n(q)^{-1}\sum_{\mathbf{k}\ll P/n(q)}|T_q(n(q)\mathbf{k})|\,|k_1k_2|^{-1/2}\end{aligned}$$

$$\ll \sum_{q=1}^{\infty} q^{-\sigma} n(q)^{-1} \sum_{\mathbf{k} \ll P/n(q)} q^2 (q, n(q)k_1)^{1/2} (q, n(q)k_2)^{1/2} |k_1 k_2|^{-1/2}$$

$$\ll \sum_{q=1}^{\infty} q^{2-\sigma} \sum_{\mathbf{k} \ll P/n(q)} (q, k_1)^{1/2} (q, k_2)^{1/2} |k_1 k_2|^{-1/2}.$$

The conditions on the original vector $\mathbf{j}$ ensure that $k_1 k_2 \neq 0$, whence

$$\sum_{\mathbf{k} \ll P/n(q)} (q, k_1)^{1/2} (q, k_2)^{1/2} |k_1 k_2|^{-1/2} \ll \{ \sum_{0 < k \ll P/n(q)} (q, k)^{1/2} k^{-1/2} \}^2.$$

Since

$$\sum_{K < k \leq 2K} (q, k) \ll K q^{\varepsilon}$$

we deduce that

$$\sum_{\mathbf{k} \ll P/n(q)} (q, k_1)^{1/2} (q, k_2)^{1/2} |k_1 k_2|^{-1/2} \ll P q^{2\varepsilon} n(q)^{-1},$$

so that

$$\sum_{\mathbf{j}} |j_1 j_2|^{-1/2} \sum_{q \in S(\mathbf{j})} q^{-\sigma} |T_q(\mathbf{j})| \ll P \sum_{q=1}^{\infty} q^{2-\sigma+2\varepsilon} n(q)^{-1}.$$

The sum over q is a product of factors

$$1 + p^{1-\sigma+2\varepsilon} + p^{3-2\sigma+4\varepsilon} + \sum_{e=3}^{\infty} p^{e(2-\sigma+2\varepsilon)} = 1 + O(p^{-1-\varepsilon})$$

providing that $\sigma \geq 7/3 + 3\varepsilon$. For such σ the corresponding product is therefore $O_{\varepsilon}(1)$.

On comparing our various estimates we now conclude that $S \ll P^{4+2\varepsilon} Q^{\sigma-2}$, and the choice $\sigma = 7/3 + 3\varepsilon$ yields $S \ll P^{9/2+7\varepsilon}$. This is clearly satisfactory for Lemma 14, if we replace ε by $\varepsilon/7$.

It remains to handle terms with $j_1 j_2 F_0(\mathbf{j}) = 0$. If $F_0(\mathbf{j}) = 0$ but $j_1 j_2 \neq 0$ then $j_i = \nu_i j$ for some integer contants ν_i, so that (37) and Lemma 16 yield

$$J_q(\mathbf{j}) \ll P^{3+\varepsilon}/|j|, \quad T_q(\mathbf{j}) \ll q^2 (q, j).$$

The terms under consideration therefore produce

$$\sum_{q \ll Q} \sum_{0 < j \ll P} q^{-2} P^{3+\varepsilon} j^{-1} q^2 (q, j) \ll P^{3+\varepsilon} \sum_{q \ll Q} d(q) \log P \ll P^{9/2+2\varepsilon},$$

which is again satisfactory.

Finally, when $j_1 = 0$, say and $j_2 = \pm j \neq 0$, we have

$$J_q(\mathbf{j}) \ll \frac{P^{13/4+\varepsilon}}{\sqrt{qj}}$$

as in (38). Moreover Lemma 16 yields

$$T_q(\mathbf{j}) \ll q^{5/2}(q,j)^{1/2}$$

in general, while if $p \not| j$ and $q = p$ we also have

$$T_q(\mathbf{j}) \ll q^2.$$

We may combine these two latter estimates to give

$$T_q(\mathbf{j}) \ll q^{5/2+\varepsilon}(q,j)m(q)^{-1/2},$$

where $m(q) = \prod p$ for those primes with $p||q$. We now have

$$\sum_{q \ll Q} \sum_{\mathbf{j} \ll P} q^{-2} T_q(\mathbf{j}) J_q(\mathbf{j}) \ll \sum_{q \ll Q} \sum_{0 < j \ll P} q^{-2} q^{5/2+\varepsilon}(q,j)m(q)^{-1/2} \frac{P^{13/4+\varepsilon}}{\sqrt{qj}}.$$

Now

$$\sum_{0 < j \ll P} (q,j) j^{-1/2} \ll q^{\varepsilon} P^{1/2},$$

giving a bound

$$P^{15/4+\varepsilon} \sum_{q \ll Q} q^{2\varepsilon} m(q)^{-1/2}.$$

However, on writing q as $q_1 q_2$ with q_1 square-free and q_2 square-full, we see that

$$\sum_{q \ll Q} m(q)^{-1/2} = \sum_{q_1 q_2 \ll Q} q_1^{-1/2} \ll \sum_{q_1 \ll Q} q_1^{-1/2} (Q/q_1)^{1/2} \ll Q^{1/2} \log Q.$$

This leads to a satisfactory bound $O(P^{9/2+5\varepsilon})$ for these terms too. This completes the proof of Lemma 13.

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