

MULTILEVEL NESTED SIMULATION FOR EFFICIENT RISK ESTIMATION: SUPPLEMENTARY MATERIAL

BY MICHAEL B. GILES AND ABDUL-LATEEF HAJI-ALI

University of Oxford

In this document, we explain in more details the model problem that was presented in the main work. We also repeat some of the discussion from the main manuscript to make the current document self-contained.

The model problem that we consider is a simple example that mimics many of the challenges of computing the probability of a large loss from a financial portfolio. Assuming we have a stock, W , following a Brownian Motion and a final payoff function, $f(x) = -x^2$ evaluated at maturity, $T = 1$, the value function for the option is

$$V(t, x) = \mathbb{E}[f(W(1)) \mid W(t) = x] = \mathbb{E}\left[-\left(x + (1-t)^{1/2}Z\right)^2\right]$$

for $t \in [0, 1]$ and Z being a standard normal variable. Note that the portfolio consisting of this one option is Delta-neutral, $\frac{\partial V}{\partial x}(0, 0) = 0$, and has negative Gamma, $\frac{\partial^2 V}{\partial x^2} = -2$. Hence a large loss is incurred with very low probability under extreme circumstances. Here, the loss is defined as the difference between future, at some risk horizon $\tau \ll 1$, and current risk-neutral portfolio expectations. That is, the loss is $V(0, 0) - V(\tau, x)$ given that the stock value at τ is x .

We are interested in the probability, η , of the portfolio loss exceeding a given loss level, L_η , i.e.,

$$(1) \quad \eta = \mathbb{E}\left[\mathbf{H}\left(V(0, 0) - V(\tau, W(\tau)) - L_\eta\right)\right].$$

Alternatively, one may specify $\eta \ll 1$ and determine the corresponding loss level, L_η , using the relation (1). Defining

$$\begin{aligned} P(y, z) &:= -\left(\tau^{1/2} y + (1-\tau)^{1/2} z\right)^2 \\ &= -\tau y^2 - 2\tau^{1/2}(1-\tau)^{1/2}yz - (1-\tau)z^2, \end{aligned}$$

we have that

$$\eta = \mathbb{E}\left[\mathbf{H}\left(\mathbb{E}\left[P(\tilde{Y}, Z)\right] - \mathbb{E}[P(Y, Z) \mid Y] - L_\eta\right)\right].$$

30 Note that only the second inner expectation in (1) is conditioned on samples
31 of the outer random variable, Y .

32 To estimate $\eta = \mathbb{E}[\mathbb{H}(\mathbb{E}[X | Y])]$ for some X , we have to estimate the two
33 inner expectations, $\mathbb{E}[P(\tilde{Y}, Z)]$ and $\mathbb{E}[P(Y, Z) | Y]$ for a given Y . We could
34 use the exact value of $\mathbb{E}[P(\tilde{Y}, Z)] = -1$ and set $X := -1 - P(Y, Z) - L_\eta$.
35 However, the variance is $\text{Var}[X | Y] = 4\tau(1 - \tau)Y^2 + 2(1 - \tau)^2 = \mathcal{O}(1)$. We
36 could also use independent samples of Z to compute both $\mathbb{E}[P(\tilde{Y}, Z)]$ and
37 $\mathbb{E}[P(Y, Z) | Y]$, setting $X := P(\tilde{Y}, \tilde{Z}) - P(Y, Z) - L_\eta$ for independent standard
38 normal variables Z, \tilde{Z} and \tilde{Y} . The variance would then be $\text{Var}[X | Y] =$
39 $2(1 - \tau)^2 + 4\tau(1 - \tau)Y^2 + 2 = \mathcal{O}(1)$.

40 Instead, we use the same samples of Z when estimating both inner expec-
41 tations. Moreover, for increased variance reduction, we also use an antithetic
42 control variate based on the fact that \tilde{Y} is identically distributed to $-\tilde{Y}$. In
43 summary, we set, for a given Y ,

$$\begin{aligned} (2) \quad X &:= \frac{1}{2} \left(P(\tilde{Y}, Z) + P(-\tilde{Y}, Z) \right) - P(Y, Z) - L_\eta \\ &= \tau(Y^2 - \tilde{Y}^2) + 2\tau^{1/2}(1 - \tau)^{1/2} YZ - L_\eta. \end{aligned}$$

45 Here, again, \tilde{Y} and Z are independent standard normal random variables.
46 The variance in this case is reduced to

$$47 \quad \sigma^2 = \text{Var}[X | Y] = 2\tau^2 + 4\tau(1 - \tau)Y^2 = \mathcal{O}(\tau).$$

48 and we can also compute analytically

$$49 \quad d = |\mathbb{E}[X | Y]| = |\tau(Y^2 - 1) - L_\eta| = |\tau(Y^2 - Y_\eta^2)|.$$

50 where $Y_\eta^2 = 1 + L_\eta/\tau$. Moreover, the cumulative distribution function (CDF)
51 of the random variable, $\mathbb{E}[X | Y]$, is

$$\begin{aligned} (3) \quad \mathbb{P}[\mathbb{E}[X | Y] \leq x] &= \mathbb{P} \left[\mathbb{E} \left[P(\tilde{Y}, Z) \right] - \mathbb{E}[P(Y, Z) | Y] - L_\eta \leq x \right] \\ &= \mathbb{P} \left[|Y| \leq \left(\frac{\tau + x + L_\eta}{\tau} \right)^{1/2} \right] \\ &= 1 - 2 \Phi \left(- \left(1 + \frac{x + L_\eta}{\tau} \right)^{1/2} \right), \end{aligned}$$

53 where Φ is the standard normal CDF and where we substituted $\mathbb{E}[P(\tilde{Y}, Z)] =$
54 -1 and $\mathbb{E}[P(Y, Z) | Y] = -\tau Y^2 - (1 - \tau)$. In particular,

$$55 \quad \eta = \mathbb{P}[\mathbb{E}[X | Y] \geq 0] = 2 \Phi \left(- \left(1 + \frac{L_\eta}{\tau} \right)^{1/2} \right).$$

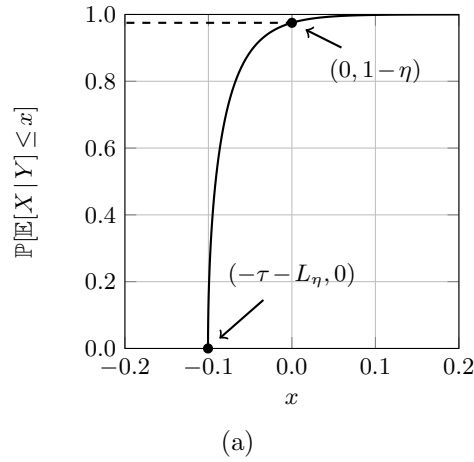


Fig 1: The cumulative distribution function of $\mathbb{E}[X|Y]$ with X as defined in (2) and Y a standard normal variable. This figure illustrates the square-root behaviour in the neighbourhood of $x = -\tau - L_\eta$. Here, we use $\tau = 0.02$ and $L_\eta \approx 0.0805$ so that $\eta = 0.025$.

56 We define the signed quantity $\bar{\delta} = \mathbb{E}[X|Y]/\sigma$. We also define the positive
 57 quantity $\delta = d/\sigma = |\bar{\delta}|$. Below, we will abuse the notation and use d, σ, δ and
 58 $\bar{\delta}$ to denote random variables or functions depending on context. That is,
 59 we will write $d(Y), \sigma(Y), \bar{\delta}(Y)$ and $\delta(Y)$ to denote corresponding functions
 60 of Y .

61 Setting $\tau = 0.02$ and $L_\eta \approx 0.0805$ so that $\eta = 0.025$, Figure 1 shows the
 62 cumulative distribution function of $\mathbb{E}[X|Y]$ and illustrates the square-root
 63 behaviour in the neighbourhood of $-\tau - L_\eta$. This figure was obtained by
 64 plotting $\mathbb{E}[X|Y] = \tau(Y^2 - 1) - L_\eta$ versus $\Phi(Y)$ for a range of values of Y .
 65 Figure 2-(a) shows $\mathbb{E}[X|Y]$, σ and $\bar{\delta}$ versus positive values of Y . Note that
 66 these quantities are even with respect to Y and monotonically increasing
 67 with respect to positive values of Y . On the other hand, Figure 2-(b) plots
 68 the CDF of $\bar{\delta}$. This is obtained by computing $\bar{\delta}$ for a range of positive values
 69 of Y and plotting it against $1 - 2\Phi(Y)$.

70 To plot the probability density function (PDF) of $\bar{\delta}$, denoted below by
 71 p , we have to compute the derivative of its CDF. Given the CDF, this can
 72 be done using a simple finite difference approximation. Another way is to
 73 compute $\bar{\delta}$ and its PDF as a function of positive y and plot them against
 74 each other. Since $\bar{\delta}$ is a monotonically increasing function of y , the PDF of

75 $\bar{\delta}$ is

76 (4)
$$p(\bar{\delta}(y)) = 2\phi(y) \left(\frac{\partial \bar{\delta}(y)}{\partial y} \right)^{-1},$$

77 as a function of y , where

78
$$\frac{\partial \bar{\delta}(y)}{\partial y} = \frac{2\tau y}{\sigma} - \frac{4\bar{\delta} y \tau(1-\tau)}{\sigma^2}$$

79 and $\phi(y) = \Phi'(y)$ is the standard normal PDF. The factor of 2 in (4) is
 80 because $\bar{\delta}(y) = \bar{\delta}(-y)$. A third way is to write y^2 as a function of $\bar{\delta}^2$. This
 81 is done by solving the equation $\bar{\delta}^2 = d^2(y)/\sigma^2(y)$ for y^2 to find the two real
 82 roots

83
$$y_1^2(\bar{\delta}) = Y_\eta^2 + 2\bar{\delta}^2 \left(\frac{1-\tau}{\tau} - \sqrt{\left(\frac{1-\tau}{\tau} \right)^2 + \frac{1}{2\bar{\delta}^2} + \frac{Y_\eta^2}{\bar{\delta}^2} \cdot \frac{1-\tau}{\tau}} \right)$$

$$y_2^2(\bar{\delta}) = Y_\eta^2 + 2\bar{\delta}^2 \left(\frac{1-\tau}{\tau} + \sqrt{\left(\frac{1-\tau}{\tau} \right)^2 + \frac{1}{2\bar{\delta}^2} + \frac{Y_\eta^2}{\bar{\delta}^2} \cdot \frac{1-\tau}{\tau}} \right)$$

84 where $Y_\eta^2 = 1 + L_\eta/\tau$. Note that $\bar{\delta}(y_1) \leq 0$ while $\bar{\delta}(y_2) \geq 0$. Then, the PDF of
 85 $\bar{\delta}$, as a function of $\bar{\delta}$, is

86
$$p(\bar{\delta}) = 2\phi(y_1(\bar{\delta})) \left(\frac{\partial \bar{\delta}(y_1(\bar{\delta}))}{\partial y} \right)^{-1} \mathbf{1}_{\bar{\delta} < 0} + 2\phi(y_2(\bar{\delta})) \left(\frac{\partial \bar{\delta}(y_2(\bar{\delta}))}{\partial y} \right)^{-1} \mathbf{1}_{\bar{\delta} \geq 0},$$

87 for any $\bar{\delta}$. Figure 2-(c) plots the PDF of δ as obtained by any of the previous
 88 three methods. Finally, Figure 2-(d) shows the PDF of δ . This is obtained
 89 by

90
$$\rho(\delta) = p(\delta) + p(-\delta).$$

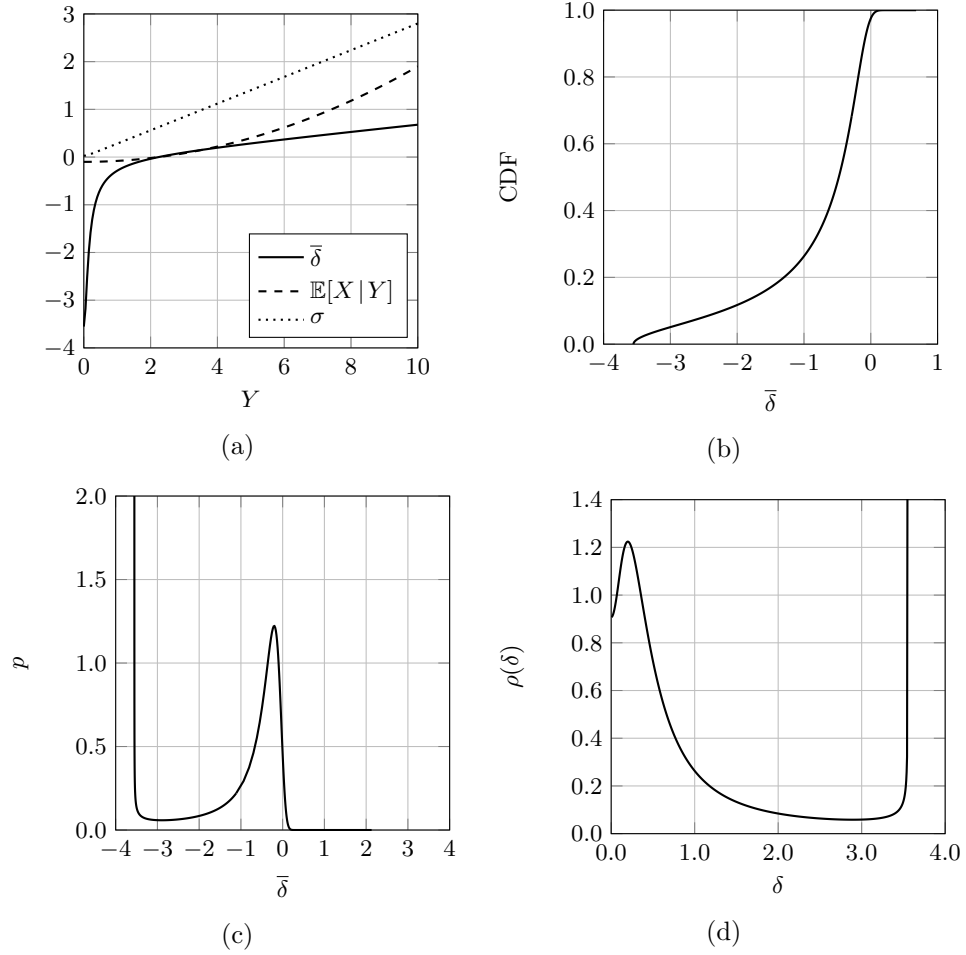


Fig 2: (a) shows that $\bar{\delta}$, $\mathbb{E}[X|Y]$ and σ are monotonically increasing with respect to Y , for positive Y . (b) and (c) show the CDF and PDF, respectively, of $\bar{\delta}$ while (d) shows the PDF of δ . Both (c) and (d) have an inverse square-root singularity, caused by the square root behaviour in the CDF of $\mathbb{E}[X|Y]$, cf. (3). For all figures, we use $\tau = 0.02$ and $L_\eta \approx 0.0805$ so that $\eta = 0.025$.