Sums of transcendental dilates

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Abstract

We show that there is an absolute constant $c > 0$ such that $|A + \lambda \cdot A| \geq e^{c\sqrt{\log |A|}}|A|$ for any finite subset A of $\mathbb{R}$ and any transcendental number $\lambda \in \mathbb{R}$. By a construction of Konyagin and Łaba, this is best possible up to the constant c.

1 Introduction

For any subset A of $\mathbb{R}$ and any $\lambda \in \mathbb{R}$, let

$$A + \lambda \cdot A = \{a + \lambda a' : a, a' \in A\}.$$

Our interest here will be in estimating the minimum size of such sums of dilates given $|A|$.

When λ is rational, say $\lambda = p/q$ with p and q coprime, a result of Bukh [3] implies that

$$|A + \lambda \cdot A| \geq (|p| + |q|)|A| - o(|A|),$$

which is best possible up to the lower-order term (though see [1] for an improvement of the lower-order term to a constant depending only on λ). The more general case where λ is algebraic has also been studied in some depth. In particular, a result of the authors [4] says that if $\lambda = (p/q)^{1/d}$ for some $p, q, d \in \mathbb{N}$, each taken as small as possible for such a representation, then

$$|A + \lambda \cdot A| \geq (p^{1/d} + q^{1/d})^d|A| - o(|A|),$$

which is again best possible up to the lower-order term. Moreover, as noted by Krachun and Petrov [7], for any fixed algebraic number λ, the minimum size of $|A + \lambda \cdot A|$ is always at most linear in $|A|$.

For λ transcendental, the picture is very different. Indeed, Konyagin and Łaba [6] showed that in this case there exists an absolute constant $c > 0$ such that

$$|A + \lambda \cdot A| \geq c\frac{\log |A|}{\log \log |A|}|A|.$$

That is, $|A + \lambda \cdot A|$ can no longer be linear in $|A|$. This result was subsequently improved by Sanders [10], by Schoen [12] and again by Sanders [11] using successive quantitative refinements

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of Freiman's theorem [5] on sets of small doubling, with Sanders' second bound saying that there exists an absolute constant $c > 0$ such that, for $|A|$ sufficiently large,

$$|A + \lambda \cdot A| \geq e^{\log^c |A|} |A|.$$

This already comes quite close to matching the best known upper bound, due to Konyagin and Łaba [6], which says that there exists $c' > 0$ and, for any fixed transcendental number λ, arbitrarily large finite subsets A of $\mathbb{R}$ such that

$$|A + \lambda \cdot A| \leq e^{c'\sqrt{\log |A|}} |A|.$$

Our main result says that this upper bound is in fact best possible up to the constant c'.

Theorem 1.1. *There is an absolute constant $c > 0$ such that*

$$|A + \lambda \cdot A| \geq e^{c\sqrt{\log |A|}} |A|$$

for any finite subset A of $\mathbb{R}$ and any transcendental number $\lambda \in \mathbb{R}$.

Before proceeding to the proof of this theorem, let us briefly look at the upper bound, which comes from considering sets of the form

$$A = \left\{ \sum_{i=1}^{m} a_i \lambda^i : (a_1, \dots, a_m) \in [n]^m \right\}.$$

This set has size n^m and

$$A + \lambda \cdot A \subset \left\{ \sum_{i=1}^{m+1} b_i \lambda^i : (b_1, \dots, b_{m+1}) \in [2n]^{m+1} \right\},$$

which has size $(2n)^{m+1}$. If we take $n = 2^m$, we have $|A| = n^m = 2^{m^2}$, so that

$$|A + \lambda \cdot A| \leq (2n)^{m+1} = 2^{(m+1)^2} \leq e^{c'\sqrt{\log |A|}} |A|$$

for some $c' > 0$, as required. This bound is reminiscent, both in its form and its proof, of Behrend's lower bound [2] for the largest subset of $[n]$ containing no three-term arithmetic progressions. Our Theorem 1.1 is arguably the first example where such a bound is known to be tight to this level of accuracy.

2 Proof of Theorem 1.1

To begin, we use a simple observation of Krachun and Petrov to recast the problem.

Lemma 2.1 (Krachun–Petrov [7])**.** *Suppose that $\lambda \in \mathbb{C}$ and A is a finite subset of $\mathbb{C}$. Then there exists $B \subset \mathbb{Q}[\lambda]$ such that $|B| = |A|$ and $|B + \lambda \cdot B| \leq |A + \lambda \cdot A|$.*

Suppose now that V is the $\mathbb{Q}$-vector space $\mathbb{Q}[\lambda]$ with basis $\{1, \lambda, \lambda^2, \ldots\}$. For any positive integer d, let $V_d \subset V$ be the d-dimensional subspace spanned by $\{1, \lambda, \lambda^2, \ldots, \lambda^{d-1}\}$, noting that $V = \bigcup_d V_d$. For any finite $A \subset V$, we must have $A \subset V_d$ for some d. Multiplication by λ therefore corresponds to taking the linear map $\Phi : V \to V$ given by the union of the maps $V_d \to V_{d+1}$ with

$$(x_1, \ldots, x_d) \mapsto (0, x_1, \ldots, x_d).$$

Thus, the problem of estimating $|A + \lambda \cdot A|$ for finite $A \subset \mathbb{R}$ and λ transcendental is equivalent to estimating $|A + \Phi(A)|$ for finite $A \subset V$. In particular, we may reformulate Theorem 1.1 in the following terms.

Theorem 2.2. *There is an absolute constant $c > 0$ such that if $A \subset V$ with $|A| = n$, then*

$$|A + \Phi(A)| \geq e^{c\sqrt{\log n}} n.$$

We will focus on proving this latter result, which bears some relation to our recent work [4] on sums of linear transformations, from here on.

Before getting to the proof proper, we first note a few additional results that we will need. The first is a discrete variant of the Brunn–Minkowski theorem taken from [4]. In what follows, for each $I \subseteq [d]$, we write $p_I : \mathbb{R}^d \to \mathbb{R}^d$ for the projection onto the coordinates indexed by I, setting all other coordinates to 0. Note that we may naturally extend the definition of p_I to V_d, and hence to V, by identifying V_d with $\mathbb{Q}^d$.

Lemma 2.3 (Conlon–Lim [4, Lemma 2.1])**.** *For any finite subsets A, B of $\mathbb{R}^d$,*

$$\sum_{I \subseteq [d]} |p_I(A + B)| \geq (|A|^{1/d} + |B|^{1/d})^d.$$

For our next result, we need the following estimate of Ruzsa [8] for the size of sumsets in $\mathbb{R}^d$. We say that a subset C of $\mathbb{R}^d$ is k-dimensional and write $\dim(C) = k$ if the dimension of the affine subspace spanned by C is k.

Lemma 2.4 (Ruzsa [8])**.** *If $A, B \subset \mathbb{R}^d$, $|A| \geq |B|$ and $\dim(A + B) = d$, then*

$$|A + B| \geq |A| + d|B| - \frac{d(d+1)}{2}.$$

For $a \in V$, write $p_k(a)$ for the vector obtained by removing the k-th coordinate from a. For $A \subset V$ and $x \in p_k(A)$, let $A_x = p_k^{-1}(x)$. We define the compression $C_k(A)$ of A along the k-th coordinate to be the set A' such that $p_k(A') = p_k(A)$ and, for each $x \in p_k(A)$, the k-th coordinates of A'_x are $0, 1, \ldots, |A_x| - 1$. It is known (see, for example, [4, Lemma 2.1]) that $|C_k(A) + C_k(B)| \leq |A + B|$ for any finite $A, B \subset V$. We say that A is compressed if $C_k(A) = A$ for all k. A compressed set $A \subset V_d$ has the property that if $(a_1, \ldots, a_d) \in A$ and $b_i \in \mathbb{Z}$ with $0 \leq b_i \leq a_i$ for all $1 \leq i \leq d$, then $(b_1, \ldots, b_d) \in A$. The next lemma will allow us to assume that A is both compressed and of low dimension when proving our main result.

Lemma 2.5. *Suppose that $A \subset V$ is finite with $|A + \Phi(A)| = K|A|$. Then there is some $d \leq 2K$ and $A' \subset V_d$ with $|A'| = |A|$ such that A' is compressed and $|A' + \Phi(A')| \leq |A + \Phi(A)|$.*

Proof. Since A is finite, $A \subset V_D$ for some D. Note that $\Phi \circ C_i = C_{i+1} \circ \Phi$ for all i. Denote by $C_{[i]}$ the composition $C_1 \circ C_2 \circ \cdots \circ C_i$. Then $C_{[D+1]}(A) = C_{[D]}(A)$ and $C_{[D+1]}(\Phi(A)) = \Phi(C_{[D]}(A))$. Thus, setting $A_1 = C_{[D]}(A)$, we have $|A_1| = |A|$ and

$$|A_1 + \Phi(A_1)| = |C_{[D]}(A) + \Phi(C_{[D]}(A))| = |C_{[D+1]}(A) + C_{[D+1]}(\Phi(A))| \leq |A + \Phi(A)|.$$

Furthermore, A_1 is compressed.

Let $e_k = \lambda^{k-1}$ be the basis vectors for $k = 1, \dots, D$. If $e_k \notin A_1$, then the k-th coordinate of every point of A_1 is 0. Let A_1' be the set formed by replacing each point $(x_1, \dots, x_{k-1}, 0, x_k, \dots, x_{D-1})$ of A_1 with the point $(x_1, \dots, x_{k-1}, x_k, \dots, x_{D-1})$, so that $A_1' \subset V_{D-1}$. We claim that $|A_1' + \Phi(A_1')| \leq |A_1 + \Phi(A_1)|$. Indeed, every point of $A_1 + \Phi(A_1)$ is of the form

$$(x_1, x_2 + y_1, x_3 + y_2, \dots, x_{k-1} + y_{k-2}, y_{k-1}, x_k, x_{k+1} + y_k, \dots, x_{D-1} + y_{D-2}, y_{D-1})$$

for some $(x_1, \dots, x_{k-1}, 0, x_k, \dots, x_{D-1}), (y_1, \dots, y_{k-1}, 0, y_k, \dots, y_{D-1}) \in A_1$, whereas every point of $A_1' + \Phi(A_1')$ is of the form

$$(x_1, x_2 + y_1, x_3 + y_2, \dots, x_{D-1} + y_{D-2}, y_{D-1}).$$

There is a clear surjection from $A_1 + \Phi(A_1)$ to $A_1' + \Phi(A_1')$ by summing and combining the k-th and $(k+1)$-th coordinates.

Repeating the above procedure whenever possible for each k, we obtain a set A' with $|A'| = |A|$, $|A' + \Phi(A')| \leq |A + \Phi(A)|$ and $A' \subset V_d$ for some d with $e_k \in A'$ for $k = 1, \dots, d$. By this last condition, A' is d-dimensional and, moreover, $A' + \Phi(A')$ is $(d+1)$-dimensional. Hence, by Lemma 2.4, we have $|A' + \Phi(A')| \geq (d+2)|A'| - \frac{(d+1)(d+2)}{2}$. Using that $|A' + \Phi(A')| \leq K|A'|$ and $|A'| \geq d+1$, we get $d \leq 2K$, as required. □

We also note the following result of Plünnecke–Ruzsa type.

Lemma 2.6. *Suppose $A \subset V$ is finite. If $|A + \Phi(A)| \leq K|A|$ for some $K > 0$, then $|(A + \Phi(A)) + \Phi(A + \Phi(A))| \leq K^{10}|A|$.*

Proof. The sum version of Ruzsa's triangle inequality [9] states that for any finite subsets X, Y, Z of an abelian group,

$$|X||Y + Z| \leq |X + Y||X + Z|.$$

Setting $X = \Phi(A)$, $Y = Z = A$ and noting that $|\Phi(A)| = |A|$, we have

$$|\Phi(A)||A + A| \leq |A + \Phi(A)||A + \Phi(A)|,$$

so that $|A + A| \leq K^2|A|$. Hence, by the Plünnecke–Ruzsa inequality, $|A + A + A + A| \leq K^8|A|$. Thus, another application of Ruzsa's triangle inequality (with $X = \Phi(A)$, $Y = A$, $Z = \Phi(A) + \Phi(A) + \Phi(A)$) yields

$$|\Phi(A)||A + \Phi(A) + \Phi(A) + \Phi(A)| \leq |A + \Phi(A)||\Phi(A) + \Phi(A) + \Phi(A) + \Phi(A)|,$$

so that $|A + \Phi(A) + \Phi(A) + \Phi(A)| \leq K^9|A|$. Applying Ruzsa's triangle inequality once more (with $X = \Phi(A)$, $Y = A + \Phi(A) + \Phi(A)$, $Z = \Phi^2(A)$), we see that

$$|\Phi(A)||A + \Phi(A) + \Phi(A) + \Phi^2(A)| \leq |A + \Phi(A) + \Phi(A) + \Phi(A)||\Phi(A) + \Phi^2(A)|,$$

so that $|A + \Phi(A) + \Phi(A) + \Phi^2(A)| \leq K^{10}|A|$, as required. □

We now come to the main novel ingredient in our proof, which is a strong upper bound for the size of the projections of any compressed $A \subset V_d$ in terms of $|A + \Phi(A)|$. Given a set $I \subseteq [d]$, we will write $\alpha(I)$ for the length of the longest set of consecutive integers in I.

Lemma 2.7. *Let $A \subset V_d$ be finite and compressed with $|A + \Phi(A)| = N$. Then, for any subset $I \subseteq [d]$,*

$$|p_I(A)| \le N^{\frac{k}{k+1}},$$

where $k = \alpha(I)$.

Proof. For any set of integers J, define $\phi(J) = \{j + 1 \mid j \in J\}$. We claim that, for any $J_1, J_2 \subset [d]$,

$$\frac{|p_{J_1}(A)||p_{J_2}(A)|}{|p_{J_1 \cap \phi(J_2)}(A)|} \le N.$$

To show this, we will exhibit an injection $p_{J_1}(A) \times p_{J_2}(A) \to p_{J_1 \cap \phi(J_2)}(A) \times (A + \Phi(A))$. Let $(x, y) \in p_{J_1}(A) \times p_{J_2}(A)$ and consider the map

$$(x, y) \mapsto (p_{J_1 \cap \phi(J_2)}(x), x + \Phi(y)).$$

Since A is compressed, $p_J(A) \subseteq A$ for every J, which easily implies that $(p_{J_1 \cap \phi(J_2)}(x), x + \Phi(y))$ is indeed in $p_{J_1 \cap \phi(J_2)}(A) \times (A + \Phi(A))$. To see that the map is injective, it is enough to observe that

$$x = p_{J_1 \cap \phi(J_2)}(x) + p_{J_1 \setminus \phi(J_2)}(x) = p_{J_1 \cap \phi(J_2)}(x) + p_{J_1 \setminus \phi(J_2)}(x + \Phi(y))$$

and

$$\Phi(y) = p_{\phi(J_2)}(\Phi(y)) = p_{\phi(J_2)}(x + \Phi(y)) - p_{\phi(J_2)}(x) = p_{\phi(J_2)}(x + \Phi(y)) - p_{J_1 \cap \phi(J_2)}(x).$$

For $i = 0, 1, \ldots, k$, let

$$I_i = \{j \in I \mid \{j, j-1, \ldots, j-i\} \subseteq I\}.$$

Then $I = I_0 \supset I_1 \supset \cdots \supset I_k = \emptyset$ and, for each $i = 0, 1, \ldots, k-1$, $I \cap \phi(I_i) = I_{i+1}$. Thus, by the claim above,

$$\frac{|p_I(A)||p_{I_i}(A)|}{|p_{I_{i+1}}(A)|} \le N.$$

Taking the product of this inequality over all $i = 0, 1 \ldots, k-1$, we get

$$|p_I(A)|^{k+1} \le N^k$$

and the lemma follows. □

We are now ready to prove our main result.

Proof of Theorem 2.2. Suppose instead that $|A + \Phi(A)| = Kn$, where $K < e^{c\sqrt{\log n}}$ for some $c > 0$ that will be fixed later. By Lemma 2.5, we may assume that A is compressed and $A \subset V_d$ with $d \le 2K$.

By Lemma 2.6, we have

$$|A + \Phi(A) + \Phi(A + \Phi(A))| \le K^{10} n.$$

Since A is compressed, so are $\Phi(A)$ and, therefore, $A+\Phi(A)$. Hence, Lemma 2.7 implies that

$$|p_I(A+\Phi(A))| \le (K^{10}n)^{\frac{k}{k+1}}$$

for any $I \subseteq [d+1]$, where $k=\alpha(I)$. But the number of $I \subseteq [d+1]$ with $\alpha(I)=k$ is at most

$$\sum_{i=1}^{d+2-k} |\{I \subseteq [d+1] \mid i, i+1, \ldots, i+k-1 \in I\}| \le (d+2)2^{d+1-k}.$$

Thus, by Lemma 2.3, we have that

$$\begin{aligned} 2^{d+1}n &\le \sum_{I \subseteq [d+1]} |p_I(A+\Phi(A))| \le \sum_{k=0}^{d+1} |\{I \subseteq [d+1] \mid \alpha(I)=k\}| \, (K^{10}n)^{\frac{k}{k+1}} \\ &\le \sum_{k=0}^{d+1} (d+2)2^{d+1-k}(K^{10}n)^{\frac{k}{k+1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} 1 &\le \sum_{k=0}^{d+1} (d+2)2^{-k}K^{\frac{10k}{k+1}}n^{-\frac{1}{k+1}} \le 2(d+2)\sum_{k=0}^{d+1} 2^{-k-1}K^{10}n^{-\frac{1}{k+1}} \\ &\le 2(d+2)\sum_{k=0}^{d+1} e^{-(k+1)\log 2+10c\sqrt{\log n}-\frac{\log n}{k+1}} \\ &\le 2(d+2)\sum_{k=0}^{d+1} e^{-2\sqrt{(\log 2)\log n}+10c\sqrt{\log n}} \qquad \left(\text{using } (k+1)\log 2+\frac{\log n}{k+1} \ge 2\sqrt{(\log 2)\log n}\right) \\ &= 2(d+2)^2 e^{(10c-2\sqrt{\log 2})\sqrt{\log n}} \le e^{(13c-2\sqrt{\log 2})\sqrt{\log n}}, \end{aligned}$$

which is a contradiction for $c=0.1$ and n sufficiently large. For smaller n, we may use the trivial estimate $|A+\Phi(A)| \ge 2|A|-1$ to choose an appropriate c that works for all n. $\square$

As a final remark, we note that the conclusion of Theorem 1.1 also holds for any finite subset A of $\mathbb{C}$ and any transcendental $\lambda \in \mathbb{C}$. Indeed, Lemma 2.1 again reduces the problem to estimating $|A+\lambda \cdot A|$ for finite $A \subset \mathbb{Q}[\lambda]$ and then to estimating $|A+\Phi(A)|$ for finite $A \subset V$, so the rest of the proof goes through without change.

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