Hypercomplex Algebraic Geometry

Dominic Joyce

published as Quart. J. Math. Oxford 49 (1998), 129–162.

1 Introduction

It is well-known that sums and products of holomorphic functions are holomorphic, and the holomorphic functions on a complex manifold form a commutative algebra over $\mathbb{C}$. The study of complex manifolds using algebras of holomorphic functions upon them is called *complex algebraic geometry*.

The purpose of this paper is to develop an analogue of complex algebraic geometry, in which the complex numbers $\mathbb{C}$ are replaced by the *quaternions* $\mathbb{H}$. The natural quaternionic analogue of a complex manifold is called a *hypercomplex manifold*. A class of $\mathbb{H}$-valued functions on hypercomplex manifolds will be defined, called *q-holomorphic functions*, that are analogues of holomorphic functions on complex manifolds. Now, the set of holomorphic functions on a complex manifold is a commutative algebra over $\mathbb{C}$. Therefore one asks: does the set of q-holomorphic functions on a hypercomplex manifold have an analogous algebraic structure, and if so, what is it?

We shall show that the q-holomorphic functions on a (noncompact) hypercomplex manifold do indeed possess a rich algebraic structure. To describe it, we shall introduce a theory of *quaternionic algebra*, which is a quaternionic analogue of real linear algebra. This theory is built on three building blocks: A$\mathbb{H}$-*modules*, the analogues of vector spaces, A$\mathbb{H}$-*morphisms*, the analogues of linear maps, and the *quaternionic tensor product*, the analogue of tensor product of real vector spaces.

As far as the author can tell, these ideas seem to be new. They enable us to construct algebraic structures over $\mathbb{H}$ as though $\mathbb{H}$ were a commutative field. Quaternionic algebra describes the algebraic structure of hypercomplex manifolds in a remarkable way, and it seems to be the natural language of *hypercomplex algebraic geometry*, the algebraic geometry of hypercomplex manifolds.

We believe that quaternionic algebra is worth studying for its own sake. It has many similarities with linear algebra over $\mathbb{R}$ or $\mathbb{C}$, which is why the analogies between complex and quaternionic theories work so well, but there are also deep differences, which give quaternionic algebra a flavour all of its own.

Quillen [12] has given a sheaf-theoretic interpretation of the ideas of quaternionic algebra, based on a previous version of this paper. He finds a contravariant equivalence between a class of A$\mathbb{H}$-modules and regular sheaves on a real form of $\mathbb{CP}^1$. Regular sheaves on $\mathbb{CP}^1$ are equivalent to representations of the *Kronecker quiver*, and out of such a representation Quillen constructs an A$\mathbb{H}$-module. Under Quillen's equivalence stable A$\mathbb{H}$-modules correspond to regular vector bundles over the real form of $\mathbb{CP}^1$.

We start in §2 by reviewing the quaternions and defining the concepts of A$\mathbb{H}$-modules and A$\mathbb{H}$-morphisms. Section 3 defines hypercomplex manifolds and q-holomorphic functions, and shows that the q-holomorphic functions on a hypercomplex manifold are an A$\mathbb{H}$-module. The key to the algebraic side of this paper is the definition in §4 of the quaternionic tensor product, and an exploration of some of its properties.

Section 5 defines H-algebras, the quaternionic analogues of commutative algebras, and shows that the q-holomorphic functions on a hypercomplex manifold form an H-algebra. We briefly discuss the problem of recovering a hypercomplex manifold from its H-algebra, which leads to the idea of an algebraic geometry of hypercomplex manifolds. In §6 hyperkähler manifolds are defined, and we explain how the H-algebra of a hyperkähler manifold acquires an additional algebraic structure, making it into an HP-algebra.

Sections 7-9 study the quaternionic algebra of finite-dimensional A$\mathbb{H}$-modules in greater depth. In §7 a series of examples are given which illustrate differences between real and quaternionic algebra. Sections 8 and 9 concern two special classes of A$\mathbb{H}$-modules, stable and semistable A$\mathbb{H}$-modules. Amongst other things, we show that the quaternionic tensor product $U \otimes_{\mathbb{H}} V$ of two stable A$\mathbb{H}$-modules is stable, and give a formula for $\dim U \otimes_{\mathbb{H}} V$. By restricting to stable A$\mathbb{H}$-modules, some of the differences between real and quaternionic algebra are resolved, and the analogy between real and quaternionic algebra becomes more complete.

Finally, §§10-12 give geometrical applications of the theory to hypercomplex and hyperkähler manifolds. Section 10 studies q-holomorphic functions on $\mathbb{H}$, and §§11 and 12 prove some new results on hyperkähler manifolds with large symmetry groups, including an algebraic construction of Kronheimer's hyperkähler metrics on coadjoint orbits [10], [11].

Acknowledgements. I would like to thank John Cernes, Peter Kronheimer, Peter Neumann, and Simon Salamon for interesting conversations.

2 A$\mathbb{H}$-modules

The *quaternions* $\mathbb{H}$ are $\mathbb{H} = \{r_0 + r_1 i_1 + r_2 i_2 + r_3 i_3 : r_0, \ldots, r_3 \in \mathbb{R}\}$, and quaternion multiplication is given by

$$i_1 i_2 = -i_2 i_1 = i_3, \quad i_2 i_3 = -i_3 i_2 = i_1, \quad i_3 i_1 = -i_1 i_3 = i_2, \quad i_1^2 = i_2^2 = i_3^2 = -1. \tag{2.1}$$

The quaternions are an associative, noncommutative algebra. If $q = r_0 + r_1 i_1 + r_2 i_2 + r_3 i_3$ then we define the *conjugate* $\overline{q}$ of q by $\overline{q} = r_0 - r_1 i_1 - r_2 i_2 - r_3 i_3$. Then $\overline{(pq)} = \overline{q}\,\overline{p}$ for $p, q \in \mathbb{H}$. The *imaginary quaternions* are $\mathbb{I} = \langle i_1, i_2, i_3 \rangle$.

Definition 2.1 The following notation will be used throughout the paper. If V is a vector space, then V^* is the dual space and $\mathrm{id} : V \to V$ the identity map. Also, $\dim V$ means the dimension of V as a *real* vector space, even if V is an $\mathbb{H}$-module. A *(left)* $\mathbb{H}$*-module* is a real vector space U with an action of $\mathbb{H}$ on the left. We write this action $(q, u) \mapsto q \cdot u$ or qu, for $q \in \mathbb{H}$ and $u \in U$. In this paper, all $\mathbb{H}$-modules will be left $\mathbb{H}$-modules. Let U be an $\mathbb{H}$-module. We define the *dual* $\mathbb{H}$*-module* $U^\times$ to be the vector space of linear maps $\alpha : U \to \mathbb{H}$ that satisfy $\alpha(qu) = q\alpha(u)$ for all $q \in \mathbb{H}$ and $u \in U$. If $q \in \mathbb{H}$ and $\alpha \in U^\times$, define $q \cdot \alpha$ by $(q \cdot \alpha)(u) = \alpha(u)\overline{q}$ for $u \in U$. Then $q \cdot \alpha \in U^\times$, and $U^\times$ is a (left) $\mathbb{H}$-module. Dual $\mathbb{H}$-modules behave just like dual vector spaces.

Now we define $A\mathbb{H}$-*modules*, which should be thought of as the quaternionic analogues of real vector spaces.

Definition 2.2 Let U be an $\mathbb{H}$-module. Let U' be a real vector subspace of U, that need not be closed under the $\mathbb{H}$-action. Define a real vector subspace $U^\dagger$ of $U^\times$ by

$$U^\dagger = \{\alpha \in U^\times : \alpha(u) \in \mathbb{I} \text{ for all } u \in U'\}. \tag{2.2}$$

Conversely, $U^\dagger$ determines U', at least for finite-dimensional U, by

$$U' = \{u \in U : \alpha(u) \in \mathbb{I} \text{ for all } \alpha \in U^\dagger\}. \tag{2.3}$$

We define an *augmented* $\mathbb{H}$*-module*, or $A\mathbb{H}$-*module*, to be a pair (U, U'), such that if $u \in U$ and $\alpha(u) = 0$ for all $\alpha \in U^\dagger$, then $u = 0$. Usually we will refer to U as an $A\mathbb{H}$-module, implicitly assuming that U' is also given. We consider $\mathbb{H}$ to be an $A\mathbb{H}$-module, with $\mathbb{H}' = \mathbb{I}$.

The condition that $u = 0$ if $\alpha(u) = 0$ for all $\alpha \in U^\dagger$ is important. Its purpose should become clear in §§3 and 4. We can interpret $U^\times$ as the dual of U as a real vector space, and then $U^\dagger$ is the annihilator of U'. Thus if U is finite-dimensional, $\dim U' + \dim U^\dagger = \dim U = \dim U^\times$. Here are the natural concepts of linear map between $A\mathbb{H}$-modules, and submodules of $A\mathbb{H}$-modules.

Definition 2.3 Let U, V be $A\mathbb{H}$-modules. Let $\phi : U \to V$ be a linear map satisfying $\phi(qu) = q\phi(u)$ for each $q \in \mathbb{H}$ and $u \in U$. Such a map is called $\mathbb{H}$*-linear*. We say that ϕ is an $A\mathbb{H}$-*morphism*, if $\phi : U \to V$ is $\mathbb{H}$-linear and $\phi(U') \subset V'$. If ϕ is an isomorphism of $\mathbb{H}$-modules and $\phi(U') = V'$, we say ϕ is an $A\mathbb{H}$-*isomorphism*. If $\phi : U \to V$ and $\psi : V \to W$ are $A\mathbb{H}$-morphisms, then $\psi \circ \phi : U \to W$ is an $A\mathbb{H}$-morphism.

Let U, V be $A\mathbb{H}$-modules and $\phi : U \to V$ an $A\mathbb{H}$-morphism. Define an $\mathbb{H}$-linear map $\phi^\times : V^\times \to U^\times$ by $\phi^\times(\beta)(u) = \beta(\phi(u))$ for $\beta \in V^\times$ and $u \in U$. Then $\phi(U') \subset V'$ implies that $\phi^\times(V^\dagger) \subset U^\dagger$. If V is an $A\mathbb{H}$-module, we say that U *is an* $A\mathbb{H}$-*submodule of* V if U is an $\mathbb{H}$-submodule of V and $U' = U \cap V'$. As $U^\dagger$ is the restriction of $V^\dagger$ to U, if $\alpha(u) = 0$ for all $\alpha \in U^\dagger$ then $u = 0$, so U is an $A\mathbb{H}$-module.

3 Hypercomplex manifolds and q-holomorphic functions

Let M be a manifold of dimension $2n$. A *complex structure* I on M is a smooth tensor I_a^b on M that satisfies the equations

$$I_a^b I_b^c = -\delta_a^c \qquad \text{and} \qquad I_c^d \nabla_b I_a^c - I_c^d \nabla_a I_b^c + I_a^c \nabla_c I_b^d - I_b^c \nabla_c I_a^d = 0, \tag{3.1}$$

where ∇ is any torsion-free connection on TM. A manifold M with a complex structure I is called a *complex manifold*. Let M be a complex manifold and $f : M \to \mathbb{C}$ a differentiable function. Write $f = f_0 + if_1$, where $f_0, f_1 : M \to \mathbb{R}$. Then f is called *holomorphic* if

$$df_0 + I(df_1) = 0 \tag{3.2}$$

on M, where (3.2) is called the Cauchy-Riemann equation.

Here are the quaternionic analogues of complex manifolds and holomorphic functions. Let M be a manifold of dimension $4n$. A *hypercomplex structure* on M is a triple (I_1, I_2, I_3) on

M, where I_j is a complex structure on M, and $I_1 I_2 = I_3$. In index notation this condition is written $(I_1)_a^b (I_2)_b^c = (I_3)_a^c$. If M has a hypercomplex structure, it is called a hypercomplex manifold. Since I_1, I_2, I_3 satisfy the quaternion relations (2.1), each tangent space $T_m M$ is an $\mathbb{H}$-module isomorphic to $\mathbb{H}^n$. For more information about hypercomplex manifolds, see for instance [13, p. 137-9], [7] and [8].

Now let M be a hypercomplex manifold, and $f : M \to \mathbb{H}$ a smooth function. Then $f = f_0 + f_1 i_1 + f_2 i_2 + f_3 i_3$, where $f_0, \ldots, f_3$ are smooth real functions. Define a 1-form $D(f)$ on M by

$$D(f) = df_0 + I_1(df_1) + I_2(df_2) + I_3(df_3). \tag{3.3}$$

We define a *q-holomorphic function on* M to be a smooth function $f : M \to \mathbb{H}$ for which $D(f) = 0$. Equation (3.3) is the natural quaternionic analogue of the Cauchy-Riemann equation (3.2).

Q-holomorphic functions on $\mathbb{H}$ were studied, a long time ago, by Fueter and his collaborators. In 1935, Fueter defined q-holomorphic functions on $\mathbb{H}$, which he called 'regular functions', and went on to develop the theory of *quaternionic analysis*, by analogy with complex analysis. This theory included analogues of Cauchy's Theorem, Cauchy's integral formula, the Laurent expansion, and so on. However, as far as the author knows, Fueter and his school did not discover the quaternionic tensor product or the theory of multiplying q-holomorphic functions that will be explained in this paper. Accounts of the theory, with references, are given by Sudbery [14] and Deavours [4].

Next we will show that the q-holomorphic functions on a hypercomplex manifold form an A$\mathbb{H}$-module, in the sense of Definition 2.2.

Definition 3.1 Let M be a hypercomplex manifold. Define A_M be the real vector space of q-holomorphic functions on M. Let $f \in A_M$ and $q \in \mathbb{H}$, and define a function $q \cdot f$ on M by $(q \cdot f)(m) = q(f(m))$ for all $m \in M$. By equation (3.3), $D(q \cdot f) = 0$ if $D(f) = 0$, so that $q \cdot f \in A_M$. This gives an action of $\mathbb{H}$ on A_M, which makes A_M into an $\mathbb{H}$-module.

Now define a real vector subspace A'_M in A_M by

$$A'_M = \{ f \in A : f(m) \in \mathbb{I} \text{ for all } m \in M \}. \tag{3.4}$$

For each $m \in M$, define $\theta_m : A_M \to \mathbb{H}$ by $\theta_m(f) = f(m)$. Then $\theta_m \in A_M^\times$, and if $f \in A'_M$ then $\theta_m(f) \in \mathbb{I}$, so that $\theta_m \in A_M^\dagger$. Suppose $f \in A_M$, and $\alpha(f) = 0$ for all $\alpha \in A_M^\dagger$. Since $\theta_m \in A_M^\dagger$, $f(m) = 0$ for each $m \in M$, and so $f = 0$. Thus we have proved that *the vector space* A_M *of q-holomorphic functions on* M *is an* A$\mathbb{H}$-*module*.

If M is a hypercomplex manifold, then it has three complex structures I_1, I_2, I_3. But these are not the only complex structures. If $a_1, a_2, a_3 \in \mathbb{R}$ with $a_1^2 + a_2^2 + a_3^2 = 1$, then $a_1 I_1 + a_2 I_2 + a_3 I_3$ is also a complex structure on M, so that there is a family of complex structures on M parametrized by the 2-sphere $\mathcal{S}^2$, which should be regarded as the unit sphere in $\mathbb{I}$.

Let $\Sigma_{j=1}^3 a_j^2 = 1$, so that $I = \Sigma_j a_j I_j$ is a complex structure on M, and let $i = \Sigma_j a_j i_j \in \mathbb{I}$. Then $i^2 = -1 \in \mathbb{H}$, and $\langle 1, i \rangle$ is a subalgebra of $\mathbb{H}$ isomorphic to $\mathbb{C}$. Let f_0, f_1 be real functions on M, and suppose that $f_0 + f_1 i$ is a holomorphic function on M with respect to I. Then $df_0 + I(df_1) = 0$ on M by (3.2). Substituting $I = \Sigma_j a_j I_j$, we see that

$$D(f_0 + a_1 f_1 i_1 + a_2 f_1 i_2 + a_3 f_1 i_3) = df_0 + I_1(a_1 df_1) + I_2(a_2 df_2) + I_3(a_3 df_3) = 0, \tag{3.5}$$

and from (3.3) we see that $f_0 + f_1 i$ is a q-holomorphic function on M.

Thus, any $\mathbb{C}$-valued function on M that is holomorphic with respect to one of the $\mathcal{S}^2$ family of complex structures, can also be regarded as a q-holomorphic $\mathbb{H}$-valued function, by embedding $\mathbb{C}$ in $\mathbb{H}$ in the appropriate way. This tells us two important things. Firstly, on small open sets in any complex manifold there are many holomorphic functions. Therefore, on small open sets of a hypercomplex manifold, there are many q-holomorphic functions. If $\dim M > 4$, the equation (3.3) is overdetermined, so one would expect few solutions or none, but this is not the case.

Secondly, the product of two holomorphic functions is holomorphic. Therefore, it is possible in some circumstances to take two q-holomorphic functions on a hypercomplex manifold, multiply them together, and get a third q-holomorphic function. So, we expect some sort of multiplicative structure on A_M, the A$\mathbb{H}$-module of q-holomorphic functions on M. However, in general the product of two q-holomorphic functions is *not* q-holomorphic. Thus, A_M is not an algebra in the simple, obvious sense. We shall explain in the next two sections how to describe the multiplicative structure on A_M.

Now we define some special A$\mathbb{H}$-modules X_q.

Definition 3.2 Let $q \in \mathbb{I}$ be nonzero. Define an A$\mathbb{H}$-module X_q by $X_q = \mathbb{H}$, and $X'_q = \{p \in \mathbb{H} : pq = -qp\}$. It is easy to show that $X'_q \subset \mathbb{I}$ and $\dim X'_q = \dim X^\dagger_q = 2$. For example, if $q = i_1$, then $X'_{i_1} = \langle i_2, i_3 \rangle \subset \mathbb{H}$.

We can use these A$\mathbb{H}$-modules to characterize the holomorphic functions in the set of q-holomorphic functions.

Lemma 3.3 *Let M be a hypercomplex manifold. Let $i \in \mathbb{I}$ satisfy $i^2 = -1$, and let I be the corresponding complex structure on M. Suppose that $f = f_0 + if_1$ is a holomorphic function on M w.r.t. I. Define a map $\phi_f : X_i \to A_M$ by $\phi_f(q) = q \cdot f$. Then ϕ_f is an* A$\mathbb{H}$*-morphism. Conversely, if $\phi : X_i \to A_M$ is an* A$\mathbb{H}$*-morphism, then $f = \phi(1) \in A_M$ takes values in $\langle 1, i \rangle \subset \mathbb{H}$, and is holomorphic w.r.t. $\mathbb{I}$.*

Proof. Suppose for simplicity that $i = i_1$ and $I = I_1$. The proof for general $i \in \mathbb{I}$ works the same way. As f is holomorphic w.r.t. I_1, $f \in A_M$ from above, and if $f \in A_M$ then $q \cdot f \in A_M$. Thus ϕ_f maps $X_{i_1} \to A_M$, and is clearly $\mathbb{H}$-linear. We must prove that $\phi_f(X'_{i_1}) \subset A'_M$. As $X'_{i_1} = \langle i_2, i_3 \rangle$, it is enough to show that $\phi_f(i_2)$ and $\phi_f(i_3)$ lie in A'_M. Now $\phi_f(i_2) = f_0 i_2 - f_1 i_3$, where f_0 and f_1 are real functions. Thus, $\phi_f(i_2)$ takes values in $\mathbb{I}$ and so $\phi_f(i_2) \in A'_M$ by (3.4). Similarly, $\phi_f(i_3) = f_1 i_2 + f_0 i_3$, so $\phi_f(i_3) \in A'_M$. Therefore ϕ_f is an A$\mathbb{H}$-morphism.

For the next part, let $\phi(1) = f_0 + f_1 i_1 + f_2 i_2 + f_3 i_3$. Since $X'_{i_1} = \langle i_2, i_3 \rangle$ and ϕ is an A$\mathbb{H}$-morphism, $\phi(i_2)$ and $\phi(i_3)$ lie in A'_M. But

$$\phi(i_2) = i_2\phi(1) = -f_2 + f_3 i_1 + f_0 i_2 - f_1 i_3, \text{ and } \phi(i_3) = i_3\phi(1) = -f_3 - f_2 i_1 + f_2 i_2 + f_0 i_3. \quad (3.6)$$

As $\phi(i_2) \in A'_M$, it takes values in $\mathbb{I}$, and so $f_2 = 0$. Similarly, $\phi(i_3)$ takes values in $\mathbb{I}$, and $f_3 = 0$. Thus $\phi(1) = f_0 + f_1 i_1$, and so $D(\phi(1)) = 0$ implies that $df_0 + I_1 df_1 = 0$. Therefore $\phi(1)$ takes values in $\langle 1, i_1 \rangle$, and is holomorphic w.r.t. I_1. ∎

4 Quaternionic tensor products

In this section we will define the *quaternionic tensor product* of two A$\mathbb{H}$-modules U and V, which is an A$\mathbb{H}$-module $U \otimes_{\mathbb{H}} V$. This is the key algebraic idea of this paper. In the analogy between quaternionic algebra and real algebra, A$\mathbb{H}$-modules correspond to vector spaces, and the quaternionic tensor product corresponds to the tensor product of vector spaces. The definition of the quaternionic tensor product is strange and difficult, and it is not obvious at first sight why it is a good analogue of the tensor product. This should become much clearer later on.

Definition 4.1 Let U be an A$\mathbb{H}$-module. Then $\mathbb{H} \otimes (U^\dagger)^*$ is an $\mathbb{H}$-module, with $\mathbb{H}$-action $p \cdot (q \otimes x) = (pq) \otimes x$. Define a map $\iota_U : U \to \mathbb{H} \otimes (U^\dagger)^*$ by $\iota_U(u) \cdot \alpha = \alpha(u)$, for $u \in U$ and $\alpha \in U^\dagger$. Then ι_U is $\mathbb{H}$-linear, so that $\iota_U(U)$ is an $\mathbb{H}$-submodule of $\mathbb{H} \otimes (U^\dagger)^*$.

Suppose $u \in \operatorname{Ker} \iota_U$. Then $\alpha(u) = 0$ for all $\alpha \in U^\dagger$, so that $u = 0$ as U is an A$\mathbb{H}$-module. Thus ι_U is injective, and $\iota_U(U) \cong U$. From (2.3), it follows that $\iota_U(U') = \iota_U(U) \cap \bigl(\mathbb{I} \otimes (U^\dagger)^*\bigr)$. Thus the A$\mathbb{H}$-module (U, U') is determined by the $\mathbb{H}$-submodule $\iota_U(U)$.

Now we define the quaternionic tensor product.

Definition 4.2 Let U, V be A$\mathbb{H}$-modules. Then $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$ is an $\mathbb{H}$-module, with $\mathbb{H}$-action $p \cdot (q \otimes x \otimes y) = (pq) \otimes x \otimes y$. Exchanging the factors of $\mathbb{H}$ and $(U^\dagger)^*$, we may regard $(U^\dagger)^* \otimes \iota_V(V)$ as a subspace of $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$. Thus $\iota_U(U) \otimes (V^\dagger)^*$ and $(U^\dagger)^* \otimes \iota_V(V)$ are $\mathbb{H}$-submodules of $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$. Define an $\mathbb{H}$-module $U \otimes_{\mathbb{H}} V$ by

$$U \otimes_{\mathbb{H}} V = \bigl(\iota_U(U) \otimes (V^\dagger)^*\bigr) \cap \bigl((U^\dagger)^* \otimes \iota_V(V)\bigr) \subset \mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*. \tag{4.1}$$

Define a vector subspace $(U \otimes_{\mathbb{H}} V)'$ by $(U \otimes_{\mathbb{H}} V)' = (U \otimes_{\mathbb{H}} V) \cap \bigl(\mathbb{I} \otimes (U^\dagger)^* \otimes (V^\dagger)^*\bigr)$. Define a linear map $\lambda_{U,V} : U^\dagger \otimes V^\dagger \to (U \otimes_{\mathbb{H}} V)^\times$ by $\lambda_{U,V}(x)(y) = y \cdot x \in \mathbb{H}$, for $x \in U^\dagger \otimes V^\dagger$, $y \in U \otimes_{\mathbb{H}} V$, where '$\cdot$' contracts together the factors of $U^\dagger \otimes V^\dagger$ and $(U^\dagger)^* \otimes (V^\dagger)^*$.

Clearly, if $x \in U^\dagger \otimes V^\dagger$ and $y \in (U \otimes_{\mathbb{H}} V)'$, then $\lambda_{U,V}(x)(y) \in \mathbb{I}$. As this holds for all $y \in (U \otimes_{\mathbb{H}} V)'$, $\lambda_{U,V}(x) \in (U \otimes_{\mathbb{H}} V)^\dagger$, so that $\lambda_{U,V}$ maps $U^\dagger \otimes V^\dagger \to (U \otimes_{\mathbb{H}} V)^\dagger$. If $y \in U \otimes_{\mathbb{H}} V$, then $\lambda_{U,V}(x)(y) = 0$ for all $x \in U^\dagger \otimes V^\dagger$ if and only if $y = 0$. Thus $U \otimes_{\mathbb{H}} V$ is an A$\mathbb{H}$-module, by Definition 2.2. This A$\mathbb{H}$-module will be called the *quaternionic tensor product of U and V*, and the operation $\otimes_{\mathbb{H}}$ will be called the *quaternionic tensor product.* When U, V are finite-dimensional, $\lambda_{U,V}$ is surjective, so that $(U \otimes_{\mathbb{H}} V)^\dagger = \lambda_{U,V}(U^\dagger \otimes V^\dagger)$.

Here are some basic properties of the operation $\otimes_{\mathbb{H}}$.

Lemma 4.3 *Let U, V, W be A$\mathbb{H}$-modules. Then there are canonical A$\mathbb{H}$-isomorphisms*

$$\mathbb{H} \otimes_{\mathbb{H}} U \cong U, \quad U \otimes_{\mathbb{H}} V \cong V \otimes_{\mathbb{H}} U, \quad \textit{and} \quad (U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W \cong U \otimes_{\mathbb{H}} (V \otimes_{\mathbb{H}} W). \tag{4.2}$$

Proof. As $\mathbb{H}^\dagger \cong \mathbb{R}$, we may identify $\mathbb{H} \otimes (\mathbb{H}^\dagger)^* \otimes (U^\dagger)^*$ and $\mathbb{H} \otimes (U^\dagger)^*$. Under this identification, it is easy to see that $\mathbb{H} \otimes_{\mathbb{H}} U$ and $\iota_U(U)$ are identified. Since $\iota_U(U) \cong U$, this gives an isomorphism $\mathbb{H} \otimes_{\mathbb{H}} U \cong U$, which is an A$\mathbb{H}$-isomorphism. The A$\mathbb{H}$-isomorphism $U \otimes_{\mathbb{H}} V \cong V \otimes_{\mathbb{H}} U$ is trivial, because the definition of $U \otimes_{\mathbb{H}} V$ is symmetric in U and V.

It remains to show that $(U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W \cong U \otimes_{\mathbb{H}} (V \otimes_{\mathbb{H}} W)$. By analogy with Definition 4.2, define an $\mathbb{H}$-submodule Z of $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^* \otimes (W^\dagger)^*$ by

$$Z = \big(\iota_U(U) \otimes (V^\dagger)^* \otimes (W^\dagger)^*\big) \cap \big((U^\dagger)^* \otimes \iota_V(V) \otimes (W^\dagger)^*\big) \cap \big((U^\dagger)^* \otimes (V^\dagger)^* \otimes \iota_W(W)\big), \quad (4.3)$$

and define $Z' = Z \cap \big(\mathbb{I} \otimes (U^\dagger)^* \otimes (V^\dagger)^* \otimes (W^\dagger)^*\big)$. Then Z is an A$\mathbb{H}$-module. Using $\lambda_{U,V}$ and $\lambda_{U\otimes_{\mathbb{H}} V,Z}$ we may construct a map from $(U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W$ to $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^* \otimes (W^\dagger)^*$, and this induces an A$\mathbb{H}$-isomorphism $(U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W \cong Z$. Similarly $Z \cong U \otimes_{\mathbb{H}} (V \otimes_{\mathbb{H}} W)$, so $(U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W \cong U \otimes_{\mathbb{H}} (V \otimes_{\mathbb{H}} W)$, as we want. ∎

Lemma 4.3 tells us that $\otimes_{\mathbb{H}}$ is commutative and associative, and that $\mathbb{H}$ acts as an identity element for $\otimes_{\mathbb{H}}$. Since $\otimes_{\mathbb{H}}$ is associative, we shall not bother to put brackets in multiple products such as $U \otimes_{\mathbb{H}} V \otimes_{\mathbb{H}} W$. Also, as $\otimes_{\mathbb{H}}$ is commutative and associative we can define symmetric and antisymmetric products of A$\mathbb{H}$-modules.

Definition 4.4 Let U be an A$\mathbb{H}$-module. Write $\bigotimes^k_{\mathbb{H}} U$ for the product $U \otimes_{\mathbb{H}} \cdots \otimes_{\mathbb{H}} U$ of k copies of U, with $\bigotimes^0_{\mathbb{H}} U = \mathbb{H}$. Then the k^{th} symmetric group S_k acts on $\bigotimes^k_{\mathbb{H}} U$ by permutation of the U factors in the obvious way. Define $S^k_{\mathbb{H}} U$ and $\Lambda^k_{\mathbb{H}} U$ to be the A$\mathbb{H}$-submodules of $\bigotimes^k_{\mathbb{H}} U$ that are symmetric and antisymmetric respectively under this action of S_k.

Here is the definition of the tensor product of two A$\mathbb{H}$-morphisms.

Definition 4.5 Let U, V, W, X be A$\mathbb{H}$-modules, and let $\phi : U \to W$ and $\psi : V \to X$ be A$\mathbb{H}$-morphisms. Then $\phi^\times(W^\dagger) \subset U^\dagger$ and $\psi^\times(X^\dagger) \subset V^\dagger$, by definition. Taking the duals gives maps $(\phi^\times)^* : (U^\dagger)^* \to (W^\dagger)^*$ and $(\psi^\times)^* : (V^\dagger)^* \to (X^\dagger)^*$. Combining these, we have a map

$$\mathrm{id} \otimes (\phi^\times)^* \otimes (\psi^\times)^* : \mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^* \to \mathbb{H} \otimes (W^\dagger)^* \otimes (X^\dagger)^*. \quad (4.4)$$

Now $U \otimes_{\mathbb{H}} V \subset \mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$ and $W \otimes_{\mathbb{H}} X \subset \mathbb{H} \otimes (W^\dagger)^* \otimes (X^\dagger)^*$. It is easy to show that $\big(\mathrm{id} \otimes (\phi^\times)^* \otimes (\psi^\times)^*\big)(U \otimes_{\mathbb{H}} V) \subset W \otimes_{\mathbb{H}} X$. Define $\phi \otimes_{\mathbb{H}} \psi : U \otimes_{\mathbb{H}} V \to W \otimes_{\mathbb{H}} X$ to be the restriction of $\mathrm{id} \otimes (\phi^\times)^* \otimes (\psi^\times)^*$ to $U \otimes_{\mathbb{H}} V$. It follows trivially from the definitions that $\phi \otimes_{\mathbb{H}} \psi$ is $\mathbb{H}$-linear and satisfies $(\phi \otimes_{\mathbb{H}} \psi)\big((U \otimes_{\mathbb{H}} V)'\big) \subset (W \otimes_{\mathbb{H}} X)'$. Thus $\phi \otimes_{\mathbb{H}} \psi$ is an A$\mathbb{H}$-morphism from $U \otimes_{\mathbb{H}} V$ to $W \otimes_{\mathbb{H}} X$. This is the *quaternionic tensor product of* ϕ *and* ψ.

Now, if U, V are A$\mathbb{H}$-modules and $u \in U$, $v \in V$, there is in general no element '$u \otimes_{\mathbb{H}} v$' in $U \otimes_{\mathbb{H}} V$ that is the product of u and v. This is a fundamental difference between the real and quaternionic tensor products, that makes the interpretation of $U \otimes_{\mathbb{H}} V$ more difficult. However, for some special elements $u \in U, v \in V$ it is possible to define an element $u \otimes_{\mathbb{H}} v \in U \otimes_{\mathbb{H}} V$. This is shown in the following Lemma, which is trivial to prove.

Lemma 4.6 *Let U, V be A$\mathbb{H}$-modules, and let $u \in U$ and $v \in V$ be nonzero. Suppose that $\alpha(u)\beta(v) = \beta(v)\alpha(u) \in \mathbb{H}$ for every $\alpha \in U^\dagger$ and $\beta \in V^\dagger$. Define an element $u \otimes_{\mathbb{H}} v$ of $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$ by $(u \otimes_{\mathbb{H}} v) \cdot (\alpha \otimes \beta) = \alpha(u)\beta(v) \in \mathbb{H}$. Then $u \otimes_{\mathbb{H}} v$ is a nonzero element of $U \otimes_{\mathbb{H}} V$.*

5 H-algebras and hypercomplex manifolds

In this section we will define the quaternionic version of a commutative algebra, which we shall call an H-algebra. Then we will show that the q-holomorphic functions on a hypercomplex manifold form an H-algebra. Here is the usual definition of a commutative algebra over $\mathbb{R}$.

Axiom A1. (*i*) A is a real vector space.

(*ii*) There is a bilinear map $\mu : A \times A \to A$, called the *multiplication map*.

(*iii*) $\mu(a,b) = \mu(b,a)$ for all $a, b \in A$. Thus μ is *commutative*.

(*iv*) $\mu\big(\mu(a,b), c\big) = \mu\big(a, \mu(b,c)\big)$ for all $a, b, c \in A$. Thus μ is *associative*.

(*v*) An *identity* $1 \in A$ is given, and $\mu(1,a) = \mu(a,1) = a$ for all $a \in A$.

Now this axiom is not in a suitable form to translate into quaternionic language. The definition involves bilinear maps, and conditions (*iii*)-(*v*) are written in terms of elements a, b, c of A. The things we understand how to translate are tensor products and linear maps. Therefore, we rewrite the axiom in the following equivalent form, replacing bilinear maps by linear maps on a tensor product, and using linear maps rather than elements of A in conditions (*iii*) and (*iv*).

Axiom A2. (*i*) A is a real vector space.

(*ii*) There is a linear map $\mu : A \otimes A \to A$, called the *multiplication map*.

(*iii*) $\Lambda^2 A \subset \operatorname{Ker} \mu$. Thus μ is *commutative*.

(*iv*) The linear maps $\mu : A \otimes A \to A$ and $\mathrm{id} : A \to A$ combine to give linear maps $\mu \otimes \mathrm{id}$ and $\mathrm{id} \otimes \mu : A \otimes A \otimes A \to A \otimes A$. Composing with μ gives linear maps $\mu \circ (\mu \otimes \mathrm{id})$ and $\mu \circ (\mathrm{id} \otimes \mu) : A \otimes A \otimes A \to A$. Then $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$. Thus μ is *associative*.

(*v*) An *identity* $1 \in A$ is given, and $\mu(1 \otimes a) = \mu(a \otimes 1) = a$ for all $a \in A$.

Now, we make a quaternionic version of Axiom A2 by replacing vector spaces by A$\mathbb{H}$-modules, linear maps by A$\mathbb{H}$-morphisms, and tensor products by quaternionic tensor products. Define an *H-algebra* (short for *Hamilton algebra*) to satisfy the following axiom.

Axiom H. (*i*) A is an A$\mathbb{H}$-module.

(*ii*) There is an A$\mathbb{H}$-morphism $\mu : A \otimes_{\mathbb{H}} A \to A$, called the *multiplication map*.

(*iii*) $\Lambda^2_{\mathbb{H}} A \subset \operatorname{Ker} \mu$. Thus μ is *commutative*.

(*iv*) The A$\mathbb{H}$-morphisms $\mu : A \otimes_{\mathbb{H}} A \to A$ and $\mathrm{id} : A \to A$ combine to give A$\mathbb{H}$-morphisms $\mu \otimes_{\mathbb{H}} \mathrm{id}$ and $\mathrm{id} \otimes_{\mathbb{H}} \mu : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \to A \otimes_{\mathbb{H}} A$. Composing with μ gives A$\mathbb{H}$-morphisms $\mu \circ (\mu \otimes_{\mathbb{H}} \mathrm{id})$ and $\mu \circ (\mathrm{id} \otimes_{\mathbb{H}} \mu) : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \to A$. Then $\mu \circ (\mu \otimes_{\mathbb{H}} \mathrm{id}) = \mu \circ (\mathrm{id} \otimes_{\mathbb{H}} \mu)$. Thus μ is *associative*.

(*v*) An *identity* $1 \in A$ is given, with $\mathbb{I} \cdot 1 \subset A'$. This implies that if $\alpha \in A^\dagger$ then $\alpha(1) \in \mathbb{R}$. Thus for each $a \in A$, $1 \otimes_{\mathbb{H}} a$ and $a \otimes_{\mathbb{H}} 1 \in A \otimes_{\mathbb{H}} A$ by Lemma 4.6. Then $\mu(1 \otimes_{\mathbb{H}} a) = \mu(a \otimes_{\mathbb{H}} 1) = a$ for each $a \in A$.

Here is an example.

Example 5.1 Let U be an A$\mathbb{H}$-module, and define $A = \bigoplus_{j=0}^{\infty} S^j_{\mathbb{H}} U$. Let $\mu_{k,l} : (S^k_{\mathbb{H}} U) \otimes_{\mathbb{H}} (S^l_{\mathbb{H}} U) \to S^{k+l}_{\mathbb{H}} U$ be the natural projection. The maps $\mu_{k,l}$ combine to give an A$\mathbb{H}$-morphism $\mu : A \otimes_{\mathbb{H}} A \to A$. Recall that $S^0_{\mathbb{H}} U = \mathbb{H}$, and define $1 \in A$ to be $1 \in S^0_{\mathbb{H}} U$. These make A into an H-algebra, the *free H-algebra generated by* U.

In the next few results we will prove that if M is a hypercomplex manifold, then the A$\mathbb{H}$-module A_M of q-holomorphic functions on M is an H-algebra. If M and N are hypercomplex manifolds, then $M \times N$ is also a hypercomplex manifold. We shall show that q-holomorphic functions on M, N and $M \times N$ are related by the quaternionic tensor product $\otimes_{\mathbb{H}}$.

Proposition 5.2 *Let M and N be hypercomplex manifolds. Then there exists a canonical, injective* A$\mathbb{H}$*-morphism* $\phi : A_M \otimes_{\mathbb{H}} A_N \to A_{M\times N}$.

Proof. Let $f \in A_M \otimes_{\mathbb{H}} A_N$. We will use f to construct a q-holomorphic function F on $M \times N$. Let $m \in M$ and $n \in N$. Then $\theta_m \in A_M^\dagger$ and $\theta_n \in A_N^\dagger$. Applying the map λ_{A_M,A_N} of Definition 4.2, $\lambda_{A_M,A_N}\big(\theta_m \otimes \theta_n\big) \in (A_M \otimes_{\mathbb{H}} A_N)^\dagger$. Define $F(m,n) = \lambda_{A_M,A_N}(\theta_m \otimes \theta_n) \cdot f$. This yields a map $F : M \times N \to \mathbb{H}$.

As each F is made from a finite number of smooth functions on M, N (see below), we see that F is smooth. Also, for each $n \in N$, the map $m \mapsto F(m,n)$ lies in A_M. Thus F is q-holomorphic in the 'M' directions. Similarly, F is q-holomorphic in the 'N' directions, so F is q-holomorphic on $M \times N$, and $F \in A_{M\times N}$. Define $\phi : A_M \otimes_{\mathbb{H}} A_N \to A_{M\times N}$ by $\phi(f) = F$. Clearly ϕ is $\mathbb{H}$-linear. It is easy to show that $F = 0$ if and only if $f = 0$, and that $F \in A'_{M\times N}$ if and only if $f \in (A_M \otimes_{\mathbb{H}} A_N)'$. Thus ϕ is an injective A$\mathbb{H}$-morphism, as we have to prove. ∎

Note that if U, V are real infinite-dimensional vector spaces, then there are several ways to define the tensor product $U \otimes V$, which can give different answers. In this paper we use the convention that every element of $U \otimes V$ is a *finite* sum $\Sigma_i u_i \otimes v_i$. In the proof above, $A_M \otimes_{\mathbb{H}} A_N$ is a subspace of $\mathbb{H} \otimes (A_M^\dagger)^* \otimes (A_N^\dagger)^*$, where $(A_M^\dagger)^*$ and $(A_N^\dagger)^*$ may be infinite-dimensional. The statement that 'each F is made from a finite number of smooth functions' in the proof uses the definition of $(A_M^\dagger)^* \otimes (A_N^\dagger)^*$ as a collection of finite sums.

The following Lemma is trivial, and the proof will be omitted.

Lemma 5.3 *Suppose M is a hypercomplex manifold, and N is a hypercomplex submanifold of M. If f is a q-holomorphic function on M, then $f|_N$ is q-holomorphic on N. Let $\rho : A_M \to A_N$ be the restriction map. Then ρ is an* A$\mathbb{H}$*-morphism.*

Let M be a hypercomplex manifold. Our goal is to make A_M into an H-algebra. First we can define the multiplication map μ on A_M.

Definition 5.4 Let M be a hypercomplex manifold. Proposition 5.2 gives an A$\mathbb{H}$-morphism $\phi : A_M \otimes_{\mathbb{H}} A_M \to A_{M\times M}$. Now the diagonal submanifold $\big\{(m,m) : m \in M\big\}$ of $M \times M$ is a hypercomplex submanifold of $M \times M$, isomorphic to M. Therefore Lemma 5.3 gives an A$\mathbb{H}$-morphism $\rho : A_{M\times M} \to A_M$. Define an A$\mathbb{H}$-morphism $\mu : A_M \otimes_{\mathbb{H}} A_M \to A_M$ by $\mu = \rho \circ \phi$.

Here is the main result of this section.

Theorem 5.5 *Let M be a hypercomplex manifold, so that A_M is an* A$\mathbb{H}$*-module. Let $1 \in A_M$ be the constant function on M with value 1, and let μ be the* A$\mathbb{H}$*-morphism given in Definition 5.4. With these definitions, A_M is an H-algebra.*

Proof. We must show that Axiom H is satisfied. Part (i) holds by Definition 3.1, and parts (ii) and (v) are trivial. For part (iii), observe that the permutation map $A_M \otimes_{\mathbb{H}} A_M \to A_M \otimes_{\mathbb{H}} A_M$ that swaps round the factors, is induced by the map $M \times M \to M \times M$ given by $(m_1, m_2) \mapsto (m_2, m_1)$. Since the diagonal submanifold is invariant under this, it follows that μ is invariant under permutation, and so $\Lambda^2_{\mathbb{H}} A_M \subset \operatorname{Ker} \mu$.

Let Δ^2_M be the 'diagonal' submanifold in $M \times M$, and let Δ^3_M be the 'diagonal' submanifold in $M \times M \times M$. We interpret part (iv) as follows. $A_M \otimes_{\mathbb{H}} A_M \otimes_{\mathbb{H}} A_M$ is a space of q-holomorphic functions on $M \times M \times M$. The maps $\mu \otimes_{\mathbb{H}} \mathrm{id}$ and $\mathrm{id} \otimes_{\mathbb{H}} \mu$ are the maps restricting to $\Delta^2_M \times M$ and $M \times \Delta^2_M$ respectively. Thus $\mu \circ (\mu \otimes_{\mathbb{H}} \mathrm{id})$ is the result of first restricting to $\Delta^2_M \times M$ and then to Δ^3_M, and $\mu \circ (\mathrm{id} \otimes_{\mathbb{H}} \mu)$ is the result of first restricting to $M \times \Delta^2_M$ and then to Δ^3_M. Clearly $\mu \circ (\mu \otimes_{\mathbb{H}} \mathrm{id}) = \mu \circ (\mathrm{id} \otimes_{\mathbb{H}} \mu)$, proving part (iv). Thus all of Axiom H applies, and A_M is an H-algebra. ∎

Theorem 5.5 is an important part of the analogy we are building between real or complex algebra and geometry, and quaternionic algebra and geometry. We know that the holomorphic functions on a complex manifold form a commutative algebra over $\mathbb{C}$, and this Theorem shows that the q-holomorphic functions on a hypercomplex manifold form an H-algebra, which is the analogue over $\mathbb{H}$ of a commutative algebra.

Next we consider the question: given an H-algebra, can we reconstruct a hypercomplex manifold from it? Let M be hypercomplex and $m \in M$. Then $\theta_m \in A^\dagger_M$, so that $\theta_m : A_M \to \mathbb{H}$ is an A$\mathbb{H}$-morphism. But $\mathbb{H}$ itself is an H-algebra, and θ_m is actually an *H-algebra morphism*, in the sense of the following definition:

Definition 5.6 Let A, B be H-algebras, and let $\phi : A \to B$ be an A$\mathbb{H}$-morphism. Write $1_A, 1_B$ for the identities and μ_A, μ_B for the multiplication maps in A, B respectively. We say ϕ is an *H-algebra morphism* if $\phi(1_A) = 1_B$ and $\mu_B \circ (\phi \otimes_{\mathbb{H}} \phi) = \phi \circ \mu_A$ as A$\mathbb{H}$-morphisms $A \otimes_{\mathbb{H}} A \to B$.

It is easy to see that in Lemma 5.3, the A$\mathbb{H}$-morphism ρ is actually an H-algebra morphism. In the special case that N is the single point $m \in M$, which is trivially a hypercomplex manifold of dimension zero, A_N is $\mathbb{H}$ and $\rho : A_M \to \mathbb{H}$ is just θ_m. This suggests a way to recover the hypercomplex manifold M from the H-algebra A_M.

Let A be an H-algebra, and define the *quaternionic variety* M_A to be the set of H-algebra morphisms $\theta : A \to \mathbb{H}$. In particularly good cases, M_A is a manifold, with a unique hypercomplex structure determined by A, and A is an H-subalgebra of A_{M_A}. However, the general situation is more complex, as M_A may be singular, or may carry a different geometric structure. The study of H-algebras A and their quaternionic varieties M_A appears to be an interesting new field, which could be called *hypercomplex algebraic geometry.*

6 Hyperkähler manifolds and HP-algebras

A metric g on a complex manifold M is called *Kähler* if

$$g_{ab} = I^c_a I^d_b g_{cd} \qquad \text{and} \qquad \nabla I = 0, \tag{6.1}$$

where ∇ is the Levi-Civita connection of g, and I is the complex structure. The 2-form $\omega_{ac} = I^c_a g_{bc}$ is a closed (1,1)-form on M called the *Kähler form.* The Kähler form ω is a symplectic form on M, so that M is a *symplectic manifold.* Let M be a symplectic manifold, and P the algebra of smooth real functions on M. Then the symplectic structure on M induces a bilinear map $\{\,,\,\} : P \times P \to P$ called the *Poisson bracket.* The algebra structure on P together with the Poisson bracket $\{\,,\,\}$ make P into a *Poisson algebra.* Poisson algebras are studied in [2].

A *hyperkähler structure* on M is a quadruple (I_1, I_2, I_3, g), where (I_1, I_2, I_3) is a hypercomplex structure, and g is a riemannian metric that is Kähler with respect to each of I_1, I_2 and I_3.

If M has a hyperkähler structure, it is called a *hyperkähler manifold*. A hyperkähler manifold has three Kähler forms $\omega_1, \omega_2, \omega_3$. Hyperkähler manifolds are the natural analogue over $\mathbb{H}$ of Kähler manifolds. For more information on hyperkähler manifolds, see [13, p. 114-123], [6], [10] and [11]. If M is a hyperkähler manifold, then the Kähler forms ω_1, ω_2 and ω_3 make M into a symplectic manifold in three different ways, and therefore induce three different Poisson brackets $\{\,,\,\}_1, \{\,,\,\}_2$ and $\{\,,\,\}_3$ on the algebra P of smooth real functions on M.

Now the previous section showed how to associate an H-algebra A_M with each hypercomplex manifold M, so that the geometry of the hypercomplex structure of M is reflected in the algebraic structure of A_M. A hyperkähler structure is a hypercomplex structure with some extra data, the metric g. It is natural to hope that the metric g on a hyperkähler manifold M might be encoded in some additional algebraic information on the H-algebra A_M. One obvious possibility is that A_M might carry some sort of quaternionic analogue of a Poisson bracket.

We shall see in this section that this is indeed true: an algebraic structure ξ can be constructed on A_M analogous to a Poisson bracket, which makes A_M into an *HP-algebra*, the quaternionic analogue of a Poisson algebra. To save space, and because we have wandered from the main subject of the paper, we will omit all proofs in this section. The proofs are elementary tensor calculations and fairly dull, and we leave them as an exercise for the reader.

We start by defining a special A$\mathbb{H}$-module Y.

Definition 6.1 Define $Y \subset \mathbb{H}^3$ by $Y = \big\{(q_1,q_2,q_3) : q_1 i_1 + q_2 i_2 + q_3 i_3 = 0\big\}$. Then $Y \cong \mathbb{H}^2$ is an $\mathbb{H}$-module. Define $Y' \subset Y$ by $Y' = \big\{(q_1,q_2,q_3) \in Y : q_j \in \mathbb{I}\big\}$. Then $\dim Y' = 5$ and $\dim Y^\dagger = 3$. Thus $\dim Y = 4j$ and $\dim Y' = 2j + r$, where $j = 2$ and $r = 1$. Define a map $\nu : Y \rightarrow \mathbb{H}$ by $\nu\big((q_1,q_2,q_3)\big) = i_1 q_1 + i_2 q_2 + i_3 q_3$. Then $\operatorname{Im}\nu = \mathbb{I}$, and $\operatorname{Ker}\nu = Y'$. But $Y/Y' \cong (Y^\dagger)^*$, so that ν induces an isomorphism $\nu : (Y^\dagger)^* \rightarrow \mathbb{I}$. Since $\mathbb{I} \cong \mathbb{I}^*$, we have $(Y^\dagger)^* \cong \mathbb{I} \cong Y^\dagger$.

Now, define an *HP-algebra*, or *Hamilton-Poisson algebra* A to satisfy Axiom H of §5 and Axioms L and P below. In Axiom L, we suppose A is an A$\mathbb{H}$-module, and in Axiom P we suppose A is an H-algebra.

Axiom L. *(i)* There is an A$\mathbb{H}$-morphism $\xi : A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} Y$ called the *Lie bracket* or *Poisson bracket*, where Y is the A$\mathbb{H}$-module of Definition 6.1.

(ii) $S^2_{\mathbb{H}} A \subset \operatorname{Ker}\xi$. Thus ξ is *antisymmetric.*

(iii) There are A$\mathbb{H}$-morphisms $\mathrm{id} \otimes_{\mathbb{H}} \xi : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} Y$ and $\xi \otimes_{\mathbb{H}} \mathrm{id} : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} Y \rightarrow A \otimes_{\mathbb{H}} Y \otimes_{\mathbb{H}} Y$. Composing gives an A$\mathbb{H}$-morphism $(\xi \otimes_{\mathbb{H}} \mathrm{id}) \circ (\mathrm{id} \otimes_{\mathbb{H}} \xi) : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} Y \otimes_{\mathbb{H}} Y$. Then $\Lambda^3_{\mathbb{H}} A \subset \operatorname{Ker}\big((\xi \otimes_{\mathbb{H}} \mathrm{id}) \circ (\mathrm{id} \otimes_{\mathbb{H}} \xi)\big)$. This is *the Jacobi identity for* ξ.

Axiom P. *(i)* If $a \in A$, we have $1 \otimes_{\mathbb{H}} a \in A \otimes_{\mathbb{H}} A$. Then $\xi(1 \otimes_{\mathbb{H}} a) = 0$.

(ii) There are A$\mathbb{H}$-morphisms $\mathrm{id} \otimes_{\mathbb{H}} \xi : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} Y$ and $\mu \otimes_{\mathbb{H}} \mathrm{id} : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} Y \rightarrow A \otimes_{\mathbb{H}} Y$. Composing gives an A$\mathbb{H}$-morphism $(\mu \otimes_{\mathbb{H}} \mathrm{id}) \circ (\mathrm{id} \otimes_{\mathbb{H}} \xi) : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} Y$. Similarly, there are A$\mathbb{H}$-morphisms $\mu \otimes_{\mathbb{H}} \mathrm{id} : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} A$ and $\xi : A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} Y$. Composing gives an A$\mathbb{H}$-morphism $\xi \circ (\mu \otimes_{\mathbb{H}} \mathrm{id}) : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} Y$. Then $\xi \circ (\mu \otimes_{\mathbb{H}} \mathrm{id}) = 2(\mu \otimes_{\mathbb{H}} \mathrm{id}) \circ (\mathrm{id} \otimes_{\mathbb{H}} \xi)$ on $S^2_{\mathbb{H}} A \otimes_{\mathbb{H}} A$. This is *the derivation property.*

Here is some motivation for these definitions. An HP-algebra is intended to be the quaternionic analogue of a Poisson algebra. Now a Poisson algebra P is a commutative algebra, so its quaternionic analogue should be an H-algebra, and satisfy Axiom H. Also, a Poisson algebra has a Poisson bracket, an antisymmetric bilinear map $\{\,,\}:P\times P\rightarrow P$. As with the multiplication map in §5, it is convenient to rewrite this as a linear map $\xi:P\otimes P\rightarrow P$. A Poisson bracket must satisfy two important conditions. Firstly, it should satisfy the *Jacobi identity*, which makes into a *Lie bracket* on P. Axiom L above gives the quaternionic analogue of the definition of a Lie bracket. Secondly, the Poisson bracket and the algebra structure of P must be compatible with each other in two ways: they must satisfy $\{1,a\}=0$ for all $a\in P$, and also $\{ab,c\}=a\{b,c\}+b\{a,c\}$ for all $a,b,c\in P$. This is called the *derivation property*. The quaternionic analogues of these conditions are given in Axiom P.

The least obvious thing about these definitions, is the use of the A$\mathbb{H}$-module Y. It would seem more natural for ξ to map $A\otimes_{\mathbb{H}}A$ to A, not $A\otimes_{\mathbb{H}}Y$. The reason for this is that a hyperkähler manifold M has three Poisson structures $\{\,,\}_1,\{\,,\}_2$ and $\{\,,\}_3$, rather than one. We may regard $A\otimes_{\mathbb{H}}Y$ as a subspace of $A\otimes(Y^\dagger)^*$, and $(Y^\dagger)^*\cong\mathbb{I}$ by Definition 6.1. Thus ξ is an antisymmetric map from $A\otimes_{\mathbb{H}}A$ to $A\otimes\mathbb{I}$, that is, a triple of antisymmetric maps from $A\otimes_{\mathbb{H}}A$ to A. These 3 antisymmetric maps should be interpreted as the 3 Poisson brackets on M.

As an example we will now construct an HP-algebra $F_{\mathfrak{g}}$ from a Lie algebra $\mathfrak{g}$. This will be useful in §§11 and 12.

Example 6.2 Let $\mathfrak{g}$ be a Lie algebra, and let $F_{\mathfrak{g}}$ be the free H-algebra $F^{\mathfrak{g}\otimes Y}$ defined in Example 5.1, generated by the A$\mathbb{H}$-module $\mathfrak{g}\otimes Y$, where Y is the A$\mathbb{H}$-module defined above. We will explain how to define an A$\mathbb{H}$-morphism $\xi:F_{\mathfrak{g}}\otimes_{\mathbb{H}}F_{\mathfrak{g}}\rightarrow F_{\mathfrak{g}}\otimes_{\mathbb{H}}Y$, that makes $F_{\mathfrak{g}}$ into an HP-algebra. Now Y is a *stable* A$\mathbb{H}$-module in the sense of §8, and using Theorem 9.1 and Proposition 9.6 of §9, it is easy to show that $\bigotimes^j_{\mathbb{H}}Y=S^j_{\mathbb{H}}Y$. In other words, in the decomposition of $\bigotimes^j_{\mathbb{H}}Y$ into components under the action of the symmetric group S_j, the only nonzero component is $S^j_{\mathbb{H}}Y$. This implies that $S^j_{\mathbb{H}}(\mathfrak{g}\otimes Y)=S^j\mathfrak{g}\otimes S^j_{\mathbb{H}}Y$, and it follows that $F_{\mathfrak{g}}=\bigoplus_{j=0}^\infty S^j\mathfrak{g}\otimes S^j_{\mathbb{H}}Y$.

As $\mathfrak{g}$ is a Lie algebra, the Lie bracket $[\,,]$ on $\mathfrak{g}$ gives a linear map $\lambda:\mathfrak{g}\otimes\mathfrak{g}\rightarrow\mathfrak{g}$, such that $\lambda(x\otimes y)=[x,y]$ for $x,y\in\mathfrak{g}$. Using λ, for $j,k\ge 1$ define $\lambda_{j,k}:S^j\mathfrak{g}\otimes S^k\mathfrak{g}\rightarrow S^{j+k-1}\mathfrak{g}$ to be the composition of maps

$$S^j\mathfrak{g}\otimes S^k\mathfrak{g}\xrightarrow{\iota_1\otimes\iota_2}\bigl(S^{j-1}\mathfrak{g}\otimes\mathfrak{g}\bigr)\otimes\bigl(\mathfrak{g}\otimes S^{k-1}\mathfrak{g}\bigr)\xrightarrow{\mathrm{id}\otimes\lambda\otimes\mathrm{id}}S^{j-1}\mathfrak{g}\otimes\mathfrak{g}\otimes S^{k-1}\mathfrak{g}\xrightarrow{\sigma}S^{j+k-1}\mathfrak{g},\tag{6.2}$$

where $\iota_1:S^j\mathfrak{g}\rightarrow S^{j-1}\mathfrak{g}\otimes\mathfrak{g}$ and $\iota_2:S^k\mathfrak{g}\rightarrow\mathfrak{g}\otimes S^{k-1}\mathfrak{g}$ are the natural inclusions, and $\sigma:S^{j-1}\mathfrak{g}\otimes\mathfrak{g}\otimes S^{k-1}\mathfrak{g}\rightarrow S^{j+k-1}\mathfrak{g}$ is the symmetrization map. Define an A$\mathbb{H}$-morphism

$$\xi_{j,k}:\bigl(S^j\mathfrak{g}\otimes S^j_{\mathbb{H}}Y\bigr)\otimes_{\mathbb{H}}\bigl(S^k\mathfrak{g}\otimes S^k_{\mathbb{H}}Y\bigr)\rightarrow\bigl(S^{j+k-1}\mathfrak{g}\otimes S^{j+k-1}_{\mathbb{H}}Y\bigr)\otimes_{\mathbb{H}}Y\tag{6.3}$$

by $\xi_{j,k}=jk\cdot\lambda_{j,k}\otimes\mathrm{id}$, where since $S^j_{\mathbb{H}}Y\otimes_{\mathbb{H}}S^k_{\mathbb{H}}Y=S^{j+k}_{\mathbb{H}}Y=S^{j+k-1}_{\mathbb{H}}Y\otimes_{\mathbb{H}}Y$, the identity map $\mathrm{id}:S^j_{\mathbb{H}}Y\otimes_{\mathbb{H}}S^k_{\mathbb{H}}Y\rightarrow S^{j+k-1}_{\mathbb{H}}Y\otimes_{\mathbb{H}}Y$ is a natural A$\mathbb{H}$-morphism.

Finally, define $\xi:F_{\mathfrak{g}}\otimes F_{\mathfrak{g}}\rightarrow F_{\mathfrak{g}}\otimes_{\mathbb{H}}Y$ to be $\xi_{j,k}$ on each $\bigl(S^j\mathfrak{g}\otimes S^j_{\mathbb{H}}Y\bigr)\otimes_{\mathbb{H}}\bigl(S^k\mathfrak{g}\otimes S^k_{\mathbb{H}}Y\bigr)$ in $F_{\mathfrak{g}}\otimes_{\mathbb{H}}F_{\mathfrak{g}}$. It can be shown that $F_{\mathfrak{g}}$ and ξ satisfy Axioms L and P above, so that $F_{\mathfrak{g}}$ is an HP-algebra. Here the Jacobi identity L(iii) for $F_{\mathfrak{g}}$ follows from the Jacobi identity satisfied by the Lie bracket λ of the Lie algebra $\mathfrak{g}$.

Next we will explain, without proofs, how to define the quaternionic Poisson bracket ξ on the H-algebra A_M of a hyperkähler manifold.

Definition 6.3 Let M be a hyperkähler manifold. Then $M \times M$ is also a hyperkähler manifold. Let $\Delta^2_M = \{(m,m) : m \in M\}$. Then Δ^2_M is a hyperkähler submanifold of $M \times M$. We shall write $M \times M = M^1 \times M^2$, using the superscripts ^{1} and ^{2} to distinguish the two factors. Let ∇ be the Levi-Civita connection on M. Define ∇^1, ∇^2 to be the lift of ∇ to the first and second factors of M in $M \times M$ respectively. Then ∇^1 and ∇^2 commute. Let ∇^{12} be the Levi-Civita connection on $M \times M$. Then $\nabla^{12} = \nabla^1 + \nabla^2$.

Let $x \in A_{M\times M}$. Then $\nabla^1\nabla^2 x \in C^\infty(\mathbb{H} \otimes T^*M^1 \otimes T^*M^2)$ over $M \times M$. Restrict $\nabla^1\nabla^2 x$ to Δ^2_M. Then $\Delta^2_M \cong M$ and $T^*M^1|_{\Delta^2_M} \cong T^*M^2|_{\Delta^2_M} \cong T^*M$. Thus $\nabla^1\nabla^2 x|_{\Delta^2_M} \in C^\infty(\mathbb{H} \otimes T^*M \otimes T^*M)$ over M. Define a linear map $\Theta : A_{M\times M} \to C^\infty(M, \mathbb{H}) \otimes \mathbb{I}$ by

$$\Theta(x) = \{g^{ab}(I_1)^c_a \nabla^1_b \nabla^2_c x|_{\Delta^2_M}\} \otimes i_1 + \{g^{ab}(I_2)^c_a \nabla^1_b \nabla^2_c x|_{\Delta^2_M}\} \otimes i_2 + \{g^{ab}(I_3)^c_a \nabla^1_b \nabla^2_c x|_{\Delta^2_M}\} \otimes i_3, \quad (6.4)$$

using index notation for tensors on M in the obvious way. Here $C^\infty(M, \mathbb{H})$ is the space of smooth $\mathbb{H}$-valued functions on M.

Here are some properties of Θ.

Proposition 6.4 *This map satisfies* $\Theta(x) \in A_M \otimes \mathbb{I}$. *Also,* $\Theta : A_{M\times M} \to A_M \otimes \mathbb{I}$ *is an* A$\mathbb{H}$-*morphism, and if* $\Theta(x) = x_1 \otimes i_1 + x_2 \otimes i_2 + x_3 \otimes i_3$ *and* $m \in M$, *then* $x_1(m)i_1 + x_2(m)i_2 + x_3(m)i_3 = 0 \in \mathbb{H}$.

Now we can define the map ξ.

Definition 6.5 Proposition 5.2 defines an A$\mathbb{H}$-morphism $\phi : A_M \otimes_{\mathbb{H}} A_M \to A_{M\times M}$. Definition 6.3 and Proposition 6.4 define an A$\mathbb{H}$-morphism $\Theta : A_{M\times M} \to A_M \otimes \mathbb{I}$. Let Y be the A$\mathbb{H}$-module of Definition 6.1. Then $(Y^\dagger)^* \cong \mathbb{I}$. Recall that $A_M \cong \iota(A_M)$, so we may identify $A_M \otimes \mathbb{I} \cong \iota(A_M) \otimes (Y^\dagger)^* \subset \mathbb{H} \otimes (A^\dagger_M)^* \otimes (Y^\dagger)^*$. Define $\xi : A_M \otimes_{\mathbb{H}} A_M \to \iota(A_M) \otimes (Y^\dagger)^*$ to be the composition $\xi = \Theta \circ \phi$.

Here is the main result of this section.

Theorem 6.6 *This* ξ *maps* $A_M \otimes_{\mathbb{H}} A_M$ *to* $A_M \otimes_{\mathbb{H}} Y$. *It is an* A$\mathbb{H}$-*morphism, and satisfies Axioms L and P. Thus, by Theorem 5.5, if* M *is a hyperkähler manifold then the vector space* A_M *of q-holomorphic functions on* M *is an HP-algebra.*

As with H-algebras and hypercomplex manifolds, given an HP-algebra A, in particularly good cases one can reconstruct a hyperkähler manifold M from A, with its full hyperkähler structure, such that A is an HP-subalgebra of A_M. Thus, hyperkähler manifolds may be constructed and studied using algebraic methods. We shall return to this idea in §12.

7 Differences between real and quaternionic algebra

The philosophy of this paper is that much algebra over $\mathbb{R}$ or $\mathbb{C}$ also works over $\mathbb{H}$, when we replace vector spaces by A$\mathbb{H}$-modules, and so on. However, quaternionic algebra also has properties rather unlike real or complex algebra, which come from the noncommutativity of the quaternions. In this section we discuss the differences between the theories, illustrating them by a series of examples.

Example 7.1 Let U, V be nonzero $\mathbb{H}$-modules, and let $U' = V' = \{0\}$. Then $U^\dagger = U^\times$ and $V^\dagger = V^\times$. Suppose that $x \in U \otimes_{\mathbb{H}} V$, and let $p, q \in \mathbb{H}$, $\alpha \in U^\dagger$ and $\beta \in V^\dagger$. Then

$$x \cdot (\alpha \otimes \beta)\overline{p}\,\overline{q} = x \cdot \big((p\alpha) \otimes \beta\big)\overline{q} = x \cdot \big((p\alpha) \otimes (q\beta)\big) = x \cdot \big(\alpha \otimes (q\beta)\big)\overline{p} = x \cdot (\alpha \otimes \beta)\overline{q}\,\overline{p}. \quad (7.1)$$

Choosing p and q such that $\overline{p}\,\overline{q} \neq \overline{q}\,\overline{p}$, we see that $x \cdot (\alpha \otimes \beta) = 0$. Thus $x = 0$, as this holds for all α, β. Therefore $U \otimes_{\mathbb{H}} V = \{0\}$, even though U, V are nonzero.

Example 7.2 Let $p, q \in \mathbb{I}$ be nonzero, and let X_p, X_q be the A$\mathbb{H}$-modules defined in Definition 3.2. It is easy to show that $X_p \otimes_{\mathbb{H}} X_q = \{0\}$ if p, q are not proportional, and $X_p \otimes_{\mathbb{H}} X_q \cong X_p$ if p, q are proportional. More generally, suppose U, V are A$\mathbb{H}$-modules, with $\dim U = 4k$, $\dim V = 4l$. It is easy to prove that $\dim(U \otimes_{\mathbb{H}} V) = 4n$, where $0 \leq n \leq kl$. However, this example shows that n can vary discontinuously under smooth variations of U', V'.

These examples show that if U and V are nonzero A$\mathbb{H}$-modules, then $U \otimes_{\mathbb{H}} V$ may be zero, and also $\dim(U \otimes_{\mathbb{H}} V)$ is not well-behaved, both of which contrast with real tensor products. However, in the next two sections we will study a subclass of A$\mathbb{H}$-modules called *stable* A$\mathbb{H}$-*modules*. If U and V are nonzero stable A$\mathbb{H}$-modules, then $U \otimes_{\mathbb{H}} V$ is nonzero, and $\dim(U \otimes_{\mathbb{H}} V)$ is given by a simple formula.

Let U be an A$\mathbb{H}$-module with $\dim U = 4k$. The condition in Definition 2.2 implies that $\dim(U^\dagger) \geq k$. But $\dim(U') + \dim(U^\dagger) = 4k$, so $\dim(U') \leq 3k$. Example 7.1 illustrates the general principle that if $\dim(U')$ is small, then quaternionic tensor products involving U tend to be small or zero. A good rule is that the most interesting A$\mathbb{H}$-modules U are those in the range $2k \leq \dim(U') \leq 3k$.

Here is another example.

Example 7.3 Define A$\mathbb{H}$-modules U, V by

$$U = \mathbb{H}^2, \quad U' = \big\langle (i_2, 0), (i_3, 0), (1, i_2), (0, i_3) \big\rangle, \quad V = \mathbb{H}, \quad \text{and} \quad V' = \langle i_2, i_3 \rangle. \quad (7.2)$$

Define an A$\mathbb{H}$-morphism $\phi : U \to V$ by $\phi\big((p, q)\big) = q$. A short calculation shows that $U \otimes_{\mathbb{H}} V$ and $V \otimes_{\mathbb{H}} V$ are both A$\mathbb{H}$-isomorphic to V, but that $\phi \otimes_{\mathbb{H}} \mathrm{id} : U \otimes_{\mathbb{H}} V \to V \otimes_{\mathbb{H}} V$ is zero.

In this example, $\phi : U \to V$ and $\mathrm{id} : V \to V$ are both surjective, and also $\phi : U' \to V'$ and $\mathrm{id} : V' \to V'$ are surjective, but $\phi \otimes_{\mathbb{H}} \mathrm{id} : U \otimes_{\mathbb{H}} V \to V \otimes_{\mathbb{H}} V$ is not surjective. Thus, if $\phi : U \to W$ and $\psi : V \to X$ are surjective A$\mathbb{H}$-morphisms, then $\phi \otimes_{\mathbb{H}} \psi : U \otimes_{\mathbb{H}} V \to W \otimes_{\mathbb{H}} X$ may not be surjective. In algebraic language, this means that the quaternionic tensor product $\otimes_{\mathbb{H}}$ is not *right-exact*. However, stable A$\mathbb{H}$-modules do satisfy a form of right-exactness, which we will not explore in this paper. The next Lemma shows that $\otimes_{\mathbb{H}}$ is *left-exact*.

Lemma 7.4 *Suppose that $\phi : U \to W$ and $\psi : V \to X$ are injective* A$\mathbb{H}$*-morphisms. Then $\phi \otimes_{\mathbb{H}} \psi : U \otimes_{\mathbb{H}} V \to W \otimes_{\mathbb{H}} X$ is an injective* A$\mathbb{H}$*-morphism.*

Proof. Consider the map $\mathrm{id} \otimes (\phi^\times)^* \otimes (\psi^\times)^*$ of (4.4). Clearly this maps $\iota_U(U) \otimes (V^\dagger)^*$ to $\iota_W(W) \otimes (X^\dagger)^*$. As $\iota_U(U) \cong U$ and $\iota_W(W) \cong W$ and the map $\phi : U \to W$ is injective, we see that the kernel of $\mathrm{id} \otimes (\phi^\times)^* \otimes (\psi^\times)^*$ on $\iota_U(U) \otimes (V^\dagger)^*$ is $\iota_U(U) \otimes \mathrm{Ker}(\psi^\times)^*$. Similarly, the kernel on $(U^\dagger)^* \otimes \iota_V(V)$ is $\mathrm{Ker}(\phi^\times)^* \otimes \iota_V(V)$. Thus the kernel of $\phi \otimes_{\mathbb{H}} \psi$ is

$$\mathrm{Ker}(\phi \otimes_{\mathbb{H}} \psi) = \big(\iota_U(U) \otimes \mathrm{Ker}(\psi^\times)^*\big) \cap \big(\mathrm{Ker}(\phi^\times)^* \otimes \iota_V(V)\big). \quad (7.3)$$

But this is contained in $\big(\iota_U(U)\cap(\mathbb{H}\otimes\mathrm{Ker}(\phi^\times)^*)\big)\otimes(V^\dagger)^*$. Now $\iota_U(U)\cap\big(\mathbb{H}\otimes\mathrm{Ker}(\phi^\times)^*\big)=0$, since if $\iota(u)$ lies in $\mathbb{H}\otimes\mathrm{Ker}(\phi^\times)^*$ then $\phi(u)=0$ in W, so $u=0$ as ϕ is injective. Thus $\mathrm{Ker}(\phi\otimes_{\mathbb{H}}\psi)=0$, and $\phi\otimes_{\mathbb{H}}\psi$ is injective. ∎

Example 7.5 Define an A$\mathbb{H}$-module U by

$$U=\mathbb{H}^2 \quad\text{and}\quad U'=\big\langle(1,1),(i_1,i_2),(i_2,i_3),(i_3,i_1)\big\rangle. \tag{7.4}$$

Define $V=\big\{(q,0):q\in\mathbb{H}\big\}\subset U$. Then V is an A$\mathbb{H}$-submodule of U with $V'=\{0\}$. Put $W=U/V$, and $W'=U'/V'\subset W$. But then W is not an A$\mathbb{H}$-module, as the condition in Definition 2.2 is not satisfied. This example shows that the quotient of an A$\mathbb{H}$-module by an A$\mathbb{H}$-submodule is not always an A$\mathbb{H}$-module.

Here are two other differences between the quaternionic and ordinary tensor products. Firstly, despite Lemma 4.6, if $u\in U$ and $v\in V$, there is in general no element '$u\otimes_{\mathbb{H}}v$' in $U\otimes_{\mathbb{H}}V$. At best, there is a real linear map from some subspace of $U\otimes V$ to $U\otimes_{\mathbb{H}}V$. Secondly, if (U,U') is an A$\mathbb{H}$-module, then the obvious definition of *dual* A$\mathbb{H}$*-module* is the pair $(U^\times,U^\dagger)$. However, this seems not to be a fruitful idea. For $(U^\times,U^\dagger)$ may not be an A$\mathbb{H}$-module at all, and even if $U^\times,V^\times$ are A$\mathbb{H}$-modules, in general $U^\times\otimes_{\mathbb{H}}V^\times$ and $(U\otimes_{\mathbb{H}}V)^\times$ are not A$\mathbb{H}$-isomorphic.

8 Stable and semistable A$\mathbb{H}$-modules

Now two special sorts of A$\mathbb{H}$-modules will be defined, called *stable* and *semistable* A$\mathbb{H}$-modules. Our aim in this paper has been to develop a strong analogy between the theories of A$\mathbb{H}$-modules and vector spaces over a field. For stable A$\mathbb{H}$-modules it turns out that this analogy is more complete than in the general case, because various important properties of the vector space theory hold for stable but not for general A$\mathbb{H}$-modules. Therefore, in applications of the theory it will often be useful to restrict to stable A$\mathbb{H}$-modules, to exploit their better behaviour. We begin with a definition.

Definition 8.1 We say that a finite-dimensional A$\mathbb{H}$-module U is *semistable* if it is generated over $\mathbb{H}$ by the subspaces $U'\cap q\,U'$ for nonzero $q\in\mathbb{I}$.

Let V be an A$\mathbb{H}$-module, and define U to be the A$\mathbb{H}$-submodule of V generated over $\mathbb{H}$ by the subspaces $V'\cap q\,V'$ for nonzero $q\in\mathbb{I}$. Then U is semistable, and contains all semistable A$\mathbb{H}$-submodules of V. We call U the *maximal semistable* A$\mathbb{H}$*-submodule of* V*.*

Lemma 8.2 *Suppose that U is semistable, with $\dim U=4j$ and $\dim U'=2j+r$, for integers j,r. Then $U'+q\,U'=U$ for generic $q\in\mathbb{I}$. Thus $r\geq 0$.*

Proof. By definition, U is generated over $\mathbb{H}$ by the subspaces $U'\cap q\,U'$. So suppose U is generated over $\mathbb{H}$ by $U'\cap q_iU'$ for $i=1,\ldots,k$, where $0\neq q_i\in\mathbb{I}$. Let $q\in\mathbb{I}$, and suppose that $qq_i\neq q_iq$ for $i=1,\ldots,k$. This is true for generic q. Define $W_i=U'\cap q_iU'$. As U is generated over $\mathbb{H}$ by the W_i, we have $U=\Sigma_{i=1}^k\mathbb{H}\cdot W_i$.

Now $W_i=q_iW_i$, and $\mathbb{H}=\langle 1,q_i,q,qq_i\rangle$ as $qq_i\neq q_iq$. Thus $\mathbb{H}\cdot W_i=W_i+q\,W_i$. But $W_i\subseteq U'$, and so $\mathbb{H}\cdot W_i\subseteq U'+q\,U'$. As $U=\Sigma_{i=1}^k\mathbb{H}\cdot W_i$, $U\subseteq U'+q\,U'$, so $U=U'+q\,U'$ for generic $q\in\mathbb{I}$, as we have to prove. Now $\dim U = 4j$ and $\dim U' = 2j + r$, so $4j + 2r \geq 4j$ as $U = U' + q\,U'$, and therefore $r \geq 0$. ∎

Next we define stable $A\mathbb{H}$-modules.

Definition 8.3 Let U be a finite-dimensional $A\mathbb{H}$-module. We say that U *is a stable* $A\mathbb{H}$-*module* if $U = U' + q\,U'$ for all nonzero $q \in \mathbb{I}$.

The point of this definition will become clear soon. Now let $q \in \mathbb{I}$ be nonzero. In §3 we defined an $A\mathbb{H}$-module X_q by $X_q = \mathbb{H}$, and $X_q' = \{p \in \mathbb{H} : pq = -qp\}$. The following properties of X_q are easy to prove.

- $X_q' \subset \mathbb{I}$ and $\dim X_q' = \dim X_q^\dagger = 2$.
- X_q is semistable, but not stable.
- There is a canonical $A\mathbb{H}$-isomorphism $X_q \otimes_{\mathbb{H}} X_q \cong X_q$.
- Let $\chi_q : X_q \to \mathbb{H}$ be the identity map on $\mathbb{H}$. Then χ_q is an $A\mathbb{H}$-morphism. Hence, if U is an $A\mathbb{H}$-module, then $\mathrm{id} \otimes_{\mathbb{H}} \chi_q$ maps $U \otimes_{\mathbb{H}} X_q$ to $U \otimes_{\mathbb{H}} \mathbb{H}$. But $U \otimes_{\mathbb{H}} \mathbb{H} \cong U$, so $\mathrm{id} \otimes_{\mathbb{H}} \chi_q : U \otimes_{\mathbb{H}} X_q \to U$ is an $A\mathbb{H}$-morphism.
- Let U be an $A\mathbb{H}$-module with $\dim U = 4j$ and $\dim U' = 2j + r$. Let $q \in \mathbb{I}$ be nonzero. Now $\langle 1, q\rangle$ is a subalgebra of $\mathbb{H}$ isomorphic to $\mathbb{C}$, which acts on $U' \cap q\,U'$. Therefore $U' \cap q\,U'$ is isomorphic to $\mathbb{C}^n$, say, and $\dim(U' \cap q\,U') = 2n$ is even.
- As $\dim(U' + q\,U') = 4j + 2r - 2n$ and $U' + q\,U' \subset U$, we have $4j + 2r - 2n \leq 4j$, so $n \geq r$. Moreover $n = r$ if and only if $U' + q\,U' = U$.
- It can be shown that $U \otimes_{\mathbb{H}} X_q \cong nX_q$, as there is an isomorphism $(U \otimes_{\mathbb{H}} X_q)' \cong U' \cap q\,U'$.
- Therefore, if U is an $A\mathbb{H}$-module with $\dim U = 4j$ and $\dim U' = 2j + r$, and $q \in \mathbb{I}$ is nonzero, then $U \otimes_{\mathbb{H}} X_q \cong nX_q$ with $n \geq r$.

  If U is *semistable*, then $U \otimes_{\mathbb{H}} X_q \cong rX_q$ for generic $q \in \mathbb{I}$, by Lemma 8.2.

  Also, U is *stable* if and only if $U \otimes_{\mathbb{H}} X_q \cong rX_q$ for all nonzero $q \in \mathbb{I}$.

Lemma 8.4 *Let V be a semistable* $A\mathbb{H}$-*module with* $\dim V = 4k$ *and* $\dim V' = 2k + r$. *Let U be the* $A\mathbb{H}$-*submodule of V generated over $\mathbb{H}$ by the subsets $V' \cap q\,V'$ for those nonzero $q \in I$ with $V \otimes_{\mathbb{H}} X_q \cong rX_q$. Then U is a stable* $A\mathbb{H}$-*submodule.*

Proof. Clearly U is semistable, by definition. As V is semistable, $V \otimes_{\mathbb{H}} X_q \cong rX_q$ for generic $q \in \mathbb{I}$. But it is easily seen that if $V \otimes_{\mathbb{H}} X_q \cong rX_q$, then $U \otimes_{\mathbb{H}} X_q \cong rX_q$. Let $\dim U = 4j$. Then $\dim U' = 2j + r$, as U is semistable and $U \otimes_{\mathbb{H}} X_q \cong rX_q$ for generic $q \in \mathbb{I}$, and so $\dim U^\dagger = 2j - r$.

Suppose for a contradiction that $p \in \mathbb{I}$ is nonzero, and $U \otimes_{\mathbb{H}} X_p \cong nX_p$, with $n > r$. Write $Y = U' \cap p\,U'$, so $\dim Y = 2n$. Define a map $\phi : U^\dagger \to (U')^*$ by $\phi(\alpha)u = \mathrm{Re}(p\alpha(u))$, for $\alpha \in U^\dagger$ and $u \in U'$, where $\mathrm{Re}(p\alpha(u))$ is the real part of $p\alpha(u) \in \mathbb{H}$.

Let $\alpha \in U^\dagger$ and $y \in Y$. Then $y \in p\,U'$, so $\alpha(py) = p\alpha(y) \in \mathbb{I}$, and $\phi(\alpha)y = 0$. Thus $\phi(\alpha)$ vanishes on Y, and $\phi(\alpha) \in Y^\circ \subset (U')^*$. Therefore ϕ is a linear map from $U^\dagger$ to Y°. As

$\dim U^\dagger = 2j - r$ and $\dim Y^\circ = \dim U' - \dim Y = 2j + r - 2n$, $\dim \operatorname{Ker} \phi \geq 2(n-r) > 0$. Choose $\alpha \neq 0$ in $\operatorname{Ker} \phi$. Then $\alpha(u) \in \mathbb{I} \cap p\,\mathbb{I}$ for all $u \in U'$.

Let $q \in \mathbb{I}$ satisfy $pq \neq qp$, and suppose $u \in U' \cap q\,U'$. Then $\alpha(u) \in \mathbb{I} \cap p\,\mathbb{I}$ and $\alpha(u) \in q\,\mathbb{I} \cap qp\,\mathbb{I}$, so $\alpha(u) = 0$ and $u \in \operatorname{Ker} \alpha$. But U is generated over $\mathbb{H}$ by subspaces $U' \cap q\,U'$, so $U \subset \operatorname{Ker} \alpha$, a contradiction as $\alpha \neq 0$. Therefore there exists no $p \in \mathbb{I}$ with $U \otimes_{\mathbb{H}} X_p = nX_p$ with $n > r$, and U is stable. ∎

Definition 8.5 Let V be a finite-dimensional, semistable A$\mathbb{H}$-module, and let U be the A$\mathbb{H}$-submodule of V generated over $\mathbb{H}$ by the subspaces $V' \cap q\,V'$ for those $q \in \mathbb{I}$ for which $V' + q\,V' = V$. By Lemma 8.4, U is stable, and it is easy to show that U contains all stable A$\mathbb{H}$-submodules of V. We call U the *maximal stable* A$\mathbb{H}$*-submodule of* V.

Here are two results relating stable and semistable A$\mathbb{H}$-modules.

Theorem 8.6 *All stable* A$\mathbb{H}$*-modules are semistable.*

Proof. Let V be a stable A$\mathbb{H}$-module with $\dim V = 4k$ and $\dim V' = 2k + r$. Let U be the maximal semistable A$\mathbb{H}$-submodule of V. We will prove that $U = V$, so V is semistable. Let $\dim U = 4j$, and let $l = k - j$. We must show $l = 0$. As U contains each subspace $V' \cap q\,V'$, we see that $U \otimes_{\mathbb{H}} X_q = V \otimes_{\mathbb{H}} X_q \cong rX_q$ for all nonzero $q \in \mathbb{I}$, so $\dim U' = 2j + r$ using Lemma 8.2.

Let $W = V/U$ and $W' = (V' + U)/U \cong V'/U'$. Then W is an $\mathbb{H}$-module with $\dim W = 4l$, and W' a real subspace with $\dim W' = 2l$. Although (W, W') need not be an A$\mathbb{H}$-module, this does not matter.

Lemma 8.7 *If* $l > 0$*, then there exists some nonzero* $q \in \mathbb{I}$ *with* $W' \cap q\,W' \neq \{0\}$*.*

Proof. Let M be the grassmannian of oriented real vector subspaces of $W \cong \mathbb{H}^l$ of dimension 2. Then M is a compact, oriented manifold of dimension $8l - 4$, which is canonically isomorphic to a nonsingular quadric Q in $\mathbb{CP}^{4l-1}$. Let N be the subset of M of subspaces lying in W'. Then N is an oriented submanifold of dimension $4l - 4$, the intersection of Q with a linear subspace $\mathbb{CP}^{2l-1}$ in $\mathbb{CP}^{4l-1}$.

Let R be the subset of W of 2-planes of the form $\langle w, qw \rangle$, where $w \in W$ and $q \in \mathbb{I}$ are nonzero. Then R is an oriented submanifold of W of dimension $4l$, and a complex submanifold of Q. Now the homology classes $[N] \in H_{4l-4}(W, \mathbb{R})$ and $[R] \in H_{4l}(W, \mathbb{R})$ are well-defined and independent of W'. Calculation shows that their intersection number is $[N] \cdot [R] = 2l$. Therefore, if $l > 0$, then N and R must intersect, so there is some nonzero subspace $\langle w, qw \rangle \subset W'$. Thus $W' \cap q\,W' \neq \{0\}$, and the Lemma is complete. ∎

Let q be as in the Lemma. Since $\dim W' = \frac{1}{2} \dim W$ and $W' \cap q\,W' \neq 0$, we see that $W' + q\,W' \neq W$. But the projection of $V' + q\,V'$ to W lies in $W' + q\,W'$. Thus $V' + q\,V' \neq V$, a contradiction as V is stable. So $l = 0$, and $U = V$, and Theorem 8.6 is proved. ∎

Proposition 8.8 *Let* V *be a finite-dimensional* A$\mathbb{H}$*-module. Then* V *is semistable if and only if* $V \cong U \oplus \bigoplus_{i=1}^{l} X_{q_i}$*, where* U *is stable and* $q_i \in \mathbb{I}$ *is nonzero.*

Proof. The 'if' part follows from Theorem 8.6 and the fact that X_{q_i} is semistable. To prove the 'only if' part, let V be semistable with $\dim V = 4k$ and $\dim V' = 2k + r$. Let U be the maximal stable $\mathrm{A}\mathbb{H}$-module of V. Then $\dim U = 4j$ and $\dim U' = 2j + r$. Let $l = k - j$. Now V is semistable, which means that it is generated over $\mathbb{H}$ by $V' \cap qV'$ for nonzero $q \in \mathbb{I}$.

Therefore we may choose nonzero $q_1, \ldots, q_l \in \mathbb{I}$ and $v_1, \ldots, v_l$ with $v_j \in V' \cap q_j V'$, such that V is generated over $\mathbb{H}$ by U and $v_1, \ldots, v_l$. But U and $v_1, \ldots, v_l$ are linearly independent over $\mathbb{H}$. Counting dimensions, we see that $V' = U' \oplus \bigoplus_{i=1}^{l} \langle v_i, q_i v_i \rangle$. Thus $V = U \oplus \bigoplus_{i=1}^{l} X_{q_i}$, where $X_{q_i} = \mathbb{H} \cdot v_i$, as we have to prove. ∎

Finally we show that generic $\mathrm{A}\mathbb{H}$-modules (U, U') with appropriate dimensions are stable or semistable. Thus there are many stable and semistable $\mathrm{A}\mathbb{H}$-modules.

Lemma 8.9 *Let j, r be integers with $0 \le r \le j$. Let $U = \mathbb{H}^j$, and let U' be a real vector subspace of U with $\dim U' = 2j + r$. For generic subspaces U', (U, U') is a semistable $\mathrm{A}\mathbb{H}$-module. If $r > 0$, for generic subspaces U', (U, U') is a stable $\mathrm{A}\mathbb{H}$-module.*

Proof. Let G be the Grassmannian of real $(2j+r)$-planes in $U \cong \mathbb{R}^{4j}$. Then $U' \in G$, and $\dim G = 4j^2 - r^2$. The condition that (U, U') be an $\mathrm{A}\mathbb{H}$-module is that $\mathbb{H} \cdot U^\dagger = U^\times$. This fails for a subset of G of codimension $4(j - r + 1)$, so for generic $U' \in G$, (U, U') is an $\mathrm{A}\mathbb{H}$-module. If $r = 0$, it can be shown that for U' outside a subset of G of codimension 2, $(U, U') \cong \Sigma_{i=1}^{j} X_{q_i}$, for $q_1, \ldots, q_j$ in $\mathbb{I}$ pairwise linearly independent. As X_{q_i} is semistable, (U, U') is semistable for generic U'.

Now suppose $r > 0$, and let $0 \ne q \in \mathbb{I}$. Then the condition $U' + q\,U' = U$ fails for a subset of G of codimension $2r + 2$. For (U, U') to be stable, this condition must hold for $q \in \mathcal{S}^2$, the unit sphere in $\mathbb{I}$. Thus (U, U') is stable outside a subset of G of codimension $2r$, so generic $\mathrm{A}\mathbb{H}$-modules are stable, and hence also semistable by Theorem 8.6. ∎

9 Quaternionic tensor products of stable $\mathrm{A}\mathbb{H}$-modules

Here is the main result of this section.

Theorem 9.1 *Let U and V be stable $\mathrm{A}\mathbb{H}$-modules with*

$$\dim U = 4j, \quad \dim U' = 2j + r, \quad \dim V = 4k, \quad \textit{and} \quad \dim V' = 2k + s. \tag{9.1}$$

Then $U \otimes_{\mathbb{H}} V$ is a stable $\mathrm{A}\mathbb{H}$-module with $\dim(U \otimes_{\mathbb{H}} V) = 4l$ and $\dim(U \otimes_{\mathbb{H}} V)' = 2l + t$, where $l = js + rk - rs$ and $t = rs$.

Proof. Define D to be the $\mathbb{H}$-module $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$, so that $D^\times = \mathbb{H} \otimes U^\dagger \otimes V^\dagger$. Define A to be the $\mathbb{H}$-submodule $\iota_U(U) \otimes (V^\dagger)^*$ of D, and B to be the $\mathbb{H}$-submodule $(U^\dagger)^* \otimes \iota_V(V)$ of D. (These were defined in §4.) Define C to be the real vector subspace $\mathbb{I} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$ of D. Then

$$\dim A = 4j(2k–s), \ \dim B = 4k(2j–r), \ \dim C = 3(2j–r)(2k–s), \ \dim D = 4(2j–r)(2k–s). \tag{9.2}$$

Define *conjugation* on D in the obvious way, by $\overline{(p \otimes \alpha \otimes \beta)} = \overline{p} \otimes \alpha \otimes \beta$ for $p \in \mathbb{H}$, $\alpha \in (U^\dagger)^*$ and $\beta \in (V^\dagger)^*$. Let $\overline{B} = \{\overline{b} : b \in B\}$ be the subspace of D conjugate to B. Then B is a real subspace of D, but not necessarily an $\mathbb{H}$-submodule. It is easy to see that

$$A \cap B = U \otimes_{\mathbb{H}} V \qquad \text{and} \qquad A \cap B \cap C = A \cap \overline{B} \cap C = (U \otimes_{\mathbb{H}} V)'. \tag{9.3}$$

Define a subspace $K_{U,V}$ of $D^\times$ by $z \in K_{U,V}$ if $z(\zeta) = 0$ whenever $\zeta \in A$ or $\zeta \in B$. Then $K_{U,V}$ is an $\mathbb{H}$-submodule of $D^\times$. Clearly $\dim K_{U,V} = \dim D - \dim(A+B)$, and $\dim(A+B) = \dim A + \dim B - \dim(A \cap B)$. As $A \cap B = U \otimes_{\mathbb{H}} V$ by (9.3), these equations and (9.2) yield $\dim(U \otimes_{\mathbb{H}} V) = 4l + \dim K_{U,V}$, where $l = js + rk - rs$. Hence $\dim(U \otimes_{\mathbb{H}} V) = 4l$ if and only if $K_{U,V} = \{0\}$. So $\dim(U \otimes_{\mathbb{H}} V) = 4l$ by the next Lemma, as we have to prove.

Lemma 9.2 *In the above, $K_{U,V} = \{0\}$.*

Proof. Suppose that W is an A$\mathbb{H}$-module, and $\phi : W \to V$ is an A$\mathbb{H}$-morphism. Then $\phi^\times : V^\dagger \to W^\dagger$, so that $\mathrm{id} \otimes \phi^\times : \mathbb{H} \otimes U^\dagger \otimes V^\dagger \to \mathbb{H} \otimes U^\dagger \otimes W^\dagger$. We have $K_{U,V} \subset \mathbb{H} \otimes U^\dagger \otimes V^\dagger$ and $K_{U,W} \subset \mathbb{H} \otimes U^\dagger \otimes W^\dagger$. It is easy to show that $(\mathrm{id} \otimes \phi^\times)(K_{U,V}) \subset K_{U,W}$. Let $0 \neq q \in \mathbb{I}$. Set $W = V \otimes_{\mathbb{H}} X_q$ and let $\phi = \mathrm{id} \otimes_{\mathbb{H}} \chi_q : W \to V$ be the A$\mathbb{H}$-morphism defined in §8.

In this case $W \cong sX_q$. The argument above shows that $K_{U,X_q} = \{0\}$ if and only if $\dim(U \otimes_{\mathbb{H}} X_q) = 4r$. But this holds automatically, as U is stable. Thus $K_{U,X_q} = \{0\}$, and $K_{U,W} = \{0\}$ as $W \cong sX_q$. Therefore $(\mathrm{id} \otimes \phi^\times)(K_{U,V}) = \{0\}$, so $K_{U,V} \subset \mathbb{H} \otimes U^\dagger \otimes \operatorname{Ker} \phi^\times$. Now V is semistable, by Theorem 8.6. Therefore V is generated by submodules $\phi(W)$ of the above type, and the intersection of the subspaces $\operatorname{Ker} \phi^\times \subset V^\dagger$ for all nonzero q, must be zero. So $K_{U,V} \subset \mathbb{H} \otimes U^\dagger \otimes \{0\}$, giving $K_{U,V} = \{0\}$, which completes the Lemma. ∎

Next we shall study the intersection $A \cap \overline{B}$. Let $x \in \iota_U(U)$ and $y \in \iota_V(V)$. Then $x = \Sigma_e p_e \otimes \alpha_e$, where $p_e \in \mathbb{H}$ and $\alpha_e \in (U^\dagger)^*$. Similarly $y = \Sigma_f q_f \otimes \beta_f$, where $q_f \in \mathbb{H}$ and $\beta_f \in (V^\dagger)^*$. Consider the element $z = \Sigma_{e,f} \overline{q}_f p_e \otimes \alpha_e \otimes \beta_f$ of D. Clearly $z = \Sigma_f (\overline{q}_f \cdot x) \otimes \beta_f$. As $\iota_U(U)$ is an $\mathbb{H}$-module, $\overline{q}_f \cdot x \in \iota_U(U)$, so $z \in \iota_U(U) \otimes (V^\dagger)^* = A$. Similarly, $\overline{z} = \Sigma_{e,f} \overline{p}_e q_f \otimes \alpha_e \otimes \beta_f$ lies in B. Thus $z \in A \cap \overline{B}$.

From $x \in \iota_U(U)$ and $y \in \iota_V(V)$ we have manufactured an element $z \in A \cap \overline{B}$. It is easy to see that this construction is bilinear in x, y, and that the set of such z is a vector subspace of $A \cap \overline{B}$ of dimension $4jk$, as $\dim \iota_U(U) = 4j$ and $\dim \iota_V(V) = 4k$. Therefore $\dim(A \cap \overline{B}) \geq 4jk$. Now $(U \otimes_{\mathbb{H}} V)' = (A \cap \overline{B}) \cap C$ by (9.3). Therefore

$$\dim(U \otimes_{\mathbb{H}} V)' \geq \dim(A \cap \overline{B}) + \dim C - \dim D \geq 4jk - (2j - r)(2k - s) = 2l + t, \tag{9.4}$$

where $l = js + rk - rs$ and $t = rs$.

Let $q \in \mathbb{I}$ be nonzero. Then $U \otimes_{\mathbb{H}} X_q \cong rX_q$ and $V \otimes_{\mathbb{H}} X_q \cong sX_q$, as U, V are stable. Therefore $(U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} X_q \cong U \otimes_{\mathbb{H}} sX_q \cong rsX_q = tX_q$, using the associativity of $\otimes_{\mathbb{H}}$. It follows that $\dim\bigl((U \otimes_{\mathbb{H}} V)' \cap q\,(U \otimes_{\mathbb{H}} V)'\bigr) = 2t$ for all nonzero $q \in \mathbb{I}$. However,

$$\dim\bigl((U \otimes_{\mathbb{H}} V)' \cap q\,(U \otimes_{\mathbb{H}} V)'\bigr) + \dim\bigl((U \otimes_{\mathbb{H}} V)' + q\,(U \otimes_{\mathbb{H}} V)'\bigr) = 2\dim(U \otimes_{\mathbb{H}} V)'. \tag{9.5}$$

Combining these facts, the equation $\dim(U \otimes_{\mathbb{H}} V) = 4l$ and the inequality $\dim(U \otimes_{\mathbb{H}} V)' \geq 2l + t$, we see that $\dim(U \otimes_{\mathbb{H}} V) = 2l + t$, as we have to prove, and that $(U \otimes_{\mathbb{H}} V)' + q\,(U \otimes_{\mathbb{H}} V)' = U \otimes_{\mathbb{H}} V$. As this holds for all nonzero $q \in \mathbb{I}$, by definition $U \otimes_{\mathbb{H}} V$ is stable. This completes the proof of Theorem 9.1. ∎

Corollary 9.3 *Let U, V be semistable* $A\mathbb{H}$*-modules. Then $U \otimes_{\mathbb{H}} V$ is semistable.*

Proof. Write $U = W \oplus \Sigma_i X_{p_i}$ and $V = X \oplus \Sigma_j X_{q_j}$ by Proposition 8.8, where W, X are stable. Now if Y is any $A\mathbb{H}$-module then $Y \otimes_{\mathbb{H}} X_q \cong nX_q$ for some n. Therefore $U \otimes_{\mathbb{H}} V = W \otimes_{\mathbb{H}} X \oplus \Sigma_k X_{r_k}$, for some collection $\{r_k\}$ of nonzero elements of $\mathbb{I}$. But $W \otimes_{\mathbb{H}} X$ is stable by Theorem 9.1, so $U \otimes_{\mathbb{H}} V$ is semistable by Proposition 8.8. ∎

Thus both stable and semistable $A\mathbb{H}$-modules form subcategories of the tensor category of $A\mathbb{H}$-modules, which are closed under the operations of connected sum and quaternionic tensor product. This is a useful feature, as in mathematical applications we can choose to restrict our attention to stable or semistable $A\mathbb{H}$-modules, which have better properties than general $A\mathbb{H}$-modules. The next Corollary is easy to prove.

Corollary 9.4 *The dimension formulae in Theorem 9.1 also hold if U is stable but V is only semistable.*

Now we will define the virtual dimension of an $A\mathbb{H}$-module.

Definition 9.5 Let U be a stable $A\mathbb{H}$-module, with $\dim U = 4j$ and $\dim U' = 2j + r$. Define the *virtual dimension of* U to be r. If V is a finite-dimensional $A\mathbb{H}$-module, let U be its maximal stable $A\mathbb{H}$-submodule, and define the virtual dimension of V to be the virtual dimension of U.

Theorem 9.1 shows that the virtual dimension of $U \otimes_{\mathbb{H}} V$ is the product of the virtual dimensions of U and V. Thus the virtual dimension is a good analogue of the dimension of a vector space, as it multiplies under $\otimes_{\mathbb{H}}$. Note also in Theorem 9.1 that $(j-r)/r + (k-s)/s = (l-t)/t$, so that the nonnegative function $U \mapsto (j-r)/r$ behaves additively under $\otimes_{\mathbb{H}}$.

We leave the proof of the next Proposition to the reader, as a (difficult) exercise.

Proposition 9.6 *Let U be a stable* $A\mathbb{H}$*-module, with* $\dim U = 4j$ *and* $\dim U' = 2j + r$*. Let n be a positive integer. Then $S^n_{\mathbb{H}} U$ and $\Lambda^n_{\mathbb{H}} U$ are stable* $A\mathbb{H}$*-modules, with* $\dim(S^n_{\mathbb{H}} U) = 4k$, $\dim(S^n_{\mathbb{H}} U)' = 2k + s$, $\dim(\Lambda^n_{\mathbb{H}} U) = 4l$ *and* $\dim(\Lambda^n_{\mathbb{H}} U)' = 2l + t$*, where*

$$k = (j-r)\binom{r+n-1}{n-1} + \binom{r+n-1}{n}, \quad s = \binom{r+n-1}{n}, \quad l = (j-r)\binom{r-1}{n-1} + \binom{r}{n}, \quad t = \binom{r}{n}. \tag{9.6}$$

10 Q-holomorphic functions on $\mathbb{H}$

Let $\mathbb{H}$ have real coordinates $(x_0, \ldots, x_3)$, so that $(x_0, \ldots, x_3)$ represents $x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$. Now $\mathbb{H}$ is naturally a hypercomplex manifold with complex structures given by

$$I_1 dx_2 = dx_3, \quad I_2 dx_3 = dx_1, \quad I_3 dx_1 = dx_2 \quad \text{and} \quad I_j dx_0 = dx_j, \quad j = 1, 2, 3. \tag{10.1}$$

The study of q-holomorphic functions on $\mathbb{H}$ is called *quaternionic analysis*, and is surveyed in [14]. In this section, as a simple worked example of our theory, we shall study the q-holomorphic polynomials on $\mathbb{H}$, finding various dimension formulae, and showing that they form an HP-algebra.

Example 10.1 First we shall determine the AH-module U of all linear q-holomorphic functions on $\mathbb{H}$. Let $q_0, \ldots, q_3 \in \mathbb{H}$, and define $u = q_0 x_0 + \cdots + q_3 x_3$ as an $\mathbb{H}$-valued function on $\mathbb{H}$. A calculation using (10.1) shows that u is q-holomorphic if and only if $q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3 = 0$. It follows that $U \cong \mathbb{H}^3$. Also, U' is the vector subspace of U with $q_j \in \mathbb{I}$ for $j = 0, \ldots, 3$. Let us identify U with $\mathbb{H}^3$ explicitly by taking (q_1, q_2, q_3) as quaternionic coordinates. Then

$$U' = \bigl\{(q_1, q_2, q_3) \in \mathbb{H}^3 : q_j \in \mathbb{I} \text{ for } j = 1, 2, 3 \text{ and } q_1 i_1 + q_2 i_2 + q_3 i_3 \in \mathbb{I}\bigr\}. \tag{10.2}$$

Thus $U' \cong \mathbb{R}^8$, and $\dim U = 4j$, $\dim U' = 2j + r$ with $j = 3$ and $r = 2$, so the virtual dimension of U is 2. This is because $\mathbb{H} \cong \mathbb{C}^2$, so the complex dimension of $\mathbb{H}$ is 2. It is easy to see that U is a stable AH-module.

Example 10.2 Let $k \geq 0$ be an integer, and let $U^{(k)}$ be the AH-module of q-holomorphic functions on $\mathbb{H}$ that are homogeneous polynomials of degree k. We shall determine $U^{(k)}$. Let $\mu : A_{\mathbb{H}} \otimes_{\mathbb{H}} A_{\mathbb{H}} \to A_{\mathbb{H}}$ be the multiplication map on $A_{\mathbb{H}}$. By Example 10.1, $U^{(1)} = U \subset A$. Thus μ induces an AH-morphism $\mu : U \otimes_{\mathbb{H}} U \to A_{\mathbb{H}}$, and composing μ with itself $k-1$ times gives $\mu^{k-1} : \bigotimes_{\mathbb{H}}^k U \to A_{\mathbb{H}}$. Clearly, $\operatorname{Im} \mu^{k-1} \subset U^{(k)}$. Also, μ^{k-1} is symmetric in the k factors of U, so it makes sense to restrict to $S^k_{\mathbb{H}} U$.

This gives an AH-morphism $\mu^{k-1} : S^k_{\mathbb{H}} U \to U^{(k)}$. It is easily seen that μ^{k-1} is injective on $S^k_{\mathbb{H}} U$. By Example 10.1, U is stable with $j = 3$ and $r = 2$. Thus Proposition 9.6 shows that $\dim S^k_{\mathbb{H}} U = 2(k+1)(k+2)$. But Sudbery [14, Thm. 7, p. 217] shows that $\dim U^{(k)} = 2(k+1)(k+2)$. Therefore μ^{k-1} is an isomorphism, and $U^{(k)} \cong S^k_{\mathbb{H}} U$.

The interpretation of Example 10.2 is simple. If V is the linear polynomials on some vector space, then $S^k V$ is the homogeneous polynomials of degree k. Here we have a quaternionic analogue of this, replacing S^k by $S^k_{\mathbb{H}}$. We have found an elegant construction of the spaces $U^{(k)}$, important in quaternionic analysis, that gives insight into their algebraic structure and dimension.

Example 10.3 Let P be the set of q-holomorphic polynomials on $\mathbb{H}$. Then $P = \bigoplus_{j=0}^{\infty} U^{(j)}$, by definition of $U^{(j)}$. Also, P is clearly an H-subalgebra of $A_{\mathbb{H}}$, the H-algebra of q-holomorphic functions on $\mathbb{H}$. Since $U^{(j)} \cong S^j_{\mathbb{H}} U$ by Example 10.2, we see that $P \cong \bigoplus_{j=0}^{\infty} S^j_{\mathbb{H}} U$. Now Example 5.1 defined the *free H-algebra* F^U generated by U, which is also given by $F^U = \bigoplus_{j=0}^{\infty} S^j_{\mathbb{H}} U$. It is easy to prove that P and F^U are isomorphic as H-algebras. The full H-algebra $A_{\mathbb{H}}$ of q-holomorphic functions on $\mathbb{H}$ is obtained by completing P, by adding in convergent power series. In the same way, the H-algebra of q-holomorphic polynomials on $\mathbb{H}^n$ is F^{nU}, the free H-algebra generated by n copies of U.

Now $\mathbb{H}$ is a hyperkähler manifold, so by Theorem 6.6, $A_{\mathbb{H}}$ is an HP-algebra. We will define an HP-algebra structure on P.

Example 10.4 Let $\xi : A_{\mathbb{H}} \otimes_{\mathbb{H}} A_{\mathbb{H}} \to A_{\mathbb{H}} \otimes_{\mathbb{H}} Y$ be the Poisson bracket on $A_{\mathbb{H}}$, and consider the restriction of ξ to $U^{(j)} \otimes_{\mathbb{H}} U^{(k)}$. From the definition of ξ, we see that ξ is bilinear in the first derivatives of the two factors. The first derivatives of polynomials of degree j, k are polynomials of degree $j-1, k-1$ respectively. Thus, ξ must send $U^{(j)} \otimes_{\mathbb{H}} U^{(k)}$ to homogeneous polynomials of degree $j+k-2$, and so ξ maps $\xi : U^{(j)} \otimes_{\mathbb{H}} U^{(k)} \to U^{(j+k-2)} \otimes_{\mathbb{H}} Y$. This implies that ξ maps $\xi : P \otimes_{\mathbb{H}} P \to P \otimes_{\mathbb{H}} Y$, and so P is an HP-subalgebra of $A_{\mathbb{H}}$.

In particular, consider $\xi : U \otimes_{\mathbb{H}} U \to U^{(0)} \otimes_{\mathbb{H}} Y$. Since ξ is antisymmetric, we may restrict to $\Lambda^2_{\mathbb{H}} U$. Now U is stable and has $j = 3$, $r = 2$, so by Proposition 9.6, we have $\dim \Lambda^2_{\mathbb{H}} U = 8$ and

$\dim(\Lambda^2_{\mathbb{H}}U)' = 5$. But these are the same dimensions as those of the A$\mathbb{H}$-module Y of Definition 6.1. In fact there is a natural isomorphism $\Lambda^2_{\mathbb{H}}U \cong Y$. Now $U^{(0)} \cong \mathbb{H}$, so that $U^{(0)} \otimes_{\mathbb{H}} Y \cong Y$. Thus there is an A$\mathbb{H}$-isomorphism $\Lambda^2_{\mathbb{H}}U \cong U^{(0)} \otimes_{\mathbb{H}} Y$.

It can be shown that ξ induces exactly this isomorphism $\Lambda^2_{\mathbb{H}}U^{(1)} \cong U^{(0)} \otimes_{\mathbb{H}} Y$. This defines ξ on a generating subspace $U^{(1)}$ for P. Using Axiom P of §6, we may extend ξ *uniquely* to all of P, because the action of ξ on the generators determines the whole action. This describes the HP-algebra structure of P.

11 Hyperkähler manifolds with symmetry groups

Let M be a hyperkähler manifold, and suppose v is a Killing vector of the hyperkähler structure on M. A *hyperkähler moment map for* v is a triple (f_1, f_2, f_3) of smooth real functions on M such that $\alpha = I_1 df_1 = I_2 df_2 = I_3 df_3$, where α is the 1-form dual to v under the metric g. Moment maps always exist if $b^1(M) = 0$, and are unique up to additive constants.

More generally, let M be a hyperkähler manifold, let G be a Lie group with Lie algebra $\mathfrak{g}$, and suppose $\rho : G \to \mathrm{Aut}(M)$ is a homomorphism from G to the group of automorphisms of the hyperkähler structure on M. Let $\rho : \mathfrak{g} \to \mathrm{Vect}(M)$ be the induced map from $\mathfrak{g}$ to the Killing vectors. Then a *hyperkähler moment map for the action* ρ *of* G is a triple (f_1, f_2, f_3) of smooth functions from M to $\mathfrak{g}^*$, such that for each $x \in \mathfrak{g}$, $(x\cdot f_1, x\cdot f_2, x\cdot f_3)$ is a hyperkähler moment map for the vector field $\rho(x)$, and in addition, (f_1, f_2, f_3) is equivariant under the action ρ of G on M and the coadjoint action of G on $\mathfrak{g}^*$.

Moment maps are a familiar part of symplectic geometry, and hyperkähler moment maps were introduced by Hitchin et al. as part of a quotient construction for hyperkähler manifolds [6], [13, pp. 118-122]. Hyperkähler moment maps will exist under quite mild conditions on M and G, for instance if $b^1(M) = 0$ and G is compact or semisimple. In this section we consider two applications of moment maps. First we will use them to construct q-holomorphic functions on hyperkähler manifolds with symmetries. Secondly, we will show that under certain conditions the moment map determines the hyperkähler structure.

Definition 11.1 Let M be a hyperkähler manifold and G a Lie group with Lie algebra $\mathfrak{g}$, and let $\rho : G \to \mathrm{Aut}(M)$ be an action of G on M preserving the hyperkähler structure. Suppose that (f_1, f_2, f_3) is a hyperkähler moment map for ρ, where $f_j : M \to \mathfrak{g}^*$ is a smooth map. Define a linear map $\phi : \mathfrak{g} \otimes Y \to C^\infty(M, \mathbb{H})$ by

$$\phi\big(x \otimes (q_1, q_2, q_3)\big) = (x \cdot f_1)q_1 + (x \cdot f_2)q_2 + (x \cdot f_3)q_3, \tag{11.1}$$

for each $x \in \mathfrak{g}$ and $(q_1, q_2, q_3) \in Y$. Here $x \cdot f_j$ is a smooth real function on M and $q_j \in \mathbb{H}$, and thus $\phi\big(x \otimes (q_1, q_2, q_3)\big)$ is a smooth $\mathbb{H}$-valued function on M, and lies in $C^\infty(M, \mathbb{H})$.

Lemma 11.2 *In the situation above,* ϕ *maps* $\mathfrak{g} \otimes Y$ *into the H-algebra* A_M *of q-holomorphic functions on* M*, and* ϕ *is an* A$\mathbb{H}$*-morphism.*

Proof. Define a function $y : M \to \mathbb{H}$ by $y = (x\cdot f_1)q_1 + (x\cdot f_2)q_2 + (x\cdot f_3)q_3$, where x, f_j and q_j are as in the Definition. Since (f_1, f_2, f_3) is a hyperkähler moment map, it follows that $I_1 d(x\cdot f_1) + I_2 d(x\cdot f_2) + I_3 d(x\cdot f_3) = 0$ on M. Also, as $(q_1, q_2, q_3) \in Y$, Definition 6.1 gives that

$q_1 i_1 + q_2 i_2 + q_3 i_3 = 0$. These two facts together imply that y is q-holomorphic on M, so that ϕ maps $\mathfrak{g} \otimes Y \to A_M$. As $(\mathfrak{g} \otimes Y)' = \mathfrak{g} \otimes Y'$, we see that $x \otimes (q_1, q_2, q_3)$ lies in $(\mathfrak{g} \otimes Y)'$ if $q_1, q_2, q_3 \in \mathbb{I}$. But then y takes values in $\mathbb{I}$, and so $y \in A'_M$. Thus ϕ maps $(\mathfrak{g} \otimes Y)'$ into A'_M. Since ϕ is clearly $\mathbb{H}$-linear, it is an A$\mathbb{H}$-morphism. ∎

It is easy to see that if U is an A$\mathbb{H}$-module, A an H-algebra, and $\phi : U \to A$ an A$\mathbb{H}$-morphism, then ϕ extends to a unique H-algebra morphism $\Phi : F^U \to A$, where F^U is the free H-algebra generated by U, as in Example 5.1. Thus, Definition 11.1 and Lemma 11.2 give an H-algebra morphism $\Phi : F^{\mathfrak{g} \otimes Y} \to A_M$. But A_M is an HP-algebra by Theorem 6.6, as M is hyperkähler, and Example 6.2 shows that $F^{\mathfrak{g} \otimes Y}$ is the HP-algebra $F_{\mathfrak{g}}$.

Now it can be proved that the H-algebra morphism $\Phi : F_{\mathfrak{g}} \to A_M$ is actually an *HP-algebra morphism*. The Lie group G acts on $\mathfrak{g}$ by the adjoint action, and this induces an action of G on $F_{\mathfrak{g}}$. Also, the action ρ of G on M preserves the hyperkähler structure, and thus it induces an action of G on A_M. So, $F_{\mathfrak{g}}$ and A_M both come equipped with G-actions, which clearly preserve the HP-algebra structures. It is easy to see that the map $\Phi : F_{\mathfrak{g}} \to A_M$ is *G-equivariant*, that is, it commutes with the two G actions. This gives the following Proposition:

Proposition 11.3 *Let M be a hyperkähler manifold, and A_M the HP-algebra of q-holomorphic functions on M. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\rho : G \to \mathrm{Aut}(M)$ be an action of G on M preserving the hyperkähler structure. Suppose (f_1, f_2, f_3) is a hyperkähler moment map for ρ. Then there exists a unique HP-algebra morphism $\Phi : F_{\mathfrak{g}} \to A_M$, where $F_{\mathfrak{g}}$ is defined by Example 6.2, and the restriction of Φ to $\mathfrak{g} \otimes Y \subset F_{\mathfrak{g}}$ is the A$\mathbb{H}$-morphism $\phi : \mathfrak{g} \otimes Y \to A_M$ of Definition 11.1. Also, Φ is equivariant with respect to the natural G-actions on $F_{\mathfrak{g}}$ and A_M.*

In our next Proposition, we show that if we know the manifold M and the hyperkähler moment maps $f_1, f_2, f_3 : M \to \mathfrak{g}^*$, then we can sometimes reconstruct the hyperkähler structure on M.

Proposition 11.4 *Let M be a hyperkähler manifold, G a Lie group with Lie algebra $\mathfrak{g}$, and $\rho : G \to \mathrm{Aut}(M)$ an action of G on M preserving the hyperkähler structure. Let (f_1, f_2, f_3) be a hyperkähler moment map for ρ, and define $F : M \to \mathfrak{g}^* \otimes \mathbb{I}$ by*

$$F(m) = f_1(m) \otimes i_1 + f_2(m) \otimes i_2 + f_3(m) \otimes i_3, \qquad \text{for } m \in M. \tag{11.2}$$

Suppose that at $p \in M$, the map $df_2|_p \oplus df_3|_p : T_pM \to \mathfrak{g}^* \oplus \mathfrak{g}^*$ is injective. Then in a neighbourhood of p, the hyperkähler structure of M is determined solely by the functions f_1, f_2, f_3 and their first derivatives, or equivalently, by the image $F(M)$ and its tangent bundle.

Proof. We first explain how to recover the complex structure I_1 on M near p from the image $F(M)$. Consider the function $f_2 + if_3 : M \to \mathfrak{g}^* \otimes \mathbb{C}$. As $df_2|_p \oplus df_3|_p$ is injective, this function embeds a neighbourhood of $p \in M$ in $\mathfrak{g}^* \otimes \mathbb{C}$. Now since $I_2 df_2 = I_3 df_3$, we have $df_2 + I_1 df_3 = 0$, and thus $f_2 + if_3$ is holomorphic with respect to I_1. As $f_2 + if_3$ embeds M near p, we may regard $f_2 + if_3$ as a set of holomorphic coordinates w.r.t. I_1, near p. But a holomorphic coordinate system determines the complex structure, and so I_1 is determined near p by $F(M)$, and in fact by its tangent bundle alone.

Now let (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3) be an oriented orthonormal basis for $\mathbb{R}^3$. Then $\sum_j a_j I_j$ is a complex structure on M, and $\sum_j b_j f_j + i \sum_j c_j f_j$ is holomorphic with respect to it. Now injectivity is an open property, so if (a_1, a_2, a_3) is sufficiently close to $(1, 0, 0)$ in $\mathbb{R}^3$, then the map $\sum_j b_j df_j|_p \oplus \sum_j c_j df_j|_p$ will be injective. Therefore by the same argument, if (a_1, a_2, a_3) is close to $(1, 0, 0)$ then we can recover the complex structure $\sum_j a_j I_j$ from the image $F(M)$ near p. Thus the image $F(M)$ determines the complex structures I_1, I_2, I_3 near p. The metric g can also be recovered using similar techniques. ∎

12 Classifying symmetric hyperkähler metrics

Suppose that M is a hyperkähler manifold with symmetry group G, and let $\Phi : F_{\mathfrak{g}} \to A_M$ be the HP-algebra morphism defined in Proposition 11.3. If the symmetry group G is sufficiently big, in some suitable sense, then Φ contains a lot of information about M and its hyperkähler structure, and we can use it to study and even classify hyperkähler manifolds with large symmetry groups. In this section we will consider hyperkähler manifolds with symmetry groups satisfying the following condition.

Condition 12.1 Let M be a connected hyperkähler manifold, G a Lie group, and $\rho : G \to \mathrm{Aut}(M)$ an action of G on M preserving the hyperkähler structure, which admits a hyperkähler moment map. Then ρ induces a map $\rho : \mathfrak{g} \to \mathrm{Vect}(M)$ from the Lie algebra $\mathfrak{g}$ of G to the vector space $\mathrm{Vect}(M)$ of vector fields on M. Write $\mathcal{S}^2 = \{i \in \mathbb{I} : i^2 = -1\}$. For each $i = a_1 i_1 + a_2 i_2 + a_3 i_3 \in \mathcal{S}^2$, define S_i to be the set of points $p \in M$ such that

$$T_pM = \big\langle \rho(x)|_p : x \in \mathfrak{g} \big\rangle + (a_1 I_1 + a_2 I_2 + a_3 I_3) \big\langle \rho(x)|_p : x \in \mathfrak{g} \big\rangle. \tag{12.1}$$

Then there exists $i \in \mathcal{S}^2$ such that S_i is dense in M.

This condition can be interpreted as follows. Let $I = a_1 I_1 + a_2 I_2 + a_3 I_3$, so that I is a complex structure on M. The condition says that for most $p \in M$, we have $T_pM = \rho(\mathfrak{g})|_p + I\big(\rho(\mathfrak{g})|_p\big)$. This means that the complexification w.r.t. I of the action of G is transitive near p, so that M looks locally like an orbit of the complexified group G^c. In particular, this is only possible if $\dim M \le 2 \dim G$, and Condition 12.1 should be interpreted as saying that the symmetry group G of M is 'sufficiently big'.

Proposition 12.2 *Suppose that M, G and ρ satisfy Condition 12.1. Let $F : M \to \mathfrak{g}^* \otimes \mathbb{I}$ be defined by* (11.2)*. Then F is a smooth map, and there is a dense open set $S_i \in M$ such that the map $F : S_i \to \mathfrak{g}^* \otimes \mathbb{I}$ is an immersion, and the hyperkähler structure on S_i is determined by its image $F(S_i)$.*

Proof. By Condition 12.1, there exists some $i \in \mathcal{S}^2$ such that the set S_i is dense in M. Now S_i is clearly open, by its definition, and F is smooth as each f_j is smooth. Suppose for simplicity that $i = i_1$, since the proof for general $i \in \mathcal{S}^2$ follows from an $SO(3)$ rotation in $\mathbb{I}$. Let $p \in S_{i_1}$, so that T_pM is spanned by the vectors $\rho(x)|_p$ and $I_1\rho(x)|_p$, for $x \in \mathfrak{g}$. Contracting these vectors with the Kähler form ω_2, we see that T_p^*M is spanned by the 1-forms $\rho(x)\cdot\omega_2|_p$ and $(I_1\rho(x))\cdot\omega_2|_p$. But $\rho(x)\cdot\omega_2 = df_2$ by definition of f_2, and similarly $(I_1\rho(x))\cdot\omega_2 = -\rho(x)\cdot\omega_3 = -x\cdot df_3$. Thus T_p^*M is spanned by the 1-forms $x\cdot df_2|_p$ and $x \cdot df_3|_p$ for $x \in \mathfrak{g}$.

Therefore, an alternative definition of the set S_{i_1} is that $p \in S_{i_1}$ if and only if the map $df_2|_p \oplus df_3|_p : T_pM \to \mathfrak{g}^* \oplus \mathfrak{g}^*$ is injective. We immediately deduce that F is an immersion on

S_{i_1}, and Proposition 11.4 applies to show that the hyperkähler structure on S_{i_1} is determined by the image $F(S_{i_1})$. This completes the proof. ∎

From Proposition 11.3, G acts on $F_{\mathfrak g}$ and A_M. Write $F_{\mathfrak g}^G$ and A_M^G for the G-invariant subspaces of $F_{\mathfrak g}$ and A_M respectively. Then

$$F_{\mathfrak g}^G = \bigoplus_{j=0}^{\infty} (S^j\mathfrak g)^G \otimes S^j_{\mathbb H} Y, \tag{12.2}$$

and $F_{\mathfrak g}^G$ and A_M^G are HP-subalgebras of $F_{\mathfrak g}$ and A_M. In fact, it can be shown that the Poisson bracket ξ of $F_{\mathfrak g}$ vanishes on $F_{\mathfrak g}^G$, so only the H-algebra structure is nontrivial. Since $\Phi : F_{\mathfrak g} \to A_M$ is G-equivariant, it follows that Φ takes $F_{\mathfrak g}^G$ to A_M^G. But if Condition 12.1 holds, then Φ takes $F_{\mathfrak g}^G$ to the constant functions, as we will now show.

Proposition 12.3 *Suppose that M, G and ρ satisfy Condition 12.1. Then the map $\Phi : F_{\mathfrak g}^G \to A_M^G$ takes each $y \in F_{\mathfrak g}^G$ to a constant function on M.*

Proof. By (12.2), it is sufficient to prove the result for $y \in (S^j\mathfrak g)^G \otimes S^j_{\mathbb H} Y$. Write $U = (S^j\mathfrak g)^G \otimes S^j_{\mathbb H} Y$. Now Y is stable, in the sense of §8. By Proposition 9.6, $S^j_{\mathbb H} Y$ is also stable, and by Theorem 8.6 it is semistable. Thus U is semistable, so that U is generated over $\mathbb H$ by the subspaces $U' \cap q\,U'$ for nonzero $q \in \mathbb I$. But we can say more: because U is stable, the subspace $U' \cap q\,U'$ for nonzero q depends real-analytically on q (in particular, the dimension remains constant). Because of this, one can show that U is generated over $\mathbb H$ by the subspaces $U' \cap i\,U'$ for i in any given nonempty open set $T \subset \mathcal S^2$.

Define $T \subset \mathcal S^2$ to be the set of $i \in \mathcal S^2$ such that the set S_i defined in Condition 12.1 is dense in M. It can be shown using transversality arguments that T is open in $\mathcal S^2$, and T contains at least one element by Condition 12.1, so T is nonempty. Let $i \in T$, and suppose that $y \in U' \cap i\,U'$. We will show that $\Phi(y)$ is constant on M.

Again we may suppose that $i = i_1$, as the proof for general i follows by an $SO(3)$ rotation of $\mathbb I$. Write

$$\Phi(y) = f_0 + f_1 i_1 + f_2 i_2 + f_3 i_3, \tag{12.3}$$

where $f_0, \dots, f_3$ are real functions on M. Now $y \in U'$. Thus $\Phi(y) \in A'_M$, so that $\Phi(y)$ takes values in $\mathbb I$, and therefore $f_0 = 0$. Similarly, $y \in i_1 U'$, so $i_1\Phi(y)$ takes values in $\mathbb I$ and $f_1 = 0$, giving $\Phi(y) = f_2 i_2 + f_3 i_3$. Since $\Phi(y)$ is q-holomorphic, we see that $df_2 = I_1 df_3$ on M.

Now $\Phi(y) \in A_M^G$, so f_2 and f_3 are G-invariant. Thus, df_2 and df_3 give zero when contracted with the vector fields $\rho(x)$ for $x \in \mathfrak g$. But $df_2 = I_1 df_3$, and so df_2 and df_3 also give zero when contracted with the vector fields $I_1\rho(x)$ for $x \in \mathfrak g$. By (12.1) these vector fields span T_pM for $p \in S_{i_1}$, so $df_2 = df_3 = 0$ on S_{i_1}. But S_{i_1} is dense in M by definition of T, so $df_2 = df_3 = 0$ on M by continuity, and as M is connected, f_2 and f_3 are constant on M. Therefore $\Phi(y)$ is constant.

We have shown that if $y \in U' \cap i\,U'$ for $i \in T$, then $\Phi(y)$ is constant. But T is dense and nonempty, and so U is generated over $\mathbb H$ by such subspaces $U' \cap i\,U'$. Thus $\Phi(y)$ is constant for y in a subset that generates U over $\mathbb H$, so $\Phi(y)$ is constant for all $y \in U$. This completes the proof. ∎

The Proposition shows that Φ maps $F_{\mathfrak g}^G$ to $\mathbb H \subset A_M$, the constant functions in A_M. Now $\mathbb H$ is an H-subalgebra of A_M, and $\Phi : F_{\mathfrak g}^G \to \mathbb H$ is an H-algebra morphism. This morphism actually contains a lot of information about the geometry of M, so much so that it is possible locally to reconstruct the manifold M with its hyperkähler structure from the H-algebra morphism $\Phi : F_{\mathfrak{g}}^G \to \mathbb{H}$. Here is the first step in this reconstruction.

Definition 12.4 Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Then $F_{\mathfrak{g}}$ and $F_{\mathfrak{g}}^G$ are H-algebras. As $Y^\dagger \cong \mathbb{I}$, we have $(\mathfrak{g} \otimes Y)^\dagger \cong \mathfrak{g}^* \otimes \mathbb{I}$. Thus each $m \in \mathfrak{g}^* \otimes \mathbb{I}$ determines an A$\mathbb{H}$-morphism $\psi_m : \mathfrak{g} \otimes Y \to \mathbb{H}$. Since $F_{\mathfrak{g}}$ is the free H-algebra $F^{\mathfrak{g}\otimes Y}$, this extends to a unique H-algebra morphism $\Psi_m : F_{\mathfrak{g}} \to \mathbb{H}$. Clearly, the restriction of Ψ_m to $F_{\mathfrak{g}}^G$ is an H-algebra morphism $\Psi_m : F_{\mathfrak{g}}^G \to \mathbb{H}$.

Now let $\Phi : F_{\mathfrak{g}}^G \to \mathbb{H}$ be a given H-algebra morphism. Define M_Φ to be the subset of $m \in \mathfrak{g}^* \otimes \mathbb{I}$ such that the restriction of Ψ_m to $F_{\mathfrak{g}}^G$ is equal to Φ. It is easy to show (using Hilbert's Nullstellensatz) that M_Φ is the zeros of a finite number of polynomials on $\mathfrak{g}^* \otimes \mathbb{I}$, so that M_Φ is a real algebraic variety, and is a manifold with singularities.

Let M, G, ρ and Φ be as in the previous Proposition. Then $\Phi : F_{\mathfrak{g}}^G \to \mathbb{H}$ is an H-algebra morphism, so that Definition 12.4 defines a variety M_Φ in $\mathfrak{g}^* \otimes \mathbb{I}$. The next result explains the relation between M and M_Φ.

Proposition 12.5 *Suppose that M, G and ρ satisfy Condition 12.1. Let $\Phi : F_{\mathfrak{g}}^G \to \mathbb{H}$ be the H-algebra morphism defined above, and let $F : M \to \mathfrak{g}^* \otimes \mathbb{I}$ be defined by* (11.2). *Then the image $F(M)$ is a subset of the real algebraic variety M_Φ defined in Definition 12.4.*

Proof. Let $m \in M$. Then Definition 3.1 defines an H-algebra morphism $\theta_m : A_M \to \mathbb{H}$. Thus $\theta_m \circ \phi : \mathfrak{g} \otimes Y \to \mathbb{H}$ is an A$\mathbb{H}$-morphism, and $\theta_m \circ \phi \in (\mathfrak{g} \otimes Y)^\dagger \cong \mathfrak{g}^* \otimes \mathbb{I}$. Let p be the element of $\mathfrak{g}^* \otimes \mathbb{I}$ corresponding to $\theta_m \circ \phi$. It follows easily from the definitions that $F(m) = p$, and $\psi_p = \theta_m \circ \phi$, and $\Psi_p = \theta_m \circ \Phi$. Consider the restriction of $\theta_m \circ \Phi$ to $F_{\mathfrak{g}}^G$. If $y \in F_{\mathfrak{g}}^G$ then $\Phi(y)$ is constant on M. But $\theta_m \circ \Phi(y)$ just evaluates $\Phi(y)$ at m, so identifying $\mathbb{H}$ with the constant $\mathbb{H}$-valued functions on M gives $\theta_m \circ \Phi(y) = \Phi(y)$. Thus $\Psi_p = \theta_m \circ \Phi = \Phi$ on $F_{\mathfrak{g}}^G$, and $p \in M_\Phi$ by definition. But $p = F(m)$, so $F(m) \in M_\Phi$ for each $m \in M$, and $F(M)$ is a subset of M_Φ, as we have to prove. ∎

It can be shown, although we will not prove it, that if $\mathfrak{g}$ is semisimple then $\dim M_\Phi = 2\dim \mathfrak{g} - 2\,\mathrm{rank}\,\mathfrak{g}$, for every H-algebra morphism $\Phi : F_{\mathfrak{g}}^G \to \mathbb{H}$. Thus, if $\dim M = 2\dim\mathfrak{g} - 2\,\mathrm{rank}\,\mathfrak{g}$, then $F(M)$ is an *open* subset of M_Φ, at least away from the singularities of M_Φ. In this case, locally $F(M)$ is equal to M_Φ. But Proposition 12.2 shows that the image $F(M)$ determines the hyperkähler structure of M in a dense subset, and hence on all of M by continuity. This proves:

Corollary 12.6 *Suppose that M, G and ρ satisfy Condition 12.1, and suppose that G is semisimple and* $\dim M = 2\dim\mathfrak{g} - 2\,\mathrm{rank}\,\mathfrak{g}$. *Then the hyperkähler structure of M is determined by the real algebraic variety M_Φ, and hence by the H-algebra morphism $\Phi : F_{\mathfrak{g}}^G \to \mathbb{H}$.*

This result suggests that given a Lie group G, if $\Phi : F_{\mathfrak{g}}^G \to \mathbb{H}$ is an H-algebra morphism, then we should be able to construct a G-invariant hyperkähler metric on some open subset of the variety M_Φ of Definition 12.4. In fact this is true in many cases, although the author has not proved that a hyperkähler metric exists on every M_Φ. Thus, for each Lie group G one may construct a natural family of hyperkähler manifolds with symmetry group G. Here is a simple example.

Example 12.7 Let G be $SO(3)$, so that $\mathfrak{g} = \mathfrak{so}(3)$. Then $\mathfrak{g}$ has a basis v_1, v_2, v_3 with

$$[v_1, v_2] = v_3, \qquad [v_2, v_3] = v_1, \qquad \text{and} \qquad [v_3, v_1] = v_2. \tag{12.4}$$

Consider the element $\alpha = v_1 \otimes v_1 + v_2 \otimes v_2 + v_3 \otimes v_3$ in $S^2\mathfrak{g}$. It is easy to show that α is invariant under the action of G on $S^2\mathfrak{g}$, so that $\alpha \in (S^2\mathfrak{g})^G$, and in fact $(S^2\mathfrak{g})^G = \langle\alpha\rangle$. Thus $\langle\alpha\rangle \otimes S^2_{\mathbb{H}}Y \subset F^G_{\mathfrak{g}}$. Write $U = \langle\alpha\rangle \otimes S^2_{\mathbb{H}}Y$, so that U is an A$\mathbb{H}$-module of $F^G_{\mathfrak{g}}$ isomorphic to $S^2_{\mathbb{H}}Y$. It turns out that U generates $F^G_{\mathfrak{g}}$ as an H-algebra, and $F^G_{\mathfrak{g}}$ is isomorphic as an H-algebra to F^U, the free H-algebra generated by U.

Now there is a 1-1 correspondence between A$\mathbb{H}$-morphisms $\phi : U \to \mathbb{H}$ and H-algebra morphisms $\Phi : F^U \to \mathbb{H}$. But A$\mathbb{H}$-morphisms $\phi : U \to \mathbb{H}$ are just elements of $U^\dagger$. As $U \cong S^2_{\mathbb{H}}Y$, and Y is stable with $\dim Y = 8$ and $\dim Y' = 5$, by Theorem 9.6 we have $\dim U = 12$ and $\dim U' = 7$, giving $\dim U^\dagger = 5$. Thus, there is a 1-1 correspondence between H-algebra morphisms $\Phi : F^G_{\mathfrak{g}} \to \mathbb{H}$, and elements $\phi \in U^\dagger \cong \mathbb{R}^5$.

There is a natural identification between $U^\dagger = \mathbb{R}^5$ and symmetric trace-free 3×3 matrices. Thus we may identify $\phi \in U^\dagger$ with the matrix (a_{ij}), where $a_{ij} = a_{ji}$ and $a_{11} + a_{22} + a_{33} = 0$. If Φ is the corresponding H-algebra morphism $\Phi : F^G_{\mathfrak{g}} \to \mathbb{H}$, then it is easy to show that the subset M_Φ of $\mathfrak{g}^* \otimes \mathbb{I}$ defined by Definition 12.4 is the set of elements $\mathbf{r}_1 \otimes i_1 + \mathbf{r}_2 \otimes i_2 + \mathbf{r}_3 \otimes i_3$ in $\mathfrak{g}^* \otimes \mathbb{I}$ satisfying the five equations

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = a_{12}, \quad \mathbf{r}_1 \cdot \mathbf{r}_3 = a_{13}, \quad \mathbf{r}_2 \cdot \mathbf{r}_3 = a_{23}, \quad \mathbf{r}_1 \cdot \mathbf{r}_1 - a_{11} = \mathbf{r}_2 \cdot \mathbf{r}_2 - a_{22} = \mathbf{r}_3 \cdot \mathbf{r}_3 - a_{33}, \tag{12.5}$$

where the scalar product on $\mathfrak{g}^*$ is given by the Killing form. These are five equations in the nine variables of $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, so that M_Φ has dimension 4. Since $\dim \mathfrak{g} = 3$ and $\operatorname{rank} \mathfrak{g} = 1$, the dimension is $2 \dim \mathfrak{g} - 2 \operatorname{rank} \mathfrak{g}$, as we claimed above.

Careful investigation shows that following the methods of Proposition 11.4, we can reconstruct a unique hyperkähler structure on a dense open set of M_Φ, for every such matrix (a_{ij}). Then Corollary 12.6 shows that every $SO(3)$-invariant hyperkähler structure on a 4-manifold M satisfying Condition 12.1 is locally isomorphic to the hyperkähler structure on some M_Φ. Now in [1], Belinskii et al. explicitly determine all hyperkähler metrics with an $SO(3)$-action of this form by solving an ODE, so our metrics must correspond to theirs.

Applying an $SO(3)$ rotation to $\mathbb{I}$ has the effect of conjugating (a_{ij}) by this matrix. Therefore, after an $SO(3)$ rotation in $\mathbb{I}$, we may suppose that (a_{ij}) is diagonal with $a_{11} \geq a_{22} \geq a_{33}$. Equation (12.5) then becomes

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_3 = \mathbf{r}_2 \cdot \mathbf{r}_3 = 0 \quad \text{and} \quad \mathbf{r}_1 \cdot \mathbf{r}_1 - a = \mathbf{r}_2 \cdot \mathbf{r}_2 - b = \mathbf{r}_3 \cdot \mathbf{r}_3 - c, \tag{12.6}$$

where $a \geq b \geq c$ and $a + b + c = 0$. Define M^+_Φ to be the subset of M_Φ where $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ form a positively oriented basis of $\mathfrak{g}^*$, and M^-_Φ to be the subset where they form a negatively oriented basis.

It turns out that the hyperkähler structure is nonsingular on $M^\pm_\Phi$ but singular on the hypersurface dividing them, and the hyperkähler structures on $M^\pm_\Phi$ have opposite orientations. There are three interesting cases:

Case (A): $a = b = c = 0$**.** In this case, $M^\pm_\Phi$ are both copies of $\mathbb{H}/\{\pm 1\}$ with the flat hyperkähler structure, meeting at the origin.

Case (B): $a > 0$**,** $b = c = -\frac{1}{2}a$**.** Here $M^\pm_\Phi$ are copies of the Eguchi-Hanson space [5], which intersect at a common $\mathcal{S}^2$ where $\mathbf{r}_2 = \mathbf{r}_3 = 0$.

Case (C): $a \geq b > c$. Here M_Φ carries one of the metrics found by Belinskii et al. [1], which has a curvature singularity on the hypersurface $\mathbf{r}_3 = 0$. Note that M_Φ is nonsingular as a submanifold at this hypersurface, even though the hyperkähler structure is singular.

In principle we could follow the construction through to find an explicit algebraic formula for the metrics and complex structures. In the same way, given a semisimple Lie group G, one can use this method to construct and classify all hyperkähler manifolds M with dimension $2 \dim \mathfrak{g} - 2 \operatorname{rank} \mathfrak{g}$ and a G-action ρ satisfying Condition 12.1.

Now Kronheimer [10], [11], Biquard [3] and Kovalev [9] have already constructed families of hyperkähler manifolds associated to Lie groups, from a completely different point of view. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and let the complexification of G be G^c with Lie algebra $\mathfrak{g}^c$. Kronheimer found that certain moduli spaces of singular G-instantons on $\mathbb{R}^4$ are hyperkähler manifolds. These moduli spaces can be identified with coadjoint orbits of G^c in $(\mathfrak{g}^c)^*$, and have hyperkähler metrics invariant under G. Kronheimer's construction worked only for certain special coadjoint orbits, and more general cases were handled by Biquard and Kovalev.

Although Kronheimer's metrics look very algebraic, their construction is in fact analytic, and the algebraic description of these metrics is not well understood. All the coadjoint orbit metrics found by Kronheimer, Biquard and Kovalev satisfy Condition 12.1. The manifolds found by Kronheimer [10], [11] have dimension $2 \dim \mathfrak{g} - 2 \operatorname{rank} \mathfrak{g}$, and so Corollary 12.6 applies to give an algebraic construction for Kronheimer's metrics. However, many of the examples of Kovalev and Biquard satisfy $\dim M < 2 \dim \mathfrak{g} - 2 \operatorname{rank} \mathfrak{g}$, so Corollary 12.6 does not apply. They can be treated algebraically, but in a more complicated way.

In the case $G = SO(3)$ of Example 12.7, case (A) gives the nilpotent orbit metric defined in [11], and case (B) gives the metrics on G^c/T^c defined in [10], where T is a maximal torus in G. However, the metrics of case (C) are not constructed by Kronheimer, Biquard or Kovalev. Thus, in general we expect that most of the metrics constructed by these algebraic methods will be new.

References

[1] V.A. Belinskii, G.W. Gibbons, D.N. Page, C.N. Pope, 'Asymptotically euclidean Bianchi IX metrics in quantum gravity', *Phys. Lett.* **76B** (1978), 433-435.

[2] K.H. Bhaskara and K. Viswanath, *Poisson algebras and Poisson manifolds*, Pitman Res. Notes in Math. 174, Longman, 1988.

[3] O. Biquard, 'Sur les équations de Nahm et la structure de Poisson des algèbres de Lie semi-simples complexes', *Math. Ann.* **304** (1996), 253-276.

[4] C.A. Deavours, 'The quaternion calculus', *Amer. Math. Monthly* **80** (1973), 995-1008.

[5] T. Eguchi and A.J. Hanson, 'Asymptotically flat solutions to Euclidean gravity', *Phys. Lett.* **74B** (1978), 249-251.

[6] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, 'Hyperkähler metrics and supersymmetry', *Commun. Math. Phys.* **108** (1987), 535-589.

[7] D.D. Joyce, 'The hypercomplex quotient and the quaternionic quotient', *Math. Ann.* **290** (1991), 323-340.

[8] D.D. Joyce, 'Compact hypercomplex and quaternionic manifolds', *J. Diff. Geom.* **35** (1992), 743-61.

[9] A. Kovalev, 'Nahm's equations and complex adjoint orbits', *Quart. J. Math. Oxford*, **47** (1996), 41-58.

[10] P.B. Kronheimer, 'A hyper-kählerian structure on coadjoint orbits of a semisimple complex group', *J. London Math. Soc.* **42** (1990), 193-208.

[11] P.B. Kronheimer, 'Instantons and the geometry of the nilpotent variety', *J. Diff. Geom.* **32** (1990), 473-490.

[12] D. Quillen, 'Quaternionic algebra and sheaves on the Riemann sphere', preprint, 1996.

[13] S.M. Salamon, *Riemannian geometry and holonomy groups*, Pitman Res. Notes in Math. 201, Longman, 1989.

[14] A. Sudbery, 'Quaternionic analysis', *Math. Proc. Camb. Phil. Soc.* **85** (1979), 199-225.

Lincoln College, Oxford.