



Differentiability and growth bounds of solutions of delay equations

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Abstract

We consider the question of eventual differentiability of the delay semigroups associated with the retarded equation $u'(t) = Au(t) + \Phi u_t$ ($t \geq 0$), where u_t is the history function, A generates an immediately norm-continuous semigroup and Φ is bounded. We show that this is determined by the rate of decay of the resolvent of A along vertical lines.

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1. Introduction

We consider an abstract delay differential equation of the form

$$u'(t) = Au(t) + \Phi u_t \quad (t \geq 0), \quad u_0 = f, \quad (\text{DDE})$$

where $u_t(\theta) = u(t + \theta)$ ($t \geq 0$, $-1 \leq \theta \leq 0$) and $f \in C([-1, 0], X)$, and we assume that A generates a C_0 -semigroup $\{T(t): t \geq 0\}$ on X and $\Phi: C([-1, 0], X) \rightarrow X$ is a bounded linear operator. One approach to the abstract theory of such equations is to construct an associated C_0 -semigroup V_Φ on the space $C([-1, 0], X)$ whose orbits correspond to mild solutions of (DDE) (see [10, Section VI.6] for this case or [4] for the similar L^p -case).

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Then general semigroup theory may be used to obtain information about the solutions of (DDE).

In this paper we use this approach to consider the question of eventual differentiability of mild solutions of (DDE), that is, the question whether the mild solution u of (DDE) corresponding to an arbitrary initial history f is a classical solution of (DDE) on $[t_0, \infty)$ for some t_0 (independent of f). This coincides with eventual differentiability of V_Φ . Pazy [15] gave a criterion for eventual differentiability of a C_0 -semigroup in terms of the resolvent of its generator. Since the resolvent of the generator of V_Φ can be represented explicitly in terms of the resolvent of A , it becomes possible to show that V_Φ is eventually differentiable if the resolvent of A decays polynomially on vertical lines (Theorem 2.3). Moreover, this condition is not only sufficient, but also necessary, for eventual differentiability (in a uniform fashion) of the semigroups V_Φ for all bounded Φ .

A very special case of (DDE) arises when $\Phi f = C(f(0))$ for some bounded operator C on X (see Example 2.4). Then (DDE) is not a delay equation at all, and our question reduces to whether the C_0 -semigroup S_C generated by $A + C$ is eventually differentiable. Renardy [17] showed that S_C is not necessarily eventually differentiable even if T is immediately differentiable, and the question has been considered further in [9] and [14]. In particular, it was shown in [9] that S_C is eventually differentiable if $\|AT(t)\|$ satisfies a certain condition for small t . In Theorem 2.6 we give a stronger condition on $\|AT(t)\|$ which is both sufficient and necessary for eventual differentiability of V_Φ when $\Phi f = f(-1)$. This condition is equivalent to a simple condition on the resolvent of A : some power of the resolvent should decay polynomially fast along vertical lines. Thus eventual differentiability of this one delay semigroup is only slightly weaker than eventual differentiability of V_Φ for all Φ .

Although differentiability of mild solutions of (DDE) seems a natural question, we found only a few papers on the subject (see [7,8,18]). Those papers work with L^p -spaces and consider situations where T is holomorphic and Φ is unbounded (but relatively bounded with respect to A in some sense) on the standard X -valued function spaces, while we have stronger assumptions on Φ but we have optimal conditions on T .

In the final section of the paper, we combine the results of Section 2.15 with techniques from [5] to give conditions under which each mild (or classical) solution of (DDE) has the property that its exponential growth bound coincides with the abscissa of holomorphy of its Laplace transform. When $A = 0$, this result was obtained previously by Huang and van Neerven [11].

2. Differentiability

We consider the retarded (delay) differential equation (DDE) under the assumptions in the first paragraph of the introduction, so A generates a C_0 -semigroup $\{T(t): t \geq 0\}$ on X and $\Phi: C([-1, 0], X) \rightarrow X$ is a bounded linear operator. It is well known (see [10, Section VI.6], for example) that there is an associated delay semigroup $\{V_\Phi(t): t \geq 0\}$ on $C([-1, 0], X)$ whose generator B_Φ is given by

$$D(B_\Phi) = \{f \in C^1([-1, 0], X): f(0) \in D(A) \text{ and } f'(0) = Af(0) + \Phi f\},$$

$$B_{\Phi} f = f'.$$

The semigroup V_{Φ} has the following properties:

$$\begin{aligned} (V_{\Phi}(t)f)(\theta) &= \begin{cases} f(t+\theta) & \text{if } t+\theta \leq 0, \\ (V_{\Phi}(t+\theta))f(0) & \text{if } t+\theta \geq 0, \end{cases} \\ (V_{\Phi}(t)f)(0) &= T(t)f(0) + (T * \Phi V_{\Phi})(t)f \\ &= T(t)f(0) + \int_0^t T(t-s)\Phi V_{\Phi}(s)f \, ds. \end{aligned} \quad (2.1)$$

The following proposition summarizes the relation between V_{Φ} and solutions of (DDE) and shows that (DDE) is well posed. A continuous function $u : [-1, \infty) \rightarrow X$ is said to be a *classical solution* of (DDE) if u has a continuous derivative on $[0, \infty)$, $u(t) \in D(A)$ for all $t \geq 0$, and (DDE) is satisfied. More generally, a continuous $u : [-1, \infty) \rightarrow X$ is a *mild solution* of (DDE) if $u_0 = f$ and $\int_0^t u(s) \, ds \in D(A)$ and

$$u(t) = u(0) + A \left(\int_0^t u(s) \, ds \right) + \int_0^t \Phi u_s \, ds \quad (2.2)$$

for all $t \geq 0$.

Proposition 2.1. *Let $f \in C([-1, 0], X)$, and define*

$$u(t) = \begin{cases} f(t) & (-1 \leq t \leq 0), \\ (V_{\Phi}(t)f)(0) & (t \geq 0). \end{cases} \quad (2.3)$$

- (1) u is the unique mild solution of (DDE).
- (2) If $f \in D(B_{\Phi})$, then u is a classical solution of (DDE).
- (3) If $u'(t)$ exists for some $t \geq 0$ then $u(t) \in D(A)$ and $u'(t) = Au(t) + \Phi u_t$.

Proof. The first two statements are standard (see [10, Corollary VI.6.3]). For the third statement, note that $s \mapsto u_s$ is continuous so it follows from (2.3) that

$$A \left(\frac{1}{h} \int_t^{t+h} u(s) \, ds \right) = \frac{1}{h} (u(t+h) - u(t)) - \frac{1}{h} \int_t^{t+h} \Phi u_s \, ds \rightarrow u'(t) - \Phi u_t$$

as $h \downarrow 0$. Since A is closed, the statement follows. \square

Corollary 2.2. *The following are equivalent:*

- (i) V_{Φ} is eventually differentiable;
- (ii) There exists $t_0 \geq 0$ such that, for each $f \in C([-1, 0], X)$, the unique mild solution of (DDE) is a classical solution on (t_0, ∞) .

Proof. (i) \Rightarrow (ii) This is immediate from (2.4) and Proposition 2.1(3).

(ii) \Rightarrow (i) Let $f \in C([-1, 0], X)$ and u be defined by (2.4). For $t > t_0 + 1$, $V_\Phi(t)f = u_t \in C^1([-1, 0], X)$ and

$$(V_\Phi(t)f)'(0) = u'(t) = Au(t) + \Phi u_t = A((V_\Phi(t)f)(0)) + \Phi(V_\Phi(t)f).$$

Thus, $V_\Phi(t)f \in D(B_\Phi)$ for every $t > t_0 + 1$ and every $f \in C([-1, 0], X)$, i.e., V_Φ is eventually differentiable. \square

It follows easily from (2.1) that V_Φ is eventually norm-continuous if T is immediately norm-continuous [10, Theorem VI.6.6]. However, V_Φ is not immediately norm-continuous (because it acts as a shift for $t + \theta \leq 0$). We shall show in Theorem 2.3 that V_Φ is eventually differentiable if

$$\|R(a + is, A)\| = O(|s|^{-\alpha}) \quad (2.4)$$

as $|s| \rightarrow \infty$ for some $\alpha > 0$ and some $a \in \mathbb{R}$. Our arguments will use Pazy's criterion for eventual differentiability which we recall here. A C_0 -semigroup S with generator B is eventually differentiable if and only if there exist constants $\beta > 0$, $c \in \mathbb{R}$ and $c' \in \mathbb{R}$ such that $\lambda \in \rho(B)$ and $\|R(\lambda, B)\| \leq c'|\operatorname{Im} \lambda|$ whenever $\operatorname{Re} \lambda \geq c - \beta \log |\operatorname{Im} \lambda|$ [15], [16, Theorem 2.4.7], [10, Theorem II.4.14]. (As usual, $\rho(B)$ is the resolvent set of B and $R(\lambda, B) = (\lambda I - B)^{-1}$. Note that if $\omega > \omega_0(S)$ (the growth bound of S) an estimate of this form automatically holds in the part of the region where $\operatorname{Re} \lambda \geq \omega$.) We shall later need the extended version of this: If there exist $\beta > 0$, $c \in \mathbb{R}$, $c' \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $\lambda \in \rho(B)$ and $\|R(\lambda, B)\| \leq c'|\operatorname{Im} \lambda|^n$ whenever $\operatorname{Re} \lambda \geq c - \beta \log |\operatorname{Im} \lambda|$, then S is eventually differentiable [15, Theorem 2.1].

Note that (2.4) is satisfied with $\alpha = 1$ if (and only if) T is holomorphic. On the other hand, (2.4) implies that T is immediately differentiable (see [16, Theorem 2.4.8], [10, Corollary II.4.15]).

We shall show that (2.4) is not only sufficient but also necessary for the following uniform concept of eventual differentiability of the semigroups V_Φ for $\|\Phi\| \leq 1$. A family of C_0 -semigroups $\{S_i: i \in I\}$, with generators B_i , is said to be *uniformly eventually differentiable* if there exists t_0 such that each T_i is differentiable on (t_0, ∞) and $\sup_{i \in I} \|B_i T_i(t)\| < \infty$ for each $t > t_0$. Pazy's criterion can be adapted to this situation. If $\sup\{\|S_i(t)\|: i \in I, t \in [0, 1]\} < \infty$, then $\{S_i\}$ is uniformly eventually differentiable if and only if there exist constants $\beta > 0$, $c \in \mathbb{R}$ and $c' \in \mathbb{R}$ such that $\lambda \in \bigcap_{i \in I} \rho(B_i)$ and $\sup_{i \in I} \|R(\lambda, B_i)\| \leq c'|\operatorname{Im} \lambda|$ whenever $\operatorname{Re} \lambda \geq c - \beta \log |\operatorname{Im} \lambda|$. This can be seen either by examining the proof of the criterion or by applying the criterion to the direct sum of all the semigroups S_i .

In order to apply Pazy's criterion to V_Φ , we recall the following description of the resolvent of B_Φ [10, Proposition VI.6.7]. Let $\lambda \in \mathbb{C}$. For $x \in X$, define $\varepsilon_\lambda \otimes x \in C([-1, 0], X)$ by $(\varepsilon_\lambda \otimes x)(\theta) = e^{\lambda\theta}x$. Define bounded linear operators Φ_λ on X and H_λ on $C([-1, 0], X)$ by

$$\begin{aligned} \Phi_\lambda(x) &= \Phi(\varepsilon_\lambda \otimes x), \\ (H_\lambda f)(\theta) &= \int_{\theta}^0 e^{\lambda(\theta-\tau)} f(\tau) d\tau. \end{aligned}$$

Now $\lambda \in \rho(B_\Phi)$ if and only if $\lambda \in \rho(A + \Phi_\lambda)$ and then

$$R(\lambda, B_\Phi)f = \varepsilon_\lambda \otimes (R(\lambda, A + \Phi_\lambda)(f(0) + \Phi H_\lambda f)) + H_\lambda f. \quad (2.5)$$

In order to establish that $\lambda \in \rho(A + \Phi_\lambda)$, recall the standard fact that if $\lambda \in \rho(A)$, $C \in \mathcal{B}(X)$, and $\|C\|\|R(\lambda, A)\| < 1$, then $\lambda \in \rho(A + C)$ and

$$R(\lambda, A + C) = R(\lambda, A) \sum_{n=0}^{\infty} (C R(\lambda, A))^n.$$

In particular,

$$\|R(\lambda, A + C)\| \leq \frac{\|R(\lambda, A)\|}{1 - \|C\|\|R(\lambda, A)\|}. \quad (2.6)$$

Applying this with C a scalar multiple of the identity operator shows that, if $|\mu - \lambda| < \frac{1}{2}\|R(\lambda, A)\|^{-1}$, then

$$\mu \in \rho(A) \quad \text{and} \quad \|R(\mu, A)\| \leq 2\|R(\lambda, A)\|. \quad (2.7)$$

In particular, this shows that the condition (2.4) is independent of a , and we shall usually take $a = 0$.

Now we give our first main result. For $C \in \mathcal{B}(X)$, define $\Phi_C \in \mathcal{B}(C([-1, 0], X), X)$ by

$$\Phi_C f = C(f(-1)).$$

Theorem 2.3. *Let A be the generator of a C_0 -semigroup. The following are equivalent:*

- (i) *There exist $\alpha > 0$, $b > 0$ and $c > 0$ such that $is \in \rho(A)$ and $\|R(is, A)\| \leq c|s|^{-\alpha}$ whenever $s \in \mathbb{R}$ and $|s| > b$;*
- (ii) *V_Φ is eventually differentiable whenever $\Phi \in \mathcal{B}(C[-1, 0], X)$, and the eventual differentiability is uniform for $\|\Phi\| \leq 1$;*
- (iii) *V_{Φ_C} is eventually differentiable whenever $C \in \mathcal{B}(X)$, uniformly for $\|C\| \leq 1$.*

Proof. (i) \Rightarrow (ii) Let $\lambda = a + is \in \mathbb{C}$. Note that

$$\|\Phi_\lambda\| \leq \|\Phi\| \max(1, e^{-a}), \quad \|H_\lambda\| \leq \max(1, e^{-a}). \quad (2.8)$$

In particular, if $\lambda \in \rho(A)$ and

$$\|\Phi\| \max(1, e^{-a}) \|R(\lambda, A)\| < \frac{1}{2}, \quad (2.9)$$

then $\lambda \in \rho(A + \Phi_\lambda)$ and

$$\|R(\lambda, A + \Phi_\lambda)\| \leq 2\|R(\lambda, A)\|,$$

by (2.6). By (2.5), $\lambda \in \rho(B_\Phi)$ and

$$\begin{aligned} \|R(\lambda, B_\Phi)\| &\leq 2 \max(1, e^{-a}) \|R(\lambda, A)\| (1 + \|\Phi\| \max(1, e^{-a})) \\ &\quad + \max(1, e^{-a}). \end{aligned} \quad (2.10)$$

Now suppose that

$$is \in \rho(A) \quad \text{and} \quad \|R(is, A)\| \leq c|s|^{-\alpha} \quad \text{whenever } |s| > b. \quad (2.11)$$

We may assume that $0 < \alpha \leq 1$. Take $\omega > \max(0, \omega_0(T))$. We may assume that

$$b > \max(1, (2c\omega)^{1/\alpha}, (4c)^{1/\alpha}, (4c\|\Phi\|)^{1/\alpha}). \quad (2.12)$$

We can choose c' such that

$$c' > \omega + \alpha \log b, \quad (2.13)$$

$$c' > \log(4c \max(1, \|\Phi\|)), \quad (2.14)$$

$$c' > \alpha \log \sigma - \frac{\sigma^\alpha}{2c} \quad (\sigma > b). \quad (2.15)$$

This is possible because $\alpha \log \sigma - \sigma^\alpha/2c \rightarrow -\infty$ as $\sigma \rightarrow \infty$.

Suppose that

$$\omega \geq a \geq c' - \alpha \log |s|. \quad (2.16)$$

Then

$$|s| > b \quad (\text{from (2.13) and (2.16)}), \quad (2.17)$$

$$\max(1, e^{-a}) \leq \max(1, e^{-c'}|s|^\alpha) \leq \frac{|s|^\alpha}{4c \max(1, \|\Phi\|)} \quad (2.18)$$

(from (2.16), (2.12), (2.17) and (2.14)),

$$-\frac{|s|^\alpha}{2c} \leq a \leq \frac{|s|^\alpha}{2c} \quad (\text{from (2.12), (2.15), (2.16) and (2.17)}), \quad (2.19)$$

$$|a| \leq \frac{1}{2} \|R(is, A)\|^{-1} \quad (\text{from (2.19) and (2.11)}). \quad (2.20)$$

Using (2.20) and replacing λ by is and μ by $a + is$ in (2.7) shows that $a + is \in \rho(A)$ and

$$\|R(a + is, A)\| \leq 2\|R(is, A)\| \leq 2c|s|^{-\alpha} \quad (\text{from (2.17) and (2.11)}). \quad (2.21)$$

Now (2.9) follows from (2.18) and (2.21). Hence $a + is \in \rho(A)$ and (2.10) and further applications of (2.18) and (2.21) give

$$\|R(a + is, B_\Phi)\| \leq 2 \frac{|s|^\alpha}{4c} \frac{2c}{|s|^\alpha} \left(1 + \frac{|s|^\alpha}{4c}\right) + \frac{|s|^\alpha}{4c} \leq 1 + \frac{|s|^\alpha}{2c} \leq c''|s|$$

for some constant c'' . By Pazy's criterion, V_Φ is eventually differentiable.

The statement that the eventual differentiability of V_Φ is uniform for $\|\Phi\| \leq 1$ follows from the fact that the above constants c' and c'' can be chosen to be independent of Φ . (Alternatively, one may apply the above result to a direct sum.)

(ii) \Rightarrow (iii) This is trivial.

(iii) \Rightarrow (i) The assumption (iii) and Pazy's criterion imply that there exist $\beta > 0$ and c such that $\lambda \in \rho(B_{\Phi_C})$ whenever $\operatorname{Re} \lambda \geq c - \beta \log |\operatorname{Im} \lambda|$ and $\|C\| \leq 1$. Thus if $\lambda = a + is$, where $a \geq c - \beta \log |s|$, then $\lambda \in \rho(A + e^{-\lambda}C)$ whenever $\|C\| \leq 1$. In particular, $\lambda \in \rho(A)$. For the purposes of a contradiction, suppose that there exists $x \in X$ with $\|x\| = 1$ and

$\|R(\lambda, A)x\| > e^a$. Then there exists $\psi \in X^*$ such that $\|\psi\| \leq 1$ and $\psi(R(\lambda, A)x) = e^\lambda$. Let $Cy = \psi(y)x$. Then C is a contraction of rank one and

$$(\lambda - (A + e^{-\lambda}C))R(\lambda, A)x = x - e^{-\lambda}CR(\lambda, A)x = 0.$$

This contradicts the fact that $\lambda \in \rho(A + e^{-\lambda}C)$. Thus $\|R(\lambda, A)\| \leq e^a$ whenever $a \geq c - \beta \log |s|$. In particular,

$$\|R(c - \beta \log |s| + is, A)\| \leq \frac{e^c}{|s|^\beta}.$$

It follows from (2.7) that $is \in \rho(A)$ and $\|R(is, A)\| \leq 2e^c |s|^{-\beta}$ if $|s|$ is so large that $|s|^\beta > 2e^c |c - \beta \log |s||$. \square

The proof of Theorem 2.3, in combination with [15] or [16, Theorem 2.4.7], shows that if (i) holds, then V_Φ is differentiable for $t > 3/\alpha$. Conversely, if V_Φ is differentiable for $t > t_0$, uniformly for $\|\Phi\| \leq 1$, then (i) holds for any $\alpha < 1/t_0$ (and some b and c). Note that it is automatic that $\alpha \leq 1$ and $t_0 \geq 1$.

The following example discusses the degenerate case when $\Phi f = C(f(0))$, i.e., there is no delay.

Example 2.4. Suppose that $\Phi f = C(f(0))$, where $C \in \mathcal{B}(X)$. Then (DDE) reduces to the simple Cauchy problem

$$u'(t) = Au(t) + Cu(t) \quad (t \geq 0), \quad u(0) = f(0).$$

The mild solutions are the orbits of the C_0 -semigroup S_C generated by $A + C$, so V_Φ is eventually differentiable if and only if S_C is. It was shown in [9] that S_C is eventually differentiable (uniformly for $\|C\| \leq 1$) if T is immediately differentiable and there are constants α and c such that

$$\|AT(t)\| \leq ct^{-\alpha/t} \quad (0 < t \leq 1). \quad (2.22)$$

It has not been shown that an estimate of this form is necessary for a given semigroup T to have the property that S_C is eventually differentiable uniformly for C in bounded subsets of $\mathcal{B}(X)$. However it has been shown in [9] that the functions $ct^{-\alpha/t}$ are optimal for the condition (2.22) to be sufficient. This involves constructions in [9] based on Renardy's example [17] of a C_0 -semigroup T with generator A on ℓ^2 and a bounded linear operator C on ℓ^2 such that T is immediately differentiable but S_C is not eventually differentiable.

There is a simple spectral condition which is necessary for the semigroups S_C to be eventually differentiable, uniformly for all contractions C . A simple modification of the proof that (iii) \Rightarrow (i) in Theorem 2.3 shows that a necessary condition is that $R(\lambda, A)$ should exist and be bounded (by 1) in a region of the form $\operatorname{Re} \lambda \geq c - \beta \log |\operatorname{Im} \lambda|$ for some $\beta > 0$ and some c . Nevertheless, the relation between the condition (2.22) and the resolvent condition (2.3) of Theorem 2.3 is not immediately obvious. It will become clearer in Theorem 2.6, where we shall show that a resolvent condition (slightly) weaker than (2.3) is equivalent to a condition significantly stronger than (2.22). Moreover, these conditions are equivalent to eventual differentiability of V_Φ in the very special case when $\Phi f = f(-1)$ (and they imply eventual differentiability of V_Φ in some other cases). We shall need the following less well-known variant of (2.6).

Proposition 2.5. Let $\lambda \in \rho(A)$, $C \in \mathcal{B}(X)$, $m \in \mathbb{N}$, and suppose that $CR(\lambda, A) = R(\lambda, A)C$ and $\|C\|^m \|R(\lambda, A)^m\| < 1$. Then $\lambda \in \rho(A + C)$ and

$$\|R(\lambda, A + C)^n\| \leq \frac{\|R(\lambda, A)^n\| \|\sum_{r=0}^{m-1} C^r R(\lambda, A)^r\|^n}{(1 - \|C\|^m \|R(\lambda, A)^m\|)^n} \quad (2.23)$$

for each $n \in \mathbb{N}$.

Proof. Since $\|(CR(\lambda, A))^m\| < 1$, $I - CR(\lambda, A)$ is invertible and

$$(I - CR(\lambda, A))^{-1} = \sum_{k=0}^{\infty} \left(\sum_{r=0}^{m-1} C^r R(\lambda, A)^r \right) (C^m R(\lambda, A)^m)^k.$$

Since

$$\lambda - (A + C) = (I - CR(\lambda, A))(\lambda - A),$$

it follows that $\lambda \in \rho(A + C)$ and $R(\lambda, A + C)^n = R(\lambda, A)^n (I - CR(\lambda, A))^{-n}$. The estimate (2.23) follows. \square

We shall use Proposition 3.4 mostly in the case when C is a scalar multiple of the identity operator and $n = 1$ or $n = m$. Then the result implies that if $\lambda \in \rho(A)$ and $|\mu - \lambda|^m < \frac{1}{2} \|R(\lambda, A)^m\|^{-1}$ (for some $m \in \mathbb{N}$) then $\mu \in \rho(A)$ and

$$\|R(\mu, A)^n\| \leq 2^n \|R(\lambda, A)^n\| \left(\sum_{r=0}^{m-1} |\mu - \lambda|^r \|R(\lambda, A)^r\| \right)^n. \quad (2.24)$$

Theorem 2.6. Let A be the generator of an immediately differentiable C_0 -semigroup T . The following are equivalent:

- (i) There exist $\alpha > 0$ and $c > 0$ such that $\|AT(t)\| \leq ct^{-\alpha}$ whenever $0 < t \leq 1$;
- (ii) There exist $m \in \mathbb{N}$, $b > 0$ and $c > 0$ such that $is \in \rho(A)$ and $\|R(is, A)^m\| \leq c|s|^{-1}$ whenever $s \in \mathbb{R}$ and $|s| > b$;
- (iii) V_Φ is eventually differentiable whenever Φ_λ and $R(\lambda, A)$ commute for all $\lambda \in \rho(A)$;
- (iv) V_Φ is eventually differentiable when $\Phi f = f(-1)$.

Proof. (i) \Rightarrow (ii) Take $m \in \mathbb{N}$ with $m > \alpha$, and take $\omega > \omega_0(T)$. Integration by parts gives

$$\begin{aligned} \|R(\omega + is, A)^m\| &= \left\| \frac{1}{(m-1)!} \int_0^\infty t^{m-1} e^{-(\omega+is)t} T(t) dt \right\| \\ &= \left\| \frac{1}{(m-1)!(\omega+is)} \int_0^\infty e^{-(\omega+is)t} ((m-1)t^{m-2} T(t) + t^{m-1} AT(t)) dt \right\| \\ &\leq \frac{1}{(m-1)!|s|} \int_0^\infty e^{-\omega t} ((m-1)t^{m-2} \|T(t)\| + t^{m-1} \|AT(t)\|) dt = \frac{c'}{|s|}, \end{aligned}$$

where c' is finite since $\|AT(t)\| \leq ct^{-\alpha}$ for $0 < t \leq 1$ and $\|AT(t)\| \leq \|AT(1)\| \|T(t-1)\|$ for $t > 1$. Now it follows from (2.24), with $\mu = is$, $\lambda = \omega + is$, $n = m$, that (ii) holds.

(ii) \Rightarrow (iii) Let $\omega > \omega_0(T)$. Since T is immediately differentiable, there exist constants c_1 and c_2 such that $\lambda \in \rho(A)$ and $\|R(\lambda, A)\| \leq c_2|s|$ whenever $\lambda = a + is$ with $\omega \geq a \geq c_1 - \log|s|$. We shall assume throughout that λ satisfies these conditions, and also that $|s| > b$ and $|a|^m < |s|/(2c)$, where m, b, c are as in (ii). Moreover, $\kappa := \sup_{|s|>b} \|R(is, A)\| < \infty$ (see [10, Corollary II.4.19] and (2.7)).

We apply (2.24) with λ replaced by is , μ replaced by $\lambda = a + is$, and $n = m$. This shows that $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)^m\| \leq 2^m \frac{c}{|s|} \left(\sum_{r=0}^{m-1} |a|^r \kappa^r \right)^m \leq \frac{c_3(1 + |a|^{m(m-1)})}{|s|} \quad (2.25)$$

for some constant c_3 (depending on m, c and κ).

Now, assume in addition that

$$2\|\Phi\|^m \max(1, e^{-ma}) c_3(1 + |a|^{m(m-1)}) < |s|. \quad (2.26)$$

Then it follows from (2.8) and (2.25) that $\|\Phi_\lambda\|^m \|R(\lambda, A)^m\| < 1/2$. Putting $C = \Phi_\lambda$ and $n = 1$ in Proposition 3.4 shows that $\lambda \in \rho(A + \Phi_\lambda)$ and

$$\|R(\lambda, A + \Phi_\lambda)\| \leq 2\|R(\lambda, A)\| \sum_{r=0}^{m-1} \|\Phi_\lambda\|^r \|R(\lambda, A)^r\| \leq c_4|s|(1 + \|\Phi_\lambda\|^{m-1}|s|^{m-1})$$

for some constant c_4 (depending on m and c_2). Using (2.5) and (2.8) and noting that $e^{-a} \leq e^{-c_1|s|}$, it follows that $\lambda \in \rho(B_\Phi)$ and

$$\begin{aligned} \|R(\lambda, B_\Phi)\| &\leq \max(1, e^{-a}) c_4|s|(1 + \|\Phi\|^{m-1} \max(1, e^{-(m-1)a})|s|^{m-1}) \\ &\quad \times (1 + \|\Phi\| \max(1, e^{-a})) + \max(1, e^{-a}) \\ &\leq c_5|s|^{2m+1} \end{aligned}$$

for some constant c_5 (depending on m, b, c_1, c_4 and $\|\Phi\|$).

This estimate holds for $\lambda = a + is$ under the assumptions that $\omega \geq a \geq c_1 - \log|s|$, $|s| > b$, $|a|^m < |s|/(2c)$ and (2.26) holds. It is possible to choose $\beta > 0$ and c_6 so that all of these are satisfied whenever $\omega \geq a \geq c_6 - \beta \log|s|$. It follows from Pazy's (extended) criterion that (iii) holds.

(iii) \Rightarrow (iv) This is trivial.

(iv) \Rightarrow (i) First we examine the form of the unique mild solution u of (DDE) in the case when $\Phi f = f(-1)$. Given $g \in C([0, 1], X)$, let $f(t) = g(t+1)$ ($-1 \leq t \leq 0$). Let $v_{-1} = g$ and $v_n(t) = u(n+t)$ ($n \in \mathbb{N}, t \in [0, 1]$). Then (DDE) becomes

$$v'_n(t) = Av_n(t) + v_{n-1}(t) \quad (n \in \mathbb{N}, t \in [0, 1]).$$

The unique mild solution is given by

$$v_n(t) = T(t)u(n) + \int_0^t T(s)v_{n-1}(t-s)ds \quad (t \in [0, 1]).$$

It follows by an easy induction that

$$u(n+t) = v_n(t) = \sum_{r=0}^n \frac{t^r}{r!} T(t) u(n-r) + \frac{1}{n!} \int_0^t s^n T(s) g(t-s) ds$$

$$(n \in \mathbb{N}, t \in [0, 1]). \quad (2.27)$$

The assumption (iv) and Proposition 2.1(3) imply that there exists $n \in \mathbb{N}$ (independent of g) such that $u(n+1) \in D(A)$. Since T is immediately differentiable, $T(1)u(n-r) \in D(A)$ for $r = 0, 1, \dots, n$. By (3.2), $(T_n * g)(1) \in D(A)$ for arbitrary $g \in C([0, 1], X)$, where $T_n(s) = s^n T(s)$ and the asterisk denotes convolution.

Now consider the map $g \mapsto A((T_n * g)(1))$ from $C([0, 1], X)$ to X . Since $g \mapsto (T_n * g)(1)$ is continuous and A is closed, the composed map has closed graph and is therefore continuous. Thus there is a constant c' such that

$$\|A((T_n * g)(1))\| \leq c' \|g\|_\infty \quad (g \in C([0, 1], X)).$$

Replacing $g(s)$ by $g(1-s)$, this gives

$$\left\| A \left(\int_0^1 s^n T(s) g(s) ds \right) \right\| \leq c' \|g\|_\infty.$$

Let $\varepsilon > 0$. Since $s \mapsto s^n AT(s)$ is norm-continuous on $[\varepsilon, 1]$, a standard approximation argument shows that

$$\int_\varepsilon^1 s^n \|AT(s)\| ds$$

$$= \sup \left\{ \left\| \int_\varepsilon^1 s^n AT(s) g(s) ds \right\| : g \in C([0, 1], X), \|g\|_\infty \leq 1, g(s) = 0 \ (s \in [0, \varepsilon]) \right\}.$$

Hence

$$\int_0^1 s^n \|AT(s)\| ds \leq c'.$$

Now fix $t \in (0, 1]$. For $s \in [t/2, t]$, $\|AT(t)\| = \|AT(s)T(t-s)\| \leq \kappa \|AT(s)\|$, where $\kappa = \sup\{\|T(\tau)\| : \tau \in [0, 1]\}$. Hence,

$$c' \geq \int_0^1 s^n \|AT(s)\| ds \geq \int_{t/2}^t (t/2)^n \|AT(t)\| / \kappa ds = \frac{t^{n+1} \|AT(t)\|}{2^{n+1} \kappa}.$$

This establishes (i) with $\alpha = n+1$ and $c = 2^{n+1} \kappa c'$. \square

The proof of Theorem 3.2 shows that if each mild solution of $u'(t) = Au(t) + u(t-1)$ is differentiable for $t \geq k \in \mathbb{N}$, then (i) holds with $\alpha = k$ (and some c) and then (ii) holds

with $m = k + 1$. However, the proof, in combination with [15, Theorem 2.1], shows only that if (ii) holds then the eventual differentiability in (iii) or (iv) holds for $t > m(2m + 3)$. It seems likely that this could be improved. In particular, there is probably a close correspondence between the values of α in (i) and m in (ii). A complimentary situation has recently been studied in [2].

Remark 2.7. There is a theory of delay semigroups based on L^p -spaces ($1 \leq p < \infty$) (see [3,4,19], for example). In many applications Φ is not bounded from $L^p([-1, 0], X)$ to X , but sometimes Φ is bounded from $C([-1, 0], X)$ to X . We assume that Φ is bounded from $C([-1, 0], X)$ to X and that there is an associated delay semigroup V_Φ on the space $X \times L^p([-1, 0], X)$. For example, this is true whenever Φ is of the form $\Phi f = \int_{-1}^0 d\eta f$ for some function $\eta: [-1, 0] \rightarrow \mathcal{B}(X)$ of bounded variation [3, Example 3.4], [12, Theorem 1.1]. In this context, Theorem 2.3 remains valid. The proof that (i) implies (ii) is almost unchanged, using the description of the resolvent of the delay semigroup given in [3, Lemma 4.1] and interpreting $\|\Phi\|$ to be the operator norm from $C([-1, 0], X)$ to X and $\|H_\lambda\|$ to be the operator norm from $L^p([-1, 0], X)$ to $C([-1, 0], X)$. If desired, the operators Φ_C in statement (iii) can be replaced by operators $\Phi \in \mathcal{B}(L^p[-1, 0], X)$ of the form $f \mapsto C(\int_{-1}^0 g(\theta)f(\theta)d\theta)$, where $C \in \mathcal{B}(X)$ and $g \in L^{p'}([-1, 0])$. Straightforward modifications of the proof that (iii) implies (i) in Theorem 2.3 show that uniform eventual differentiability of such delay semigroups implies (i).

3. Growth bounds

Suitable regularity of a C_0 -semigroup T has the important consequence that the exponential growth bound $\omega_0(T)$ of T is determined by the spectral bound $s(A)$ of A . It is well known that $s(A) \leq \omega_0(T)$ in general and that $s(A) = \omega_0(T)$ if T is eventually norm-continuous [10, Corollary IV.3.11] (a more general result can be found in [13, Corollary 1.4]).

For $x \in X$, let $u_x(t) = T(t)x$ ($t \geq 0$). Then u_x is the unique mild solution of the Cauchy problem

$$u'(t) = Au(t) \quad (t \geq 0), \quad u(0) = x,$$

and it is a classical solution if and only if $x \in D(A)$. The Laplace transform is given by $\hat{u}_x(\lambda) = R(\lambda, A)x$ for $\operatorname{Re} \lambda > \omega_0(T)$. Moreover,

$$\begin{aligned} \omega_0(T) &= \sup\{\omega_0(u_x): x \in X\}, \\ s(A) &= \sup\{\operatorname{hol}(\hat{u}_x): x \in X\}, \end{aligned}$$

where $\omega_0(u)$ and $\operatorname{hol}(\hat{u})$ are the growth bound of u and the abscissa of holomorphy of \hat{u} , respectively (see [1, Sections 1.4, 5.1]). This raises the question whether $\omega_0(u_x) = \operatorname{hol}(\hat{u}_x)$ for each initial value $x \in X$, if T is eventually norm-continuous. We do not know the complete answer to this question, but the following partial answers are obtained from the theory of the non-analytic growth bound developed in [5].

Proposition 3.1.

- (1) If T is eventually differentiable, then $\omega_0(u_x) = \text{hol}(\hat{u}_x)$ for all $x \in X$.
 (2) If T is eventually norm-continuous, then $\omega_0(u_x) = \text{hol}(\hat{u}_x)$ whenever $x \in D((\lambda - A)^\alpha)$ for some $\lambda > \omega_0(T)$ and some $\alpha > 0$.

Proof. (1) This is immediate from [5, Theorem 5.7].

(2) By [6, Corollary 3.3], the pseudo-spectral bound $s_0^\infty(A)$ of A is $-\infty$. The claim now follows as in [5, Theorem 5.8]. \square

We can ask the same question about mild solutions u of (DDE), i.e., functions of the form $u(t) = (V_\Phi(t)f)(0)$ ($t \geq 0$). The Laplace transform of u is then given by

$$\hat{u}(\lambda) = (R(\lambda, B_\Phi)f)(0) = R(\lambda, A + \Phi_\lambda)(f(0) + \Phi H_\lambda f).$$

An affirmative answer was given in [11] when $A = 0$, and the case when A is bounded follows. The following extends this to a much wider class of generators A .

Theorem 3.2. Suppose that there exist $\alpha > 0$, $b > 0$ and $c > 0$ such that $is \in \rho(A)$ and $\|R(is, A)\| \leq c|s|^{-\alpha}$ whenever $s \in \mathbb{R}$ and $|s| > b$. Then $\omega_0(u) = \text{hol}(\hat{u})$ for each mild solution u of (DDE).

Proof. It is always true that $\text{hol}(\hat{u}) \leq \omega_0(u)$.

Let u be a mild solution of (DDE), $f = u_0$ and $v(t) = V_\Phi(t)f$ ($t \geq 0$). Then $u(t) = v(t)(0)$, so $\omega_0(u) \leq \omega_0(v)$. By Theorem 2.3 and Proposition 3.1(1), $\omega_0(v) = \text{hol}(\hat{v})$. So it suffices to show that $\text{hol}(\hat{v}) \leq \text{hol}(\hat{u})$ (then equality holds).

Suppose that \hat{u} extends holomorphically to $H_\omega := \{\lambda \in \mathbb{C} : \text{Re } \lambda > \omega\}$ for some $\omega \in \mathbb{R}$. Let $\theta \in [-1, 0]$. Then

$$v(t)(\theta) = \begin{cases} f(t + \theta) & \text{if } t + \theta \leq 0, \\ u(t + \theta) & \text{if } t + \theta > 0. \end{cases}$$

For $\text{Re } \lambda > \omega_0(V_\Phi)$,

$$\begin{aligned} \hat{v}(\lambda)(\theta) &= \int_0^{-\theta} f(t + \theta)e^{-\lambda t} dt + \int_{-\theta}^{\infty} u(t + \theta)e^{-\lambda t} dt \\ &= \int_{\theta}^0 f(t)e^{-\lambda(t-\theta)} dt + \int_0^{\infty} u(t)e^{-\lambda(t-\theta)} dt \\ &= \int_{\theta}^0 f(t)e^{-\lambda(t-\theta)} dt + e^{\lambda\theta} \hat{u}(\lambda). \end{aligned}$$

The final formula extends holomorphically to H_ω , and these extensions are uniformly bounded for $\theta \in [-1, 0]$ and λ in any compact subset of H_ω . It follows from [1, Corollary A.4] that \hat{v} extends to a holomorphic function from H_ω to $C([-1, 0], X)$. This completes the proof. \square

For classical solutions of (DDE), the equality of $\omega_0(u)$ and $\text{hol}(\hat{u})$ can be proved under weaker assumptions on T ; for example, it suffices that T is immediately norm-continuous.

Theorem 3.3. *Suppose that $\lim_{|s| \rightarrow \infty} \|R(a + is, A)\| = 0$ for some $a > \omega_0(T)$. Then $\omega_0(u) = \text{hol}(\hat{u})$ for each classical solution u of (DDE).*

Proof. First note that, by (2.7), for any $a \in \mathbb{R}$, $a + is \in \rho(A)$ whenever $|s|$ is sufficiently large and $\lim_{|s| \rightarrow \infty} \|R(a + is, A)\| = 0$. By (2.10), $a + is \in \rho(B_\Phi)$ and

$$\begin{aligned} \|R(a + is, B_\Phi)\| &\leq 2 \max(1, e^{-a}) \|R(a + is, A)\| (1 + \|\Phi\| \max(1, e^{-a})) \\ &\quad + \max(1, e^{-a}) \end{aligned}$$

whenever $|s|$ is sufficiently large. Thus, $s_0^\infty(B_\Phi) = -\infty$.

Let u be a classical solution of (DDE), and let

$$w(t)(\theta) = u(t + 1 + \theta) \quad (t \geq 0, -1 \leq \theta \leq 0).$$

Then w is a classical solution of

$$w'(t) = B_\Phi w(t) \quad (t \geq 0), \quad w(0) = u_1,$$

so that $u_1 \in D(B_\Phi)$ and $w(t) = V_\Phi(t)u_1$. Using notation and results from [5, pp. 131, 138, 141, 144, 152] and taking $\mu \in \rho(A)$, it follows that $\zeta(w) \leq \zeta(V_\Phi(\cdot)R(\mu, B_\Phi)) \leq s_0^\infty(B_\Phi) = -\infty$, so $\omega_0(w) = \text{hol}(\hat{w})$ by [5, Proposition 2.4].

Since $u(t) = w(t)(-1)$, $\omega_0(u) \leq \omega_0(w)$. Furthermore, for $\text{Re } \lambda$ large,

$$\begin{aligned} \hat{w}(\lambda)(\theta) &= \int_0^\infty u(t + 1 + \theta) e^{-\lambda t} dt = \int_{1+\theta}^\infty u(s) e^{-\lambda(s-1-\theta)} ds \\ &= e^{\lambda(1+\theta)} \left(\hat{u}(\lambda) - \int_0^{1+\theta} u(s) e^{-\lambda s} ds \right). \end{aligned}$$

As in the proof of Theorem 3.2, it follows that $\text{hol}(\hat{w}) \leq \text{hol}(\hat{u}) \leq \omega_0(u)$, and this completes the proof. \square

Remark 3.4. An open question is whether immediate norm-continuity of T is sufficient to imply that $\omega_0(u) = \text{hol}(\hat{u})$ for all mild solutions u of (DDE). This would follow if the non-analytic growth bound of an eventually norm-continuous semigroup is always $-\infty$ (see [5, Section 5]).

Note added in proof

After this paper was completed, the author noticed that a result of M.G. Crandall and A. Pazy [J. Math. Mech. 18 (1968/1969) 1007–1016] shows that condition (i) of Theorem 2.6 implies condition (i) of Theorem 2.3.

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