SOME PROBLEMS IN COMMUTATIVE ALGEBRA.

(THE COUSIN COMPLEX IN COMMUTATIVE ALGEBRA.)

by

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ABSTRACT

In this thesis, the word "ring" will always mean "commutative, (associative,) Noetherian ring with a non-zero multiplicative identity".

An arbitrary module M over a ring A determines in a natural, but mildly complicated, fashion a certain complex:-

$$\underline{C(M)}: \quad 0 \longrightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \ldots \longrightarrow M^n \xrightarrow{d^n} \ldots$$

of A-modules and -homomorphisms. This thesis is concerned with the construction of $\underline{C(M)}$, with some of its properties and uses, and with some natural extensions of this theory. I have called $\underline{C(M)}$ the Cousin Complex for M: this seems appropriate because it is, in fact, the commutative algebra analogue of the Cousin Complex of §2 of Chapter IV of Hartshorne (12).

One can, of course, regard A as a module over itself and construct $\underline{C(A)}$. The power of the Cousin Complex stems mainly from simple characterizations of Cohen-Macaulay rings and Gorenstein rings in terms of the Cousin Complex. A ring A is Cohen-Macaulay if and only if $\underline{C(A)}$ is exact, and A is Gorenstein if and only if $\underline{C(A)}$ provides an injective (resp. minimal injective) resolution for A. Indeed, it was while looking for some sort of natural minimal injective resolution for a Gorenstein ring that I stumbled upon the existence of $\underline{C(A)}$ for an arbitrary ring A, and it was only later (considerably later, in fact, for my ability to work with algebraic geometry in general and schemes in particular is limited) that I realised $\underline{C(A)}$ was, in fact, the analogue of the Cousin Complex of the Affine scheme of the ring A as defined in Hartshorne (12).

While familiarity with the fundamental ideas of commutative algebra

and homological algebra is assumed, a survey of those basic standard results and definitions which are important for the subsequent work is given in the first two Chapters: Chapter 1 is devoted to commutative algebra and Chapter 2 to homological algebra. The main purposes of this are to establish notation and to attempt to leave the way open for comparatively unimpeded progress through the discussion of the Cousin Complex and its applications in Chapters 3–12. The experienced reader may like to begin at Chapter 3 and refer to Chapters 1 and 2 when necessary

The construction and properties of the Cousin Complex provide the subject matter of Chapter 3, and Chapters 4 and 5 contain the characterizations of Cohen-Macaulay rings and Gorenstein rings in terms of the Cousin Complex. I have, of course, made considerable use of the existing literature on Cohen-Macaulay rings and Gorenstein rings, and two papers to which I have frequently referred are Rees (24) and Bass(5). Chapter 4 actually examines the Cousin Complex for Cohen-Macaulay modules: the generalization from rings to modules is, in this case, fairly straightfoward, and I have given the theory in the more general case.

While several authors have studied Cohen-Macaulay modules (for example, see Chap. IV of Serre (26)), I have found no mention of "Gorenstein module" in the literature, and I have defined Gorenstein modules (over a ring A) as those non-zero, finitely-generated A-modules M for which $\underline{C(M)}$ provides a minimal injective resolution. In Chapter 6, I have examined some of the properties of Gorenstein modules and developed various characterizations of them: I have looked for characterizations which reduce to some of Bass' characterizations of Gorenstein rings in the particular case when M = A. (See Fundamental Theorem of §1 of Bass(5).

This has not been as easy as might have been expected, mainly for the following reason. Suppose for the moment that A is a local ring. In (3.3) of (5), Bass showed that, if M is a non-zero, finitely-generated A-module of finite injective dimension, then $\text{inj.dim.}_A M = \text{codh}_A A$ (not necessarily $\text{codh}_A M$). This result has particularly pleasant consequences when M = A: one is able to deduce that a local ring of finite injective dimension (as a module over itself) must be Cohen-Macaulay. However, in the situation of Bass' result mentioned above, it is not necessarily true that M is Cohen-Macaulay. Consequently, while Bass was able to characterize Gorenstein local rings as those local rings A for which $\text{inj.dim.}_A A$ is finite, there are non-zero, finitely-generated modules M over a local ring A which are not Gorenstein modules but for which $\text{inj.dim.}_A M$ is finite. The generalization of this characterization is more complicated and less obvious. Difficulties also arise when one attempts to generalize some of Bass' other characterizations of Gorenstein rings.

Bass himself conjectured (also in (5)) that, for a local ring A, there exist non-zero, finitely-generated A-modules M of finite injective dimension only if A is Cohen-Macaulay. I have perhaps added some weight to this conjecture by showing that, if A is local and there exists a Gorenstein A-module, then A has to be a Cohen-Macaulay ring.

The remarkably simple characterizations of Cohen-Macaulay rings and Gorenstein rings in terms of the Cousin Complex seem adequate reason for suspecting that the Cousin Complex may help in attempts to classify (commutative Noetherian) rings which are not necessarily Cohen-Macaulay. Chapter 7 contains a few brief skirmishes in this direction. As it seemed appropriate to examine the positions of Gorenstein and Cohen-

Macaulay local rings in the general heirarchy of local rings, I have included a brief survey of some well-known special classes of local rings (Regular, Complete Intersections, Gorenstein, and Cohen-Macaulay) together with examples to illustrate the distinctions between these classes. Consequently, only a small part of the work in Chapter 7 is original.

Chapter 8 is an illustration of the use of the Cousin Complex. The situation studied in (15) by Iversen is ideally suited for Cousin Complex arguments, and it turns out that the theory of the Cousin Complex yields more information than Iversen was able to obtain.

As mentioned above, the construction of the Cousin Complex is mildly complicated. In certain situations it becomes desirable to compare two Cousin Complexes, or variations thereof, and these comparisons become very complicated. I have tried to describe these comparisons as simply and clearly as possible, and I feel that the results justify these somewhat unpleasant computations.

One such comparison occurs in Chapter 9. Suppose A is a ring. In Chapter 6 it gradually becomes evident that it would be helpful to compare, for a Gorenstein A-module M and an $x \in A$ which is such that $x.M \neq M$ and x is not a zero-divisor on M, minimal injective resolutions for the A-module M and the A/(x)-module M/xM, and that the existing machinery is inadequate for this purpose. This problem is overcome in Chapter 9 by comparing the Cousin Complexes involved in a more general situation, and then using the fact the the Cousin Complex provides a minimal injective resolution for a Gorenstein module. A further body of information about Gorenstein modules results.

The work in Chapter 10 developed from an attempt to use the Cousin Complex to strengthen a result of Murthy which appears in (20): there he proves that, if the ring A is a unique factorization domain which is a homomorphic image of a regular local ring, then A is Gorenstein if and only if A is Cohen-Macaulay. My initial aim (which was achieved) was to show that a Cohen-Macaulay unique factorization domain which is a homomorphic image of a Gorenstein local ring must itself be Gorenstein.

However, it gradually became clear that the method of proof I was using had strong connections with the theory of Gorenstein modules developed in Chapters 6 & 9. Suppose that A is a Cohen-Macaulay ring which is a homomorphic image of a Gorenstein local ring B, and suppose that the Krull Dimensions of B and A are n and m respectively. Then $W = \mathrm{Ext}_B^{n-m}(A,B)$, which has a natural structure as an A-module, is a Gorenstein A-module. I am very grateful to Professor D. Rees for pointing out to me that the injective dimension of W as an A-module is finite: after realising this, it is easy to see that W is a Gorenstein A-module. So, although Chapter 10 does contain the above-mentioned strengthening of Murthy's result, together with some allied results, its main purpose is to prepare the ground for the uniqueness Theorems of Chapter 12.

As mentioned above, Chapter 5 contains a proof that, for there to exist a Gorenstein module over a local ring, it is necessary that the ring should be Cohen-Macaulay. The work in Chapter 10 provides a partial converse of this result, for it shows that there exist Gorenstein modules over Cohen-Macaulay local rings which are homomorphic images of Gorenstein local rings. In particular, using Cohen's Theorem, the existence of Gorenstein modules over complete Cohen-Macaulay local rings is established

The existence and uniqueness of Gorenstein modules over Gorenstein local rings is easy: the Gorenstein modules are precisely the non-zero, finitely-generated free modules. This is proved at the beginning of Chapter 12. However, the main purpose of Chapter 12 is the complete determination of all Gorenstein modules over a complete local Cohen-Macaulay ring. The two main results of Chapter 12 are the following:-

Let A be a complete local Cohen-Macaulay ring of Krull Dimension m. Suppose B is a Gorenstein local ring of Krull Dimension n such that there exists a surjective ring homomorphism $\phi : B \longrightarrow A$. Then:-

(i) $W = \mathrm{Ext}_B^{n-m}(A,B)$, which has a natural structure as an A-module, depends, up to A-isomorphism, only on A and not on B;

(ii) the Gorenstein A-modules are precisely those A-modules isomorphic to the direct sum of a (strictly positive) finite number of copies of W.

I would, of course, like to be able to prove the corresponding results for a Cohen-Macaulay local ring which is not necessarily complete. Unfortunately, there are two strong technical reasons why the arguments presented in Chapter 12 will not work in the case of a non-complete Cohen-Macaulay local ring.

I have not yet mentioned Chapter 11: this is devoted to developing the machinery necessary for my proofs of the main Theorems in Chapter 12. In fact, I need several of the consequences in commutative algebra of the Local Cohomology Theory developed by Hartshorne in (13). Hartshorne's definitions of the Local Cohomology Groups are formulated in the language of preschemes: it seems inappropriate to quote such general results in a thesis on commutative algebra, and so I have developed (from scratch almost) an analogous theory entirely within the framework of functors on

categories of modules over a (commutative, Noetherian) ring. While Hartshorne's work has obviously been a general guide in developing this theory, some of the necessary intermediate results turned out to be quite difficult to prove, and my eventual proofs bear little or no resemblance to Hartshorne's proofs of the analogous results in (13).

The end of a proof (or absence of a proof) is indicated thus:- //

ORIGINAL WORK AND ACKNOWLEDGEMENTS

Original Work.

With the exception of the first two introductory Chapters, which contain little of nothing new, I have adopted the following convention (the usual one) for indicating which results I claim as original.

If no reference appears by a result, then to the best of my knowledge and belief, it has not previously appeared in the literature, and I claim it as original.

If a result has appeared in the literature, I indicate the reference alongside the statement. (A list of references appears at the end of the thesis.) I have included proofs of such previously known results if I can find only statements and no proofs in the existing literature, or if I have a proof which is different from and (to me at least) more comprehensible than, those appearing in the literature.

I should perhaps add here that there is comparatively little original work in Chapter 7, and that, in Chapter 11, while the proofs of the results are most certainly original, I was guided in the general development of the theory by Hartshorne's development of his Local Cohomology Theory in (13).

Acknowledgements.

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CHAPTER 1

Notation, Definitions, and Standard Results in Commutative Algebra.

In this thesis, the word "ring" will always mean "commutative, Noetherian ring with a non-zero identity". If A is a ring, "A-module" will always mean "unitary A-module" (i.e. $1.x = x$ for all x in the module).

While familiarity with the fundamental ideas of commutative algebra is assumed, a survey of those basic results and definitions which are important for the subsequent work is included in this Chapter. Chapter 2 is devoted in a similar manner to homological algebra. A proof of any result quoted in this Chapter can be found in at least one of (1), (7), (9), (21), (26), (27).

(1.1) If X and Y are subsets of a set S, $X - Y$ will denote set-theoretic difference, i.e. $X - Y = \{s \in S \mid s \in X \,\&\, s \notin Y\}$.

Let A be a ring. If N is a submodule of the A-module M, and Y is a subset of M, $(N:Y)_A$ denotes the ideal $\{a \in A \mid a.y \in N \text{ for all } y \in Y\}$ of A. In particular, if $x \in M$, $(0:x)_A$ denotes the annihilator of x. If E is a subset of A, $(N:E)_M$ denotes the submodule $\{m \in M \mid e.m \in N \text{ for all } e \in E\}$ of M.

"f.-g." will be used as an abbreviation for "finitely-generated".

I shall indicate that a diagram of A-modules and -homomorphisms, or of rings and ring homomorphisms, commutes by writing "(C)" by the side of it.

If $f:M \to N$ is a homomorphism of A-modules, ker f will denote the kernel of f, im $f = f(M)$ the image of f, coker $f = N/f(M)$ the cokernel of f, and coim $f = M/\ker f$ the coimage of f.

A three term sequence $L \xrightarrow{f} M \xrightarrow{g} N$ of A-modules and

-homomorphisms is called a complex if $g.f = 0$: when that is the case $\ker g/\text{im } f$ is called the homology module at M. The sequence is said to be exact if $\ker g = \text{im } f$: I shall indicate this by writing (E) by the sequence, thus:- $(E)\ L \xrightarrow{f} M \xrightarrow{g} N.$

More generally, a sequence of A-modules and -homomorphisms

$\underline{C}$:.... $X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow$ .... which may be finite, infinite, or semi-infinite, is called a <u>complex</u> if $d^n.d^{n-1} = 0$ for all n. $H^n = H^n(\underline{C}) = \ker d^n/\text{im } d^{n-1}$ is called the <u>nth homology module</u> of $\underline{C}$. $\underline{C}$ is said to be exact at X^n if $H^n(\underline{C}) = 0$, and <u>exact</u> if $H^i(\underline{C}) = 0$ for all i. I should indicate that $\underline{C}$ was exact by writing (E) before or after the complex. It is sometimes convenient to write X_n instead of X^{-n}.

Suppose that $\underline{D}$: $\ldots\longrightarrow D^{n-1} \xrightarrow{d^{n-1}} D^n \longrightarrow \ldots$ and

$\underline{E}$: $\ldots\longrightarrow E^{n-1} \xrightarrow{e^{n-1}} E^n \longrightarrow \ldots$ are

complexes of A-modules and -homomorphisms. Then by a <u>translation or homomorphism of complexes</u> $\underline{\Psi} : \underline{D} \longrightarrow \underline{E}$ will be meant a family $\{\chi^n\}_{n\in Z}$ (Z denoting the integers) of A-homomorphisms $\chi^n : D^n \longrightarrow E^n$, such that, for each $n \in Z$,

$$\begin{array}{ccc} D^n & \xrightarrow{d^n} & D^{n+1} \\ \downarrow{\scriptstyle \chi^n} & & \downarrow{\scriptstyle \chi^{n+1}} \\ E^n & \xrightarrow{e^n} & E^{n+1} \end{array} \qquad (C)$$

. If each χ^n is an isomorphism, I shall say that $\underline{\Psi}$ is an <u>isomorphism of complexes</u>.

<u>(1.2)Rings and Modules of Fractions</u>. (For more details, see Chap. III of (1).)

Let A be a ring. A multiplicatively closed subset (m.c.s.) of A is a subset S of A such that $1 \in S$, $0 \notin S$, and $s,s' \in S$ imply $s.s' \in S$. S determines the <u>ring of fractions</u> $S^{-1}A$ which has, for its elements,

equivalence classes of elements of A x S under the equivalence relation $\sim$ on A x S defined by:- $(a,s) \sim (a',s')$ if and only if there is an $s'' \in S$ with $s''(s'.a - s.a') = 0$. a/s is used to denote the equivalence class (under $\sim$) to which (a,s) belongs.

The rules for addition and multiplication in $S^{-1}A$ are given by:-

$a/s + a'/s' = (s'.a + s.a')/s.s'$ and $a/s \,.\, a'/s' = a.a'/s.s'$.

$S^{-1}A$ is a ring; $\phi : A \longrightarrow S^{-1}A$ defined by $\phi(a) = a/1$ is a ring homomorphism; the extension, by means of ϕ, to $S^{-1}A$ of an ideal $\underline{a}$ of A will be written $S^{-1}\underline{a}$ or $\underline{a}.S^{-1}A$; every ideal of $S^{-1}A$ is an extended ideal; and $\underline{p} \longleftrightarrow S^{-1}\underline{p}$ gives a 1-1 onto correspondence between the prime ideals of A which don't meet S and the prime ideals of $S^{-1}A$.

Similarly, if M is an A-module, the equivalence classes of elements of M x S under the equivalence relation $\sim$ on M x S defined by:-

$(m,s) \sim (m',s')$ if and only if there is an $s'' \in S$ with $s''(s'm - sm') = 0$, form an $S^{-1}A$-module under the operations (where m/s is used to denote the equivalence class (under $\sim$) to which (m,s) belongs):-

$m/s + m'/s' = (s'm + sm')/ss'$ and $a/s.m/s' = am/ss'$.

This $S^{-1}A$-module is called the <u>module of fractions</u> of M w.r.t. S, and is written $S^{-1}M$. One can, of course, regard it as an A-module by means of $\phi : A \longrightarrow S^{-1}A$

<u>(1.3)</u> $S^{-1}A$, when regarded as an A-module by means of ϕ, is <u>flat</u>, for, if M is an arbitrary A-module,

$S^{-1}M \xrightarrow{\mu_M} M \otimes_A S^{-1}A$ as $S^{-1}A$-modules under $m/s \longmapsto m \otimes 1/s$.

Let μ_M be this isomorphism. μ_M is such that, if $f : M \longrightarrow N$ is an A-module homomorphism and $S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$ is the $S^{-1}A$-module homomorphism defined

by:- $(S^{-1}f)(m/s) = f(m)/s$, then

$$\begin{array}{ccc} S^{-1}M & \xrightarrow{S^{-1}f} & S^{-1}N \\ \downarrow{\scriptstyle \mu_M} & & \downarrow{\scriptstyle \mu_N} \\ M\otimes_A S^{-1}A & \xrightarrow[f\otimes 1]{} & N\otimes_A S^{-1}A \end{array} \qquad (C)$$

(1.4)If $\underline{p}$ is a prime ideal of A, A - $\underline{p}$ is an m.c.s. of A, and the corresponding ring of fractions is denoted by $A_{\underline{p}}$. Similarly, if M is an A-module, the corresponding module of fractions is denoted by $M_{\underline{p}}$.

If M is an A-module, the support of M, written Supp(M) or $\mathrm{Supp}_A(M)$, is the set $\{\underline{p}$ a prime ideal of A $\mid M_{\underline{p}} \neq 0\}$.

Observe that M = 0 if and only if Supp(M) is empty.

(1.5)Lemma. Suppose that N and P are submodules of the A-module M and that P is f.-g. Suppose $\underline{a}$ is an ideal of A (and so is f.-g.). Let S be an m.c.s. of A, $a \in A$, and $x \in M$. Then:-

(i) $S^{-1}((N:P)_A) = (S^{-1}N:S^{-1}P)_{S^{-1}A}$; (ii) $S^{-1}((N:\underline{a})_M) = (S^{-1}N:S^{-1}\underline{a})_{S^{-1}M}$

(iii) $S^{-1}((N:x)_A) = (S^{-1}N:x/1)_{S^{-1}A}$; (iv) $S^{-1}((N:a)_M) = (S^{-1}N:a/1)_{S^{-1}M}$

and, in particular, $S^{-1}((0:x)_A) = (0:x/1)_{S^{-1}A}$. //

(1.6)Corollary. $\mathrm{Supp}_{S^{-1}A}(S^{-1}M) = \{S^{-1}\underline{p} \mid \underline{p} \in \mathrm{Supp}_A(M)$ and $\underline{p}\cap S = \emptyset\}$.

Proof. If $S^{-1}\underline{p} \in \mathrm{Supp}_{S^{-1}A}(S^{-1}M)$, with $\underline{p}$ a prime of A and $\underline{p}\cap S = \emptyset$, then, for some $m \in M$, $(0:m/1)_{S^{-1}A} \subseteq S^{-1}\underline{p}$, and so it follows from (1.5) that $(0:m)_A \subseteq \underline{p}$.

Conversely, if $\underline{p} \in \mathrm{Supp}_A(M)$ and $\underline{p}\cap S = \emptyset$, there is an $m \in M$ such that $(0:m)_A \subseteq \underline{p}$. By (1.5), $(0:m/1)_{S^{-1}A} \subseteq S^{-1}\underline{p}$, which therefore belongs to $\mathrm{Supp}_{S^{-1}A}(S^{-1}M)$. //

(1.7)Note that, if M is an A-module such that multiplication by each element of some m.c.s. S of A gives an automorphism of M, then M has a

natural structure as an $S^{-1}A$-module: define $(a/s).m$ to be the unique element m' of M for which $s.m' = a.m$.

For convenience, I include here two rather technical Lemmas which will be needed later.

(1.8)Lemma. Suppose $\underline{a}$ is an ideal of A and S is an m.c.s. of A s.t. $\underline{a} \cap S = \emptyset$. If $\bar{S}$ denotes the image of S under $A \longrightarrow A/\underline{a}$, $\bar{S}$ is an m.c.s. of $A/\underline{a}$ and:- $S^{-1}A/S^{-1}\underline{a} \cong (\bar{S})^{-1}(A/\underline{a})$.

If N is an A-module s.t. $\underline{a}.N = 0$ and the above two rings are identified by means of the above isomorphism, $S^{-1}N$ and $(\bar{S})^{-1}(N)$ are isomorphic modules over this ring.

Proof. $S^{-1}\underline{a}$ is the kernel of the induced surjective ring homomorphism $S^{-1}A \longrightarrow (\bar{S})^{-1}(A/\underline{a})$, so $g: S^{-1}A/S^{-1}\underline{a} \longrightarrow (\bar{S})^{-1}(A/\underline{a})$ defined by $g(\widetilde{a/s}) = \bar{a}/\bar{s}$ (where $\sim$ is used to denote natural images in $S^{-1}A/S^{-1}\underline{a}$ and $\bar{}$ is used similarly for $A/\underline{a}$) is a ring isomorphism.

Note that

$$\begin{array}{ccc} & A/\underline{a} & \\ \swarrow & & \searrow \\ S^{-1}A/S^{-1}\underline{a} & \xrightarrow[\cong]{g} & (\bar{S})^{-1}(A/\underline{a}) \end{array} \qquad (C)$$

(where the other two homomorphisms are obvious).

N is naturally an $A/\underline{a}$-module: hence $(\bar{S})^{-1}N$. $S^{-1}N$ is annihilated by $S^{-1}\underline{a}$, and hence is an $S^{-1}A/S^{-1}\underline{a}$-module. It is clear that $n/s \leftrightarrow n/\bar{s}$ provides the required isomorphism. //

(1.9)Lemma. If S is an m.c.s. of A, X is an $S^{-1}A$-module, and $\underline{p}$ is a prime of A disjoint from S, then one can regard X as an A-module and form $X_{\underline{p}}$. One can also form $X_{S^{-1}\underline{p}}$. These both have natural $S^{-1}A$-module structures, and as such are isomorphic.

Proof. $S \subseteq A - \underline{p}$, and

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ S^{-1}A & \longrightarrow & A_{\underline{p}} \end{array} \quad (C),$$

where the bottom homomorphism is the obvious $a/s \longmapsto a/s$.

$h: X_{\underline{p}} \longrightarrow X_{S^{-1}\underline{p}}$ defined by $h(x/t) = x/(t/1)$ provides the required isomorphism. //

(1.10) Primary decomposition. I require the theory of primary decomposition for non-zero, f.-g. modules over a ring A. There is a full account of this in Chap. I of Serre (26).

Let A be a ring. I shall write $r(\underline{a})$ (or $r_A(\underline{a})$) for the radical of an ideal $\underline{a}$ of A.

A f.-g. A-module M is s.t.b. coprimary if and only if $M \neq 0$, and whenever $a \in A$ is a zero-divisor on M, there exists a positive integer N s.t. $a^N.M = 0$. In this case, $(0:M)_A$ is a primary ideal, and if $r_A(0:M) = \underline{p}$, one says that M is $\underline{p}$-coprimary. If N is a proper submodule of a f.-g. A-module M, N can be expressed as a finite intersection of submodules of M, $N = \bigcap_{i=1}^{n} Q_i$ s.t.:-

(i) M/Q_i is $\underline{p}_i$-coprimary for $i = 1,\ldots,n$;

(ii) $\underline{p}_1,\ldots,\underline{p}_n$ are n distinct prime ideals of A;

& (iii) no Q_i is redundant in the intersection, i.e. $Q_j \not\supseteq \bigcap_{\substack{i=1 \\ i\neq j}}^{n} Q_i$ for each $j = 1,\ldots,n$. Such an expression is called a minimal primary decomposition for N in M. These expressions have the following uniqueness properties:-

(i) The number of terms in the intersection and the prime ideals

$\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ are uniquely determined by N and M. These primes are called the <u>essential primes of N in M</u>. Clearly, they are also the essential primes of 0 in M/N. They are also called the <u>associated primes of M/N</u>, and the set which they form is written Ass(M/N) (or $\mathrm{Ass}_A(M/N)$).

(ii) The minimal elements of the set Ass(M/N) w.r.t. inclusion of ideals are called the <u>isolated</u> primes; the remaining ones are called <u>embedded</u>. The terms Q_i corresponding to isolated primes are also uniquely determined by N and M.

<u>(1.11)</u> In fact, the primes of Ass(M/N) can be characterized in the following way:-

<u>Proposition</u>.(Proposition 7 of Chap.I of (<u>26</u>).) If X is a non-zero, f.-g. A-module, then $\mathfrak{p} \in \mathrm{Ass}(X)$ if and only if there is a $y \in X$ s.t. $(0:y)_A = \mathfrak{p}$ (so that X has a submodule isomorphic to the cyclic module $A/\mathfrak{p}$). //

<u>(1.12)Corollary</u>.(See Corollary 4 on p.I-19 of (<u>26</u>).) Suppose M is a non-zero, f.-g. A-module. Then there is a finite chain of submodules of M:-

$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_r = M$ such that, for each $i = 1,\ldots,r$, $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of A. //

<u>(1.13)</u> The primes of Ass(M) are intimately related to the set of zero-divisors on M:-

<u>Propsition</u>.(Corollary 1 to Proposition 7 of Chap. I of (<u>26</u>).) Suppose M is a non-zero, f.-g. A-module. Then $\bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p} = \{ a \in A \mid a \text{ is a zero-divisor on } M \}$. //

Now suppose that S is an m.c.s. of A. I need to know the effect of the operation "S^{-1}" on primary decomposition.

If the f.-g. A-module M is $\underline{p}$-coprimary, and $\underline{p} \cap S = \emptyset$, $S^{-1}M$ is $S^{-1}\underline{p}$-coprimary. If $\underline{p}$ meets S, $S^{-1}M = 0$.

If $N = \bigcap_{i=1}^{n} Q_i$ is a minimal p.d. for the proper submodule N of the f.-g. A-module M, with M/Q_i $\underline{p}_i$-coprimary, and if S does not meet $\underline{p}_1,\ldots,\underline{p}_m$ but does meet $\underline{p}_{m+1},\ldots,\underline{p}_n$, then:-

$S^{-1}N = \bigcap_{i=1}^{m} S^{-1}Q_i$ is a minimal p.d. for $S^{-1}N$ in $S^{-1}M$.

$S^{-1}M/S^{-1}Q_i \cong S^{-1}(M/Q_i)$ is $S^{-1}\underline{p}_i$-coprimary for $i = 1,\ldots,m$.

Consequently:-

(1.14) $\mathrm{Ass}_{S^{-1}A}(S^{-1}M/S^{-1}N) = \mathrm{Ass}_{S^{-1}A}(S^{-1}(M/N)) = \left\{ S^{-1}\underline{p} \;\middle|\; \underline{p} \in \mathrm{Ass}_A(M/N) \text{ and } S \cap \underline{p} = \emptyset \right\}$.

(1.15) Primary Decomposition for Proper Ideals of the Ring A.

Accounts of this are available in Chapter IV of (1), Chapter I of (21), and Chapter IV of (27). However, this theory is just the particular case of the above in which M = A and N is a proper ideal of A: $A/\underline{q}$ is a $\underline{p}$-coprimary A-module if and only if $\underline{q}$ is a $\underline{p}$-primary ideal.

Note that if $\underline{a} \subseteq \underline{b} \subsetneq A$ are proper ideals of A, and $\bar{c}$ denotes the image in $A/\underline{a}$ of the ideal $\underline{c} \supseteq \underline{a}$ of A, then a minimal p.d. $\underline{b} = \bigcap_{i=1}^{n} \underline{q}_i$ with $\underline{q}_i$ $\underline{p}_i$-primary gives rise to a minimal p.d. $\bar{\underline{b}} = \bigcap_{i=1}^{n} \bar{\underline{q}}_i$ with $\bar{\underline{q}}_i$ $\bar{\underline{p}}_i$-primary.

If S is an m.c.s. of A, and $S \cap \underline{p} = \emptyset$, $\underline{q} \leftrightarrow S^{-1}\underline{q}$ gives a 1-1 onto correspondence between the $\underline{p}$-primary ideals of A and the $S^{-1}\underline{p}$-primary ideals of $S^{-1}A$. Let $\underline{b}^c$ denote the contraction to A of an ideal $\underline{b}$ of $S^{-1}A$. A minimal p.d. for a proper ideal $\underline{a}$ of A:-

$\underline{a} = \bigcap_{i=1}^{n} \underline{q}_i$, with $\underline{q}_i$ $\underline{p}_i$-primary, $\underline{p}_i \cap S = \emptyset$ for $i = 1,\ldots,m$ and

$\underline{p}_j$ meeting S for $j = m+1,\ldots,n$ gives rise to a minimal p.d.:-
$(S^{-1}\underline{a})^c = \bigcap_{i=1}^{m} \underline{q}_i$.

Recall also the definition of irreducible ideal: a proper ideal of A is s.t.b. <u>irreducible</u> if and only if it cannot be expressed as the intersection of two strictly larger ideals. As is well known (again see (1), (21), or (27)), every proper ideal of A can be expressed as a finite intersection of irreducible ideals, and every irreducible ideal is primary: indeed, these facts are used in proving that every proper ideal of A has a primary decomposition.

<u>(1.16)Applications of Primary Decomposition</u>. Again, for further details and proofs see Chap. I of Serre (26). Let A be a ring.

<u>Proposition and Definition</u>. Let M be a f.-g. A-module. The following conditions on a prime ideal $\underline{p}$ of A are equivalent:-

(i) $\underline{p} \in \mathrm{Ass}(M/N)$ for some proper submodule N of M;

(ii) $M_{\underline{p}} \neq 0$ (i.e. $\underline{p} \in \mathrm{Supp}(M)$);

(iii) $\underline{p} \supseteq (0:M)_A$;

(iv) $\underline{p}$ contains a prime ideal belonging to Ass(M).

The set of prime ideals of A satisfying these conditions is called the <u>variety associated with M</u>, and is denoted V(M) or $V_A(M)$.

<u>Proof</u>. Proposition 9 of Chap. I of (26). //

Thus V(A) denotes the set of prime ideals of A. If M is f.-g., then V(M) = Supp(M).

<u>(1.17) Lemma.</u> If M and N are f.-g. A-modules, $V(M \otimes_A N) = V(M) \cap V(N)$.

<u>Proof</u>. See Proposition 10 of Chap. I of (26). //

Corollary. If M is a f.-g. A-module and $\underline{a}$ is an ideal of A, $M/\underline{a}M \xrightarrow{\sim} M\otimes_A A/\underline{a}$ under $\phi : \bar{m} \longmapsto m\otimes\tilde{1}$ (where $^-$ and $^\sim$ denote natural images in $M/\underline{a}M$ and $A/\underline{a}$ resp.), and so $V_A(M/\underline{a}M) = V_A(M)\cap V_A(A/\underline{a})$. //

(1.18) As a consequence of this Corollary, I make the following

Definition. If M is a f.-g. A-module, and ideal $\underline{a}$ of A is s.t.b. proper w.r.t. M or M-proper if $M/\underline{a}M \neq 0$.

Observe that $\underline{a}$ is M-proper if and only if $V(M)\cap V(A/\underline{a}) \neq \emptyset$.

(1.19)Proposition. If S is a m.c.s. of A and M a f.-g. A-module, $V_{S^{-1}A}(S^{-1}M) = \left\{ S^{-1}\underline{p} \;\middle|\; \underline{p} \in V_A(M) \text{ and } \underline{p}\cap S = \emptyset \right\}$.

Proof. Since $V_A(M) = \mathrm{Supp}_A(M)$, and $V_{S^{-1}A}(S^{-1}M) = \mathrm{Supp}_{S^{-1}A}(S^{-1}M)$, this follows from (1.6). //

(1.20)Proposition. If M is a f.-g. A-module, M is of finite length if and only if V(M) consists of maximal ideals (in which case V(M) must be finite).

Proof. See Propostion 13 of Chap. I of (26). //

Consequently, if A is local with maximal ideal $\underline{m}$ and $\underline{q}$ is a proper ideal of A, $A/\underline{q}$ is an A-module of finite length if and only if $\underline{q}$ is $\underline{m}$-primary.

Dimension Theory.

The subsequent work will depend heavily on some results from Dimension Theory. I start with the well-known Definitions.

(1.21)Definitions. Let A be a ring. A chain of prime ideals of A is a finite ascending chain $\underline{p}_0 \subsetneq \underline{p}_1 \subsetneq \cdots \subsetneq \underline{p}_n$ of prime ideals with proper inclusions. (Such a chain is said to be saturated if, for each i = 1,...,n,

there is no prime ideal $\underline{q}$ of A satisfying $\underline{p}_{i-1} \subsetneq \underline{q} \subsetneq \underline{p}_i$.) The <u>length</u> of the chain is the number of links, in this case n.

The <u>Krull Dimension of A</u> (written K-Dim A) is defined to be the supremum of lengths of chains of prime ideals of A if this supremum exists, and ∞ if it does not.

If $\underline{p}$ is a prime ideal of A, the <u>height</u> of $\underline{p}$, written ht $\underline{p}$, is the supremum of lengths of chains of prime ideals of A which terminate with $\underline{p}$ (i.e. have $\underline{p}$ as their largest term). Thus ht $\underline{p}$ = K-Dim $A_{\underline{p}}$. (See (1.2).)

Similarly, the <u>coheight</u> of $\underline{p}$ (coht $\underline{p}$) is the supremum of lengths of chains of primes of A which commence with $\underline{p}$, if this supremum exists, and ∞ if it does not. Thus coht $\underline{p}$ = K-Dim $A/\underline{p}$.

<u>(1.22)</u> I make some similar Definitions for modules:-

<u>Definitions</u>. Let A be a ring and $M \neq 0$ an A-module. The <u>Krull Dimension of M</u>, written K-Dim$_A$M or K-Dim M, is the supremum of lengths of chains of prime ideals of Supp(M) if this exists, and ∞ if it does not.

If $\underline{p} \in$ Supp(M), the <u>M-height</u> of $\underline{p}$ (written $\mathrm{ht}_M \underline{p}$) is the supremum of lengths of chains of prime ideals of Supp(M) terminating with $\underline{p}$. From (1.6), $\mathrm{ht}_M \underline{p}$ = K-Dim$_{A_{\underline{p}}} M_{\underline{p}}$.

<u>(1.23)</u> I now come to the Fundamental Theorm of Dimension Theory. The form in which I require it is the following:-

<u>Theorem</u>. Suppose A is a local ring and M is a non-zero, f.-g., A-module. Then K-Dim M is finite and is equal to the least (non-negative) integer r for which there is a proper ideal $\underline{x}$ generated by r elements with $M/\underline{x}M$ an A-module of finite length. (Observe that, since $\underline{x}$ is proper, $M/\underline{x}M \neq 0$, by Nakayama's Lemma.)

Proof. See Chap. III-B of Serre (26). //

(1.24)Corollary. If, in the situation of (1.23), $a \in \underline{m}$, the maximal ideal of A, and $\underline{a} \not\subseteq$ any minimal element of Supp(M) = V(M), then $K\text{-Dim}_A(M/aM) = K\text{-Dim}_A M - 1$.

Proof. By (1.17), $V(M/aM) = V(M) \cap V(A/(a))$, so $K\text{-Dim}_A(M/aM) \leqslant K\text{-Dim}_A M - 1$. On the other hand, using the Theorm, it is clear that $K\text{-Dim}_A M \leqslant K\text{-Dim}_A(M/aM) + 1$, for if $(x_1,\ldots,x_r)$ is s.t. $(M/aM)/(x_1,\ldots,x_r)(M/aM)$ is of finite length, then $M/(a,x_1,\ldots,x_r)M$ is of finite length. //

(1.25)Consequences of Fundamental Theorem of Dimension Theory.

Suppose A is an arbitrary ring and M an arbitrary non-zero, f.-g. A-module.

For each $\underline{p} \in V(A)$, $K\text{-Dim } A_{\underline{p}} = ht\,\underline{p}$ is finite.

Thus the prime ideals of A satisfy the descending chain condition, and so, if $N \neq 0$ is any A-module, $K\text{-Dim}_A N$ is the supremum of lengths of saturated chains of prime ideals of Supp(N) if this exists, and ∞ if it does not.

If $\underline{p} \in V(M) = Supp(M)$, $ht_M \underline{p}$ is finite and $\leqslant ht\,\underline{p}$.

If $\underline{a}$ is an M-proper ideal of A, $Ass(M/aM) \subseteq V(M)$: define the M-height of $\underline{a}$ ($ht_M \underline{a}$) to be $\inf\left\{ ht_M \underline{p} \,\middle|\, \underline{p} \in V(M) \text{ and } \underline{p} \supseteq \underline{a} \right\}$. It follows that $ht_M \underline{a} = \inf\left\{ ht_M \underline{p} \,\middle|\, \underline{p} \in Ass(M/\underline{a}M) \right\}$.

In particular, if $\underline{a}$ is a proper ideal of A, the height of $\underline{a}$ is given by:- $ht\,\underline{a} = \inf\left\{ ht\,\underline{p} \,\middle|\, \underline{p} \in V(A) \text{ and } \underline{p} \supseteq \underline{a} \right\} = \inf\left\{ ht\,\underline{p} \,\middle|\, \underline{p} \in Ass(A/\underline{a}) \right\}$

(1.26)Lemma. Suppose that $\underline{x} = (x_1,\ldots,x_r)$ is an M-proper ideal of A generated by r elements, and that $\underline{p}$ is an isolated prime ideal of $M/\underline{x}M$. Then ($\underline{p} \in V(M)$ and) $ht_M \underline{p} \leqslant r$.

Proof. $\underline{p}$ is a minimal element of $\mathrm{Ass}(M/\underline{x}M)$, and so $\underline{p}A_{\underline{p}}$ is the only element of $\mathrm{Ass}_{A_{\underline{p}}}(M_{\underline{p}}/(x_1/1,\ldots,x_r/1)M_{\underline{p}})$ (by (1.14)). Thus, by (1.20), $M_{\underline{p}}/(x_1/1,\ldots,x_r/1)M_{\underline{p}}$ is an $A_{\underline{p}}$-module of finite length. Therefore $\mathrm{ht}_M\underline{p} = \text{K-Dim}_{A_{\underline{p}}} M_{\underline{p}} \leqslant r$, by (1.23). //

(1.27) I shall also be very much concerned with unmixed ideals.

Definition. A proper ideal $\underline{a}$ of A is s.t.b. unmixed if all the primes of $\mathrm{Ass}(A/\underline{a})$ have the same height. Clearly, such an $\underline{a}$ can have no embedded primes for primary decomposition in A.

More generally, if M is a non-zero, f.-g. A-module, an M-proper ideal $\underline{a}$ of A is s.t.b. M-unmixed if all the primes of $\mathrm{Ass}(M/\underline{a}M)$ have the same M-height. Again, for such an $\underline{a}$, $M/\underline{a}M$ can have no embedded primes.

Some of the well-known Theorems of the Theory of Integral Dependence, some properties of Unique Factorization Domains, and some facts about complete local rings will be involved in some of the examples and applications.

In the Theory of Integral Dependence I shall need the following results from Atiyah and Macdonald (1):- (5.8), (5.10) ("Lying Over" Theorem), (5.11) ("Going Up" Theorem), and (5.16) ("Going Down" Theorem). An account of the Theory of Integral Dependence is also contained in Chap. V (Vol I) of Zariski and Samuel (27).

(1.28)(Noetherian) Unique Factorization Domains.

Let A be an integral domain (which is not a field).

Recall that a non-zero element y of A is s.t.b. irreducible if it is

not a unit and if $y = u.v$ for $u, v \in A$ implies that one of u and v is a unit. Thus, if $y \neq 0$, y is irreducible if and only if the ideal (y) is maximal among proper principal ideals of A.

Definition. A is s.t.b. a Unique Factorization Domain (U.F.D.) if and only if there is a subset P of A consisting of non-zero, non-unit elements s.t., if $0 \neq x \in A$, x can be expressed in one and only one way as $x = u.\prod_{p \in P} p^{n(p)}$, where each n(p) is a non-negative integer, $n(p) = 0$ for all but a finite number of p's in P, and u is a unit.

Observe that the elements of P must be irreducible.

Since I am assuming that A is an arbitrary Noetherian integral domain, it follows that, even if A is not a U.F.D., every non-zero, non-unit element of A can be expressed as a finite product of irreducible factors (for otherwise there is an infinite strictly ascending chain of principal ideals).

I shall need:-

(1.29)Proposition. Let A be an integral domain which is not a field. Then, if A is a U.F.D., every unmixed ideal of A of height 1 is principal.

Proof. I use the notation of the Definition in (1.28). I first show that the only primes of height 1 are the ideals of the set $\{(p) \mid p \in P\}$.

First of all, (0) is prime. It is clear that, if $p \in P$, (p) is prime. Since $(p) \supsetneq (0)$, (1.26) shows that $ht(p) = 1$.

Suppose conversely that $\underline{p}$ is a prime of height 1. If $0 \neq x \in \underline{p}$, factorizing x in terms of the elements of P shows that some p in P belongs to $\underline{p}$. It follows that $\underline{p} = (p)$.

Next, I show that, if $p \in P$, the (p)-primary ideals are precisely

those which form the set $\{(p^n) \mid n = 1,2,\ldots\}$. Use the unique factorization to see that each (p^n) is (p)-primary.

Conversely, if $\underline{q}$ is a (p)-primary ideal of A, and n is the least integer k s.t. $p^k \in \underline{q}$, then $\underline{q} = (p^n)$. For if not, and $x \in \underline{q} - (p^n)$, then $x = u.p^h.p_1^{n_1}.\ldots.p_r^{n_r}$ with $p_1,\ldots,p_r \in P$, all different and different from p, $n_i \geqslant 0$ for each $i = 1,\ldots,r$, $0 \leqslant h < n$, and u a unit. Since $p_i \neq p$ for $i = 1,\ldots,r$, $p_i \notin (p)$, and so, since $\underline{q}$ is (p)-primary, $u.p^h \in \underline{q}$. Thus $p^h \in \underline{q}$, contradicting the definition of n.

Thus, if $\underline{a}$ is an unmixed ideal of A of height 1, there exist $p_1,\ldots,p_r \in P$ and positive integers $n_1,\ldots,n_r$ s.t. $\underline{a} = \bigcap_{i=1}^{r} (p_i^{n_i})$. It follows that $\underline{a} = (p_1^{n_1}.\ldots.p_r^{n_r})$. //

(1.30)Complete Local Rings. (For further details see Chap. X of Atiyah and Macdonald (1), Cohen (7), Chap. V of Northcott (21), or Chap. II of Serre (26).)

Suppose A is a local ring with maximal ideal $\underline{m}$. Then $\bigcap_{i=1}^{\infty} \underline{m}^i = 0$ (by, for example, Theorem 3 of Chap. III of Northcott (21)). Consequently, a metric $d: A \times A \longrightarrow R$ (the field of real numbers) can be defined as follows:-

(i) If $a,a' \in A$ and $a \neq a'$, there is a non-negative integer i s.t. $a - a' \in \underline{m}^i$ but $a - a' \notin \underline{m}^{i+1}$ (where $\underline{m}^0 = A$, of course): define $d(a,a') = 1/2^i$.

(ii) Define, for $a \in A$, $d(a,a) = 0$.

A, furnished with the resulting metric topology (called the $\underline{m}$-adic topology) becomes a topological ring. Convergence of a sequence of elements in such a local ring A will always mean convergence with respect

to the $\underline{m}$-adic topology. A is said to be a complete local ring if and only if A regarded as a metric space as above is a complete metric space.

An arbitrary local ring A with maximal ideal $\underline{m}$ can be embedded in a complete local ring $\hat{A}$ with maximal ideal $\hat{\underline{m}}$ s.t.:-

(i) a sequence of elements in A converges to zero in A if and only if it converges to zero in $\hat{A}$;

& (ii) every element of $\hat{A}$ is the limit of a sequence of elements of A.

$\hat{A}$ is uniquely determined (up to unique ring isomorphism over A) by A, and is called the completion of A. (I shall consistently use $\hat{A}$ to denote the completion of a local ring A.)

(1.31) I shall also need the following facts about the completion of a local ring A. Let $\underline{m}$ be the maximal ideal of A, and, if $\underline{a}$ is an ideal of A, let $\underline{a}\hat{A}$ denote the extension of $\underline{a}$ to $\hat{A}$. Then:-

(i) $\underline{m}\hat{A} = \hat{\underline{m}}$, the maximal ideal of $\hat{A}$ (see Prop. 6 of Chap. V of (21));

(ii) for each integer $s > 0$, $(\hat{\underline{m}})^s \cap A = \underline{m}^s$ (see Prop. 6 of Chap. V of (21));

(iii) if $\underline{a}$ is an ideal of A, $(\underline{a}\hat{A}) \cap A = \underline{a}$ (see Prop. 6 of Chap. V of (21));

(iv) $\hat{A}$, when regarded as an A-module by means of the natural embedding $A \xrightarrow{\subseteq} \hat{A}$, is flat (see Prop. (10.14) of (1) or Corollary 2 of Theorem 1 of Chap. II of Serre (26));

& (v) K-Dim A = K-Dim $\hat{A}$ (see Theorem 8 of Chap. V of (21)).

(1.32) Direct Systems and Direct Limits. (For further details see, for example, Chap. VIII of Eilenberg and Steenrod (9).) Let A be a ring. I need a little of the theory of direct limits of A-modules and -homom-

-orphisms for use in Chapters 10, 11, and 12: I establish the necessary notation here.

A <u>directed set</u> is a set D together with a relation $<$ in D which satisfies the 3 conditions:-

(i) $\alpha<\alpha$ for each $\alpha \in D$ (i.e. $<$ is reflexive);

(ii) if $\alpha,\beta,\gamma \in D$ and $\alpha<\beta$, $\beta<\gamma$, then $\alpha<\gamma$ (i.e. $<$ is transitive);

& (iii) for each pair α,β in D, there exists a $\delta \in D$ s.t. $\alpha<\delta$ & $\beta<\delta$.

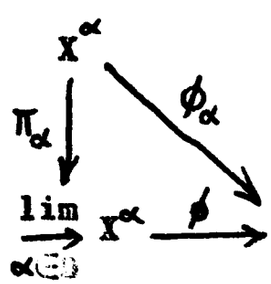
A <u>direct system</u> of A-modules and -homomorphisms $\{X,\pi\}$ over a directed set D is a function which attaches to each $\alpha \in D$ an A-module X^α and to each pair α,β in D with $\alpha<\beta$ an A-homomorphism $\pi_\alpha^\beta : X^\alpha \longrightarrow X^\beta$ s.t., for each $\alpha \in D$, π_α^α is the identity on X^α, and, for $\alpha<\beta<\gamma$ in D, $\pi_\beta^\gamma . \pi_\alpha^\beta = \pi_\alpha^\gamma$.

Given such a direct system $\{X,\pi\}$ over a directed set D, for each pair α,β in D with $\alpha<\beta$ and each $x^\alpha \in X^\alpha$, the element $\pi_\alpha^\beta x^\alpha - x^\alpha$ of $\bigoplus_{\alpha \in D} X^\alpha$ is called a <u>relation</u>. The <u>direct limit</u> of the direct system $\{X,\pi\}$ over D, written $\varinjlim_{\alpha \in D} X^\alpha$, is the A-module $\bigoplus_{\alpha \in D} X^\alpha / Y$, where Y is the submodule of $\bigoplus_{\alpha \in D} X^\alpha$ generated by all relations.

For each $\alpha \in D$, $\pi_\alpha : X^\alpha \longrightarrow \varinjlim_{\alpha \in D} X^\alpha$ is the composition of the canonical homomorphisms $X^\alpha \longrightarrow \bigoplus_{\alpha \in D} X^\alpha$ and $\bigoplus_{\alpha \in D} X^\alpha \longrightarrow \varinjlim_{\alpha \in D} X^\alpha$. If $\alpha<\beta$ in D, $\pi_\beta . \pi_\alpha^\beta = \pi_\alpha$.

Furthermore, $\varinjlim_{\alpha \in D} X^\alpha$ and the family $\left\{ \pi_\alpha : X^\alpha \longrightarrow \varinjlim_{\alpha \in D} X^\alpha \right\}_{\alpha \in D}$ have the following universal property:-

Given an A-module W and a family $\{\phi_\alpha : X^\alpha \longrightarrow W\}_{\alpha \in D}$ of A-homomorphisms s.t., for $\alpha<\beta$ in D, $\phi_\beta . \pi_\alpha^\beta = \phi_\alpha$, then there exists a unique homomorphism $\phi : \varinjlim_{\alpha \in D} X^\alpha \longrightarrow W$ s.t., for each $\alpha \in D$,



(C)

Consequently, if the A-module T and the family $\{\gamma_\alpha : X^\alpha \to T\}_{\alpha \in D}$ of A-homomorphisms with, for $\alpha < \beta$ in D, $\gamma_\beta \cdot \pi_\alpha^\beta = \gamma_\alpha$, have the analagous universal property, then there is a unique isomorphism $\gamma : \varinjlim_{\alpha \in D} X^\alpha \to T$ s.t., for each $\alpha \in D$,

$$(C) \qquad \begin{array}{ccc} X^\alpha & & \\ \downarrow{\scriptstyle \pi_\alpha} & \searrow^{\gamma_\alpha} & \\ \varinjlim_{\alpha \in D} X^\alpha & \xrightarrow[\cong]{\gamma} & T \end{array} \quad .$$

(1.33)Lemma. Let $\{X, \pi\}$ be a direct system of A-modules and -homomorphisms over the directed set D. Then:-

(i) (Lemma 4.3 of Chap. VIII of (9).) If $x \in \varinjlim_{\alpha \in D} X^\alpha$, then there exists $\alpha \in D$ and an $x^\alpha \in X^\alpha$ s.t. $\pi_\alpha(x^\alpha) = x$.

(ii) (Lemma 4.4 of Chap. VIII of (9).) If $x^\gamma \in X^\gamma$ ($\gamma \in D$) and $\pi_\gamma(x^\gamma) = 0$, then there is a $\delta > \gamma$ in D s.t. $\pi_\gamma^\delta(x^\gamma) = 0$. //

(1.34)Observation. (This appears as Example 4.9 of Chap. VIII of (9).) Suppose M is an A-module. Let Φ be the set of all finite subsets of M, and for $\alpha \in \Phi$, let N^α be the submodule of M generated by the elements of α. Define a relation $<$ in Φ by the rule:-

$\alpha < \beta$ if and only if $N^\alpha \subseteq N^\beta$; and when this is the case, let $\pi_\alpha^\beta : N^\alpha \to N^\beta$ be the inclusion. Φ is clearly a directed set, and $\{N, \pi\}$ is clearly a direct system over Φ. Furthermore, it is easy to see that $\varinjlim_{\alpha \in \Phi} N^\alpha \cong M$. Hence the imprecise but suggestive statement: " an A-module is the direct limit of its f.-g. submodules". //

(1.35)Observation. (This appears as Example 4.10 of Chap. VIII of (9).) Suppose $\{H_j\}_{j \in J}$ is a family of A-modules and H is their direct sum. Let Φ be the collection of finite subsets of J, and let $\alpha < \beta$ in Φ if and

only if $\alpha \subseteq \beta$. If $\alpha \in \Phi$, define $N^\alpha = \bigoplus_{j \in \alpha} H_j$, and if $\alpha < \beta$ in Φ, define $\pi_\alpha^\beta : N^\alpha \longrightarrow N^\beta$ to be the obvious natural inclusion. Then Φ is clearly a directed set, and $\{N, \pi\}$ is a direct system over Φ. It is easy to see that $\varinjlim_{\alpha \in \Phi} N^\alpha \cong H$. Hence the imprecise but suggestive statement: " a direct sum is the direct limit of finite direct sums". //

CHAPTER 2

Notation, Definitions, Techniques, and Standard Results in Homological Algebra.

(Reminder: "ring" means "commutative, Noetherian ring with a non-zero identity".)

As was indicated at the beginning of Chapter 1, a large part of this Chapter is devoted to a survey of those basic results and definitions in homological algebra which are important for the subsequent work. However, at the end of the Chapter, I have included details of the more sophisticated homological algebra ideas which are developed in paragraphs 4 & 6 of Hartshorne (13). These ideas are needed only in the final two Chapters of this thesis, i.e. Chapters 11 and 12, and do not really belong under the broad heading of "basic results", but the alternative to their inclusion in this Chapter was to incorporate them in Chapter 11, and I feel the interruption would have detracted from the development of the main theme of Chapters 11 & 12.

All the homological algebra which appears in this thesis is within the framework of functors on categories of modules over a (commutative Noetherian) ring. The functors which occur by far the most frequently are Extension functors. In fact, it is only in Chapters 11 & 12 that other functors are first used extensively: these other functors are all functors of a single variable, and their properties are easily established by refering to either Cartan and Eilenberg (6) or Northcott (22) as and when necessary. Consequently, the emphasis in the first part of this Chapter is on the properties of Extension functors.

(2.1) Let A be a ring. Let $\underline{\mathcal{C}}_A$, $\underline{\mathcal{C}}_A^f$, $\underline{\mathcal{C}}_A^l$, denote, respectively, the category of A-modules and A-homomorphisms, the category of A-modules which are f.-g. and all A-homomorphisms between them, and the category of A-modules of finite length and all A-homomorphisms between them. Then, if M and N are A-modules, $\mathrm{Hom}_A(M,N)$ is an A-module and is an additive, A-linear functor from $\underline{\mathcal{C}}_A \times \underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$. (To say that $\mathrm{Hom}_A(M,N)$ is *additive* means that, if f_1, f_2, g_1, g_2 are A-homomorphisms such that, for each $i = 1,2$, f_i & g_i have the same range and domain, then:-

$$\mathrm{Hom}_A(f_1+g_1, f_2) = \mathrm{Hom}_A(f_1, f_2) + \mathrm{Hom}_A(g_1, f_2) \text{ and}$$

$$\mathrm{Hom}_A(f_1, f_2+g_2) = \mathrm{Hom}_A(f_1, f_2) + \mathrm{Hom}_A(f_1, g_2).$$

To say that $\mathrm{Hom}_A(M,N)$ is *A-linear* means that, if $a \in A$, then:-

$$\mathrm{Hom}_A(af_1, g_1) = a.\mathrm{Hom}_A(f_1, g_1) = \mathrm{Hom}_A(f_1, ag_1). \quad)$$

As well as being A-linear, $\mathrm{Hom}_A(M,N)$ is left exact and is contravariant in M and covariant in N. The functors $\mathrm{Ext}_A^i(M,N)$ are the right derived functors of $\mathrm{Hom}_A(M,N)$. The theory of derived functors is developed in Chap. V of Cartan and Eilenberg (6) and Chap. 6 of Northcott (22). While Cartan and Eilenberg is the standard reference for homological algebra, I have found parts of Northcott's book helpful for applications to commutative algebra, and, in fact, a variation of Theorem 3 of Chap. 6 of the latter work will serve as a good starting point.

(2.2) Properties of Extension Functors.

For each $n = 0, \pm 1, \pm 2, \ldots$, $\mathrm{Ext}_A^n(M,N)$ is the nth right derived functor of $\mathrm{Hom}_A(M,N)$. Thus $\mathrm{Ext}_A^n(M,N)$ is an additive, A-linear functor from $\underline{\mathcal{C}}_A \times \underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$, contravariant in M and covariant in N. $\mathrm{Ext}_A^n(M,N) = 0$ if $n < 0$ for all A-modules M and N.

Since $\mathrm{Hom}_A(M,N)$ is left exact, $\mathrm{Hom}_A(M,N)$ and $\mathrm{Ext}_A^0(M,N)$ are naturally equivalent functors. I shall use this natural equivalence without further comment.

Given exact sequences of A-modules:-

(E) $0 \to M' \to M \to M'' \to 0$ and (E) $0 \to N' \to N \to N'' \to 0$, then, for each $n \geqslant 0$, there exist well-defined A-module connecting homomorphisms:-

$$\mathrm{Ext}_A^n(M',N) \longrightarrow \mathrm{Ext}_A^{n+1}(M'',N) \quad \text{and} \quad \mathrm{Ext}_A^n(M,N'') \longrightarrow \mathrm{Ext}_A^{n+1}(M,N')$$

and these connecting homomorphisms make the sequences:-

(2.2)(A) (E) $0 \to \mathrm{Hom}_A(M'',N) \to \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(M',N) \to \mathrm{Ext}_A^1(M'',N) \to \ldots$

$\ldots \longrightarrow \mathrm{Ext}_A^{n-1}(M',N) \longrightarrow \mathrm{Ext}_A^n(M'',N) \to \mathrm{Ext}_A^n(M,N) \to \mathrm{Ext}_A^n(M',N) \to \mathrm{Ext}_A^{n+1}(M'',N) \ldots$

and

(2.2)(B) (E) $0 \to \mathrm{Hom}_A(M,N') \to \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(M,N'') \to \mathrm{Ext}_A^1(M,N') \to \ldots$

$\ldots \longrightarrow \mathrm{Ext}_A^{n-1}(M,N'') \longrightarrow \mathrm{Ext}_A^n(M,N') \to \mathrm{Ext}_A^n(M,N) \to \mathrm{Ext}_A^n(M,N'') \to \mathrm{Ext}_A^{n+1}(M,N') \ldots$

of A-modules and A-homomorphisms exact.

Moreover, if (2.2)(A) and (2.2)(B) are regarded as belonging to the translation category of (right semi-infinite) exact sequences of A-modules and A-homomorphisms, then (2.2)(A) is a contravariant functor of (E) $0 \to M' \to M \to M'' \to 0$ and a covariant functor of the module N, while (2.2)(B) is a contravariant functor of the module M and a covariant functor of (E) $0 \to N' \to N \to N'' \to 0$.

(2.3)Methods of Calculating $\mathrm{Ext}_A^n(M,N)$.

See, for example, Theorem 9 of Chap. 7 of Northcott (22).

(A) Let $\underline{X}$: $\ldots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow 0$

be a projective resolution of M, so that there is an A-homomorphism $P_0 \longrightarrow M$ with :-

(E) $\ldots\longrightarrow P_{n+1}\longrightarrow P_n\longrightarrow P_{n-1}\longrightarrow\ldots\longrightarrow P_0\longrightarrow M\longrightarrow 0.$

Then $\mathrm{Ext}_A^n(M,N)$ is isomorphic to the nth homology module of the complex:-

$0\rightarrow \mathrm{Hom}_A(P_0,N)\rightarrow \mathrm{Hom}_A(P_1,N)\rightarrow\ldots\longrightarrow \mathrm{Hom}_A(P_n,N)\rightarrow \mathrm{Hom}_A(P_{n+1},N)\longrightarrow\ldots.$

(B) Let $\underline{Y}$: $0\longrightarrow I^0\longrightarrow I^1\longrightarrow\ldots\longrightarrow I^n\longrightarrow I^{n+1}\longrightarrow\ldots$

be an injective resolution of N, so that thereis an A-homomorphism

$N\longrightarrow I^0$ with:-

(E) $0\longrightarrow N\longrightarrow I^0\longrightarrow I^1\longrightarrow\ldots\longrightarrow I^n\longrightarrow I^{n+1}\longrightarrow\ldots.$

Then $\mathrm{Ext}_A^n(M,N)$ is isomorphic to the nth homology module of the complex:-

$0\rightarrow \mathrm{Hom}_A(M,I^0)\rightarrow \mathrm{Hom}_A(M,I^1)\rightarrow\ldots\longrightarrow \mathrm{Hom}_A(M,I^n)\rightarrow \mathrm{Hom}_A(M,I^{n+1})\longrightarrow\ldots.$

(2.4) It follows from (2.3)(A) that $\mathrm{Ext}_A^n(M,N)$ is f.-g. if both M and N are.

(2.5) Also from (2.3)(A), an A-module P is projective if and only if $\mathrm{Ext}_A^1(P,C)=0$ for all A-modules C. Recall the Definition of Homological dimension:-

Definition. A non-zero A-module M is said to have homological dimension equal to n if and only if:-

(i) there is an exact sequence of the form:-

(E) $0\longrightarrow P_n\longrightarrow P_{n-1}\longrightarrow\ldots\longrightarrow P_0\longrightarrow M\longrightarrow 0$, where each P_i is projective;

& (ii) there is no exact sequence of this type with fewer terms..

In this case, I write $\mathrm{hd}_A M = n$. If no such sequence of the above type exists, I write $\mathrm{hd}_A M = \infty$. I can now state:-

(2.6)Proposition. If M is a non-zero A-module and n an integer ≥ 0, the following conditions are equivalent:-

(i) $\mathrm{hd}_A M \leq n$;

(ii) $\mathrm{Ext}_A^{n+1}(M,C) = 0$ for all A-modules C;

(iii) $\mathrm{Ext}_A^{n}(M,C)$ is a right exact functor of the variable module C;

& (iv) given (E) $0 \longrightarrow X \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \ldots P_0 \longrightarrow M \longrightarrow 0$

with P_i projective for each $i = 0,\ldots,n-1$, the module X is projective.

Proof. See Proposition 2.1 of Chap. VI of Cartan and Eilenberg (6) (or § 7.5 of Northcott (22)). //

(2.7)Corollary. If M is a non-zero, f.-g. A-module with $\mathrm{hd}_A M = n$, there exists an exact sequence (E)$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$ in which each P_i is projective and f.-g. for $i = 0,\ldots,n$.

Proof. Use "(i) implies (iv)" of (2.6). //

It follows from (2.6) that if a non-zero A-module M has $\mathrm{hd}_A M = n$, then for some A-module C, $\mathrm{Ext}_A^n(M,C) \neq 0$. For f.-g. modules over local rings, this result can be improved, and as I am going to need this improvement in Chapter 6, I include a proof of it here. Some Lemmas are needed first.

(2.8)Lemma. Let A be a local ring with maximal ideal $\underline{m}$, and M a f.-g. A-module. Let $k = A/\underline{m}$, and let $\bar{}$ be used to denote natural images in both $A/\underline{m}$ and $N/\underline{m}N$. Then $x_1,\ldots,x_s$ generate N if and only if $\bar{x}_1,\ldots,\bar{x}_s$ generate $N/\underline{m}N$ as a k-vector-space.

Proof. One way round this is obvious, so suppose $\bar{x}_1,\ldots,\bar{x}_s$ generate $N/\underline{m}N$ as a k-space. Then, if L is the submodule of N generated by $x_1,\ldots,x_s$, $N = \underline{m}N + L$. Thus $N/L = (\underline{m}N + L)/L = \underline{m}(N/L)$, so Nakayama's Lemma completes the proof. //

(Incidentally, Nakayama's Lemma will be used frequently in this thesis: a proof of it can be found in Proposition (...6) of Atiyah and Macdonald(1).)

(2.9)Theorem. Every f.-g. projective module over a local ring is free.

Proof. See Theorem 12 of Chap. 9 of Northcott (22). //

In (2.7), it was observed that, for f.-g. modules of finite homological dimension, (over a ring A,) there exist minimal length projective resolutions with f.-g. terms. An improvement of this result is available for local rings:-

(2.10)Lemma. Suppose A is local with maximal ideal $\underline{m}$ and the non-zero, f.-g. A-module N has $hd_A N = n$ (≥ 1). Then there exists a free resolution:-

(E) $0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$ of N s.t. each F_i ($i = 0,\dots,n$) is f.-g. and $\operatorname{im}(F_{i+1} \longrightarrow F_i) \subseteq \underline{m}F_i$ for $i = 0,\dots,n-1$.

Proof. Let $x_1,\dots,x_r$ be a set of generators for N with the property that no proper subset also generates N. Let $\bar{}$ be used to denote natural images in $N/\underline{m}N$. By (2.8), $\bar{x}_1,\dots,\bar{x}_r$ form a basis for the $A/\underline{m}$-vector-space $N/\underline{m}N$. Let F_0 be an r-generator free A-module, with $z_1,\dots,z_r$ as free generators, and let $f_0 : F_0 \to N$ be the A-module epimorphism determined by $z_i \mapsto x_i$ for $i = 1,\dots,r$. Then the induced epimorphism $F_0/\underline{m}F_0 \longrightarrow N/\underline{m}N$ must be an isomorphism of $A/\underline{m}$-spaces and so $K_0 = \ker f_0 \subseteq \underline{m}F_0$.

Repeating this process for K_0, I obtain a f.-g. free A-module F_1 and an epimorphism $f_1' : F_1 \longrightarrow K_0$ s.t. $K_1 = \ker f_1' \subseteq \underline{m}F_1$. Let $f_1 : F_1 \to F_0$ be the composite $F_1 \xrightarrow{f_1'} K_0 \subseteq F_0$. Then (E) $F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \longrightarrow 0$ and $\ker f_1 \subseteq \underline{m}F_1$, $f_1 F_1 = \ker f_0 \subseteq \underline{m}F_0$.

It is obvious that repetition of this process will produce:-

(E) $F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \longrightarrow 0$,

where $F_{n-1},\dots,F_0$ are all f.-g. and free, $\ker f_{n-1} \subseteq \underline{m}F_{n-1}$, and $f_{i+1}F_{i+1} = \ker f_i \subseteq \underline{m}F_i$ for each $i = 0,\dots,n-2$. Let $F_n = \ker f_{n-1}$. F_n is f.-g.,

is projective by (2.6), and so is free by (2.9). Thus the Lemma is proved.
//

Now for the improvement of (2.6) for f.-g. modules over local rings:-

(2.11)Proposition. (Auslander and Goldman (3), Proposition (4.10).)
Suppose that A is a local ring with maximal ideal $\underline{m}$ and that N is a non-zero f.-g. A-module s.t. $hd_A N = n$, a non-negative integer. Then, if X is any non-zero f.-g. A-module, $Ext^n_A(N,X) \neq 0$.

Proof. If $n = 0$, this is obvious, in view of (2.9). So suppose $n > 0$, and let (E) $0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow N \longrightarrow 0$ provide a free resolution for N s.t. each F_i is f.-g. and free ($i = 0,\ldots,n$), and $\mathrm{im}(F_{i+1} \longrightarrow F_i) \subseteq \underline{m} F_i$ for $i = 0,\ldots,n-1$. Such exists by (2.10).

Suppose, then, that for some A-module X which is non-zero and f.-g., $Ext^n_A(N,X) = 0$. By (2.3)(A), it follows that (E) $\mathrm{Hom}_A(F_{n-1},X) \longrightarrow \mathrm{Hom}_A(F_n,X) \longrightarrow 0$. Thus every homomorphism $F_n \longrightarrow X$ can be factored through F_{n-1}. But F_n is non-zero and free, and $\mathrm{im}(F_n \longrightarrow F_{n-1}) \subseteq \underline{m} F_{n-1}$. Therefore $X = \underline{m} X$, a contradiction to Nakayama's Lemma. //

(2.12)Injective Dimension.

I shall be very much concerned with injective dimension in later Chapters. However, a sensible discussion of this isn't really possible until after the development of the theory of injective modules. This will be surveyed and discussed in Chapter 5, and only the definition of injective dimension is given here. Let A be a ring.

Definition. A non-zero A-module M is said to have injective dimension equal to n if and only if:-

(i) there is an exact sequence of the form:-

(E) $0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \ldots \longrightarrow I^n \longrightarrow 0$, where each

I^j is injective;

& (ii) there is no exact sequence of this type with fewer terms.

In this case I write $\text{inj.dim.}_A M = n$. If no such sequence of the above type exists, I write $\text{inj.dim.}_A M = \infty$.

Connected Sequences of Functors.

These seem an adequate tool for coping with various situations which occur in applications of Extension functors and other functors to Noetherian rings. For more details than given here see Chap. III, § 4 of Cartan and Eilenberg (6) and § 6.5 of Northcott (22). Let A and B denote two rings.

(2.13)Definition. (A) Suppose that $\{T^n(M)\}_{n\geqslant 0}$ is a sequence of covariant functors from some category of A-modules $\underline{\mathcal{D}}$ (for example, $\underline{\mathcal{D}}$ might be $\underline{\mathcal{C}}_A$ or $\underline{\mathcal{C}}_A^f$) to $\underline{\mathcal{C}}_B$, and suppose also that for each exact sequence of $\underline{\mathcal{D}}$

(E) $0 \to M' \to M \to M'' \to 0$, there is defined a connecting B-homomorphism $T^n(M'') \longrightarrow T^{n+1}(M')$, for each $n \geqslant 0$, s.t. the sequence:-

(2.13)(A) $0 \to T^0(M') \to T^0(M) \to T^0(M'') \to T^1(M') \to \ldots \longrightarrow T^{n-1}(M'') \longrightarrow T^n(M')$ $\longrightarrow T^n(M) \to T^n(M'') \to \ldots$ is a complex.

If (2.13)(A), when regarded as belonging to the translation category of (right semi-infinite) complexes of B-modules and -homomorphisms, is a covariant functor of the exact sequence (E) $0 \to M' \to M \to M'' \to 0$, then $\{T^n\}_{n\geqslant 0}$ is s.t.b. a connected right sequence of covariant functors.

(B) The modifications for contravariant functors are easy. The exact sequence (E) $0 \to M' \to M \to M'' \to 0$ of $\underline{\mathcal{D}}$ gives rise to $T^n(M') \longrightarrow T^{n+1}(M'')$ for each $n \geqslant 0$, and the complex:-

(2.13)(B) $0 \to T^0(M'') \to T^0(M) \to T^0(M') \to T^1(M'') \to \ldots \longrightarrow T^{n-1}(M') \longrightarrow T^n(M'')$

$\longrightarrow T^n(M) \longrightarrow T^n(M') \longrightarrow \ldots$

is required to be a contravariant functor of the exact sequence (E) $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$. If these conditions are satisfied, $\{T^n\}_{n \geqslant 0}$ is s.t.b. a <u>connected right sequence of contravariant functors</u>.

(C) The modifications for connected left sequences of covariant functors are straightfoward. Suppose that $\{T_n(M)\}_{n \geqslant 0}$ is a sequence of covariant functors from some category $\underline{\mathcal{A}}$ of A-modules to $\underline{\mathcal{C}}_B$, and suppose also that each exact sequence (E) $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\underline{\mathcal{A}}$ gives rise to a connecting B-homomorphism $T_{n+1}(M'') \longrightarrow T_n(M')$ for each $n \geqslant 0$ s.t. the sequence:-

(2.13)(C) $\ldots \longrightarrow T_{n+1}(M'') \longrightarrow T_n(M') \longrightarrow T_n(M) \longrightarrow T_n(M'') \longrightarrow T_{n-1}(M') \longrightarrow \ldots$

$\longrightarrow T_1(M'') \longrightarrow T_0(M') \longrightarrow T_0(M) \longrightarrow T_0(M'') \longrightarrow 0$ is a complex.

If (2.13)(C), when regarded as belonging to the translation category of (left semi-infinite) complexes of B-modules and -homomorphisms, is a covariant functor of the exact sequence (E) $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, then $\{T_n\}_{n \geqslant 0}$ is s.t.b. a <u>connected left sequence of covariant functors</u>.

(D) The modifications to (C) required for the definition of <u>connected left sequences of contravariant functors</u> are left to the reader.

(2.14)Definition. (A) Suppose $\{T^n\}_{n \geqslant 0}, \{U^n\}_{n \geqslant 0}$ are two connected right sequences of covariant functors from $\underline{\mathcal{A}}$ to $\underline{\mathcal{C}}_B$, and suppose $\Phi = \{\phi^n\}_{n \geqslant 0}$ is a sequence of natural transformations $\phi^n : T^n \rightsquigarrow U^n$. Φ is said to be a <u>homomorphism</u> of $\{T^n\}_{n \geqslant 0}$ into $\{U^n\}_{n \geqslant 0}$ if and only if, for each (E) $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\underline{\mathcal{A}}$ and each $n \geqslant 0$

(C)
$$\begin{array}{ccc} T^n(M'') & \longrightarrow & T^{n+1}(M') \\ \downarrow{\scriptstyle \phi^n} & & \downarrow{\scriptstyle \phi^{n+1}} \\ U^n(M'') & \longrightarrow & U^{n+1}(M') \end{array}$$

.

If, in addition, each ϕ^n is a natural equivalence, Φ is s.t.b. an isomorphism.

(B) The modifications to (A) necessary for the case of connected right sequences of contravariant functors are obvious. For each

(E) $0 \to M' \to M \to M'' \to 0$ in $\underline{\mathcal{D}}$ and each $n \geq 0$,

$$\begin{array}{ccc} T^n(M') & \longrightarrow & T^{n+1}(M'') \\ \downarrow{\scriptstyle \phi^n} & & \downarrow{\scriptstyle \phi^{n+1}} \\ U^n(M') & \longrightarrow & U^{n+1}(M'') \end{array} \qquad (C)$$

(C) Suppose $\{T_n\}_{n\geq 0}$ and $\{U_n\}_{n\geq 0}$ are two connected left sequences of covariant functors from $\underline{\mathcal{D}}$ to $\underline{\mathcal{C}}_B$, and suppose $\Phi = \{\phi_n\}_{n\geq 0}$ is a sequence of natural transformations $\phi_n : T_n \rightsquigarrow U_n$. Φ is s.t.b. a homomorphism of $\{T_n\}_{n\geq 0}$ into $\{U_n\}_{n\geq 0}$ if and only if, for each

(E) $0 \to M' \to M \to M'' \to 0$ in $\underline{\mathcal{D}}$ and each $n \geq 0$,

$$\begin{array}{ccc} T_{n+1}(M'') & \longrightarrow & T_n(M') \\ \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_n} \\ U_{n+1}(M'') & \longrightarrow & U_n(M') \end{array} \qquad (C).$$

If, in addition, each ϕ_n is a natural equivalence, Φ is s.t.b. an isomorphism.

(D) The modifications to (C) necessary for the case of connected left sequences of contravariant functors are obvious, and are left to the reader.

Given two connected sequences of functors, it is sometimes possible to extend a natural transformation of the two "zero-index functors" to a homomorphism of the connected sequences. There are several Theorems describing situations in which this is possible in § 6.5 of Northcott (22), and I shall use some of these, such as the Corollary to Theorem 10 of Chap. 6 of (22), later in the thesis. However, there are two Theorems

of this type which I shall need later but which don't actually appear in (22). Consequently, I state and prove themand their Corollaries in this Chapter, even though their proofs are very similar to some of the proofs in § 6.5 of (22).

(2.15)Theorem. Suppose A and B are rings. Let $\{T_n\}_{n\geq 0}$ and $\{U_n\}_{n\geq 0}$ be connected left sequences of covariant functors from $\underline{\mathcal{C}}_A^f$ to $\underline{\mathcal{C}}_B$, and let $\phi_0 : T_0 \rightsquigarrow U_0$ be a given natural transformation. Suppose further that, whenever (E) $0 \to L \to P \to M \to 0$ in $\underline{\mathcal{C}}_A^f$, with P an A-projective module, then (E) $0 \to U_{n+1}(M) \longrightarrow U_n(L) \to U_n(P)$ for every $n \geq 0$. In these circumstances, there exists a unique extension of ϕ_0 to a homomorphism $\Phi : \{T_n\}_{n\geq 0} \rightsquigarrow \{U_n\}_{n\geq 0}$ of connected sequences.

Proof. I shall construct (ϕ_0), ϕ_1, ϕ_2,... by an inductive process, and, consequently, I assume inductively that natural transformations ϕ_0, ϕ_1,..., ϕ_n which commute with the connecting homomorphisms have already been constructed. Suppose M & M' are two f.-g. A-modules. Suppose P,P' are f.-g., projective A-modules and L,L' are f.-g. A-modules s.t. there are sequences:- (E) $0 \to L \to P \to M \to 0$ & (E) $0 \to L' \to P' \to M' \to 0$.

Now

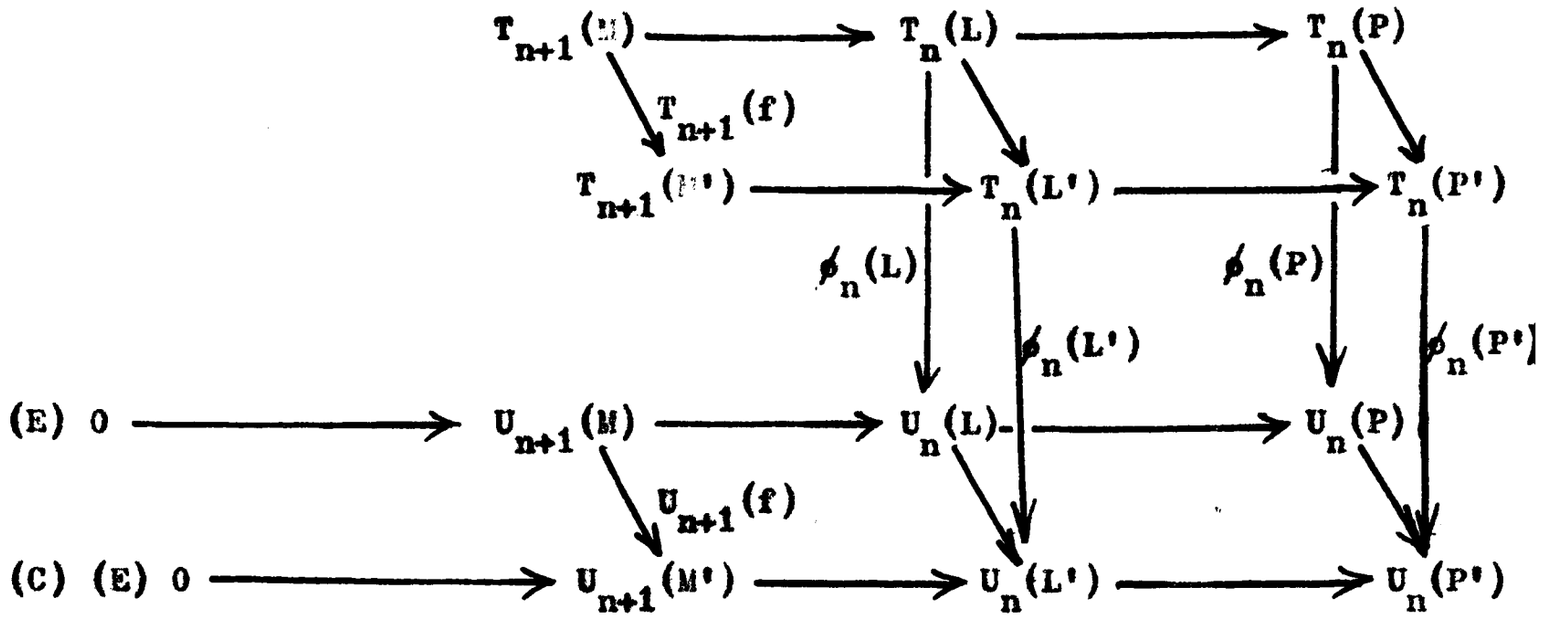
$$\begin{array}{ccccccc} & & T_{n+1}(M) & \longrightarrow & T_n(L) & \longrightarrow & T_n(P) \\ & & \vdots & & \downarrow \phi_n & & \downarrow \phi_n \\ (E)\ 0 & \longrightarrow & U_{n+1}(M) & \longrightarrow & U_n(L) & \longrightarrow & U_n(P) \qquad (C) \end{array}$$

There is a unique B-homomorphism $\phi_{n+1}(M) : T_{n+1}(M) \to U_{n+1}(M)$ s.t. the extended diagram commutes, and this proves, in particular, that ϕ_{n+1} (if it exists) is unique.

Now let $f : M \longrightarrow M'$ be a given A-module homomorphism. Since P is projective, there exist A-homomorphisms $P \to P'$ and $L \longrightarrow L'$ s.t.:-

$$\begin{array}{ccccccccc} & 0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \quad (E) \\ & & & \downarrow & & \downarrow & & \downarrow f & & \\ (C) & 0 & \longrightarrow & L' & \longrightarrow & P' & \longrightarrow & M' & \longrightarrow & 0 \quad (E)\ . \end{array}$$

From the hypotheses and the inductive hypotheses:-



Now this diagram remains commutative when it is extended by the unique B-homs. $T_{n+1}(M) \longrightarrow U_{n+1}(M)$ and $T_{n+1}(M') \longrightarrow U_{n+1}(M')$ which will make the extended long vertical faces commute. In other words:-

$$\begin{array}{ccc} T_{n+1}(M) & \xrightarrow{T_{n+1}(f)} & T_{n+1}(M') \\ \downarrow & & \downarrow \\ U_{n+1}(M) & \xrightarrow{U_{n+1}(f)} & U_{n+1}(M') \end{array} \qquad (1) \qquad (C)$$

For the moment, take $M = M'$ and f to be the identity. The horizontal B-homs.in (1) are then also identities, and so this shows that $T_{n+1}(M)$ $U_{n+1}(M)$ does not depend on the choice of P and L. I am thus justified in denoting this homomorphism by $\phi_{n+1}(M)$.

Returning to (1) as it applies in the general case, I see that the $\phi_{n+1}(M)$ provide a natural transformation $\phi_{n+1}: T_{n+1} \rightsquigarrow U_{n+1}$.

Thus, suppose (E) $0 \to N' \to N \to N'' \to 0$ in $\underline{\mathcal{C}}^f_A$. Let L",P" be f.-g. A-modules with P" A-projective s.t. (E) $0 \to L'' \to P'' \to N'' \to 0$. Since P" is A-projective, there exist A-homs. $P'' \longrightarrow N$, $L'' \longrightarrow N'$ s.t.:-

$$\begin{array}{ccccccccc} (E) & 0 & \longrightarrow & L'' & \longrightarrow & P'' & \longrightarrow & N'' & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \downarrow 1_{N''} & & \\ (C)\ (E) & 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array} .$$

From this I deduce:-

$$\begin{array}{ccc} & & T_n(L'') \\ & \nearrow & \downarrow \\ T_{n+1}(N'') & & \\ & \searrow & \\ (C) & & T_n(N') \end{array} \qquad \begin{array}{ccc} & & U_n(L'') \\ & \nearrow & \downarrow \\ U_{n+1}(N'') & & \\ & \searrow & \\ (C) & & U_n(N') \end{array} \qquad (2)$$

Also, by the construction of $\phi_{n+1}(N'')$,

$$\begin{array}{ccccccc} & & T_{n+1}(N'') & \longrightarrow & T_n(L'') & \longrightarrow & T_n(P'') \\ & & \downarrow \phi_{n+1}(N'') & & \downarrow \phi_n(L'') & & \downarrow \phi_n(P'') \\ (E)\ 0 & \longrightarrow & U_{n+1}(N'') & \longrightarrow & U_n(L'') & \longrightarrow & U_n(P'') \quad (C) \end{array} \qquad (3)$$

Finally, since ϕ_n is a natural transformation:-

$$\begin{array}{ccc} T_n(L'') & \longrightarrow & T_n(N') \\ \downarrow \phi_n(L'') & & \downarrow \phi_n(N') \\ (C)\ U_n(L'') & \longrightarrow & U_n(N') \end{array} \qquad (4)$$

Combining (2), (3), and (4) it follows that:-

$$\begin{array}{ccc} T_{n+1}(N'') & \longrightarrow & T_n(N') \\ \downarrow \phi_{n+1}(N'') & & \downarrow \phi_n(N') \\ (C)\ U_{n+1}(N'') & \longrightarrow & U_n(N') \end{array} \quad .$$

So ϕ_{n+1} has all the required properties, and the Theorem follows by induction. //

(2.16)Corollary. With the hypotheses of (2.15), suppose that ϕ_0 is a natural equivalence, and that, whenever (E) $0 \to L \to P \to M \to 0$ in $\underline{\mathcal{C}}_A^f$ with P A-projective, then (E) $0 \to T_{n+1}(M) \to T_n(L) \to T_n(P)$ and (E) $0 \to U_{n+1}(M) \to U_n(L) \to U_n(P)$ for all $n \geq 0$. Then the unique extension of ϕ_0 to a homomorphism $\Phi : \{T_n\}_{n\geq 0} \rightsquigarrow \{U_n\}_{n\geq 0}$ of connected sequences is an isomorphism.

Proof. Let $\gamma_0 : U_0 \rightsquigarrow T_0$ be the equivalence inverse to ϕ_0, and let Ψ =

$\{\chi_n\}_{n\geq 0}: \{U_n\}_{n\geq 0} \rightsquigarrow \{T_n\}_{n\geq 0}$ be the unique extension of χ_0 to a homomorphism of connected sequences. (I use (2.15) here.) Then (with an obvious notation) $\Psi.\Phi : \{T_n\}_{n\geq 0} \rightsquigarrow \{T_n\}_{n\geq 0}$ is a homomorphism of connected sequences extending the identity equivalence $T_0 \rightsquigarrow T_0$. Thus, by the uniqueness part of (2.15), $\chi_n . \phi_n : T_n \rightsquigarrow T_n$ is the identity equivalence for each value of $n \geq 0$. Similarly, $\phi_n . \chi_n : U_n \rightsquigarrow U_n$ is the identity equivalence for each value of $n \geq 0$. Thus χ_n and ϕ_n are inverse equivalences. //

(2.17)Theorem. Let $\{T^n\}_{n\geq 0}$ and $\{U^n\}_{n\geq 0}$ be connected right sequences of contravariant functors from $\underline{\mathcal{C}}_A^f$ to $\underline{\mathcal{C}}_B$ (where A and B are two rings, of course) and let $\phi^0 : T^0 \rightsquigarrow U^0$ be a given natural transformation. Suppose further that, whenever (E) $0 \to L \to P \to M \to 0$ in $\underline{\mathcal{C}}_A^f$ with P an A-projective module, then (E) $T^n(P) \longrightarrow T^n(L) \longrightarrow T^{n+1}(M) \longrightarrow 0$ for every $n \geq 0$. In these circumstances, there exists a unique extension of ϕ^0 to a homomorphism $\Phi : \{T^n\}_{n\geq 0} \rightsquigarrow \{U^n\}_{n\geq 0}$ of connected sequences.

Proof. I shall construct (ϕ^0), ϕ^1, ϕ^2,... by an inductive process, and consequently, I assume inductively that natural transformations ϕ^0, ϕ^1,..., ϕ^n which commute with the connecting homomorphisms in the sense of (2.14)(B), have already been constructed. Suppose M and M' are two f.-g. A-modules. Suppose P,P' are f.-g. projective A-modules and L,L' are f.-g. A-modules s.t. there are sequences:-

(E) $0 \to L \to P \to M \to 0$ and (E) $0 \to L' \to P' \to M' \to 0$.

Now
$$\begin{array}{ccccccc} T^n(P) & \longrightarrow & T^n(L) & \longrightarrow & T^{n+1}(M) & \longrightarrow & 0 \quad (E) \\ \downarrow{\scriptstyle \phi^n} & & \downarrow{\scriptstyle \phi^n} & & \vdots & & \\ U^n(P) & \longrightarrow & U^n(L) & \longrightarrow & U^{n+1}(M) & & \quad (C). \end{array}$$

There is a unique B-module homomorphism $\phi^{n+1}(M):T^{n+1}(M)\longrightarrow U^{n+1}(M)$ s.t. the extended diagram commutes, and this proves in particular that ϕ^{n+1} (if it exists) is unique.

Now let $f:M\to M'$ be a given A-module homomorphism. Since P is projective, there exist A-homomorphisms $P\to P'$ and $L\to L'$ s.t.:-

$$\begin{array}{ccccccccc} & 0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & M & \longrightarrow 0 \quad (E) \\ & & & \downarrow & & \downarrow & & \downarrow f & \\ (C) & 0 & \longrightarrow & L' & \longrightarrow & P' & \longrightarrow & M' & \longrightarrow 0 \quad (E) \end{array}$$

From the hypotheses and the inductive hypotheses,

$$\begin{array}{ccccccccc} & T^n(P') & \longrightarrow & T^n(L') & \longrightarrow & T^{n+1}(M') & \longrightarrow & 0 & (E) \\ & T^n(P) & \longrightarrow & T^n(L) & \longrightarrow & T^{n+1}(M) & \longrightarrow & 0 & (E) \\ \phi^n(P') & \downarrow & \phi^n(L') & \downarrow & \phi^n(L) & \downarrow & & & \\ & U^n(P') & \longrightarrow & U^n(L') & \longrightarrow & U^{n+1}(M') & & & \\ (C) & U^n(P) & \longrightarrow & U^n(L) & \longrightarrow & U^{n+1}(M) & & & \end{array}$$

Now this diagram remains commutative when it is extended by the unique homomorphisms $T^{n+1}(M')\to U^{n+1}(M')$ and $T^{n+1}(M)\to U^{n+1}(M)$ which will make the extended long vertical faces commute. In other words,

$$\begin{array}{ccc} T^{n+1}(M') & \xrightarrow{T^{n+1}(f)} & T^{n+1}(M) \\ \downarrow & & \downarrow \\ (C)\ U^{n+1}(M') & \xrightarrow{U^{n+1}(f)} & U^{n+1}(M) \end{array} \qquad (1)$$

For the moment, take $M = M'$ and f to be the identity. The horizontal homs. in (1) are then also identities, and so this shows that $T^{n+1}(M)\longrightarrow U^{n+1}(M)$ does not depend on the choice of P & L. I am thus justified in denoting this homomorphism by $\phi^{n+1}(M)$.

It follows from (1), as it applies in the general case, that the

$\phi^{n+1}(N)$ provide a natural transformation $\phi^{n+1}: T^{n+1} \rightsquigarrow U^{n+1}$.

Thus, suppose (E) $0 \to N' \to N \to N'' \to 0$ in $\underline{\mathcal{C}}_A^f$. Let L'', P'' be f.-g. A-modules with P'' projective s.t. (E) $0 \to L'' \to P'' \to N'' \to 0$. Since P'' is projective, there exist homs. $P'' \to N$, $L'' \to N'$ s.t.:-

$$\begin{array}{ccccccccc} (E)\ 0 & \longrightarrow & L'' & \longrightarrow & P'' & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow 1_{N''} & & \\ (C)\ (E)\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array} .$$

From this I deduce (recall definition (2.13)):-

$$\begin{array}{ccc} T^n(N') & & \\ \downarrow & \searrow & \\ (C)\ T^n(L'') & \nearrow & T^{n+1}(N'') \end{array} \qquad \begin{array}{ccc} U^n(N') & & \\ \downarrow & \searrow & \\ (C)\ U^n(L'') & \nearrow & U^{n+1}(N'') \end{array} \qquad (2)$$

Also, by the construction of $\phi^{n+1}(N'')$,

$$\begin{array}{ccccccc} T^n(P'') & \longrightarrow & T^n(L'') & \longrightarrow & T^{n+1}(N'') & \longrightarrow & 0 \quad (E) \\ \downarrow \phi^n(P'') & & \downarrow \phi^n(L'') & & \downarrow \phi^{n+1}(N'') & & \\ (C)\ U^n(P'') & \longrightarrow & U^n(L'') & \longrightarrow & U^{n+1}(N'') & & \end{array} \qquad (3)$$

Finally, since ϕ^n is a natural transformation,

$$\begin{array}{ccc} T^n(N') & \longrightarrow & T^n(L'') \\ \downarrow \phi^n(N') & & \downarrow \phi^n(L'') \\ (C)\ U^n(N') & \longrightarrow & U^n(L'') \end{array} \qquad (4)$$

Combining (2),(3), and (4), it follows that

$$\begin{array}{ccc} T^n(N') & \longrightarrow & T^{n+1}(N'') \\ \downarrow \phi^n(N') & & \downarrow \phi^{n+1}(N'') \\ (C)\ U^n(N') & \longrightarrow & U^{n+1}(N'') \end{array} .$$

So ϕ^{n+1} has all the required properties, and the Theorem follows by induction. //

(2.18)Corollary. With the hypotheses of (2.17), suppose that, in addition,

ϕ^0 is a natural equivalence, and that, whenever (E) $0 \to L \to P \to M \to 0$ in $\underline{\mathcal{C}}_A^f$ with P projective, then:-

(E) $T^n(P) \to T^n(L) \to T^{n+1}(M) \longrightarrow 0$ & (E) $U^n(P) \to U^n(L) \to U^{n+1}(M) \longrightarrow 0$

for each $n \geqslant 0$. Then the unique extension of ϕ^0 to a homomorphism $\Phi : \{T^n\}_{n\geqslant 0} \rightsquigarrow \{U^n\}_{n\geqslant 0}$ of connected sequences is an isomorphism.

Proof. Let $\chi^0 : U^0 \rightsquigarrow T^0$ be the equivalence inverse to ϕ^0, and let Ψ be the unique extension of χ^0 to a homomorphism $\{U^n\}_{n\geqslant 0} \rightsquigarrow \{T^n\}_{n\geqslant 0}$ of connected sequences. Then (with an obvious notation) $\Psi.\Phi : \{T^n\}_{n\geqslant 0} \rightsquigarrow \{T^n\}_{n\geqslant 0}$ is a homomorphism of connected sequences extending the identity equivalence $T^0 \rightsquigarrow T^0$. Thus, by the uniqueness part of (2.17), $\chi^n.\phi^n : T^n \rightsquigarrow T^n$ is the identity equivalence for each $n \geqslant 0$. Similarly, $\phi^n.\chi^n : U^n \rightsquigarrow U^n$ is the identity equivalence for each value of $n \geqslant 0$. Thus ϕ^n and χ^n are inverse equivalences. //

(2.19)Corollary. Suppose A is a ring, M is a f.-g. A-module, and $\{C_j\}_{j\in J}$ is a family of A-modules. Then $\operatorname{Ext}_A^n(M, \bigoplus_{j\in J} C_j) \cong \bigoplus_{j\in J} \operatorname{Ext}_A^n(M, C_j)$ for each $n \geqslant 0$.

Proof. Observe that, if $N \in \underline{\mathcal{C}}_A^f$, and $T^n(N) = \bigoplus_{j\in J} \operatorname{Ext}_A^n(N, C_j)$ and $U^n(N) = \operatorname{Ext}_A^n(N, \bigoplus_{j\in J} C_j)$, $\{T^n\}_{n\geqslant 0}$ and $\{U^n\}_{n\geqslant 0}$ are connected right sequences of contravariant functors from $\underline{\mathcal{C}}_A^f$ to $\underline{\mathcal{C}}_A$. (I put $T^0(N) = \bigoplus_{j\in J} \operatorname{Hom}_A(N, C_j)$ and $U^0(N) = \operatorname{Hom}_A(N, \bigoplus_{j\in J} C_j)$: recall that $\operatorname{Hom}_A(X,Y)$ and $\operatorname{Ext}_A^0(X,Y)$ are naturally equivalent functors. (See (2.2).)) I plan to use (2.18), and to this end I construct a natural equivalence $\phi^0 : T^0 \rightsquigarrow U^0$: it is clear that the other conditions of (2.18) are satisfied.

If N is a f.-g. A-module, define $\phi^0(N) : T^0(N) \longrightarrow U^0(N)$ by the rule:-

if $f \in \bigoplus_{j\in J} \mathrm{Hom}_A(N,C_j)$ and the jth component of f is $f_j \in \mathrm{Hom}_A(N,C_j)$, (so that all but a finite number of these f_j's will be zero,) let $\phi^0(N)(f):N \longrightarrow \bigoplus_{j\in J} C_j$ be the homomorphism g s.t. $p_j.g = f_j$ for all $j \in J$ (where $p_k:\bigoplus_{j\in J} C_j \longrightarrow C_k$ is the natural projection).

It is clear that $\phi^0(N)$ is an A-module monomorphism. To show that $\phi^0(N)$ is epimorphic (and so an isomorphism), let $g \in \mathrm{Hom}_A(N, \bigoplus_{j\in J} C_j)$. Since N is f.-g., g(N) will be contained in a submodule $\bigoplus_{j\in I} C_j$ of $\bigoplus_{j\in J} C_j$, where I is a finite subset of J, and so only a finite number of $p_j.g = f_j: N\longrightarrow C_j$ will be non-zero. Then $g = \phi^0(N)(f)$, where the jth component of f is f_j. Hence $\phi^0(N)$ is epimorphic.

Since it is clear that the $\phi^0(N)$ constitute a natural transformation of functors: $\phi^0:T^0 \rightsquigarrow U^0$, they provide a natural equivalence, and so the Corollary follows from (2.18). //

(2.20) (2.19) brings to an end the discussion of basic definitions and results, and the rest of this Chapter is devoted to the development of more sophisticated ideas contained in §§ 4 & 6 of Hartshorne (13). In fact, (13) contains statements and proofs (or sketches of proofs) of almost all the results in the remainder of Chapter 2. My reasons for giving such a detailed account of these results are, firstly, that I wish to amplify the proofs given in (13), and, secondly, that I want it to be clear that the work in this thesis is all within the broad framework of categories of modules over (commutative, Noetherian) rings and does not depend in some subtle way on any general results about

preschemes. However, I should add that, in Chapter 11, I shall need to apply a general homological algebra result of Grothendieck, which appears in (11), to the particular case in which the category concerned is the category of modules over a (commutative, Noetherian) ring.

For the rest of this Chapter, A will denote a ring. I now begin the description of some of the ideas in Hartshorne (13) about functors on categories of A-modules. As all such functors to which I shall subsequently apply these ideas will also be additive and A-linear, I shall restrict myself to such functors in this account. A full subcategory of $\underline{\mathcal{C}}_A$ is a category $\underline{\mathcal{D}}$ whose objects are objects of $\underline{\mathcal{C}}_A$, and which is such that, if $M \& N \in \underline{\mathcal{D}}$, all elements of $\mathrm{Hom}_A(M,N)$ are morphisms in $\underline{\mathcal{D}}$. It thus makes sense to speak of additive, A-linear functors from $\underline{\mathcal{D}}$ to $\underline{\mathcal{C}}_A$.

(2.21)Lemma. (See Lemma 4.1 of Hartshorne (13).) Suppose $\underline{\mathcal{D}}$ is a full subcategory of $\underline{\mathcal{C}}_A$ which is such that A itself is an object of $\underline{\mathcal{D}}$. Let $T: \underline{\mathcal{D}} \rightsquigarrow \underline{\mathcal{C}}_A$ be a contravariant, additive, A-linear functor. Then there is a natural transformation of functors (from $\underline{\mathcal{D}}$ to $\underline{\mathcal{C}}_A$):-

$$\phi : T \rightsquigarrow \mathrm{Hom}_A(\ ,T(A)).$$

Proof. Let M be an A-module which is an object of $\underline{\mathcal{D}}$. For each $m \in M$, let $\varepsilon_m : A \longrightarrow M$ be the A-homomorphism defined by $\varepsilon_m(a) = a.m$ for all $a \in A$. Since $A \in \underline{\mathcal{D}}$ and $\underline{\mathcal{D}}$ is a full subcategory of $\underline{\mathcal{C}}_A$, ε_m is a morphism of $\underline{\mathcal{D}}$.

$T(\varepsilon_m): T(M) \longrightarrow T(A)$ is an A-homomorphism. Suppose $x \in T(M)$. Define $\phi(M)(x): M \longrightarrow T(A)$ by the rule:- $(\phi(M)(x))(m) = T(\varepsilon_m)(x)$.

There are three things to check. Firstly, I show that $\phi(M)(x) \in \mathrm{Hom}_A(M,T(A))$. Suppose $m_1, m_2 \in M$ and $a_1, a_2 \in A$. Then:-

$$(\phi(M)(x))(a_1m_1 + a_2m_2) = T(\varepsilon_{a_1m_1+a_2m_2})(x)$$

$$= T(a_1\varepsilon_{m_1} + a_2\varepsilon_{m_2})(x)$$
$$= (a_1T(\varepsilon_{m_1}) + a_2T(\varepsilon_{m_2}))(x)$$
$$= a_1(\phi(M)(x))(m_1) + a_2(\phi(M)(x))(m_2).$$

Thus $\phi(M)(x) \in \mathrm{Hom}_A(M,T(A))$.

Secondly, I show that $\phi(M):T(M) \longrightarrow \mathrm{Hom}_A(M,T(A))$ is an A-homomorphism. Suppose $x_1,x_2 \in T(M)$ and $a_1,a_2 \in A$. Let m be an arbitrary element of M. Then:-

$$(\phi(M)(a_1x_1 + a_2x_2))(m) = T(\varepsilon_m)(a_1x_1 + a_2x_2)$$
$$= a_1T(\varepsilon_m)(x_1) + a_2T(\varepsilon_m)(x_2)$$
$$= (a_1\phi(M)(x_1) + a_2\phi(M)(x_2))(m), \text{ as}$$

was to be shown.

Finally, I show that the $\phi(M)$ constitute a natural transformation of functors. Suppose M and N are objects of $\underline{\mathcal{D}}$, and $g:M \longrightarrow N$ an A-homomorphism. Consider the diagram:-

$$\begin{array}{ccc} T(N) & \xrightarrow{\phi(N)} & \mathrm{Hom}_A(N,T(A)) \\ {\scriptstyle T(g)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_A(g,1_{T(A)})} \\ T(M) & \xrightarrow{\phi(M)} & \mathrm{Hom}_A(M,T(A)) \end{array} \qquad (1)$$

Suppose $y \in T(N)$. I must show that $(\mathrm{Hom}_A(g,1_{T(A)}).\phi(N))(y) = (\phi(M).T(g))(y)$. Let $m \in M$. Then, defining $\mu_n:A \longrightarrow N$, for $n \in N$, to be the A-homomorphism defined by $\mu_n(a) = a.n$ for all $a \in A$,

$((\mathrm{Hom}_A(g,1_{T(A)}).\phi(N))(y))(m) = T(\mu_{g(m)})(y)$. But $((\phi(M).T(g))(y))(m) = T(\varepsilon_m)(T(g)(y))$. Also

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon_m} & M \\ & {\scriptstyle \mu_{g(m)}}\searrow & \downarrow{\scriptstyle g} \\ & & N \end{array} \quad (C),$$

and so $T(\mu_{g(m)}) = T(g.\varepsilon_m) = T(\varepsilon_m).T(g)$. Thus diagram (1) commutes. The Lemma is now proved. //

(2.22)Proposition. (See Proposition 4.2 of Hartshorne (13).) Suppose

$T: \underline{\mathcal{C}}_A^f \rightsquigarrow \underline{\mathcal{C}}_A$ is a contravariant, additive, A-linear functor. Then the natural transformation of functors of (2.21), $\phi: T \rightsquigarrow \mathrm{Hom}_A(\ ,T(A))$, is a natural equivalence if and only if T is left exact.

Proof. ($\Rightarrow$) If ϕ is a natural equivalence, then T is left exact since $\mathrm{Hom}_A(\ ,T(A))$ is.

($\Leftarrow$) Suppose T is left exact, and M is a finitely-generated A-module. I must show that $\phi(M): T(M) \to \mathrm{Hom}_A(M,T(A))$ is an isomorphism. Let, for each $N \in \underline{\mathcal{C}}_A^f$, $T'(N) = \mathrm{Hom}_A(N,T(A))$.

Now if $x \in T(A)$, $(\phi(A)(x))(1) = T(\varepsilon_1)(x)$, and since $\varepsilon_1: A \to A$ is the identity, $(\phi(A)(x))(1) = x$. Thus $\phi(A): T(A) \to \mathrm{Hom}_A(A,T(A))$ is an isomorphism. Since T and T' are additive, it follows that $\phi(F)$ is an isomorphism whenever F is a f.-g. free A-module. (To see this, use, for example, Theorem 2 of Chap.3 of Northcott(22) and the "5-Lemma".) Now there exist f.-g. free A-modules F_0 & F_1 and:- (E) $F_1 \to F_0 \to M \to 0$.

Consequently:-

$$\begin{array}{ccccccc} (E)\ 0 & \longrightarrow & T(M) & \longrightarrow & T(F_0) & \longrightarrow & T(F_1) \\ & & \downarrow \phi(M) & & \cong \downarrow \phi(F_0) & & \cong \downarrow \phi(F_1) \\ (E)\ 0 & \longrightarrow & T'(M) & \longrightarrow & T'(F_0) & \longrightarrow & T'(F_1) \end{array} \qquad (C),$$

since T and T' are left exact. Hence $\phi(M)$ is an isomorphism, as was to be proved. //

(2.23)Corollary. (See Corollary 4.3 of Hartshorne (13).) There is a 1-1 onto correspondence between the isomorphism classes of A-modules and the natural equivalence classes of contravariant, left exact, additive, A-linear functors from $\underline{\mathcal{C}}_A^f$ to $\underline{\mathcal{C}}_A$ given by:-

$M \longmapsto \mathrm{Hom}_A(\ ,M)$ for $M \in \underline{\mathcal{C}}_A$

& $T \longmapsto T(A)$ for a contravariant, left exact, additive, A-linear

functor $T: \underline{\mathcal{C}}_A^f \rightsquigarrow \underline{\mathcal{C}}_A$.

Note that, in this correspondence, the (classes of) exact functors correspond to the (classes of) injective A-modules, for if I is injective, $\mathrm{Hom}_A(\ ,I)$ is exact, while if T (of the above form) is exact, then, by (2.22), $\mathrm{Hom}_A(\ ,T(A)): \underline{\mathcal{C}}_A^f \rightsquigarrow \underline{\mathcal{C}}_A$ is exact, so that $X^{T(A)}$ is injective by Theorem (3.2) of Chap. I of Cartan and Eilenberg (6) (using the fact that every ideal of A is f.-g.). //

(2.24) (2.21) - (2.23) are the results I require from § 4 of (13): I now jump to § 6 of (13) and fill in the details of Hartshorne's account of Yoneda's interpretation of the Extension modules.

If $\underline{K}$: $\ldots K^{n-1} \xrightarrow{d_K^{n-1}} K^n \xrightarrow{d_K^n} K^{n+1} \longrightarrow \ldots$ and

$\underline{L}$: $\ldots L^{n-1} \xrightarrow{d_L^{n-1}} L^n \xrightarrow{d_L^n} L^{n+1} \longrightarrow \ldots$ are

two complexes of A-modules and -homomorphisms, a *morphism of $\underline{K}$ into $\underline{L}$ of degree s* is a family $\underline{f} = \{f^n\}_{n\in Z}$ of A-homomorphisms $f^n: K^n \longrightarrow L^{n+s}$ s.t., for each $n \in Z$, $f^{n+1}.d_K^n = d_L^{n+s}.f^n$. I denote by $\mathrm{Hom}^s(\underline{K},\underline{L})$ the set of morphisms of $\underline{K}$ into $\underline{L}$ of degree s, endowed with the obvious A-module structure. Two such morphisms, $\underline{f}$ and $\underline{g}$, are s.t.b. *homotopic* if and only if there exists a collection $\{p^n\}_{n\in Z}$ of A-homomorphisms $p^n: K^n \longrightarrow L^{n+s-1}$ s.t., for all $n \in Z$, $f^n - g^n = d_L^{n+s-1}.p^n + p^{n+1}.d_K^n$. I shall denote the fact that $\underline{f}$ and $\underline{g}$ are homotopic by writing $\underline{f} \sim \underline{g}$.

If $\underline{f} \sim \underline{g}$, $\underline{h} \in \mathrm{Hom}^s(\underline{K},\underline{L})$, and $a \in A$, then clearly $\underline{f} + \underline{h} \sim \underline{g} + \underline{h}$ and $a.\underline{f} \sim a.\underline{g}$. Since $\sim$ is an equivalence relation on $\mathrm{Hom}^s(\underline{K},\underline{L})$, it follows that the A-module structure on $\mathrm{Hom}^s(\underline{K},\underline{L})$ induces an A-module structure on the set of equivalence classes under $\sim$ of elements of

$\mathrm{Hom}^s(\underline{K},\underline{L})$: I denote by $\underset{\sim}{H}^s(\underline{K},\underline{L})$ the resulting A-module.

Now suppose that M is an A-module, $\underline{K}$ is a complex of A-modules and -homomorphisms s.t. $K^n = 0$ for $n < 0$, and $\varepsilon : M \longrightarrow K^0$ is an A-hom. s.t. (E) $0 \longrightarrow M \longrightarrow K^0 \xrightarrow{d_K^0} K^1$. Suppose $\underline{L}$ is any complex of A-mods. and -homs. ("complex in $\mathcal{C}_A$" will henceforth be used as an abbreviation for this). I denote by $\mathrm{Hom}_A(M,\underline{L})$ the complex induced from $\underline{L}$ by applying the functor $\mathrm{Hom}_A(M,\ )$ to it.

Given an $\underline{f} = \{f^n\}_{n\in Z} \in \mathrm{Hom}^s(\underline{K},\underline{L})$, $f^0.\varepsilon : M \longrightarrow L^s$ is an element of $\mathrm{Hom}_A(M,L^s)$; since $d_L^s.f^0.\varepsilon = f^1.d_K^0.\varepsilon = 0$, $f^0.\varepsilon \in \ker(\mathrm{Hom}_A(M,L^s) \xrightarrow{d_L^{s}*} \mathrm{Hom}_A(M,L^{s+1}))$ (where '*'s are used to denote induced homomorphisms). Thus $f^0.\varepsilon$ determines $\overline{f^0.\varepsilon} \in H^s(\mathrm{Hom}_A(M,\underline{L}))$ (where '$^-$' is used to denote natural images in $H^s(\mathrm{Hom}_A(M,\underline{L}))$ of elements of $\ker(\mathrm{Hom}_A(M,L^s) \xrightarrow{d_L^{s}*} \mathrm{Hom}_A(M,L^{s+1}))$).

Suppose that $\underline{f} \sim \underline{g} \in \mathrm{Hom}^s(\underline{K},\underline{L})$. Then there exists a family $\{p^n\}_{n\in Z}$ of A-homs. s.t. $p^n : K^n \longrightarrow L^{n+s-1}$, and $f^0 - g^0 = d_L^{s-1}.p^0 + p^1.d_K^0$.

Therefore $f^0.\varepsilon - g^0.\varepsilon = d_L^{s-1}.p^0.\varepsilon$, and so $\overline{f^0.\varepsilon} = \overline{g^0.\varepsilon}$. Thus $\overline{f^0.\varepsilon}$ depends only on the homotopy class of $\underline{f}$: in this way, I construct a map $\phi^s : \underset{\sim}{H}^s(\underline{K},\underline{L}) \longrightarrow H^s(\mathrm{Hom}_A(M,\underline{L}))$. It is clear that ϕ^s is an A-homomorphism.

(2.25)Theorem. (See Theorem E of § 6 of Hartshorne (13).) If

(E) $0 \longrightarrow M \xrightarrow{\varepsilon} K^0 \xrightarrow{d_K^0} K^1 \longrightarrow \ldots \longrightarrow K^n \xrightarrow{d_K^n} K^{n+1} \longrightarrow \ldots$

and the constituent modules of $\underline{L}$ are all injective (using the notation of (2.24)), then, for each $s \in Z$, ϕ^s is an isomorphism.

Proof. If $\underline{f} \in \mathrm{Hom}^s(\underline{K},\underline{L})$, I shall use $\tilde{\underline{f}}$ to denote the equivalence

class in $\underset{\sim}{H}^s(\underline{K},\underline{L})$ to which $\underline{f}$ belongs. I first show that ϕ^s is monomorphic. Suppose $\underline{f} \in \mathrm{Hom}^s(\underline{K},\underline{L})$ is s.t. $\phi^s(\tilde{\underline{f}}) = 0$. Consequently, there exists $g \in \mathrm{Hom}_A(M,L^{s-1})$ s.t. $d_L^{s-1}.g = f^0.\varepsilon$. I aim, of course, to show that $\underline{f} \sim \underline{0} \in \mathrm{Hom}^s(\underline{K},\underline{L})$.

$$\begin{array}{lccccccccccccc}
(E)\ 0 \longrightarrow & M & \xrightarrow{\varepsilon} & K^0 & \xrightarrow{d_K^0} & K^1 & \longrightarrow \cdots \longrightarrow & K^n & \xrightarrow{d_K^n} & K^{n+1} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & \\
(C)\ \cdots \longrightarrow & L^{s-1} & \xrightarrow[d_L^{s-1}]{} & L^s & \xrightarrow[d_L^s]{} & L^{s+1} & \longrightarrow \cdots \longrightarrow & L^{s+n} & \xrightarrow[d_L^{s+n}]{} & L^{s+n+1} & \longrightarrow \cdots
\end{array} \qquad (1)$$

I need to find a family $\{p^n\}_{n\in Z}$ of A-homomorphisms s.t. $p^n : K^n \longrightarrow L^{n+s-1}$ and, for $n \geqslant 0$, $f^n = d_L^{n+s-1}.p^n + p^{n+1}.d_K^n$. If $n < 0$, define p^n to be zero.

L^{s-1} is A-injective. Let $p^0 : K^0 \longrightarrow L^{s-1}$ be s.t. $p^0.\varepsilon = g$. Then $(f^0 - d_L^{s-1}.p^0)$ vanishes on $\varepsilon(M)$, and so induces an A-homomorphism $\overline{g^0} : K^0/\varepsilon(M) \longrightarrow L^s$. Since the top row in diagram (1) is exact and L^s is injective, there exists $p^1 : K^1 \longrightarrow L^s$ s.t. $p^1.d_K^0 = (f^0 - d_L^{s-1}.p^0)$, so that:-

$$f^0 = d_L^{s-1}.p^0 + p^1.d_K^0.$$

Suppose, inductively, that $n \geqslant 1$, and $p^0,\ldots,p^n$ have been constructed so that $f^i = d_L^{i+s-1}.p^i + p^{i+1}.d_K^i$, for $0 \leqslant i < n$. In particular, $f^{n-1} = d_L^{n+s-2}.p^{n-1} + p^n.d_K^{n-1}$.

Now $(f^n - d_L^{n+s-1}.p^n).d_K^{n-1} = f^n.d_K^{n-1} - d_L^{n+s-1}.f^{n-1} = 0$. Thus $f^n - d_L^{n+s-1}.p^n$ vanishes on $d_K^{n-1}(K^{n-1})$, and so induces $\overline{g^n} : K^n/d_K^{n-1}(K^{n-1}) \longrightarrow L^{s+n}$. Since L^{s+n} is injective, and the top row in diagram (1) is exact, there exists $p^{n+1} : K^{n+1} \longrightarrow L^{s+n}$ s.t. $p^{n+1}.d_K^n = f^n - d_L^{n+s-1}.p^n$. This completes the inductive step, so the family $\{p^j\}_{j\in Z}$ can be constructed by induction. Thus $\underline{f} \sim \underline{0}$, and ϕ^s is monomorphic.

I now show that ϕ^s is epimorphic. Suppose that $z \in \mathrm{Hom}_A(M,L^s)$ is s.t. $d_L^s.z = 0$: I shall show that there exists $\underline{f} \in \mathrm{Hom}^s(\underline{K},\underline{L})$ s.t. $\phi^s(\tilde{\underline{f}}) = \bar{z}$.

$z(M) \subseteq \ker d_L^s$, so z induces $z' : M \to \ker d_L^s$. Consider:-

$$\begin{array}{lccccccccccc} (E)\ 0 \longrightarrow & M & \xrightarrow{\varepsilon} & K^0 & \xrightarrow{d_K^0} & K^1 & \xrightarrow{d_K^1} & \cdots & \longrightarrow & K^n & \xrightarrow{d_K^n} & K^{n+1} \longrightarrow \cdots \\ & \downarrow{\scriptstyle z'} & \searrow{\scriptstyle z} & & & & & & & & & \\ (C)\quad 0 \longrightarrow & \ker d_L^s & \xrightarrow{\subseteq} & L^s & \xrightarrow{d_L^s} & L^{s+1} & \xrightarrow{d_L^{s+1}} & \cdots & \longrightarrow & L^{s+n} & \xrightarrow{d_L^{s+n}} & L^{s+n+1} \longrightarrow \cdots \end{array} \qquad (2)$$

By, for example, Theorem 13 of Chapter 5 of Northcott (22), there exists a family $\{f^n\}_{n \geqslant 0}$ of A-homomorphisms s.t. $f^n : K^n \to L^{s+n}$ for each $n \geqslant 0$, $f^0 . \varepsilon = z$, and, for $n > 0$, $f^n . d_K^{n-1} = d_L^{n-1} . f^{n-1}$. Define $f^n = 0$ if $n < 0$, and then $\underline{f} = \{f^n\}_{n \in \mathbb{Z}} \in \mathrm{Hom}^s(\underline{K}, \underline{L})$ and $\phi^s(\bar{f}) = \bar{z}$. Thus ϕ^s is an epimorphism, and therefore an isomorphism. //

I intend to use (2.25) in the special case in which $\underline{K}$ & $\underline{L}$ provide injective resolutions for M and N respectively. Unfortunately, however, (2.25) as it stands is inadequate for my purposes: I must establish that the isomorphism ϕ^s of (2.25) is "natural" in the sense explained in the following Proposition.

(2.26) Proposition. Suppose that $\alpha : M \to M'$ is a homomorphism of A-modules, N is an A-module, and s is a non-negative integer. Let

$$(E)\ 0 \longrightarrow M \xrightarrow{\varepsilon} K^0 \xrightarrow{d_K^0} K^1 \longrightarrow \cdots \longrightarrow K^n \xrightarrow{d_K^n} K^{n+1} \longrightarrow \cdots,$$

$$(E)\ 0 \longrightarrow M' \xrightarrow{\varepsilon'} J^0 \xrightarrow{d_J^0} J^1 \longrightarrow \cdots \longrightarrow J^n \xrightarrow{d_J^n} J^{n+1} \longrightarrow \cdots,$$

$$\&(E) 0 \longrightarrow N \xrightarrow{\eta} L^0 \xrightarrow{d_L^0} L^1 \longrightarrow \cdots \longrightarrow L^n \xrightarrow{d_L^n} L^{n+1} \longrightarrow \cdots$$

be injective resolutions for M, M', & N resp.

Then α induces (in an unambiguous manner) an A-homomorphism $\alpha^* : \underset{\sim}{H}^s(\underline{J}, \underline{L}) \to \underset{\sim}{H}^s(\underline{K}, \underline{L})$ s.t.:-

$$\begin{array}{ccc} \underset{\sim}{H}^s(\underline{J}, \underline{L}) & \xrightarrow[\cong]{\phi^s(M')} & H^s(\mathrm{Hom}_A(M', \underline{L})) \\ \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^*} \\ \underset{\sim}{H}^s(\underline{K}, \underline{L}) & \xrightarrow[\cong]{\phi^s(M)} & H^s(\mathrm{Hom}_A(M, \underline{L})) \end{array} \qquad (1)\ (C)$$

where, of course, $\alpha^*:H^s(\mathrm{Hom}_A(M',\underline{L}))\longrightarrow H^s(\mathrm{Hom}_A(M,\underline{L})$ is the A-homomorphism induced by α, and $\phi^s(M)$, $\phi^s(M')$ are the isomorphisms given in ((2.24) &) (2.25).

Proof. By, for example, the Corollary to Theorem 13 of Chapter 5 of Northcott (22), there exists a family $\gamma=\{\gamma^n\}_{n\geq 0}$ of A-homomorphisms s.t. $\gamma^n:K^n\longrightarrow J^n$ and:-

$$\begin{array}{ccccccccccccccc}
(E)\ 0 & \longrightarrow & M & \xrightarrow{\varepsilon} & K^0 & \xrightarrow{d_K^0} & K^1 & \longrightarrow & \cdots & \longrightarrow & K^n & \xrightarrow{d_K^n} & K^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma^0} & & \downarrow{\scriptstyle\gamma^1} & & & & \downarrow{\scriptstyle\gamma^n} & & \downarrow{\scriptstyle\gamma^{n+1}} & & \\
(C)\ (E)\ 0 & \longrightarrow & M' & \xrightarrow{\varepsilon'} & J^0 & \xrightarrow{d_J^0} & J^1 & \longrightarrow & \cdots & \longrightarrow & J^n & \xrightarrow{d_J^n} & J^{n+1} & \longrightarrow & \cdots,
\end{array}$$

and, furthermore, γ is determined up to homotopy. Then γ induces a homomorphism $\gamma^*:\mathrm{Hom}^s(\underline{J},\underline{L})\longrightarrow\mathrm{Hom}^s(\underline{K},\underline{L})$ defined by $\gamma^*(\underline{f})=\underline{f}\cdot\gamma$. Furthermore, if $\underline{f}\sim\underline{g}$ in $\mathrm{Hom}^s(\underline{J},\underline{L})$, then $\underline{f}\cdot\gamma\sim\underline{g}\cdot\gamma$ in $\mathrm{Hom}^s(\underline{K},\underline{L})$. Consequently, γ^* induces an A-homomorphism $\underset{\sim}{H}^s(\underline{J},\underline{L})\longrightarrow\underset{\sim}{H}^s(\underline{K},\underline{L})$, but since γ is determined by α up to homotopy, this homomorphism is unambiguously determined by α, and so it can be denoted by $\alpha^*:\underset{\sim}{H}^s(\underline{J},\underline{L})\longrightarrow\underset{\sim}{H}^s(\underline{K},\underline{L})$. It is now easy to check that diagram (1) commutes. //

As mentioned previously, I plan to use (2.25) in the particular case in which $\underline{K}$ and $\underline{L}$ provide injective resolutions for M and N respectively. In that case, of course, $H^s(\mathrm{Hom}_A(M,\underline{L}))\cong\mathrm{Ext}_A^s(M,N)$, but, again, the mere fact that these two A-modules are isomorphic is insufficient: I need the result that the functors $H^s(\mathrm{Hom}_A(\ ,\underline{L}))$ and $\mathrm{Ext}_A^s(\ ,N)$ from $\mathcal{C}_A$ to $\underline{\mathcal{C}}_A$ are naturally equivalent, and this is proved in the next Proposition. The proof, which is hinted at at the end of §7.4 of Northcott (22) provides an example of the use of connected sequences of functors.

(2.27)Proposition. Let N be an A-module, and let $\underline{L}$ be an injective resolution for N. Then the functors $H^s(\mathrm{Hom}_A(\ ,\underline{L}))$ and $\mathrm{Ext}^s_A(\ ,N)$ from $\mathcal{C}_A$ to $\mathcal{C}_A$ are naturally equivalent for each $s \geq 0$.

Proof. I show first that $H^0(\mathrm{Hom}_A(\ ,\underline{L}))$ and $\mathrm{Hom}_A(\ ,N)$ are naturally equivalent. There is an A-homomorphism $\eta: N \longrightarrow L^0$ s.t.:-

(E) $0 \longrightarrow N \xrightarrow[\eta]{} L^0 \xrightarrow[d_L^0]{} L^1 \longrightarrow \ldots$. Let $f: M_1 \longrightarrow M_2$ be a homomorphism of A-modules. I use '*'s to denote induced homomorphisms. Since $\mathrm{Hom}_A(\ ,\ )$ is left exact,

$$\begin{array}{ccccccc}
(E)\ 0 \longrightarrow & \mathrm{Hom}_A(M_2,N) & \xrightarrow{\eta^*} & \mathrm{Hom}_A(M_2,L^0) & \xrightarrow{(d_L^0)^*} & \mathrm{Hom}_A(M_2,L^1) & \\
& \downarrow f^* & & \downarrow f^* & & \downarrow f^* & (1) \\
(E)\ 0 \longrightarrow & \mathrm{Hom}_A(M_1,N) & \xrightarrow{\eta^*} & \mathrm{Hom}_A(M_1,L^0) & \xrightarrow{(d_L^0)^*} & \mathrm{Hom}_A(M_1,L^1) & (C)
\end{array}$$

Now $H^0(\mathrm{Hom}_A(M_2,\underline{L})) = \ker((d_L^0)^*: \mathrm{Hom}_A(M_2,L^0) \longrightarrow \mathrm{Hom}_A(M_2,L^1))$ and $H^0(\mathrm{Hom}_A(M_1,\underline{L})) = \ker((d_L^0)^*: \mathrm{Hom}_A(M_1,L^0) \longrightarrow \mathrm{Hom}_A(M_1,L^1))$. It follows from diagram (1) that η induces a natural equivalence between the functors $\mathrm{Hom}_A(\ ,N)$ and $H^0(\mathrm{Hom}_A(\ ,\underline{L}))$.

Next suppose that (E) $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\mathcal{C}_A$. Since L^i is injective for each $i \geq 0$, this exact sequence induces:-

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(E)\ 0 \longrightarrow & \mathrm{Hom}_A(M'',L^0) & \longrightarrow & \mathrm{Hom}_A(M,L^0) & \longrightarrow & \mathrm{Hom}_A(M',L^0) & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
(E)\ 0 \longrightarrow & \mathrm{Hom}_A(M'',L^1) & \longrightarrow & \mathrm{Hom}_A(M,L^1) & \longrightarrow & \mathrm{Hom}_A(M',L^1) & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
 & \vdots & & \vdots & & \vdots & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
(E)\ 0 \longrightarrow & \mathrm{Hom}_A(M'',L^n) & \longrightarrow & \mathrm{Hom}_A(M,L^n) & \longrightarrow & \mathrm{Hom}_A(M',L^n) & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
 & \vdots & & \vdots & & \vdots & & & (C)
\end{array}$$

Now it follows from Theorem 5 of Chap. 4 of Northcott (21) that, for each $i \geqslant 0$, there exists a connecting homomorphism:-

$H^i(\mathrm{Hom}_A(M', \underline{L})) \longrightarrow H^{i+1}(\mathrm{Hom}_A(M'', \underline{L}))$, and these are s.t.:-

(E) $0 \longrightarrow H^0(\mathrm{Hom}_A(M'', \underline{L})) \longrightarrow H^0(\mathrm{Hom}_A(M, \underline{L})) \longrightarrow H^0(\mathrm{Hom}_A(M', \underline{L})) \longrightarrow$
$H^1(\mathrm{Hom}_A(M'', \underline{L})) \longrightarrow \ldots\ldots \longrightarrow H^{n-1}(\mathrm{Hom}_A(M', \underline{L})) \longrightarrow H^n(\mathrm{Hom}_A(M'', \underline{L})) \longrightarrow$
$H^n(\mathrm{Hom}_A(M, \underline{L})) \longrightarrow H^n(\mathrm{Hom}_A(M', \underline{L})) \longrightarrow H^{n+1}(\mathrm{Hom}_A(M'', \underline{L})) \longrightarrow \ldots\ldots$;

and, furthermore, if this long exact sequence is regarded as belonging to the translation category of (right semi-infinite) complexes of A-modules and -homomorphisms, it is a contravariant functor of

(E) $0 \to M' \to M \to M'' \to 0$.

Thus $\{H^n(\mathrm{Hom}_A(\ , \underline{L}))\}_{n \geqslant 0}$ and $\{\mathrm{Ext}^n_A(\ , N)\}_{n \geqslant 0}$ are connected right sequences of contravariant functors from $\underline{\mathscr{C}}_A$ to $\underline{\mathscr{C}}_A$. Also, whenever P is a projective A-module, $\mathrm{Ext}^n_A(P,N) = 0$ and $H^n(\mathrm{Hom}_A(P, \underline{L})) = 0$ whenever $n > 0$ (for $H^n(\mathrm{Hom}_A(P, \underline{L})) \cong \mathrm{Ext}^n_A(P,N)$). It now follows immediately from the Corollary to Theorem 11 of Chapter 6 of Northcott (21) that the functors $H^n(\mathrm{Hom}_A(\ , \underline{L}))$ and $\mathrm{Ext}^n_A(\ , N)$ from $\underline{\mathscr{C}}_A$ to $\underline{\mathscr{C}}_A$ are naturally equivalent (for each $n \geqslant 0$). //

(2.28) Corollary. Let s be a non-negative integer, and let $\tau^s : H^s(\mathrm{Hom}_A(\ , \underline{L})) \rightsquigarrow \mathrm{Ext}^s_A(\ , N)$ be the natural equivalence given in (2.27). Using the notation of (2.24) - (2.26), let $\nu^s(M) : \underset{\sim}{H}^s(\underline{K}, \underline{L}) \xrightarrow{\cong} \mathrm{Ext}^s_A(M,N)$ be the composition of $\phi^s(M) : \underset{\sim}{H}^s(\underline{K}, \underline{L}) \xrightarrow{\cong} H^s(\mathrm{Hom}_A(M, \underline{L}))$ and $\tau^s(M) : H^s(\mathrm{Hom}_A(M, \underline{L})) \xrightarrow{\cong} \mathrm{Ext}^s_A(M,N)$. Define $\nu^s(M')$ similarly. Then:-

(C)
$$\begin{array}{ccccc}
\underset{\sim}{H}^s(\underline{J}, \underline{L}) & \xrightarrow[\cong]{\phi^s(M')} & H^s(\mathrm{Hom}_A(M', \underline{L})) & \xrightarrow[\cong]{\tau^s(M')} & \mathrm{Ext}^s_A(M', N) \\
\Big\downarrow{\scriptstyle \alpha^*} & & \Big\downarrow{\scriptstyle \alpha^*} & & \Big\downarrow{\scriptstyle \mathrm{Ext}^s_A(\alpha, 1_N)} \\
\underset{\sim}{H}^s(\underline{K}, \underline{L}) & \xrightarrow[\cong]{\phi^s(M)} & H^s(\mathrm{Hom}_A(M, \underline{L})) & \xrightarrow[\cong]{\tau^s(M)} & \mathrm{Ext}^s_A(M, N)
\end{array}$$

with $\nu^s(M')$: $\underset{\sim}{H}^s(\underline{J}, \underline{L}) \xrightarrow{\cong} \mathrm{Ext}^s_A(M', N)$ and $\nu^s(M)$: $\underset{\sim}{H}^s(\underline{K}, \underline{L}) \xrightarrow{\cong} \mathrm{Ext}^s_A(M, N)$.

//

(2.29) Definition. Let X,Y,& Z be three A-modules. A map $k:X \times Y \longrightarrow Z$ is said to be a pairing if and only if, whenever, $x,x' \in X$, $y,y' \in Y$, and $a,a' \in A$,

$$k(ax + a'x',y) = a.k(x,y) + a'.k(x',y) \text{ and}$$

$$k(x,ay + a'y') = a.k(x,y) + a'.k(x,y').$$

Notice that a pairing induces natural A-homomorphisms $Y \longrightarrow \mathrm{Hom}_A(X,Z)$ and $X \longrightarrow \mathrm{Hom}_A(Y,Z)$. The pairing is said to be perfect if and only if both these homomorphisms are isomorphisms.

Conversely, if $\phi : Y \longrightarrow \mathrm{Hom}_A(X,Z)$ is an A-homomorphism, ϕ defines a pairing $k:X \times Y \longrightarrow Z$ given by $k(x,y) = (\phi(y))(x)$. Similarly, an A-homomorphism $\chi : X \longrightarrow \mathrm{Hom}_A(Y,Z)$ defines a pairing $k':X \times Y \longrightarrow Z$ given by $k'(x,y) = (\chi(x))(y)$.

(2.30) It seems appropriate to preface the next Proposition with a few remarks about right derived functors.

Let $T:\underline{\mathcal{C}}_A \rightsquigarrow \underline{\mathcal{C}}_A$ be an additive, A-linear, covariant, left exact functor. Let R^iT denote the ith right derived functor of T: for any A-module M, $R^iT(M) = 0$ if $i < 0$; each R^iT is an additive, A-linear, covariant functor from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$; and, as T is left exact, R^0T is naturally equivalent to T.

Suppose $\alpha : M \longrightarrow M'$ is a homomorphism of A-modules, and

$$(E)\quad 0 \longrightarrow M \xrightarrow{\varepsilon} K^0 \xrightarrow{d_K^0} K^1 \longrightarrow \cdots \longrightarrow K^n \xrightarrow{d_K^n} K^{n+1} \longrightarrow \cdots \quad \&$$

$$(E)\quad 0 \longrightarrow M' \xrightarrow{\varepsilon'} J^0 \xrightarrow{d_J^0} J^1 \longrightarrow \cdots \longrightarrow J^n \xrightarrow{d_J^n} J^{n+1} \longrightarrow \cdots$$

are injective resolutions for M & M' respectively. By the Corollary to Theorem 13 of Chap. 5 of Northcott (22), there exists a family $\gamma = \{\gamma^n\}_{n \geq 0}$ of A-homomorphisms s.t. $\gamma^n : K^n \longrightarrow J^n$ and:-

$$\begin{array}{ccccccccccccc}
(E)\ 0 & \longrightarrow & M & \xrightarrow{\varepsilon} & K^0 & \xrightarrow{d_K^0} & K^1 & \longrightarrow \cdots \longrightarrow & K^n & \xrightarrow{d_K^n} & K^{n+1} & \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma^0} & & \downarrow{\scriptstyle \gamma^1} & & \downarrow{\scriptstyle \gamma^n} & & \downarrow{\scriptstyle \gamma^{n+1}} & & \\
(C)\ (E)'\ 0 & \longrightarrow & M' & \longrightarrow & J^0 & \xrightarrow{d_J^0} & J^1 & \longrightarrow \cdots \longrightarrow & J^n & \xrightarrow{d_J^n} & J^{n+1} & \longrightarrow \cdots ,
\end{array} \qquad (1)$$

and γ is determined by α up to homotopy.

Now the functors R^iT depend on the choice of a collection of injective resolutions, one for each module in $\underline{\mathcal{C}}_A$: such a chosen collection is called a defining system for the right derived functors of T. The effect of changing the defining system is to produce another system of right derived functors of T, $*R^iT$, which are such that, for each $i \geqslant 0$, R^iT and $*R^iT$ are naturally equivalent.

Suppose that the injective resolutions for M and M' above belong to the relevant defining system. Consider the diagram induced from (1):-

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & T(K^0) & \longrightarrow & T(K^1) & \longrightarrow \cdots \longrightarrow & T(K^n) & \xrightarrow{T(d_K^n)} & T(K^{n+1}) & \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle T(\gamma^0)} & & \downarrow{\scriptstyle T(\gamma^1)} & & \downarrow{\scriptstyle T(\gamma^n)} & & \downarrow{\scriptstyle T(\gamma^{n+1})} & & \\
(C)\ 0 & \longrightarrow & T(J^0) & \longrightarrow & T(J^1) & \longrightarrow \cdots \longrightarrow & T(J^n) & \xrightarrow{T(d_J^n)} & T(J^{n+1}) & \longrightarrow \cdots .
\end{array} \qquad (2)$$

$\gamma \in \mathrm{Hom}^0(\underline{K}, \underline{J})$, and consequently $T(\gamma) = \{T(\gamma^n)\}_{n\geqslant 0} \in \mathrm{Hom}^0(T(\underline{K}), T(\underline{J}))$, are determined by α to within a homotopy, and so the homomorphisms induced on the homology modules in (2) are determined unambiguously by α. For each $i \geqslant 0$, $R^iT(M) = H^i(T(\underline{K}))$, $R^iT(M') = H^i(T(\underline{J}))$, and if $\alpha^{*(i)}: H^i(T(\underline{K})) \longrightarrow H^i(T(\underline{J}))$ is the A-homomorphism induced by α as described above, then $\alpha^{*(i)} = R^iT(\alpha): R^iT(M) \longrightarrow R^iT(M')$.

(2,31) Proposition. (See Proposition 6.1 of Hartshorne (13).) Let $T: \underline{\mathcal{C}}_A \rightsquigarrow \underline{\mathcal{C}}_A$ be an additive, A-linear, covariant, left exact functor. Suppose M & N are A-modules. Then, for each $i \geqslant 0$, and $s \geqslant 0$, there is a pairing:-

$$R^iT(M) \times \mathrm{Ext}_A^s(M,N) \longrightarrow R^{i+s}T(N) .$$

Furthermore, if $\chi^{i,s}(M):\mathrm{Ext}_A^s(M,N) \longrightarrow \mathrm{Hom}_A(R^iT(M),R^{i+s}T(N))$ denotes the A-homomorphism induced by the above pairing, then, regarding N as fixed, the $\chi^{i,s}(M)$ constitute a natural transformation of the functor $\mathrm{Ext}_A^s(\ ,N)$ to $\mathrm{Hom}_A(R^iT(\ ),R^{i+s}T(N))$ (from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$).

Proof. Suppose that $\alpha:M \longrightarrow M'$ is a homomorphism of A-modules. Let:-

$$(E)\quad 0 \longrightarrow M \xrightarrow{\varepsilon} K^0 \xrightarrow{d_K^0} K^1 \longrightarrow \cdots \longrightarrow K^n \xrightarrow{d_K^n} K^{n+1} \longrightarrow \cdots$$

$$(E)\quad 0 \longrightarrow M' \xrightarrow{\varepsilon'} J^0 \xrightarrow{d_J^0} J^1 \longrightarrow \cdots \longrightarrow J^n \xrightarrow{d_J^n} J^{n+1} \longrightarrow \cdots$$

$$\&(E)\quad 0 \longrightarrow N \xrightarrow{\eta} L^0 \xrightarrow{d_L^0} L^1 \longrightarrow \cdots \longrightarrow L^n \xrightarrow{d_L^n} L^{n+1} \longrightarrow \cdots$$

provide injective resolutions, for M,M',& N respectively, which belong to the defining system for the right derived functors of T.

I can now apply the work in (2.24) - (2.30), and I use the notation established therein. In view of Corollary (2.28) and the remarks contained in Definition (2.29), it is sufficient to construct (for an A-module M having injective resolution $\underline{K}$ as above) an A-homomorphism:-

$\lambda^{i,s}(M):\underset{\sim}{H}^s(\underline{K},\underline{L}) \longrightarrow \mathrm{Hom}_A(R^iT(M),R^{i+s}T(N))$ which is s.t., with the above notation:-

$$\begin{array}{ccc} \underset{\sim}{H}^s(\underline{J},\underline{L}) & \xrightarrow{\lambda^{i,s}(M')} & \mathrm{Hom}_A(R^iT(M'),R^{i+s}T(N)) \\ \Big\downarrow{\scriptstyle \alpha^*} & & \Big\downarrow{\scriptstyle \mathrm{Hom}_A(R^iT(\alpha),1_{R^{i+s}T(N)})} \\ \underset{\sim}{H}^s(\underline{K},\underline{L}) & \xrightarrow{\lambda^{i,s}(M)} & \mathrm{Hom}_A(R^iT(M),R^{i+s}T(N)) \end{array} \qquad (1)\ (C)$$

Note that I have chosen $\underline{K}$, $\underline{J}$, & $\underline{L}$ in such a way that $R^iT(M)$, $R^iT(M')$, $R^{i+s}T(N)$, & $R^iT(\alpha)$ may be calculated from them directly. I define $\lambda^{i,s}(M)$ as follows:-

Suppose $\underline{f} = \{f^n\}_{n\in\mathbb{Z}} \in \mathrm{Hom}^s(\underline{K},\underline{L})$. Then $T(\underline{f}) = \{T(f^n)\}_{n\in\mathbb{Z}} \in \mathrm{Hom}^s(T(\underline{K}),T(\underline{L}))$, and so $T(\underline{f})$ induces an A-homomorphism:-

$f^{**}: H^i(T(\underline{K})) \longrightarrow H^{i+s}(T(\underline{L}))$, i.e. $f^{**}: R^iT(M) \longrightarrow R^{i+s}T(N)$.

If $\underline{g} = \{g^n\}_{n\in\mathbb{Z}} \in \mathrm{Hom}^s(\underline{K},\underline{L})$ and $\underline{g} \sim \underline{f}$, then it is easy to see that $T(\underline{g}) \sim T(\underline{f})$ in $\mathrm{Hom}^s(T(\underline{K}),T(\underline{L}))$ (since T is additive and A-linear). Consequently, $f^{**} = g^{**}$. Thus the map $\underline{f} \mapsto f^{**}$ from $\mathrm{Hom}^s(\underline{K},\underline{L})$ to $\mathrm{Hom}_A(R^iT(M),R^{i+s}T(N))$ induces a map $\lambda^{i,s}(M): \underset{\sim}{H}^s(\underline{K},\underline{L}) \longrightarrow \mathrm{Hom}_A(R^iT(M),R^{i+s}T(N))$ such that $\lambda^{i,s}(M)(\tilde{\underline{f}}) = f^{**}$. That $\lambda^{i,s}(M)$ is actually an A-homomorphism follows easily from the facts that T is additive and A-linear.

It is now straightfoward to check that diagram (1) commutes. //

(2.32)Corollary. Let M be an A-module and N a fixed A-module. Suppose i & s are non-negative integers. Let $\Theta^{i,s}(M): R^iT(M) \longrightarrow \mathrm{Hom}_A(\mathrm{Ext}_A^s(M,N),R^{i+s}T(N))$ be the A-homomorphism induced by the pairing given in (2.31). Then the $\Theta^{i,s}(M)$ constitute a natural transformation of the functor R^iT to $\mathrm{Hom}_A(\mathrm{Ext}_A^s(\ ,N),R^{i+s}T(N))$ (from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$).

Proof. This is an easy consequence of (2.31) and the remarks included in Definition (2.29). //

CHAPTER 3

The Cousin Complex: Construction and Basic Properties.

(Reminder: "ring" means "commutative Noetherian ring with a non-zero identity".)

Throughout this Chapter, A will denote a (fixed) ring.

(3.1)Definition. Suppose U is a subset of V(A) (see (1.16)). A subset U' of U is s.t.b. low w.r.t. U if and only if, for all $\underline{p} \in U'$ and $\underline{q} \in V(A)$ such that $\underline{q} \subsetneq \underline{p}$, $\underline{q} \notin U$, i.e. if and only if each element of U' is minimal (w.r.t. inclusion) in U.

One could, of course, express this definition in terms of the geometric concept of "specialization", but I've chosen the above terminology because it is convenient and suggestive(recall the definition of the height of a prime ideal (1.21)), and I have no plans for geometric applications.

Example. For $i \geqslant 0$, define $U^i = \left\{\underline{p} \in V(A) \,\middle|\, \mathrm{ht}\,\underline{p} \geqslant i\right\}$. Then $U^i - U^{i+1} = \left\{\underline{p} \in V(A) \,\middle|\, \mathrm{ht}\,\underline{p} = i\right\}$ is low w.r.t. U^i.

(3.2)Proposition. Suppose $U' \subseteq U$ are subsets of V(A) such that U - U' is low w.r.t. U. Let M be an A-module s.t. $\mathrm{Supp}(M) \subseteq U$, and for each $\underline{p} \in U - U'$, let $g_{\underline{p}} : M \longrightarrow M_{\underline{p}}$ denote the natural A-module homomorphism given by $g_{\underline{p}}(m) = m/1$. Then, if $m \in M$, $g_{\underline{p}}(m) = 0$ for all but a finite number of prime ideals $\underline{p} \in U - U'$.

Proof. Suppose $\underline{p} \in U - U'$ is s.t. $g_{\underline{p}}(m) = m/1 \neq 0$. Then $\underline{p} \in \mathrm{Supp}(Am) \subseteq \mathrm{Supp}(M)$, and since Am is f.-g., $\mathrm{Supp}(Am) = V(Am)$ (see (1.16)).

If $\underline{p}$ were not a minimal element of V(Am), there would be a prime $\underline{q} \subsetneq \underline{p}$ with $\underline{q} \in V(Am) \subseteq \mathrm{Supp}(M) \subseteq U$. But this is impossible, since $\underline{p} \in$

$U - U'$ which is low w.r.t. U. Thus $\underline{p}$ is a minimal element of $V(Am)$, and so (by (1.16)) belongs to the finite set $Ass(Am)$. //

(3.3)Corollary. Consequently, there is an A-module homomorphism $\mathcal{E}: M \longrightarrow \bigoplus_{\underline{p} \in U-U'} M_{\underline{p}}$ for which, if $m \in M$, the component of $\mathcal{E}(m)$ in the direct summand $M_{\underline{p}}$ is $m/1$. //

I shall examine $\ker \mathcal{E}$ and $\operatorname{coker} \mathcal{E}$, but first:-

(3.4)Remark. If X is a subset of $V(A)$ and N is an A-module, $Supp(N) \subseteq X$ if and only if, for all $\underline{p} \in V(A) - X$ and for all $n \in N$, $(0:n)_A \not\subseteq \underline{p}$. In particular, $\underline{p} \notin Supp(N)$ if and only if, for all $n \in N$, $(0:n)_A \not\subseteq \underline{p}$. //

(3.5)Proposition. With the notation of (3.2) and (3.3),

(i) $Supp(\ker \mathcal{E}) \subseteq U'$;

(ii) if $\underline{p}$ is an arbitrary element of $U - U'$ and $M_{\underline{p}}$ is regarded as an $A_{\underline{p}}$-module, the annihilator of any non-zero element of $M_{\underline{p}}$ is $\underline{p}A_{\underline{p}}$-primary;

& (iii) $Supp(\operatorname{coker} \mathcal{E}) \subseteq U'$.

Proof. (i) Since $\ker \mathcal{E} \subseteq M$, $Supp(\ker \mathcal{E}) \subseteq Supp(M) \subseteq U$. But if $m \in \ker \mathcal{E}$, and $\underline{p} \in U - U'$, $m/1 = 0$ in $M_{\underline{p}}$, and so $(0:m)_A \not\subseteq \underline{p}$. Now use (3.4).

(ii) Suppose $\underline{p} \in U - U'$, and m/t ($m \in M$, $t \in A - \underline{p}$) is an arbitrary non-zero element of $M_{\underline{p}}$. Since $1/t$ is a unit in $A_{\underline{p}}$, $(0:m/t)_{A_{\underline{p}}} = (0:m/1)_{A_{\underline{p}}}$. Thus, by (1.5), $(0:m/t)_{A_{\underline{p}}} = ((0:m)_A).A_{\underline{p}}$. As this is a proper ideal of $A_{\underline{p}}$, $(0:m)_A \subseteq \underline{p}$, and in view of (1.15) it is sufficient to show that $\underline{p}$ is the only prime ideal of A which contains $(0:m)_A$ and is contained in $\underline{p}$. But if $\underline{q}$ is prime and $(0:m)_A \subseteq \underline{q} \subseteq \underline{p}$, $\underline{q} \in Supp(M)$ (by (3.4)), and so $\underline{q} \in U$. Therefore $\underline{q} = \underline{p}$, since $U - U'$ is low w.r.t. U.

Consequently, when $M_{\underline{p}}$ is regarded as an A-module, m/t is annihilated

by some power of $\underline{p}$.

(iii) Coker $\varepsilon = M_0/\varepsilon(M)$, where $M_0 = \bigoplus_{\underline{p}\in U-U'} M_{\underline{p}}$, and so coker ε is a quotient of M_0. Thus $\mathrm{Supp}(\mathrm{coker}\ \varepsilon) \subseteq \mathrm{Supp}(M_0)$.

Suppose $\underline{q} \notin \mathrm{Supp}(M)$, so that $M_{\underline{q}} = 0$. Using the isomorphism $N_{\underline{q}} \cong N\otimes_A A_{\underline{q}}$ of (1.3), $(M_0)_{\underline{q}} \cong \bigoplus_{\underline{p}\in U-U'} (M\otimes_A A_{\underline{p}})\otimes_A A_{\underline{q}}$, and since $(M\otimes_A A_{\underline{p}})\otimes_A A_{\underline{q}} \cong (M\otimes_A A_{\underline{q}})\otimes_A A_{\underline{p}} = 0$, $(M_0)_{\underline{q}} = 0$, and $\underline{q} \notin \mathrm{Supp}(M_0)$.

Thus $\mathrm{Supp}(M_0) \subseteq \mathrm{Supp}(M)$, and so $\mathrm{Supp}(\mathrm{coker}\ \varepsilon) \subseteq U$. So, by (3.4), I need only show that for every $x \in M_0$ and every $\underline{p} \in U - U'$, $(\varepsilon M{:}x)_A \notin \underline{p}$.

Suppose, then, that an arbitrary element x of $M_0 = \bigoplus_{\underline{p}\in U-U'} M_{\underline{p}}$ has non-zero components m_i/t_i in $M_{\underline{p}_i}$ $(m_i \in M,\ t_i \in A - \underline{p}_i)$ for $i = 1,\ldots,n$, and zero components in the other direct summands. By part (ii), there is a positive integer k_i s.t. $\underline{p}_i^{k_i} \subseteq (0{:}m_i/t_i)_A$, $(i = 1,\ldots,n)$. I shall show that $(\varepsilon M{:}x)_A$ is not contained in any element $\underline{p}$ of $U - U'$. There are two cases to consider.

(a) $\underline{p}$ is not equal to any of $\underline{p}_1,\ldots,\underline{p}_n$.

In this case $\bigcap_{j=1}^{n} \underline{p}_j \notin \underline{p}$ (for otherwise $\underline{p}_j \subseteq \underline{p}$ for some $j \in \{1,\ldots,n\}$, and since $\underline{p}$ and $\underline{p}_j$ both $\in U - U'$, $\underline{p}_j = \underline{p}$). Thus, there is an $a \in \bigcap_{j=1}^{n} \underline{p}_j - \underline{p}$ and $N > 0$ s.t. $\underline{p}_j^N(m_j/t_j) = 0$ for each $j = 1,\ldots,n$. It follows that $a^N.x = 0$, so $a^N \in (\varepsilon M{:}x)_A - \underline{p}$.

(b) $\underline{p}$ is equal to one of $\underline{p}_1,\ldots,\underline{p}_n$, say $\underline{p}_1$.

By reasoning similar to that in (a), there exists an element $a \in \bigcap_{j=2}^{n} \underline{p}_j - \underline{p}_1$ and an integer $N > 0$ s.t. t_1a^Nx has, for its component in $M_{\underline{p}_1}$, $a^Nm_1/1$, and, for all its other components, 0.

Suppose $\varepsilon(a^N m_1)$ has non-zero components, as well as in $M_{\mathfrak{p}_1}$, in $M_{\mathfrak{p}_2'}, \ldots, M_{\mathfrak{p}_m'}$. (If $t_1 a^N x = \varepsilon(a^N m_1)$, the proof is complete.) Then there is a $b \in \bigcap_{j=2}^{m} \mathfrak{p}_j' - \mathfrak{p}_1$ and an integer $K > 0$ s.t. $\varepsilon(b^K a^N m_1) = b^K t_1 a^N x$, and so $b^K t_1 a^N \in (\varepsilon M : x)_A - \mathfrak{p}$. (Recall that $\mathfrak{p} = \mathfrak{p}_1$.)

Thus, in both cases, $(\varepsilon M : x)_A \not\subseteq \mathfrak{p}$, and so $\mathrm{Supp}(\mathrm{coker}\ \varepsilon) \subseteq U'$ by (3.4). //

(3.6) The Construction of the Cousin Complex for an A-module M.

Define, for integers $i \geq 0$, $U^i(M) = \{\mathfrak{p} \in \mathrm{Supp}(M) \mid \mathrm{ht}_M \mathfrak{p} \geq i\}$. (See (1.22) for definition of M-height.) Note that:-

$U^0(M) = \mathrm{Supp}(M)$;

$U^{i+1}(M) \subseteq U^i(M)$ and $U^i(M) - U^{i+1}(M) = \{\mathfrak{p} \in \mathrm{Supp}(M) \mid \mathrm{ht}_M \mathfrak{p} = i\}$;

$U^i(M) - U^{i+1}(M)$ is low w.r.t. $U^i(M)$.

If I define $M^{-2} = 0$, $M^{-1} = M$, and $d^{-2}: M^{-2} \longrightarrow M^{-1}$ to be the zero homomorphism, then $\mathrm{coker}\ d^{-2} = M$, and so $\mathrm{Supp}(\mathrm{coker}\ d^{-2}) \subseteq U^0(M)$. The construction of the Cousin Complex will be by induction: $M^{-2} \xrightarrow{d^{-2}} M^{-1}$ provides the first two terms and a basis for the induction:-

Suppose inductively that $n \geq 0$ and a complex:-

$M^{-2} \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} M^0 \longrightarrow \cdots \longrightarrow M^{n-2} \xrightarrow{d^{n-2}} M^{n-1}$ with the property that $\mathrm{Supp}(\mathrm{coker}\ d^{n-2}) \subseteq U^n(M)$ has been constructed.

Apply (3.2), (3.3), & (3.5) to the module $N = \mathrm{coker}\ d^{n-2}$ and the subsets $U^{n+1}(M) \subseteq U^n(M)$ of $V(A)$. Define $N_0 = M^n = \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Supp}(M) \\ \mathrm{ht}_M \mathfrak{p} = n}} (\mathrm{coker}\ d^{n-2})_{\mathfrak{p}}$, using the convention that the direct sum of an empty family of A-modules is zero. By (3.2) & (3.3), there is an A-homomorphism $\varepsilon: N \longrightarrow N_0$ for which, if $n \in N$, the component of $\varepsilon(n)$ in the direct summand $N_{\mathfrak{p}}$ is $n/1$. (If $N_0 = 0$,

I take ε as zero, of course.) Now define $d^{n-1}:M^{n-1}\longrightarrow M^n$ to be the composition of the natural epimorphism $M^{n-1}\longrightarrow$ coker d^{n-2} and $\varepsilon:N\longrightarrow N_0$ ($N_0 = M^n$, $N =$ coker d^{n-2}). It is clear that $d^{n-1}.d^{n-2} = 0$, and that coker d^{n-1} = coker ε, so by (3.5), Supp(coker d^{n-1}) $\subseteq U^{n+1}(M)$. (This is still true, obviously, if $M^n = 0$.)

Thus the inductive step is complete, and so, using this induction, I construct a complex $\underline{C(M)}$ (= $\underline{C_A(M)}$):-

$$0\longrightarrow M\xrightarrow{d^{-1}}M^0\xrightarrow{d^0}M^1\longrightarrow\ldots\ldots\longrightarrow M^n\xrightarrow{d^n}M^{n+1}\longrightarrow\ldots\ldots \qquad (1).$$

I have called this the <u>Cousin Complex for M</u> because it is, in fact, the commutative algebra analogue of the Cousin Complex of §2 of Chap. IV of Hartshorne (12). I shall consistently use the notation (1) for the Cousin Complex for M: in addition, it will frequently be useful to write $M^{-2} = 0$, $M^{-1} = M$, and $d^{-2}:M^{-2}\longrightarrow M^{-1}$ for the zero homomorphism, as in the above construction. For $i \geqslant -1$, I shall denote the ith homology module (see (1.1)) of $\underline{C(M)}$ by $H^i(M)$.

(3.7) Properties of the Cousin Complex for an A-module M.

(i) If $M^n = 0$ for some $n \geqslant -1$, $M^{n+r} = 0$ for each $r > 0$. If M has finite Krull Dimension, say $K\text{-}Dim_A M = s$, then $U^{s+1}(M)$ must be empty, and so $M^{s+1} = 0$. Thus $M^{s+r} = 0$ for each $r > 0$.

(ii) If $n \geqslant 0$, $M^n = \bigoplus_{\substack{\underline{p}\in Supp(M)\\ ht_M\underline{p} = n}} (\text{coker } d^{n-2})_{\underline{p}}$. In particular,

$$M^0 = \bigoplus_{\substack{\underline{p}\in Supp(M)\\ ht_M\underline{p} = 0}} M_{\underline{p}}.$$

(iii) If $n \geqslant 0$, $x \in M^{n-1}$, and $^-$ is used to denote natural images in coker d^{n-2}, $d^{n-1}(x)$ has, for its component in the direct summand

$(\text{coker } d^{n-2})_{\underline{p}}$ of M^n (where $\underline{p} \in \text{Supp}(M)$ and $\text{ht}_M \underline{p} = n$), $(\bar{x})/1$. In particular, if $m \in M$ and $\underline{p} \in \text{Supp}(M)$ with $\text{ht}_M \underline{p} = 0$, $d^{-1}(m)$ has $m/1$ for its component in the direct summand $M_{\underline{p}}$ of M^0.

(iv) If $n \geqslant 0$, $\text{Supp}(\text{coker } d^{n-2}) \subseteq U^n(M)$.

(v) Consequently, if $n \geqslant 0$, $y \in \text{coker } d^{n-2}$, and $\underline{p} \in V(A) - U^n(M)$, then $(0:y)_A \not\subseteq \underline{p}$ (by (3.4)).

(vi) If $n \geqslant 0$, and $\underline{p} \in \text{Supp}(M)$ has M-height n, then every element of $(\text{coker } d^{n-2})_{\underline{p}}$ (when this is regarded as an A-module) is annihilated by some power of $\underline{p}$. This follows from the construction of the Cousin Complex and (3.5)(ii). In particular, taking $n = 0$, every element of $M_{\underline{p}}$, $\underline{p}$ being a minimal element of $\text{Supp}(M)$, is annihilated by some power of $\underline{p}$.

(vii) If $n \geqslant 0$, $H^{n-1}(M) \subseteq \text{coker } d^{n-2}$, and $\text{Supp}(H^{n-1}(M)) \subseteq U^{n+1}(M)$. This follows from the construction of the Cousin Complex and (3.5)(i).

(viii) Consequently, if $n \geqslant 0$ $y \in H^{n-1}(M)$, and $\underline{p} \in V(A) - U^{n+1}(M)$, then $(0:y)_A \not\subseteq \underline{p}$ (by (3.4)).

(ix) If $n \geqslant -1$ and $a \in A$ is s.t. $a.M^n = 0$, then $a.M^{n+k} = 0$ for each $k \geqslant 0$. This follows (by induction on k) from the construction of the Cousin Complex: since $\text{coker } d^{n-1}$ is a quotient of M^n, it too is annihilated by a, and so the same is true of $M^{n+1} = \bigoplus_{\substack{\underline{p} \in \text{Supp}(M) \\ \text{ht}_M \underline{p} = n+1}} (\text{coker } d^{n-1})_{\underline{p}}$, i.e. $a.M^{n+1} = 0$.

(x) If $n \geqslant -1$, and $a \in A$ is s.t. $a.M^n = M^n$, then $a.M^{n+k} = M^{n+k}$ for each $k \geqslant 0$. Again, this follows, by induction on k, from the construction of the Cousin Complex: $a.(\text{coker } d^{n-1}) = (\text{coker } d^{n-1})$, and so $a.M^{n+1} = M^{n+1}$.

(xi) Consequently, if M is non-zero and f.-g., and a is not a zero-divisor on M, then $a.M^i = M^i$ for each $i \geqslant 0$. For, by (1.11) and (1.13), a does not belong to any prime of $\mathrm{Ass}(M)$, and so, by (1.16), a does not belong to any minimal element of $V(M)$. Thus, for each $\underline{p} \in V(M) = \mathrm{Supp}(M)$ of M-height 0, $a \in A - \underline{p}$. It follows that multiplication by a provides an A-automorphism of M^0, and so the result follows from (x).

(3.8) For the remainder of Chapter 3, I am going to examine the Cousin Complex in the context of "ring extensions", and the notation introduced below will be maintained for the rest of the Chapter.

Suppose that A and B are rings and that $f: A \longrightarrow B$ is a ring homomorphism. (I demand that $f(1_A) = 1_B$.) B can be regarded as an A-module by means of f, and, for an arbitrary A-module N, N_B will denote $N \otimes_A B$ regarded as a B-module in the obvious way.

Let M be an A-module, and let:-

$\underline{C(M)}: 0 \longrightarrow M \xrightarrow{d^{-1}} M^0 \longrightarrow \ldots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \ldots$ denote the Cousin Complex for M. $\underline{C(M)} \otimes_A B$ will denote the complex of B-modules and B-homomorphisms:-

$$0 \longrightarrow M_B \xrightarrow{d^{-1} \otimes 1_B} (M^0)_B \longrightarrow \ldots \longrightarrow (M^n)_B \xrightarrow{d^n \otimes 1_B} (M^{n+1})_B \ldots,$$

where 1_B denotes the identity on B. Let the Cousin Complex for the B-module M_B be:-

$\underline{C_B(M_B)}: 0 \longrightarrow M_B \xrightarrow{e^{-1}} (M_B)^0 \xrightarrow{e^0} \ldots \longrightarrow (M_B)^n \xrightarrow{e^n} (M_B)^{n+1} \longrightarrow \ldots$

Note that $\underline{C(M)} \otimes_A B$ & $\underline{C_B(M_B)}$ have the same initial term, namely M_B. Under certain conditions, one can construct a homomorphism of complexes of B-modules $\Psi: \underline{C(M)} \otimes_A B \longrightarrow \underline{C_B(M_B)}$ such that $\gamma^{-1}: M_B \longrightarrow M_B$ is the identity. The conditions I shall impose in order to obtain Ψ will seem rather

technical: however, they are satisfied in a variety of different situations, and I feel that the resulting applications justify the dry, technical discussion needed to construct Ψ.

If $\underline{a}$ is an ideal of A, $\underline{a}^e$ will denote the extension of $\underline{a}$ to the ideal $f(\underline{a}).B$ of B. If $\underline{b}$ is an ideal of B, $\underline{b}^c$ will denote the contraction $f^{-1}(\underline{b})$ of $\underline{b}$ in A. $\underline{b}^c$ is an ideal of A, If $\underline{q}$ is a prime ideal of B, $\underline{q}^c = \underline{p}$ is a prime ideal of A, since $A/\underline{p}$ can be embedded as a subring in $B/\underline{q}$.

(3.9)Lemma. Suppose $n \geqslant 0$, and $^-$, $\tilde{}$ are used to denote natural images in coker d^{n-2} and coker$(d^{n-2}\otimes 1_B)$ respectively. If $m \in M^n$, the notation:-

$m = (\overline{m}_{\underline{p}}/s_{\underline{p}})$ will be used to denote that, for $\underline{p} \in$ Supp(M) of M-height n, the component of m in the direct summand $(\text{coker } d^{n-2})_{\underline{p}}$ of M^n is $\overline{m}_{\underline{p}}/s_{\underline{p}}$, so that all but a finite number of the $\overline{m}_{\underline{p}}/s_{\underline{p}}$ will be zero. (Obvious extensions of this notation will be used without further comment.)

$\bigoplus_{\substack{\underline{p}\in\text{Supp}(M)\\ \text{ht}_M\underline{p}=n}} (\text{coker}(d^{n-2}\otimes 1_B))_{\underline{p}}$ has, of course, a natural structure as a B-module given by $b.(\tilde{n}_{\underline{p}}/s_{\underline{p}}) = (\widetilde{bn}_{\underline{p}}/s_{\underline{p}})$ (where $n_{\underline{p}} \in (M^{n-1})_B$, of course). As such, there is a natural B-module isomorphism:-

$$\alpha^n : (M^n)_B \xrightarrow{\cong} \bigoplus_{\substack{\underline{p}\in\text{Supp}(M)\\ \text{ht}_M\underline{p}=n}} (\text{coker}(d^{n-2}\otimes 1_B))_{\underline{p}} \text{ which is such that:-}$$

$$\alpha^n((\overline{m}_{\underline{p}}/s_{\underline{p}})\otimes b) = (\widetilde{(m_{\underline{p}}\otimes b)}/s_{\underline{p}}).$$

Proof. The first part is straightfoward: to form $(\text{coker}(d^{n-2}\otimes 1_B))_{\underline{p}}$, one regards the B-module coker$(d^{n-2}\otimes 1_B)$ as an A-module in the natural way (which, of course, gives the same A-module structure as that induced by $f:A\longrightarrow B$). It is easy to see that $(\text{coker}(d^{n-2}\otimes 1_B))_{\underline{p}}$ has a B-module

structure given by, if $b \in B$, $x \in (M^{n-1})_B$, and $s \in A - \underline{p}$,

$b.(\tilde{x}/s) = (b.\tilde{x})/s = \widetilde{(b.x)}/s$.

To show that α^n is a well-defined B-module isomorphism, I shall exhibit it as a composition of obvious A-module isomorphisms $\mathcal{S}$ and then observe that it is also a B-module homomorphism.

First, from (E) $M^{n-2} \xrightarrow{d^{n-2}} M^{n-1} \longrightarrow \operatorname{coker} d^{n-2} \longrightarrow 0$,

I obtain:-

$$\begin{array}{ccccccc} (E)\ (M^{n-2})_B & \xrightarrow{d^{n-2}\otimes 1_B} & (M^{n-1})_B & \longrightarrow & (\operatorname{coker} d^{n-2})\otimes_A B & \longrightarrow & 0 \\ \| & & \| & & \vartheta \downarrow \cong & & \\ (E)\ (M^{n-2})_B & \xrightarrow{d^{n-2}\otimes 1_B} & (M^{n-1})_B & \longrightarrow & \operatorname{coker}(d^{n-2}\otimes 1_B) & \longrightarrow & 0\ (C). \end{array}$$

It follows that there is a unique A- and B-module isomorphism $\vartheta:(\operatorname{coker} d^{n-2})\otimes_A B \longrightarrow \operatorname{coker}(d^{n-2}\otimes 1_B)$ which will make the completed diagram commute. In fact, $\vartheta(\bar{m}\otimes b) = \widetilde{m\otimes b}$. For each $\underline{p} \in \operatorname{Supp}(M)$ of M-height n, there is an A-module isomorphism induced from ϑ:-

$$\vartheta_{\underline{p}}:((\operatorname{coker} d^{n-2})\otimes_A B)_{\underline{p}} \xrightarrow{\cong} (\operatorname{coker}(d^{n-2}\otimes 1_B))_{\underline{p}}.$$

The direct sum of these isomorphisms gives an A-isomorphism:-

$$\phi^n: \bigoplus_{\substack{\underline{p}\in \operatorname{Supp}(M) \\ \operatorname{ht}_M \underline{p} = n}} ((\operatorname{coker} d^{n-2})\otimes_A B)_{\underline{p}} \xrightarrow{\cong} \bigoplus_{\substack{\underline{p}\in \operatorname{Supp}(M) \\ \operatorname{ht}_M \underline{p} = n}} (\operatorname{coker}(d^{n-2}\otimes 1_B))_{\underline{p}}$$

which is s.t. $\phi^n(((\bar{m}_{\underline{p}}\otimes b_{\underline{p}})/s_{\underline{p}})) = (\widetilde{(m_{\underline{p}}\otimes b_{\underline{p}})}/s_{\underline{p}})$.

Also, there is an A-isomorphism $\beta^n:(M^n)_B = \left[\bigoplus_{\substack{\underline{p}\in \operatorname{Supp}(M) \\ \operatorname{ht}_M \underline{p} = n}} (\operatorname{coker} d^{n-2})_{\underline{p}}\right] B$

$$\xrightarrow{\cong} \bigoplus_{\substack{\underline{p}\in \operatorname{Supp}(M) \\ \operatorname{ht}_M \underline{p} = n}} ((\operatorname{coker} d^{n-2})_{\underline{p}}\otimes_A B) \xrightarrow{\cong} \bigoplus_{\substack{\underline{p}\in \operatorname{Supp}(M) \\ \operatorname{ht}_M \underline{p} = n}} ((\operatorname{coker} d^{n-2}\otimes_A B)\otimes_{A_{\underline{p}}})$$

$$\xrightarrow{\cong} \bigoplus_{\substack{\underline{p}\in \operatorname{Supp}(M) \\ \operatorname{ht}_M \underline{p} = n}} ((\operatorname{coker} d^{n-2}\otimes_A B)_{\underline{p}}, \text{ which is s.t. } \beta^n((\bar{m}_{\underline{p}}/s_{\underline{p}})\otimes b) =$$

$(\bar{m}_{\underline{p}}\otimes b)/s_{\underline{p}}$. α^n is the composition of β^n and the isomorphism ϕ^n above.

(Of course, in examining ρ^n, I have used (1.3). Not every element of $(M^n)_B$ is of the form $(\overline{m}_{\underline{p}}/s_{\underline{p}})\otimes b$, of course, but the elements of this form generate it.)

This shows that α^n is a homomorphism of Abelian groups which is both 1-1 and onto. It is easy to see that it is a B-module homomorphism. //

It is necessary to compare $\mathrm{Supp}_B(M_B)$ with $\mathrm{Supp}_A(M)$.

(3.10)Lemma. If $\underline{q} \in \mathrm{Supp}_B(M_B)$, then $\underline{p} = \underline{q}^c \in \mathrm{Supp}_A(M)$.

Proof. f induces a ring homomorphism $f':A_{\underline{p}} \longrightarrow B_{\underline{q}}$ defined by $f'(a/s) = f(a)/f(s)$. If $\sigma:A \longrightarrow A_{\underline{p}}$ and $\rho:B \longrightarrow B_{\underline{q}}$ are the natural ring homomorphisms, then:-

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \sigma\downarrow & & \downarrow\rho \\ A_{\underline{p}} & \xrightarrow{f'} & B_{\underline{q}} \end{array} \qquad (C).$$

If $\underline{q} \in \mathrm{Supp}_B(M_B)$, $(M\otimes_A B)\otimes_B B_{\underline{q}} \neq 0$. Thus, when $B_{\underline{q}}$ is regarded as an A-module by means of $\rho.f = f'.\sigma$, $M\otimes_A B_{\underline{q}} \neq 0$. Therefore $M\otimes_A(A_{\underline{p}}\otimes_{A_{\underline{p}}} B_{\underline{q}}) \neq 0$, and so $M\otimes_A A_{\underline{p}} \neq 0$, i.e. $\underline{p} \in \mathrm{Supp}_A(M)$. //

(3.11)Theorem. Suppose that A, B, f, and M satisfy the following two conditions:-

(X1) If $\underline{q} \in \mathrm{Supp}_B(M_B)$, then (from (3.10), $\underline{q}^c \in \mathrm{Supp}_A(M)$ and) $\mathrm{ht}_M \underline{q}^c \leqslant \mathrm{ht}_{M_B} \underline{q}$.

(X2) If $\underline{p} \in \mathrm{Supp}_A(M)$, the set $\mathcal{E}(\underline{p}) = \left\{\underline{q} \in \mathrm{Supp}_B(M_B) \;\middle|\; \underline{q}^c = \underline{p} \text{ and } \mathrm{ht}_{M_B}\underline{q} = \mathrm{ht}_M \underline{p}\right\}$ is finite.

Then there is a homomorphism $\mathcal{H}:\underline{C(M)}\otimes_A B \longrightarrow \underline{C_B(M_B)}$ of complexes of B-modules s.t. $\chi^{-1}:M_B \longrightarrow M_B$ is the identity.

Proof. As usual, it will be convenient to write $M^{-2} = 0$, etc. Define $\chi^{-2} = 0:(M^{-2})_B \longrightarrow (M_B)^{-2}$ and $\chi^{-1} = 1_{M_B}:(M^{-1})_B \longrightarrow (M_B)^{-1}$, where 1_{M_B}

denotes the identity on M_B. I shall construct a family $\{\chi^n\}_{n\geq -2}$ of B-homomorphisms with the required properties by induction on n, χ^{-2} and χ^{-1} providing a basis for this induction.

Suppose inductively, then, that $n \geq 0$ and B-homomorphisms $\chi^{-2} = 0$, $\chi^{-1} = 1_{M_B}, \chi^0, \ldots, \chi^{n-1}$ such that:-

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M_B & \xrightarrow{d^{-1}\otimes 1_B} & (M^0)_B & \longrightarrow & \cdots & \longrightarrow & (M^{n-2})_B & \xrightarrow{d^{n-2}\otimes 1_B} & (M^{n-1})_B \\ \| & & \| & & \downarrow \chi^0 & & & & \downarrow \chi^{n-2} & & \downarrow \chi^{n-1} \\ 0 & \longrightarrow & M_B & \xrightarrow{e^{-1}} & (M_B)^0 & \longrightarrow & \cdots & \longrightarrow & (M_B)^{n-2} & \xrightarrow{e^{n-2}} & (M_B)^{n-1} \end{array} \quad (C)$$

have been constructed.

The above diagram induces a B-homomorphism $\overline{\chi^{n-1}}$: coker$(d^{n-2}\otimes 1_B) \longrightarrow$ coker e^{n-2}, where $\overline{\chi^{n-1}}(\tilde{x}) = \widetilde{\chi^{n-1}(x)}$ (where the second $\sim$ denotes natural images in coker e^{n-2}).

Recall the B-isomorphism of (3.9):-

$$\alpha^n : (M^n)_B \longrightarrow \bigoplus_{\substack{\underline{p}\in \mathrm{Supp}(M) \\ \mathrm{ht}_M \underline{p} = n}} (\mathrm{coker}(d^{n-2}\otimes 1_B))_{\underline{p}}.$$

For each $\underline{p} \in \mathrm{Supp}(M)$ with M-height n, construct a B-module homomorphism $\gamma_{\underline{p}} : (\mathrm{coker}(d^{n-2}\otimes 1_B))_{\underline{p}} \longrightarrow \bigoplus_{\underline{q}\in \mathcal{E}(\underline{p})} (\mathrm{coker}(d^{n-2}\otimes 1_B))_{\underline{q}}$ as follows:-

If $\mathcal{E}(\underline{p}) = \emptyset$, define $\gamma_{\underline{p}} = 0$. If $\mathcal{E}(\underline{p}) \neq \emptyset$ and $x \in \mathrm{coker}(d^{n-2}\otimes 1_B)$, $s \in A - \underline{p}$, define $\gamma_{\underline{p}}(x/s)$ to be the element whose component in the direct summand $(\mathrm{coker}(d^{n-2}\otimes 1_B))_{\underline{q}}$ is $x/f(s)$. This is a sensible definition of a B-homomorphism: it is here that I use the finiteness of $\mathcal{E}(\underline{p})$.

Form the direct sum of the $\gamma_{\underline{p}}$'s to obtain a B-hom.:-

$$\lambda^n : \bigoplus_{\substack{\underline{p}\in \mathrm{Supp}(M) \\ \mathrm{ht}_M \underline{p} = n}} (\mathrm{coker}(d^{n-2}\otimes 1_B))_{\underline{p}} \longrightarrow \bigoplus_{\substack{\underline{p}\in \mathrm{Supp}(M) \\ \mathrm{ht}_M \underline{p} = n \\ \underline{q}\in \mathcal{E}(\underline{p})}} (\mathrm{coker}(d^{n-2}\otimes 1_B))_{\underline{q}}$$

But $\bigcup_{\substack{\underline{p}\in Supp(M)\\ ht_M \underline{p} = n}} \mathcal{E}(\underline{p})$ is a set of primes of $Supp_B(M_B)$ all of which have M_B-height n. So there is a B-homomorphism:-

$$\mu^n: \bigoplus_{\substack{\underline{p}\in Supp(M)\\ ht_M \underline{p} = n\\ \underline{q}\in \mathcal{E}(\underline{p})}} (coker(d^{n-2}\otimes 1_B))_{\underline{q}} \longrightarrow \bigoplus_{\substack{\underline{q}\in Supp_B(M_B)\\ ht_{M_B} \underline{q} = n}} (coker(d^{n-2}\otimes 1_B))_{\underline{q}}.$$

Finally, $\widetilde{\chi^{n-1}}$ induces a B-homomorphism:-

$$\nu^n: \bigoplus_{\substack{\underline{q}\in Supp_B(M_B)\\ ht_{M_B} \underline{q} = n}} (coker(d^{n-2}\otimes 1_B))_{\underline{q}} \longrightarrow \bigoplus_{\substack{\underline{q}\in Supp_B(M_B)\\ ht_{M_B} \underline{q} = n}} (coker\ e^{n-2})_{\underline{q}} = (M_B)^n.$$

Define $\chi^n: (M^n)_B \longrightarrow (M_B)^n$ to be the composition $\nu^n.\mu^n.\lambda^n.\alpha^n$.

It remains to check that:-

$$\begin{array}{ccc} (M^{n-1})_B & \xrightarrow{d^{n-1}\otimes 1_B} & (M^n)_B \\ {\scriptstyle \chi^{n-1}}\downarrow & & \downarrow{\scriptstyle \chi^n} \\ (C)\quad (M_B)^{n-1} & \xrightarrow{e^{n-1}} & (M_B)^n \end{array} \qquad (1),$$

and to check this it is sufficient to show that $e^{n-1}\chi^{n-1}(y) = \chi^n(d^{n-1}\otimes 1_B)(y)$ for elements y of $(M^{n-1})_B$ of the form $x\otimes b$ with $x \in M^{n-1}$ and $b \in B$. Suppose $y \in (M^{n-1})_B$ is of this form. Then it is straightfoward to check that, if $\underline{q} \in Supp_B(M_B)$ & $ht_{M_B} \underline{q} = n$,

the component of $e^{n-1}\chi^{n-1}(y)$ in $(coker\ e^{n-2})_{\underline{q}}$ is $\widetilde{\chi^{n-1}(y)}/1$ for all such $\underline{q}$

the component of $\chi^n(d^{n-1}\otimes 1_B)(y)$ in $(coker\ e^{n-2})_{\underline{q}}$ is $\widetilde{\chi^{n-1}(y)}/1$ if $\underline{q} \in \mathcal{E}(\underline{q}^c)$

the component of $\chi^n(d^{n-1}\otimes 1_B)(y)$ in $(coker\ e^{n-2})_{\underline{q}}$ is 0 if $\underline{q} \notin \mathcal{E}(\underline{q}^c)$.

(Observe that $\underline{q} \in \mathcal{E}(\underline{p})$ for some $\underline{p} \in Supp(M)$ if and only if $\underline{q} \in \mathcal{E}(\underline{q}^c)$.)

Thus, from condition (X1), the only way in which an element $\underline{q} \in Supp_B(M_B)$ can fail to belong to $\mathcal{E}(\underline{q}^c)$ is for $ht_M \underline{q}^c$ to be strictly less that $ht_{M_B} \underline{q}$.

Thus, it is sufficient to show that, if $\underline{q} \in Supp_B(M_B)$ and $ht_{M_B} \underline{q} = n$ but

$ht_M \mathfrak{q}^c < n$, then $\widetilde{\chi^{n-1}(y)}/1 = 0$ in $(\operatorname{coker} e^{n-2})_{\mathfrak{q}}$. This I now do.

$y = x \otimes b$ with $x \in M^{n-1}$ and $b \in B$ (see above remarks). Since $ht_M \mathfrak{q}^c < n$, $\mathfrak{q}^c \in V(A) - U^n(M)$, so by (3.7)(v), $(d^{n-2}M^{n-2} : x) \not\subseteq \mathfrak{q}^c$. Thus there is an element $a \in A - \mathfrak{q}^c$ s.t. $a.x = d^{n-2}(z)$ for some $z \in M^{n-2}$. $f(a) \notin \mathfrak{q}$, &:-

$$f(a).\chi^{n-1}(y) = \chi^{n-1}(f(a).(x \otimes b)) = \chi^{n-1}(x \otimes f(a).b) = \chi^{n-1}(a.x \otimes b)$$
$$= \chi^{n-1}((d^{n-2} \otimes 1_B)(z \otimes b)) = e^{n-2}\chi^{n-2}(z \otimes b).$$

Therefore $f(a)\widetilde{\chi^{n-1}(y)} = 0$, and so, since $f(a) \notin \mathfrak{q}$, $\widetilde{\chi^{n-1}(y)}/1 = 0$ in $(\operatorname{coker} e^{n-2})_{\mathfrak{q}}$.

Consequently, diagram (1) commutes and the inductive step is complete. Thus the Theorem is proved by induction. //

In the applications of this Theorem, it will be useful to maintain the notation of (3.9) and (3.11). One useful observation can already be made:-

(3.12)Remark. If $\mathfrak{p} \in \operatorname{Supp}(M)$, $ht_M \mathfrak{p} = n$, and $\mathfrak{p}^e = B$, then $(\operatorname{coker}(d^{n-2} \otimes 1_B))_{\mathfrak{p}} = 0$, and consequently $\bigoplus_{\substack{\mathfrak{p} \in \operatorname{Supp}(M) \\ ht_M \mathfrak{p} = n}} (\operatorname{coker}(d^{n-2} \otimes 1_B))_{\mathfrak{p}} =$

$$\bigoplus_{\substack{\mathfrak{p} \in \operatorname{Supp}(M) \\ ht_M \mathfrak{p} = n \\ \mathfrak{p}^e \neq B}} (\operatorname{coker}(d^{n-2} \otimes 1_B))_{\mathfrak{p}}.$$

Proof. From the proof of (3.9), $\operatorname{coker}(d^{n-2} \otimes 1_B) \cong (\operatorname{coker} d^{n-2}) \otimes_A B$, so it is sufficient to show that $(\operatorname{coker} d^{n-2})_{\mathfrak{p}} \otimes_A B = 0$. Suppose $x \in (\operatorname{coker} d^{n-2})_{\mathfrak{p}}$. By (3.7)(vi), there is a positive integer N s.t. $\mathfrak{p}^N.x = 0$. Since $\mathfrak{p}^e = B$, there are elements $b_1, \ldots, b_k \in B$, $a_1, \ldots, a_k \in \mathfrak{p}^N$ $(k \geq 1)$ s.t. $\sum_{i=1}^{k} b_i.f(a_i) = 1$. Then, for $b \in B$, $1.(x \otimes b) = \sum_{i=1}^{k} b_i f(a_i)(x \otimes b) =$

$$\sum_{i=1}^{k} (a_i x) \otimes (b_i b) = 0.$$

Thus $(\operatorname{coker}(d^{n-2} \otimes 1_B))_{\mathfrak{p}} = 0$. //

I am now ready to reap the benefits of the hard and unpleasant work of (3.9) and (3.11). Probably the most useful application is one in which substantial simplifications are available, but there is at least one application in which the full generality of (3.11) is needed. I deal first with the case in which $B = S^{-1}A$, where S is an m.c.s. of A.

(3.13)First Application of Theorem (3.11). Let A be a ring, S an m.c.s. of A, M an A-module, $\underline{C(M)}$: $0 \longrightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} \ldots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \ldots$ the Cousin Complex for M, and:-

$\underline{C_{S^{-1}A}(S^{-1}M)}$: $0 \longrightarrow S^{-1}M \xrightarrow{e^{-1}} (S^{-1}M)^0 \xrightarrow{e^0} \ldots \longrightarrow (S^{-1}M)^n \xrightarrow{e^n} (S^{-1}M)^{n+1} \ldots$ the Cousin Complex for the $S^{-1}A$-module $S^{-1}M$. Then $\underline{C_{S^{-1}A}(S^{-1}M)}$ and

$\underline{S^{-1}(C(M))}$: $0 \longrightarrow S^{-1}M \xrightarrow{S^{-1}(d^{-1})} S^{-1}(M^0) \longrightarrow \ldots \longrightarrow S^{-1}(M^n) \xrightarrow{S^{-1}(d^n)} S^{-1}(M^{n+1}) \ldots$ (recall (1.3)) are isomorphic complexes of $S^{-1}A$-modules and -homomorphisms.

Proof. It is clear from (1.3) that $S^{-1}(\underline{C(M)})$ and $\underline{C(M)} \otimes_A S^{-1}A$ are isomorphic complexes of $S^{-1}A$-modules and -homomorphisms. So in (3.11) I take $B = S^{-1}A$ and $f: A \longrightarrow S^{-1}A$ to be the natural homomorphism. I must check that conditions (X1) and (X2) are satisfied: I use (1.6) to see that this is so.

If $\underline{q} = S^{-1}\underline{p} \in \mathrm{Supp}_{S^{-1}A}(S^{-1}M)$ (with $\underline{p} \in V(A)$: recall (1.2)), then $\underline{p} \in \mathrm{Supp}_A(M)$, $\underline{p} \cap S = \emptyset$, and $\mathrm{ht}_M \underline{p} = \mathrm{ht}_{S^{-1}M} \underline{q}$, so (X1) is satisfied.

If $\underline{p} \in \mathrm{Supp}(M)$, $\mathcal{E}(\underline{p}) = \emptyset$ if $\underline{p} \cap S \neq \emptyset$, while if $\underline{p} \cap S = \emptyset$, $\mathcal{E}(\underline{p}) = \{S^{-1}\underline{p}\}$. Thus (X2) is satisfied. So I may apply (3.11).

Next I show that the homomorphism of complexes $\Psi: \underline{C(M)} \otimes_A S^{-1}A \longrightarrow \underline{C_{S^{-1}A}(S^{-1}M)}$ which is obtained from (3.11) is an isomorphism. χ^{-1} clearly is an isomorphism, so I suppose, inductively, that $n \geq 0$, and $\chi^{-1}, \chi^0, \ldots, \chi^{n-1}$ have all been shown to be isomorphisms. Immediately I see that $\overline{\chi^{n-1}}$ is an

isomorphism, so (using the notation of (3.11)), ν^n is an isomorphism. α^n is an isomorphism anyway (see (3.9)). Since every element $\underline{q}$ of $\mathrm{Supp}_{S^{-1}A}(S^{-1}M)$ with $S^{-1}M$-height n belongs to $\mathcal{E}(\underline{q}^c)$, μ^n is an isomorphism. To show λ^n is an isomorphism, it is sufficient to show that, if $\underline{p} \in \mathrm{Supp}(M)$ with M-height n, the homomorphism:-

$$\gamma_{\underline{p}} : (\mathrm{coker}(d^{n-2} \otimes 1_{S^{-1}A}))_{\underline{p}} \longrightarrow \bigoplus_{\underline{q} \in \mathcal{E}(\underline{p})} (\mathrm{coker}(d^{n-2} \otimes 1_{S^{-1}A}))_{\underline{q}} \qquad (1)$$

is an isomorphism.

Two cases arise. If $\underline{p} \cap S \neq \emptyset$, $S^{-1}\underline{p} = S^{-1}A$, so by (3.12) $(\mathrm{coker}(d^{n-2} \otimes 1_{S^{-1}A}))_{\underline{p}} = 0$. But in this case $\mathcal{E}(\underline{p}) = \emptyset$, so both sides of (1) are zero and $\gamma_{\underline{p}}$ is trivially isomorphic.

Secondly, if $\underline{p} \cap S = \emptyset$, $\mathcal{E}(\underline{p}) = \{S^{-1}\underline{p}\}$, and the map $\gamma_{\underline{p}}$ of (1) is precisely the isomorphism of (1.9). (Observe that there is no ambiguity in the manner in which $(\mathrm{coker}(d^{n-2} \otimes 1_{S^{-1}A}))_{\underline{p}}$ is regarded as an $S^{-1}A$-module, for if $x \in \mathrm{coker}(d^{n-2} \otimes 1_{S^{-1}A})$, and $t \in A - \underline{p}$, $a \in A$, and $s \in S$, $((a/s).x)/t = ((a/1).x)/s.t$.)

(This result was the reason for including (1.9).)

Thus $\varkappa^n$ is an isomorphism, and so, by induction, Ψ is an isomorphism of complexes. //

(3.13) will be very useful in the subsequent work: I shall use it most frequently in the case when S is the complement of a prime ideal. There's another observation that can be made about this case.

(3.14)Remark. Suppose M is an A-module and $\underline{p} \in \mathrm{Supp}(M)$ has M-height n. Then, if $i > n$, $(M_{\underline{p}})^i = 0$, while $(M_{\underline{p}})^n \cong (\mathrm{coker}\, d^{n-2})_{\underline{p}}$.

Proof. The first part is obvious from (1.6) and the construction of the Cousin Complex, since no prime in $\mathrm{Supp}_{A_{\underline{p}}}(M_{\underline{p}})$ can have $M_{\underline{p}}$-height greater

than n.

For the second part, I use (3.13): $(M_{\underline{p}})^n \cong (M^n)_{\underline{p}} \cong$ $\bigoplus_{\substack{\underline{q} \in \mathrm{Supp}(M) \\ \mathrm{ht}_M \underline{q} = n}} ((\mathrm{coker}\ d^{n-2})_{\underline{q}}) \otimes_A A_{\underline{p}}$. But, if $\underline{q} \in \mathrm{Supp}(M)$, $\underline{q} \neq \underline{p}$, and $\mathrm{ht}_M \underline{q} = n$, $\underline{q} \not\subseteq \underline{p}$, so, by the proof of (3.12), $(M^n)_{\underline{p}} \cong (\mathrm{coker}\ d^{n-2})_{\underline{p}} \otimes_A A_{\underline{p}} \cong (\mathrm{coker}\ d^{n-2})_{\underline{p}}$. //

(3.15) Second Application of Theorem (3.11). Suppose that A, B, and f are such that B when regarded as an A-module by means of f is flat (e.g. $B = S^{-1}A$ for an m.c.s. S of A). This situation has been studied recently by Iversen (15). If I take M = A, then $M \otimes_A B = A \otimes_A B$ is B-isomorphic to B: Supp(M) becomes V(A) and $\mathrm{Supp}_B(M \otimes_A B)$ becomes V(B).

Suppose $\underline{q} \in V(B)$. Then by Theorem 2 part (1) of (15), $\mathrm{ht}_A \underline{q}^c \leq \mathrm{ht}_B \underline{q}$. Thus condition (X1) is satisfied.

To see that (X2) is satisfied, suppose $\underline{p} \in V(A)$. Then if $\underline{q} \in \mathcal{E}(\underline{p})$, $\underline{q}^c = \underline{p}$ and $\mathrm{ht}_B \underline{q} = \mathrm{ht}_A \underline{p}$, and so, by Theorem 2 part (1) of (15) again, $\underline{q}$ is a minimal element of $\mathrm{Ass}_B(B/\underline{p}^e)$. Consequently, $\mathcal{E}(\underline{p})$ is finite. Thus Theorem (3.11) is applicable.

As this situation will be considered in Chapter 8, I'll leave further discussion until then.

(3.16) Third Application of Theorem (3.11). Suppose A and B are integral domains with A a subring of B and $f: A \longrightarrow B$ the inclusion. Suppose that B is integral over A and that A is integrally closed. Again, take M = A.

Suppose that $\underline{q} \in V(B)$ and $\underline{q}^c = \underline{p}$. If $\mathrm{ht}_B \underline{q} = n$ and $\underline{q} = \underline{q}_n \supsetneq \underline{q}_{n-1} \supsetneq \cdots \supsetneq \underline{q}_0$ is a maximal saturated chain of primes of B, then, by (5.9) of Atiyah and Macdonald (1), $\underline{p} = \underline{q}_n^c \supsetneq \underline{q}_{n-1}^c \supsetneq \cdots \supsetneq \underline{q}_0^c$

so $ht_A \underline{p} \geq ht_B \underline{q}$. On the other hand, use the "Going Down" Theorem, (5.16) of (1) to see that $ht_A \underline{p} \leq ht_B \underline{q}$. Thus $ht_A \underline{p} = ht_B \underline{q}$ and (X1) is satisfied.

To show that (X2) is satisfied, I show that, for $\underline{q} \in V(B)$ and $\underline{p} \in V(A)$, $\underline{q}^c = \underline{p}$ if and only if $\underline{q}$ is a minimal element of $Ass_B(B/\underline{p}^e)$. If $\underline{q}^c = \underline{p}$, and $\underline{q} \supsetneq \underline{q}' \supseteq \underline{p}^e$ with $\underline{q}' \in Ass_B(B/\underline{p}^e)$, then $\underline{p} = \underline{q}^c \supseteq \underline{q}'^c \supseteq \underline{p}^{ec} \supseteq \underline{p}$, a contradiction to (5.9) of (1). Thus $\underline{q}$ is a minimal element of $Ass_B(B/\underline{p}^e)$.

Conversely, if $\underline{q}$ is a minimal element of $Ass_B(B/\underline{p}^e)$, $\underline{q}^c \supseteq \underline{p}^{ec} \supseteq \underline{p}$. If $\underline{q}^c \supsetneq \underline{p}$, then, by the "Going Down" Theorem again, there is a prime $\underline{q}'$ of B s.t. $\underline{q}' \subsetneq \underline{q}$ and $\underline{q}'^c = \underline{p}$. Therefore $\underline{p}^e \subseteq \underline{q}' \subsetneq \underline{q}$, a contradiction. Thus $\underline{q}^c = \underline{p}$.

It follows that for $\underline{p} \in V(A)$, $\mathcal{E}(\underline{p})$ is finite, and so (X2) is satisfied. Theorem (3.11) can now be applied, Again, further remarks about this situation will be made in Chapter 8.

(3.17)Fourth Application of Theorem (3.11). Suppose that A is a ring, M is a f.-g. A-module, and $\underline{b}$ is an ideal of A contained in $(0:M)_A$. By (3.7)(ix), $\underline{b}$ annihilates all the terms of $\underline{C(M)}$, and so M and the terms and homomorphisms of $\underline{C(M)}$ can be regarded as $A/\underline{b}$-modules and homomorphisms. When this is done, $\underline{C_A(M)}$ provides the Cousin Complex for the $A/\underline{b}$-module M (up to isomorphism of complexes).

Proof. First note that, if N is any A-module annihilated by $\underline{b}$, then the natural epimorphism $A \longrightarrow A/\underline{b}$ induces an isomorphism $N \otimes_A A \xrightarrow{\cong} N \otimes_A A/\underline{b}$ (since the image of $N \otimes_A \underline{b}$ in (E) $N \otimes_A \underline{b} \longrightarrow N \otimes_A A \longrightarrow N \otimes_A A/\underline{b} \longrightarrow 0$ is zero). Thus the complexes $\underline{C(M)}$ and $\underline{C(M)} \otimes_A A/\underline{b}$ are isomorphic (both as complexes of A-modules and as complexes of $A/\underline{b}$-modules.)

Thus I take $B = A/\underline{b}$ and $f:A \longrightarrow B$ to be the natural surjective ring homomorphism. Write M' for M when regarded as an $A/\underline{b}$-module. Since M is f.-g., $\mathrm{Supp}_A(M) = V(M)$: observe that, by (1.16), any $\underline{p} \in V_A(M)$ must contain $\underline{b}$. (1.8) implies that $V_B(M') = \left\{\underline{p}/\underline{b} \,\middle|\, \underline{p} \in V_A(M)\right\}$.

If $\underline{q} \in V_B(M')$, $\mathrm{ht}_{M'}\underline{q} = \mathrm{ht}_M \underline{q}^c$, and if $\underline{p} \in V_A(M)$, $\xi(\underline{p}) = \left\{\underline{p}/\underline{b}\right\}$. Since $M \otimes_A B \cong M'$, it follows that conditions (X1) and (X2) are satisfied, and there is, by (3.11), a homomorphism of complexes of B-modules:-

$$\Psi : \underline{C_A(M)} \otimes_A A/\underline{b} \longrightarrow \underline{C_{A/\underline{b}}(M')}$$

To see that Ψ is an isomorphism, note that, if $n \geqslant 0$, μ^n is an isomorphism, while if $\underline{p} \in V_A(M)$ has M-height n, $\gamma_{\underline{p}} : (\mathrm{coker}(d^{n-2} \otimes 1_{A/\underline{b}}))_{\underline{p}} \longrightarrow (\mathrm{coker}(d^{n-2} \otimes 1_{A/\underline{b}}))_{\underline{p}/\underline{b}}$ is a B-isomorphism by (1.8) again. It follows easily by induction that Ψ is an isomorphism of complexes. //

When M is a f.-g. A-module, $V(M) = V(A/(0:M)_A)$ (by (1.16)): one may ask whether there is a relation between $\underline{C_A(M)}$ & $\underline{C_A(A/(0:M)_A)}$.

(3.18)Proposition. Suppose that A is a ring, M is a non-zero, f.-g., A-module, and $\underline{a} = (0:M)_A$. Then there is an isomorphism of complexes:-

$$\Psi : \underline{C_A(A/\underline{a})} \otimes_A M \longrightarrow \underline{C_A(M)}.$$

Proof. Suppose $\underline{C_A(M)} : 0 \longrightarrow M \xrightarrow{e^{-1}} M^0 \longrightarrow \ldots \longrightarrow M^n \xrightarrow{e^n} M^{n+1} \ldots$ and $\underline{C_A(A/\underline{a})} : 0 \longrightarrow A/\underline{a} \xrightarrow{d^{-1}} B^0 \longrightarrow \ldots \longrightarrow B^n \xrightarrow{d^n} B^{n+1} \longrightarrow \ldots$ are the Cousin Complexes for M and $A/\underline{a}$ (= B) respectively. Since $\underline{a}.M = 0$, (1.17) provides an isomorphism $\chi^{-1} : B^{-1} \otimes_A M \xrightarrow{\cong} M^{-1}$, and χ^{-1} and $0 : B^{-2} \otimes_A M \longrightarrow M^{-2}$ provide a basis for the following induction.

I suppose inductively that $n \geqslant 0$ and I have constructed isomorphisms $\chi^{-2}, \chi^{-1}, \ldots, \chi^{n-1}$ s.t.:-

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & B\otimes_A M & \xrightarrow{d^{-1}\otimes 1_M} & B^0\otimes_A M & \longrightarrow \cdots \longrightarrow & B^{n-2}\otimes_A M & \xrightarrow{d^{n-2}\otimes 1_M} & B^{n-1}\otimes_A M \\
\| & & \chi^{-2}\ \cong\downarrow\chi^{-1} & & \cong\downarrow\chi^{0} & & \cong\downarrow\chi^{n-2} & & \cong\downarrow\chi^{n-1} \\
0 & \longrightarrow & M & \xrightarrow{e^{-1}} & M^0 & \longrightarrow \cdots \longrightarrow & M^{n-2} & \xrightarrow{e^{n-2}} & M^{n-1}
\end{array}
\qquad (C)
$$

1_M, of course, denotes the identity map on M. $V_A(M) = V_A(B)$. $B^n\otimes_A M = \left[\bigoplus_{\substack{\underline{p}\in V(M)\\ ht_M\underline{p}=n}} (\operatorname{coker} d^{n-2})_{\underline{p}}\right]\otimes_A M$. So there is a natural isomorphism:-

$$\rho^n : B^n\otimes_A M \xrightarrow{\cong} \bigoplus_{\substack{\underline{p}\in V(M)\\ ht_M\underline{p}=n}} ((\operatorname{coker} d^{n-2})\otimes_A M)_{\underline{p}}.$$

Now, if $\sim$ and $^-$ are used to denote natural images in $\operatorname{coker}(d^{n-2}\otimes 1_M)$ and $\operatorname{coker} d^{n-2}$ resp., there is an isomorphism:-

$$w^n : \operatorname{coker} d^{n-2}\otimes_A M \xrightarrow{\cong} \operatorname{coker}(d^{n-2}\otimes 1_M) \text{ s.t., if } x\in B^{n-1}, \& m\in M, \quad w^n(\bar{x}\otimes m) = \widetilde{x\otimes m}.$$

χ^{n-2} and χ^{n-1} induce an isomorphism:-

$$\overline{\chi^{n-1}} : \operatorname{coker}(d^{n-2}\otimes 1_M) \xrightarrow{\cong} \operatorname{coker} e^{n-2}, \text{ and so } \overline{\chi^{n-1}}.w^n \text{ induces:-}$$

$$\sigma^n : \bigoplus_{\substack{\underline{p}\in V(M)\\ ht_M\underline{p}=n}} ((\operatorname{coker} d^{n-2})\otimes_A M)_{\underline{p}} \xrightarrow{\cong} \bigoplus_{\substack{\underline{p}\in V(B)\\ ht_B\underline{p}=n}} (\operatorname{coker} e^{n-2})_{\underline{p}} = M^n.$$

Define $\chi^n = \sigma^n.\rho^n : B^n\otimes_A M \xrightarrow{\cong} M^n$. χ^n has the required commutativity properties and the result follows by induction. //

CHAPTER 4

A Characterization of Cohen-Macaulay Modules in Terms of the Cousin Complex.

(Reminder: "ring" means "commutative Noetherian ring with a non-zero identity".)

Throughout this Chapter, A will denote a (fixed) ring.

The chief aim of this Chapter is to prove that a non-zero, f.-g., A-module M is Cohen-Macaulay if and only if its Cousin Complex C(M) is exact (so that, in particular, A is a Cohen-Macaulay ring if and only if C(A) is exact). However, before a proof of this can be achieved, a discussion of some of the properties of Cohen-Macaulay modules is necessary.

In (23) and (24), Rees introduces the notion of grade and discusses its important connections with Cohen-Macaulay rings. For a non-zero, f.-g., A-module M, an easy and natural generalization of grade, called M-grade, is available: the theory of the M-grade is very relevant to the theory of Cohen-Macaulay modules, and will be helpful both in this Chapter and later in the thesis. Consequently, (and also because I have been unable to find such an account in the existing literature) this Chapter begins with an exposition of the theory of the M-grade.

(4.1) Definition. Let M be an A-module. I shall use $\underline{x} = (x_1,\ldots,x_r)$ to denote both a sequence $x_1,\ldots,x_r$ of elements of A and the ideal they generate. $\underline{x}$ is s.t.b. an M-sequence (of length r) if $\underline{x}M \neq M$ and x_i is not a zero-divisor on $M/(x_1,\ldots,x_{i-1}).M$ for each $i = 1,\ldots,r$ (so that, in particular, x_1 is not a zero-divisor on M).

(4.2)Proposition. (Stated as Proposition (2.9) of Bass (5).) Let M & N be two A-modules. Suppose $\underline{x} = (x_1,\ldots,x_n)$ is an M-sequence contained in $(0:N)_A$. Then:- $\mathrm{Ext}^i_A(N,M) = 0$ if $i < n$,

& $\mathrm{Ext}^n_A(N,M) \cong \mathrm{Hom}_A(N,M/\underline{x}M)$.

Proof. I use induction on n. Suppose first that n = 1.

From (E) $0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$ I deduce

(E) $0 \longrightarrow \mathrm{Hom}_A(N,M) \xrightarrow{x_1} \mathrm{Hom}_A(N,M) \longrightarrow \mathrm{Hom}_A(N,M/x_1M) \longrightarrow \mathrm{Ext}^1_A(N,M) \xrightarrow{x_1} \mathrm{Ext}^1_A(N,M) \longrightarrow \ldots$.

Because each $\mathrm{Ext}^i_A(\ ,\ )$ is an A-linear functor (see (2.1) and (2.2)), the induced homomorphisms $\mathrm{Hom}_A(N,M) \longrightarrow \mathrm{Hom}_A(N,M)$ and $\mathrm{Ext}^i_A(N,M) \longrightarrow \mathrm{Ext}^i_A(N,M)$ are all multiplication by x_1. Since $x_1 \in (0:N)_A$, each of these induced homomorphisms must be zero. Therefore $\mathrm{Hom}_A(N,M) = 0$ and $\mathrm{Hom}_A(N,M/x_1M) \cong \mathrm{Ext}^1_A(N,M)$.

Suppose, inductively, that the result is true when $n = k-1 \geqslant 1$, and suppose $\underline{x} = (x_1,\ldots,x_k)$ is an M-sequence contained in $(0:N)_A$. Let $M' = M/(x_1,\ldots,x_{k-1})M$: by the inductive hypothesis, $\mathrm{Ext}^i_A(N,M) = 0$ if $i < k-1$ and $\mathrm{Ext}^{k-1}_A(N,M) \cong \mathrm{Hom}_A(N,M')$.

Now (x_k) is an M'-sequence, so by the case n = 1 (proved above), $\mathrm{Hom}_A(N,M') = 0$. Reconsider the long exact sequence:-

(E) $0 \longrightarrow \mathrm{Hom}_A(N,M) \xrightarrow{x_1} \mathrm{Hom}_A(N,M) \longrightarrow \mathrm{Hom}_A(N,M/x_1M) \longrightarrow \ldots$

$\ldots \longrightarrow \mathrm{Ext}^{k-1}_A(N,M) \longrightarrow \mathrm{Ext}^{k-1}_A(N,M/x_1M) \longrightarrow \mathrm{Ext}^k_A(N,M) \xrightarrow{x_1} \mathrm{Ext}^k_A(N,M)$..

By the above, $\mathrm{Ext}^{k-1}_A(N,M) = 0$, and so $\mathrm{Ext}^k_A(N,M) \cong \mathrm{Ext}^{k-1}_A(N,M/x_1M)$.

But $(x_2,\ldots,x_k)$ is an M/x_1M-sequence, so, by the inductive hypothesis, $\mathrm{Ext}^{k-1}_A(N,M/x_1M) \cong \mathrm{Hom}_A(N,M/\underline{x}M)$. Thus $\mathrm{Ext}^i_A(N,M) = 0$ if $i < k$ and $\mathrm{Ext}^k_A(N,M) \cong \mathrm{Hom}_A(N,M/\underline{x}M)$. The result follows by induction. //

(4.3)Lemma. (Lemma 1.31 of Rees (24).) Suppose A is an integral domain and M is a f.-g. A-module with $(0:M)_A = 0$. Then $\mathrm{Hom}_A(M,A) \neq 0$. //

(4.4)Lemma. (This result is stated on p.12 of Bass (5), and a special case of it is proved in Lemma 1.33 of Rees (24).) Let A be a ring and M and N non-zero, f.-g. A-modules s.t. $(0:N)_A = \underline{a}$ consists of zero-divisors on M. Then $\mathrm{Hom}_A(N,M) \neq 0$.

Proof. By (1.13), $\underline{a} \subseteq \bigcup_{\underline{p} \in \mathrm{Ass}(M)} \underline{p}$, so $\underline{a}$ is contained in some element $\underline{p}$ of Ass(M). By (1.11), M has a submodule isomorphic to $A/\underline{p}$, so to show $\mathrm{Hom}_A(N,M) \neq 0$, it is sufficient to show $\mathrm{Hom}_{A/\underline{p}}(N/\underline{p}N, A/\underline{p}) \neq 0$. (Both $N/\underline{p}N$ and $A/\underline{p}$ have natural $A/\underline{p}$-module structures.) Lemma (4.3) will complete the proof if I can show $(0:N/\underline{p}N)_{A/\underline{p}} = 0$, i.e. $(\underline{p}N:N)_A = \underline{p}$. Clearly $(\underline{p}N:N)_A \supseteq \underline{p}$.

Suppose $u_1,\ldots,u_n$ are a set of generators of N, and $x \in (\underline{p}N:N)_A$. Then $x.u_i = \sum_{j=1}^{n} a_{ij}.u_j$, $(a_{ij} \in \underline{p},\ i = 1,\ldots,n, j = 1,\ldots,n)$. Thus $\left|\delta_{ij}.x - a_{ij}\right|.u_k = 0$ for $k = 1,\ldots,n$ (where $\left|b_{ij}\right|$ is used to denote the determinant of the square matrix (b_{ij}), and δ_{ij} denotes the "Kronecker Delta"). Thus $\left|\delta_{ij}.x - a_{ij}\right| \in (0:N)_A = \underline{a} \subseteq \underline{p}$. Expanding the determinant, $x^n \in \underline{p} + (0:N)_A \subseteq \underline{p}$, so $x \in \underline{p}$. Therefore $(\underline{p}N:N)_A = \underline{p}$, and (4.3) completes the proof. //

For a f.-g. Amodule M, recall the definition of M-proper ideals of A ((1.18)). Recall also the definition, for an M-proper ideal $\underline{a}$, of the M-height of $\underline{a}$ (see (1.25)).

(4.5)Proposition. Let A be a ring, and M a non-zero, f.-g., A-module. Suppose $\underline{x} = (x_1,\ldots,x_r)$ is an M-sequence contained in the M-proper

ideal $\underline{a}$. Then $r \leqslant \mathrm{ht}_M \underline{a}$ (which, of course, is finite).

Proof. $M/\underline{x}M \neq 0$, so $\underline{x}$ is M-proper and $\underline{x} \subseteq \underline{a}$. Thus it is sufficient to show that $r \leqslant \mathrm{ht}_M \underline{x}$: for this I use induction on r.

If $\mathrm{ht}_M(x_1)$ were 0, x_1 would belong to a minimal element of $V(M)$, and so, by (1.16) and (1.13), x_1 would be a zero-divisor on M, a contradiction. Thus $1 \leqslant \mathrm{ht}_M(x_1)$.

Suppose, inductively, that $1 \leqslant k < r$ and $\mathrm{ht}_M(x_1,\ldots,x_k) \geqslant k$. If $\mathrm{ht}_M(x_1,\ldots,x_{k+1}) \not\geqslant k+1$, the inductive hypothesis implies that $\mathrm{ht}_M(x_1,\ldots,x_{k+1}) = k$. Thus $(x_1,\ldots,x_k) \subseteq (x_1,\ldots,x_{k+1}) \subseteq \underline{p}$, where $\underline{p}$ is a prime of $V(M)$ of M-height k. Since $\mathrm{ht}_M(x_1,\ldots,x_k) = k$, $\underline{p}$ must be a minimal element of $V(M) \cap V(A/(x_1,\ldots,x_k)) = V(M/(x_1,\ldots,x_k).M)$ (see (1.17) and (1.25)). Thus, by (1.16), $\underline{p} \in \mathrm{Ass}(M/(x_1,\ldots,x_k).M)$.

But $x_{k+1} \in \underline{p}$, and so, by (1.13), x_{k+1} is a zero-divisor on $M/(x_1,\ldots,x_k).M$, a contradiction. Therefore $\mathrm{ht}_M(x_1,\ldots,x_{k+1}) \geqslant k+1$, and the result follows by induction. //

(4.6) Proposition and Definition. Let M be a non-zero, f.-g., A-module and $\underline{a}$ an M-proper ideal of A. Then all maximal M-sequences contained in $\underline{a}$ have the same length, namely the least integer r such that $\mathrm{Ext}^r_A(A/\underline{a}, M) \neq 0$. I define this integer to be the M-grade of $\underline{a}$, and write $\mathrm{grade}_M \underline{a} = r$. Also $\mathrm{grade}_M \underline{a} \leqslant \mathrm{ht}_M \underline{a}$.

Proof. By (4.5) any M-sequence contained in $\underline{a}$ will have length not greater than $\mathrm{ht}_M \underline{a}$. Let $\underline{x} = (x_1,\ldots,x_r)$ be a maximal M-sequence contained in $\underline{a}$. By (4.2), $\mathrm{Ext}^i_A(A/\underline{a}, M) = 0$ if $i < r$ and $\mathrm{Ext}^r_A(A/\underline{a}, M) \cong \mathrm{Hom}_A(A/\underline{a}, M/\underline{x}M)$. Since $\underline{x}$ is a maximal M-sequence contained in $\underline{a}$, $\underline{a}$ consists of zero-divisors on the non-zero module $M/\underline{x}M$ (for otherwise

$M/\underline{x}M = \underline{a}.(M/\underline{x}M)$, and so $M = \underline{a}M$, a contradiction). Thus by (4.4), $\mathrm{Hom}_A(A/\underline{a}, M/\underline{x}M) \neq 0$, and so $\mathrm{Ext}^r_A(A/\underline{a}, M) \neq 0$. It follows from (4.5) that $\mathrm{grade}_M \underline{a} \leqslant \mathrm{ht}_M \underline{a}$. //

(4.7)Corollary. Under the hypotheses of (4.6), every M-sequence contained in $\underline{a}$ can be extended to a maximal M-sequence contained in $\underline{a}$ (and this will have $\mathrm{grade}_M \underline{a}$ terms). //

Having introduced the notion of M-grade, I now establish contact with the well-known concept of homological codimension.

If A is a local ring and M is a non-zero, f.-g., A-module, Nakayama's Lemma implies that every proper ideal of A is M-proper. In particular, the maximal ideal of A is M-proper.

(4.8)Definition. If A is a local ring with maximal ideal $\underline{m}$, and M is a non-zero, f.-g., A-module, the homological codimension, or codimension, of M (written $\mathrm{codh}_A M$) is defined to be $\mathrm{grade}_M \underline{m}$, i.e. the maximum length of an M-sequence contained in $\underline{m}$.

Since every M-sequence must be contained in $\underline{m}$, $\mathrm{codh}_A M$ is the length of every maximal M-sequence.

(4.9)Definition. If A is local with maximal ideal $\underline{m}$, a non-zero, f.-g., A-module M is s.t.b. Cohen-Macaulay (C.-M.) if and only if $\mathrm{codh}_A M = \text{K-Dim}_A M$, i.e. if and only if $\mathrm{grade}_M \underline{m} = \mathrm{ht}_M \underline{m}$.

More generally, if A is a (not necessarily local) ring, a non-zero, f.-g., A-module M is s.t.b. C.-M. if and only if $M_{\underline{m}}$ is a C.-M. $A_{\underline{m}}$-module in the above sense for each maximal ideal $\underline{m} \in V(M)$.

An arbitrary ring A is s.t.b. C.-M. if and only if it is C.-M. when

regarded as a module over itself.

Of course, many authors have studied C.-M. rings (for example, see Rees (24)). C.-M. modules are discussed in Chap.IV of Serre (26). Before proceeding I need one further (temporary) definition.

(4.10)Definition. Let A be a ring. A non-zero, f.-g., A-module M will be called special if and only if $\operatorname{grade}_M \underline{a} = \operatorname{ht}_M \underline{a}$ for all M-proper ideals $\underline{a}$.

The immediate aim is to show that the special A-modules are precisely the C.-M. A-modules.

(4.11)Theorem. Let A be a ring and M a non-zero, f.-g., A-module. Then M is special if and only if every M-sequence generates an M-unmixed ideal. (See (1.27) for definition of "M-unmixed".)

Proof. ($\Leftarrow$) Suppose every M-sequence generates an M-unmixed ideal, and that $\underline{a}$ is an M-proper ideal with $\operatorname{grade}_M \underline{a} = k \leqslant \operatorname{ht}_M \underline{a}$ (see (4.6)). Let $\underline{x} = (x_1,\ldots,x_k)$ be a maximal M-sequence contained in $\underline{a}$. Then $\underline{a}$, since it is M-proper, consists of zero-divisors on $M/\underline{x}M$, so $\underline{a}$ is contained in some prime $\underline{p}$ of $\operatorname{Ass}(M/\underline{x}M)$. By hypothesis, all the primes of $\operatorname{Ass}(M/\underline{x}M)$ have the same M-height, and by (1.26), this M-height must be $\leqslant k$. Therefore $\operatorname{ht}_M \underline{a} \leqslant k$, and so $\operatorname{grade}_M \underline{a} = \operatorname{ht}_M \underline{a}$.

($\Rightarrow$) Suppose $\underline{x} = (x_1,\ldots,x_r)$ is an M-sequence and $\underline{p} \in \operatorname{Ass}(M/\underline{x}M)$. $\operatorname{Ass}(M/\underline{x}M) \subseteq V(M/\underline{x}M) = V(M) \cap V(A/\underline{x})$, so $\underline{p} \in V(M) \cap V(A/\underline{p})$, and consequently $\underline{p}$ is M-proper (see (1.16) - (1.18)). Since $\underline{p}$ consists entirely of zero-divisors on $M/\underline{x}M$, $\underline{x}$ is a maximal M-sequence contained in $\underline{p}$. Thus $\operatorname{grade}_M \underline{p} = r$, and so, by hypothesis, $\operatorname{ht}_M \underline{p} = r$. //

(4.12)Proposition. (This result is contained in Theorem 2 and its Corollary

of Chap.IV of Serre (26).) Suppose A is a local ring with maximal ideal $\underline{m}$ and the non-zero, f.-g., A-module M is C.-M. with $K\text{-}Dim_A M = n$. Then, if $1 \leqslant k \leqslant n$, and $\underline{x} = (x_1,\ldots,x_k)$ is an M-sequence, $M/\underline{x}M$ is C.-M. with Krull Dimension $n - k$.

Proof. By (4.7) there exist $x_{k+1},\ldots,x_n \in \underline{m}$ s.t. $(x_1,\ldots,x_k,x_{k+1},\ldots,x_n)$ is an M-sequence, and so (by (4.5)) certainly $K\text{-}Dim_A(M/\underline{x}M) \geqslant n - k$.

If $K\text{-}Dim_A(M/\underline{x}M) > n - k$, there exists a $\underline{p} \in Ass(M/\underline{x}M)$ s.t. $coht\ \underline{p} > n - k$. Since $\underline{p}$ is M-proper and $\underline{p} \supseteq \underline{x}$, $ht_M \underline{p} \geqslant ht_M \underline{x} \geqslant grade_M \underline{x} = k$. Thus $ht_M \underline{p} + coht\ \underline{p} > n$, a contradiction.

So $K\text{-}Dim_A(M/\underline{x}M) = n - k$ and $M/\underline{x}M$ is C.-M. //

(4.13)Lemma. (See p.IV-15 of Serre (26).) Suppose A is local, with maximal ideal $\underline{m}$, and that M is a non-zero, f.-g., A-module. Suppose $x_1 \in \underline{m}$ and is not a zero-divisor on M, and let $M_1 = M/x_1M$. Then, if $d_0 = \inf\{coht\ \underline{p} \mid \underline{p} \in Ass(M)\}$ and $d_1 = \inf\{coht\ \underline{p} \mid \underline{p} \in Ass(M_1)\}$, $d_0 \geqslant d_1 + 1$.

Proof. Let $\underline{p} \in Ass(M)$ be such that $coht\ \underline{p} = d_0$. From

$$(E)\quad 0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M_1 \longrightarrow 0$$

I deduce

$$(E)\quad 0 \longrightarrow Hom_A(A/\underline{p},M) \xrightarrow{x_1} Hom_A(A/\underline{p},M) \longrightarrow Hom_A(A/\underline{p},M_1)\ .$$

Since $Hom_A(A/\underline{p},M)$ is non-zero and f.-g., $Hom_A(A/\underline{p},M_1) \neq 0$ (by Nakayama's Lemma). So there exists $0 \neq y \in M_1$ s.t. $\underline{p}$ annihilates y. Since x_1 annihilates the whole of M_1, $\underline{p} + (x_1) \subseteq (0:y)_A$. So $\underline{p} + (x_1)$ is contained in some prime $\underline{q}$ of $Ass(M_1)$. But $x_1 \notin \underline{p}$, so $coht\ \underline{p} \geqslant 1 + coht\ \underline{q} \geqslant 1 + d_1$. Thus $d_0 \geqslant 1 + d_1$. //

(4.14)Theorem. (Proposition 13 of Chap.IV of Serre (26).) Suppose A is local with maximal ideal $\underline{m}$, and the non-zero, f.-g., A-module M is C.-M.

of Krull Dimension n. Then all the primes of Ass(M) have coheight n, and so, in particular, M has no embedded primes.

Proof. Suppose $\underline{p} \in \mathrm{Ass}(M)$. Certainly coht $\underline{p} \leqslant n$.

Let $\underline{x} = (x_1,\ldots,x_n)$ be a maximal M-sequence. Let $M_0 = M$, and for $1 \leqslant i \leqslant n$, $M_i = M/(x_1,\ldots,x_i)M$, and let $d_i = \inf\{\text{coht } \underline{p} \mid \underline{p} \in \mathrm{Ass}(M_i)\}$ for each $i = 1,\ldots,n$. Use (4.13) n times to see that:-

$d_i \geqslant d_{i+1} + 1$ for $0 \leqslant i \leqslant n-1$. Adding these n inequalities, $d_0 \geqslant d_n + n \geqslant n$. Thus coht $\underline{p} \geqslant d_0 \geqslant n$. //

(4.15)Corollary. If $1 \leqslant k \leqslant n$ and $\underline{x} = (x_1,\ldots,x_k)$ is an M-sequence (with the hypotheses of (4.14)), all the primes of $\mathrm{Ass}(M/\underline{x}M)$ have coheight n-k.

Proof. (4.12) and (4.14). //

Next, an elementary observation about M-grade:-

(4.16)Lemma. Suppose M is a non-zero, f.-g., A-module and $\underline{a}$ is an M-proper ideal. Then $\mathrm{grade}_M \underline{a} = \inf\{\mathrm{grade}_M \underline{p} \mid \underline{p} \in V(M) \text{ and } \underline{p} \supseteq \underline{a}\}$.

Proof. Observe that, if a prime ideal $\underline{p}$ of A belongs to V(M), then $\underline{p}$ is M-proper. It is clear that

$$\mathrm{grade}_M \underline{a} \leqslant \inf\{\mathrm{grade}_M \underline{p} \mid \underline{p} \in V(M) \text{ and } \underline{p} \supseteq \underline{a}\}.$$

On the other hand, if $\underline{x}$ is a maximal M-sequence contained in $\underline{a}$, $\underline{a}$ is contained in some prime $\underline{p}$ in $\mathrm{Ass}(M/\underline{x}M)$. $\underline{p}$ consists entirely of zero-divisors on $M/\underline{x}M$, $\underline{p} \in V(M)$, $\underline{x} \subseteq \underline{p}$, and so $\underline{x}$ is a maximal M-sequence contained in $\underline{p}$. Therefore $\mathrm{grade}_M \underline{a} = \mathrm{grade}_M \underline{p}$. //

(4.17)Corollary. The non-zero, f.-g., A-module M is special if and only if $\mathrm{grade}_M \underline{p} = \mathrm{ht}_M \underline{p}$ for each prime $\underline{p} \in V(M)$.

Proof. (1.25) and (4.16). //

(4.18) Proposition. If A is local, a non-zero, f.-g., A-module M is C.-M. if and only if it is special.

Proof. Let $\underline{m}$ be the maximal ideal of A.

($\Leftarrow$) Obvious from the definitions. (Every proper ideal is M-proper, of course.)

($\Rightarrow$) By (4.17), it is sufficient to show $\operatorname{grade}_M \underline{p} = \operatorname{ht}_M \underline{p}$ for each prime $\underline{p} \in V(M)$. Suppose $\underline{p} \in V(M)$ and $\operatorname{grade}_M \underline{p} = k \leqslant \operatorname{ht}_M \underline{p}$. Let $\underline{x} = (x_1, \ldots, x_k)$ be a maximal M-sequence contained in $\underline{p}$. Let $n = \text{K-Dim}_A M$. By (4.15), all the primes of $\operatorname{Ass}(M/\underline{x}M)$ have coheight $n - k$. $\underline{p}$ is contained in some element $\underline{q}$ of $\operatorname{Ass}(M/\underline{x}M)$, so if $\operatorname{ht}_M \underline{p} > k$, it follows that $\operatorname{ht}_M \underline{p} + \operatorname{coht} \underline{p} > k + \operatorname{coht} \underline{q} = n$, a contradiction. Therefore $\operatorname{grade}_M \underline{p} = k = \operatorname{ht}_M \underline{p}$. //

(4.19) Theorem. Suppose A is a ring and M is a non-zero, f.-g., A-module. The following three conditions on M are equivalent:-

(i) M is special.

(ii) $M_{\underline{p}}$ is a special $A_{\underline{p}}$-module for each $\underline{p} \in V(M)$.

(iii) $M_{\underline{m}}$ is a special $A_{\underline{m}}$-module for each maximal ideal $\underline{m} \in V(M)$.

Proof. (i)$\Rightarrow$(ii) Suppose $\underline{p} \in V(M)$, so that $\underline{p}$ is M-proper. $\operatorname{grade}_M \underline{p} = \operatorname{ht}_M \underline{p}$ by hypothesis: suppose their common value is k. Let $\underline{x} = (x_1, \ldots, x_k)$ be a maximal M-sequence contained in $\underline{p}$. Using (1.5), it is easy to see that $(x_1/1, \ldots, x_k/1)$ is an $M_{\underline{p}}$-sequence contained in the maximal ideal $\underline{p}A_{\underline{p}}$ of $A_{\underline{p}}$. Now $\text{K-Dim}_{A_{\underline{p}}} M_{\underline{p}} = \operatorname{ht}_M \underline{p}$ (by (1.6)) $= k \leqslant \operatorname{codh}_{A_{\underline{p}}} M_{\underline{p}} \leqslant \text{K-Dim}_{A_{\underline{p}}} M_{\underline{p}}$. Thus $M_{\underline{p}}$ is a C.-M. $A_{\underline{p}}$-module, and so is special by (4.18).

(ii)$\Rightarrow$(iii) Obvious.

(iii)$\Rightarrow$(i) Suppose M is not special. By (4.11), there is an

M-sequence $\underline{x} = (x_1,\ldots,x_k)$ which is such that $\underline{x}$ is not M-unmixed. Now $ht_M \underline{x} = k$, by (4.5) and (1.26), so there exists $\underline{p} \in Ass(M/\underline{x}M)$ with $ht_M \underline{p} > k$. Let $\underline{m}$ be a maximal ideal of A containing $\underline{p}$. Localising at $\underline{m}$ (and using (1.5) again), $(x_1/1,\ldots,x_k/1)$ is an $M_{\underline{m}}$-sequence contained in $\underline{p}A_{\underline{m}}$. (Note that $\underline{m} \in V_A(M/\underline{x}M)$.) Now $\underline{p} \in Ass(M/\underline{x}M)$, so, by (1.14), $\underline{p}A_{\underline{m}} \in Ass_{A_{\underline{m}}}((M/\underline{x}M)_{\underline{m}}) = Ass_{A_{\underline{m}}}(M_{\underline{m}}/(x_1/1,\ldots,x_k/1)M_{\underline{m}})$. Therefore $(x_1/1,\ldots,x_k/1)$ is a maximal $M_{\underline{m}}$-sequence contained in $\underline{p}A_{\underline{m}}$. Thus $grade_{M_{\underline{m}}} \underline{p}A_{\underline{m}} = k < ht_M \underline{p} = ht_{M_{\underline{m}}} \underline{p}A_{\underline{m}}$, a contradiction to the fact that $M_{\underline{m}}$ is a special $A_{\underline{m}}$-module. Thus M is special. //

(4.20)Corollary. Let A be a ring, and M a non-zero, f.-g., A-module. Then M is Cohen-Macaulay if and only if it is special.

Proof. (4.18), (4.19), and Definition (4.8). //

Although I need no further information about Cohen-Macaulay modules for the proof of the main Theorem of this Chapter, I do need two Lemmas about the Cousin Complex. Both are vitally important for this and other applications of the Cousin Complex.

(4.21)Lemma. Suppose M is an A-module and the Cousin Complex for M is:-

$$C(M): 0 \longrightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \ldots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \ldots$$

Let $r \geqslant 0$ be an integer. If L is a f.-g. A-module such that, whenever $\underline{p} \in Supp(M)$ has M-height r, $(0:L)_A \not\subseteq \underline{p}$, then for each $i \geqslant 0$,

$$Ext^i_A(L,M^r) = 0.$$

Proof. Recall that $M^{-2} = 0$, $M^{-1} = M$, and $d^{-2} = 0 : M^{-2} \longrightarrow M^{-1}$.

Let $\underline{a} = (0:L)_A$. $M = \bigoplus_{\substack{\underline{p} \in Supp(M) \\ ht_M \underline{p} = r}} (coker\ d^{r-2})_{\underline{p}}$. Thus, since L is f.-g.,

by (2.18), $\mathrm{Ext}^i_A(L,M^r) \cong \bigoplus_{\substack{\underline{p}\in \mathrm{Supp}(M)\\ \mathrm{ht}_M \underline{p} = r}} \mathrm{Ext}^i_A(L,(\mathrm{coker}\ d^{r-2})_{\underline{p}})$.

If $\underline{p} \in \mathrm{Supp}(M)$ has M-height r, there exists $a \in \underline{a}$ such that $a \notin \underline{p}$. Multiplication by a on L is zero, and on $(\mathrm{coker}\ d^{r-2})_{\underline{p}}$ is an automorphism. So, since $\mathrm{Ext}^i_A(\ ,\ )$ is an A-linear functor ((2.2)), multiplication by a on $\mathrm{Ext}^i_A(L,(\mathrm{coker}\ d^{r-2})_{\underline{p}})$ is both zero and an automorphism. Thus $\mathrm{Ext}^i_A(L,(\mathrm{coker}\ d^{r-2})_{\underline{p}}) = 0$, and so $\mathrm{Ext}^i_A(L,M^r) = 0$. //

Remark. Note the particular case of (4.21) in which $L = A/\underline{a}$, where $\underline{a}$ is an ideal of A such that, whenever $\underline{p} \in \mathrm{Supp}(M)$ has M-height r, $\underline{a} \not\subseteq \underline{p}$.

(4.22)Lemma. Partially Exact Cousin Complex Argument. (P.E.C.C.A.)

Suppose $n > 0$ and the Cousin Complex $C(M)$ for the A-module M is exact at $M = M^{-1}, M^0, \ldots, M^{n-2}$. Suppose L is a f.-g. A-module s.t. $(0:L)_A$ is not contained in any $\underline{p} \in \mathrm{Supp}(M)$ having M-height $\leqslant n-1$. Then $\mathrm{Ext}^i_A(L,M) = 0$ if $i \leqslant n-1$ and $\mathrm{Ext}^n_A(L,M) \cong \mathrm{Hom}_A(L,\mathrm{coker}\ d^{n-2})$.

Proof. The exact sequence (E) $0 \longrightarrow M^{-1} \xrightarrow{d^{-1}} M^0 \to \ldots M^{n-2} \xrightarrow{d^{n-2}} M^{n-1}$ gives rise to short exact sequences:-

(E) $0 \longrightarrow M^{-1} \longrightarrow M^0 \longrightarrow \mathrm{coker}\ d^{-1} \to 0$ (i) (Note that $M^{-1} = \mathrm{coker}\, d^{-2}$.)

(E) $0 \longrightarrow \mathrm{coker}\ d^{-1} \to M^1 \longrightarrow \mathrm{coker}\ d^0 \to 0$ (ii)

$\vdots$

(E) $0 \longrightarrow \mathrm{coker}\ d^{n-4} \to M^{n-2} \longrightarrow \mathrm{coker}\ d^{n-3} \to 0$ (n-1)

(E) $0 \longrightarrow \mathrm{coker}\ d^{n-3} \to M^{n-1} \longrightarrow \mathrm{coker}\ d^{n-2} \to 0$ (n).

By (4.21) $\mathrm{Ext}^i_A(L,M^r) = 0$ for all $i \geqslant 0$ and each r s.t. $0 \leqslant r \leqslant n-1$. Consequently, the long exact sequences derived from (i),...,(n) imply, respectively,:-

$\mathrm{Hom}_A(L,M) = 0$, and, for $k > 0$, $\mathrm{Ext}^k_A(L,M) \cong \mathrm{Ext}^{k-1}_A(L,\mathrm{coker}\ d^{-1})$;

$\mathrm{Hom}_A(L,\mathrm{coker}\ d^{-1}) = 0$, &, for $k>0$, $\mathrm{Ext}^k_A(L,\mathrm{coker}\ d^{-1}) \cong \mathrm{Ext}^{k-1}_A(L,\mathrm{coker}\ d^0)$;

$\vdots$

$\mathrm{Hom}_A(L,\mathrm{coker}\ d^{n-3}) = 0$, &, for $k>0$, $\mathrm{Ext}^k_A(L,\mathrm{coker}\ d^{n-3}) \cong \mathrm{Ext}^{k-1}_A(L,\mathrm{coker}\ d^{n-2})$

The result stated in the Lemma follows. Note also, that if $k > n$,

$\mathrm{Ext}^k_A(L,M) \cong \mathrm{Ext}^{k-n}_A(L,\mathrm{coker}\ d^{n-2})$. //

The argument in the proof of the Lemma is just as important as the result: both will be used many times. "P.E.C.C.A." will refer to both the Lemma and the argument of its proof. Note the particular case of (4.22) in which $L = A/\underline{a}$, where $\underline{a}$ is an ideal of A such that, whenever $\underline{p} \in \mathrm{Supp}(M)$ has M-height $\leqslant n-1$, $\underline{a} \not\subseteq \underline{p}$.

And now, at last, the long-awaited main Theorem of Chapter 4:-

(4.23)Theorem. Let A be a ring and M a non-zero, f.-g., A-module with Cousin Complex $C(M)$. Then M is Cohen-Macaulay if and only if $C(M)$ is exact.

Proof. ($\Leftarrow$) Suppose $C(M)$ is exact. I need to show that for each maximal ideal $\underline{m} \in V(M)$, $\mathrm{grade}_{M_{\underline{m}}} \underline{m}A_{\underline{m}} = \mathrm{ht}_{M_{\underline{m}}} \underline{m}A_{\underline{m}}$. Suppose $\underline{m} \in V(M)$, $\underline{m}$ is maximal, and $\mathrm{ht}_{M_{\underline{m}}} \underline{m}A_{\underline{m}} = n$. I may assume $n > 0$, for otherwise there is nothing to prove. By (3.13), $C_{A_{\underline{m}}}(M_{\underline{m}})$ is exact. $\underline{m}A_{\underline{m}}$ is not contained in any prime of $V_{A_{\underline{m}}}(M_{\underline{m}})$ having $M_{\underline{m}}$-height $\leqslant n-1$. By P.E.C.C.A. (4.22), $\mathrm{Ext}^i_{A_{\underline{m}}}(A_{\underline{m}}/\underline{m}A_{\underline{m}}, M_{\underline{m}}) = 0$ if $0 \leqslant i \leqslant n-1$, so $\mathrm{grade}_{M_{\underline{m}}} \underline{m}A_{\underline{m}} = n$ and $M_{\underline{m}}$ is a Cohen-Macaulay $A_{\underline{m}}$-module.

($\Rightarrow$) Suppose M is Cohen-Macaulay. I first show that $C(M)$ is exact at $M = M^{-1}$. Suppose not. Then there exists $0 \neq y \in H^{-1}(M) \subseteq M = \mathrm{coker}\ d^{-2}$.

Consequently, $\mathrm{Hom}_A(A/(0:y)_A, M) \neq 0$. But by (3.7)(viii), if $\underline{p} \in V(A)-U^1(M)$, $(0:y)_A \nsubseteq \underline{p}$. Thus $(0:y)_A$ is an M-proper ideal (since it is contained in some maximal ideal which will belong to $U^1(M) \subseteq V(M)$) of M-height $\geqslant 1$. Since M is C.-M., $\mathrm{grade}_M((0:y)_A) = \mathrm{ht}_M((0:y)_A) \geqslant 1$, and so $\mathrm{Hom}_A(A/(0:y)_A, M) = 0$, a contradiction. Thus C(M) is exact at $M = M^{-1}$.

Now suppose, inductively, that $n \geqslant 0$, and that C(M) has been shown to be exact at $M = M^{-1}, M^0, \ldots, M^{n-1}$. I shall show that $H^n(M) = 0$. Again, suppose not, so that there exists $0 \neq y \in H^n(M) \subseteq \mathrm{coker}\ d^{n-1}$. Consequently, $\mathrm{Hom}_A(A/(0:y)_A, \mathrm{coker}\ d^{n-1}) \neq 0$. But, by (3.7)(viii), if $\underline{p} \in V(A) - U^{n+2}(M)$, $(0:y)_A \nsubseteq \underline{p}$. Thus $(0:y)_A$ is an M-proper ideal of M-height $\geqslant n+2$. Since M is C.-M., $\mathrm{grade}_M((0:y)_A) = \mathrm{ht}_M((0:y)_A)$ (by (4.20)), so $\mathrm{grade}_M((0:y)_A) \geqslant n+2$. Therefore $\mathrm{Ext}^i_A(A/(0:y)_A, M) = 0$ if $0 \leqslant i \leqslant n+1$.

But the ideal $(0:y)_A$ is not contained in any $\underline{p} \in \mathrm{Supp}(M)$ having M-height $\leqslant n$, and consequently, by P.E.C.C.A. (4.22), $\mathrm{Ext}^{n+1}_A(A/(0:y)_A, M) \cong \mathrm{Hom}_A(A/(0:y)_A, \mathrm{coker}\ d^{n-1})$, which is non-zero, as remarked above. Thus I have a contradiction, and so $H^n(M) = 0$. Therefore C(M) is exact at M^n, also, and so, by induction, the Theorem is proved. //

(4.24)Corollary. If M is a C.-M. A-module (so that of necessity M is non-zero and f.-g.) and $\underline{p} \in V(M)$ has M-height n, then $(\mathrm{coker}\ d^{n-2})_{\underline{p}} \neq 0$.

Proof. Since C(M) is exact, so also is $C_{A_{\underline{p}}}(M_{\underline{p}})$ (by (3.13)) so $M_{\underline{p}}$ is a C.-M. $A_{\underline{p}}$-module of Krull Dimension n. By (3.14), $(\mathrm{coker}\ d^{n-2})_{\underline{p}} \cong (M_{\underline{p}})^n$, so it is sufficient to show that, if A is local with maximal ideal $\underline{m}$ and M is a C.-M. A-module of Krull Dimension n, then $M^n \neq 0$. In this case, $\mathrm{grade}_M \underline{m} = n$. By P.E.C.C.A. (4.22), $\mathrm{Ext}^n_A(A/\underline{m}, M) \cong \mathrm{Hom}_A(A/\underline{m}, \mathrm{coker}\ d^{n-2})$. But $M^n \cong \mathrm{coker}\ d^{n-2}$, so since $\mathrm{Ext}^n_A(A/\underline{m}, M) \neq 0$, it follows that $M^n \neq 0$.//

CHAPTER 5

A Characterization of Gorenstein Rings in Terms of the Cousin Complex.

(Reminder: "ring" means "commutative Noetherian ring with a non-zero identity".)

Throughout this Chapter, A will denote a (fixed) ring.

One can, of course, regard A as a module over itself and form C(A): it follows from (4.23) that A is Cohen-Macaulay if and only if C(A) is exact. The main result of this Chapter is that A is Gorenstein if and only if C(A) provides an injective (resp. minimal injective) resolution for A. The proof of this result relies heavily on the work of Bass in (5), in which he characterizes Gorenstein rings in several ways. A review of some of the basic definitions and Lemmas will be helpful both here and later in the thesis.

An A-module monomorphism $f:M \longrightarrow N$ is s.t.b. essential (and N is then s.t.b. an essential extension of M (by f)) if and only if every non-zero submodule of N intersects f(M) non-trivially. N is s.t.b. an injective envelope for M if, in addition, N is an injective A-module. The facts that every A-module M has an injective envelope, and that any two such injective envelopes for M are isomorphic (even over M) are proved in Eckmann and Schopf (8).

I shall find it convenient to say that an A-homomorphism $f:M \longrightarrow N$ is essential if and only if every non-zero submodule of N intersects f(M) non-trivially, i.e. if and only if the induced monomorphism $M/\ker f \longrightarrow N$ is essential. An injective resolution:-

$$(E)\quad 0 \longrightarrow M \xrightarrow{\varepsilon^{-1}} Q^0 \xrightarrow{\varepsilon^0} Q^1 \longrightarrow \cdots \longrightarrow Q^n \xrightarrow{\varepsilon^n} \cdots$$

for an A-module M is s.t.b. a minimal injective resolution for M if and only if all the homomorphisms $\varepsilon^{-1},\varepsilon^{0},\ldots,\varepsilon^{n},\ldots$ are essential.

Every A-module M has a minimal injective resolution, and these resolutions have strong uniqueness properties. In particular, $Q^i = E^i(M)$ is determined up to isomorphism by M. The fact that $\text{inj.dim.}_A M < n$ if and only if $E^n(M) = 0$ is stated on p.19 of Bass (4) and follows from the following Lemma. Observe that if $E^n(M) = 0$, then necessarily $E^{n+i}(M) = 0$ for each $i > 0$

(5.1)Lemma. Let $\underline{X}: 0 \longrightarrow M \xrightarrow{\sigma^{-1}} E^0(M) \xrightarrow{\sigma^0} \ldots \longrightarrow E^n(M) \xrightarrow{\sigma^n} E^{n+1}(M) \ldots$ (E)

be a minimal injective resolution for M and:-

$$\underline{Y}: 0 \longrightarrow M \xrightarrow{\mu^{-1}} I^0 \xrightarrow{\mu^0} I^1 \longrightarrow \ldots \longrightarrow I^n \xrightarrow{\mu^n} I^{n+1} \longrightarrow \ldots \qquad (E)$$

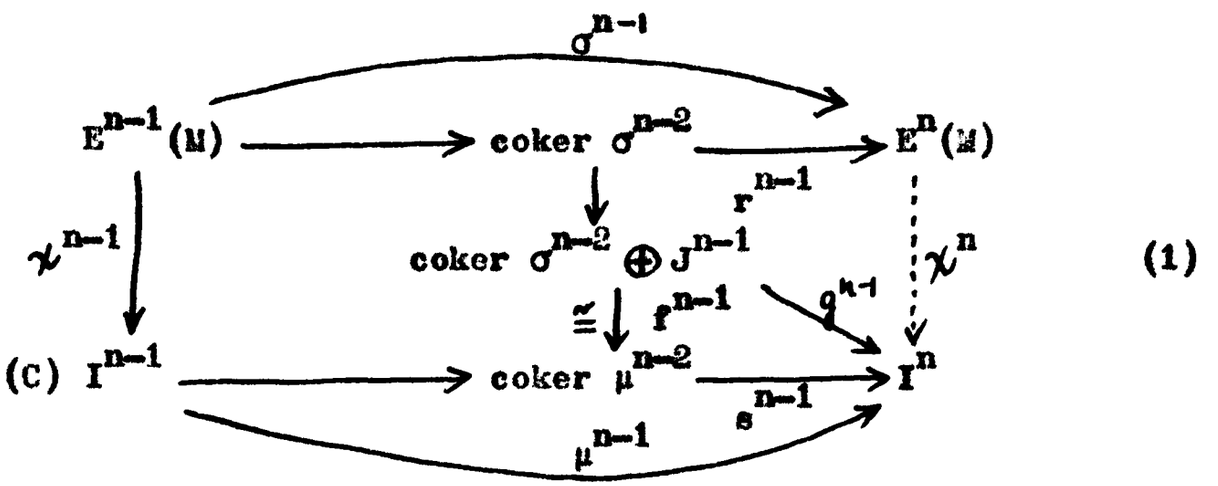
an arbitrary injective resolution for M. Then there is a homomorphism $\Psi = \{\chi^n\}_{n \geqslant -1} : \underline{X} \longrightarrow \underline{Y}$ of complexes such that $\chi^{-1}: M \longrightarrow M$ is the identity and all the other constituent homomorphisms $\chi^n : E^n(M) \longrightarrow I^n$ are monomorphisms.

Proof. Recall that an injective module is a direct summand of every module which contains it. Economy will be achieved by writing $E^{-1}(M) = I^{-1} = M$, $E^{-2}(M) = I^{-2} = 0$, $\sigma^{-2} = 0 : E^{-2}(M) \longrightarrow E^{-1}(M)$, and $\mu^{-2} = 0 : I^{-2} \longrightarrow I^{-1}$. Let $\chi^{-2} = 0 : E^{-2}(M) \longrightarrow I^{-2}$, $\chi^{-1} = 1_M : E^{-1}(M) \longrightarrow I^{-1}$. Then:-

$$\text{(C)} \quad \begin{array}{ccc} E^{-2}(M) & \xrightarrow{\sigma^{-2}} & E^{-1}(M) \\ \| & & \| \\ I^{-2} & \xrightarrow{\mu^{-2}} & I^{-1} \end{array}$$

provides a basis for the following induction.

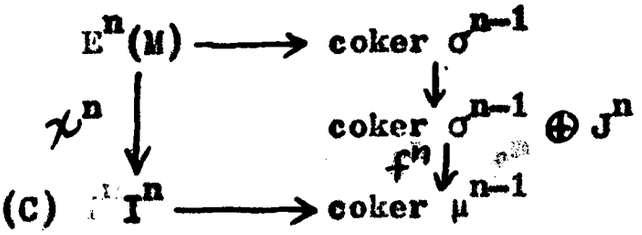
I suppose, inductively, that $n \geqslant 0$, and I have constructed $\chi^{-2},\chi^{-1},\ldots,\chi^{n-1}$ with the required properties, and that there is an injective A-module J^{n-1} and an isomorphism $f^{n-1} : \text{coker } \sigma^{n-2} \oplus J^{n-1} \longrightarrow \text{coker } \mu^{n-2}$ such that:-



I shall use the notation of diagram (1) without further comment: the unnamed homomorphisms are all the obvious natural ones. r^{n-1} and s^{n-1} are monomorphic. Let $g^{n-1} = s^{n-1}.f^{n-1}$, so that g^{n-1} is monomorphic also.

As I^n is injective, diagram (1) can be completed with χ^n so that the extended diagram commutes. Since the middle column and s^{n-1} are monomorphic and r^{n-1} is an essential monomorphism, χ^n will be monomorphic. Thus $\chi^n(E^n(M))$ and $g^{n-1}(J^{n-1})$ are injective submodules of I^n, and they generate their direct sum: for otherwise $\chi^n(E^n(M)) \cap g^{n-1}(J^{n-1})$ will be a non-zero submodule of the essential extension $\chi^n(E^n(M))$ of $\chi^n r^{n-1}(\text{coker } \sigma^{n-2}) = g^{n-1}(\text{coker } \sigma^{n-2})$, and so will intersect $g^{n-1}(\text{coker } \sigma^{n-2})$ non-trivially, and this, of course, can't happen.

Next, $\mu^{n-1}(I^{n-1}) = \chi^n\sigma^{n-1}(E^{n-1}(M)) \oplus g^{n-1}(J^{n-1})$. Since $\chi^n(E^n(M))$, $g^{n-1}(J^{n-1})$, and I^n are injective, there is an injective A-module J^n such that $I^n = \chi^n(E^n(M)) \oplus g^{n-1}(J^{n-1}) \oplus J^n$. It is now clear how to construct an A-isomorphism f^n:coker $\sigma^{n-1} \oplus J^n \xrightarrow{\cong}$ coker μ^{n-1} such that:-



This completes the inductive step, and the Lemma follows by induction.

//

The proof of this Lemma is more complicated than might have been expected: more straightfoward approaches don't seem (to me at least) to provide the necessary information that, at each stage, the induced homomorphisms on cokernels are monomorphic.

(5.2) As I am dealing with commutative Noetherian rings, a very satisfactory theory of injective modules is available: see Gabriel (10) and Matlis (17). If M is an A-module, I denote by E(M) (or $E_A(M)$) its injective envelope.

(i) A direct sum of a family of A-modules is injective if and only if each of the summands is injective. (See §4 of (10).)

(ii) The operations of forming direct sums and of forming injective envelopes "commute". (See Proposition (2.1) of (17).)

(iii) Every injective A-module can be expressed as a direct sum of indecomposable injective A-modules. (See Theorem (2.5) of (17).)

(iv) The decompositions of (iii) are unique in the sense of Krull-Remak-Schmidt-Azumaya. (See §4 of (10).)

(v) There is a 1-1 onto correspondence between the prime ideals $\underline{p}$ of A and the isomorphism types of non-zero indecomposable injective A-modules given by $\underline{p} \longleftrightarrow E(A/\underline{p})$. (See Proposition (3.1) of (17).)

(vi) Every element of $E(A/\underline{p})$ (where $\underline{p}$ is a prime ideal of A) is annihilated by some power of $\underline{p}$, and multiplication by an element $a \in A - \underline{p}$ provides an automorphism of $E(A/\underline{p})$. (See Lemma (3.2)(2) & Theorem (3.4) of (17).)

(vii) A is Artinian if and only if every indecomposable injective A-module is finitely-generated. (See Theorem (3.11) of (17).)

If M is an A-module, I denote by $E^i(M)$ $(= E^i_A(M))$ the ith term in a minimal injective resolution for M:-

(E) $0 \longrightarrow M \longrightarrow E^0(M) \longrightarrow E^1(M) \longrightarrow \ldots \longrightarrow E^i(M) \longrightarrow E^{i+1}(M) \longrightarrow \ldots$.

Thus $E^0(M) = E(M)$. $E^i(M)$ (for each $i \geqslant 0$) is determined up to isomorphism by M, and so, by means of the above structure theory, one may define (see p.11 of Bass (5)) cardinals $\mu^i(\underline{p}, M)$ by the equation:-

$E^i(M) \cong \bigoplus_{\underline{p} \in V(A)} \mu^i(\underline{p}, M)E(A/\underline{p})$, where $\oplus \mu E$ denotes a direct sum of μ copies of E.

Suppose S is a m.c.s. of A: I shall need to use the facts that the operation "S^{-1}" preserves injective envelopes and minimal injective resolutions. The proof of these facts is provided by a sequence of Lemmas.

(5.3)Lemma. (Lemma 1.2(a) of Bass (4).) Suppose S is an m.c.s. of A, and E is an $S^{-1}A$-module. Then E is A-injective if and only if E is $S^{-1}A$-injective.

Proof. ($\Rightarrow$) Suppose E is A-injective and (E) $0 \longrightarrow M_1 \xrightarrow{\beta} M_2$, with $f : M_1 \to E$, is a diagram of $S^{-1}A$-modules and -homomorphisms. Regard them all as A-modules and -homomorphisms. Then there is an A-homomorphism $h: M_2 \longrightarrow E$ s.t. the extended diagram commutes. It is easy to see that h is an $S^{-1}A$-homomorphism.

($\Leftarrow$) Suppose E is $S^{-1}A$-injective and (E) $0 \longrightarrow M_1 \xrightarrow{\alpha} M_2$, with $f : M_1 \to E$, is a diagram of A-modules and -homomorphisms. $\phi : S^{-1}E \longrightarrow E$ defined by

$\phi(e/s) = (1/s).e$ is a (well-defined) $S^{-1}A$-isomorphism, and the diagram:-

(E)
$$\begin{array}{ccccc} 0 & \longrightarrow & S^{-1}M_1 & \xrightarrow{S^{-1}\alpha} & S^{-1}M_2 \\ & & \downarrow{\scriptstyle S^{-1}f} & \swarrow{\scriptstyle h} & \\ & & S^{-1}E & & \\ & & \downarrow{\scriptstyle \phi}\ {\scriptstyle \cong} & & \\ & & E & & \end{array}$$
(C)

can be completed with an $S^{-1}A$-homomorphism $h: S^{-1}M_2 \longrightarrow E$ such that the completed diagram commutes. $g: M_2 \longrightarrow E$ defined by $g(m_2) = h(m_2/1)$ is an A-homomorphism s.t. $g.\alpha = f$. //

(5.4)Lemma. If S is an m.c.s. of A and $\underline{p}$ is a prime ideal of A which does not meet S, then $E_A(A/\underline{p})$ has a natural structure as an $S^{-1}A$-module, and as such is isomorphic to $E_{S^{-1}A}(S^{-1}A/S^{-1}\underline{p})$. (Also $E_{S^{-1}A}(S^{-1}A/S^{-1}\underline{p})$ when regarded as an A-module is isomorphic to $E_A(A/\underline{p})$.)

Proof. $S \subseteq A - \underline{p}$, so by (5.2)(vi), multiplication by an element $s \in S$ provides an automorphism of $E_A(A/\underline{p})$. So by (1.7), $E_A(A/\underline{p})$ has a natural structure as an $S^{-1}A$-module: $(a/s).m$, for $m \in E_A(A/\underline{p})$, $a \in A$, $s \in S$, is defined to be the unique element $m' \in E_A(A/\underline{p})$ for which $s.m' = a.m$. By (5.3), $E_A(A/\underline{p})$ is an (indecomposable) injective $S^{-1}A$-module: multiplication by elements of $S^{-1}A - S^{-1}\underline{p}$ produces an automorphism, and multiplication by elements of $S^{-1}\underline{p}$ does not. Thus, by (5.2)(v) & (vi), the indecomposable injective $S^{-1}A$-module $E_A(A/\underline{p})$ must be isomorphic to $E_{S^{-1}A}(S^{-1}A/S^{-1}\underline{p})$.

One can see the second part directly as follows. There is an essential $S^{-1}A$-monomorphism $\alpha: S^{-1}A/S^{-1}\underline{p} \longrightarrow E_{S^{-1}A}(S^{-1}A/S^{-1}\underline{p})$. When regarded as an A-monomorphism, α will still be essential. $E_{S^{-1}A}(S^{-1}A/S^{-1}\underline{p})$

is A-injective by (5.3). Also, the natural A-monomorphism $\beta: A/\underline{p} \to S^{-1}A/S^{-1}\underline{p}$ is essential. Since the composition of two essential monomorphisms is essential, the result follows. //

(5.5)Corollary. (See Lemma (1.2)(b) of Bass (4).) If E is an injective A-module, $S^{-1}E$ is both A-injective and $S^{-1}A$-injective. If $\underline{p}$ is a prime ideal of A, $S^{-1}(E_A(A/\underline{p})) = 0$ if $\underline{p}$ meets S, while $S^{-1}(E_A(A/\underline{p})) \cong E_{S^{-1}A}(S^{-1}A/S^{-1}\underline{p})$ if $\underline{p} \cap S = \emptyset$.

Proof. I deal with the second part first. If $S \cap \underline{p} \neq \emptyset$, then since every element of $E(A/\underline{p})$ is annihilated by some power of $\underline{p}$ (by (5.2)(vi)), $S^{-1}(E(A/\underline{p})) = 0$. If $S \cap \underline{p} = \emptyset$, $S \subseteq A - \underline{p}$, so, by (5.2)(vi) again, multiplication by any element of S provides an automorphism of $E(A/\underline{p})$. Using (5.4), it follows that, when $E(A/\underline{p})$ is given its natural $S^{-1}A$-module structure, $S^{-1}(E(A/\underline{p})) \cong E(A/\underline{p})$ (both as A- and $S^{-1}A$-modules) Thus, by (5.4), $S^{-1}(E(A/\underline{p})) \cong E_{S^{-1}A}(S^{-1}A/S^{-1}\underline{p})$.

The first part is now obvious from (5.2)(i), (iii), & (v), since $E \cong \bigoplus_{j \in J} E(A/\underline{p}_j)$, where $\{\underline{p}_j\}_{j \in J}$ is some family of prime ideals of A. Thus $S^{-1}E \cong \bigoplus_{j \in J} S^{-1}(E(A/\underline{p}_j))$. ((5.3) is used here, of course.) //

(5.6)Lemma. Let N be an A-module which is an essential extension of its submodule $C = \{x \in N \mid x/1 = 0 \text{ in } S^{-1}N\}$. Then $N = C$.

Proof. If $C = 0$ the result is obvious, so suppose $0 \subsetneq C \subsetneq N$. Let $\underline{b}^c$ denote the contraction to A of an ideal $\underline{b}$ of $S^{-1}A$.

Let $y \in N - C$, and $\underline{a} = (0:y)_A$. Then $(S^{-1}\underline{a})^c = (C:y)_A \subsetneq A$. Thus, by (1.15), there is a $\underline{p} \in \mathrm{Ass}(A/\underline{a}) = \mathrm{Ass}(Ay)$ which does not meet S. By (1.11), there is an $h \in A$ s.t. $(0:hy)_A = \underline{p}$. Then hy is a non-zero

element of N s.t. $A(hy) \cap C = 0$ (since $\underline{p} \cap S = \emptyset$), a contradiction. Thus $C = N$. //

(5.7)Lemma. (Corollary (1.3) of Bass (4).) Let the A-module N be an essential extension of its submodule M. Then $S^{-1}N$ is an essential extension of its ($S^{-1}A$-)submodule $S^{-1}M$.

Proof. If $S^{-1}N = 0$, the result is obvious, so assume $S^{-1}N \neq 0$. Let $C = \{x \in N \mid x/1 = 0 \text{ in } S^{-1}N\}$. Now every $S^{-1}A$-submodule of $S^{-1}N$ is of the form $S^{-1}P$ for some submodule P of N. Let $S^{-1}P$ be a non-zero submodule of $S^{-1}N$, and suppose $S^{-1}M \cap S^{-1}P$ $(= S^{-1}(M \cap P)) = 0$. Thus $M \cap P \subseteq C$. Now P is an essential extension of $M \cap P$, and so an essential extension of $C \cap P = \{x \in P \mid x/1 = 0 \text{ in } S^{-1}P\}$. Thus, by (5.6), $C \cap P = P$, and so $P \subseteq C$. Therefore $S^{-1}P = 0$, a contradiction. Thus $S^{-1}M \cap S^{-1}P \neq 0$, and $S^{-1}N$ is an essential extension of $S^{-1}M$. //

(5.8)Corollary. (Corollary (1.3) of Bass (4).) If S is an m.c.s. of A, the operation "S^{-1}" preserves both injective envelopes and minimal injective resolutions.

Proof. (1.3), (5.5), and (5.7). //

(5.9)Corollary. (Corollary (2.4) of Bass (5).) If M is an A-module and $\underline{p}$ is a prime ideal of A which does not meet the m.c.s. S of A, then:-

$$\mu^i(S^{-1}\underline{p}, S^{-1}M) = \mu^i(\underline{p}, M) \text{ for each } i \geq 0.$$

Proof. (5.8) and (5.5). //

It will be convenient, both in the proof of the main result of this Chapter and in Chapter 6, to have a statement of Bass' Fundamental Theorem readily available. Recall that, if A is a local ring of Krull

Dimension d, a <u>system of parameters</u> for A is a set of d elements which generate an $\underline{m}$-primary ideal of A (where $\underline{m}$ is the maximal ideal of A). These always exist, by (1.23) and (1.20).

<u>(5.10)Theorem and Definition. Bass' Fundamental Theorem.</u> (See §§1-4 of (5).) A ring A is called a <u>Gorenstein ring</u> if and only if it satisfies the following equivalent conditions:-

(i) For each prime (resp., maximal) ideal $\underline{p}$, $A_{\underline{p}}$ is a Cohen-Macaulay ring in which some system of parameters generates an irreducible ideal.

(ii) For each prime (resp., maximal) ideal $\underline{p}$, every system of parameters in $A_{\underline{p}}$ generates an irreducible ideal.

(iii) An ideal of height h generated by h elements is unmixed, and all of its primary components are irreducible.

(iv) For all f.-g. A-modules M and all $j \geqslant 0$ for which $\mathrm{Ext}^j_A(M,A) \neq 0$, $\mathrm{grade}_A(0:\mathrm{Ext}^j_A(M,A)) \geqslant j$.

(v) For each prime (resp., maximal) ideal $\underline{p}$, $\mathrm{inj.dim.}_{A_{\underline{p}}} A_{\underline{p}}$ is finite.

(vi) For each $h \geqslant 0$, $E^h(A) \cong \bigoplus_{\substack{\underline{p} \in V(A) \\ \mathrm{ht}\,\underline{p} = h}} E(A/\underline{p})$ (i.e. $\mu^i(\underline{p},A) = \delta_{i,\mathrm{ht}\,\underline{p}}$ for all $\underline{p} \in V(A)$ (Kronecker delta)).

If, moreover, K-Dim A is finite and equal to n, the above conditions are all equivalent to:-

(vii) $\mathrm{inj.dim.}_A A$ is finite, in which case $\mathrm{inj.dim.}_A A = n$. //

(A comparison of my definition of "A-grade" and Bass' definition of "grade" on p.12 of (5), together with an examination of Corollary (2.11) of (5), show that condition (iv) above is equivalent to Bass' condition (d) in his statement of the Theorem.)

The local version of Bass' Fundamental Theorem is as follows:-

(5.11)Theorem. (Theorem (4.1) of (5).) Let A be a local ring with maximal ideal $\underline{m}$ and Krull Dimension n. The following conditions on A are equivalent:-

(i) $\text{inj.dim.}_A A$ is finite (resp., = n).

(ii) A is a Cohen-Macaulay ring in which some system of parameters generates an irreducible ideal.

(iii) Every system of parameters in A generates an irreducible ideal.

(iv) $\mu^i(\underline{m},A) = \begin{cases} 0 & \text{for } i < n \\ 1 & \text{for } i = n \end{cases}$.

(v) A is a Cohen-Macaulay ring and there is a natural isomorphism:-

$\text{Ext}^n_A(M,A) \cong \text{Hom}_A(M,E(A/\underline{m}))$ for all A-modules M of finite length.

(vi) $\mu^i(\underline{p},A) = \delta_{i,\text{ht}\,\underline{p}}$ for all primes $\underline{p} \in V(A)$. //

(Note that (5.10)(vi) is the same as (5.11)(vi).)

Before stating the main Theorem of this Chapter, I make an easy observation about the Cousin Complex.

(5.12)Proposition. Let the Cousin Complex for the A-module M be:-

$$\underline{C(M)}: 0 \longrightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots .$$

Then the homomorphisms $d^{-1},d^0,\ldots,d^n,\ldots$ are all essential, and so, if $\underline{C(M)}$ provides an injective resolution for M, it provides a minimal injective resolution.

Proof. As usual, write $M^{-2} = 0$, $M^{-1} = M$, and $d^{-2} = 0: M^{-2} \longrightarrow M^{-1}$. Suppose $n \geqslant 0$ and $0 \neq x \in M^n = \bigoplus_{\substack{\underline{p} \in \text{Supp}(M) \\ \text{ht}_M \underline{p} = n}} (\text{coker } d^{n-2})_{\underline{p}}$. Suppose x has non-zero components in the direct summands $(\text{coker } d^{n-2})_{\underline{p}_j}$ for $j = 1,\ldots,n$ (each $\underline{p}_j$

being a prime of Supp(M) having M-height n), and zero components elsewhere. By (3.7)(v), $(d^{n-1}M^{n-1}:x)_A \not\subseteq \underline{p}_1$, so there is an $a \in A - \underline{p}_1$ such that $a.x \in d^{n-1}M^{n-1}$. But a cannot annihilate the non-zero component of x in the summand $(\operatorname{coker} d^{n-2})_{\underline{p}_1}$. Thus $a.x \neq 0$, and so d^{n-1} is essential. //

(5.13)Theorem. Let A be a ring. Then A is Gorenstein if and only if the Cousin Complex C(A) for A provides an injective resolution (resp., minimal injective resolution) for A.

Proof. ($\Leftarrow$) Suppose C(A) provides an injective resolution for A. By (5.12), this will be a minimal injective resolution. By (3.13) and (5.8), the Cousin Complex for $A_{\underline{p}}$, where $\underline{p}$ is any prime ideal of A, provides a minimal injective resolution for $A_{\underline{p}}$. ($A_{\underline{p}}$ is here regarded as a module over itself, of course.) But the Cousin Complex for a local ring has only a finite number of non-zero terms, and so $\operatorname{inj.dim.}_{A_{\underline{p}}} A_{\underline{p}}$ is finite for all $\underline{p} \in V(A)$. Thus, by (5.10)(v), A is Gorenstein.

($\Rightarrow$) Suppose A is Gorenstein, and let:-

$C(A): 0 \longrightarrow A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \longrightarrow \cdots \longrightarrow A^n \xrightarrow{d^n} \cdots$ be the Cousin Complex for A. By (5.10), A is Cohen-Macaulay, so by (4.23), C(A) is exact. I shall show that $A^0, A^1, \ldots, A^n, \ldots$ are all A-injective by induction on n. Note that, from (5.10)(vi) and (5.9), for each $\underline{p} \in V(A)$, $A_{\underline{p}}$ is Gorenstein.

First of all, $A^0 = \bigoplus_{\substack{\underline{p} \in V(A) \\ \operatorname{ht} \underline{p} = 0}} A_{\underline{p}}$. By (5.11)(i), since, if $\operatorname{ht} \underline{p} = 0$, $A_{\underline{p}}$ is an Artin local ring which is Gorenstein, $A_{\underline{p}}$ is an injective $A_{\underline{p}}$-module, and so is an injective A-module by (5.3). Thus, by (5.2)(i), A^0 is an

injective A-module.

Now I suppose, inductively, that $n \geqslant 0$ and I have shown that $A^0, A^1, \ldots, A^n$ are all injective. $A^{n+1} = \bigoplus_{\substack{\underline{p} \in V(A) \\ \mathrm{ht}\,\underline{p} = n+1}} (\mathrm{coker}\ d^{n-1})_{\underline{p}}$. It is sufficient, by (5.2)(i), to show that, for any prime $\underline{p}$ of height n+1, $(\mathrm{coker}\ d^{n-1})_{\underline{p}}$ is A-injective. For this, by (5.3), it will suffice to show $(\mathrm{coker}\ d^{n-1})_{\underline{p}}$ is $A_{\underline{p}}$-injective. But, by (3.14), $(\mathrm{coker}\ d^{n-1})_{\underline{p}}$ is isomorphic to the (n+1)th term in the Cousin Complex for $A_{\underline{p}}$ over itself.

Let $B = A_{\underline{p}}$, and let:-

$\underline{C(B)}: 0 \longrightarrow B \xrightarrow{e^{-1}} B^0 \longrightarrow \ldots \longrightarrow B^n \xrightarrow{e^n} B^{n+1} \longrightarrow 0$ (E) be the Cousin Complex for B as a module over itself. By (3.13) and (5.5), $B^0, B^1, \ldots, B^n$ are all B-injective. B is Gorenstein and so is Cohen-Macaulay: $\underline{C(B)}$ is exact. K-Dim B = n+1. By (5.12), (E) $0 \longrightarrow B \xrightarrow{e^{-1}} B^0 \xrightarrow{e^0} B^1 \longrightarrow \ldots \xrightarrow{e^{n-1}} B^n$ can be extended to a minimal injective resolution for B. But, by (5.10)(vi), a minimal injective resolution for B has the (n+2)nd term zero, and the (n+1)th term isomorphic to $E_B(B/\underline{m})$, where $\underline{m}$ is the maximal ideal of B. Thus coker $e^{n-1} \cong E_B(B/\underline{m})$. But $B^{n+1} =$ coker e^{n-1}.

Therefore $(\mathrm{coker}\ d^{n-1})_{\underline{p}} \cong E_{A_{\underline{p}}}(A_{\underline{p}}/\underline{p}A_{\underline{p}})$ when regarded as an $A_{\underline{p}}$-module, so, when regarded as an A-module, $(\mathrm{coker}\ d^{n-1})_{\underline{p}} \cong E_A(A/\underline{p})$ (by (5.4)). In particular, $(\mathrm{coker}\ d^{n-1})_{\underline{p}}$ is A-injective, and so the inductive step is complete. The Theorem follows by induction. //

(5.14)Corollary. If $\underline{p}$ is a prime ideal of the Gorenstein ring A and $\mathrm{ht}\,\underline{p} = n$, then (with the usual Cousin Complex notation) $(\mathrm{coker}\ d^{n-2})_{\underline{p}} \cong E_A(A/\underline{p})$.

Proof. If $n > 0$, this follows from the above proof. If $n = 0$, $(\mathrm{coker}\ d^{n-2})_{\underline{p}}$

$= A_{\underline{p}}$ which must be isomorphic to $E_A(A/\underline{p})$ when regarded as an A-module (since it is isomorphic to $E_{A_{\underline{p}}}(A_{\underline{p}}/\underline{p}A_{\underline{p}})$ as an $A_{\underline{p}}$-module, by (5.10)(vi). //

CHAPTER 6

Gorenstein Modules: Basic Properties and Characterizations.

(Reminder: "ring" means "commutative Noetherian ring with a non-zero identity".)

Throughout this Chapter, A will denote a ring. A will only be assumed to be local when this is explicitly stated, but, when this is the case, $\underline{m}$ will be consistently used to denote the maximal ideal of A.

In view of the main results of Chapters 4 and 5, it seems natural to examine those non-zero, f.-g. A-modules M for which $\underline{C(M)}$ provides a minimal injective resolution. It also seems natural to call these modules "Gorenstein": although several authors have discussed Cohen-Macaulay modules, I have been unable to find any mention of "Gorenstein module" in the literature.

The purpose of this Chapter is to investigate the properties of these Gorenstein modules, and to attempt to characterize them. I have tried to find generalizations of Bass' characterizations of Gorenstein rings given in (5.10) and (5.11) (i.e. Bass' Fundamental Theorem). This has not been as easy as might have been expected: perhaps one reason for this is that, while a local ring A for which $\text{inj.dim.}_A A$ is finite is forced to be Cohen-Macaulay, a non-zero, f.-g., module M over a local ring A with $\text{inj.dim.}_A M$ finite need not be Cohen-Macaulay. Some of Bass' characterizations (e.g. (5.10)(i),(ii), & (iii)) seem to have no sensible generalization to the case of Gorenstein modules; others (e.g. (5.10)(iv)) have straightfoward generalizations; and yet others (e.g. (5.10)(v)) have natural, but non-obvious generalizations.

As might be expected, I shall need many of the preliminary Lemmas employed by Bass for his characterizations of Gorenstein rings. As they are important for the subsequent argument, I give a survey of them. But first, the definition:-

(6.1)Definition. Let M be a non-zero, f.-g., A-module. M is s.t.b. Gorenstein if and only if the Cousin Complex for M provides a (minimal - see (5.12)) injective resolution for M.

Next, a useful technical result:-

(6.2)Proposition. Let S be an m.c.s. of A. Suppose that N is a f.-g. A-module. Then the connected right sequences of covariant functors from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_{S^{-1}A}$, $\{\mathrm{Ext}^n_{S^{-1}A}(S^{-1}N, S^{-1}(\ ))\}_{n\geq 0}$ and $\{S^{-1}(\mathrm{Ext}^n_A(N,\ ))\}_{n\geq 0}$ are isomorphic. In particular, if M is any A-module, $\mathrm{Ext}^n_{S^{-1}A}(S^{-1}N, S^{-1}M) \cong S^{-1}(\mathrm{Ext}^n_A(N,M))$.

Proof. I plan to use the Corollary to Theorem 10 of Chap.6 of Northcott (22). By (5.5), all I need show is that the functors $\mathrm{Hom}_{S^{-1}A}(S^{-1}N, S^{-1}(\ ))$ and $S^{-1}(\mathrm{Hom}_A(N,\ ))$ are naturally equivalent.

Let M be any A-module, and define $\phi_N(M): S^{-1}(\mathrm{Hom}_A(N,M)) \longrightarrow \mathrm{Hom}_{S^{-1}A}(S^{-1}N, S^{-1}M)$ by $\phi_N(M)(f/s) = (1/s).S^{-1}(f)$ (for $f \in \mathrm{Hom}_A(N,M)$ and $s \in S$). It is straightfoward to check that $\phi_N(M)$ is a well-defined $S^{-1}A$-monomorphism which actually constitutes a natural transformation of functors. All that remains is to show that each $\phi_N(M)$ is an epimorphism.

Let F_1, F_2 be f.-g. free A-modules s.t. $(E)\ F_2 \longrightarrow F_1 \longrightarrow N \longrightarrow 0$.

Then
$$\begin{array}{lccccc} (E)\ 0 \longrightarrow & S^{-1}(\mathrm{Hom}_A(N,M)) & \longrightarrow & S^{-1}(\mathrm{Hom}_A(F_1,M)) & \longrightarrow & S^{-1}(\mathrm{Hom}_A(F_2,M)) \\ & \downarrow \phi_N(M) & & \downarrow \phi_{F_1}(M) & & \downarrow \phi_{F_2}(M) \\ (C) \quad (E)\ 0 \longrightarrow & \mathrm{Hom}_{S^{-1}A}(S^{-1}N, S^{-1}M) & \to & \mathrm{Hom}_{S^{-1}A}(S^{-1}F_1, S^{-1}M) & \to & \mathrm{Hom}_{S^{-1}A}(S^{-1}F_2, S^{-1}M). \end{array}$$

Since it is easy to see that $\phi_F(M)$ is epimorphic and so isomorphic for each M when F is a f.-g. free A-module, the result follows from the above commutative diagram. //

Now I give some of Bass' results and Lemmas.

(6.3)Lemma. (See Remark (1), p.520, of Matlis (17).) If E is an injective A-module and $\underline{a}$ is an ideal of A, $\mathrm{Hom}_A(A/\underline{a}, E)$ is an injective $A/\underline{a}$-module. If $\underline{p}$ is a prime ideal of A and $\underline{a} \subseteq \underline{p}$, $\mathrm{Hom}_A(A/\underline{a}, E_A(A/\underline{p})) \cong E_{A/\underline{a}}((A/\underline{a})/(\underline{p}/\underline{a}))$ (as $A/\underline{a}$-modules). If $\underline{a} \not\subseteq \underline{p}$, $\mathrm{Hom}_A(A/\underline{a}, E_A(A/\underline{p})) = 0$.

Proof. $\mathrm{Hom}_A(A/\underline{a}, E) \cong (0:\underline{a})_E$. Suppose

$$\begin{array}{ccccc} (E)0 & \longrightarrow & M & \xrightarrow{f} & N \\ & & \downarrow{\scriptstyle h} & & \\ & & (0:\underline{a})_E & & \end{array}$$

is a diagram of $A/\underline{a}$-modules and -homomorphisms. These can be regarded as A-modules and -homomorphisms, and so there exists an A-homomorphism $g: N \longrightarrow E$ such that

$$\begin{array}{ccccc} (E)0 & \longrightarrow & M & \xrightarrow{f} & N \\ & & \downarrow{\scriptstyle h} & & \\ & & (0:\underline{a})_E & & \\ & & \downarrow & \swarrow{\scriptstyle g} & \\ & & E & & \end{array} \qquad (C).$$

But $g(N) \subseteq (0:\underline{a})_E$, and $g: N \longrightarrow (0:\underline{a})_E$ is also an $A/\underline{a}$-homomorphism.

For the second part, just observe that the $A/\underline{a}$-module $(0:\underline{a})_E$ is an essential extension of the $A/\underline{a}$-module $A/\underline{p} \cong (A/\underline{a})/(\underline{p}/\underline{a})$, while the third part is an obvious consequence of (5.2)(vi). //

(6.4)Lemma. (Theorem (2.2) of Bass (4).) Let:-

$$(E)0 \longrightarrow M \xrightarrow{e^{-1}} E^0(M) \xrightarrow{e^0} \cdots \longrightarrow E^n(M) \xrightarrow{e^n} E^{n+1}(M) \longrightarrow \cdots$$

be a minimal injective resolution for the A-module M, let $D = e^0(E^0(M))$, and suppose $x \in A$ is not a zero-divisor on M and is not a zero-divisor

on A. Then the induced complex of A/(x)-modules:-

(*) $0 \longrightarrow \mathrm{Hom}_A(A/(x),D) \longrightarrow \mathrm{Hom}_A(A/(x),E^1(M)) \longrightarrow \ldots \longrightarrow \mathrm{Hom}_A(A/(x),E^n(M)) \ldots$

provides a minimal injective resolution for the A/(x)-module $\mathrm{Hom}_A(A/(x),D) \cong M/xM$.

Proof. I first show that (*) is exact. From (E) $0 \longrightarrow D \longrightarrow E^1(M) \longrightarrow E^2(M)$, (*) is exact at $\mathrm{Hom}_A(A/(x),D)$ and $\mathrm{Hom}_A(A/(x),E^1(M))$. If $n > 1$, the homology module of the induced complex:-

$$\mathrm{Hom}_A(A/(x),E^{n-1}(M)) \longrightarrow \mathrm{Hom}_A(A/(x),E^n(M)) \longrightarrow \mathrm{Hom}_A(A/(x),E^{n+1}(M))$$

is isomorphic to $\mathrm{Ext}^n_A(A/(x),M)$, and the latter can easily be seen to be zero by using (E) $0 \longrightarrow A \xrightarrow{x} A \longrightarrow A/(x) \longrightarrow 0$ and the fact that A is a projective A-module. Thus (*) is exact.

Next, from (E) $0 \longrightarrow A \xrightarrow{x} A \longrightarrow A/(x) \longrightarrow 0$, I deduce, for an arbitrary injective A-module I,

(E) $0 \longrightarrow \mathrm{Hom}_A(A/(x),I) \longrightarrow \mathrm{Hom}_A(A,I) \xrightarrow{x} \mathrm{Hom}_A(A,I) \longrightarrow 0$,

and so $I = x.I$. Using this and the fact that (*) is exact, it is now straightfoward to check that the homomorphisms of (*) (other than the first zero one) are all essential. It follows from (6.3) that $\mathrm{Hom}_A(A/(x),E^i(M))$ is an A/(x)-injective module for $i \geq 1$. Thus, it remains only to show that $\mathrm{Hom}_A(A/(x),D) \cong M/xM$.

From the above, $E^0(M) = x.E^0(M)$. Since x is not a zero-divisor on M, $(0:x)_{E^0(M)} \cap e^{-1}(M) = 0$, and so $(0:x)_{E^0(M)} = 0$ since e^{-1} is an essential monomorphism. Thus multiplication by x produces an automorphism of $E^0(M)$, and this induces an isomorphism:-

$E^0(M)/e^{-1}(M) \cong E^0(M)/e^{-1}(x.M)$. Therefore $D \cong E^0(M)/x.(e^{-1}(M))$, and so:-

$\mathrm{Hom}_A(A/(x),D) \cong ((x.(e^{-1}(M)):x)_{E^0(M)})/x.(e^{-1}(M)) = e^{-1}(M)/e^{-1}(xM) \cong M/xM$.

//

(6.5)Corollary to Lemma (6.4). (Corollary (2.6) of Bass (5).) Under the hypotheses of (6.4), if x belongs to the prime ideal $\underline{p}$ of A, then for all $i \geqslant 0$, $\mu^i(\underline{p}/(x),M/xM) = \mu^{i+1}(\underline{p},M)$.

Proof. (6.3) and (6.4), together with the fact that $\mathrm{Hom}_A(A/(x), \bigoplus_{i\in I} C_i) \cong \bigoplus_{i\in I} \mathrm{Hom}_A(A/(x),C_i)$ for any family $\{C_i\}_{i\in I}$ of A-modules (by (2.19)). //

The following Lemma will be useful in applications of (6.4) & (6.5).

(6.6)Lemma. Suppose A is local and M & N are non-zero, f.-g., A-modules. Let $r = \min(\mathrm{codh}_A M, \mathrm{codh}_A N)$ (see (4.8)). Then any sequence which is both an M-sequence and an N-sequence can be extended to a maximal such, and this will have r terms.

Proof. Assume $r > 0$, and the $\underline{x}' = (x_1,\ldots,x_k)$ is both an M-sequence and an N-sequence, where $0 \leqslant k < r$. Since $k < \mathrm{codh}_A M = \mathrm{grade}_M \underline{m}$, and every M-sequence can be extended to a maximal M-sequence which must have $\mathrm{grade}_M \underline{m}$ terms (by (4.7)), $\underline{m}$ does not consist entirely of zero-divisors on $M/\underline{x}'M$. Thus $\underline{m} \notin \bigcup_{\underline{p}\in \mathrm{Ass}(M/\underline{x}'M)} \underline{p}$. Similarly, $\underline{m} \notin \bigcup_{\underline{p}\in \mathrm{Ass}(N/\underline{x}'N)} \underline{p}$. Thus $\underline{m} \notin \bigcup_{\underline{p}\in \mathrm{Ass}(M/\underline{x}'M)\cup \mathrm{Ass}(N/\underline{x}'N)} \underline{p}$, so there exists an $x_{k+1} \in \underline{m} - \bigcup_{\underline{p}\in \mathrm{Ass}(M/\underline{x}'M)\cup \mathrm{Ass}(N/\underline{x}'N)} \underline{p}$. Then (using Nakayama's Lemma), $(x_1,\ldots,x_k,x_{k+1})$ is both an M-sequence and an N-sequence. Now use induction. //

(6.7)Lemma. (Lemma (2.7) of Bass (5).) Let M be an A-module, $\underline{p} \in V(A)$, and let $k(\underline{p}) = A_{\underline{p}}/\underline{p}A_{\underline{p}}$ be the residue field of the local ring $A_{\underline{p}}$. Then:- $\mu^i(\underline{p},M) = \dim_{k(\underline{p})} \mathrm{Ext}^i_{A_{\underline{p}}}(k(\underline{p}),M_{\underline{p}}) = \dim_{k(\underline{p})} (\mathrm{Ext}^i_A(A/\underline{p},M))_{\underline{p}}$. (Vector space dimension.) In particular, $\mu^i(\underline{p},M) > 0$ only if $\underline{p} \in \mathrm{Supp}(M)$,

and if M is f.-g., $\mu^i(\underline{p},M)$ is finite for each $i \geqslant 0$ and for each $\underline{p} \in V(A)$.

Proof. The fact that $\mathrm{Ext}^i_{A_{\underline{p}}}(k(\underline{p}),M_{\underline{p}}) \cong (\mathrm{Ext}^i_A(A/\underline{p},M))_{\underline{p}}$ follows from (6.2). By (5.9), $\mu^i(\underline{p},M) = \mu^i(\underline{p}A_{\underline{p}},M_{\underline{p}})$. Thus it is sufficient to assume A is local and to show $\mu^i(\underline{m},M) = \dim_k(\mathrm{Ext}^i_A(k,M))$, where $k = A/\underline{m}$.

Let $(E)\ 0 \longrightarrow M \xrightarrow{e^{-1}} E^0 \xrightarrow{e^0} E^1 \longrightarrow \cdots \longrightarrow E^i \xrightarrow{e^i} E^{i+1} \longrightarrow \cdots$ be a minimal injective resolution for M, and let $T^i = \{x \in E^i \mid \underline{m}x = 0\}$ for each $i \geqslant 0$. There is an isomorphism of complexes:-

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(k,E^0) & \longrightarrow & \mathrm{Hom}_A(k,E^1) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_A(k,E^i) & \longrightarrow & \cdots \\ & & \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & & \\ (C)\ 0 & \longrightarrow & T^0 & \longrightarrow & T^1 & \longrightarrow & \cdots & \longrightarrow & T^i & \longrightarrow & \cdots \end{array}$$

and $\mathrm{Ext}^i_A(k,M)$ is isomorphic to the ith homology module of either complex.

Note that $e^iT^i = 0$. For if $0 \neq x \in T^i$, Ax must intersect $\ker e^i$ non-trivially, and since Ax is simple, $Ax \subseteq \ker e^i$. Thus the complexes in the above diagram have zero differentiation, and $\mathrm{Ext}^i_A(k,M) \cong \mathrm{Hom}_A(k,E^i) \cong T^i$ for each $i \quad 0$.

Now $E^i \cong \bigoplus_{\underline{p} \in V(A)} \mu^i(\underline{p},M)E_A(A/\underline{p})$, so by (2.19) and (5.2)(vi), $\mathrm{Hom}_A(k,E^i) \cong \bigoplus \mu^i(\underline{m},M)\mathrm{Hom}_A(k,E_A(k))$. But, by Theorem (3.4)(4) of Matlis (17), $\mathrm{Hom}_A(k,E_A(k)) \cong (0:\underline{m})_{E_A(k)} \cong k$, so $\mu^i(\underline{m},M) = \dim_k \mathrm{Hom}_A(k,E^i)$, as required. //

(6.8)Corollary. Suppose M is a non-zero, f.-g., A-module. Then M is Cohen-Macaulay if and only if, for each $\underline{p} \in V(M)$ and $i < \mathrm{ht}_M\underline{p}$, $\mu^i(\underline{p},M) = 0$.

Proof. It follows from (4.19) and (4.20) that M is Cohen-Macaulay if and only if $M_{\underline{p}}$ is a C.-M. $A_{\underline{p}}$-module for each $\underline{p} \in V(M)$. But, for $\underline{p} \in V(M)$, $M_{\underline{p}}$ is C.-M. if and only if $\mathrm{codh}_{A_{\underline{p}}} M_{\underline{p}} = \text{K-Dim}_{A_{\underline{p}}} M_{\underline{p}}$, i.e. if and only if

$\operatorname{grade}_{M_{\underline{p}}} \underline{p}A_{\underline{p}} = \operatorname{ht}_M \underline{p}$, i.e. if and only if $\operatorname{Ext}^i_{A_{\underline{p}}}(k(\underline{p}), M_{\underline{p}}) = 0$ for $i < \operatorname{ht}_M \underline{p}$.

Now use (6.7). //

(6.9)Corollary. (Note Corollary (2.12) of Bass (5), which gives the following result in the particular case in which M = A.) Suppose M is a non-zero, f.-g., A-module and $\underline{p} \in V(M)$. Then $\operatorname{codh}_{A_{\underline{p}}} M_{\underline{p}}$ is the least integer r for which $\mu^r(\underline{p}, M) > 0$.

Proof. $\underline{p}A_{\underline{p}}$ is certainly $M_{\underline{p}}$-proper, and $\operatorname{codh}_{A_{\underline{p}}} M_{\underline{p}}$ is the least integer r for which $\operatorname{Ext}^r_{A_{\underline{p}}}(k(\underline{p}), M_{\underline{p}}) \neq 0$. Now use (6.7). //

(6.10)Lemma. (Lemma (3.1) of Bass (5).) Suppose that M is a f.-g. A-module and $\underline{q} \subsetneq \underline{p}$ are primes of A with no prime between them. Then $\mu^h(\underline{q}, M) > 0$ implies $\mu^{h+1}(\underline{p}, M) > 0$. //

(6.11)Corollary. Suppose A is a local ring and the non-zero, f.-g., A-module M is C.-M. with Krull Dimension n. Then every maximal saturated chain of primes in $V_A(M)$ has length n.

Proof. Let $\underline{p}_0 \subsetneq \underline{p}_1 \subsetneq \cdots \subsetneq \underline{p}_k = \underline{m}$ be a maximal saturated chain of primes of $V_A(M)$, so that $k \leqslant n$. $\underline{p}_0 \in \operatorname{Ass}_A(M)$, so $\underline{p}_0 \cdot A_{\underline{p}_0} \in \operatorname{Ass}_{A_{\underline{p}_0}}(M_{\underline{p}_0})$ (by (1.14)), so $\mu^0(\underline{p}_0, M) > 0$ by (1.11) and (6.7). By (6.10), $\mu^k(\underline{p}_k, M) > 0$, i.e. $\mu^k(\underline{m}, M) > 0$. Thus, by (6.8), k = n. //

(6.12)Corollary. (Corollary (3.2) of Bass (5).) If M is a f.-g. A-module such that $\operatorname{inj.dim.}_A M = r < \infty$, then $r \geqslant \text{K-Dim}_A M$ and $\mu^r(\underline{p}, M) > 0$ only if $\underline{p}$ is maximal. //

(6.13)Lemma. (Lemma (3.3) of Bass (5).) Suppose A is local and that M is

a non-zero, f.-g. A-module such that $\text{inj.dim.}_A M = r < \infty$. Then $r = \text{codh}_A A$, and so:- $\text{codh}_A M \leqslant (\text{ht}_M \underline{m} =)\ \text{K-Dim}_A M \leqslant \text{inj.dim.}_A M = \text{codh}_A A$.

Proof. If $\underline{p} \in V(A)$ is such that $\mu^r(\underline{p}, M) > 0$, $\underline{p} = \underline{m}$ by (6.11), so $\text{Ext}^r_A(k, M) \neq 0$, where $k = A/\underline{m}$. Since $\text{inj.dim.}_A M = r$, $\text{Ext}^{r+1}_A(X, M) = 0$ for every A-module X, so that $\text{Ext}^r_A(\ , M)$ is a right exact functor. Consequently, if N is any A-module with a submodule isomorphic to k, it follows from (E) $0 \longrightarrow k \longrightarrow M$ that $\text{Ext}^r_A(N, M) \neq 0$.

Suppose $s = \text{codh}_A A$, and $\underline{x}$ is a maximal A-sequence. By, e.g., Theorem 20 of Chapter 9 of Northcott (22), $\text{hd}_A(A/\underline{x}) = s$. Since $\underline{x}$ is a maximal A-sequence, $\underline{m} \in \text{Ass}_A(A/\underline{x})$, so $\text{Ext}^r_A(A/\underline{x}, M) \neq 0$, and thus $r \leqslant s$. But, by (2.11), $\text{Ext}^s_A(A/\underline{x}, M) \neq 0$, so $s \leqslant r$. Therefore $\text{inj.dim.}_A M = \text{codh}_A A$. //

It now becomes obvious, by putting $M = A$ in (6.13), that if a local ring A has finite injective dimension, then A has to be Cohen-Macaulay. However, if M is a non-zero, f.-g., A-module with $\text{inj.dim.}_A M$ finite, M need not be Cohen-Macaulay. For example, let A be a regular local ring such that $\text{K-Dim } A \geqslant 1$. Then, by the work of Auslander and Buchsbaum in (2), A has finite global dimension, n say. (To say that A has finite global dimension n means that $n = \sup\{\text{hd}_A M \mid M \text{ an A-module}\}$.) Consequently, by (2.6), $\text{Ext}^{n+1}_A(M, N) = 0$ for every pair of A-modules M and N, so $\text{inj.dim.}_A M \leqslant n$ for every A-module M (by (6.7)). Let $M = A/\underline{m} \oplus A$. $V(M) = V(A)$, so $\text{K-Dim}_A M \geqslant 1$, while $\underline{m} \in \text{Ass}_A(M)$, so $\text{codh}_A M = 0$. Thus M has finite injective dimension but is not Cohen-Macaulay.

Actually, I have now catalogued all those results of Bass which I shall need, and I can begin the examination of Gorenstein modules. First of all it's worth making the following observations:-

(6.14)Corollary to (6.13). Suppose M is a non-zero, f.-g., A-module and $\underline{p} \in V(M)$. Suppose also that inj.dim.$_{A_{\underline{p}}} M_{\underline{p}}$ is finite. Then, if $\mu^i(\underline{p},M) > 0$, $\operatorname{codh}_{A_{\underline{p}}} M_{\underline{p}} \leqslant i \leqslant \operatorname{codh}_{A_{\underline{p}}} A_{\underline{p}}$, while for i at either extremity of this range, $\mu^i(\underline{p},M) > 0$.

Proof. By (5.9), $\mu^i(\underline{p},M) = \mu^i(\underline{p}A_{\underline{p}}, M_{\underline{p}})$, and by (6.13) inj.dim.$_{A_{\underline{p}}} M_{\underline{p}} = \operatorname{codh}_{A_{\underline{p}}} A_{\underline{p}}$. Now use (6.9) and (6.12). //

(6.15)Lemma. Suppose M is a non-zero, f.-g., A-module and $\underline{p} \in V(M)$. Suppose also that for i > some constant N, $\mu^i(\underline{p},M) = 0$. Then inj.dim.$_{A_{\underline{p}}} M_{\underline{p}}$ is finite and $\leqslant N$.

Proof. $\mu^i(\underline{p},M) = \mu^i(\underline{p}A_{\underline{p}}, M_{\underline{p}})$, so I assume A is local and $\underline{p} = \underline{m}$. If $E^{N+1}(M) \neq 0$, then for some $\underline{q} \in V(M)$, $\mu^{N+1}(\underline{q},M) > 0$. Let $\underline{q} = \underline{q}_0 \subsetneqq \underline{q}_1 \subsetneqq \cdots \subsetneqq \underline{q}_k = \underline{m}$ be a saturated chain of primes of V(M). By (6.10), $\mu^{N+k+1}(\underline{m},M) > 0$, a contradiction. Thus $E^{N+1}(M) = 0$ and inj.dim.$_A M \leqslant N$. //

(6.16)Corollary. Suppose M is a non-zero, f.-g., A-module and $\underline{p} \in V(M)$. Suppose also that $\mu^i(\underline{p},M) = 0$ if and only if $i \neq K$. Then inj.dim.$_{A_{\underline{p}}} M_{\underline{p}}$ is finite, and:- $\operatorname{codh}_{A_{\underline{p}}} M_{\underline{p}} = \operatorname{ht}_M \underline{p} = \text{inj.dim.}_{A_{\underline{p}}} M_{\underline{p}} = \operatorname{codh}_{A_{\underline{p}}} A_{\underline{p}} = K$.

Proof. Obvious from (6.15), (6.14), and (6.13). //

Next, some further introductory Lemmas to set the stage.

(6.17)Lemma. Suppose N and M are non-zero f.-g. A-modules s.t. $T = \operatorname{Ext}^j_A(N,M) \neq 0$. Then $V(T) \subseteq V(M)$ and so $\underline{a} = (0:T)_A$ is M-proper.

Proof. T is f.-g., and, if $\underline{p} \in V(A)$, $T_{\underline{p}} = \operatorname{Ext}^j_{A_{\underline{p}}}(N_{\underline{p}}, M_{\underline{p}})$ (by (6.2)).

Thus, if $T_{\underline{p}} \neq 0$, then $M_{\underline{p}} \neq 0$, and so $V(T) \subseteq V(M)$. (1.18) then shows that $\underline{a}$ is M-proper. //

I shall need a result of Matlis (see (19)) on Artinian modules (i.e. modules which satisfy the descending chain condition on submodules).

(6.18)Definitions. (1 of Matlis (19).) An A-module M is said to have maximal orders if and only if, for each $x \in M$, the only prime ideals containing $(0:x)_A$ are maximal.

The socle of an A-module M is the submodule of M generated by those elements of M which are annihilated by some maximal ideal of A. If A is local, socle of $M = (0:\underline{m})_M \cong \mathrm{Hom}_A(A/\underline{m}, M)$.

(6.19)Proposition. (Proposition 3 of Matlis (19).) Let M be an A-module. The following conditions are equivalent:-

(i) M is Artinian.

(ii) M is isomorphic to a submodule of an A-module of the form $E_A(A/\underline{m}_1) \oplus E_A(A/\underline{m}_2) \oplus \ldots \oplus E_A(A/\underline{m}_n)$, where $\underline{m}_1, \ldots, \underline{m}_n$ are maximal ideals of A.

(iii) M has maximal orders and finitely generated socle. //

(6.20)Corollary. If M is an Artinian A-module, then all the terms $E^i(M)$ for $i \geq 0$ of a minimal injective resolution for M are Artinian. //

It is obvious that Gorenstein modules form a particularly nice class of Cohen-Macaulay modules: I therefore make a closer examination of certain properties of Cohen-Macaulay modules.

(6.21)Proposition. Suppose A is local and the non-zero, f.-g., A-module M is C.-M. with $K\text{-}\dim_A M = n$. Then M^n is Artinian, and, if

(E) $0 \longrightarrow M^n \longrightarrow E^0(M^n) \longrightarrow E^1(M^n) \longrightarrow \ldots \longrightarrow E^r(M^n) \longrightarrow \ldots$

is a minimal injective resolution for M^n, $E^r(M^n) \cong \oplus \mu^{r+n}(\underline{m}, M) E_A(A/\underline{m})$.

Proof. By (3.7)(vi), every element of M^n is annihilated by some power of $\underline{m}$, so M^n has maximal orders. By (4.24), $M^n \neq 0$. By P.E.C.C.A. (4.22), $\mathrm{Ext}^n_A(A/\underline{m}, M) \cong \mathrm{Hom}_A(A/\underline{m}, M^n)$, so the socle of M^n is f.-g. Thus M^n is Artinian by (6.19). By (6.20), $E^i(M^n)$ is Artinian for each $i \geqslant 0$, so by (6.19) and the structure theory for injective A-modules, each $E^i(M^n)$ is isomorphic to a direct sum of a finite number of copies of $E_A(A/\underline{m})$.

Thus $E^i(M^n) \cong \oplus \mu^i(\underline{m}, M^n) E_A(A/\underline{m})$, and, by (6.7), $\mu^i(\underline{m}, M^n) = \dim_k \mathrm{Ext}^i_A(k, M^n)$, where $k = A/\underline{m}$.

But, by P.E.C.C.A. (4.22), $\mathrm{Ext}^i_A(k, M^n) \cong \mathrm{Ext}^{i+n}_A(k, M)$, so $\mu^i(\underline{m}, M^n) = \mu^{i+n}(\underline{m}, M)$. //

(6.22)Lemma. Suppose that A is local and that the non-zero, f.-g., A-module M is C.-M. with $K\text{-}\dim_A M = n$. Then the functors $\mathrm{Ext}^n_A(\ , M)$ and $\mathrm{Hom}_A(\ , M^n)$ from $\underline{\mathcal{C}}^1_A$ to $\underline{\mathcal{C}}_A$ are naturally equivalent.

Proof. If L is an A-module of finite length, $(0:L)_A$ is $\underline{m}$-primary (by (1.20)) and so $(0:L)_A$ is not contained in any prime $\underline{p} \in V(M)$ which has M-height $< n$. Now apply P.E.C.C.A. (4.22) to obtain, for an A-module L of finite length, natural isomorphisms:-

$$\mathrm{Ext}^{n-1}_A(L, \mathrm{coker}\ d^{-1}) \xrightarrow{\cong} \mathrm{Ext}^n_A(L, M)$$
$$\mathrm{Ext}^{n-2}_A(L, \mathrm{coker}\ d^{0}) \xrightarrow{\cong} \mathrm{Ext}^{n-1}_A(L, \mathrm{coker}\ d^{-1})$$
$$\vdots$$
$$\mathrm{Ext}^1_A(L, \mathrm{coker}\ d^{n-3}) \xrightarrow{\cong} \mathrm{Ext}^2_A(L, \mathrm{coker}\ d^{n-4})$$
$$\mathrm{Hom}_A(L, M^n) \xrightarrow{\cong} \mathrm{Ext}^1_A(L, \mathrm{coker}\ d^{n-3})$$

(with the usual Cousin Complex notation).

That these isomorphisms actually constitute natural transformations of functors follows from the properties of Extension functors described in the final paragraph of (2.2). //

The next Theorem commences the characterization of Gorenstein modules.

(6.23)Theorem. Let M be a non-zero, f.-g., A-module. Then the following conditions are equivalent:-

(i) M is Gorenstein.

(ii) For each $j \geqslant 0$ and f.-g. A-module N for which $\mathrm{Ext}^j_A(N,M) \neq 0$, $\underline{a} = (0:\mathrm{Ext}^j_A(N,M))_A$ is (M-proper and) such that $\mathrm{grade}_M \underline{a} \geqslant j$.

(iii) For each $\underline{p} \in V(M)$, $\mathrm{codh}_{A_{\underline{p}}} M_{\underline{p}} = \mathrm{ht}_M \underline{p} = \mathrm{inj.dim.}_{A_{\underline{p}}} M_{\underline{p}} = \mathrm{codh}_{A_{\underline{p}}} A_{\underline{p}}$.

(iv) For each $\underline{p} \in V(M)$, $\mu^i(\underline{p},M) = 0$ if and only if $i \neq \mathrm{ht}_M \underline{p}$.

Proof. Let the Cousin Complex for M be:-

$$\underline{C(M)}: 0 \longrightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots .$$

As usual, I write $M^{-1} = M$, $M^{-2} = 0$, and $d^{-2} = 0: M^{-2} \longrightarrow M^{-1}$.

(i)$\Rightarrow$(ii) If M is Gorenstein, $\underline{C(M)}$ provides a minimal injective resolution for M, so, in particular, M is Cohen-Macaulay (by (4.23)).

Suppose $j \geqslant 0$, and N is a f.-g. A-module with $T = \mathrm{Ext}^j_A(N,M) \neq 0$. By (6.17), $\underline{a} = (0:T)_A$ is M-proper. Since M is C.-M., $\mathrm{grade}_M \underline{a} = \mathrm{ht}_M \underline{a}$, so it is sufficient to show that $\underline{a}$ is not contained in any $\underline{p} \in V(M)$ having M-height $< j$. For this, in turn, it is sufficient to show that, if $\underline{p} \in V(M)$ has M-height $< j$, $T_{\underline{p}} = 0$ (since T is f.-g.), i.e. to show that $\mathrm{Ext}^j_{A_{\underline{p}}}(N_{\underline{p}}, M_{\underline{p}}) = 0$. By (5.8), (3.13), (3.14), & (4.24), $M_{\underline{p}}$ is a Gorenstein $A_{\underline{p}}$-module with $\mathrm{inj.dim.}_{A_{\underline{p}}} M_{\underline{p}} = \mathrm{ht}_M \underline{p} < j$, so clearly

$\mathrm{Ext}^j_{A_{\underline{p}}}(N_{\underline{p}}, M_{\underline{p}}) = 0$

(ii)$\Rightarrow$(iii) Suppose $\underline{p} \in V(M)$. By (6.9) $\mu^r(\underline{p},M) > 0$ for at least one positive integer r. Suppose j is any positive integer such that $\mu^j(\underline{p},M) > 0$. By (6.7), $(\mathrm{Ext}^j_A(A/\underline{p},M))_{\underline{p}} \neq 0$, so $\mathrm{Ext}^j_A(A/\underline{p},M) = T \neq 0$ and $\underline{p} \supseteq \underline{a} = (0:T)_A$. But clearly $\underline{p} \subseteq \underline{a}$. Thus $\underline{a} = \underline{p}$. By hypothesis, $\mathrm{grade}_M \underline{a} \geqslant j$.

So $\mu^j(\underline{p},M) > 0$ implies $j \leqslant \mathrm{grade}_M \underline{p}$. It follows from (6.15) that $\mathrm{inj.dim.}_{A_{\underline{p}}} M_{\underline{p}}$ is finite and $\leqslant \mathrm{grade}_M \underline{p}$, and so by (6.13):-

$$\mathrm{grade}_M \underline{p} = \mathrm{ht}_M \underline{p} = \mathrm{inj.dim.}_{A_{\underline{p}}} M_{\underline{p}} = \mathrm{codh}_{A_{\underline{p}}} A_{\underline{p}}.$$

This is true for each $\underline{p} \in V(M)$, so by (4.17) M is C.-M. Thus, for each $\underline{p} \in V(M)$, $\mathrm{codh}_{A_{\underline{p}}} M_{\underline{p}} = \mathrm{ht}_M \underline{p} = \mathrm{inj.dim.}_{A_{\underline{p}}} M_{\underline{p}} = \mathrm{codh}_{A_{\underline{p}}} A_{\underline{p}}$.

(iii)$\Rightarrow$(iv) This follows immediately from (6.13) and (6.14).

(iv)$\Rightarrow$(i) By (6.8), M is C.-M., so $\underline{C(M)}$ is exact: it remains only to show that, for each $n \geqslant 0$, $M^n = \bigoplus_{\substack{\underline{p} \in V(M) \\ \mathrm{ht}_M \underline{p} = n}} (\mathrm{coker}\ d^{n-2})_{\underline{p}}$ is injective.

For this it is sufficient to show that, for each $\underline{p} \in V(M)$, $(\mathrm{coker}\ d^{n-2})_{\underline{p}}$ is $A_{\underline{p}}$-injective (where $n = \mathrm{ht}_M \underline{p}$). By localising at such a $\underline{p} \in V(M)$ (and using (3.14) & (4.24)), it is sufficient to assume that A is local, M is C.-M., $\text{K-Dim}_A M = \mathrm{ht}_M \underline{m} = n$, and $\mu^i(\underline{m},M) = 0$ if and only if $i \neq n$, and then to show that M^n is A-injective.

By (6.21), since $\mu^{n+1}(\underline{m},M) = 0$, $E^1(M^n) = 0$, so consequently M^n is injective. //

(6.24)Corollary. Suppose M is a non-zero, f.-g., A-module. Then the following statements are equivalent:-

(i) M is Gorenstein.

(ii) $M_{\underline{p}}$ is a Gorenstein $A_{\underline{p}}$-module for each $\underline{p} \in V(M)$.

(iii) $M_{\underline{m}}$ is a Gorenstein $A_{\underline{m}}$-module for each maximal $\underline{m} \in V(M)$.

Proof. (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are obvious.

(iii)$\Rightarrow$(i) Suppose $\underline{p} \in V(M)$, and let $\underline{m}$ be a maximal ideal of A which contains $\underline{p}$, so $\underline{m} \in V(M)$. Since $M_{\underline{m}}$ is a Gorenstein $A_{\underline{m}}$-module, by (6.23)(iv), $\mu^i(\underline{p}A_{\underline{m}}, M_{\underline{m}}) = 0$ if and only if $i \neq \mathrm{ht}_{M_{\underline{m}}} \underline{p}A_{\underline{m}}$. Thus $\mu^i(\underline{p}, M) = 0$ if and only if $i \neq \mathrm{ht}_M \underline{p}$, and so, by (6.23), M is Gorenstein. //

Observe that condition (6.23)(ii) is the natural generalization of condition (5.10)(iv), but that (6.23)(iv) is not the obvious generalization of (5.10)(vi).

(6.25)Proposition. Suppose the non-zero, f.-g., A-module M is Gorenstein, and that $\underline{x} = (x_1, \ldots, x_k)$ is both an M-sequence and an A-sequence. Then the $A/\underline{x}$-module $M/\underline{x}M$ is Gorenstein.

Proof. It is clearly sufficient to prove this when $k = 1$. Let $\underline{p}^* = \underline{p}/(x_1)$, $A^* = A/(x_1)$, and $M^* = M/x_1M$, where $\underline{p}$ is an arbitrary prime of A containing x_1.

Suppose $\underline{p}^* \in V_{A^*}(M^*)$. Then $\underline{p}^* \supseteq (0:M^*)_{A^*}$, so $\underline{p} \supseteq (x_1M:M)_A \supseteq (0:M)_A$. Thus $\underline{p} \in V_A(M)$.

By (6.23)(iv), $\mu^i(\underline{p}, M) = 0$ if and only if $i \neq \mathrm{ht}_M \underline{p}$. By (6.5), $\mu^i(\underline{p}^*, M^*) = 0$ if and only if $i \neq \mathrm{ht}_M \underline{p} - 1$. So, by (6.16), $\mathrm{inj.dim}_{A^*_{\underline{p}^*}} M^*_{\underline{p}^*}$ is finite and:-

$$\mathrm{codh}_{A^*_{\underline{p}^*}} M^*_{\underline{p}^*} = \mathrm{ht}_{M^*} \underline{p}^* = \mathrm{inj.dim}_{A^*_{\underline{p}^*}} M^*_{\underline{p}^*} = \mathrm{codh}_{A^*_{\underline{p}^*}} A^*_{\underline{p}^*}.$$

Since this is true for each $\underline{p} \in V(A)$ which contains x_1 and which is

s.t. $\underline{p}^* \in V_{A^*}(M^*)$, it follows from (6.23) that M^* is a Gorenstein A^*-module. //

(6.26)Corollary. Suppose A is local, and the non-zero, f.-g., A-module M is Gorenstein with $K\text{-Dim}_A M = n$. Then, if $\underline{x}$ is maximal w.r.t. the property of being both an M-sequence and an A-sequence, $M/\underline{x}M$ is an injective $A/\underline{x}$-module; also A is a Cohen-Macaulay ring.

Proof. By (6.23), $\text{codh}_A M = K\text{-Dim}_A M = \text{inj.dim.}_A M = \text{codh}_A A = n$. Bearing in mind (6.6), let $\underline{x} = (x_1,\ldots,x_n)$ be both an M-sequence and an A-sequence. By (6.25), $M/\underline{x}M$ is a Gorenstein $A/\underline{x}$-module, and $K\text{-Dim}_A M/\underline{x}M = 0$, by (4.12).

Thus $\text{codh}_{A/\underline{x}} M/\underline{x}M = K\text{-Dim}_{A/\underline{x}} M/\underline{x}M = \text{inj.dim.}_{A/\underline{x}} M/\underline{x}M = \text{codh}_{A/\underline{x}} A/\underline{x} = 0$.

In particular, $M/\underline{x}M$ is a non-zero, f.-g., injective $A/\underline{x}$-module.

Write $M^* = M/\underline{x}M$, $A^* = A/\underline{x}$, and $\underline{m}^* = \underline{m}/\underline{x}$. Since $K\text{-Dim}_{A^*} M^* = 0$, M^* is an A^*-module of finite length. By (6.19), M^* must be isomorphic to a direct sum of a finite number of copies of $E_{A^*}(A^*/\underline{m}^*)$. Since M^* is of finite length, there is an integer $N > 0$ s.t. $\underline{m}^{*N}.E_{A^*}(A^*/\underline{m}^*) = 0$. Thus, by (3.4), (3.9), & (3.10) of Matlis (17), $\underline{m}^{*N}/\underline{m}^{*N+1} = 0$. Thus, by Nakayama's Lemma, $\underline{m}^{*N} = 0$, and $A^* = A/\underline{x}$ is an Artin local ring.

Thus $\underline{x}$ is an $\underline{m}$-primary ideal of A, and so $n \geq K\text{-Dim } A$. Thus A is Cohen-Macaulay. //

I am now ready to state and prove the main Theorem.

(6.27)Theorem. Suppose that A is local, M is a non-zero, f.-g., A-module such that $K\text{-Dim}_A M = \text{ht}_M \underline{m} = n$, and that the Cousin Complex for M is:-

$\underline{C(M)}: 0 \longrightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \ldots \longrightarrow M^{n-1} \xrightarrow{d^{n-1}} M^n \longrightarrow 0$ (with,

as usual, $M^{-1} = M$, $M^{-2} = 0$, and $d^{-2} = 0 : M^{-2} \longrightarrow M^{-1}$). Then the following statements are equivalent:-

(i) M is Gorenstein.

(ii) For each $\underline{p} \in V(M)$, $\mu^i(\underline{p}, M) = 0$ if and only if $i \neq \mathrm{ht}_M \underline{p}$.

(iii) $\mu^i(\underline{m}, M) = 0$ if and only if $i \neq \mathrm{ht}_M \underline{m} = n$.

(iv) M is Cohen-Macaulay and M^n is injective.

(v) M is Cohen-Macaulay and there is an Artinian injective A-module I and a natural equivalence of the functors $\mathrm{Ext}^n_A(\ ,M)$ and $\mathrm{Hom}_A(\ ,I)$ from $\underline{\mathcal{C}}^1_A$ to $\underline{\mathcal{C}}_A$.

(vi) $\mathrm{codh}_A M = \text{K-Dim}_A M = \mathrm{inj.dim.}_A M = \mathrm{codh}_A A$.

(vii) $\mathrm{codh}_A M = \text{K-Dim}_A M = \mathrm{inj.dim.}_A M = \mathrm{codh}_A A = \text{K-Dim } A$ (so A is Cohen-Macaulay).

(viii) $\mathrm{inj.dim.}_A M$ is finite, and every A-sequence is an M-sequence.

(ix) For every (resp., for some) sequence $\underline{x}$ in A, maximal w.r.t. the property of being both an M-sequence and A-sequence, $M/\underline{x}M$ is an injective $A/\underline{x}$-module.

(x) For each $j \geqslant 0$ and f.-g. A-module N s.t. $\mathrm{Ext}^j_A(N,M) \neq 0$, $\mathrm{grade}_M((0:\mathrm{Ext}^j_A(N,M))_A) \geqslant j$. $((0:\mathrm{Ext}^j_A(N,M))_A$ is M-proper, of course.)

Comments. I have already compared (6.27)(ii) with (5.10)(vi). (6.27)(v) is analogous to (5.11)(v), Although (6.27)(viii) reduces to (5.11)(i) when M = A, it is not the most natural generalization of this condition: as I explained earlier, "$\mathrm{inj.dim.}_A M$ finite" does not even force M to be Cohen-Macaulay. Again, I have already pointed out that (6.27)(x) is a perfect generalization of (5.10)(iv).

Proof. (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(x) are proved in (6.23).

(i)$\Rightarrow$(vii) follows from (6.23) and (6.26). (vii)$\Rightarrow$(vi) is obvious.

(vi)$\Rightarrow$(iii) is immediate from (6.14). (iii)$\Rightarrow$(iv) follows from (6.8) and (6.21). Similarly, (iv)$\Rightarrow$(iii) follows from (6.8) and (6.21). (iv)$\Rightarrow$(v) is immediate from (6.21) and (6.22) on putting $M^n = I$.

(iii)$\Rightarrow$(i) Suppose $\underline{p} \in V(M)$. If there are two distinct integers k and h such that $\mu^k(\underline{p},M) > 0$ and $\mu^h(\underline{p},M) > 0$, then, by considering a saturated chain of prime ideals from $\underline{p}$ to $\underline{m}$, and using (6.10), there are two distinct integers k+r and h+r for which $\mu^{k+r}(\underline{m},M) > 0$ and $\mu^{h+r}(\underline{m},M) > 0$, a contradiction. Thus (see (6.8)), $\operatorname{codh}_{A_{\underline{p}}} M_{\underline{p}}$ is the only integer j for which $\mu^j(\underline{p},M) > 0$.

Thus, by (6.16), $\operatorname{codh}_{A_{\underline{p}}} M_{\underline{p}} = \operatorname{ht}_M \underline{p} = \operatorname{inj.dim.}_{A_{\underline{p}}} M_{\underline{p}} = \operatorname{codh}_{A_{\underline{p}}} A_{\underline{p}}$, and M is Gorenstein, by (6.23).

(v)$\Rightarrow$(iv) By (6.22), there is a natural equivalence between the functors $\operatorname{Hom}_A(\ ,M^n)$ and $\operatorname{Hom}_A(\ ,I)$ from $\mathcal{C}_A^1$ to $\mathcal{C}_A$ given by, say,

$\phi(L):\operatorname{Hom}_A(L,M^n) \xrightarrow{\cong} \operatorname{Hom}_A(L,I)$ for a module L of finite length. The aim is to show $M^n \cong I$.

Define A-module isomorphisms:-

$\chi_k:\operatorname{Hom}_A(A/\underline{m}^k,M^n) \xrightarrow{\cong} (0:\underline{m}^k)_{M^n}$ by $\chi_k(f) = f(\bar{1})$;

$\theta_k:\operatorname{Hom}_A(A/\underline{m}^k,I) \xrightarrow{\cong} (0:\underline{m}^k)_I$ by $\theta_k(f) = f(\bar{1})$;

for each $k > 0$. ($\bar{1}$ is the natural image of 1 in $A/\underline{m}^k$.)

If $0 < k \leqslant h$, I deduce from the natural homomorphism $A/\underline{m}^h \longrightarrow A/\underline{m}^k$,

$$\begin{array}{ccccccc}
(0:\underline{m}^k)_{M^n} & \xrightarrow[\cong]{\chi_k^{-1}} & \operatorname{Hom}_A(A/\underline{m}^k,M^n) & \xrightarrow[\cong]{\phi(A/\underline{m}^k)} & \operatorname{Hom}_A(A/\underline{m}^k,I) & \xrightarrow[\cong]{\theta_k} & (0:\underline{m}^k)_I \\
\cap\downarrow & & \downarrow & & \downarrow & & \cap\downarrow \\
(0:\underline{m}^h)_{M^n} & \xrightarrow[\cong]{\chi_h^{-1}} & \operatorname{Hom}_A(A/\underline{m}^h,M^n) & \xrightarrow[\cong]{\phi(A/\underline{m}^h)} & \operatorname{Hom}_A(A/\underline{m}^h,I) & \xrightarrow[\cong]{\theta_h} & (0:\underline{m}^h)_I
\end{array} \quad (C)$$

Thus, for each $k > 0$, define $\lambda_k:(0:\underline{m}^k)_{M^n} \xrightarrow{\cong} (0:\underline{m}^k)_I$ by:-

$\lambda_k = \theta_k \cdot \phi(A/\underline{m}^k) \cdot \chi_k^{-1}$. For $k \leqslant h$:-

$$\begin{array}{ccc} (0:\underline{m}^k)_{M^n} & \xrightarrow[\cong]{\lambda_k} & (0:\underline{m}^k)_I \\ \cap\downarrow & & \cap\downarrow \\ (0:\underline{m}^h)_{M^n} & \xrightarrow[\cong]{\lambda_h} & (0:\underline{m}^h)_I \end{array} \qquad (C)$$

By (3.7)(vi), $M^n = \bigcup_{k=1}^{\infty}(0:\underline{m}^k)_{M^n}$. $I = \bigcup_{k=1}^{\infty}(0:\underline{m}^k)_I$ by (6.19).

Define $\lambda: M^n \longrightarrow I$ by the rule:- if $x \in M^n$ and $\underline{m}^k.x = 0$, $\lambda(x) = \lambda_k(x)$. It is clear that λ is a well-defined A-module isomorphism. Thus M^n is injective.

I have now proved the equivalence of conditions (i),(ii),(iii),(iv),(v),(vi),(vii),&(x). Only (viii) and (ix) remain outstanding.

(i)$\Rightarrow$(ix) is proved in (6.26).

(ix)$\Rightarrow$(vi) Suppose $\underline{x} = (x_1,\ldots,x_r)$ is a sequence maximal w.r.t. the property of being both an A-sequence and an M-sequence, and which is such that $M/\underline{x}M$ is an injective $A/\underline{x}$-module. By (6.6), $r = \min(\operatorname{codh}_A M, \operatorname{codh}_A A)$. By (6.13), $\operatorname{codh}_{A/\underline{x}} A/\underline{x} = 0$, so every element of $\underline{m}$ is a zero-divisor on $A/\underline{x}$ and $r = \operatorname{codh}_A A$.

By the already proved equivalence (i)$\Leftrightarrow$(vi) and (6.13), $M/\underline{x}M$ is a Gorenstein $A/\underline{x}$-module. Thus $\mu^i(\underline{m}/\underline{x}, M/\underline{x}M) = 0$ if and only if $i > 0$. By (6.5), therefore, $\mu^i(\underline{m}, M) \neq 0$ for precisely one value of i, so by (6.16), $\operatorname{codh}_A M = \text{K-Dim}_A M = \text{inj.dim.}_A M = \operatorname{codh}_A A$.

(i)$\Rightarrow$(viii) It is clear that $\text{inj.dim.}_A M$ is finite. Let $\underline{x} = (x_1,\ldots,x_k)$ be an A-sequence. $x_1 \in \underline{m}$, so $M/x_1M \neq 0$. I need to show that $x_1 \notin \bigcup_{\underline{p}\in\text{Ass}(M)} \underline{p}$. Since M is C.-M., $\text{Ass}(M) = \left\{\underline{p} \in V(M) \;\middle|\; \text{ht}_M \underline{p} = 0\right\}$ (by (4.14)). Thus, if $\underline{p} \in \text{Ass}(M)$, $M_{\underline{p}}$ is a Gorenstein $A_{\underline{p}}$-module with $\text{K-Dim}_{A_{\underline{p}}} M_{\underline{p}} = 0$. Thus $A_{\underline{p}}$ is C.-M. with $\text{K-Dim } A_{\underline{p}} = 0$, and so $\underline{p}$ is a minimal prime of A. Consequently, since x_1 is not a zero-divisor on A,

$x_1 \notin \underline{p}$. Hence x_1 is not a zero-divisor on M.

By (6.25), M/x_1M is a Gorenstein $A/(x_1)$-module, and the result now follows by induction.

(viii)$\Longrightarrow$(vi) is obvious from (6.13). //

(6.23) (A not necessarily local) and (6.27) (A local) constitute my characterizations of Gorenstein modules.

(6.28)Remark. While in a Gorenstein local ring every system of parameters generates an irreducible ideal (see (5.11)(iii)), it is not necessarily true that, for a Gorenstein module M over a local ring A and a proper ideal $\underline{x}$ of A generated by $K\text{-}Dim_A M$ elements and s.t. $M/\underline{x}M$ has finite length, $\underline{x}M$ is an irreducible submodule of M (i.e. cannot be expressed as the intersection of two strictly larger submodules of M). The following simple example shows this:-

Let A be any Artin local ring, and let $E = E_A(A/\underline{m})$. Let $M = E_1 \oplus E_2$, where $E_1 \cong E_2 \cong E$. M is (non-zero and) f.-g. by (5.2)(vii). M is injective: it is clear that M is Gorenstein with $K\text{-}Dim_A M = 0$, but that the zero submodule of M is not irreducible in M.

(6.29)Remark. In (6.26), I showed that, for there to exist a Gorenstein module over a local ring A, it is necessary for A to be Cohen-Macaulay. The above simple example shows that it is not necessary for A to be Gorenstein, as there exist Artin local rings which are not Gorenstein (e.g. $k[X,Y]/(X,Y^2)\cap(X^2,Y)$, where k is a field and X and Y are independent indeterminates).

(6.30)Remarks on Bass' Conjecture. The fact that, for there to exist a

Gorenstein module over a local ring A, it is necessary for A to be Cohen-Macaulay perhaps adds some weight to a conjecture made by Bass in (5):

Conjecture A. (Bass on p.14 of (5).) If A is local and there is a non-zero, f.-g., A-module M of finite injective dimension, then A is Cohen-Macaulay. (The contention of this Conjecture in the particular case in which $\operatorname{codh}_A A = 1$ has been recently proved in Theorem 2.1 of Levin and Vasconcelos (16).)

I have already given an example (see the remarks following Lemma (6.13)) of a non-zero, f.-g., module over a (regular) local ring which has finite injective dimension but which is not Cohen-Macaulay, and so certainly not Gorenstein. So, as a possible intermediate stage in an attempt to prove Conjecture A, it might be advisable to try to establish:-

Conjecture B. If A is local and there is a non-zero, f.-g., Cohen-Macaulay A-module M of finite injective dimension, then A is Cohen-Macaulay.

To see that I haven't already proved Conjecture B, again consider a regular local ring A of Krull Dimension $n \geqslant 1$. Then $A/\underline{m}$ is Cohen-Macaulay, and $\operatorname{inj.dim.}_A(A/\underline{m}) = n$ (by, e.g., Theorem 6 of Matlis (19)). Thus:- $\operatorname{codh}_A(A/\underline{m}) = \text{K-Dim}_A(A/\underline{m}) < \operatorname{inj.dim.}_A(A/\underline{m}) = \operatorname{codh}_A A$.

It is possible to make some positive observations on Conjectures A & B. Suppose A is local and the non-zero, f.-g., A-module M has finite injective dimension. Then (by (6.13)):-

$$s = \operatorname{codh}_A M \leqslant \text{K-Dim}_A M \leqslant \operatorname{inj.dim.}_A M = \operatorname{codh}_A A = n.$$

By (6.6), there exists $\underline{x} = (x_1, \ldots, x_s)$ which is both an M-sequence and an A-sequence. By repeated application of (6.4), $\operatorname{inj.dim.}_{A/\underline{x}} M/\underline{x}M$ is finite, and:-

$$0 = \operatorname{codh}_{A/\underline{x}} M/\underline{x}M \leqslant \text{K-Dim}_{A/\underline{x}} M/\underline{x}M \leqslant \operatorname{inj.dim.}_{A/\underline{x}} M/\underline{x}M = \operatorname{codh}_{A/\underline{x}} A/\underline{x}.$$

If M is C.-M., then it follows from (4.12) that $K\text{-}Dim_A(M/\underline{x}M) = 0$, so $K\text{-}Dim_{A/\underline{x}}(M/\underline{x}M) = 0$, and $M/\underline{x}M$ is a C.-M. $A/\underline{x}$-module (and is of finite length). Since, if $A/\underline{x}$ is C.-M., so also is A, it follows that, in attempting to prove either of Conjectures A & B, one may assume that $codh_A M = 0$.

Now consider Conjecture B, in which it is assumed at the outset that M is C.-M. As remarked above, I may assume $codh_A M = 0$, and then:-

$0 = codh_A M = K\text{-}Dim_A M \leqslant inj.dim._A M = codh_A A$. When this is done, M is annihilated by some power of $\underline{m}$, and M has a natural structure as a module over $\hat{A}$, the completion of A. For if $x \in A$, $\hat{a} \in \hat{A}$, and (a_n) is a sequence of elements of A converging to $\hat{a}$, then, using (1.31), the sequence $(a_n x)$ of elements of M is ultimately constant, and furthermore this ultimate constant value does not depend on the choice of sequence (a_n) of elements of A converging to $\hat{a}$. Define $\hat{a}.x$ to be the ultimate constant value of the sequence $(a_n x)$: this gives M the structure of an $\hat{A}$-module. Also, it follows easily from (1.31) that, as an $\hat{A}$-module, M has finite length (and is non-zero).

By Proposition 8 of Matlis (19), $inj.dim._{\hat{A}} M = inj.dim._A M$, and by Proposition 10 of Chap.IV of Serre (26), A is C.-M. if and only if $\hat{A}$ is. Since $K\text{-}Dim_{\hat{A}} M = 0$, it may be assumed that A is complete. But then, by Proposition 6 of Matlis (19) and Corollary 4.3 of Matlis (17), $Hom_A(M, E_A(A/\underline{m}))$ is a non-zero A-module of finite length having homological dimension equal to $inj.dim._A M$. Thus, in order to prove Conjecture B, it is sufficient to prove:-

Conjecture C. If A is a complete local ring and there is a non-zero A-module M of finite length having finite homological dimension, then A is Cohen-Macaulay.

CHAPTER 7

Some Special Classes of (Commutative Noetherian) Rings.

(Reminder: "ring" means "commutative Noetherian ring with a non-zero identity".)

In view of the remarkably simple characterizations of Cohen-Macaulay and Gorenstein rings in terms of the Cousin Complex (see Chapters 4 & 5), it seems reasonable to suspect that the Cousin Complex might be useful in attempts to classify (commutative Noetherian) rings which are not necessarily Cohen-Macaulay. Even Cohen-Macaulay rings are very well-behaved from a Cousin Complex point of view, for a ring is Cohen-Macaulay if and only if its Cousin Complex is exact, and so there is plenty of scope for examining weaker conditions on the Cousin Complex for a ring. I feel that, in order to try to put any discussion along these lines in its true perspective, it would help to point out that the condition that a ring should be Cohen-Macaulay is itself comparatively weak. I have illustrated this by examining the general position of Cohen-Macaulay local rings in the general heirarchy of local rings: I have discussed the classes of Regular local rings, Local Complete Intersections, Gorenstein local rings, and Cohen-Macaulay local rings, together with examples to illustrate the distinctions between these (well-known) special classes.

Unfortunately, my attempts to develop new and useful classifications of rings which are not necessarily Cohen-Macaulay have so far met with only limited success, mainly because the methods used in Chapters 4 & 5 are practically useless for dealing with a Cousin Complex which is not even partially exact. However, it is shown in this Chapter that the rings

characterized in Proposition (6.1) of Bass (5) and in Theorem (2.1) of Ishikawa (14) can also be characterized by means of the Cousin Complex, and this leads on to the definition, in Cousin Complex terms, of several similar, but new, classes of rings.

Also mentioned in this Chapter are one or two Conjectures concerning the Cousin Complex. My own examinations of these Conjectures have not yet progressed beyond the stage of brief initial skirmishes, mainly in low Dimensions, and I have made very little progress towards establishing these Conjectures.

The nett effect is that this Chapter contains a lot of discussion and Examples and only a little original work: it is little more than a light-hearted interlude before the hard work on Cousin Complex applications and further properties of Gorenstein modules in the remainder of the thesis.

(7.1) Regular Local Rings.

Regular local rings form the nicest class of local rings. If A is a local ring with Krull Dimension d and maximal ideal $\underline{m}$, A is s.t.b. regular if and only if $\underline{m}$ can be generated by d elements,, i.e. if and only if $\underline{m}$ can be generated by some system of parameters. Equivalently, A is regular if and only if the dimension of $\underline{m}/\underline{m}^2$ as a vector space over $A/\underline{m}$ is precisely d. There is a fine account of the classical theory of regular local rings in Chapters VI and V of Northcott (21).

Suppose A is a regular local ring with K-Dim A = d, and let $\{u_1,\ldots,u_d\}$ be a set of d elements which generate $\underline{m}$. Then $0 \subsetneq (u_1) \subsetneq (u_1,u_2) \subsetneq \ldots \subsetneq (u_1,\ldots,u_d) = \underline{m}$ is a chain of prime ideals of A: it clearly must be saturated. (For proof of this see §4.6 of Northcott (21).) In particular

$(u_1,\ldots,u_d)$ is an A-sequence, and since $\underline{m}$ is irreducible and $\operatorname{codh}_A A$ = K-Dim A = d, A is Gorenstein by (5.11).

In (2), Auslander and Buchsbaum obtained a homological characterization of regular local rings: a local ring is regular if and only if it has finite global dimension. (This was mentioned in Chapter 6.)

(7.2)Examples of Regular Local Rings.

Any field is a regular local ring. Less trivially, any discrete valuation ring (i.e. a (Noetherian) local domain of Krull Dimension 1 which is integrally closed (in its field of fractions)) is a regular local ring. The ring of formal power series $A = k[[X_1,\ldots,X_n]]$ in n indeterminates $X_1,\ldots,X_n$ over a field k is a regular local ring. To see that A is Noetherian and is local with maximal ideal $\underline{m} = (X_1,\ldots,X_n)$ see, e.g., §5.4 of Northcott (21). Consequently, K-Dim $A \leqslant n$, by (1.23). But $0 \subsetneqq (X_1) \subsetneqq (X_1,X_2) \subsetneqq \ldots \subsetneqq (X_1,\ldots,X_n) = \underline{m}$ is obviously a chain of prime ideals of A, so K-Dim $A \geqslant n$. Hence A is regular.

(7.3)Local Complete Intersections.

A local ring A is s.t.b. a local complete intersection if and only if, for some regular local ring R and ideal $\underline{x}$ of R generated by an R-sequence, A is ring isomorphic to $R/\underline{x}$.

Clearly a regular local ring is a local complete intersection. Suppose R is a regular local ring and $\underline{x} = (x_1,\ldots,x_k)$ is an R-sequence. From (7.1), R is Gorenstein, and so, by (5.11) and (6.4), $R/\underline{x}$ is Gorenstein. Consequently, every local complete intersection is Gorenstein.

(7.4)Example of a Local Complete Intersection which is not Regular. (This

example is due to I. Kaplansky and appears as Remark 1 on p.61 of Matlis (18).)

Let $A = k[[X,Y]]/(Y^2 - X^3)$, where k is a field. By (7.2), $k[[X,Y]]$ is a regular local ring with maximal ideal (X,Y). $Y^2 - X^3 \neq 0$ in the integral domain $k[[X,Y]]$, so, by (7.3), A is a local complete intersection with maximal ideal $(X,Y)/(Y^2 - X^3)$ and Krull Dimension 1. A is not regular because, as is easy to see, its maximal ideal cannot be generated by one element.

A is, in fact, an integral domain. To see this, let $\phi : k[[X,Y]] \to k[[T]]$ (the ring of formal power series in one indeterminate T) be the substitution homomorphism given by $\phi(X) = T^2$, $\phi(Y) = T^3$. (The fact that one does obtain a ring homomorphism in this manner is proved on pp.135-6 of §1, Chap.VII, Vol.II of Zariski and Samuel (27).)

Clearly $(X^3 - Y^2) \subseteq \ker\phi$: I shall show $\ker\phi = (X^3 - Y^2)$. Suppose $\alpha \in \ker\phi$. Then, using the fact that $k[[X,Y]]$ is a complete local ring (see §1, Chap.VII, Vol.II of (27) for proof of this), it can be shown that there is an element $\beta \in (X^3 - Y^2)$ such that:-

$\alpha + \beta = f(X) + Y.g(X)$, where f(X) and g(X) are power series in X.

Suppose $f(X) = c_0 + c_1X + c_2X^2 + \ldots + c_nX^n + \ldots$,

and $g(X) = d_0 + d_1X + d_2X^2 + \ldots + d_nX^n + \ldots$. Then:-

$\phi(\alpha+\beta) = c_0 + c_1T^2 + d_0T^3 + c_2T^4 + d_1T^5 + \ldots + c_nT^{2n} + d_{n-1}T^{2n+1} + \ldots$

Since $\alpha \in \ker\phi$ and $\beta \in (X^3 - Y^2) \subseteq \ker\phi$, $\phi(\alpha+\beta) = 0$, so $c_0 = c_1 = \ldots = c_n = \ldots = 0$ and $d_0 = d_1 = \ldots = d_n = \ldots = 0$. Thus $\alpha+\beta = 0$, and so $\alpha = -\beta \in (X^3 - Y^2)$. Therefore $\ker\phi = (X^3 - Y^2)$.

Consequently, A is isomorphic to a subring of the integral domain $k[[T]]$, and so A is an integral domain. Thus A is a (Noetherian) local

domain of Krull Dimension 1 which is a local complete intersection but which is not regular (and so not integrally closed).

As remarker in (7.3), every local complete intersection is Gorenstein: however, I give:-

(7.5)Example of a Gorenstein Local Ring which is not a Local Complete Intersection. (This example is due to D.G.Northcott, and appears on p.29 of Bass (4).)

Let $R = k[[X,Y,Z]]$ be the formal power series ring over a field k in three indeterminates X,Y,& Z, and let $\underline{a}$ be the ideal $(X^2-Y^2, Y^2-Z^2, XY, YZ, ZX)$. $A = R/\underline{a}$ is a local ring. I shall show that it is Gorenstein, but not a local complete intersection.

Let $\underline{m} = (X,Y,Z)$. Observe that $X^2-Z^2 \in \underline{a}$, $X^3 = X.(X^2-Y^2) - Y.(XY) \in \underline{a}$, and similarly $Y^3, Z^3 \in \underline{a}$. Since $XY, YZ,$ & $ZX \in \underline{a}$, $\underline{m}^3 \subseteq \underline{a}$, so $\underline{a}$ is $\underline{m}$-primary. Consequently, A is an Artin local ring, and so to show it is Gorenstein it is sufficient, by (5.11)(ii), to show that its zero ideal is irreducible. I prove that $\underline{a}$ is an irreducible ideal of R. Note that $\underline{a} \subsetneq \underline{m}^2$, for $X^2 \notin \underline{a}$. I shall show that any ideal of R which properly contains $\underline{a}$ must contain $\underline{m}^2$. Suppose $R \supsetneq \underline{b} \supsetneq \underline{a}$ ($\underline{b}$ being an ideal of R, of course).

Since $\underline{a} \supseteq \underline{m}^3$, and $XY, YZ, ZX \in \underline{a}$, there are elements $a,b,c,d,e,f \in k$ s.t. $aX + bY + cZ + dX^2 + eY^2 + fZ^2 \in \underline{b} - \underline{a}$. Therefore $aX + bY + cZ + (d+e+f)X^2 \in \underline{b} - \underline{a}$. If one of a,b, or c is non-zero, say a, then $aX^2 \in \underline{b}$ and $\underline{m}^2 \subseteq \underline{b}$. If $a = b = c = 0$, $(d+e+f) \neq 0$, so again $X^2 \in \underline{b}$, and so $\underline{m}^2 \subseteq \underline{b}$, as was to be shown.

Thus A is Gorenstein. Let $\underline{n} = \underline{m}/\underline{a}$.

Suppose A is a local complete intersection, i.e. suppose there is a regular local ring C with maximal ideal $\underline{r}$ and Krull Dimension s, and a surjective ring homomorphism $\phi : C \longrightarrow A$ such that $\ker \phi = \underline{w}$ is generated by a C-sequence $(w_1,\ldots,w_s)$ (of length s, necessarily). Note that $\phi(\underline{r}) = \underline{n}$. '$\bar{}$'s will be used to denote natural images in residue class rings. Recall from Lemma (2.8) that, if D is a local ring with maximal ideal $\underline{d}$, an ideal $\underline{e}$ of D is generated by $c_1,\ldots,c_t$ if and only if $\bar{c}_1,\ldots,\bar{c}_t$ generate $\underline{e}/\underline{d}.\underline{e}$ as a vector space over $D/\underline{d}$.

It is clear that $\{\bar{X},\bar{Y},\bar{Z}\}$ is a minimal generating set for the maximal ideal $\underline{n}$ of A. Consequently, $\bar{\bar{X}},\bar{\bar{Y}},\bar{\bar{Z}}$ in $\underline{n}/\underline{n}^2$ are linearly independent over $A/\underline{n}$ (by (2.8)). Let u_1,u_2,u_3 be elements of $\underline{r}$ s.t. $\phi(u_1) = \bar{X}$, $\phi(u_2) = \bar{Y}$, $\phi(u_3) = \bar{Z}$. Note that $\bar{u}_1,\bar{u}_2,\bar{u}_3$ in $\underline{r}/\underline{r}^2$ are linearly independent over $C/\underline{r}$: for if $c_1u_1 + c_2u_2 + c_3u_3 \in \underline{r}^2$, $\phi(c_1)\bar{X} + \phi(c_2)\bar{Y} + \phi(c_3)\bar{Z} \in \underline{n}^2$, and so $\phi(c_1), \phi(c_2), \phi(c_3) \in \underline{n}$. Thus $\dim_{C/\underline{r}}(\underline{r}/\underline{r}^2) \geqslant 3$, and so $s \geqslant 3$. Also, by (2.8), $\{u_1,u_2,u_3\}$ can be extended by $\{u_4,\ldots,u_s\}$ to a set of s generators for $\underline{r}$. Since $\underline{r} = (u_1,u_2,u_3) + \underline{w}$, it may be assumed that $u_4,\ldots,u_s$ all belong to $\underline{w}$.

Then $\underline{w} \supseteq \underline{g} = (u_1^2-u_2^2, u_2^2-u_3^2, u_2u_3, u_3u_1, u_1u_2, u_4, u_5, u_6, \ldots, u_s)$. I shall show that $\underline{w} = \underline{g}$. To do this, it is sufficient to show that an element of $\underline{w}$ of the form $c_1u_1 + c_2u_2 + c_3u_3$ belongs to $\underline{g}$. So suppose $\phi(c_1u_1+c_2u_2+c_3u_3) = 0$. Then $\phi(c_1)\bar{X} + \phi(c_2)\bar{Y} + \phi(c_3)\bar{Z} = 0$, and so $\phi(c_1), \phi(c_2), \phi(c_3) \in \underline{n}$, i.e. $c_1,c_2,c_3 \in \underline{r}$. Thus it suffices to show that an element of $\underline{w}$ of the form $a_1u_1^2+a_2u_2^2+a_3u_3^2$ belongs to $\underline{g}$. But $a_1u_1^2+a_2u_2^2+a_3u_3^2 = (a_1+a_2+a_3)u_1^2 - a_2(u_1^2-u_2^2) - a_3(u_1^2-u_3^2)$, and so $(a_1+a_2+a_3)u_1^2 \in \underline{w}$. Thus $\phi(a_1+a_2+a_3)\bar{X}^2 = 0$, and, as seen above, this means $\phi(a_1+a_2+a_3) \in \underline{n}$. Consequently $(a_1+a_2+a_3)u_1^2 \in \underline{g}$, and $\underline{w} = \underline{g}$.

All I need show now is that none of the s+2 generators in $\underline{w}$ = $(u_1^2-u_2^2, u_2^2-u_3^2, u_2u_3, u_3u_1, u_1u_2, u_4, u_5, \ldots, u_s)$ is superfluous (for $\underline{w}$ can be generated by s elements and any two minimal generating sets for $\underline{w}$ must have the same number of elements). Clearly, none of $u_4, u_5, \ldots, u_s$ is superfluous, and if any of the others, say $u_1^2-u_2^2$, was superfluous, there would be a relation of the form:-

$$(\bar{u}_1^2-\bar{u}_2^2) + \bar{a}_2(\bar{u}_2^2-\bar{u}_3^2) + \bar{a}_3\bar{u}_1\bar{u}_2 + \bar{a}_4\bar{u}_2\bar{u}_3 + \bar{a}_5\bar{u}_3\bar{u}_1 = 0$$

in the regular local ring $C/(u_4, u_5, \ldots, u_s)$ of Krull Dimension 3 which has $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ for its maximal ideal. But this contradicts Theorem 3 of Chapter 4 of Northcott (21). Thus $\underline{w}$ has a minimal generating set of s+2 elements and can be generated by s elements, a contradiction.

So A is not a complete intersection.

From (5.11)(ii), every Gorenstein local ring is Cohen-Macaulay: however, I give:-

(7.6)Example of a Cohen-Macaulay Local Ring which is not Gorenstein.

Let $A = k[[X,Y]]/(X^2,Y) \cap (Y^2,X)$, where k is a field. This is an Artin local ring with zero ideal which is not irreducible, since $(X^2,Y) \not\subseteq (Y^2,X)$ and $(Y^2,X) \not\subseteq (X^2,Y)$ in $k[[X,Y]]$. By (5.11)(iii), A is not Gorenstein: however, any Artin local ring has to be Cohen-Macaulay.

So one has the following symbolic diagram for local rings:-

Regular Local $\Longrightarrow$ Local Complete Intersection $\Longrightarrow$ Gorenstein Local $\Longrightarrow$ Cohen-Macaulay Local ,

and none of the implications can be reversed.

Local rings which are not Cohen-Macaulay are, of course, prevalent:-

(7.7)Examples of Local Rings which are not Cohen-Macaulay.

If A is a local ring with maximal ideal $\underline{m}$, and either:-

(i) K-Dim A = 1 and A is not an integral domain,

or (ii) K-Dim A $\geqslant$ 2,

then there is a homomorphic image of A which is not Cohen-Macaulay.

Proof. In either case there exists a prime ideal $\underline{p}$ of A s.t. $0 \neq \underline{p} \subsetneq \underline{m}$. Now $\bigcap_{n=1}^{\infty} \underline{m}^n = 0$ (by, e.g., Theorem 3 of Chapter 3 of Northcott (21)), so for some positive integer n, $\underline{p} \not\subseteq \underline{m}^n$. $\underline{m}^n$ is $\underline{m}$-primary, so $\underline{a} = \underline{p} \cap \underline{m}^n$ is a minimal primary decomposition for $\underline{a}$. Thus, if B = A/$\underline{a}$, B is a local ring of Krull Dimension at least 1 but whose maximal ideal belongs to $\mathrm{Ass}_B(B/(0))$, and so consists entirely of zero-divisors. Thus $\mathrm{codh}_B B = 0 <$ K-Dim B. //

Cohen-Macaulay rings, which form the worst of the four special classes of rings considered above, are very well-behaved from a Cousin Complex point of view: a ring is Cohen-Macaulay if and only if its Cousin Complex is exact. Consequently, it seems reasonable to conjecture that the examination of weaker conditions on the Cousin Complex for a ring might lead to new and helpful classifications of rings which may not be Cohen-Macaulay. One of the main tools used in Chapter 4 to characterize Cohen-Macaulay modules in terms of the Cousin Complex was Lemma (4.22) (P.S.C.C.A.), and consequently, a good starting point for this discussion is the examination, for a ring A, of those non-zero, f.-g., A-modules whose Cousin Complexes are exact up to a certain point. As usual, A denotes a ring.

(7.8) Theorem. Let M be a non-zero, f.-g., A-module and n an integer s.t. $0 \leqslant n \leqslant \mathrm{K\text{-}Dim}_A M$. Let the Cousin Complex for M be:-

$$C(M): 0 \longrightarrow M \xrightarrow{e^{-1}} M^0 \xrightarrow{e^0} M^1 \longrightarrow \cdots \longrightarrow M^r \xrightarrow{e^r} M^{r+1} \longrightarrow \cdots .$$

Then $\underline{C(M)}$ is exact at $M^{-1} = M$, M^0, M^1,..., & M^{n-1} if and only if every M-sequence of length $\leqslant n$ generates an M-unmixed ideal.

Proof. By (4.5) and (1.23), if $\underline{x} = (x_1,\ldots,x_k)$ is an M-sequence, any minimal element of $\mathrm{Ass}_A(M/\underline{x}M)$ has M-height k.

($\Longrightarrow$) Suppose $0 \leqslant k \leqslant n$ and $\underline{x} = (x_1,\ldots,x_k)$ is an M-sequence. Suppose $\underline{x}$ is not M-unmixed, so that there is a $\underline{p} \in \mathrm{Ass}_A(M/\underline{x}M)$ with $\mathrm{ht}_M\underline{p} > k$. Since $\underline{p}$ consists entirely of zero-divisors on $M/\underline{x}M$ (and is M-proper) $\mathrm{grade}_M\underline{p} = k$. But $\underline{p}$ is not contained in any prime of $V(M)$ of M-height $\leqslant k$, and:-

(E) $0 \longrightarrow M^{-1} \xrightarrow{e^{-1}} M^0 \longrightarrow \ldots. \longrightarrow M^{k-1} \xrightarrow{e^{k-1}} M^k$. I use P.E.C.C.A. (4.22) to deduce that $\mathrm{Ext}^k_A(A/\underline{p},M) = 0$, a contradiction to the fact that $\mathrm{grade}_M\underline{p} = k$. Thus $\underline{x}$ is M-unmixed.

($\Longleftarrow$) I show first that $\underline{C(M)}$ is exact at M^{-1}. Suppose not, and let $0 \neq y \in H^{-1}(M) = \ker e^{-1}$. Let $\underline{a} = (0:y)_A$. By (3.7)(viii) and the fact that there is a maximal ideal of A containing $\underline{a}$, $\underline{a}$ is M-proper and has M-height $\geqslant 1$. Since (0) is M-unmixed, $\underline{a}$ does not consist entirely of zero-divisors on M, and so $\mathrm{grade}_M\underline{a} \geqslant 1$. Consequently $\mathrm{Hom}_A(Ay,M) = 0$: but this is a contradiction since $H^{-1}(M) \subseteq M$. Thus $H^{-1}(M) = 0$, and so $\underline{C(M)}$ is exact at M^{-1}. This provides a basis for the following induction.

I suppose, inductively, that $0 \leqslant k < n$ and I have shown that:-

(E) $0 \longrightarrow M^{-1} \xrightarrow{e^{-1}} M^0 \xrightarrow{e^0} M^1 \longrightarrow \ldots. \longrightarrow M^{k-1} \xrightarrow{e^{k-1}} M^k$. Suppose $\underline{C(M)}$ is not exact at M^k, so there is an element $0 \neq y \in H^k(M)$. Let $\underline{a} = (0:y)_A$. By (3.7)(vii) & (viii), $\underline{a}$ is M-proper and has M-height $\geqslant k+2$. By P.E.C.C.A. (4.22), $\mathrm{grade}_M\underline{a} \geqslant k+1$, so there exists an M-sequence $\underline{x} = (x_1,\ldots,x_{k+1}) \subseteq \underline{a}$. By hypothesis, $\underline{x}$ is M-unmixed, so $\underline{a}$ does not consist entirely of zero-divisors on $M/\underline{x}M$. Thus $\mathrm{grade}_M\underline{a} \geqslant k+2$. But, by

P.E.C.C.A. (4.22), $\mathrm{Ext}_A^{k+\ell}(A/\underline{a}, M) \cong \mathrm{Hom}_A(A/\underline{a}, \mathrm{coker}\ e^{k-1}) \cong \mathrm{Hom}_A(Ay, \mathrm{coker}\ e^{k-1}) \neq 0$ since $H^k(M) \subseteq \mathrm{coker}\ e^{k-1}$. This is a contradiction, and so $\underline{C(M)}$ must be exact at M^k.

The Theorem follows by induction. //

(7.9)Remark. In connection with the case n = 0 of Theorem (7.8), it is probably worth pointing out that, if M is a non-zero, f.-g., A-module with Cousin Complex: $0 \longrightarrow M \xrightarrow{e^{-1}} M^0 \longrightarrow \ldots$, and if $0 = \bigcap_{i=1}^{k} Q_i$ is a minimal primary decomposition for 0 in M, with M/Q_i $\mathfrak{p}_i$-coprimary for each $i = 1,\ldots,k$, $\mathfrak{p}_1,\ldots,\mathfrak{p}_m$ $(m \leqslant k)$ the minimal elements of $\mathrm{Ass}_A(M)$, and $\mathfrak{p}_{m+1},\ldots,\mathfrak{p}_k$ the embedded primes of $\mathrm{Ass}_A(M)$, then $\ker e^{-1} = \bigcap_{i=1}^{m} Q_i$.

In particular, if A is a ring with Cousin Complex:-
$0 \longrightarrow A \xrightarrow{d^{-1}} A^0 \longrightarrow \ldots$, then $\ker d^{-1}$ is equal to the intersection of the isolated primary components of the zero ideal. (These isolated primary components are, of course, uniquely determined: see (1.10).)

(7.10)Examples. (i) If A is an integral domain, then clearly $\underline{C(A)}$ is exact at $A^{-1} = A$.

(ii) If A is an integrally closed integral domain, then Proposition 9 of Chap.III of Serre (26) and (7.8) show that $\underline{C(A)}$ is exact at A^{-1} and A^0.

Note that, when A is an integral domain, $A^0 = K$, the field of fractions of A. K is, of course, an injective K-module, and so, by (5.3), K is an injective A-module. In view of this and Theorem (7.8), it seems that it may be worthwhile to examine those rings A for which the Cousin Complex is exact at A^{-1}, $A^0,\ldots,A^{n-1}$ and for which A^n is injective.

(7.11)Theorem. Suppose that $0 \leqslant n \leqslant$ K-Dim A, and that the Cousin Complex for A is:- $\underline{C(A)}: 0 \longrightarrow A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \longrightarrow \ldots$. Then the following statements are equivalent:-

(i) $\underline{C(A)}$ is exact at $A^{-1}, A^0, \ldots, A^{n-1}$ and A^n is injective;

(ii) $\underline{C(A)}$ is exact at $A^{-1}, A^0, \ldots, A^{n-1}$ and, for all primes $\underline{p}$ of height n, $A_{\underline{p}}$ is Gorenstein;

& (iii) the ideal generated by every A-sequence of length $\leqslant n$ is unmixed, and the (uniquely determined) primary components of the ideal generated by each A-sequence of length n are irreducible.

Proof. (i)$\Rightarrow$(ii) If $\underline{p}$ is a prime of height n, then, using (3.13), (5.5), & (6.27), $A_{\underline{p}}$ is a Gorenstein $A_{\underline{p}}$-module, and so is a Gorenstein ring.

(ii)$\Rightarrow$(iii) By (7.8), the ideal generated by any A-sequence of length $\leqslant n$ is unmixed. Let $\underline{x}$ be an A-sequence of length n, $\underline{p} \in \mathrm{Ass}(A/\underline{x})$ (so that $\mathrm{ht}\,\underline{p} = n$), and $\underline{q}$ the (uniquely determined) $\underline{p}$-primary component of $\underline{x}$. It is sufficient to show that $\underline{q}A_{\underline{p}}$ is an irreducible ideal of $A_{\underline{p}}$, and this follows from (5.11) since $A_{\underline{p}}$ is Gorenstein and $\underline{q}A_{\underline{p}} = \underline{x}A_{\underline{p}}$ is an ideal of $A_{\underline{p}}$ generated by a system of parameters.

(iii)$\Rightarrow$(i) By (7.8), $\underline{C(A)}$ is exact at $A^{-1}, A^0, \ldots, A^{n-1}$. It remains only to show that A^n is injective.

Now $A^n = \bigoplus_{\substack{\underline{p} \in V(A) \\ \mathrm{ht}\,\underline{p} = n}} (\mathrm{coker}\, d^{n-2})_{\underline{p}}$, and so it is sufficient to show that, for each $\underline{p} \in V(A)$ of height n, $(\mathrm{coker}\, d^{n-2})_{\underline{p}}$ is $A_{\underline{p}}$-injective (by (5.3)). So suppose $\underline{p} \in V(A)$ has height n. By P.E.C.C.A. (4.22), $\mathrm{grade}_A \underline{p} = n$, so there exists an A-sequence $\underline{x}$ of length n contained in $\underline{p}$. By hypothesis, since $\underline{p}$ is a minimal element of $\mathrm{Ass}_A(A/\underline{x})$, $\underline{x}A_{\underline{p}}$ is an irreducible ideal of $A_{\underline{p}}$ generated by a system of parameters. Since it

is obvious, by localising $\underline{C(A)}$, that $A_{\underline{p}}$ is Cohen-Macaulay, it follows from (5.11) that $A_{\underline{p}}$ is Gorenstein. Thus by (3.14) and (5.13), $(\operatorname{coker} d^{n-2})_{\underline{p}}$ is $A_{\underline{p}}$-injective. //

(7.12)Remark. Note that the rings characterized by Theorem (7.11) in the particular case of that Theorem in which n = 0 are precisely the rings characterized in Proposition (6.1) of Bass (5) and in Theorem 2.4 of Ishikawa (14). Suppose that A is such a ring, and that:-

$\underline{C(A)}: 0 \longrightarrow A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \longrightarrow \ldots$ is the Cousin Complex for A. Then $A^0 \cong K$, the full ring of quotients of A, as I now show.

Let $\underline{p}_1, \ldots, \underline{p}_n$ be the minimal prime ideals of A. Since 0 in A is unmixed, these are precisely the elements of Ass(A). $K = S^{-1}A$, where $S = \{x \in A \mid x \text{ not a zero-divisor in } A\} = A - \bigcup_{i=1}^{n} \underline{p}_i$. $A^0 = \bigoplus_{i=1}^{n} A_{\underline{p}_i}$.

Define $\varepsilon: A \longrightarrow K$ by $\varepsilon(a) = a/1$. ε is an A-monomorphism. There is also a (well-defined) A-homomorphism $\alpha: K \longrightarrow A^0$ given by $\alpha(a/s) = (a/s, \ldots, a/s)$. Clearly

$$(E)\quad 0 \longrightarrow A \xrightarrow{\varepsilon} K \qquad \text{with } d^{-1}: A \longrightarrow A^0 \text{ and } \alpha: K \longrightarrow A^0 \quad (C).$$

Now d^{-1} and ε are essential monomorphisms (using (5.12)), so α is an essential monomorphism. But K is A-injective (by Proposition (6.1) of Bass (5)), and so α is an isomorphism. Thus $\alpha: K \xrightarrow{\cong} A^0$.

This Remark suggests a name for the rings characterized in Theorem (7.11)

(7.13)Definition. A ring A is s.t.b. a Bass-Ishikawa-n ring if and only if it satisfies the equivalent conditions of Theorem (7.11).

Thus the rings characterized in Proposition (6.1) of Bass (5) and in Theorem 2.4 of Ishikawa (14) are precisely the Bass-Ishikawa-0 rings.

Note also that, if A is local with Krull Dimension n, then A is Gorenstein if and only if A is Bass-Ishikawa-n (using (6.27), (5.13), and the fact that, if B is a local ring with K-Dim B = s, the Cousin Complex for B must be exact at the sth term).

(7.14)Examples. (i) If A is an integral domain, then $A^0 = K$, the field of fractions of K. K is clearly K-injective, and so is A-injective by (5.3) Thus a (Noetherian) domain is Bass-Ishikawa-0.

(ii) If A is an integrally closed integral domain, then, as was pointed out in (7.10)(ii), C(A) is exact at $A^{-1} = A$ and A^0. It also follows from Proposition 9 of Chap.III of Serre (26) that, for each prime $\underline{p}$ of A of height 1, $A_{\underline{p}}$ is a discrete valuation ring, i.e. a regular local ring of Krull Dimension 1. Thus, by (7.11)(ii), A is Bass-Ishikawa-1.

However, not every (Noetherian) integral domain which is Bass-Ishikawa-1 is integrally closed. Example (7.4) provides an example of a local domain of Krull Dimension 1 which is Gorenstein (and so Bass-Ishikawa-1) but which is not integrally closed.

(iii) Any geometric local ring is Bass-Ishikawa-0.

(7.15)Further Conjectures. Suppose, as usual, that A is a ring with Cousin Complex:-

$$C(A): 0 \longrightarrow A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \longrightarrow \cdots \longrightarrow A^n \xrightarrow{d^n} A^{n+1} \longrightarrow \cdots .$$

The meaning of the condition "$H^{-1}(A) = 0$" is given precisely in Theorem (7.8): $H^{-1}(A) = 0$ if and only if the zero ideal of A is unmixed. I have not, as yet, been able to discover if there is any equally agreeable interpretation of the condition on A that the ith homology

module, for some $i \geqslant 0$, of $\underline{C(A)}$ should be zero, mainly because I have been depending heavily on Lemma (4.22) (P.S.C.C.A.), and for this to be applicable it is necessary that the Cousin Complex should be exact at all terms up to a certain point.

There are many further questions one might ask about the Cousin Complex for a ring: some such questions may turn out to be interesting topics for further study. There are two Conjectures about the Cousin Complex which, I feel, are worth mentioning here: my own examinations of these Conjectures have not yet progressed beyond the stage of brief initial skirmishes, mainly in low Dimensions, and I've so far made only slight progress towards establishing them.

Firstly, if K-Dim A is finite, then $A^n = 0$ whenever $n >$ K-Dim A. (See (3.7)(i).) On the other hand, if A is a Cohen-Macaulay ring and $-1 \leqslant n \leqslant$ K-Dim A, then it follows from (4.24) that $A^n \neq 0$. I am inclined to conjecture that this result is true even when A is not Cohen-Macaulay:-

Conjecture D. If A is a ring and $-1 \leqslant n \leqslant$ K-Dim A, then the nth term of the Cousin Complex for A is non-zero.

Secondly, I feel that, if A is such that the terms $A^0, A^1, \ldots, A^n, \ldots$ of its Cousin Complex are all injective, then it should be possible to prove that A must be Gorenstein, i.e. to prove that its Cousin Complex is also exact. Examinations of low Dimensional cases tend to confirm this view, although this problem seems to be interrelated with the problem concerned in Conjecture D.

Conjecture E. If A is a ring with Cousin Complex:-

$$\underline{C(A)}: 0 \longrightarrow A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \longrightarrow \ldots \longrightarrow A^n \xrightarrow{d^n} A^{n+1} \longrightarrow \ldots ,$$

and if A^0, A^1,..., A^n,... are all injective A-modules, then A is a Gorenstein ring.

If this Conjecture is false, then the condition concerned in this Conjecture determines a class of rings which strictly contains the class of Gorenstein rings, but which neither contains nor is contained in the class of all Cohen-Macaulay rings.

CHAPTER 8

Applications of the Cousin Complex to the Theory of Flat Extensions of (Commutative Noetherian) Rings.

(Reminder: "ring" means "commutative, Noetherian ring with non-zero identity".)

Throughout this Chapter, A and B will denote rings, and $f:A \longrightarrow B$ will be a ring homomorphism (so that, in particular, $f(1_A) = 1_B$). B can be regarded as an A-module by means of f, and, for an arbitrary A-module N, N_B will denote $N \otimes_A B$ regarded as a B-module in the obvious way. If $\underline{a}$ is an ideal of A, $\underline{a}^e$ will denote the extension of $\underline{a}$ to B, and if $\underline{b}$ is an ideal of B, $\underline{b}^c$ will denote the contraction $f^{-1}(\underline{b})$ of $\underline{b}$ in A.

Recall that certain aspects of this situation were considered in the second half of Chapter 3. I plan to exploit Theorem (3.11), but before I can do this I need a Lemma similar to Lemma (4.21).

(8.1)Lemma. Suppose M is an A-module, and the Cousin Complex for M is:-

$$C(M): 0 \longrightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \dots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \dots .$$

Let $r \geqslant 0$ be an integer. Suppose L is a f.-g. B-module such that, whenever $\underline{p} \in \mathrm{Supp}_A(M)$ has M-height r, $((0:L)_B)^c \not\subseteq \underline{p}$. Then, for each $i \geqslant 0$,

$$\mathrm{Ext}^i_B(L, (M^r)_B) = 0.$$

Proof. As usual, I write $M^{-1} = M$, $M^{-2} = 0$, $d^{-2} = 0: M^{-2} \longrightarrow M^{-1}$.

$$(M^r)_B = \left[\bigoplus_{\substack{\underline{p} \in \mathrm{Supp}(M) \\ \mathrm{ht}_M \underline{p} = r}} (\mathrm{coker}\ d^{r-2})_{\underline{p}} \right] \otimes_A B \cong \bigoplus_{\substack{\underline{p} \in \mathrm{Supp}(M) \\ \mathrm{ht}_M \underline{p} = r}} (\mathrm{coker}\ d^{r-2})_{\underline{p}} \otimes_A B.$$

Thus, by (2.19), $\mathrm{Ext}^i_B(L, (M^r)_B) \cong \bigoplus_{\substack{\underline{p} \in \mathrm{Supp}(M) \\ \mathrm{ht}_M \underline{p} = r}} \mathrm{Ext}^i_B(L, (\mathrm{coker}\ d^{r-2})_{\underline{p}} \otimes_A B)$.

If $\underline{p} \in \mathrm{Supp}_A(M)$ has M-height r, then $((0:L)_B)^c \not\subseteq \underline{p}$, so there exists $a \in ((0:L)_B)^c - \underline{p}$. Then multiplication by a on $(\mathrm{coker}\ d^{r-2})_{\underline{p}}$ provides an automorphism, so multiplication by f(a) on the B-module $(\mathrm{coker}\ d^{r-2})_{\underline{p}} \otimes_A B$ also provides an automorphism. But $f(a) \in (0:L)_B$, so multiplication by f(a) on L is zero. Thus, since $\mathrm{Ext}^i_B(\ ,\ )$ is a B-linear functor ((2.2)), multiplication by f(a) on $\mathrm{Ext}^i_B(L, (\mathrm{coker}\ d^{r-2})_{\underline{p}} \otimes_A B)$ is both zero and an automorphism. Thus $\mathrm{Ext}^i_B(L, (\mathrm{coker}\ d^{r-2})_{\underline{p}} \otimes_A B) = 0$, and so $\mathrm{Ext}^i_B(L, (M^r)_B) = 0$. //

(8.2)Lemma. Suppose M is an A-module, and the Cousin Complex for M is:-

$$\underline{C(M)}: 0 \longrightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots .$$

Consider the complex $\underline{C(M)} \otimes_A B$: if $n \geq 0$ is an integer, $x \in \mathrm{coker}(d^{n-2} \otimes 1_B)$, and $\underline{p} \in V(A) - U^n(M)$ (see Chapter 3 for definition), then $((0:x)_B)^c \not\subseteq \underline{p}$.

Proof. Use $\bar{}$ to denote natural images in $\mathrm{coker}(d^{n-2} \otimes 1_B)$. x will be of the form $\sum_{i=1}^{r} \overline{m_i \otimes b_i}$, where $m_i \in M^{n-1}$ and $b_i \in B$ $(i = 1,\ldots,r)$. By (3.7)(v), $(d^{n-2}M^{n-2} : m_i)_A \not\subseteq \underline{p}$ for each $i = 1,\ldots,r$, so there exists $a \in \bigcap_{i=1}^{r} (d^{n-2}M^{n-2} : m_i)_A - \underline{p}$. Then $f(a) \in (0:x)_B$, so $a \in ((0:x)_B)^c - \underline{p}$. //

The following Proposition is one on which many of my calculations will be based.

(8.3)Proposition. Suppose M is an A-module such that conditions (X1) and (X2) of Theorem (3.11) are satisfied. Let $\Psi = \{\psi^n\}_{n \geq -1} : \underline{C(M)} \otimes_A B \longrightarrow \underline{C_B(M_B)}$ be the homomorphism of complexes of Theorem (3.11). Using the notation employed in (3.8) - (3.12), let, for each $n \geq -1$, $\overline{\psi^n} : \mathrm{coker}(d^{n-1} \otimes 1_B) \longrightarrow \mathrm{coker}\ e^{n-1}$ be the B-homomorphism induced by Ψ. Then, if $\underline{C_B(M_B)}$ is exact, $\overline{\psi^{-1}}, \overline{\psi^0}, \ldots, \overline{\psi^n}, \ldots$ are all monomorphisms, and consequently $\underline{C(M)} \otimes_A B$ is exact.

Proof. Since the induced homomorphisms on cokernels themselves induce homomorphisms on the corresponding homology modules, and $H^n(\underline{C(M)\otimes_A B}) \subseteq \operatorname{coker}(d^{n-1}\otimes 1_B)$ and $H^n(\underline{C_B(M_B)}) \subseteq \operatorname{coker} e^{n-1}$ for each $n \geqslant -1$, it is clearly sufficient to show that $\overline{\chi^{-1}}, \overline{\chi^0}, \ldots, \overline{\chi^n}, \ldots$ are all monomorphisms. It is clear that $\overline{\chi^{-1}} = \chi^{-1} : M_B \to M_B$ is the identity, and so certainly a monomorphism.

$$
\begin{array}{cccccccccccc}
 & 0 & \longrightarrow & M_B & \xrightarrow{d^{-1}\otimes 1_B} & (M^0)_B & \longrightarrow & \cdots & \longrightarrow & (M^n)_B & \xrightarrow{d^n\otimes 1_B} & (M^{n+1})_B \cdots \\
(C) & \downarrow{\scriptstyle \chi^{-2}=0} & & \| {\scriptstyle \chi^{-1}=1_{M_B}} & & \downarrow{\scriptstyle \chi^0} & & & & \downarrow{\scriptstyle \chi^n} & & \downarrow{\scriptstyle \chi^{n+1}} \\
(E) & 0 & \longrightarrow & M_B & \xrightarrow[e^{-1}]{} & (M_B)^0 & \longrightarrow & \cdots & \longrightarrow & (M_B)^n & \xrightarrow[e^n]{} & (M_B)^{n+1} \cdots
\end{array}
$$

I suppose, inductively, that $n \geqslant 0$ and I have shown $\overline{\chi^{-1}}, \ldots, \overline{\chi^{n-1}}$ are all monomorphisms. By the remarks above, $\underline{C(M)\otimes_A B}$ is exact at M_B $(= (M^{-1})_B)$, $(M^0)_B, \ldots, (M^{n-1})_B$. So:-

$$
\begin{array}{ccccccccc}
(E) & 0 \longrightarrow & (M^{-1})_B & \longrightarrow & (M^0)_B & \longrightarrow & \operatorname{coker}(d^{-1}\otimes 1_B) & \longrightarrow & 0 \\
 & & \chi^{-1}\| & & \chi^0\downarrow & & \overline{\chi^0}\downarrow & & \\
(C)\ (E) & 0 \longrightarrow & (M_B)^{-1} & \longrightarrow & (M_B)^0 & \longrightarrow & \operatorname{coker} e^{-1} & \longrightarrow & 0 \\
(E) & 0 \longrightarrow & \operatorname{coker}(d^{-1}\otimes 1_B) & \longrightarrow & (M^1)_B & \longrightarrow & \operatorname{coker}(d^0\otimes 1_B) & \longrightarrow & 0 \\
 & & \overline{\chi^0}\downarrow & & \chi^1\downarrow & & \overline{\chi^1}\downarrow & & \\
(C)\ (E) & 0 \longrightarrow & \operatorname{coker} e^{-1} & \longrightarrow & (M_B)^1 & \longrightarrow & \operatorname{coker} e^0 & \longrightarrow & 0 \\
 & & \vdots & & & & & & \\
(E) & 0 \longrightarrow & \operatorname{coker}(d^{n-2}\otimes 1_B) & \longrightarrow & (M^n)_B & \longrightarrow & \operatorname{coker}(d^{n-1}\otimes 1_B) & \longrightarrow & 0 \\
 & & \overline{\chi^{n-1}}\downarrow & & \chi^n\downarrow & & \overline{\chi^n}\downarrow & & \\
(C)\ (E) & 0 \longrightarrow & \operatorname{coker} e^{n-2} & \longrightarrow & (M_B)^n & \longrightarrow & \operatorname{coker} e^{n-1} & \longrightarrow & 0
\end{array}
\qquad \Big\} (A)
$$

Suppose that x is a non-zero element of $\operatorname{coker}(d^{n-1}\otimes 1_B)$: I wish to

show that $\overline{\gamma^n}(x) \neq 0$. By (8.2), $((0:x)_B)^c$ is not contained in any $\underline{p} \in \mathrm{Supp}_A(M)$ of M-height $\leqslant n$, so by (8.1) $\mathrm{Ext}^i_B(Bx,(M^r)_B) = 0$ for all $i \geqslant 0$ and $0 \leqslant r \leqslant n$.

By condition (X1), $(0:x)_B$ is not contained in any $\underline{q} \in \mathrm{Supp}_B(M_B)$ of M_B-hieght $\leqslant n$. Thus, by (4.21), $\mathrm{Ext}^i_B(Bx,(M_B)^r) = 0$ for all $i \geqslant 0$ and $0 \leqslant r \leqslant n$.

Now, using those properties of the Extension functors described in the last paragraph of (2.2), I derive from (A):-

$$\begin{array}{ccccc}
\mathrm{Hom}_B(Bx,\mathrm{coker}(d^{n-1}\otimes 1_B)) & \xrightarrow[\cong]{} & \mathrm{Ext}^1_B(Bx,\mathrm{coker}(d^{n-2}\otimes 1_B)) & \xrightarrow{\cong} & \cdots. \\
\downarrow{\scriptstyle (\overline{\gamma^n})^*} & & \downarrow{\scriptstyle (\overline{\gamma^{n-1}})^*} & & \\
\mathrm{Hom}_B(Bx,\mathrm{coker}\ e^{n-1}) & \xrightarrow{\cong} & \mathrm{Ext}^1_B(Bx,\mathrm{coker}\ e^{n-2}) & \xrightarrow{\cong} & \cdots.
\end{array} \tag{C}$$

$$\begin{array}{ccccc}
\cdots. \xrightarrow[\cong]{} \mathrm{Ext}^n_B(Bx,\mathrm{coker}(d^{-1}\otimes 1_B)) & \xrightarrow[\cong]{} & \mathrm{Ext}^{n+1}_B(Bx,M_B) & & \\
\downarrow{\scriptstyle (\overline{\gamma^0})^*} & & \| & & \\
\cdots. \xrightarrow{\cong} \mathrm{Ext}^n_B(Bx,\mathrm{coker}\ e^{-1}) & \xrightarrow{\cong} & \mathrm{Ext}^{n+1}_B(Bx,M_B) & & (C).
\end{array}$$

(The "*"s denote induced homomorphisms, of course.)

Thus $(\overline{\gamma^n})^*:\mathrm{Hom}_B(Bx,\mathrm{coker}(d^{n-1}\otimes 1_B)) \longrightarrow \mathrm{Hom}_B(Bx,\mathrm{coker}\ e^{n-1})$ is an isomorphism. Since $x \neq 0$, the inclusion $Bx \xrightarrow{\subseteq} \mathrm{coker}(d^{n-1}\otimes 1_B)$ is non-zero, and so the image of this under $(\overline{\gamma^n})^*$ is non-zero. Therefore the composition $Bx \xrightarrow{\subseteq} \mathrm{coker}(d^{n-1}\otimes 1_B) \xrightarrow{\overline{\gamma^n}} \mathrm{coker}\ e^{n-1}$ is non-zero, and so $\overline{\gamma^n}$ is a monomorphism.

The Proposition now follows by induction. //

Sometimes one can see that some of the induced homomorphisms on cokernels are even isomorphisms:-

(8.4)Proposition. Assume the hypotheses of Proposition (8.3). Suppose $n \geqslant 0$. Then $\overline{\gamma^n}$ is an isomorphism if the following condition is satisfied:-

(X3(n)) if $\underline{q} \in \mathrm{Supp}_B(M_B)$ has M_B-height $\geqslant n+1$, then $\mathrm{ht}_M \underline{q}^c \geqslant n+1$.

Proof. By (8.3), $\overline{\chi^{-1}}, \overline{\chi^0}, \ldots, \overline{\chi^r}, \ldots$ are all monomorphisms: I must show that $\overline{\chi^n}$ is an epimorphism. Let y be a non-zero element of coker e^{n-1}, and let $\underline{b} = (0:y)_B$. By (3.7)(v), $\underline{b}$ is an M_B-proper ideal of B not contained in any $\underline{q} \in \mathrm{Supp}_B(M_B)$ of M_B-height $\leqslant n$. Let $\underline{b} = \bigcap_{i=1}^{p} \underline{b}_i$, with each $\underline{b}_i$ $\underline{q}_i$-primary, be a minimal p.d. for $\underline{b}$ in B. Again by (3.7)(v), each $\underline{q}_i$ $(i = 1,\ldots,p)$ belongs to $\mathrm{Supp}_B(M_B)$ and has M_B-height $\geqslant n+1$.

Suppose $\underline{p} \in V(A)$ contains $\underline{b}^c$. $\underline{b}^c = \bigcap_{i=1}^{p} \underline{b}_i{}^c$. For each $i = 1,\ldots,p$ there is a positive integer h_i s.t. $\underline{q}_i{}^{h_i} \subseteq \underline{b}_i \subseteq \underline{q}_i$. Now $(\underline{q}_i{}^c)^{h_i} \subseteq (\underline{q}_i{}^{h_i})^c \subseteq \underline{b}_i{}^c$, so $\bigcap_{i=1}^{p} (\underline{q}_i{}^c)^{h_i} \subseteq \underline{p}$. Consequently, for some i s.t. $1 \leqslant i \leqslant p$, $(\underline{q}_i{}^c)^{h_i} \subseteq \underline{p}$, so $\underline{q}_i{}^c \subseteq \underline{p}$. By (3.10), $\underline{q}_i{}^c \in \mathrm{Supp}_A(M)$, and by hypothesis, $\mathrm{ht}_M \underline{q}_i{}^c \geqslant n+1$. Thus $\underline{p} \in \mathrm{Supp}_A(M)$ and $\mathrm{ht}_M \underline{p} \geqslant n+1$.

Thus $\underline{b}$ is not contained in any $\underline{q} \in \mathrm{Supp}_B(M_B)$ of M_B-height $\leqslant n$, and $\underline{b}^c$ is not contained in any $\underline{p} \in \mathrm{Supp}_A(M)$ of M-height $\leqslant n$.

By (8.1), $\mathrm{Ext}^i_B(By, (M^r)_B) = 0$ for all $i \geqslant 0$ and $0 \leqslant r \leqslant n$.

By (4.21), $\mathrm{Ext}^i_B(By, (M_B)^r) = 0$ for all $i \geqslant 0$ and $0 \leqslant r \leqslant n$.

Thus, again using properties of the Extension functors described in the last paragraph of (2.2), I derive from the commutative diagrams (A) of the proof of (8.3):-

$$
\begin{array}{ccccccc}
\mathrm{Hom}_B(By, \mathrm{coker}(d^{n-1}\otimes 1_B)) & \xrightarrow{\cong} & \mathrm{Ext}^1_B(By, \mathrm{coker}(d^{n-2}\otimes 1_B)) & \xrightarrow{\cong} & \cdots & & \\
\downarrow (\overline{\chi^n})^* & & \downarrow (\overline{\chi^{n-1}})^* & & & & \\
\mathrm{Hom}_B(By, \mathrm{coker}\ e^{n-1}) & \xrightarrow{\cong} & \mathrm{Ext}^1_B(By, \mathrm{coker}\ e^{n-2}) & \xrightarrow{\cong} & \cdots & & \\
\cdots & \xrightarrow{\cong} & \mathrm{Ext}^n_B(By, \mathrm{coker}(d^{-1}\otimes 1_B)) & \xrightarrow{\cong} & \mathrm{Ext}^{n+1}_B(By, M_B) & & \\
 & & \downarrow (\overline{\chi^0})^* & & \| & & \\
\cdots & \xrightarrow{\cong} & \mathrm{Ext}^n_B(By, \mathrm{coker}\ e^{-1}) & \xrightarrow{\cong} & \mathrm{Ext}^{n+1}_B(By, M_B) & & (C).
\end{array}
\tag{C}
$$

(Again, the "*"s denote induced homomorphisms.)

Thus $(\overline{\gamma^n})^*:\mathrm{Hom}_B(By,\mathrm{coker}(d^{n-1}\otimes 1_B)) \longrightarrow \mathrm{Hom}_B(By,\mathrm{coker}\ e^{n-1})$ is an isomorphism, so there is a homomorphism $\alpha:By \longrightarrow \mathrm{coker}(d^{n-1}\otimes 1_B)$ s.t. $\overline{\gamma^n}\circ\alpha$ is the inclusion $By \xrightarrow{\subseteq} \mathrm{coker}\ e^{n-1}$. Thus $\overline{\gamma^n}(\alpha(y)) = y$.

Consequently, $\overline{\gamma^n}$ is epimorphic, and so is an isomorphism. //

(8.5)Corollary. Again assume the hypotheses of Proposition (8.3). If, for each $\mathfrak{q} \in \mathrm{Supp}_B(M_B)$, $\mathrm{ht}_{M_B}\mathfrak{q} = \mathrm{ht}_M \mathfrak{q}^c$, then Ψ is an isomorphism of complexes.

Proof. By (8.4), each $\overline{\gamma^n}$ $(n \geqslant -1)$ is an isomorphism. Now use the "5-Lemma" (noting that both $C_B(M_B)$ and $C(M)\otimes_A B$ are exact) to see that each γ^n is an isomorphism. //

(8.6)Corollary. In the situation of Corollary (8.5), and using the notation of the statement and proof of Theorem (3.11), for each $\mathfrak{p} \in \mathrm{Supp}_A(M)$ of M-height n, $(\mathrm{coker}\ d^{n-2})_{\mathfrak{p}}\otimes_A B \cong \bigoplus_{\mathfrak{q}\in\mathcal{E}(\mathfrak{p})} (\mathrm{coker}\ e^{n-2})_{\mathfrak{q}}$.

Proof. Recall that if $\mathfrak{p} \in \mathrm{Supp}_A(M)$, $\mathcal{E}(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Supp}_B(M_B) \mid \mathfrak{q}^c = \mathfrak{p}\ \&\ \mathrm{ht}_{M_B}\mathfrak{q} = \mathrm{ht}_M\mathfrak{p}\}$, and so $\bigcup_{\substack{\mathfrak{p}\in\mathrm{Supp}_A(M)\\ \mathrm{ht}_M\mathfrak{p}=n}} \mathcal{E}(\mathfrak{p})$ is precisely the set of primes of $\mathrm{Supp}_B(M_B)$ having M_B-height n.

Using the notation of the proof of Theorem (3.11), since Ψ is an isomorphism of complexes, γ^n,α^n,μ^n, & ν^n are all isomorphisms, so for each $\mathfrak{p}\in\mathrm{Supp}_A(M)$ of M-ht. n, $\gamma_{\mathfrak{p}}:(\mathrm{coker}(d^{n-2}\otimes 1_B))_{\mathfrak{p}} \longrightarrow \bigoplus_{\mathfrak{q}\in\mathcal{E}(\mathfrak{p})} (\mathrm{coker}(d^{n-2}\otimes 1_B))_{\mathfrak{q}}$ must be an isomorphism. The result follows, since $\overline{\gamma^{n-1}}:\mathrm{coker}(d^{n-2}\otimes 1_B) \xrightarrow{\cong} \mathrm{coker}\ e^{n-2}$ and (from the proof of Lemma (3.9)) $(\mathrm{coker}(d^{n-2}\otimes 1_B))_{\mathfrak{p}} \cong (\mathrm{coker}\ d^{n-2})_{\mathfrak{p}}\otimes_A B$ (as B-modules). //

(8.7)Example. Suppose A and B are integral domains with A a subring of B and $f:A \longrightarrow B$ the inclusion. Suppose that B is integral over A and that A is integrally closed. Take M = A. Then $M_B \cong B$ as B-modules.

In (3.16), it was noted that conditions (X1) and (X2) are satisfied in this situation, and that if $\mathfrak{q} \in V(B)$, $ht_B \mathfrak{q} = ht_A \mathfrak{q}^c$. Thus, if B is Cohen-Macaulay (a necessary and sufficient condition for $C_B(B)$ to be exact) the results of (8.3) - (8.6) may be applied.

By the "Lying Over" Theorem ((5.10) of Atiyah and Macdonald (1)) every $\mathfrak{p} \in V(A)$ is the contraction of some $\mathfrak{q} \in V(B)$. Thus, if both A and B are Gorenstein (for example, if A = Z, the ring of integers, and B is the integral closure of Z in a finite algebraic extension of the rational field $\mathbb{Q}$) it follows that, for each $\mathfrak{p} \in V(A)$,

$$E_A(A/\mathfrak{p}) \otimes_A B \cong \bigoplus_{\substack{\mathfrak{q} \in V(B) \\ \mathfrak{q}^c = \mathfrak{p}}} E_B(B/\mathfrak{q}) \text{ , and so is a non-zero injective}$$

B-module. (Use (8.6) and (5.14).) Thus, in this case, if E is any non-zero injective A-module, $E \otimes_A B$ is a non-zero injective B-module.

(8.8) For the rest of this Chapter I shall assume that B, when regarded as an A-module by means of f, is flat. In a recent paper, (15), Iversen showed that, in this situation, the functor $_ \otimes_A B$ from $\mathcal{C}_A$ to $\mathcal{C}_B$ preserves essential monomorphisms, and from this fact he was able to deduce several interesting results, some concerning Gorenstein rings and Cohen-Macaulay rings. I shall show how some of Iversen's results and some new results not obtained by Iversen can be deduced from an examination of the Cousin Complexes involved. The work depends heavily on Iversen's work, and the Cousin Complex proofs that I give of some of his results are longer than

his proofs: the value of this approach lies in the fact that it yields more information than Iversen was (apparently) able to obtain.

For ease of reference, I catalogue those of Iversen's results which I shall need to use.

(8.9)Theorem. (Theorem 1 of Iversen (15)) The functor $_\otimes_A B$ from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_B$ preserves essential monomorphisms. //

(8.10)Lemma. (Lemma 2 of Iversen (15).) Suppose $\underline{p} \in V(A)$. If $\underline{q} \in \mathrm{Ass}_B(B/\underline{p}^e)$, $\underline{q}^c = \underline{p}$. //

(8.11)Proposition. (Proposition 3 of Iversen (15).) Suppose $\underline{q} \in V(B)$ and M is a f.-g. A-module. Then:-

..(i) $\underline{q} \in \mathrm{Ass}_B(M_B)$ if and only if $\underline{q}^c \in \mathrm{Ass}_A(M)$ and $\underline{q} \in \mathrm{Ass}_B(B/\underline{q}^{ce})$;

& (ii) $\underline{q}$ is a minimal element of $\mathrm{Ass}_B(M_B)$ if and only if $\underline{q}^c$ is a minimal element of $\mathrm{Ass}_A(M)$ and $\underline{q}$ is a minimal element of $\mathrm{Ass}_B(B/\underline{q}^{ce})$. //

(8.12)Lemma. (Lemma 3 of Iversen (15).) Let $\underline{p}_0 \subsetneq \underline{p}_1 \subsetneq \dots \subsetneq \underline{p}_n$ be a chain of prime ideals of A, and suppose $\underline{q}_n \in V(B)$ is such that $\underline{q}_n^c = \underline{p}_n$. Then there exists primes $\underline{q}_0 \subsetneq \underline{q}_1 \subsetneq \dots \subsetneq \underline{q}_n$ of B such that $\underline{q}_i^c = \underline{p}_i$ for each $i = 0,\dots,n$. //

(8.13)Theorem. (Theorem 2, part (1), of Iversen (15).) If $\underline{q}$ is a prime ideal of B, $\mathrm{ht}_A \underline{q}^c \leqslant \mathrm{ht}_B \underline{q}$, and $\mathrm{ht}_A \underline{q}^c = \mathrm{ht}_B \underline{q}$ if and only if $\underline{q}$ is a minimal element of $\mathrm{Ass}_B(B/\underline{q}^{ce})$. //

The results which suggest that an examination of the Cousin Complexes involved might be profitable are:-

(8.14) (Corollary 1 to Theorem 2 of Iversen (15).) Suppose $\underline{m}^e \neq B$ for

each maximal ideal $\underline{m}$ of A, and let $\underline{p}$ be a prime ideal of A. Then:-

(i) if B is Cohen-Macaulay, so is A, and $\underline{p}^e$ in B has no embedded prime ideals;

(ii) if B is Gorenstein, so is A, and the (unique) primary components of $\underline{p}^e$ in B are all irreducible. //

(8.15) (Corollary 2 to Theorem 2 of Iversen (15).) Suppose A and B are local with maximal ideals $\underline{m}$ and $\underline{n}$ respectively, and suppose that $\underline{m}^e = \underline{n}$. Then A is Gorenstein if and only if B is Gorenstein. //

It is worthwhile to examine the condition "$\underline{m}^e \neq B$ for each maximal ideal $\underline{m}$ of A" in greater detail.

(8.16) Lemma. (This appears as an Exercise for Chapter II of Atiyah and Macdonald (1).) Suppose N is a B-module. Regarding N as an A-module by means of f, form $N \otimes_A B$. Then the A-module homomorphism $g: N \longrightarrow N \otimes_A B$ defined by $g(n) = n \otimes 1$ is a monomorphism, and g(N) is a direct summand of $N \otimes_A B$.

Proof. Define $p: N \otimes_A B \longrightarrow N$ by $p(n \otimes b) = b.n$. It is easy to see that this is a well-defined A-module homomorphism, and that $p.g = 1_N$. Thus g is monomorphic, and $N \otimes_A B = g(N) \oplus \ker p$. //

(8.17) Proposition and Definition. (This appears as an Exercise for Chapter III of Atiyah and Macdonald (1).) (N.B. that B, when regarded as an A-module by means of f, is flat.) B is s.t.b. faithfully flat over A if and only if the following equivalent conditions are satisfied:-

(i) $\underline{a}^{ec} = \underline{a}$ for all ideals $\underline{a}$ of A;

(ii) for each maximal ideal $\underline{m}$ of A, $\underline{m}^e \neq B$;

(iii) if M is a non-zero A-module, $M \otimes_A B \neq 0$;

& (iv) for each A-module M, the A-module homomorphism $g: M \longrightarrow M \otimes_A B$ defined by $g(m) = m \otimes 1$ is a monomorphism.

Proof. (i)$\Rightarrow$(ii) If $\underline{m}^e = B$, $\underline{m}^{ec} = A \neq \underline{m}$, a contradiction.

(ii)$\Rightarrow$(iii) Let $x \in M$, $x \neq 0$, and let $M' = Ax = A/\underline{a}$ where $\underline{a} = (0:x)_M$. Since B is flat, it is sufficient to show $A/\underline{a} \otimes_A B \neq 0$. Again, since B is flat, it follows from (E) $0 \longrightarrow \underline{a} \longrightarrow A \longrightarrow A/\underline{a} \longrightarrow 0$ that:-

$$(E)\quad 0 \longrightarrow \underline{a} \otimes_A B \longrightarrow A \otimes_A B \longrightarrow A/\underline{a} \otimes_A B \longrightarrow 0$$

$$\begin{array}{ccc} g \downarrow \cong & & h \downarrow \cong \\ \underline{a}^e & \subseteq & B \end{array} \qquad (C),$$

where $h: A \otimes_A B \to B$ is the canonical A- (& B-) isomorphism, and $g: \underline{a} \otimes_A B \to \underline{a}^e$ is the obvious natural epimorphism. Since $\underline{a} \neq A$, $\underline{a}$ is contained in some maximal ideal of A, so by hypothesis $\underline{a}^e \neq B$. Thus $A/\underline{a} \otimes_A B \neq 0$.

(iii)$\Rightarrow$(iv) Let $N = \ker g$, so that $(E)\ 0 \longrightarrow N \longrightarrow M \xrightarrow{g} M \otimes_A B$. Since B is flat, it follows that $(E)\ 0 \to N \otimes_A B \to M \otimes_A B \xrightarrow{g \otimes 1_B} (M \otimes_A B) \otimes_A B$ If $h: (M \otimes_A B) \otimes_A B \longrightarrow (M \otimes_A B) \otimes_A B$ is the natural A-module isomorphism which is s.t. $h((m \otimes b) \otimes b') = (m \otimes b') \otimes b$, then it follows from (8.16) that $h.(g \otimes 1_B)$ is monomorphic. Thus $g \otimes 1_B$ is monomorphic, $N \otimes_A B = 0$, and so $N = 0$ by hypothesis.

(iv)$\Rightarrow$(i) If $\underline{c}$ is any ideal of A, then the natural epimorphism $\underline{c} \otimes_A B \longrightarrow \underline{c}^e$ is an isomorphism, since B is A-flat.

Let $\underline{a}$ be any ideal of A. $\underline{a} \subseteq \underline{a}^{ec}$, certainly. From:-

$$(E)\quad 0 \longrightarrow \underline{a} \xrightarrow{\subseteq} \underline{a}^{ec} \longrightarrow \underline{a}^{ec}/\underline{a} \longrightarrow 0 \quad \text{it follows that:-}$$

$$(E)\quad 0 \longrightarrow \underline{a} \otimes_A B \longrightarrow \underline{a}^{ec} \otimes_A B \longrightarrow (\underline{a}^{ec}/\underline{a}) \otimes_A B \longrightarrow 0$$

$$(C)\qquad \begin{array}{ccc} \cong \downarrow & & \cong \downarrow \\ \underline{a}^e & \subseteq & \underline{a}^{ece} \end{array}$$

But $\underline{a}^e = \underline{a}^{ece}$, so $(\underline{a}^{ec}/\underline{a}) \otimes_A B = 0$. Consequently, by hypothesis, $\underline{a}^{ec} = \underline{a}$. //

(8.18)Corollary. If B is faithfully flat over A, f is injective.

Proof. f is the composition of the natural A-homomorphism $g:A \longrightarrow A\otimes_A B$ defined by $g(a) = a\otimes 1 = 1\otimes f(a)$ and the natural A- and B-isomorphism $h:A\otimes_A B \longrightarrow B$ defined by $h(a\otimes b) = f(a).b$. //

As might be anticipated, I plan to use Theorem (3.11) and the consequences of this Theorem developed in the first part of this Chapter. However, I must first make sure that conditions (X1) and (X2) are satisfied. In (3.15) I showed that these conditions are satisfied when M = A: I want to check that they are satisfied for an arbitrary f.-g. A-module.

(8.19)Lemma. If $\mathfrak{q} \in V(B)$ and $\mathfrak{p} = \mathfrak{q}^c$, the induced ring homomorphism $f':A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{q}}$ defined by $f'(a/s) = f(a)/f(s)$ makes $B_{\mathfrak{q}}$ faithfully flat over $A_{\mathfrak{p}}$.

Proof. Let $\sigma:A \longrightarrow A_{\mathfrak{p}}$ and $\rho:B \longrightarrow B_{\mathfrak{q}}$ be the natural ring homomorphisms. Then:-

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \sigma\downarrow & & \downarrow\rho \\ A_{\mathfrak{p}} & \xrightarrow{f'} & B_{\mathfrak{q}} \end{array} \qquad (C).$$

Suppose (E) $0 \longrightarrow N \longrightarrow M$ is an exact sequence of $A_{\mathfrak{p}}$-modules and homomorphisms. By means of σ one may regard this as an exact sequence of A-modules and -homs. Then since B is A-flat and $B_{\mathfrak{q}}$ is B-flat,

(E) $0 \longrightarrow N\otimes_A B \longrightarrow M\otimes_A B$ & (E)$0 \longrightarrow (N\otimes_A B)\otimes_B B_{\mathfrak{q}} \longrightarrow (M\otimes_A B)\otimes_B B_{\mathfrak{q}}$.

Thus (E)$0 \longrightarrow N\otimes_A B_{\mathfrak{q}} \longrightarrow M\otimes_A B_{\mathfrak{q}}$, where $B_{\mathfrak{q}}$ is regarded as an A-module by means of $\rho.f = f'.\sigma$. Thus:-

(E) $0 \longrightarrow (N\otimes_A A_{\mathfrak{p}})\otimes_A B_{\mathfrak{q}} \longrightarrow (M\otimes_A A_{\mathfrak{p}})\otimes_A B_{\mathfrak{q}}$, so since N & M are $A_{\mathfrak{p}}$-modules, (E) $0 \longrightarrow N\otimes_A B_{\mathfrak{q}} \longrightarrow M\otimes_A B_{\mathfrak{q}}$.

Thus $B_{\mathfrak{q}}$ is a flat $A_{\mathfrak{p}}$-module. Since $\mathfrak{p}^e \subseteq \mathfrak{q}$, $f'(\mathfrak{p}A_{\mathfrak{p}}) \subseteq \mathfrak{q}B_{\mathfrak{q}}$, so by (8.17), f' makes $B_{\mathfrak{q}}$ faithfully flat over $A_{\mathfrak{p}}$. //

(8.20)Lemma. If M is an A-module, $\mathrm{Supp}_B(M_B) = \left\{\mathfrak{q} \in V(B) \mid \mathfrak{q}^c \in \mathrm{Supp}_A(M)\right\}$.

Proof. By (3.10), L.H.S. R.H.S. Suppose $\mathfrak{q} \in V(B)$ is s.t. $M \otimes_A A_{\mathfrak{q}^c} \neq 0$. Then, by (8.19) and (8.17), $(M \otimes_A A_{\mathfrak{q}^c}) \otimes_{A_{\mathfrak{q}^c}} B_{\mathfrak{q}} \neq 0$, so $M \otimes_A B_{\mathfrak{q}} \neq 0$ ($B_{\mathfrak{q}}$ being regarded as an A-module by means of $\rho.f = f'.\sigma$), and so finally $(M \otimes_A B) \otimes_B B_{\mathfrak{q}} \neq 0$. //

I need a result for f.-g. modules analogous to (8.13).

(8.21)Proposition. Suppose that M is a non-zero f.-g. A-module, $\mathfrak{q} \in V_B(M_B)$, and $\mathfrak{p} = \mathfrak{q}^c$ (and so belongs to $V_A(M)$ by (8.20)). Then $\mathrm{ht}_M \mathfrak{p} \leqslant \mathrm{ht}_{M_B} \mathfrak{q}$ and $\mathrm{ht}_M \mathfrak{p} = \mathrm{ht}_{M_B} \mathfrak{q}$ if and only if $\mathfrak{q}$ is a minimal element of $\mathrm{Ass}_B(B/\mathfrak{p}^e)$.

Proof. Suppose $\mathrm{ht}_M \mathfrak{p} = k$, and let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_k = \mathfrak{p}$ be a saturated chain of prime ideals of $V_A(M)$. By (8.12), there exists a chain $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \dots \subsetneq \mathfrak{q}_k = \mathfrak{q}$ of primes of B s.t. $\mathfrak{q}_i^c = \mathfrak{p}_i$ for each $i = 0, \dots, k$. By (8.20), each $\mathfrak{q}_i \in V_B(M_B)$ ($i = 0, \dots, k$), so it follows that $\mathrm{ht}_{M_B} \mathfrak{q} \geqslant \mathrm{ht}_M \mathfrak{p}$.

Next, suppose $\mathrm{ht}_M \mathfrak{p} = \mathrm{ht}_{M_B} \mathfrak{q}$. Certainly, $\mathfrak{p}^e \subseteq \mathfrak{q}$, so $\mathfrak{q}$ contains a minimal element $\mathfrak{q}'$ of $\mathrm{Ass}_B(B/\mathfrak{p}^e)$. By (8.10), $\mathfrak{q}'^c = \mathfrak{p}$, and so by (8.20), $\mathfrak{q}' \in V_B(M_B)$. Thus, by the first part, $\mathrm{ht}_M \mathfrak{p} \leqslant \mathrm{ht}_{M_B} \mathfrak{q}' \leqslant \mathrm{ht}_{M_B} \mathfrak{q} = \mathrm{ht}_M \mathfrak{p}$. Since $\mathfrak{q}' \subseteq \mathfrak{q}$, $\mathfrak{q}' = \mathfrak{q}$, and so $\mathfrak{q}$ is a minimal element of $\mathrm{Ass}_B(B/\mathfrak{p}^e)$.

Suppose, conversely, that $\mathfrak{q}$ is a minimal element of $\mathrm{Ass}_B(B/\mathfrak{p}^e)$. Let $f' : A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{q}}$ be the ring homomorphism induced by f. (see (8.19).) By (8.19), f' makes $B_{\mathfrak{q}}$ faithfully flat over $A_{\mathfrak{p}}$. $\mathfrak{q}B_{\mathfrak{q}}$ is a minimal element of

$\mathrm{Ass}_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}/\mathfrak{p}^{e}.B_{\mathfrak{q}})$, and $\mathfrak{p}^{e}.B_{\mathfrak{q}} = (f'(\mathfrak{p}A_{\mathfrak{p}})).B_{\mathfrak{q}}$. By (1.19), $\mathfrak{q}B_{\mathfrak{q}} \in V_{B_{\mathfrak{q}}}((M_B)_{\mathfrak{q}})$, $\mathfrak{p}A_{\mathfrak{p}} \in V_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$, $\mathrm{ht}_{(M_B)_{\mathfrak{q}}}(\mathfrak{q}B_{\mathfrak{q}}) = \mathrm{ht}_{M_B}\mathfrak{q}$, and $\mathrm{ht}_{M_{\mathfrak{p}}}(\mathfrak{p}A_{\mathfrak{p}}) = \mathrm{ht}_{M}\mathfrak{p}$. Finally, $(M\otimes_A B)\otimes_B B_{\mathfrak{q}} \cong (M\otimes_A A_{\mathfrak{p}})\otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}}$ as $B_{\mathfrak{q}}$-modules, so it may be assumed that A and B are local with maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$ respectively.

Thus $\mathrm{ht}_M\mathfrak{p} = \text{K-Dim}_A M = k$, and $\mathfrak{p}^e$ is a $\mathfrak{q}$-primary ideal of B. By (1.23), there exist k elements $x_1,\ldots,x_k$ of $\mathfrak{p}$ s.t. $(0:M)_A + (x_1,\ldots,x_k)$ is a $\mathfrak{p}$-primary ideal of A. By (8.11)(i), since $((0:M)_A)^e \subseteq (0:M_B)_B$, $(0:M_B)_B + (f(x_1),\ldots,f(x_k))$ is a $\mathfrak{q}$-primary ideal of B.

Thus $\text{K-Dim}\,(B/(0:M_B)_B) = \text{K-Dim}_B(M_B) \leqslant k$, so by the first part, $\mathrm{ht}_{M_B}\mathfrak{q} = k = \mathrm{ht}_M\mathfrak{p}$. //

(8.22)Corollary. If M is a non-zero, f.-g., A-module, then A,B,f, & M satisfy the conditions (X1) and (X2) of Theorem (3.11).

Proof. It is immediate from (8.21) that (X1) is satisfied. Suppose $\mathfrak{p} \in V_A(M)$, and $\mathfrak{q} \in \mathcal{E}(\mathfrak{p})$, so that $\mathfrak{q} \in V_B(M_B)$, $\mathfrak{q}^c = \mathfrak{p}$ and $\mathrm{ht}_{M_B}\mathfrak{q} = \mathrm{ht}_M\mathfrak{p}$. By (8.21), $\mathfrak{q}$ is a minimal element of $\mathrm{Ass}_B(B/\mathfrak{p}^e)$. Consequently, $\mathcal{E}(\mathfrak{p})$ is finite, and (X2) is satisfied. //

Thus, Theorem (3.11) may now be applied: however, I first give, for the sake of completeness, the following well-known

(8.23)Lemma. (See Theorem 1 of Chapter 6 of Northcott (22).) Suppose that $M' \xrightarrow{f} M \xrightarrow{g} M''$ is a complex of A-module and homomorphisms with homology module H(M). Let $H(M_B)$ be the homology module of the induced complex $M'_B \xrightarrow{f\otimes 1_B} M_B \xrightarrow{g\otimes 1_B} M''_B$. Then the B-modules $(H(M))_B$ & $H(M_B)$

are isomorphic.

Proof. The inclusions $f(M') \subseteq \ker g \subseteq M$ induce B-monomorphisms:-

$\alpha : (f(M'))_B \longrightarrow M_B$, $\beta : (\ker g)_B \longrightarrow M_B$, and $\gamma : (f(M'))_B \longrightarrow (\ker g)_B$ such that:-

$$\begin{array}{ccc} (f(M'))_B & \xrightarrow{\gamma} & (\ker g)_B \\ \alpha \downarrow & & \downarrow \beta \\ M_B & \xrightarrow[1_{M_B}]{} & M_B \end{array} \quad (C) .$$

It will be sufficient to show that $\alpha((f(M'))_B) = (f \otimes 1_B)(M'_B)$ and $\beta((\ker g)_B) = \ker(g \otimes 1_B)$, for then:-

$$\begin{array}{ccccccccc} (E) & 0 & \longrightarrow & (f(M'))_B & \xrightarrow{\gamma} & (\ker g)_B & \longrightarrow & (H(M))_B & \longrightarrow 0 \\ & & & \cong \downarrow & & \cong \downarrow & & & \\ (C)\ (E) & 0 & \longrightarrow & (f \otimes 1_B)(M'_B) & \xrightarrow{\subseteq} & \ker(g \otimes 1_B) & \longrightarrow & H(M_B) & \longrightarrow 0 \end{array} ,$$

from which the result is obvious.

From $(E)\ 0 \longrightarrow \ker g \xrightarrow{\subseteq} M \xrightarrow{g} M''$ I deduce:-

$(E)\ 0 \longrightarrow (\ker g)_B \xrightarrow{\beta} M_B \xrightarrow{g \otimes 1_B} (M'')_B$, so $\beta((\ker g)_B) = \ker(g \otimes 1_B)$.

$(E)\ M' \xrightarrow{f} M \longrightarrow \operatorname{coker} f \longrightarrow 0$ induces a B-isomorphism $\delta : (\operatorname{coker} f)_B \xrightarrow{\cong} \operatorname{coker}(f \otimes 1_B)$ such that:-

$$\begin{array}{ccccccccc} (E) & 0 & \longrightarrow & (\operatorname{Im} f)_B & \xrightarrow{\alpha} & M_B & \longrightarrow & (\operatorname{coker} f)_B & \longrightarrow 0 \\ & & & & & \| & & \cong \downarrow \delta & \\ (C)\ (E) & 0 & \longrightarrow & \operatorname{Im}(f \otimes 1_B) & \xrightarrow{\subseteq} & M_B & \longrightarrow & \operatorname{coker}(f \otimes 1_B) & \longrightarrow 0 \end{array} ,$$

and so $\alpha((f(M')_B) = (f \otimes 1_B)(M'_B)$, and the result follows. //

I can now, at last, state and prove the main results of this Chapter.

(8.24)Theorem. Suppose that B is faithfully flat over A and that M is a non-zero, f.-g., A-module. Then M_B is a non-zero, f.-g., B-module (using (8.17)). If M_B is a Cohen-Macaulay B-module, M is a Cohen-Macaulay A-module, and, for each $\underline{p} \in V_A(M)$, $B/\underline{p}^e$ has only isolated associated prime ideals.

Proof. By (8.22), the conditions (X1) and (X2) of Theorem (3.11) are satisfied. By (8.3), $\underline{C(M)}\otimes_A B$ is exact, where $\underline{C(M)}$ denotes the Cousin Complex for the A-module M. Thus, for each $i \geqslant -1$, $H^i(M)\otimes_A B = 0$, by (8.23), so, since B is faithfully flat over A, $H^i(M) = 0$ for each $i \geqslant -1$. Thus M is Cohen-Macaulay.

Now suppose $\underline{p} \in V_A(M)$ and $\underline{q} \in Ass_B(B/\underline{p}^e)$. By (8.10), $\underline{q}^c = \underline{p}$, so $\underline{q} \in V_B(M_B)$ by (8.20). By (8.21), it is sufficient, for the second part of this Theorem, to show that $ht_M \underline{p} = ht_{M_B} \underline{q}$. By (8.19), the induced ring homomorphism $f' : A_{\underline{p}} \longrightarrow B_{\underline{q}}$ makes $B_{\underline{q}}$ faithfully flat over $A_{\underline{p}}$, and it is sufficient to assume that A and B are local with maximal ideals $\underline{p}$ and $\underline{q}$ respectively.

Let $ht_M \underline{p} = n$, $ht_{M_B} \underline{q} = n+k \geqslant n$. I must show that $k = 0$. Suppose not. Use the notation of Theorem (3.11), so that $\mathcal{T} = \{\chi^n\}_{n\geqslant -1} : \underline{C(M)}\otimes_A B \to \underline{C_B(M_B)}$ denotes the homomorphism of complexes with $\chi^{-1} = 1_{M_B} : M_B \longrightarrow M_B$ whose existence is given by Theorem (3.11). Therefore:-

$$\begin{array}{lccccccccccccc}
(E)\,0 \longrightarrow & (M^{-1})_B & \to & (M^0)_B & \to \cdots \to & (M^{n-1})_B & \to & (M^n)_B & \to 0 \to \cdots \to & 0 & \longrightarrow & 0 \\
& \| & & \downarrow \chi^0 & & \downarrow \chi^{n-1} & & \downarrow \chi^n & \quad \downarrow \chi^{n+1} & \downarrow \chi^{n+k} & & \\
(C)\ (E)\,0 \longrightarrow & (M_B)^{-1} & \to & (M_B)^0 & \to \cdots \to & (M_B)^{n-1} & \to & (M_B)^n & \to (M_B)^{n+1} \cdots \to & (M_B)^{n+k} & \to & 0.
\end{array}$$

Since M is C.-M., $grade_M \underline{p} = ht_M \underline{p} = n$, and so, by P.E.C.C.A. (4.22), $Ext^n_A(A/\underline{p}, M) \cong Hom_A(A/\underline{p}, M^n) \neq 0$. Thus there exists $x \in M^n$ s.t. $(0:x)_A = \underline{p}$. In $(M^n)_B$, $(0:x\otimes 1)_B = \underline{p}^e$. Since $\underline{q} \in Ass_B(B/\underline{p}^e)$, it follows from (1.11) that there exists $y \in (M^n)_B$ such that $(0:y)_B = \underline{q}$. Thus $Hom_B(B/\underline{q}, (M^n)_B) \neq 0$.

Since the top row of the above diagram is exact and $\underline{q}^c = \underline{p}$, I may

apply (8.1) n times to obtain $\mathrm{Ext}^n_B(B/\underline{q}, M_B) \cong \mathrm{Hom}_B(B/\underline{q}, (M^n)_B)$. Thus $\mathrm{grade}_{M_B} \underline{q} \leqslant n < n+k = \mathrm{ht}_{M_B} \underline{q}$, a contradiction since M_B is a Cohen-Macaulay B-module.

Thus, for each $\underline{p} \in V_A(M)$, $B/\underline{p}^e$ has only isolated associated prime ideals. //

The particular case in which M = A is worth stating:-

(8.25)Corollary. (This result appears as Corollary 1, part(i), to Theorem 2 of Iversen (15): see (8.14).) Suppose B is faithfully flat over A and B is Cohen-Macaulay. Then A is Cohen-Macaulay, and for each $\underline{p} \in V(A)$, $\underline{p}^e$ has no embedded prime ideals in B. //

A partial converse of (8.24) is available:-

(8.26)Theorem. Suppose that A and B are local with maximal ideals $\underline{m}$ and $\underline{n}$ respectively, that $\underline{m}^e$ is $\underline{n}$-primary, that M is a non-zero, f.-g., Cohen-Macaulay A-module, and that $\mathrm{ht}_M \underline{m} = r$. Then M_B is a non-zero, f.-g., Cohen-Macaulay B-module, $\mathrm{ht}_{M_B} \underline{n} = r$, and (with the usual notation) $\chi^r : (M^r)_B \longrightarrow (M_B)^r$ is an isomorphism.

Proof. First observe from (8.17) that B is faithfully flat over A, so that M_B is a non-zero, f.-g., B-module. By (8.10) & (8.21), $\mathrm{ht}_{M_B} \underline{n} = r$, so from Chapter 4, I need only show that $\mathrm{grade}_{M_B} \underline{n} = r$ in order to show that M_B is Cohen-Macaulay. Since M is C.-M. and B is A-flat, $\underline{C(M)} \otimes_A B$ is exact, and I may use (8.1) r times to see that, if $0 \leqslant j \leqslant r-1$, $\mathrm{Ext}^j_B(B/\underline{n}, M_B) = 0$, as was to be proved.

Thus M_B is a Cohen-Macaulay B-module, M satisfies conditions (X1) and

(X2) of (3.11), and I may apply the techniques of Propositions (8.3) and (8.4). Observe that condition (X3(r-1)) is satisfied, so Proposition (8.4) shows that the induced homomorphism $\overline{\chi^{r-1}}:\operatorname{coker}(d^{r-2}\otimes 1_B)\longrightarrow \operatorname{coker} e^{r-2}$ is an isomorphism (with the usual notation). It follows that $\chi^r:(M^r)_B\longrightarrow (M_B)^r$ is an isomorphism. //

Again, the particular case in which M = A reads as follows:-

(8.27)Corollary. Suppose that A and B are local with maximal ideals $\underline{m}$ and $\underline{n}$ respectively, that $\underline{m}^e$ is $\underline{n}$-primary, and that K-Dim A = r. Then, if A is Cohen-Macaulay, so also is B, K-Dim B = r, and $\chi^r:A^r\otimes_A B\longrightarrow B^r$ is a B-isomorphism. //

Before looking at Gorenstein rings and modules I must prove:-

(8.28)Lemma. Suppose that B is faithfully flat over A, and that the A-module G is such that $G\otimes_A B$ is an injective B-module. Then G is an injective A-module.

Proof. Let P be an A-module s.t. $(E)\ 0\longrightarrow G\xrightarrow{f} E_A(G)\longrightarrow P\longrightarrow 0$. From this it follows that $(E)\ 0\longrightarrow G\otimes_A B\xrightarrow{f\otimes 1_B} E_A(G)\otimes_A B\longrightarrow P\otimes_A B\longrightarrow 0$. Since f is essential, it follows,from (8.9), that $f\otimes 1_B$ is an essential monomorphism. Since $G\otimes_A B$ is B-injective, $f\otimes 1_B$ is an isomorphism, and $P\otimes_A B = 0$. Since B is faithfully flat over A, P = 0, and G is A-injective. //

(8.29)Theorem. Suppose that A and B are local with maximal ideals $\underline{m}$ and $\underline{n}$ respectively, and that $\underline{n}\in \operatorname{Ass}_B(B/\underline{m}^e)$. Suppose also that M is a non-zero, f.-g., A-module for which M_B is a Gorenstein B-module. Then M is Gorenstein, and, if $\operatorname{ht}_M\underline{m} = \text{K-Dim}_A M = r$, $\mu^r(\underline{m},M).\mu^0(\underline{n},B/\underline{m}^e) = \mu^r(\underline{n},M_B)$, and $E_A(A/\underline{m})\otimes_A B\cong \oplus \mu^0(\underline{n},B/\underline{m}^e)E_B(B/\underline{n})$ (and so is B-injective).

Proof. It follows immediately from (8.17) that B is faithfully flat over A. By (8.24), M is a C.-M. A-module and $\underline{m}^e$ is $\underline{n}$-primary. I can now apply (8.26) to see that $(M^r)_B \cong (M_B)^r$, and since M_B is Gorenstein, $(M^r)_B$ is B-injective, so M^r is A-injective by (8.28). Consequently, by (6.27), M is a Gorenstein A-module.

Using (6.27) and the fact that $\mathrm{ht}_{M_B}\underline{n} = r$, it follows that $(M_B)^r \cong \oplus\mu^r(\underline{n},M_B).E_B(B/\underline{n})$. On the other hand, $M^r \cong \oplus\mu^r(\underline{m},M)E_A(A/\underline{m})$, so is $(M^r)_B \cong \oplus\mu^r(\underline{m},M).(E_A(A/\underline{m})\otimes_A B)$. Since $(M^r)_B$ is B-injective, $E_A(A/\underline{m})\otimes_A B$ is B-injective by (5.2)(i). Using (8.9), $E_A(A/\underline{m})\otimes_A B \cong E_B(A/\underline{m}\otimes_A B) \cong E_B(B/\underline{m}^e)$ as B-modules. Since $\underline{m}^e$ is $\underline{n}$-primary, $E_B(B/\underline{m}^e) \cong \oplus\mu^0(\underline{n},B/\underline{m}^e).E_B(B/\underline{n})$, so the result follows. //

(8.30)Corollary. Suppose that B is faithfully flat over A and that M is a non-zero, f.-g., A-module. If M_B is a Gorenstein B-module, then M is a Gorenstein A-module.

Proof. By (6.24), it is sufficient to prove that $M_{\underline{p}}$ is a Gorenstein $A_{\underline{p}}$-module for each $\underline{p} \in V_A(M)$. Choose $\underline{p} \in V_A(M)$, and $\underline{q} \in \mathrm{Ass}_B(B/\underline{p}^e)$. Then $\underline{q}^c = \underline{p}$ by (8.10), $\underline{q} \in V_B(M_B)$ by (8.20), f induces $f':A_{\underline{p}} \longrightarrow B_{\underline{q}}$ which makes $B_{\underline{q}}$ faithfully flat over $A_{\underline{p}}$, and $(M_B)_{\underline{q}} \cong M_{\underline{p}}\otimes_{A_{\underline{p}}} B_{\underline{q}}$ (as $B_{\underline{q}}$-modules) is a Gorenstein $B_{\underline{q}}$-module. Since $\underline{p}^e.B_{\underline{q}} = f'(\underline{p}A_{\underline{p}}).B_{\underline{q}}$, it follows from (8.29) that $M_{\underline{p}}$ is a Gorenstein $A_{\underline{p}}$-module. //

A sort of partial converse of (8.29) is available:-

(8.31)Theorem. Suppose that A and B are local with maximal ideals $\underline{m}$ and $\underline{n}$ respectively, that $\underline{m}^e$ is $\underline{n}$-primary, that the non-zero, f.-g., A-module M is Gorenstein, and that $\mathrm{ht}_M\underline{m} = r$. Then the (non-zero, f.-g.,) B-module

M_B is Gorenstein if and only if $E_A(A/\underline{m})\otimes_A B$ is an injective B-module.

Proof. By (8.17), B is faithfully flat over A, so that M_B is a non-zero, f.-g., B-module.

If M_B is a Gorenstein B-module, then, by (8.29), $E_A(A/\underline{m})\otimes_A B$ is an injective B-module.

Suppose, conversely, that $E_A(A/\underline{m})\otimes_A B$ is an injective B-module. By (8.26), M_B is a C.-M. B-module, $\mathrm{ht}_{M_B}\underline{n} = r$, and (with the usual notation) $\chi^*:(M^r)_B \longrightarrow (M_B)^r$ is an isomorphism. But, by (6.27), $M^r \cong \oplus \mu^r(\underline{m},M)E_A(A/\underline{m})$ so $(M^r)_B \cong \oplus \mu^r(\underline{m},M)(E_A(A/\underline{m})\otimes_A B)$, as B-modules, so $(M_B)^r$ is an injective B-module. Thus, by (6.27), M_B is a Gorenstein B-module. //

(8.32)Corollary. Suppose that C is a local ring and that $\hat{C}$ is the completion of C. Suppose M is a non-zero, f.-g., C-module. Then M is Gorenstein if and only if $M\otimes_C\hat{C}$ is a Gorenstein $\hat{C}$-module.

Proof. Let $\underline{c}$ be the maximal ideal of C. From (1.31), $\hat{C}$, when regarded as a C-module by means of the natural embedding $C \xrightarrow{\subseteq} \hat{C}$, is flat, and $\underline{c}^e = \hat{\underline{c}}$, the maximal ideal of $\hat{C}$. The Corollary will follow immediately from (8.29) and (8.31) once I have shown that $E_C(C/\underline{c})\otimes_C\hat{C}$ is an injective $\hat{C}$-module. This I now do.

By (5.2)(vi), every element of $E_C(C/\underline{c})$ is annihilated by some power of $\underline{c}$. Thus, if $x \in E_C(C/\underline{c})$, $\hat{c} \in \hat{C}$, and (c_n) is a sequence of elements of C which converge to $\hat{c}$ in $\hat{C}$, then there exists a positive integer N s.t. $m,n \geqslant N$ implies $c_m x = c_n x$. (For, if h is a positive integer, $(\hat{\underline{c}})^h \cap C = \underline{c}^h$, by (1.31)(ii).) Thus, for large r, $c_r x$ is ultimately constant: it is easy to check that defining $\hat{c}x$ to be this ultimate constant value is a sensible definition of an action of $\hat{C}$ on $E_C(C/\underline{c})$, and that, under this action,

$E_C(C/\underline{c})$ becomes a $\hat{C}$-module.

Next, I show that $E_C(C/\underline{c})\otimes_C\hat{C}$ is $\hat{C}$-isomorphic to $E_C(C/\underline{c})$. Define $\phi : E_C(C/\underline{c})\otimes_C\hat{C} \longrightarrow E_C(C/\underline{c})$ to be the C-homomorphism which is s.t. $\phi(x\otimes\hat{c}) = \hat{c}.x$ $(x \in E_C(C/\underline{c}),\ \hat{c} \in \hat{C})$: ϕ is also a $\hat{C}$-homomorphism. Define $\psi : E_C(C/\underline{c}) \longrightarrow E_C(C/\underline{c})\otimes_C\hat{C}$ by $\psi(x) = x\otimes 1$ for $x \in E_C(C/\underline{c})$.

Suppose $x \in E_C(C/\underline{c})$, and $N > 0$ is an integer s.t. $\underline{c}^N.x = 0$. Let $\hat{c} \in \hat{C}$, and suppose (c_n) is a sequence of elements of C which converges to $\hat{c}$ in $\hat{C}$. Then there exists $H > 0$ s.t. $n \geqslant H$ implies $c_n - \hat{c} \in (\underline{\hat{c}})^N$, and $\hat{c}.x = c_n.x$ for any $n \geqslant H$. Therefore, choosing an $n \geqslant H$, $\psi(\hat{c}x) = \psi(c_n x) = c_n x\otimes 1 = x\otimes c_n$.

Now $(\underline{\hat{c}})^N = (\underline{c}^N)^e$, so since $\underline{c}^N.x = 0$, $x\otimes(c_n - \hat{c}) = 0$. Thus $x\otimes c_n = x\otimes\hat{c}$, and $\psi(\hat{c}x) = x\otimes\hat{c} = \hat{c}(\psi(x))$. Thus ψ is a $\hat{C}$-module homomorphism.

Clearly, $\phi.\psi$ is the identity on $E_C(C/\underline{c})$, while if $x \in E_C(C/\underline{c})$ and $\hat{c} \in \hat{C}$, $(\psi.\phi)(x\otimes\hat{c}) = (\psi.\phi)(\hat{c}(x\otimes 1)) = \hat{c}(\psi.\phi(x\otimes 1)) = \hat{c}.(x\otimes 1) = x\otimes\hat{c}$. Thus ϕ and ψ are inverse $\hat{C}$-isomorphisms, and it remains only to show that $E_C(C/\underline{c})$ is an injective $\hat{C}$-module.

To prove this, I reason as in the proof of Theorem 3.6. of Matlis (17). Since $\hat{C}$ is C-flat, $C/\underline{c}\otimes_C\hat{C} \cong \hat{C}/\underline{\hat{c}}$ as $\hat{C}$-modules, and since the $\hat{C}$-monom. $C/\underline{c}\otimes_C\hat{C} \longrightarrow E_C(C/\underline{c})\otimes_C\hat{C}$ induced from $C/\underline{c} \subseteq E_C(C/\underline{c})$ is essential (by (8.9)), it follows from the above that the composition $C/\underline{c}\otimes_C\hat{C} \longrightarrow E_C(C/\underline{c})\otimes_C\hat{C} \xrightarrow{\phi} E_C(C/\underline{c})$ makes the $\hat{C}$-module $E_C(C/\underline{c})$ an essential extension of $\hat{C}/\underline{\hat{c}}$.

Let F be the $\hat{C}$-injective envelope of $E_C(C/\underline{c})$. Then $F \cong E_{\hat{C}}(\hat{C}/\underline{\hat{c}})$, so that every element of F is annihilated by some power of $\underline{\hat{c}}$. As C-modules, $E_C(C/\underline{c})$ is a direct summand of F, so there is a C-submodule X of F s.t. $E_C(C/\underline{c})\oplus X = F$. Since every element of F is annihilated by some power of

$\hat{\underline{c}}$, X is also a $\hat{C}$-submodule of F. But F is indecomposable as a $\hat{C}$-module, so X = 0 and $E_C(C/\underline{c}) = F$. Thus $E_C(C/\underline{c})$ is an injective $\hat{C}$-module, as was to be proved. //

My remarks for the remainder of this Chapter will be confined to Gorenstein rings.

(8.33)Theorem. (This result appears as Corollary 1, part (ii), to Theorem 2 of Iversen (15): see (8.14).) Suppose B is faithfully flat over A and that B is Gorenstein. Then A is Gorenstein, and if $\underline{p} \in V(A)$, $B/\underline{p}^e$ has no embedded primes and the (unique) primary components of $\underline{p}^e$ in B are all irreducible.

Proof. The first two parts follow from (8.30) and (8.25) respectively. To prove the final result, let $\underline{q}$ be an arbitrary element of $Ass_B(B/\underline{p}^e)$, so $\underline{q}^c = \underline{p}$. To show that the $\underline{q}$-primary component of $\underline{p}^e$ is irreducible, it is sufficient to show that $\underline{p}^e.B_{\underline{q}}$ is an irreducible $\underline{q}.B_{\underline{q}}$-primary ideal of $B_{\underline{q}}$.

Observe from the second part that $\underline{q}$ is a minimal element of $Ass_B(B/\underline{p}^e)$. As usual, f induces $f': A_{\underline{p}} \longrightarrow B_{\underline{q}}$ which makes $B_{\underline{q}}$ faithfully flat over $A_{\underline{p}}$, $\underline{p}^e.B_{\underline{q}} = f'(\underline{p}A_{\underline{p}}).B_{\underline{q}}$, and $A_{\underline{p}}$ and $B_{\underline{q}}$ are both Gorenstein rings.

Suppose $r = ht_A \underline{p} = K\text{-Dim } A_{\underline{p}}$. In this situation Theorem (8.29) may be applied. Since $ht_B \underline{q} = K\text{-Dim } B_{\underline{q}} = r$, it follows from (5.11) that $\mu^r(\underline{q}B_{\underline{q}}, B_{\underline{q}}) = 1$, so $\mu^0(\underline{q}B_{\underline{q}}, B_{\underline{q}}/f'(\underline{p}A_{\underline{p}}).B_{\underline{q}}) = 1$. It now follows from Theorem 2.3 of Matlis (17) that the $\underline{q}B_{\underline{q}}$-primary ideal $\underline{p}^e.B_{\underline{q}}$ is irreducible, as was to be shown. //

(8.34)Theorem. Suppose that A and B are local with maximal ideals $\underline{m}$ and $\underline{n}$ respectively, and that $\underline{m}^e$ is irreducible and $\underline{n}$-primary. Then A is

Gorenstein if and only if B is Gorenstein, and when both rings are Gorenstein, $E_A(A/\underline{m})\otimes_A B \cong E_B(B/\underline{n})$ (as B-modules).

Remark. The "if and only if" part of this Theorem is (almost) contained in Corollary 2 to Theorem 2 of Iversen (15). (See (8.15).) The relation $E_A(A/\underline{m})\otimes_A B \cong E_B(B/\underline{n})$ between the injective envelopes which can be obtained using this approach is one which Iversen was (apparently) unable to obtain.

Proof. Observe that B is faithfully flat over A, that $\underline{n}^c = \underline{m}$, and that K-Dim A = K-Dim B (= r, say) (by (8.10) & (8.13)).

($\Leftarrow$) This is part of (8.33).

($\Rightarrow$) By (8.27), B is Cohen-Macaulay, and $\mathcal{H}^r:(A^r)_B \longrightarrow B^r$ is an isomorphism. By (5.11), $A^r \cong E_A(A/\underline{m})$, so that $E_B((A^r)_B) \cong$ (using (8.9)) $E_B(A/\underline{m}\otimes_A B) \cong E_B(B/\underline{m}^e)$. Since $\underline{m}^e$ is $\underline{n}$-primary and irreducible, $E_B(B/\underline{m}^e) \cong E_B(B/\underline{n})$ by Proposition 3.1 of Matlis (17). Thus $\mu^0(\underline{n},B^r) = 1$. But, by P.E.C.C.A. (4.22), $\mathrm{Hom}_B(B/\underline{n},B^r) \cong \mathrm{Ext}^r_B(B/\underline{n},B)$, so $\mu^r(\underline{n},B) = 1$. Consequently, by (6.8) & (5.11), B is Gorenstein, and $B^r \cong E_B(B/\underline{n})$. Since $\mathcal{H}^r$ is an isomorphism, it follows that $E_A(A/\underline{m})\otimes_A B \cong E_B(B/\underline{n})$. //

CHAPTER 9

Use of the Cousin Complex to Obtain Further Properties of Gorenstein Modules.

(Reminder: "ring" means "commutative Noetherian ring with a non-zero identity".)

Throughout this Chapter, A will denote a ring. A will only be assumed to be local when this is explicitly stated, but, when this is the case, $\underline{m}$ will be consistently used to denote the maximal ideal of A.

Suppose, for the moment, that A is local and that the non-zero, f.-g., A-module M is Gorenstein. Then Theorem (6.27) shows that A must be Cohen-Macaulay, that:-

$\operatorname{codh}_A M = \text{K-Dim}_A M = \text{inj.dim.}_A M = \operatorname{codh}_A A = \text{K-Dim A}$, and that every A-sequence is an M-sequence. Perhaps it is also the case that every M-sequence is an A-sequence; perhaps every prime ideal in V(A) belongs to V(M): the theory of Gorenstein modules developed in Chapter 6 does not rule out either of these possibilities. One sensible plan of campaign for attempting to discuss these questions would be to consider M-sequences which are not necessarily A-sequences, and, unfortunately, the machinery developed so far seems inadequate for this purpose. For instance (6.4) & (6.5) cannot be used in this situation. ((6.4) & (6.5) are splendid for dealing with rings, of course, for to use them when M = A, it is only necessary for x not to be a zero-divisor on A.)

However, for an arbitrary ring A, a non-zero, f.-g., A-module M, and an $x \in A$ s.t. (x) is an M-sequence, there is a strong connection between $\underline{C_A(M)}$ and $\underline{C_A(M/xM)}$ which is particularly nice when M is Cohen-Macaulay.

In this Chapter, this relationship is examined, and then exploited to prove, among other things, that for a Gorenstein module M over a local ring A, $(0:M)_A = 0$ and every M-sequence is an A-sequence.

I assume, until further notice, that M is a non-zero, f.-g., A-module, and $x \in A$ is such that (x) is an M-sequence. Write $N = M/xM$ and let the Cousin Complexes for the A-modules M and N be:-

$$\underline{C(M)}: 0 \longrightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \dots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \dots \quad \&$$

$$\underline{C(N)}: 0 \longrightarrow N \xrightarrow{e^{-1}} N^0 \xrightarrow{e^0} N^1 \longrightarrow \dots \longrightarrow N^n \xrightarrow{e^n} N^{n+1} \longrightarrow \dots .$$

(9.1)Lemma. If $\underline{p} \in V_A(M/xM)$, $ht_{M/xM}\underline{p} = ht_M\underline{p} - 1$.

Proof. $V_A(M/xM) = V_A(M) \cap V_A(A/(x))$, so $x \in \underline{p}$. $x/1 \in \underline{p}A_{\underline{p}}$ is not a zero-divisor on $M_{\underline{p}}$, $ht_{M/xM}\underline{p} = K\text{-}Dim_{A_{\underline{p}}}(M_{\underline{p}}/(x/1)M_{\underline{p}})$, and $ht_M\underline{p} = K\text{-}Dim_{A_{\underline{p}}} M_{\underline{p}}$. Now use (1.24). //

I shall use "*" to denote the functor $Hom_A(A/(x), \ )$ from $\mathcal{C}_A$ to $\mathcal{C}_A$. Note that, if $f: P \longrightarrow Q$ is a homomorphism of A-modules, there are natural isomorphisms $\phi(P)$, $\phi(Q)$ s.t.:-

$$\begin{array}{ccc} P^* & \xrightarrow{f^*} & Q^* \\ \phi(P)\downarrow\cong & & \cong\downarrow\phi(Q) \\ (0:x)_P & \xrightarrow{f|} & (0:x)_Q \end{array} \qquad (C).$$

This will help in some of the calculations.

(9.2)Lemma. Multiplication by x provides:-

(i) a monomorphism on M;

(ii) an automorphism of M^0;

& (iii) an epimorphism on coker d^{-1}, M^1, M^2,..., M^n,... .

Proof. (i) is obvious because x is not a zero-divisor on M. (ii) follows because x does not belong to any minimal prime of V(M). (iii) then

follows from (3.7)(x) & (xi). //

(9.3)Lemma. There is an epimorphism $\varkappa^{-1}:N \longrightarrow (\text{coker } d^{-1})^*$ which is an isomorphism when C(M) is exact at M.

Proof. From (9.2), I deduce

$$\begin{array}{ccccccccc} & & M & \xrightarrow{d^{-1}} & M^0 & \longrightarrow & \text{coker } d^{-1} & \longrightarrow & 0 \quad (E) \\ & & x\downarrow\cong & & x\downarrow\cong & & \phi\downarrow\cong & & \\ (C) & & xM & \xrightarrow[d^{-1}]{} & M^0 & \longrightarrow & M^0/xd^{-1}(M) & \longrightarrow & 0 \quad (E), \end{array}$$

so this diagram can be completed with an isomorphism ϕ such that the extended diagram commutes.

Thus $(\text{coker } d^{-1})^* \cong (0:x)_{M^0}/x.d^{-1}(M) = (x.d^{-1}(M):x)_{M^0}/x.d^{-1}(M) = d^{-1}(M)/x.d^{-1}(M)$ by (9.2)(ii). Finally, d^{-1} induces an epimorphism $M/xM \longrightarrow d^{-1}(M)/x.d^{-1}(M)$ which is an isomorphism if d^{-1} is a monomorphism. //

Let $d^0:\text{coker } d^{-1} \longrightarrow M^1$ be the obvious homomorphism induced by d^0, and let C*(x,M) denote the induced complex:-

$$C^*(x,M): 0 \longrightarrow (\text{coker } d^{-1})^* \xrightarrow{d^{0*}} (M^1)^* \xrightarrow{d^{1*}} (M^2)^* \longrightarrow \ldots \longrightarrow (M^n)^* \xrightarrow{d^{n*}} \ldots$$

The immediate aim is to incorporate $\varkappa^{-1}$ in a homomorphism of complexes $\Psi:C(N) \longrightarrow C^*(x,M)$, $\Psi = \{\varkappa^n\}_{n \geqslant -1}$, which is such that each $\varkappa^n:N^n \longrightarrow (M^{n+1})^*$ $(n \geqslant 0)$ is an epimorphism. I shall find it convenient to employ again the notation $y = (\overline{y_{\mathfrak{p}}}/s_{\mathfrak{p}})$ for an element $y \in N^n$ to signify that, for a $\mathfrak{p} \in V(N)$ of N-height n, the component of y in the direct summand $(\text{coker } e^{n-2})_{\mathfrak{p}}$ of N^n is $\overline{\overline{y_{\mathfrak{p}}}}/s_{\mathfrak{p}}$, so that all but a finite number of the $\overline{y_{\mathfrak{p}}}/s_{\mathfrak{p}}$ will be zero.

(9.4)Lemma. For $n \geqslant 0$, the map:-

$$\lambda^n: \bigoplus_{\substack{\mathfrak{p} \in V(N) \\ ht_N \mathfrak{p} = n}} ((\text{coker } d^{n-1})^*)_{\mathfrak{p}} \longrightarrow (M^{n+1})^* \quad \text{defined by:-}$$

$\lambda^n((f_{\underline{p}}/s_{\underline{p}})) = f \in (M^{n+1})^*$, where, if $\bar{1}$ denotes the identity of $A/(x)$, $f(\bar{1}) = (y_{\underline{p}}/t_{\underline{p}})$ with, for each $\underline{p} \in V(M)$ of M-height $n+1$,

$$y_{\underline{p}}/t_{\underline{p}} = \begin{cases} 0 & \text{if } x \notin \underline{p} \\ f_{\underline{p}}(\bar{1})/s_{\underline{p}} & \text{if } x \in \underline{p} \end{cases}$$ is a (well-defined) A-isomorphism.

Proof. I shall exhibit λ^n as the composition of several obvious natural isomorphisms. Bear in mind (9.1), and note that, if $\underline{p} \in V(N)$ has N-height n, $\underline{p} \in V(M)$ also and $\mathrm{ht}_M \underline{p} = n+1$.

$$\bigoplus_{\substack{\underline{p} \in V(N) \\ \mathrm{ht}_N \underline{p} = n}} ((\mathrm{coker}\ d^{n-1})^*)_{\underline{p}} \cong \bigoplus_{\substack{\underline{p} \in V(N) \\ \mathrm{ht}_N \underline{p} = n}} ((0:x)_{\mathrm{coker}\ d^{n-1}})_{\underline{p}} = \text{(by (1.5))}$$

$$\bigoplus_{\substack{\underline{p} \in V(N) \\ \mathrm{ht}_N \underline{p} = n}} (0:x)(\mathrm{coker}\ d^{n-1})_{\underline{p}}$$ (remember that each $(\mathrm{coker}\ d^{n-1})_{\underline{p}}$ is here being considered as an A-module) $$\cong \bigoplus_{\substack{\underline{p} \in V(N) \\ \mathrm{ht}_N \underline{p} = n}} \mathrm{Hom}_A(A/(x), (\mathrm{coker}\ d^{n-1})_{\underline{p}}).$$

Now, from (9.1) and the fact that, if $\underline{p} \in V(M)$ has M-height $n+1$ and $x \notin \underline{p}$, $\mathrm{Hom}_A(A/(x), (\mathrm{coker}\ d^{n-1})_{\underline{p}}) = 0$, it follows that

$$\bigoplus_{\substack{\underline{p} \in V(N) \\ \mathrm{ht}_N \underline{p} = n}} \mathrm{Hom}_A(A/(x), (\mathrm{coker}\ d^{n-1})_{\underline{p}}) = \bigoplus_{\substack{\underline{p} \in V(M) \\ \mathrm{ht}_M \underline{p} = n+1}} \mathrm{Hom}_A(A/(x), (\mathrm{coker}\ d^{n-1})_{\underline{p}}),$$

and, by (2.19), this last module is isomorphic to $(M^{n+1})^*$. λ^n is the composition of these isomorphisms. //

(9.5) Lemma. The natural induced homomorphisms $\mu^0 : \mathrm{coker}(d^0)^* \longrightarrow (\mathrm{coker}\ d^0)^*$ and, for $n > 0$, $\mu^n : \mathrm{coker}(d^n)^* \longrightarrow (\mathrm{coker}\ d^n)^*$ are all epimorphisms. If $C(M)$ is exact, they are all isomorphisms.

Proof. μ^0 is (uniquely) defined by:-

$$\begin{array}{ccccccc} (\text{coker } d^{-1})^* & \xrightarrow{(d^{0\prime})^*} & (M^1)^* & \longrightarrow & (\text{coker}(d^{0\prime})^*) & \longrightarrow & 0 \quad (E) \\ \| & & \| & & \downarrow \mu^0 & & \\ (\text{coker } d^{-1})^* & \xrightarrow{(d^{0\prime})^*} & (M^1)^* & \longrightarrow & (\text{coker } d^0)^* & & \quad (C) \end{array},$$

where the bottom row is induced from (E) $(\text{coker } d^{-1}) \xrightarrow{d^{0\prime}} M^1 \longrightarrow (\text{coker } d^0)$.

To show that μ^0 is epimorphic it is thus sufficient to show that the induced homomorphism $(M^1)^* \longrightarrow (\text{coker } d^0)^*$ is epimorphic, and for this it is sufficient to show $(0:x)_{M^1} \longrightarrow (0:x)_{\text{coker } d^0}$ is epimorphic. So suppose $m \in M^1$ and $x.\bar{m} = 0$ (where, of course, $\bar{}$ is used to denote natural images in coker d^0). Then $x.m \in d^0M^0$, so for some $z \in M^0$, $x.m = d^0(z)$. By (9.2)(ii) there is a $z' \in M^0$ s.t. $x.z' = z$. Then $x.(m - d^0(z')) = 0$ and $\overline{m - d^0(z')} = \bar{m}$. Thus $(0:x)_{M^1} \longrightarrow (0:x)_{\text{coker } d^0}$ is epimorphic.

If C(M) is exact, then (E) $0 \longrightarrow \text{coker } d^{-1} \xrightarrow{d^{0\prime}} M^1 \longrightarrow \text{coker } d^0$, so (E) $0 \longrightarrow (\text{coker } d^{-1})^* \xrightarrow{(d^{0\prime})^*} (M^1)^* \longrightarrow (\text{coker } d^0)^*$, and μ^0 is an isomorphism.

Similarly, if $n > 0$, μ^n is (uniquely) defined by:-

$$\begin{array}{ccccccc} (M^n)^* & \xrightarrow{(d^n)^*} & (M^{n+1})^* & \longrightarrow & (\text{coker } d^{n*}) & \longrightarrow & 0 \quad (E) \\ \| & & \| & & \downarrow \mu^n & & \\ (M^n)^* & \xrightarrow{(d^n)^*} & (M^{n+1})^* & \longrightarrow & (\text{coker } d^n)^* & & \quad (C) \end{array},$$

where the bottom row is induced from (E) $M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \text{coker } d^n \longrightarrow 0$.

To show μ^n is epimorphic, it is sufficient to show that $(0:x)_{M^{n+1}} \longrightarrow (0:x)_{\text{coker } d^n}$ is epimorphic and the proof of this is entirely similar to that for the case $n = 0$: use the fact that $x.M^n = M^n$ ((9.2)(iii)).

Suppose C(M) is exact. To show μ^n is an isomorphism it is clearly sufficient to show (E) $(M^n)^* \xrightarrow{(d^n)^*} (M^{n+1})^* \longrightarrow (\text{coker } d^n)^*$, so I show (E) $(0:x)_{M^n} \xrightarrow{d^n|} (0:x)_{M^{n+1}} \longrightarrow (0:x)_{\text{coker } d^n}$.

Suppose $z \in M^n$ and $x.(d^n(z)) = 0$. Then $d^n(x.z) = 0$, so, since $\underline{C(M)}$ is exact, $x.z = d^{n-1}(y)$ for some $y \in M^{n-1}$. $n > 0$, so $x.M^{n-1} = M^{n-1}$ (by (9.2)). Thus $y = x.y'$ for some $y' \in M^{n-1}$. So $x.(z - d^{n-1}(y')) = 0$ and $d^n(z - d^{n-1}(y')) = d^n(z)$. Thus μ^n is an isomorphism. //

I can now construct Ψ:-

(9.6)Theorem. There is a homomorphism of complexes $\Psi = \{\chi^n\}_{n \geq -1}$, $\Psi : \underline{C(N)} \longrightarrow \underline{C^*(x,M)}$ such that χ^{-1} is the epimorphism of (9.3) and each $\chi^n : N^n \longrightarrow (M^{n+1})^*$ for $n \geqslant 0$ is epimorphic.

If $\underline{C(M)}$ is exact, Ψ is an isomorphism of complexes.

Proof. For ease of proof write $L^{-2} = 0$, $L^{-1} = (\operatorname{coker} d^{-1})^*$, $L^n = (M^{n+1})^*$ for each $n \geqslant 0$, and $h^{-2} = 0 : L^{-2} \longrightarrow L^{-1}$, $h^{-1} = (d^0{}')^*$, $h^n = (d^{n+1})^*$ for $n \geqslant 0$. Then

$$\begin{array}{ccc} N^{-2} & \longrightarrow & N^{-1} \\ \| & & \downarrow \chi^{-1} \\ L^{-2} & \longrightarrow & L^{-1} \end{array}$$

(C) provides a basis for the following induction.

I suppose, inductively, that $n \geqslant 0$, and I have constructed epimorphisms $\chi^{-1}, \ldots, \chi^{n-1}$ such that:-

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & N^{-1} & \xrightarrow{e^{-1}} & N^0 & \xrightarrow{e^0} & N^1 & \longrightarrow \ldots \longrightarrow & N^{n-2} & \xrightarrow{e^{n-2}} & N^{n-1} \\ \| & & \downarrow \chi^{-1} & & \downarrow \chi^0 & & \downarrow \chi^1 & & \downarrow \chi^{n-2} & & \downarrow \chi^{n-1} \\ 0 & \longrightarrow & L^{-1} & \xrightarrow{h^{-1}} & L^0 & \xrightarrow{h^0} & L^1 & \longrightarrow \ldots \longrightarrow & L^{n-2} & \xrightarrow{h^{n-2}} & L^{n-1} \end{array} \quad (C)$$

χ^{n-1} induces an epimorphism $\overline{\chi^{n-1}} : \operatorname{coker} e^{n-2} \longrightarrow \operatorname{coker} h^{n-2}$, which, in turn, induces an epimorphism:-

$$\vartheta^n : N^n = \bigoplus_{\substack{\underline{p} \in V(N) \\ \operatorname{ht}_N \underline{p} = n}} (\operatorname{coker} e^{n-2})_{\underline{p}} \longrightarrow \bigoplus_{\substack{\underline{p} \in V(N) \\ \operatorname{ht}_N \underline{p} = n}} (\operatorname{coker} h^{n-2})_{\underline{p}}.$$

Two cases now arise. Firstly, if $n = 0$, $\operatorname{coker} h^{n-2} = (\operatorname{coker} d^{-1})^*$, so

define $\varkappa^0 : N^0 \longrightarrow L^0 = (M^1)^*$ by $\varkappa^0 = \lambda^0 . \vartheta^0$ (see (9.4)) and check (easily) that

$$\begin{array}{ccc} N & \xrightarrow{e^{-1}} & N^0 \\ {\scriptstyle \varkappa^{-1}}\downarrow & & \downarrow{\scriptstyle \varkappa^0} \\ (\operatorname{coker} d^{-1})^* & \longrightarrow & (M^1)^* \end{array} \quad (C) \ .$$

(Remember that, if $\underline{p} \in V(M)$ with M-height 1 and $x \notin \underline{p}$, then $(0:x)_{(\operatorname{coker} d^{-1})_{\underline{p}}} = 0$.) Since λ^0 is an isomorphism, $\varkappa^0$ is an epimorphism.

The second case is when $n > 0$. From the epimorphism $\mu^{n-1} : \operatorname{coker} h^{n-2} \longrightarrow (\operatorname{coker} d^{n-1})^*$ of (9.5), I deduce a further epimorphism:-

$$\nu^n : \bigoplus_{\substack{\underline{p} \in V(N) \\ \operatorname{ht}_N \underline{p} = n}} (\operatorname{coker} h^{n-2})_{\underline{p}} \longrightarrow \bigoplus_{\substack{\underline{p} \in V(N) \\ \operatorname{ht}_N \underline{p} = n}} ((\operatorname{coker} d^{n-1})^*)_{\underline{p}} .$$

Define $\varkappa^n : N^n \longrightarrow (M^{n+1})^* \ (= L^n)$ to be the epimorphism $\lambda^n . \nu^n . \vartheta^n$ (see (9.4)). Remembering that, if $\underline{p} \in V(M)$ with M-height $n+1$ is such that $x \notin \underline{p}$, $(0:x)_{(\operatorname{coker} d^{n-1})_{\underline{p}}} = 0$, it is then straightfoward to check that:-

$$\begin{array}{ccc} N^{n-1} & \xrightarrow{e^{n-1}} & N^n \\ {\scriptstyle \varkappa^{n-1}}\downarrow & & \downarrow{\scriptstyle \varkappa^n} \\ (M^n)^* & \xrightarrow{d^{n*}} & (M^{n+1})^* \end{array} \quad (C) \ .$$

The first part of the Theorem now follows by induction.

Now suppose $\underline{C(M)}$ is exact. By (9.3), $\varkappa^{-1}$ is isomorphic. By (9.5), μ^i $(i \geqslant 0)$ are all isomorphisms, and so therefore ν^i $(i \geqslant 1)$ are all isomorphisms. λ^i $(i \geqslant 0)$ are always isomorphisms. It is now an easy inductive argument to see that each $\varkappa^n$ $(n \geqslant 0)$ must also be an isomorphism.

Thus, when $\underline{C(M)}$ is exact, Ψ is an isomorphism. //

(9.7)Observation. The modules of the complex $\underline{C^*(x,M)}$ are all annihilated by x; by (3.7)(ix), the modules of $\underline{C(N)}$ are all annihilated by x. Consequently, one may regard $\underline{C^*(x,M)}$ and $\underline{C(N)}$ as complexes of $A/(x)$-modules

and -homomorphisms, and each $\varkappa^n$ $(n \geqslant -1)$ as an $A/(x)$-homomorphism. When this is done, $\underline{C(N)}$ gives the Cousin Complex for the $A/(x)$-module M/xM (by (3.17)).

(9.8)Corollary. If $\underline{C(M)}$ is exact, so also is $\underline{C^*(x,M)}$.

Proof. If $\underline{C(M)}$ is exact, M is Cohen-Macaulay (by (4.23)), and $\Psi : \underline{C(N)} \longrightarrow \underline{C^*(x,M)}$ is an isomorphism (by (9.6)). But $N = M/xM$ is C.-M., by localising at each maximal ideal of $V_A(M/xM)$ and using (4.12), so $\underline{C(N)}$ and therefore $\underline{C^*(x,M)}$ are exact. //

Next, the result for which I have been working:-

(9.9)Corollary. Suppose the non-zero, f.-g., A-module M is Gorenstein, and $x \in A$ is such that (x) is an M-sequence. Then M/xM is a Gorenstein $A/(x)$-module.

Proof. By (9.6) and (9.7), $\underline{C^*(x,M)}$ regarded as a complex of $A/(x)$-modules and -homomorphisms, is isomorphic to $\underline{C_{A/(x)}(M/xM)}$. By (9.8), $\underline{C^*(x,M)}$ is exact. Since M is Gorenstein, $M^0, M^1, \ldots, M^n, \ldots$ are all A-injective. By (6.3), $(M^1)^*, (M^2)^*, \ldots, (M^{n+1})^*, \ldots$ are all $A/(x)$-injective. Thus M/xM is a Gorenstein $A/(x)$-module. //

I now use (9.9) to obtain further interesting information about Gorenstein modules.

(9.10)Theorem. Suppose A is local and the non-zero, f.-g., A-module M is Gorenstein. Then $(0:M)_A = 0$ (and so $V(M) = V(A)$).

Remark. Using somewhat different ideas and methods, Levin and Vasconcelos have recently shown in Theorem 4.1 of (16) that if A is a local ring and M is a non-zero, f.-g., A-module of finite injective dimension, then

$(0:M)_A$ is either zero or contains an element which is not a zero-divisor on A.

Proof. I first show that $V(M) = V(A)$. To do this it is clearly sufficient to show that any minimal element of $V(A)$ must belong to $V(M)$. Suppose this is not so, and let $\underline{q}$ be a minimal element of $V(A)$ which does not belong to $V(M)$.

Let $\underline{p}_1,\ldots,\underline{p}_k$ be the minimal elements of $V(M)$. Since $M_{\underline{p}_i}$ is a Gorenstein $A_{\underline{p}_i}$-module, $ht_M\underline{p}_i = ht_A\underline{p}_i$ (by (6.27)), so $ht_A\underline{p}_i = 0$ for each $i = 1,\ldots,k$. Since M is Cohen-Macaulay, $\underline{p}_1,\ldots,\underline{p}_k$ are precisely the elements of $Ass(M)$ (by (4.11) and (4.20)).

Thus, there exists an element $a \in \underline{q} - \bigcup_{\underline{p}\in Ass(M)} \underline{p}$, and so a is not a zero-divisor on M. Also, since $a \in \underline{m}$, $M/aM \neq 0$, and by (9.9), M/aM is a Gorenstein $A/(a)$-module. Let $n = K\text{-}Dim_A M$.

Since $V_{A/(a)}(M/aM) = \left\{\underline{p}/(a) \;\middle|\; \underline{p} \in V_A(M/aM) = V_A(M)\cap V_A(A/(a))\right\}$, $K\text{-}Dim_{A/(a)} M/aM = n-1$ by (1.24). By (6.27), A and $A/(a)$ are Cohen-Macaulay and $K\text{-}Dim\ A = n$, $K\text{-}Dim\ A/(a) = n-1$. By (4.14), $coht\ \underline{q} = n$, and since $a \in \underline{q}$, this contradicts $K\text{-}Dim\ A/(a) = n-1$.

Thus $V(M) = V(A)$, and so, if $\underline{p} \in V(A)$, $ht_A\underline{p} = ht_M\underline{p}$. Before proving that $(0:M)_A = 0$, recall the definition of the Nth symbolic prime power $\underline{p}^{(N)}$ $(N \geq 1)$ of a prime ideal $\underline{p}$ of A: $\underline{p}^{(N)}$ is the contraction in A of the ideal $\underline{p}^N.A_{\underline{p}}$ of $A_{\underline{p}}$, and so $\underline{p}^{(N)}$ is a $\underline{p}$-primary ideal of A. For further details see, e.g. §3.2 of Northcott (21).

Let $\underline{a} = (0:M)_A$. Let $\underline{p}_1,\ldots,\underline{p}_k$ be the minimal elements of $V(A)$. Since A is Cohen-Macaulay, these are precisely the elements of $Ass(A)$.

For each $i = 1,\ldots,k$, let N_i be the least integer r_i s.t. $\underline{p}_i^{r_i}.A_{\underline{p}_i} = 0$. (Each $A_{\underline{p}_i}$ is an Artin local ring.) It is easy to see that $0 = \bigcap_{i=1}^{k} \underline{p}_i^{(N_i)}$ is the (unique) minimal p.d. for 0 in A. To show $\underline{a} = 0$, it is thus sufficient to show $\underline{a} \subseteq \underline{p}_i^{(N_i)}$ for each $i = 1,\ldots,k$.

By (3.7)(ix), $\underline{a}.M^0 = 0$. But, by (6.27), $M^0 \cong \bigoplus_{i=1}^{k} \mu^0(\underline{p}_i,M)E_A(A/\underline{p}_i)$, and each $\mu^0(\underline{p}_i,M)$ $(i = 1,\ldots,k)$ is strictly positive. Thus $\underline{a}.E_A(A/\underline{p}_i) = 0$ for each $i = 1,\ldots,k$, and so it follows by Theorem (3.4) of Matlis (17) that $\underline{a} \subseteq \underline{p}_i^{(N_i)}$ for each $i = 1,\ldots,k$. Thus $(0:M)_A = 0$. //

(9.11)Corollary. If A is local and the non-zero, f.-g., A-module M is Gorenstein, then a sequence $\underline{x}$ is an A-sequence if and only if it is an M-sequence.

Proof. By (6.27), every A-sequence is an M-sequence, so suppose $n = \text{K-Dim}_A M$ and $(x_1,\ldots,x_n)$ is an M-sequence. By (9.9), M/x_1M is a Gorenstein $A/(x_1)$-module, so it is clearly sufficient to show that x_1 is not a zero-divisor on A. But this is now easy: x_1 does not belong to any minimal element of $V(M) = V(A)$, and since A is C.-M., the minimal elements of V(A) are precisely the elements of Ass(A). //

I shall finish this Chapter with an examination of the properties of an arbitrary ring A for which there is a Gorenstein A-module. For the rest of this Chapter, M denotes a Gorenstein A-module.

(9.12)Corollary (of (9.10)). If a prime ideal $\underline{q}$ is contained in some $\underline{p} \in V(M)$, then $\underline{q} \in V(M)$.

Proof. $M_{\underline{p}}$ is a Gorenstein $A_{\underline{p}}$-module, so, by (9.10), $\underline{q}A_{\underline{p}} \in V_{A_{\underline{p}}}(M_{\underline{p}})$.

Now use (1.19). //

I now establish some more notation. Let $\underline{p}_1,\ldots,\underline{p}_m$ be the minimal elements of V(M). By (6.27), since $M_{\underline{p}_i}$ is a Gorenstein $A_{\underline{p}_i}$-module for each $i = 1,\ldots,m$, $\mathrm{ht}_A \underline{p}_i = 0$, and $\underline{p}_1,\ldots,\underline{p}_m$ are all minimal elements of V(A). Let $\underline{p}_{m+1},\ldots,\underline{p}_n$ be the remaining minimal elements of V(A), and let $\underline{p}_{n+1},\ldots,\underline{p}_k$ be the embedded prime ideals of Ass(A). It is clear that no $\underline{p}_j$, $j = n+1,\ldots,k$ can belong to V(M) (for $A_{\underline{p}_j}$ cannot be C.-M. for such a $\underline{p}_j$).

(9.13)Corollary (of (9.12)). $V(A) - V(M) = \bigcup_{i=m+1}^{n} V(A/\underline{p}_i)$. //

If I denote by Spec(A) V(A) furnished with the Zariski topology, then (9.13) implies that V(M) is both an open and closed subset of Spec(A), so unless V(M) = V(A), Spec(A) is disconnected and it can be expected that A is isomorphic to the direct product of two other rings. I now look for an explicit description of such a direct decomposition.

Suppose $0 = \bigcap_{i=1}^{k} \underline{q}_i$ is a minimal p.d. for 0 in A, with each $\underline{q}_i$ $\underline{p}_i$-primary: $\underline{q}_1,\ldots,\underline{q}_n$ are uniquely determined, of course. Let $\underline{a}_1 = \bigcap_{i=1}^{m} \underline{q}_i$ and $\underline{a}_2 = \bigcap_{j=m+1}^{k} \underline{q}_j$.

(9.14)Lemma. The natural ring homomorphism $A \longrightarrow A/\underline{a}_1 \times A/\underline{a}_2$ (direct product) is an isomorphism.

Proof. It is clearly injective, since $\underline{a}_1 \cap \underline{a}_2 = 0$, and the fact that it is surjective follows from (9.13), since no maximal ideal of A can contain $\underline{a}_1 + \underline{a}_2$, so that $\underline{a}_1 + \underline{a}_2 = A$. //

This, of course, was to be expected from (9.13): however, it is

possible to say more:-

(9.15)Lemma. $\underline{a}_1 = (0:M)_A$ and $A/\underline{a}_1 = A/(0:M)_A$ is a Cohen-Macaulay ring.

Proof. Let $\underline{a} = (0:M)_A$. I first prove that $A/\underline{a}$ is C.-M. To do this it is sufficient to show that $(A/\underline{a})_{\underline{p}/\underline{a}}$ is a Cohen-Macaulay local ring for each prime $\underline{p} \in V_A(A/\underline{a}) = V_A(M)$. But, by (1.8), $(A/\underline{a})_{\underline{p}/\underline{a}} \cong A_{\underline{p}}/\underline{a}A_{\underline{p}}$, and $\underline{a}A_{\underline{p}} = ((0:M)_A).A_{\underline{p}} = (0:M_{\underline{p}})_{A_{\underline{p}}}$. Since $\underline{p} \in V(M)$, $M_{\underline{p}}$ is a Gorenstein $A_{\underline{p}}$-module, so $(0:M_{\underline{p}})_{A_{\underline{p}}} = 0$ by (9.10) and $A_{\underline{p}}$ is C.-M. by (6.27). Thus $(A/\underline{a})_{\underline{p}/\underline{a}}$ is C.-M. for each $\underline{p} \in V_A(A/\underline{a})$ and so $A/\underline{a}$ is a Cohen-Macaulay ring.

Consequently, $\underline{a}$ has no embedded prime ideals, and so $\mathrm{Ass}_A(A/\underline{a}) = \{\underline{p}_1,\ldots,\underline{p}_m\}$. Let $\underline{a} = \bigcap_{i=1}^{m} \underline{q}_i'$ be the unique minimal p.d. for $\underline{a}$ in A, with $\underline{q}_i'$ $\underline{p}_i$-primary for $i = 1,\ldots,m$.

For each $i = 1,\ldots,m$, $\underline{a}A_{\underline{p}_i} = \underline{q}_i'A_{\underline{p}_i}$, and $\underline{a}A_{\underline{p}_i} = (0:M_{\underline{p}_i})_{A_{\underline{p}_i}} = 0$, by (9.10) since $M_{\underline{p}_i}$ is a Gorenstein $A_{\underline{p}_i}$-module. Thus $\underline{q}_i'A_{\underline{p}_i} = 0$. But, from the choice $0 = \bigcap_{i=1}^{k} \underline{q}_i$ for a minimal p.d. for 0 in A, $0 = \underline{q}_iA_{\underline{p}_i}$. Thus $\underline{q}_i' = \underline{q}_i$ for each $i = 1,\ldots,m$, and so $\underline{a} = \underline{a}_1$. //

Collecting the results of this investigation:-

(9.16)Theorem. If the non-zero, f.-g., A-module M is Gorenstein (A not necessarily local), then $A/(0:M)_A$ is a Cohen-Macaulay ring, and A is naturally isomorphic to the direct product of $A/(0:M)_A$ and a second (commutative Noetherian) ring. //

A sort of partial converse to Theorem (9.16) is available:-

(9.17)Theorem. Suppose the ring A is such that there exists a non-zero, f.-g., A-module M which is Gorenstein. Then, if C is an arbitrary (commutative Noetherian) ring, there is a Gorenstein $(A/(0:M)_A \times C)$-module.

Remark. By (9.15), $A/(0:M)_A$ has to be Cohen-Macaulay. Since, as is easy to see, the direct product of two rings is C.-M. if and only if each of the two factors is, this Theorem shows that there exist Gorenstein modules over non-Cohen-Macaulay rings.

Proof. Let $B = A/(0:M)_A$. M can be regarded as a B-module in an obvious natural way, and it is easy to see that M then becomes a Gorenstein B-module. (Use (3.7)(ix), (3.17), & (6.3).) Make M into a $B \times C$-module by means of the natural projection $\pi: B \times C \longrightarrow B$, so that $(b,c).m = b.m$.

Since $V_B(M) = V(B)$, it is easy to see that $V_{B\times C}(M) = \{\underline{p} \times C \mid \underline{p} \in V_B(M)\}$. Now the rings $(B \times C)_{\underline{p}\times C}$ and $B_{\underline{p}}$ (where $\underline{p}$ is a prime of $V_B(M)$) are isomorphic under $(b,0)/(s,0) \longleftrightarrow b/s$, and when these rings are identified by means of this isomorphism, the $(B \times C)_{\underline{p}\times C}$-module $M_{\underline{p}\times C}$ and the $B_{\underline{p}}$-module $M_{\underline{p}}$ are isomorphic under $m/(s,0) \longleftrightarrow m/s$. Thus, by (6.24), M is a Gorenstein $B \times C$-module. //

CHAPTER 10

Some Properties of Cohen-Macaulay (Local) Rings which are Homomorphic Images of Gorenstein Local Rings.

(Reminder: "ring" means "commutative, Noetherian ring with a non-zero identity".)

In (20), Murthy shows that, if A is a Unique Factorization Domain which is a quotient of a regular local ring, then A is Gorenstein if and only if A is Cohen-Macaulay. The work in this Chapter developed out of a straightfoward attempt to adapt the argument Murthy used in (20) to prove that if A is a U.F.D. which is a quotient of a Gorenstein local ring, then A is Gorenstein if and only if it is Cohen-Macaulay. (A regular local ring is, of course, Gorenstein: see (7.1).) However, it gradually became clear that the ideas I was using in order to obtain this strengthening of Murthy's result have strong connections with the theory of Gorenstein modules. Consequently, although this Chapter does contain the above-mentioned strengthening of Murthy's result, together with some allied results, its main purpose is to prepare some of the ground for the existence and uniqueness Theorems concerning Gorenstein modules over complete Cohen-Macaulay local rings which appear in Chapter 12.

(10.1)Observation. Suppose that A and B are rings, $\phi : B \longrightarrow A$ is a surjective ring homomorphism, and $\underline{b} = \ker \phi$.

Suppose, furthermore, that T is an additive, B-linear functor of n variables from $\mathcal{C}_B \times \mathcal{C}_B \times \ldots \times \mathcal{C}_B$ (n times) to $\mathcal{C}_B$, that $M_1, \ldots, M_{n-1}$ are B-modules and X is a fixed A-module. X can be regarded as a B-module by "restriction of scalars" using ϕ. Let N be the B-module

$T(M_1,M_2,\ldots,M_{i-1},X,M_i,\ldots,M_{n-1})$. Then N has an A-module structure, as I now show.

Suppose $n \in N$, $a \in A$, & b,b' are 2 elements of B such that $\phi(b) = \phi(b') = a$. Then $b - b' \in \ker\phi$. Since T is B-linear, the B-endomorphism of N provided by multiplication by $b - b'$ is zero, since multiplication by $b - b'$ on X is zero. Thus $b.n = b'.n$. Define $a.n$ to be $b.n$: this is now unambiguous, and defines an A-module structure on N which will henceforth be referred to as the "natural" A-module structure on N.

Let $U(M_1,\ldots,M_{i-1},M_i,\ldots,M_{n-1}) = T(M_1,\ldots,M_{i-1},X,M_i,\ldots,M_{n-1})$, and, whenever $f_1,\ldots,f_{n-1}$ are B-homomorphisms in $\underline{\mathcal{C}}_B$, let $U(f_1,\ldots,f_{i-1},f_i,\ldots,f_{n-1}) = T(f_1,\ldots,f_{i-1},1_X,f_i,\ldots,f_{n-1})$, where 1_X denotes the identity map on X. When this is done U becomes an additive functor of n-1 variables from $\underline{\mathcal{C}}_B \times \underline{\mathcal{C}}_B \times \ldots \times \underline{\mathcal{C}}_B$ (n-1 times) to $\underline{\mathcal{C}}_A$. These ideas will be used frequently in the remainder of the thesis.

For the remainder of Chapter 10, let B be a Gorenstein local ring with maximal ideal $\underline{n}$ and Krull Dimension n, A a Cohen-Macaulay local ring with maximal ideal $\underline{m}$ and Krull Dimension m, $\phi : B \longrightarrow A$ a surjective ring homomorphism, and $\underline{h} = \ker\phi$.

(10.2)Proposition. (The argument used here is similar to that used in Lemma 11 of Serre (25).) Suppose the non-zero, f.-g., B-module M is Cohen-Macaulay with $K\text{-}Dim_B M = codh_B M = n-h$. Then $Ext_B^i(M,B) = 0$ if $i \neq h$, and $M' = Ext_B^h(M,B)$ is again a Cohen-Macaulay B-module with $codh_B M' = n-h$. Furthermore, if $\underline{x} = (x_1,\ldots,x_k)$ $(k \leqslant n-h)$ is an M-sequence, it is also an M'-sequence, and: $M'/\underline{x}M' \cong Ext_B^{h+k}(M/\underline{x}M,B)$.

Proof. Proceed by induction on $s = n-h = codh_B M = K\text{-}Dim_B M$.

If $s = 0$, M is of finite length, and so is annihilated by some power of $\underline{n}$, the maximal ideal of B. Thus, by (4.21) and P.E.C.C.A. (4.22), $\mathrm{Ext}^i_B(M,B) = 0$ if $i < n$, and $\mathrm{Ext}^n_B(M,B) = \mathrm{Hom}_B(M, E_B(B/\underline{n}))$. Since $\mathrm{inj.dim.}_B B = n$, $\mathrm{Ext}^i_B(M,B) = 0$ if $i > n$. Since M is of finite length, its injective envelope is isomorphic to a direct sum of a finite number of copies of $E_B(B/\underline{n})$, and so $\mathrm{Hom}_B(M, E_B(B/\underline{n})) \neq 0$. Thus $\mathrm{Ext}^n_B(M,B)$ is a non-zero module of finite length, and so is Cohen-Macaulay of homological codimension 0. Also, every M-sequence has zero length.

Now suppose, inductively, that $s > 0$, and that the assertion of the Proposition is true for any C.-M. B-module N with $\mathrm{codh}_B N = s-1$. Suppose M is C.-M. with $\mathrm{codh}_B M = s$. Since $s > 0$, $\underline{n}$ contains at least one element which is not a zero-divisor on M. Let $\underline{x} = (x_1,\ldots,x_k)$ $(0 < k \leqslant s)$ be an M-sequence. By (4.12), $N = M/x_1M$ is a C.-M. B-module with $\mathrm{codh}_B N = s-1$. $(x_2,\ldots,x_k)$ is an N-sequence, and so, by the inductive hypothesis, $\mathrm{Ext}^i_B(N,B) = 0$ if $i \neq h+1$, and if $N' = \mathrm{Ext}^{h+1}_B(N,B)$, N' is C.-M., $\mathrm{codh}_B N' = n-h-1 = s-1$, $(x_2,\ldots,x_k)$ is an N'-sequence, and:-

$N'/(x_2,\ldots,x_k)N' \cong \mathrm{Ext}^{h+k}_B(N/(x_2,\ldots,x_k)N,B) \cong \mathrm{Ext}^{h+k}_B(M/\underline{x}M,B)$.

From (E) $0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$, it follows that, for each $i \geqslant 0$, (E) $\mathrm{Ext}^i_B(N,B) \longrightarrow \mathrm{Ext}^i_B(M,B) \xrightarrow{x_1} \mathrm{Ext}^i_B(M,B) \longrightarrow \mathrm{Ext}^{i+1}_B(N,B)$.

Using the inductive hypothesis, if $i \neq h$, $\mathrm{Ext}^i_B(M,B) = x_1.\mathrm{Ext}^i_B(M,B)$, and so $\mathrm{Ext}^i_B(M,B) = 0$ by Nakayama's Lemma.

Thus (E) $0 \longrightarrow \mathrm{Ext}^h_B(M,B) \xrightarrow{x_1} \mathrm{Ext}^h_B(M,B) \longrightarrow \mathrm{Ext}^{h+1}_B(N,B) \longrightarrow 0$.

This shows that x_1 is not a zero-divisor on $M' = \mathrm{Ext}^h_B(M,B)$, that $M' \neq 0$, and that $M'/x_1M' \cong N'$. By the inductive hypothesis, $\underline{x}$ is an M'-sequence.

Since $\mathrm{codh}_B N' = s-1$, $\mathrm{codh}_B M' = s$. Since $(0:M)_B \subseteq (0:M')_B$, $\text{K-Dim}_B M' \leqslant$

$K\text{-}Dim_B M = s$, so M' is C.-M. with $codh_B M' = s$. Finally,

$$M'/\underline{x}M' \cong N'/(x_2,\ldots,x_k).N' \cong Ext_B^{h+k}(M/\underline{x}M,B).$$

The result now follows by induction. //

Since A is a C.-M. local ring, it follows that A, when regarded as a B-module by means of ϕ, is a C.-M. B-module with $codh_B A = K\text{-}Dim_B A = m$. (10.2) shows that $Ext_B^i(A,B) = 0$ whenever $i \neq n-m$, and that $Ext_B^{n-m}(A,B)$ is a C.-M. B-module of homological codimension m. However, using the remarks in (10.1), $Ext_B^{n-m}(A,B)$ has a natural structure as an A-module: W will be used, for the remainder of this Chapter, to denote $Ext_B^{n-m}(A,B)$ endowed with this natural A-module structure. It follows that W is a C.-M. A-module with $codh_A W = K\text{-}Dim_A W = m$.

The next step in a straightfoward adaption of the argument used by Murthy in (20) would be to show that, when A is an integral domain, W is isomorphic to A or to some ideal of A which is unmixed of height 1. However, I have been able to show that W is isomorphic to an ideal of A if and only if A is Bass-Ishikawa-0, and that when this is the case, either $W \cong A$, or W is isomorphic to an ideal of A which is unmixed of height 1. (Recall that an integral domain is Bass-Ishikawa-0.) These results, although stronger than is actually necessary for a straightfoward adaption of Murthy's argument, seem worth including, but before I can prove them I must establish several preliminary results which, however, are also necessary for the development of the general theory.

I am very grateful to Professor D. Rees for pointing out to me the following result.

(10.3)Proposition. $Inj.dim._A W$ is finite.

Proof. By (10.2), $\mathrm{Ext}^i_B(A,B) = 0$ if and only if $i \neq n-m$.

Let $\underline{C(B)}: 0 \longrightarrow B \xrightarrow{d^{-1}} B^0 \xrightarrow{d^0} B^1 \longrightarrow \cdots \longrightarrow B^{n-1} \xrightarrow{d^{n-1}} B^n \longrightarrow 0$ denote the Cousin Complex for B, and consider the complex:-

$(*)\ 0 \longrightarrow \mathrm{Hom}_B(A,B^0) \to \mathrm{Hom}_B(A,B^1) \to \cdots \to \mathrm{Hom}_B(A,B^{n-m-1}) \to \mathrm{Hom}_B(A,B^{n-m}) \longrightarrow \mathrm{Hom}_B(A,B^{n-m+1}) \to \cdots \to \mathrm{Hom}_B(A,B^n) \to 0$ of A-modules and -homomorphisms (see (10.1)) induced from $\underline{C(B)}$. Since $\underline{C(B)}$ provides an injective resolution for B, (*) is exact except at $\mathrm{Hom}_B(A,B^{n-m})$ and the homology module at this term is (A- & B-)isomorphic to W. But $\underline{b} = (0:A)_B$, and by (6.11), the height of any minimal prime ideal of $\underline{b}$ in B must be n-m, and so by (4.21), $\mathrm{Hom}_B(A,B^i) = 0$ for $0 \leqslant i < n-m$. Thus, it follows from (*) that there is an exact sequence:-

$(E)\ 0 \longrightarrow W \to \mathrm{Hom}_B(A,B^{n-m}) \longrightarrow \mathrm{Hom}_B(A,B^{n-m+1}) \to \cdots \to \mathrm{Hom}_B(A,B^n) \longrightarrow 0.$

Using the ideas of the proof of the first part of Lemma (6.3), it is easy to see that $\mathrm{Hom}_B(A,B^{n-m}),\ldots,\mathrm{Hom}_B(A,B^n)$ are all injective A-modules. //

(10.4)Corollary. W is a Gorenstein A-module, and if $\underline{p}$ is a prime ideal of A, $\mu^i(\underline{p},W) = \delta_{i,\mathrm{ht}\,\underline{p}}$ (Kronecker delta).

Proof. By (10.2) & (10.3), W is a non-zero, f.-g., A-module of finite injective dimension, and again by (10.2), every A-sequence (of elements of A) is a W-sequence. Now use (6.27)(viii) to see that W is a Gorenstein A-module.

Suppose that $\underline{p}$ is a prime ideal of A, say $\underline{p} = \phi(\underline{q})$, where $\underline{q}$ is a prime ideal of B containing $\underline{b}$. By (6.11), $\mathrm{ht}_A\underline{p} = \mathrm{ht}_B\underline{q} - (n-m)$ (since $\mathrm{ht}_A\underline{p} + \mathrm{coht}\,\underline{p} = m$, $\mathrm{ht}_B\underline{q} + \mathrm{coht}\,\underline{q} = n$, and $\mathrm{coht}\,\underline{p} = \mathrm{coht}\,\underline{q}$). By (9.10), $(0:W)_A = 0$, and so, by (6.27)(ii), $\mu^i(\underline{p},W) = 0$ if and only if $i \neq \mathrm{ht}_A\underline{p}$ $(= \mathrm{ht}_W\underline{p})$. Thus it remains only to show that $\mu^{\mathrm{ht}\,\underline{p}}(\underline{p},W) = 1$.

I use the notation of the proof of (10.3). By (5.13) & (5.10), if $i \geqslant n-m$, $B^i \cong \bigoplus_{\substack{\underline{q}' \in V(B) \\ \operatorname{ht}\underline{q}' = i}} E_B(B/\underline{q}')$, and so, by (2.19), (6.11), and (a straightfoward adaption of) (6.3), $\operatorname{Hom}_B(A,B^i) \cong \bigoplus_{\substack{\underline{p}' \in V(A) \\ \operatorname{ht}\underline{p}' = i-(n-m)}} E_A(A/\underline{p}')$.

Applying (5.1) to the injective resolution for W obtained in the proof of (10.3), and using the structure theory for injective A-modules described in (5.2), it follows that $\mu^{\operatorname{ht}\underline{p}}(\underline{p},W) \leqslant 1$. But $\mu^{\operatorname{ht}\underline{p}}(\underline{p},W) > 0$. Therefore $\mu^i(\underline{p},W) = \delta_{i,\operatorname{ht}\underline{p}}$ for each $i \geqslant 0$. //

The following (well-known) Proposition helps to provide greater access to the module W.

(10.5)Proposition. (See Theorem 2.1 of Rees (23) and Theorem 6 of Chapter 9 of Northcott (22).) Suppose C is a ring, M a C-module, and $\underline{x} = (x_1,..,x_k)$ is both a C-sequence and an M-sequence. Let $D = C/\underline{x}$; for each D-module N, let $U^n(N) = \operatorname{Ext}^n_D(N,M/\underline{x}M)$ and $T^n(N) = \operatorname{Ext}^{n+k}_C(N,M)$. (N is regarded as a C-module by "restriction of scalars" using the natural surjective ring homomorphism $C \longrightarrow C/\underline{x} = D$, and $\operatorname{Ext}^{n+k}_C(N,M)$ is annihilated by $\underline{x}$, and so has a natural structure as a D-module.) Then the connected right sequences $\{T^n\}_{n\geqslant 0}$ and $\{U^n\}_{n\geqslant 0}$ of contravariant functors from $\mathcal{C}_D$ to $\mathcal{C}_D$ are isomorphic.

Proof. I shall use the Corollary to Theorem 11 of Chapter 6 of Northcott (22). Suppose $f:N \longrightarrow N'$ is a homomorphism of D-modules. '*'s will be used to denote induced homomorphisms, and considerable use will be made of the properties of Extension functors described in the last paragraph of (2.2).

By (4.2), $\operatorname{Ext}^{k-1}_C(N'',M) = 0$ for all D-modules N'', and

$$(E)\quad 0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0 \text{ induces:-}$$

$$\begin{array}{ccccccc}
(E)\ 0 & \longrightarrow & \mathrm{Ext}_C^{k-1}(N',M/x_1M) & \longrightarrow & \mathrm{Ext}_C^{k}(N',M) & \xrightarrow{x_1} & \mathrm{Ext}_C^{k}(N',M) \\
 & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\
(E)\ 0 & \longrightarrow & \mathrm{Ext}_C^{k-1}(N,M/x_1M) & \longrightarrow & \mathrm{Ext}_C^{k}(N,M) & \xrightarrow{x_1} & \mathrm{Ext}_C^{k}(N,M) \qquad (C)\ .
\end{array}$$

Since the multiplications by x_1 in this diagram are both zero, it follows that the connecting homomorphisms $\mathrm{Ext}_C^{k-1}(N,M/x_1M) \longrightarrow \mathrm{Ext}_C^{k}(N,M)$ provide a natural equivalence between the functors $\mathrm{Ext}_C^{k-1}(\ ,M/x_1M)$ and $\mathrm{Ext}_C^{k}(\ ,M)$ from $\underline{\mathcal{C}}_D$ to $\underline{\mathcal{C}}_D$.

Since, for each i s.t. $0 < i < k$, $M/(x_1,\ldots,x_i)M \Big/ x_{i+1}(M/(x_1,\ldots,x_i)M) = M/(x_1,\ldots,x_{i+1})M$, $(x_{i+1},\ldots,x_k)$ is an $M/(x_1,\ldots,x_i)M$-sequence, and the argument given above did not use the fact that $\underline{x}$ is a C-sequence, this argument can be repeated a further k-1 times to obtain natural equivalences between the functors, from $\underline{\mathcal{C}}_D$ to $\underline{\mathcal{C}}_D$,

$$\mathrm{Ext}_C^{k-2}(\ ,M/(x_1,x_2)M) \quad \text{and} \quad \mathrm{Ext}_C^{k-1}(\ ,M/x_1M)$$

$$\vdots$$

$$\mathrm{Hom}_C(\ ,M/\underline{x}M) \quad \text{and} \quad \mathrm{Ext}_C^{1}(\ ,M/(x_1,\ldots,x_{k-1})M)\ .$$

Combining these natural equivalences, I obtain a natural equivalence between the functors $\mathrm{Ext}_C^{k}(\ ,M)$ and $\mathrm{Hom}_C(\ ,M/\underline{x}M)$ from $\underline{\mathcal{C}}_D$ to $\underline{\mathcal{C}}_D$: however, if N is a D-module, $\mathrm{Hom}_C(N,M/\underline{x}M) = \mathrm{Hom}_D(N,M/\underline{x}M)$.

Thus, U^0 and T^0 from $\underline{\mathcal{C}}_D$ to $\underline{\mathcal{C}}_D$ are naturally equivalent, and it remains only to show that, for each projective D-module P and all $n \geqslant 0$, $U^{n+1}(P) = T^{n+1}(P) = 0$. It is obvious that $U^{n+1}(P) = 0$. Since each T^{n+1} is an additive functor, and any projective D-module is a direct summand of a free D-module, it is sufficient to show that $T^{n+1}(F) = 0$ for each free

D-module F and all $n \geqslant 0$.

I first prove this when F = D: in this case it is sufficient to prove that $\mathrm{hd}_C D \leqslant k$. I prove by induction on h that $\mathrm{hd}_C(C/(x_1,\ldots,x_h)) \leqslant h$ for $0 \leqslant h \leqslant k$: this is obvious when h = 0. If $0 < l \leqslant k$ and $\mathrm{hd}_C(C/(x_1,\ldots,x_{l-1})) \leqslant l-1$, then:-

(E) $0 \longrightarrow C/(x_1,..,x_{l-1}) \xrightarrow{x_l} C/(x_1,..,x_{l-1}) \longrightarrow C/(x_1,\ldots,x_l) \longrightarrow 0$

induces, for an arbitrary C-module Y,

(E)$\ldots\longrightarrow \mathrm{Ext}^l_C(C/(x_1,..,x_{l-1}),Y) \longrightarrow \mathrm{Ext}^{l+1}_C(C/(x_1,..,x_l),Y) \longrightarrow$

$\mathrm{Ext}^{l+1}_C(C/(x_1,..,x_{l-1}),Y) \longrightarrow \ldots$, and since

$\mathrm{hd}_C(C/(x_1,\ldots,x_{l-1})) \leqslant l-1$, it follows that $\mathrm{Ext}^{l+1}_C(C/(x_1,\ldots,x_l),Y) = 0$ for all C-modules Y. Therefore $\mathrm{hd}_C(C/(x_1,\ldots,x_l)) \leqslant l$ by (2.6).

Now suppose F is an arbitrary free D-module, say $F \cong \bigoplus_{i\in I} D^{(i)}$, where $D^{(i)} \cong D$ for each $i \in I$. Thus $\mathrm{hd}_C D^{(i)} \leqslant k$ for each $i \in I$. Let

(E) $0 \longrightarrow P^{(i)}_k \longrightarrow P^{(i)}_{k-1} \longrightarrow \ldots \longrightarrow P^{(i)}_0 \longrightarrow D^{(i)} \longrightarrow 0$ be a projective resolution for $D^{(i)}$ when regarded as a C-module (for each $i \in I$). (Thus, for each $i \in I$ and j s.t. $0 \leqslant j \leqslant k$, $P^{(i)}_j$ is a projective C-module.)

Form the direct sum, taken over all $i \in I$, of these exact sequences to see that $\mathrm{hd}_C F \leqslant k$, since the direct sum of a family of projective C-modules is again projective. It follows that $T^{n+1}(F) = 0$ for all $n \geqslant 0$.

The result now follows from the Corollary to Theorem 11 of Chapter 6 of Northcott (22). //

Applying (10.5) to the rings A & B, I obtain:-

(10.6)Corollary, Suppose $\underline{x} = (x_1,\ldots,x_k)$ is a B-sequence contained in $\underline{b} = \ker\phi$. Then ϕ induces a surjective ring homomorphism $\phi' : B/\underline{x} \longrightarrow A$. $\mathrm{Ext}^{n-m-k}_{B/\underline{x}}(A, B/\underline{x})$ can be regarded as as A-module in the manner indicated

in (10.1), and, when this is done,

$W = \mathrm{Ext}_B^{n-m}(A,B) \cong \mathrm{Ext}_{B/\underline{x}}^{n-m-k}(A,B/\underline{x})$ as A-modules. //

Suppose $\mathrm{ht}\,\underline{b} = k$, and $\underline{q} \in \mathrm{Ass}_B(B/\underline{b})$ has height h. Since B and $B/\underline{b}$ are C.-M., it follows from (6.11) that $\mathrm{coht}\,\underline{q} = m$ and $\mathrm{ht}\,\underline{q} + \mathrm{coht}\,\underline{q} = h + m = n$. Thus $h = n-m$. Since B is C.-M., $\mathrm{grade}_B \underline{b} = \mathrm{ht}\,\underline{b}$, so let $\underline{x}$ be a B-sequence of length n-m contained in $\underline{b}$. Then Corollary (10.6) shows that, when $\mathrm{Hom}_{B/\underline{x}}(A,B/\underline{x})$ is regarded as an A-module using the induced ~~surjective~~ surjective ring homomorphism $\phi' : B/\underline{x} \longrightarrow A$ in the manner indicated in (10.1), then W is A-isomorphic to $\mathrm{Hom}_{B/\underline{x}}(A,B/\underline{x})$. This is useful, for $\mathrm{Hom}_{B/\underline{x}}(A,B/\underline{x})$ is often more accessible than $\mathrm{Ext}_B^{n-m}(A,B)$. Note also that $B/\underline{x}$ is a Gorenstein local ring of Krull Dimension m, by (9.9) and (4.12). Thus one effect of Corollary (10.6) is that, when attempting to establish results about W, it is often possible to assume that $n = m$.

With these ideas in mind, I now establish the results mentioned previously concerning a necessary and sufficient condition for W to be isomorphic to an ideal of A.

(10.7)Proposition. If W is A-isomorphic to a proper ideal of A, then that ideal is unmixed and of height 1.

Proof. Suppose that W is A-isomorphic to the ideal $\phi(\underline{a})$ of A, where $\underline{a}$ is an ideal of B such that $\underline{b} \subseteq \underline{a} \subsetneq B$.

Regarding W and A as B-modules in the obvious manners, there is an exact sequence (E) $0 \longrightarrow W \longrightarrow A \longrightarrow B/\underline{a} \longrightarrow 0$ (*) of B-modules and -homomorphisms. Using (10.2) (and the remarks immediately following its proof), both A and W are Cohen-Macaulay B-modules of homological

codimension m, and $\mathrm{Ext}^i_B(A,B) = 0 = \mathrm{Ext}^i_B(W,B)$ if $i \neq n-m$. Consequently, using (*), $\mathrm{Ext}^i_B(B/\underline{a},B) = 0$ whenever $i \neq n-m$ or $n-m+1$, and:-

(E) $0 \to \mathrm{Ext}^{n-m}_B(B/\underline{a},B) \to \mathrm{Ext}^{n-m}_B(A,B) \to \mathrm{Ext}^{n-m}_B(W,B) \to \mathrm{Ext}^{n-m+1}_B(B/\underline{a},B) \to 0$.

Thus $\mathrm{grade}_B \underline{a} = n-m$ or $n-m+1$, and since B is Cohen-Macaulay, $\mathrm{grade}_B \underline{a} = \mathrm{ht}_B \underline{a}$.

Suppose $\mathrm{ht}_B \underline{a} = n-m$. Using (6.11), it follows that $\not{p}(\underline{a}) \subseteq \not{p}(\underline{p}_1)$, a minimal prime ideal of the zero ideal of A (where $\underline{p}_1$ is an ideal of B containing $\underline{b}$). Suppose $\underline{p}_2, \ldots, \underline{p}_s$ are prime ideals of B containing $\underline{b}$ s.t. $\not{p}(\underline{p}_2), \ldots, \not{p}(\underline{p}_s)$ are the other associated prime ideals of 0 in A. (These must all be minimal since A is Cohen-Macaulay.) Let $\underline{q}_1, \ldots, \underline{q}_s$ be ideals of B containing $\underline{b}$ s.t. $0 = \bigcap_{i=1}^{s} \not{p}(\underline{q}_i)$, with $\not{p}(\underline{q}_i)$ $\not{p}(\underline{p}_i)$-primary for each $i = 1, \ldots, s$, is a minimal p.d. for 0 in A. There exists an integer $N > 0$ s.t. $(\not{p}(\underline{p}_1))^N \subseteq \not{p}(\underline{q}_1)$. Now if $\underline{c} = \bigcap_{i=2}^{s} \underline{q}_i$, $\not{p}(\underline{c}) = \bigcap_{i=2}^{s} \not{p}(\underline{q}_i) \neq 0$, and $(\not{p}(\underline{a}))^N . \not{p}(\underline{c}) = 0$. If R is the least positive integer k such that $(\not{p}(\underline{a}))^k . \not{p}(\underline{c}) = 0$, it follows that:-

$0 \neq (\not{p}(\underline{a}))^{R-1} . \not{p}(\underline{c}) \subseteq (0 : \not{p}(\underline{a}))_A = (0:W)_A$. But, by (10.4), W is a Gorenstein A-module, and so $(0:W)_A = 0$ by (9.10), a contradiction. Thus $\mathrm{grade}_B \underline{a} = \mathrm{ht}_B \underline{a} = n-m+1$, $\mathrm{Ext}^{n-m}_B(B/\underline{a},B) = 0$, and $\mathrm{Ext}^{n-m+1}_B(B/\underline{a},B) \neq 0$.

Now $\mathrm{codh}_B A = \mathrm{codh}_B W = m$. By (4.6) and (4.8), m is therefore the least integer i s.t. $\mathrm{Ext}^i_B(B/\underline{n},A) \neq 0$, and the least integer j s.t. $\mathrm{Ext}^j_B(B/\underline{n},W) \neq 0$. Thus, from (*), it follows that the least integer i s.t. $\mathrm{Ext}^i_B(B/\underline{n},B/\underline{a}) \neq 0$ is $\geqslant m-1$, i.e. $\mathrm{codh}_B B/\underline{a} \geqslant m-1$. Since $\mathrm{ht}_B \underline{a} = n-m+1$, $\text{K-Dim}_B B/\underline{a} \leqslant m-1$. Thus $B/\underline{a}$ is a Cohen-Macaulay B-module with $\mathrm{codh}_B B/\underline{a} = \text{K-Dim}_B B/\underline{a} = m-1$. Thus $B/\underline{a}$ and $A/\not{p}(\underline{a})$ are Cohen-Macaulay rings of

Krull Dimension m-1. Therefore, by (4.14), any prime in $\mathrm{Ass}_A(A/\phi(\underline{a}))$ has coheight m-1, and so, by (6.11), has A-height 1. Thus $\phi(\underline{a})$ is unmixed of height 1. //

(10.8)Theorem. W is A-isomorphic to an ideal of A if and only if A is Bass-Ishikawa-0.

Proof. ($\Longrightarrow$) Suppose W is isomorphic to an ideal of A. If $W \cong A$, then it follows from (10.4) that A is a Gorenstein ring and so is certainly Bass-Ishikawa-0. (See (7.13) for the Definition of a Bass-Ishikawa-0 ring.)

So suppose W is not isomorphic to A. Then, by (10.7), there is an A-module monomorphism $f:W \longrightarrow A$ s.t. $f(W)$ is a proper ideal of A which is unmixed of height 1. Since A is Cohen-Macaulay, there exists $r \in f(W)$ s.t. r is not a zero-divisor on A. It follows easily that f is an essential monomorphism. Consequently, $E_A(W) \cong E_A(A)$. But, by (10.4), $E_A(W) \cong \bigoplus_{\substack{\underline{p} \in V(A) \\ \mathrm{ht}\,\underline{p}=0}} E_A(A/\underline{p})$. Consequently, by (5.11), for each $\underline{p} \in V(A)$ of A-height 0, $A_{\underline{p}}$ is Gorenstein. Since A is Cohen-Macaulay, it follows from (7.11) & (7.13) that A is Bass-Ishikawa-0.

($\Longleftarrow$) Suppose, conversely, that A is Bass-Ishikawa-0. By the remarks immediately after (10.6), I may assume that m = n.

Thus $W \cong \mathrm{Hom}_B(A,B)$ as A-modules. Now $B \subseteq K$, the full ring of quotients of B, and by (7.12), K is B-isomorphic to $\bigoplus_{\substack{\underline{q} \in V(B) \\ \mathrm{ht}_B\,\underline{q}=0}} E_B(B/\underline{q})$. Using (2.10), W is B-isomorphic and A-isomorphic (when all the modules concerned are regarded as A-modules in the manner indicated in (10.1)) to a submodule of

$\bigoplus_{\substack{\mathfrak{q}\in V(B)\\ ht_B\mathfrak{q}=0}} \mathrm{Hom}_B(A, E_B(B/\mathfrak{q}))$. Applying (a straightfoward adaption of) (6.3), together with (6.11), it follows that W is A-isomorphic to a submodule of $\bigoplus_{\substack{\mathfrak{p}\in V(A)\\ ht_A\mathfrak{p}=0}} E_A(A/\mathfrak{p})$. (Recall that both A & B are C.-M.) But since A is Bass-Ishikawa-0, it follows from (7.11), (7.12), & (7.13) that $\bigoplus_{\substack{\mathfrak{p}\in V(A)\\ ht_A\mathfrak{p}=0}} E_A(A/\mathfrak{p}) \cong L$, the full ring of quotients of A.

Now $L = S^{-1}A$, where $S = \{s \in A \mid s \text{ is not a zero-divisor on } A\}$, and, by the above, W is isomorphic to a f.-g. A-submodule N of L. Suppose N is generated by $a_1/s_1,\ldots,a_t/s_t$ ($a_i \in A$, $s_i \in S$ for each $i = 1,\ldots,t$). Then $N \subseteq A.(1/(s_1\ldots s_t)) \subseteq L$, and $A.(1/(s_1\ldots s_t)) \cong A$. Thus W is A-isomorphic to an ideal of A. //

The next step in a straightfoward adaption of the argument used by Murthy in (20), would be to adapt the proof of Serre's result which appears as Proposition (5.1) of Bass (5) to show that, if $W \cong A$, then A is Gorenstein. However, I have already shown in (10.4), that W is a Gorenstein A-module, so it is now obvious that if $W \cong A$, then A is Gorenstein. In fact, I now have at my disposal all the tools necessary to prove the strengthening of Murthy's result mentioned in the introduction to this Chapter.

(10.9)Theorem. Suppose that B is a Gorenstein local ring, A is a Unique Factorization Domain, and $\phi : B \longrightarrow A$ is a surjective ring homomorphism. Then A is Gorenstein if and only if A is Cohen-Macaulay.

Proof. A is, of course, a local ring, and since, by (5.11), every

Gorenstein local ring is Cohen-Macaulay, one half of the claim of the Theorem is immediate.

Suppose, therefore, that A is Cohen-Macaulay. I use the notation which has been standard for most of this Chapter. Since A is an integral domain, A is Bass-Ishikawa-0. (See (7.14)(i).) By (10.8), W is A-isomorphic to an ideal of A, and, by (10.7), it follows that either $W \cong A$, or W is isomorphic to a proper ideal of A which is unmixed and of height 1. But, by (1.29), any (proper) unmixed ideal of A of height 1 is principal, and so $W \cong A$ in either case. But, by (10.4), W is a Gorenstein A-module, and so A is a Gorenstein ring. //

As was indicated in the introduction to this Chapter, the results of this Chapter arose out of a straightfoward attempt to establish Theorem (10.9), but some of the intermediate results seem, to me at least, to be more worthwhile than Theorem (10.9). In particular, the fact that W is a Gorenstein A-module, proved in (10.4), is fundamental to the work of Chapter 12: for one thing, it establishes the existence of a Gorenstein module over a Cohen-Macaulay local ring which is a homomorphic image of a Gorenstein local ring. In case the study of Cohen-Macaulay local rings which are homomorphic images of Gorenstein local rings seems a little artificial, it seems worthwhile to make the following

(10.10)Observation. Any complete Cohen-Macaulay local ring is a homomorphic image of a Gorenstein local ring.

Proof. By Corollary 2 to Theorem 15 of Cohen (7), every complete local ring is a homomorphic image of a regular local ring, and regular local rings are Gorenstein. (See(7.1).) //

Hence (10.4) and (10.10) together establish the existence of Gorenstein modules over any complete Cohen-Macaulay local ring. However, before I can exploit these ideas to the full, I must first develop the necessary machinery, and so Chapter 11 is devoted to the study of a commutative algebra analogue of the Local Cohomology Theory developed by Hartshorne in (13).

CHAPTER 11

Local Cohomology Theorey in Commutative Algebra.

(Reminder: "ring" means "commutative Noetherian ring with a non-zero identity".)

The aim of this Chapter is to develop, for use in Chapter 12, a commutative algebra analogue of the Local Cohomology Theory developed by Hartshorne in (13). The alternative to including an account of such a theory would have been to quote, from (13), commutative algebra consequences of Hartshorne's work as and when necessary: however, Hartshorne's definitions of the Local Cohomology Groups are formulated in the language of preschemes, and it seems inappropriate to quote such general results in a thesis on commutative algebra.

The work in this Chapter is developed (more or less from scratch) entirely within the framework of functors on categories of modules over a (commutative Noetherian) ring. While Hartshorne's work has obviously been a general guide in developing this theory, some of the necessary intermediate results turned out to be quite difficult to prove, and my eventual proofs bear little or no resemblance to Hartshorne's proofs of the analogous results in (13).

Throughout this Chapter, A will denote a ring, and $\underline{a}$ will denote an ideal of A.

(11.1) Definition. For an arbitrary A-module M, define:-

$$L_{\underline{a}}(M) = \{ m \in M \mid \text{there exists an integer } N > 0 \text{ s.t. } \underline{a}^N.m = 0 \}$$
$$= \bigcup_{k=1}^{\infty} (0 : \underline{a}^k)_M .$$

$L_{\underline{a}}(M)$ is a submodule of M. If $f:M \longrightarrow N$ is a homomorphism of A-modules, define $L_{\underline{a}}(f):L_{\underline{a}}(M) \longrightarrow L_{\underline{a}}(N)$ to be the restriction of f to $L_{\underline{a}}(M)$. Then $L_{\underline{a}}:\underline{\mathcal{C}}_A \rightsquigarrow \underline{\mathcal{C}}_A$ is a covariant functor. $L_{\underline{a}}$ is the <u>local cohomology functor</u> from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$, <u>w.r.t. the ideal a of A</u>. It is clear that $L_{\underline{a}}$ is an additive, A-linear functor, and it is easy to see that it is left exact. So (see (2.30)), if $R^iL_{\underline{a}}$ denotes the ith right derived functor of $L_{\underline{a}}$, $R^0L_{\underline{a}}$ and $L_{\underline{a}}$ are naturally equivalent functors. (Note that each $R^iL_{\underline{a}}$ $(i \geqslant 0)$ is an additive, A-linear, covariant functor from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$.)

<u>(11.2)The Functors $\varinjlim_k \operatorname{Ext}^n_A(A/\underline{a}^k, \ ):\underline{\mathcal{C}}_A \rightsquigarrow \underline{\mathcal{C}}_A$</u>. Suppose $T:\underline{\mathcal{C}}_A \times \underline{\mathcal{C}}_A \rightsquigarrow \underline{\mathcal{C}}_A$ is an additive A-linear functor of two variables which is contravariant in the first variable and covariant in the second.

The positive integers with the usual ordering form a directed set N. (See (1.32).) If $i,j \in N$ with $i \leqslant j$, $\underline{a}^j \subseteq \underline{a}^i$, and the natural A-homomorphism $A/\underline{a}^j \longrightarrow A/\underline{a}^i$ induces, for an arbitrary A-module M, $\pi_i^j(M):T(A/\underline{a}^i,M) \longrightarrow T(A/\underline{a}^j,M)$ (an A-module homomorphism). Furthermore, the $T(A/\underline{a}^i,M)$ and $\pi_i^j(M)$ form a direct system of A-modules and -homs. over N, and so $\varinjlim_k T(A/\underline{a}^k,M)$ can be formed.

Similarly, and A-module homomorphism $f:M \longrightarrow N$ induces a homomorphism $T(A/\underline{a}^i,f):T(A/\underline{a}^i,M) \longrightarrow T(A/\underline{a}^i,N)$ for each $i \in N$, s.t., if $i \leqslant j$,

$$\begin{array}{ccc} T(A/\underline{a}^i,M) & \xrightarrow{T(A/\underline{a}^i,f)} & T(A/\underline{a}^i,N) \\ \downarrow{\scriptstyle \pi_i^j(M)} & & \downarrow{\scriptstyle \pi_i^j(N)} \\ T(A/\underline{a}^j,M) & \xrightarrow{T(A/\underline{a}^j,f)} & T(A/\underline{a}^j,N) \end{array} \qquad (C).$$

Consequently, f induces $\varinjlim_k T(A/\underline{a}^k,f):\varinjlim_k T(A/\underline{a}^k,M) \longrightarrow \varinjlim_k T(A/\underline{a}^k,N)$.

It is now easy to check that, in this way, $\varinjlim_k T(A/\underline{a}^k, \ )$ becomes a covariant, additive, A-linear functor from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$. (Again, see (1.32).)

In particular, I can use these ideas to define covariant, additive, A-linear functors $\varinjlim_k \mathrm{Hom}_A(A/\underline{a}^k, \ ): \underline{\mathcal{C}}_A \rightsquigarrow \underline{\mathcal{C}}_A$, and, for each $n \geqslant 0$, $\varinjlim_k \mathrm{Ext}^n_A(A/\underline{a}^k, \ ): \underline{\mathcal{C}}_A \rightsquigarrow \underline{\mathcal{C}}_A$. Furthermore, if (E) $0 \to M' \hookrightarrow M \twoheadrightarrow M'' \to 0$ in $\underline{\mathcal{C}}_A$, using the properties of Extension functors described in the last paragraph of (2.2), I obtain, for $i \leqslant j$,

$$\begin{array}{ccc} \mathrm{Hom}_A(A/\underline{a}^i, M'') & \longrightarrow & \mathrm{Ext}^1_A(A/\underline{a}^i, M') \\ \downarrow & & \downarrow \\ \mathrm{Hom}_A(A/\underline{a}^j, M'') & \longrightarrow & \mathrm{Ext}^1_A(A/\underline{a}^j, M') \end{array} \qquad \text{(C)} \quad \text{and}$$

$$\begin{array}{ccc} \mathrm{Ext}^n_A(A/\underline{a}^i, M'') & \longrightarrow & \mathrm{Ext}^{n+1}_A(A/\underline{a}^i, M') \\ \downarrow & & \downarrow \\ \mathrm{Ext}^n_A(A/\underline{a}^j, M'') & \longrightarrow & \mathrm{Ext}^{n+1}_A(A/\underline{a}^j, M') \end{array} \qquad \text{(C)} \quad \text{for } n > 0.$$

(The vertical homomorphisms in each case are the natural ones forming the relevant direct system, while the horizontal ones are all connecting homomorphisms.) These diagrams induce A-module homomorphisms:- $\varinjlim_k \mathrm{Hom}_A(A/\underline{a}^k, M'') \longrightarrow \varinjlim_k \mathrm{Ext}^1_A(A/\underline{a}^k, M')$ and, for $n > 0$, $\varinjlim_k \mathrm{Ext}^n_A(A/\underline{a}^k, M'') \longrightarrow \varinjlim_k \mathrm{Ext}^{n+1}_A(A/\underline{a}^k, M')$. These "connecting homomorphisms" make $\left\{\varinjlim_k \mathrm{Ext}^n_A(A/\underline{a}^k, \ )\right\}_{n \geqslant 0}$ into a connected right sequence of covariant functors from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$.

(11.3)Theorem. The functors $L_{\underline{a}}$ and $\varinjlim_k \mathrm{Hom}_A(A/\underline{a}^k, \ )$ from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$ are naturally equivalent, and, for $n > 0$, the functors $R^n L_{\underline{a}}$ and

$\varinjlim_k \mathrm{Ext}^n_A(A/\underline{a}^k, \ )$ from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$ are naturally equivalent.

Proof. I plan to apply the Corollary to Theorem 10 of Chapter 6 of Northcott (22) to the connected right sequences of covariant functors from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$:- $\left\{\varinjlim_k \mathrm{Ext}^n_A(A/\underline{a}^k, \ )\right\}_{n\geqslant 0}$ and $\left\{R^n L_{\underline{a}}\right\}_{n\geqslant 0}$ (interpreting $\varinjlim_k \mathrm{Ext}^0_A(A/\underline{a}^k, \ )$ as $\varinjlim_k \mathrm{Hom}_A(A/\underline{a}^k, \ )$ with which it is naturally equivalent, and $R^0 L_{\underline{a}}$ as $L_{\underline{a}}$ with which it is naturally equivalent). It is clear that, if $n > 0$ and I is an injective A-module, then $\varinjlim_k \mathrm{Ext}^n_A(A/\underline{a}^k, I) = 0 = R^n L_{\underline{a}}(I)$.

Suppose (E) $0 \to M' \to M \to M'' \to 0$ in $\underline{\mathcal{C}}_A$. Then, by Theorem 5.4 of Chapter VIII of Eilenberg and Steenrod (9), the resulting complex:-

$$0 \longrightarrow \varinjlim_k \mathrm{Hom}_A(A/\underline{a}^k, M') \longrightarrow \varinjlim_k \mathrm{Hom}_A(A/\underline{a}^k, M) \longrightarrow \varinjlim_k \mathrm{Hom}_A(A/\underline{a}^k, M'')$$
$$\longrightarrow \varinjlim_k \mathrm{Ext}^1_A(A/\underline{a}^k, M') \longrightarrow \cdots \longrightarrow \varinjlim_k \mathrm{Ext}^{n-1}_A(A/\underline{a}^k, M'') \longrightarrow$$
$$\varinjlim_k \mathrm{Ext}^n_A(A/\underline{a}^k, M') \longrightarrow \varinjlim_k \mathrm{Ext}^n_A(A/\underline{a}^k, M) \longrightarrow \varinjlim_k \mathrm{Ext}^n_A(A/\underline{a}^k, M'') \longrightarrow$$
$$\varinjlim_k \mathrm{Ext}^{n+1}_A(A/\underline{a}^k, M') \longrightarrow \cdots$$ is, in fact, exact.

Consequently, it is sufficient to show that $L_{\underline{a}}$ and $\varinjlim_k \mathrm{Hom}_A(A/\underline{a}^k, \ )$ from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$ are naturally equivalent functors.

For each A-module M and $k \in \mathbb{N}$, define an A-module isomorphism $\chi_k(M): \mathrm{Hom}_A(A/\underline{a}^k, M) \xrightarrow{\cong} (0:\underline{a}^k)_M$ by:- $\chi_k(M)(f) = f(\bar{1})$, where $\bar{1}$ denotes the natural image of 1 in $A/\underline{a}^k$. If $0 < i \leqslant j$ are integers, let $\alpha^j_i(M): (0:\underline{a}^i)_M \longrightarrow (0:\underline{a}^j)_M$ be the natural inclusion. Then:-

$$\begin{array}{ccc} \mathrm{Hom}_A(A/\underline{a}^i, M) & \xrightarrow[\chi_i(M)]{\cong} & (0:\underline{a}^i)_M \\ \downarrow \pi^j_i(M) & & \downarrow \alpha^j_i(M) \\ \mathrm{Hom}_A(A/\underline{a}^j, M) & \xrightarrow[\chi_j(M)]{\cong} & (0:\underline{a}^j)_M \end{array} \qquad (C).$$

Thus the $\varkappa_k(M)$ induce an isomorphism $\varkappa(M):\varinjlim_k \mathrm{Hom}_A(A/\underline{a}^k,M) \xrightarrow{\cong} L_{\underline{a}}(M)$.

Furthermore, if $f:M\longrightarrow M''$ is a homomorphism of A-modules, and '*'s denote induced homomorphisms, it follows from:-

$$\begin{array}{ccccc}
\mathrm{Hom}_A(A/\underline{a}^i,M) & \xrightarrow{\varkappa_i(M)} & (0:\underline{a}^i)_M & & \\
\downarrow{\scriptstyle \Pi_i^j(M)} \quad \searrow^{f^*} & & \quad\searrow^{f^*} & & \\
\mathrm{Hom}_A(A/\underline{a}^j,M) & \xrightarrow{\varkappa_j(M)} & (0:\underline{a}^j)_M & & \\
\downarrow{\scriptstyle \Pi_i^j(M'')} & \searrow^{f^*} & \downarrow{\scriptstyle \alpha_i^j(M)} & \searrow^{f^*} & \\
\mathrm{Hom}_A(A/\underline{a}^i,M'') & \xrightarrow{\varkappa_i(M'')} & (0:\underline{a}^i)_{M''} & & \\
\downarrow{\scriptstyle \Pi_i^j(M'')} & & \downarrow{\scriptstyle \alpha_i^j(M'')} & & \\
\mathrm{Hom}_A(A/\underline{a}^j,M'') & \xrightarrow[\varkappa_j(M'')]{} & (0:\underline{a}^j)_{M''} & & \qquad (C)
\end{array}$$

(where $i \leqslant j$) that:-

$$\begin{array}{ccc}
\varinjlim_k \mathrm{Hom}_A(A/\underline{a}^k,M) & \xrightarrow[\cong]{\varkappa(M)} & L_{\underline{a}}(M) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle L_{\underline{a}}(f)} \\
\varinjlim_k \mathrm{Hom}_A(A/\underline{a}^k,M'') & \xrightarrow[\cong]{\varkappa(M'')} & L_{\underline{a}}(M'')
\end{array} \qquad (C),$$

and so the $\varkappa(M)$ constitute a natural equivalence of functors.

Hence, using the Corollary to Theorem 10 of Chapter 6 of Northcott (22), the Theorem is proved. //

Next I show that the functor $L_{\underline{a}}:\underline{C}_A \longrightarrow \underline{C}_A$ "commutes with the formation of direct limits".

(11.4)Proposition. Suppose that a set D together with a relation $<$ in D is a directed set. (See (1.32).) Suppose that $\{N,\Pi\}$ is a direct system of A-modules and -homomorphisms over D. Then the functor $L_{\underline{a}}$ induces a direct system of A-modules and -homomorphisms $\{L_{\underline{a}}(N),L_{\underline{a}}(\Pi)\}$ over D, and a

natural A-module isomorphism $\theta : \varinjlim_{\alpha\in D} L_{\underline{a}}(N^\alpha) \xrightarrow{\cong} L_{\underline{a}}(\varinjlim_{\alpha\in D} N^\alpha)$.

Proof. If $\alpha<\beta$ in D, $L_{\underline{a}}$ induces $L_{\underline{a}}(\pi_\alpha^\beta): L_{\underline{a}}(N^\alpha) \longrightarrow L_{\underline{a}}(N^\beta)$, and it is clear that the $L_{\underline{a}}(N^\alpha)$ and $L_{\underline{a}}(\pi_\alpha^\beta)$ form a direct system of A-modules and -homomorphisms $\left\{L_{\underline{a}}(N), L_{\underline{a}}(\pi)\right\}$ over D. Let $N^\infty = \varinjlim_{\alpha\in D} N^\alpha$, and let, for each $\alpha \in D$, $\pi_\alpha : N^\alpha \longrightarrow N^\infty$ be the natural A-homomorphism, so that if $\alpha<\beta$ in D, $\pi_\beta . \pi_\alpha^\beta = \pi_\alpha$.

$L_{\underline{a}}$ induces, for each $\alpha \in D$, $L_{\underline{a}}(\pi_\alpha): L_{\underline{a}}(N^\alpha) \longrightarrow L_{\underline{a}}(N^\infty)$ such that, if $\alpha<\beta$ in D, $L_{\underline{a}}(\pi_\beta).L_{\underline{a}}(\pi_\alpha^\beta) = L_{\underline{a}}(\pi_\alpha)$. Consequently, these homomorphisms induce a (uniquely determined) homomorphism $\vartheta : \varinjlim_{\alpha\in D} L_{\underline{a}}(N^\alpha) \longrightarrow L_{\underline{a}}(N^\infty)$ s.t. for each $\alpha \in D$,

$$\begin{array}{ccc} L_{\underline{a}}(N^\alpha) & & \\ {\scriptstyle \phi_\alpha}\downarrow & \searrow^{L_{\underline{a}}(\pi_\alpha)} & \\ \varinjlim_{\alpha\in D} L_{\underline{a}}(N^\alpha) & \xrightarrow{\vartheta} & L_{\underline{a}}(N^\infty) \end{array} \qquad (C)$$

(where the vertical homomorphism ϕ_α is the natural one). (See (1.32).) I shall show that ϑ is an isomorphism.

Suppose $x \in \varinjlim_{\alpha\in D} L_{\underline{a}}(N^\alpha)$ is such that $\vartheta(x) = 0$. By (1.33)(i), there exists $\alpha \in D$ and an $x^\alpha \in L_{\underline{a}}(N^\alpha)$ s.t. $\phi_\alpha(x^\alpha) = x$. Therefore, $L_{\underline{a}}(\pi_\alpha)(x^\alpha) = 0$. Since $L_{\underline{a}}(\pi_\alpha)$ is the restriction of π_α to $L_{\underline{a}}(N^\alpha)$, $\pi_\alpha(x^\alpha) = 0$. By (1.33)(ii), there exists a $\beta>\alpha$ in D s.t. $\pi_\alpha^\beta(x^\alpha) = 0$. But $x^\alpha \in L_{\underline{a}}(N^\alpha)$, so $L_{\underline{a}}(\pi_\alpha^\beta)(x^\alpha) = 0$. It follows that $(\phi_\beta . L_{\underline{a}}(\pi_\alpha^\beta))(x^\alpha) = \phi_\alpha(x^\alpha) = 0$. Thus $x = 0$ and ϑ is monomorphic.

It remains to show that ϑ is epimorphic. So suppose $y \in L_{\underline{a}}(N^\infty)$. By (1.33)(i), there exists an $\alpha \in D$ and a $y^\alpha \in N^\alpha$ s.t. $\pi_\alpha(y^\alpha) = y$. Now there is an integer $K > 0$ s.t. $\underline{a}^K.y = 0$. Since A is Noetherian, $\underline{a}^K$ is f.-g., say by $a_1,\ldots,a_r$. Thus $\pi_\alpha(a_i y^\alpha) = 0$ for each $i = 1,\ldots,r$, and so, by

(1.33)(ii), for each $i = 1,\dots,r$, there exists $\delta_i > \alpha$ in D s.t. $\pi_\alpha^{\delta_i}(a_i y^\alpha) = 0$. Choose $\delta \in D$ s.t. $\delta > \delta_i$ for each $i = 1,\dots,r$. Then $\underline{a}^K . \pi_\alpha^\delta(y^\alpha) = 0$, and so $\pi_\alpha^\delta(y^\alpha) \in L_{\underline{a}}(N^\delta)$.

Now $y = \pi_\alpha(y^\alpha) = (\pi_\delta . \pi_\alpha^\delta)(y^\alpha) = L_{\underline{a}}(\pi_\delta)(\pi_\alpha^\delta(y^\alpha))$ (since $L_{\underline{a}}(\pi_\delta)$ is merely the restriction of π_δ to $L_{\underline{a}}(N^\delta)$), and so $y = \Phi(\phi_\delta(\pi_\alpha^\delta(y^\alpha)))$.

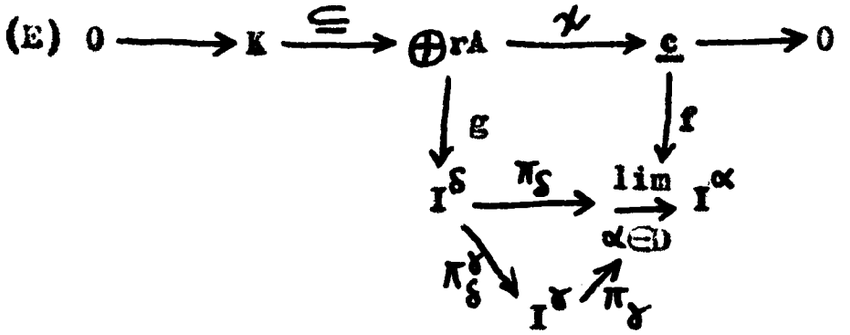
Thus Φ is an epimorphism and the Proposition is proved. //

I wish to appeal to a result of Grothendieck, but first I must prove the following

(11.5)**Proposition.** (This result appears as Exercise 8 to Chapter I of Cartan and Eilenberg (6).) Suppose that a set D together with a relation $<$ in D is a directed set. Suppose also that $\{I,\pi\}$ is a direct system of A-modules and -homomorphisms over D such that, for each $\alpha \in D$, I^α is injective. Then $\varinjlim_{\alpha\in D} I^\alpha$ is also an injective A-module.

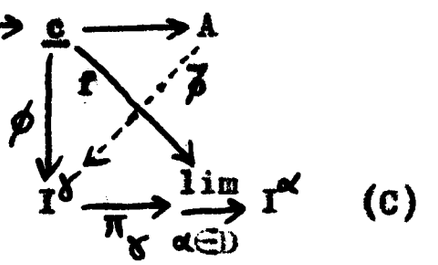
Proof. I use the criterion of Theorem 3.2 of Chapter I of Cartan and Eilenberg (6). Suppose that $\underline{c}$ is an ideal of A, and $f : \underline{c} \longrightarrow \varinjlim_{\alpha\in D} I^\alpha$ is an A-homomorphism: I have to show that f can be extended to a homomorphism $h : A \longrightarrow \varinjlim_{\alpha\in D} I^\alpha$. Since A is Noetherian, $\underline{c}$ is f.-g., say by $c_1,\dots,c_r$. By (1.33)(i), there exists $\delta \in D$ and $n_1^\delta,\dots,n_r^\delta \in I^\delta$ s.t. $\pi_\delta(n_i^\delta) = f(c_i)$ for each $i = 1,\dots r$. ($\pi_\delta : I^\delta \longrightarrow \varinjlim_{\alpha\in D} I^\alpha$ is, of course, the natural homomorphism.)

Let $e_1,\dots,e_r$ be free generators of the free A-module $\oplus rA$, let $\psi : \oplus rA \longrightarrow \underline{c}$ be the A-epimorphism defined by $\psi(e_i) = c_i$ for $i = 1,\dots,r$, and let $K = \ker \psi$. Define $g : \oplus rA \longrightarrow I^\delta$ by $g(e_i) = n_i^\delta$ for $i = 1,\dots,r$. Then:-



(C) (where $\delta < \gamma \in D$).

Now K is a f.-g. A-module, generated by, say, $z_1,\ldots,z_t$, and $\pi_\delta(g(z_i)) = 0$ for all $i = 1,\ldots,t$. Using (1,33)(ii), there exists $\delta < \gamma \in D$ such that $\pi^\gamma_\delta(g(z_i)) = 0$ for each $i = 1,\ldots,t$. Consider $\pi^\gamma_\delta.g : \oplus rA \longrightarrow I^\gamma$. $K \subseteq \ker(\pi^\gamma_\delta.g)$, so $\pi^\gamma_\delta.g$ induces a homomorphism $\phi : \underline{c} \longrightarrow I^\gamma$ s.t. $\phi(c_i) = \pi^\gamma_\delta(n_i^\delta)$ for each $i = 1,\ldots,r$. Thus:-



Since I^γ is injective, there exists an A-homomorphism $\bar{\phi} : A \longrightarrow I^\gamma$ extending ϕ. Define $h = \pi_\gamma.\bar{\phi} : A \longrightarrow \varinjlim_{\alpha\in D} I^\alpha$: this extends f as required. //

(11.6)Theorem. Suppose that $\{N,\pi\}$ is a direct system of A-modules and -homomorphisms over the directed set $\{D,<\}$. Then, for each $i \geqslant 0$, the natural A-homomorphisms:- $\varinjlim_{\alpha\in D} R^iL_{\underline{a}}(N^\alpha) \longrightarrow R^iL_{\underline{a}}(\varinjlim_{\alpha\in D} N^\alpha)$ are all isomorphisms.

Proof. Since the Abelian category $\mathcal{C}_A$ satisfies the axiom AB 5) of Chap. I of Grothendieck (11), and possesses a generator (see §1.9 of (11)), the result follows immediately from the statement and proof of Proposition 3.6.2 of (11) together with (11.4) and (11.5). //

So far, the development of the theory of the functor $L_{\underline{a}}$ has been purely technical: however, I've now reached the stage at which more

interesting results can be proved.

(11.7) Theorem. Suppose M is an A-module of finite Krull Dimension s. Then:-

$R^i L_{\underline{a}}(M) = 0$ whenever $i > s$.

Proof. This will be by induction on s.

Suppose, firstly, that $s = 0$. It follows from (1.34) & (11.6) that it is sufficient to prove the result when M is a non-zero, f.-g., A-module. By (1.12), there is a finite chain of submodules of M:-

$0 \subsetneq M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots \subsetneq M_r = M$ such that, for each $i = 1,\dots,r$, $M_i/M_{i-1} \cong A/\underline{p}_i$ for some prime ideal $\underline{p}_i$ of A. Clearly, each of $\underline{p}_1,\dots,\underline{p}_r \in V_A(M)$, so, since $K\text{-}\dim_A M = 0$, each of $\underline{p}_1,\dots,\underline{p}_r$ must be maximal. It is now merely an easy inductive argument to see that it is sufficient to prove that $R^i L_{\underline{a}}(A/\underline{m}) = 0$ whenever $i > 0$ and $\underline{m}$ is a maximal ideal of A.

Consider a minimal injective resolution for $A/\underline{m}$:-

$$(E)\quad 0 \longrightarrow A/\underline{m} \longrightarrow E_A^0(A/\underline{m}) \longrightarrow E_A^1(A/\underline{m}) \longrightarrow \dots \longrightarrow E_A^n(A/\underline{m}) \longrightarrow \dots .$$

By (6.7), if $\underline{p}$ is a prime ideal of A different from $\underline{m}$, then $\mu^i(\underline{p}, A/\underline{m}) = 0$ for all $i \geqslant 0$, while $\mu^i(\underline{m}, A/\underline{m})$ is finite for all $i \geqslant 0$. Thus, for each $i \geqslant 0$, $E_A^i(A/\underline{m}) \cong \oplus \mu^i(\underline{m}, A/\underline{m}) E_A(A/\underline{m})$. Two cases now arise.

(a) When $\underline{a} \not\subseteq \underline{m}$.

In this case, there exists $x \in \underline{a} - \underline{m}$, and, for all $j > 0$, multiplication by x^j provides an automorphism of $E_A(A/\underline{m})$ (by (5.2)(vi)), and so an automorphism of $E_A^i(A/\underline{m})$ for each $i \geqslant 0$. Thus $L_{\underline{a}}(E_A^i(A/\underline{m})) = 0$ for each $i \geqslant 0$, and consequently $R^i L_{\underline{a}}(A/\underline{m}) = 0$ for all $i \geqslant 0$.

(b) When $\underline{a} \subseteq \underline{m}$.

By (5.2)(vi), every element of $E_A(A/\underline{m})$ is annihilated by some power of $\underline{m}$, and so, since $\underline{a} \subseteq \underline{m}$, $L_{\underline{a}}(E_A^i(A/\underline{m})) = E_A^i(A/\underline{m})$ for each $i \geqslant 0$. Consequently, $R^i L_{\underline{a}}(A/\underline{m}) = 0$ for all $i > 0$.

This concludes the proof for the case $s = 0$.

Suppose, inductively, that $k > 0$ and that whenever N is an A-module with $K\text{-Dim}_A N = s < k$, then $R^i L_{\underline{a}}(N) = 0$ for all $i > s$. Suppose M is an A-module with Krull Dimension k. Since any submodule of M will have Krull Dimension $\leqslant k$, it follows from (1.34) and (11.6) that it is sufficient to assume that M is (non-zero and) f.-g. By (1.12), there is a finite chain of submodules of M:- $0 = M_0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \ldots \subsetneqq M_r = M$ s.t., for each $i = 1,\ldots,r$, $M_i/M_{i-1} \cong A/\underline{p}_i$ for some prime ideal $\underline{p}_i$ of A. Clearly, each of $\underline{p}_1,\ldots,\underline{p}_r \in V_A(M)$, and so it follows easily that it is sufficient to prove that, if $\underline{p}$ is a prime ideal of A of coheight k, then $R^i L_{\underline{a}}(A/\underline{p}) = 0$ for all $i > k$. (The inductive hypothesis is used here, of course.)

Let K be the field of fractions of the integral domain $A/\underline{p}$. K is an $A/\underline{p}$-module. Regard $A/\underline{p}$ and K as A-modules by means of the natural homomorphism $A \longrightarrow A/\underline{p}$. By (3.17), the inclusion $(E)0 \longrightarrow A/\underline{p} \xrightarrow[\subseteq]{d^{-1}} K$ is the beginning of the Cousin Complex for the A-module $A/\underline{p}$ (up to isomorphism of complexes of A-modules). Let $H = \operatorname{coker} d^{-1}$, so that:-

(E) $0 \longrightarrow A/\underline{p} \xrightarrow[\subseteq]{d^{-1}} K \longrightarrow H \longrightarrow 0$ (*). I show that $R^i L_{\underline{a}}(K) = 0$ for all $i > 0$. Two cases arise.

(a) When $\underline{a} \not\subseteq \underline{p}$.

In this case, there exists $x \in \underline{a} - \underline{p}$, and multiplication by x on K provides an automorphism of K. Consequently, if $i \geqslant 0$, since $R^i L_{\underline{a}}$ is an A-linear functor, multiplication by x on $R^i L_{\underline{a}}(K)$ provides an automorphism of $R^i L_{\underline{a}}(K)$. But, if $z \in R^i L_{\underline{a}}(K)$, then (from the construction of $R^i L_{\underline{a}}(K)$ by means of applying the functor $L_{\underline{a}}$ to an injective resolution of K), it follows that there is an integer $j > 0$ s.t. $\underline{a}^j.z = 0$. Thus $x^j.z = 0$, and so $z = 0$. Thus $R^i L_{\underline{a}}(K) = 0$ for all $i \geqslant 0$.

(b) When $\underline{a} \subseteq \underline{p}$.

Consider a minimal injective resolution for the A-module K:-

(E) $0 \longrightarrow K \longrightarrow E_A^0(K) \longrightarrow E_A^1(K) \longrightarrow \ldots \longrightarrow E_A^j(K) \longrightarrow \ldots$.

If $\underline{q}$ is a prime ideal of A s.t., for some $i \geqslant 0$, $\mu^i(\underline{q}, K) > 0$, then, by (6.7), $\underline{q} \in \mathrm{Supp}_A(K) = V_A(A/\underline{p})$, so $\underline{q} \supseteq \underline{p}$. Thus, by (5.2)(vi), since $\underline{a} \subseteq \underline{p}$, every element of $E_A^j(K)$ is annihilated by some power of $\underline{a}$ (for each $j \geqslant 0$), and consequently $L_{\underline{a}}(E_A^j(K)) = E_A^j(K)$. Thus $R^i L_{\underline{a}}(K) = 0$ for all $i > 0$.

Thus, in any case, $R^i L_{\underline{a}}(K) = 0$ for all $i > 0$.

Now consider again the exact sequence (*). By (3.7)(iv), $K\text{-}\dim_A H \leqslant k-1$. Thus, by the inductive hypothesis, $R^i L_{\underline{a}}(H) = 0$ for all $i \geqslant k$. But (*) induces:-

(E) $\ldots \longrightarrow R^k L_{\underline{a}}(K) \longrightarrow R^k L_{\underline{a}}(H) \longrightarrow R^{k+1} L_{\underline{a}}(A/\underline{p}) \longrightarrow R^{k+1} L_{\underline{a}}(K) \longrightarrow R^{k+1} L_{\underline{a}}(H) \longrightarrow R^{k+2} L_{\underline{a}}(A/\underline{p}) \longrightarrow \ldots$, and since $R^i L_{\underline{a}}(K) = 0$ for all $i > 0$ and $R^i L_{\underline{a}}(H) = 0$ for all $i \geqslant k$, it follows that $R^i L_{\underline{a}}(A/\underline{p}) = 0$ whenever $i > k$. This completes the proof of the inductive step.

The Theorem now follows by induction. //

I am now going to study the functor $L_{\underline{a}}$ in the context of "ring extensions". Suppose that (A &) B are rings and $f: A \longrightarrow B$ is a ring homomorphism (so that $f(1_A) = 1_B$). I shall use the notation of (3.8) concerning extensions in B of ideals of A and contractions in A of ideals of B. An arbitrary B-module can, of course, be regarded as an A-module by "restriction of scalars", and the next three results are concerned with this idea.

(11.8)Lemma. Suppose that Q is an indecomposable injective B-module. When

Q is regarded as an A-module by means of f, $R^iL_{\underline{a}}(Q) = 0$ for all $i > 0$.

Proof. Suppose $Q \cong E_B(B/\underline{q})$, where $\underline{q}$ is a prime ideal of B. Two cases arise.

(a) When $\underline{a} \not\subseteq \underline{q}^c$.

In this case there exists $a \in \underline{a}$ s.t. $f(a) \notin \underline{q}$. Thus multiplication by a on Q provides an A-automorphism, and since each $R^iL_{\underline{a}}$ is an A-linear functor, multiplication by a on $R^iL_{\underline{a}}(Q)$ provides an A-automorphism (for each $i \geqslant 0$). But, if $z \in R^iL_{\underline{a}}(Q)$, there exists a positive integer J s.t. $\underline{a}^J.z = 0$, and so $a^J.z = 0$. Therefore $z = 0$. Thus $R^iL_{\underline{a}}(Q) = 0$ for all $i \geqslant 0$.

(b) When $\underline{a} \subseteq \underline{q}^c$.

Consider a minimal injective resolution for Q regarded as an A-module:-

$$(E)\quad 0 \longrightarrow Q \longrightarrow E_A^0(Q) \longrightarrow E_A^1(Q) \longrightarrow \dots \longrightarrow E_A^i(Q) \longrightarrow \dots .$$

If $\underline{p}$ is a prime ideal of A s.t., for some $i \geqslant 0$, $\mu^i(\underline{p}, Q) > 0$, then $\underline{p} \in \mathrm{Supp}_A(Q)$ (by (6.7)). This implies that $\underline{p} \supseteq \underline{q}^c$, for otherwise there exists $x \in \underline{q}^c - \underline{p}$, so that $f(x) \in \underline{q}$ and every element of Q is annihilated by some power of x, and so $\underline{p} \notin \mathrm{Supp}_A(Q)$, a contradiction.

Thus, using (5.2)(vi), for each $i \geqslant 0$, every element of $E_A^i(Q)$ is annihilated by some power of $\underline{a}$, and so $L_{\underline{a}}(E_A^i(Q)) = E_A^i(Q)$. It follows that $R^iL_{\underline{a}}(Q) = 0$ for all $i > 0$.

Thus, in any case, $R^iL_{\underline{a}}(Q) = 0$ for all $i > 0$. //

(11.9)Corollary. Suppose that I is an injective B-module. When I is regarded as an A-module by means of f, $R^iL_{\underline{a}}(I) = 0$ for all $i > 0$.

Proof. By (5.2)(iii), I is B-isomorphic (and so A-isomorphic) to a direct sum of indecomposable injective B-modules. Since, for each $i > 0$, $R^iL_{\underline{a}}$ is an additive functor, it follows from (11.8) that $R^iL_{\underline{a}}(J) = 0$ whenever

$i > 0$ and J is an injective B-module which is isomorphic to a direct sum of a finite number of indecomposable injective B-modules. The result now follows from (1.35) and (11.6). //

(11.10)Theorem. The connected right sequences of covariant functors from $\mathscr{C}_B$ to $\mathscr{C}_A$, $\{R^i L_{\underline{a}}\}_{i\geq 0}$ and $\{R^i L_{\underline{a}} e\}_{i\geq 0}$, are isomorphic. (If N is a B-module, then, for the first sequence of functors, N is regarded as an A-module by means of f, and then the A-module $R^i L_{\underline{a}}(N)$ is formed, while, for the second sequence of functors, the B-module $R^i L_{\underline{a}} e(N)$ is itself regarded as an A-module by means of f.)

Proof. I shall use the Corollary to Theorem 10 of Chapter 6 of Northcott (22). Note that, if $h:N \longrightarrow P$ is a homomorphism of B-modules, it is, of course, a homomorphism of A-modules when N & P are regarded as A-modules by means of f. If I is an injective B-module, then certainly $R^i L_{\underline{a}} e(I) = 0$ when $i > 0$, while $R^i L_{\underline{a}}(I) = 0$ when $i > 0$ by (11.9). Thus it remains only to show that the functors $L_{\underline{a}}:\mathscr{C}_B \longrightarrow \mathscr{C}_A$ and $L_{\underline{a}} e:\mathscr{C}_B \longrightarrow \mathscr{C}_A$ are naturally equivalent.

Suppose, then, that $g:M \longrightarrow N$ is a homomorphism of B-modules. $L_{\underline{a}}(M) = \{m \in M \mid$ there is a positive integer n s.t. $\underline{a}^n.m = 0\}$, and $L_{\underline{a}} e(M) = \{m \in M \mid$ there is a positive integer n s.t. $(\underline{a}^e)^n.m = 0\}$, regarded as an A-module. It is easy to see that $L_{\underline{a}}(M) = L_{\underline{a}} e(M)$, and that $L_{\underline{a}}(g)$ and $L_{\underline{a}} e(g):L_{\underline{a}}(M) \longrightarrow L_{\underline{a}}(N)$ are both merely the restriction of g to the submodule $L_{\underline{a}}(M)$ of M. The result now follows. //

I shall also need the following result, which is also concerned with connected sequences of functors.

(11.11)Theorem. Suppose that S is an m.c.s. of A. Then the connected right sequences of covariant functors from $\mathscr{C}_A$ to $\mathscr{C}_{S^{-1}A}$; $\{S^{-1}(R^iL_{\underline{a}}(\ ))\}_{i\geq 0}$ and $\{R^iL_{S^{-1}\underline{a}}(S^{-1}(\ ))\}_{i\geq 0}$, are isomorphic.

Proof. Again, I shall use the Corollary to Theorem 10 of Chapter 6 of Northcott (22). By (5.5), it is sufficient to prove that the functors $S^{-1}(L_{\underline{a}}(\ ))$ and $L_{S^{-1}\underline{a}}(S^{-1}(\ ))$ from $\mathscr{C}_A$ to $\mathscr{C}_{S^{-1}A}$ are naturally equivalent.

Suppose, then, that $g:M\longrightarrow N$ is a homomorphism of A-modules. Let $\phi(M):L_{\underline{a}}(M)\longrightarrow M$ be the natural inclusion. Then:-

$$\begin{array}{ccc} S^{-1}(L_{\underline{a}}(M)) & \xrightarrow{S^{-1}(\phi(M))} & S^{-1}(M) \\ \downarrow{\scriptstyle S^{-1}(L_{\underline{a}}(g))} & & \downarrow{\scriptstyle S^{-1}(g)} \\ S^{-1}(L_{\underline{a}}(N)) & \xrightarrow{S^{-1}(\phi(N))} & S^{-1}(N) \end{array}$$

(C), so it clearly is sufficient to show that, for an A-module M, $S^{-1}(\phi(M))(S^{-1}(L_{\underline{a}}(M))) = L_{S^{-1}\underline{a}}(S^{-1}(M))$ (*)

Suppose $x/s \in S^{-1}M$ is s.t. $x \in L_{\underline{a}}(M)$. Then there exists a positive integer n s.t. $\underline{a}^n.x = 0$. Since $(S^{-1}\underline{a})^n = S^{-1}(\underline{a}^n)$, it follows immediately that $(S^{-1}\underline{a})^n.(x/s) = 0$, and so L.H.S. $\subseteq$ R.H.S. in (*).

Conversely, suppose $x/s \in L_{S^{-1}\underline{a}}(S^{-1}M)$. Then there exists a positive integer n s.t. $(S^{-1}\underline{a})^n.(x/s) = 0$. Suppose $\underline{a}^n$ is generated by $b_1,\ldots,b_r$. Since $(S^{-1}\underline{a})^n = S^{-1}(\underline{a}^n)$, it follows that, for each $i = 1,\ldots,r$, there exists $s_i \in S$ s.t. $s_ib_ix = 0$. Let $t = s_1.....s_r$. Then $\underline{a}^n(tx) = 0$, and $x/s = tx/ts \in$ L.H.S. of (*). Thus R.H.S. $\subseteq$ L.H.S. in (*).

This completes the proof of the Theorem. //

Recall that the latter part of Chapter 2 (from (2.20) onwards) was devoted to a detailed exposition of some of the homological algebra ideas of §§4 & 6 of Hartshorne (13), culminating in some results about an

additive, A-linear, covariant, left exact functor T from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$. These ideas have been included for use in Chapter 12, where, among other things, the above-mentioned results will be applied to the additive, A-linear, covariant, left exact functor $L_{\underline{a}}$ from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$ (for certain choices of A and $\underline{a}$). Consequently, it seems appropriate to end this rather technical discussion of the functor $L_{\underline{a}}$ with a statement of the results of (2.31) & (2.32) in the particular case in which the additive, A-linear, covariant, left exact functor from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$ concerned is $L_{\underline{a}}$.

(11.12)Corollary. (of (2.31) and (2.32)). Suppose M and N are A-modules. Then, for each $i \geqslant 0$ and $s \geqslant 0$, there is a pairing:-

$$R^iL_{\underline{a}}(M) \times \mathrm{Ext}^s_A(M,N) \longrightarrow R^{i+s}L_{\underline{a}}(N).$$

Furthermore, if $\varkappa^{i,s}(M):\mathrm{Ext}^s_A(M,N) \longrightarrow \mathrm{Hom}_A(R^iL_{\underline{a}}(M), R^{i+s}L_{\underline{a}}(N))$ and $\vartheta^{i,s}(M):R^iL_{\underline{a}}(M) \longrightarrow \mathrm{Hom}_A(\mathrm{Ext}^s_A(M,N), R^{i+s}L_{\underline{a}}(N))$ are the A-homomorphisms induced by the above pairing, then, regarding N as fixed,:-

(i) the $\varkappa^{i,s}(M)$ constitute a natural transformation of the functor $\mathrm{Ext}^s_A(\ ,N)$ to $\mathrm{Hom}_A(R^iL_{\underline{a}}(\ ), R^{i+s}L_{\underline{a}}(N))$ (from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$);

& (ii) the $\vartheta^{i,s}(M)$ constitute a natural transformation of the functor $R^iL_{\underline{a}}$ to $\mathrm{Hom}_A(\mathrm{Ext}^s_A(\ ,N), R^{i+s}L_{\underline{a}}(N))$ (from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$). //

CHAPTER 12

Applications of Local Cohomology Theory and the Study of Gorenstein Modules over Complete Cohen-Macaulay Local Rings.

(Reminder:"ring" means "commutative Noetherian ring with a non-zero identity".)

The main aim of this Chapter is the complete determination of all Gorenstein modules over a complete Cohen-Macaulay local ring. Recall that the existence of Gorenstein modules over any complete Cohen-Maculay local ring was established by (10.4) and (10.10). The main results of this Chapter are as follows:-

Let A be a complete Cohen-Macaulay local ring of Krull dimension m. Bearing in mind (10.10), let B be a Gorenstein local ring of Krull Dimension n such that there exists a surjective ring homomorphism $\phi : B \to A$. Then:-

(i) $W = \mathrm{Ext}_B^{n-m}(A,B)$, which has a natural structure as a Gorenstein A-module (see Chapter 10), depends, up to A-isomorphism, only on A and not on B;

& (ii) the Gorenstein A-modules are precisely those A-modules isomorphic to the direct sum of a (strictly positive) finite number of copies of W.

As well as the work in Chapter 10, two of the main tools used to obtain these results are the local cohomology functors discussed in Chapter 11, and the "duality" expressed in Corollary 4.3 of Matlis (17). I should like, of course, to be able to prove the corresponding results for a not necessarily complete Cohen-Macaulay local ring. Unfortunately, there are two strong technical reasons why the arguments used in this Chapter will not work in the case of a non-complete Cohen-Macaulay local ring.

As a prelude to the proofs of (i) and (ii), I give a discussion of Gorenstein modules over a Gorenstein local ring A. In this case, the existence of Gorenstein A-modules is obvious, for A itself is a Gorenstein A-module. Also, the complete determination of all Gorenstein A-modules is comparatively easy: in fact, I shall prove that the Gorenstein A-modules are precisely the (non-zero) finitely-generated free A-modules.

I start with an easy Lemma.

(12.1)Lemma. Let R be local ring with maximal ideal $\underline{r}$ and Krull Dimension r. Suppose M_1 & M_2 are non-zero, f.-g., R-modules, and $M = M_1 \oplus M_2$. Then:-

(i) For each non-negative integer i and each $\underline{p} \in V(R)$,

$\mu^i(\underline{p},M) = \mu^i(\underline{p},M_1) + \mu^i(\underline{p},M_2)$;

(ii) M is Cohen-Macaulay of Krull Dimension s if and only if each of M_1 and M_2 is Cohen-Macaulay of Krull Dimension s;

& (iii) M is Gorenstein if and only if each of M_1 and M_2 is Gorenstein.

Proof. (i) Suppose $i \geqslant 0$ is an integer and $\underline{p} \in V(R)$. Then $M_{\underline{p}} \cong M_{1\underline{p}} \oplus M_{2\underline{p}}$ as $R_{\underline{p}}$-modules. Let $k(\underline{p})$ denote the residue field of the local ring $R_{\underline{p}}$. By (2.19), $\mathrm{Ext}^i_{R_{\underline{p}}}(k(\underline{p}),M_{\underline{p}}) \cong \mathrm{Ext}^i_{R_{\underline{p}}}(k(\underline{p}),M_{1\underline{p}}) \oplus \mathrm{Ext}^i_{R_{\underline{p}}}(k(\underline{p}),M_{2\underline{p}})$, so by (6.7), $\mu^i(\underline{p},M) = \mu^i(\underline{p},M_1) + \mu^i(\underline{p},M_2)$.

(ii) Let $k = R/\underline{r}$. By (2.19), for each $i \geqslant 0$, $\mathrm{Ext}^i_R(k,M) \cong \mathrm{Ext}^i_R(k,M_1) \oplus \mathrm{Ext}^i_R(k,M_2)$, while, if $\underline{p} \in V(R)$, $M_{\underline{p}} \cong M_{1\underline{p}} \oplus M_{2\underline{p}}$ as $R_{\underline{p}}$-modules. It follows that:-

$\mathrm{codh}_R M = \min.(\mathrm{codh}_R M_1, \mathrm{codh}_R M_2)$, and $V(M) = V(M_1) \cup V(M_2)$.

Suppose M is Cohen-Macaulay with $\mathrm{codh}_R M = \text{K-Dim}_R M = s$. Then $\mathrm{codh}_R M_1 \geqslant s$, $\mathrm{codh}_R M_2 \geqslant s$, while $\text{K-Dim}_R M_1 \leqslant s$, $\text{K-Dim}_R M_2 \leqslant s$. Thus each of M_1 and M_2 is Cohen-Macaulay of Krull Dimension s.

Conversely, suppose M_1 and M_2 are both Cohen-Macaulay of Krull Dimension s. Then $\operatorname{codh}_R M = s$: Suppose $K\text{-}\dim_R M = \operatorname{ht}_M \underline{r} > s$. Then there is a prime ideal $\underline{q}$ of $V_R(M)$ s.t. $\operatorname{coht} \underline{q} > s$. $\underline{q}$ must belong to one of $V_R(M_1)$ and $V_R(M_2)$, thus contradicting the fact that $K\text{-}\dim_R(M_1) = s = K\text{-}\dim_R(M_2)$. Thus $K\text{-}\dim_R M = s$, and M is Cohen-Macaulay.

(iii) Suppose M is Gorenstein. In particular, M is Cohen-Macaulay of Krull Dimension r, and so, by part (ii), each of M_1 & M_2 is Cohen-Macaulay of Krull Dimension r. Thus $\operatorname{ht}_{M_1} \underline{r} = \operatorname{ht}_{M_2} \underline{r} = \operatorname{ht}_M \underline{r} = r$. By part (i), for each $i \geqslant 0$, $\mu^i(\underline{r}, M) = \mu^i(\underline{r}, M_1) + \mu^i(\underline{r}, M_2)$. It now follows from (6.27)(iii) and (6.9) that each of M_1 & M_2 is Gorenstein.

Conversely, if each of M_1 and M_2 is Gorenstein, then, by part (ii) (and (6.27)), M is Cohen-Macaulay of Krull Dimension r, and $\operatorname{ht}_M \underline{r} = \operatorname{ht}_{M_1} \underline{r} = \operatorname{ht}_{M_2} \underline{r} = r$. It follows from part (i) and (6.27)(iii) that M is Gorenstein. //

I now use the ideas of §2 of Levin and Vasconcelos (16) to determine all Gorenstein modules over a Gorenstein local ring.

(12.2)Theorem. (See Theorem 2.2 of Levin and Vasconcelos (16).) Let A be Gorenstein local ring Krull Dimension n and with maximal ideal $\underline{m}$. Then the Gorenstein A-modules are precisely the non-zero, f.-g., free A-modules.

Proof. Let M be a Gorenstein A-module, so that, in particular, $\operatorname{codh}_A M = K\text{-}\dim_A M = \operatorname{inj.dim.}_A M = \operatorname{codh}_A A = K\text{-}\dim A = n$, and $(0:M)_A = 0$, (and, of course, M is a non-zero, f.-g., A-module).

If $n = 0$, then A is an Artin local ring whose only prime ideal is $\underline{m}$, and both A & M are injective A-modules (by (6.27)). Since M is of finite

length it is isomorphic to a direct sum of a finite number of copies of $E_A(A/\underline{m})$, and by (5.11), $A \cong E_A(A/\underline{m})$. Thus M is a f.-g. free A-module.

Thus I suppose $n > 0$. Let F be a f.-g. free A-module s.t. there exists an A-epimorphism $\phi : F \longrightarrow M$, and let $K = \ker \phi$, so that:-

(E) $0 \longrightarrow K \xrightarrow{\subseteq} F \xrightarrow{\phi} M \longrightarrow 0$ (*). If $K = 0$, ϕ is an isomorphism, and M is free, so suppose $K \neq 0$.

By (12.1), F is Gorenstein, so both F & M are Cohen-Macaulay A-modules of Krull Dimension n. It follows that K is Cohen-Macaulay of Krull Dimension n. Since $\text{inj.dim.}_A M = \text{inj.dim.}_A F = n$, $\text{Ext}_A^{n+2}(A/\underline{p}, K) = 0$ for all prime ideals $\underline{p}$ of A, and so it follows from (6.7) that $\text{inj.dim.}_A K$ is finite. By (6.13) and (6.27), K is a Gorenstein A-module.

Next, I show that $\text{Ext}_A^1(M,K) = 0$, and to do this I prove, by induction on r, that if N is a non-zero, f.-g., A-module s.t. there exists an N-sequence of length r, then $\text{Ext}_A^{n-i}(N,K) = 0$ for all i s.t. $0 \leqslant i < r$. Recall that $\text{inj.dim.}_A K = n$.

When $r = 1$, suppose (x_1) is an N-sequence. From:-

(E) $0 \longrightarrow N \xrightarrow{x_1} N \longrightarrow N/x_1N \longrightarrow 0$, I deduce:-

(E) $\text{Ext}_A^n(N/x_1N,K) \longrightarrow \text{Ext}_A^n(N,K) \xrightarrow{x_1} \text{Ext}_A^n(N,K) \longrightarrow 0$. Since N and K are f.-g., it follows from Nakayama's Lemma that $\text{Ext}_A^n(N,K) = 0$.

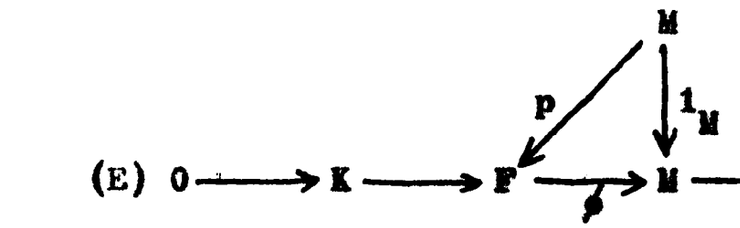
Now suppose, inductively, that $1 < k \leqslant n$, and that the result is true when $1 \leqslant r < k$. Suppose N is a f.-g. A-module and $(x_1,\ldots,x_k)$ is an N-sequence. By the inductive hypothesis, since $(x_1,\ldots,x_{k-1})$ is an N-sequence and $(x_2,\ldots,x_k)$ is an N/x_1N-sequence, $\text{Ext}_A^{n-i}(N,K) = 0$ for $0 \leqslant i < k-1$ and $\text{Ext}_A^{n-(k-2)}(N/x_1N,K) = 0$. From:-

(E) $0 \longrightarrow N \xrightarrow{x_1} N \longrightarrow N/x_1N \longrightarrow 0$, I deduce:-

(E) $\text{Ext}_A^{n-(k-1)}(N/x_1N,K) \longrightarrow \text{Ext}_A^{n-(k-1)}(N,K) \xrightarrow{x_1} \text{Ext}_A^{n-(k-1)}(N,K) \longrightarrow 0$,

so that, again by Nakayama's Lemma, $\mathrm{Ext}_A^{n-(k-1)}(N,K) = 0$. Thus the inductive step is complete, and the (intermediate) result follows by induction. Therefore $\mathrm{Ext}_A^1(M,K) = 0$.

Consequently, from (*), I deduce (E)$\mathrm{Hom}_A(M,F) \longrightarrow \mathrm{Hom}_A(M,M) \longrightarrow 0$, so there is an A-homomorphism $p : M \longrightarrow F$ s.t. $\phi \cdot p = 1_M$.



. Thus (*) splits, and so M is projective. Since A is local and M is f.-g., it follows from (2.9) that M is a f.-g. free A-module.

Thus any Gorenstein A-module is a f.-g. free A-module, and, of course, it is obvious from (12.1) that any (non-zero) f.-g. free A-module is Gorenstein. //

I now begin to marshall the ammunition for the attack on the main problems of this Chapter.

(12.3) I am going to need to use the "duality" developed by Matlis in (17): this is partially expressed in Corollary 4.3 of (17). More precisely, suppose A is a complete local ring with maximal ideal $\underline{m}$, and let $E = E_A(A/\underline{m})$. Let $\underline{\mathcal{C}}_A^d$ denote the category of Artinian A-modules and all A-homomorphisms between them: recall that $\underline{\mathcal{C}}_A^f$ denotes the category of f.-g. A-modules and all A-homomorphisms between them. Let $D : \underline{\mathcal{C}}_A^{op} \longrightarrow \underline{\mathcal{C}}_A$ be the additive, A-linear, contravariant, exact functor $\mathrm{Hom}_A(\ ,E)$. In (17), Matlis shows that, if $M \in \underline{\mathcal{C}}_A^f$, $D(M) \in \underline{\mathcal{C}}_A^d$, while if $M \in \underline{\mathcal{C}}_A^d$, $D(M) \in \underline{\mathcal{C}}_A^f$.

Also, for an A-module M, let $\lambda(M) : M \longrightarrow DD(M) = \mathrm{Hom}_A(\mathrm{Hom}_A(M,E),E)$ be the A-homomorphism defined by $(\lambda(M)(m))(f) = f(m)$ (where $m \in M$, $f \in D(M)$).

The $\lambda(M)$ constitute a natural transformation of the identity functor to the functor DD (from $\underline{\mathcal{C}}_A$ to $\underline{\mathcal{C}}_A$). Now if $M \in \underline{\mathcal{C}}_A^f$, $DD(M) \in \underline{\mathcal{C}}_A^f$, while if $M \in \underline{\mathcal{C}}_A^d$, $DD(M) \in \underline{\mathcal{C}}_A^d$. The $\lambda(M)$ actually provide a natural equivalence of the identity functor and the functor DD from $\underline{\mathcal{C}}_A^f$ to $\underline{\mathcal{C}}_A^f$, and also provide a natural equivalence of the identity functor and the functor DD from $\underline{\mathcal{C}}_A^d$ to $\underline{\mathcal{C}}_A^d$. These two facts are not explicitly stated in (17), but they are easy consequences of Matlis' work in (17).

For, if M is a f.-g. A-module, then there exist f.-g. free A-modules F_0 and F_1 and an exact sequence $(E)F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$, while if N is an Artinian A-module, there exist, by (6.19), Artinian injective A-modules I^0 and I^1 (so that each of I^0 and I^1 will be isomorphic to a direct sum of a finite number of copies of E) and an exact sequence $(E)0 \rightarrow N \rightarrow I^0 \rightarrow I^1$. Consequently, it is sufficient to check that $\lambda(A)$ and $\lambda(E)$ are isomorphisms, and this follows easily from the fact that every A-endomorphism of E can be realized by multiplication by exactly one element of A. (See Theorem 3.7 of Matlis (17).)

Next, I apply the local cohomology functors studied in Chapter 11 to particular situations.

(12.4)Lemma. Let A be a Gorenstein local ring with maximal ideal $\underline{m}$ and Krull Dimension n. Then $R^i L_{\underline{m}}(A) = 0$ if $i \neq n$, and $R^n L_{\underline{m}}(A) \cong E_A(A/\underline{m})$.

Proof. By (5.13), the Cousin Complex for A, C(A), provides a minimal injective resolution for A, so I can use C(A) to calculate the $R^i L_{\underline{m}}(A)$. Let the Cousin Complex for A be:-

$$\underline{C(A)}: 0 \longrightarrow A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \longrightarrow A^{n-1} \xrightarrow{d^{n-1}} A^n \longrightarrow 0 \longrightarrow \cdots (E).$$

By (5.11) and (5.2)(vi), $L_{\underline{m}}(A^i) = 0$ for $0 \leq i < n$, and $L_{\underline{m}}(A^n) = A^n \cong$

$E_A(A/\underline{m})$. The result is now obvious. //

(12.5)Proposition. (See Theorem 6.3 of Hartshorne (13).) Let A be a Gorenstein local ring with maximal ideal $\underline{m}$ and Krull Dimension n. If M is an A-module, let:-

$\vartheta^n(M):R^nL_{\underline{m}}(M) \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(M,A),R^nL_{\underline{m}}(A))$ be the A-homomorphism induced by the pairing of (11.12) (with i = n and s = 0). Then $\vartheta^n(M)$ is an isomorphism whenever M is finitely-generated.

Proof. By (11.7), $R^{n+1}L_{\underline{m}}(X) = 0$ for all A-modules X, so $R^nL_{\underline{m}}:\underline{\mathcal{C}}_A \longrightarrow \underline{\mathcal{C}}_A$ is right exact. On the other hand, $R^nL_{\underline{m}}(A) \cong E_A(A/\underline{m})$ by (12.4), so $U = \mathrm{Hom}_A(\mathrm{Hom}_A(\ ,A),R^nL_{\underline{m}}(A)):\underline{\mathcal{C}}_A \longrightarrow \underline{\mathcal{C}}_A$ is also right exact. Suppose M is a f.-g. A-module: then there exist f.-g. free A-modules F_0 and F_1 and an exact sequence:- $(E)\ F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$.

By (11.12), the $\vartheta^n(X)$ (for $X \in \underline{\mathcal{C}}_A$) constitute a natural transformation of the functor $R^nL_{\underline{m}}$ to the functor U, so that:-

$$\begin{array}{ccccccccc}
(E) & R^nL_{\underline{m}}(F_1) & \longrightarrow & R^nL_{\underline{m}}(F_0) & \longrightarrow & R^nL_{\underline{m}}(M) & \longrightarrow & 0 & \\
 & \downarrow\vartheta^n(F_1) & & \downarrow\vartheta^n(F_0) & & \downarrow\vartheta^n(M) & & & \\
(E) & U(F_1) & \longrightarrow & U(F_0) & \longrightarrow & U(M) & \longrightarrow & 0 & (C)
\end{array}$$

It is thus sufficient to show that $\vartheta^n(F)$ is isomorphic whenever F is a f.-g. free A-module, and since both $R^nL_{\underline{m}}$ and U are additive A-linear functors, it is therefore sufficient to show that:-

$\vartheta^n(A):R^nL_{\underline{m}}(A) \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(A,A),R^nL_{\underline{m}}(A))$ is an isomorphism.

To do this, it will certainly be sufficient to show that, for each $x \in R^nL_{\underline{m}}(A)$, $(\vartheta^n(A)(x))(1_A) = x$ (where 1_A denotes the identity map on A) and for this in turn it will be sufficient to show that the image of $(x,1_A)$ is x in the pairing of (11.12):- $R^nL_{\underline{m}}(A) \times \mathrm{Hom}_A(A,A) \longrightarrow R^nL_{\underline{m}}(A)$.

To establish this, I must refer to the proof of Proposition (2.31) in which the above pairing first appears. I use the Cousin Complex for A, $\underline{C(A)} = \underline{C}$, which, by (5.13), provides a minimal injective resolution for A:-

$$\underline{C(A)} = \underline{C}: 0 \longrightarrow A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \longrightarrow \dots \longrightarrow A^{n-1} \xrightarrow{d^{n-1}} A^n \longrightarrow 0 \dots$$

Let $1_{\underline{C}}$ denote the identity morphism (of degree 0) $\in \mathrm{Hom}^0(\underline{C}, \underline{C})$: $\overline{1}_{\underline{C}}$ denotes the equivalence class in $\underset{\sim}{H}^0(\underline{C}, \underline{C})$ to which it belongs. (See (2.24) for Definitions.) First note that the image of $\overline{1}_{\underline{C}}$ under the composition of the natural A-isomorphisms of (2.26) and (2.27):-

$\underset{\sim}{H}^0(\underline{C}, \underline{C}) \longrightarrow H^0(\mathrm{Hom}_A(A, \underline{C})) \longrightarrow \mathrm{Hom}_A(A,A)$ is just 1_A. It now follows easily from the proof of (2.31) that the image of $(x, 1_A)$ in the pairing $R^n L_{\underline{m}}(A) \times \mathrm{Hom}_A(A,A) \longrightarrow R^n L_{\underline{m}}(A)$ is x, as required. //

(12.6)Lemma. Suppose that A is a local ring with maximal ideal $\underline{m}$, and that the non-zero, f.-g., A-module M is Cohen-Macaulay s.t. $K\text{-}\mathrm{Dim}_A M = s$. Let the Cousin Complex for M be:-

$$(E)\ \underline{C(M)}: 0 \longrightarrow M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \dots \longrightarrow M^{s-1} \xrightarrow{d^{s-1}} M^s \longrightarrow 0 \dots .$$

Then $R^s L_{\underline{m}}(M) \cong M^s$.

Proof. $\underline{C(M)}$ is exact, of course, and M^s is the final non-zero term of $\underline{C(M)}$. I derive from $\underline{C(M)}$ a family of s short exact sequences:-

$$\left.\begin{array}{lll} (E)\ 0 \longrightarrow M \longrightarrow M^0 \longrightarrow \mathrm{coker}\ d^{-1} \longrightarrow 0 & (i) & \\ (E)\ 0 \longrightarrow \mathrm{coker}\ d^{-1} \longrightarrow M^1 \longrightarrow \mathrm{coker}\ d^0 \longrightarrow 0 & (ii) & \\ \vdots & & \\ (E)\ 0 \longrightarrow \mathrm{coker}\ d^{s-3} \longrightarrow M^{s-1} \longrightarrow M^s \longrightarrow 0 & (s) & \end{array}\right\} (1)$$

If $i \geqslant 0$ and $0 \leqslant j < s$, then by (4.21), $\mathrm{Ext}^i_A(A/\underline{m}^k, M^j) = 0$, But, by (11.3), $R^i L_{\underline{m}}(M^j) \cong \varinjlim_k \mathrm{Ext}^i_A(A/\underline{m}^k, M^j) = 0$. Thus $R^i L_{\underline{m}}(M^j) = 0$ whenever

$i \geqslant 0$ and $0 \leqslant j < s$.

I can now deduce from equations (1):-

$$R^s L_{\underline{m}}(M) \cong R^{s-1} L_{\underline{m}}(\operatorname{coker} d^{-1})$$
$$R^{s-1} L_{\underline{m}}(\operatorname{coker} d^{-1}) \cong R^{s-2} L_{\underline{m}}(\operatorname{coker} d^{0})$$
$$\vdots$$
$$R^1 L_{\underline{m}}(\operatorname{coker} d^{s-3}) \cong L_{\underline{m}}(M^s)$$

, so that $R^s L_{\underline{m}}(M) \cong L_{\underline{m}}(M^s)$. By the construction of the Cousin Complex and (3.7)(vi), every element of M^s is annihilated by some power of $\underline{m}$, so that $L_{\underline{m}}(M^s) = M^s$ and the result follows. //

(12.7)Proposition. Let A be a Gorenstein local ring with maximal ideal $\underline{m}$ and Krull Dimension n. Then the connected left sequences of covariant functors from $\mathcal{C}_A^f$ to $\mathcal{C}_A$, $\left\{R^{n-i} L_{\underline{m}}\right\}_{i \geqslant 0}$ and $\left\{\operatorname{Hom}_A(\operatorname{Ext}_A^i(\ ,A), E_A(A/\underline{m}))\right\}_{i \geqslant 0}$, are isomorphic.

Proof. Let $D: \mathcal{C}_A \rightsquigarrow \mathcal{C}_A$ denote the functor $\operatorname{Hom}_A(\ , E_A(A/\underline{m}))$. Recall that, if X is any A-module, $R^j L_{\underline{m}}(X) = 0$ whenever $j > n$ (by (11.7)), and so both $R^n L_{\underline{m}}$ and $D(\operatorname{Hom}_A(\ ,A))$ are right exact. (I interpret $\operatorname{Ext}_A^0(\ ,\ )$ as $\operatorname{Hom}_A(\ ,\ )$, of course.)

By (11.12), (12.4), and (12.5), $R^n L_{\underline{m}}$ and $D(\operatorname{Hom}_A(\ ,A))$ from $\mathcal{C}_A^f$ to $\mathcal{C}_A$ are naturally equivalent. I plan to use Corollary (2.16). Bearing in mind that $R^0 L_{\underline{m}}$ and $L_{\underline{m}}$ are naturally equivalent, it is sufficient to show that whenever $i > 0$ and P is a f.-g. projective A-module, $D(\operatorname{Ext}_A^i(P,A)) = 0$ and $R^{n-i} L_{\underline{m}}(P) = 0$. The first of these is obvious, so I deal with the second.

By (2.9), P is free, and since each $R^{n-i} L_{\underline{m}}$ is an additive A-linear functor, it is sufficient to show that $R^j L_{\underline{m}}(A) = 0$ whenever $0 \leqslant j < n$.

By (11.3), $R^j L_{\underline{m}}(A) \cong \varinjlim_k \mathrm{Ext}^j_A(A/\underline{m}^k, A)$. But A is Cohen-Macaulay, and $\mathrm{ht}_A \underline{m}^k = n$. Thus, for each $k \geqslant 1$, $\mathrm{Ext}^j_A(A/\underline{m}^k, A) = 0$. Thus $R^j L_{\underline{m}}(A) = 0$, and the Proposition now follows from (2.16). //

I am now in a position to apply these ideas to obtain some interesting results. As in Chapter 10, the central object of study will be rings which are homomorphic images of Gorenstein local rings. From now until further notice, let A be complete local ring with maximal ideal $\underline{m}$ and Krull Dimension m, B a Gorenstein local ring with maximal ideal $\underline{n}$ and Krull Dimension n, $\phi : B \longrightarrow A$ a surjective ring homomorphism, and $\underline{b} = \ker \phi$. (Note that, by Corollary 2 to Theorem 15 of Cohen (7), any complete local ring can be expressed as a homomorphic image of a Gorenstein local ring in this manner.) The more interesting results will concern the case in which A is Cohen-Macaulay, but I'm not going to assume this for the moment: in fact A will only be assumed to be Cohen-Macaulay when this is explicitly stated. Considerable use will be made of the ideas of (10.1).

(12.8) Theorem. Suppose M is a f.-g. A-module. Then, for each $i \geqslant 0$, $\mathrm{Ext}^{n-i}_B(M,B)$, endowed with its natural A-module structure (see (10.1)), is A-isomorphic to $\mathrm{Hom}_A(R^i L_{\underline{m}}(M), E_A(A/\underline{m}))$.

Proof. Let $D : \mathcal{C}_A \rightsquigarrow \mathcal{C}_A$ be the functor $\mathrm{Hom}_A(\ , E_A(A/\underline{m}))$.

When M is regarded as a B-module by means of ϕ, it is f.-g., and so by (12.7), $R^i L_{\underline{n}}(M) \cong \mathrm{Hom}_B(\mathrm{Ext}^{n-i}_B(M,B), E_B(B/\underline{n}))$ (*) as B-modules. $R^i L_{\underline{n}}(M)$, $\mathrm{Ext}^{n-i}_B(M,B)$, and $\mathrm{Hom}_B(\mathrm{Ext}^{n-i}_B(M,B), E_B(B/\underline{n}))$ all have natural A-module structures, and the isomorphism of (*) is an A-isomorphism.

By (11.10), $R^i L_{\underline{n}}(M)$ is B-isomorphic, and consequently A-isomorphic, to

$R^i L_{\underline{m}}(M)$. On the other hand, since $\underline{b}$ annihilates $\mathrm{Ext}^{n-i}_B(M,B)$, $\mathrm{Hom}_B(\mathrm{Ext}^{n-i}_B(M,B), E_B(B/\underline{n})) \cong \mathrm{Hom}_B(\mathrm{Ext}^{n-i}_B(M,B), (0:\underline{b})_{E_B(B/\underline{n})})$, (as A- and B-modules), and $\mathrm{Hom}_B(\mathrm{Ext}^{n-i}_B(M,B), (0:\underline{b})_{E_B(B/\underline{n})}) =$ $\mathrm{Hom}_A(\mathrm{Ext}^{n-i}_B(M,B), (0:\underline{b})_{E_B(B/\underline{n})})$ when $(0:\underline{b})_{E_B(B/\underline{n})}$ is given its obvious A-module structure. Using the ideas of the proof of Lemma (6.3), it is easy to see that $(0:\underline{b})_{E_B(B/\underline{n})}$ is A-isomorphic to $E_A(A/\underline{m})$.

Thus $R^i L_{\underline{m}}(M)$ is A-isomorphic to $D(\mathrm{Ext}^{n-i}_B(M,B))$. $\mathrm{Ext}^{n-i}_B(M,B)$ is a f.-g. A-module, so, since A is complete, I can use Matlis' duality (see (12.3)) to deduce that $\mathrm{Ext}^{n-i}_B(M,B)$ is A-isomorphic to $D(R^i L_{\underline{m}}(M))$, as required. //

One of the two results which form the main aims of this Chapter is an easy consequence of this Theorem.

(12.9)Corollary and Definition. Suppose the complete local ring A is Cohen-Macaulay. Then $W = \mathrm{Ext}^{n-m}_B(A,B)$, endowed with its natural A-module structure, is a Gorenstein A-module by (10.4) and, up to A-isomorphism, depends only on A and not on the choice of B.

W is defined to be the canonical Gorenstein A-module. Note that, by (12.8), W is A-isomorphic to $\mathrm{Hom}_A(R^m L_{\underline{m}}(A), E_A(A/\underline{m}))$. Since A is Cohen-Macaulay of Krull Dimension m, it follows from (12.6) that W is A-isomorphic to $\mathrm{Hom}_A(A^m, E_A(A/\underline{m}))$, where A^m is the final non-zero term in the Cousin Complex for A. //

Returning now to the case in which the complete local ring A is not necessarily Cohen-Macaulay, and letting $D:\mathcal{C}_A^{mod} \to \mathcal{C}_A$ denote the functor

$\mathrm{Hom}_A(\ ,E_A(A/\underline{m}))$, I am going to consider certain properties of the functors $T^i:\underline{\mathcal{C}}_A \rightsquigarrow \underline{\mathcal{C}}_A$, where, for each $i \geqslant 0$, $T^i = D.R^iL_{\underline{m}}$. Note that, by (12.8), if M is any f.-g. A-module, $T^i(M) \cong \mathrm{Ext}^{n-i}_B(M,B)$ (as A-modules).

(12.10)Proposition. (See Propositions 6.4 & 6.6 of Hartshorne (13).)
Suppose that M is a non-zero, f.-g., A-module of Krull Dimension s. Then:-

(i) $T^i(M) = 0$ and $R^iL_{\underline{m}}(M) = 0$ whenever $i > s$;

(ii) $T^i(M)$ is a f.-g. A-module;

(iii) if $i \geqslant 0$ and $T^i(M) \neq 0$, $K\text{-}\mathrm{Dim}_A(T^i(M)) \leqslant i$;

(iv) $R^sL_{\underline{m}}(M) \neq 0$ and $K\text{-}\mathrm{Dim}_A(T^s(M)) = s$;

(v) $\mathrm{Ass}_A(T^s(M)) = \left\{\underline{p} \in \mathrm{Ass}_A(M) \,\middle|\, \mathrm{coht}\,\underline{p} = s\right\}$.

Proof. (i) This is immediate from (11.7).

(ii) This is immediate from (12.8), since $T^i(M)$ is A-isomorphic to $\mathrm{Ext}^{n-i}_B(M,B)$.

(iii) Suppose $i \geqslant 0$ and $T^i(M) \neq 0$. It is clearly sufficient to show that $K\text{-}\mathrm{Dim}_B(\mathrm{Ext}^{n-i}_B(M,B)) \leqslant i$, and since B is Gorenstein, this is an obvious consequence of (5.10)(iv).

(iv) As in (iii), it is sufficient to show that $\mathrm{Ext}^{n-s}_B(M,B) \neq 0$ and $K\text{-}\mathrm{Dim}_B(\mathrm{Ext}^{n-s}_B(M,B)) = s$. From (iii) $K\text{-}\mathrm{Dim}_B(\mathrm{Ext}^{n-s}_B(M,B)) \leqslant s$. Also, it is clear that $K\text{-}\mathrm{Dim}_B M = s$.

Suppose that $\underline{p}$ is a (minimal)element of $V_B(M)$ having coheight s. $M_{\underline{p}}$ is a non-zero $B_{\underline{p}}$-module of finite length, and, using (6.11), $B_{\underline{p}}$ is a Gorenstein local ring of Krull Dimension n-s. Now $(\mathrm{Ext}^{n-s}_B(M,B))_{\underline{p}} \cong \mathrm{Ext}^{n-s}_{B_{\underline{p}}}(M_{\underline{p}},B_{\underline{p}})$ (by (6.2)), and this in turn is isomorphic to $\mathrm{Hom}_{B_{\underline{p}}}(M_{\underline{p}},E_{B_{\underline{p}}}(B_{\underline{p}}/\underline{p}B_{\underline{p}}))$ by (5.11)(v). This latter module is certainly non-zero, for by (6.19) $E_{B_{\underline{p}}}(M_{\underline{p}})$ is isomorphic to a direct sum of a

finite number of copies of $E_{B_{\underline{p}}}(B_{\underline{p}}/\underline{p}B_{\underline{p}})$. Thus $\mathrm{Ext}^{n-s}_B(M,B) \neq 0$, $\underline{p} \in V_B(\mathrm{Ext}^{n-s}_B(M,B))$, and so $\text{K-Dim}_B(\mathrm{Ext}^{n-s}_B(M,B)) = s$, as required.

(v) If X is a non-zero, f.-g., A-module, $\underline{b} \subseteq \underline{q}'$, a prime ideal of B, and $\phi(\underline{q}') = \underline{p}'$ (which is therefore a prime ideal of A), then it follows easily from (1.11) that $\underline{p}' \in \mathrm{Ass}_A(X)$ if and only if $\underline{q}' \in \mathrm{Ass}_B(X)$. Since any prime of $\mathrm{Ass}_B(X)$ must contain $\underline{b}$, it is sufficient, in order to prove (v), to prove that:-

$$\mathrm{Ass}_B(\mathrm{Ext}^{n-s}_B(M,B)) = \left\{\underline{p} \in \mathrm{Ass}_B(M) \,\middle|\, \mathrm{coht}\,\underline{p} = s\right\}.$$

From the above proof of (iv), $\left\{\underline{p} \in \mathrm{Ass}_B(M) \,\middle|\, \mathrm{coht}\,\underline{p} = s\right\} \subseteq \mathrm{Ass}_B(\mathrm{Ext}^{n-s}_B(M,B))$.

Let the Cousin Complex for the Gorenstein local ring B be:-

$$\underline{C(B)}: 0 \longrightarrow B \xrightarrow{d^{-1}} B^0 \xrightarrow{d^0} B^1 \longrightarrow \cdots \longrightarrow B^{n-1} \xrightarrow{d^{n-1}} B^n \longrightarrow 0 \quad (E).$$

Since $\text{K-Dim}_B M = s$, $(0:M)_B$ is not contained in any prime of V(B) of height $\leqslant n-s-1$ (using (6.11) again). Thus, by P.H.C.C.A. (4.22), $\mathrm{Ext}^{n-s}_B(M,B) \cong \mathrm{Hom}_B(M, \mathrm{coker}\, d^{n-s-2})$ and the latter is isomorphic to a submodule of $\mathrm{Hom}_B(M, B^{n-s})$.

By (2.19), $\mathrm{Hom}_B(M,B^{n-s}) \cong \bigoplus_{\substack{\underline{q} \in V(B) \\ \mathrm{ht}\,\underline{q} = n-s}} \mathrm{Hom}_B(M, (\mathrm{coker}\, d^{n-s-2})_{\underline{q}})$. Suppose $\underline{p} \in \mathrm{Ass}_B(\mathrm{Ext}^{n-s}_B(M,B))$. Certainly, $\underline{p} \in V_B(M)$, and by part (iv), $\mathrm{coht}\,\underline{p} \leqslant s$. Suppose $\mathrm{coht}\,\underline{p} < s$, so that, by (6.11), $\mathrm{ht}\,\underline{p} > n-s$. By (1.11), $\mathrm{Ext}^{n-s}_B(M,B)$, and consequently $\bigoplus_{\substack{\underline{q} \in V(B) \\ \mathrm{ht}\,\underline{q} = n-s}} \mathrm{Hom}_B(M, (\mathrm{coker}\, d^{n-s-2})_{\underline{q}})$, has a B-submodule isomorphic to $B/\underline{p}$. But, if $\underline{q} \in V(B)$ has height n-s, multiplication by any element not in $\underline{q}$ provides an automorphism of $\mathrm{Hom}_B(M, (\mathrm{coker}\, d^{n-s-2})_{\underline{q}})$, and this leads to a contradiction.

Thus $\operatorname{coht}\underline{p} = s$, and $\underline{p}$ is a minimal element of $V_B(M)$. Thus $\operatorname{Ass}_B(\operatorname{Ext}_B^{n-s}(M,B)) \subseteq \left\{\underline{p} \in \operatorname{Ass}_B(M) \;\middle|\; \operatorname{coht}\underline{p} = s\right\}$. //

(12.11)Corollary. (See Proposition 6.6(8) of Hartshorne (13).) There is a natural transformation of functors from $\underline{\mathcal{C}}_A^f$ to $\underline{\mathcal{C}}_A$:-

χ :Identity $\rightsquigarrow T^mT^m$.

Proof. Since, by (11.7), $R^{m+1}L_{\underline{m}}(X) = 0$ for all A-modules X, $T^m:\underline{\mathcal{C}}_A^f \rightsquigarrow \underline{\mathcal{C}}_A$ is a contravariant, left exact, additive, A-linear functor. By (2.22), T^m and $\operatorname{Hom}_A(\ ,T^m(A))$ are naturally equivalent functors from $\underline{\mathcal{C}}_A^f$ to $\underline{\mathcal{C}}_A$. But, if M is a f.-g. A-module, $T^m(M)$ is a f.-g. A-module by (12.10). It follows that T^mT^m and $\operatorname{Hom}_A(\operatorname{Hom}_A(\ ,T^m(A)),T^m(A))$ are naturally equivalent functors from $\underline{\mathcal{C}}_A^f$ to $\underline{\mathcal{C}}_A$, and so it is sufficient to exhibit a natural transformation of functors from $\underline{\mathcal{C}}_A^f$ to $\underline{\mathcal{C}}_A$:- λ:Identity $\rightsquigarrow \operatorname{Hom}_A(\operatorname{Hom}_A(\ ,T^m(A)),T^m(A))$.

If M is a f.-g. A-module, define $\lambda(M):M \longrightarrow \operatorname{Hom}_A(\operatorname{Hom}_A(M,T^m(A)),T^m(A))$ by $(\lambda(M)(m))(f) = f(m)$ for $m \in M$, $f \in \operatorname{Hom}_A(M,T^m(A))$. It is then easy to check that:- (i) $\lambda(M)(m) \in \operatorname{Hom}_A(\operatorname{Hom}_A(M,T^m(A)),T^m(A))$;

(ii) $\lambda(M)$ is an A-module homomorphism;

& (iii) the $\lambda(M)$ constitute a natural transformation of functors as required. //

(12.12)Corollary. (See Proposition 6.6(8) of Hartshorne (13).) Suppose that M is a non-zero, f.-g., A-module having Krull Dimension m. Then the A-homomorphism of (12.11), $\chi(M):M \longrightarrow T^mT^m(M)$ is a monomorphism if and only if all the primes in $\operatorname{Ass}_A(M)$ have coheight m.

Proof. Suppose first that $\chi(M)$ is monomorphic, and $\underline{p} \in \operatorname{Ass}_A(M)$. Using (1.11), $\underline{p} \in \operatorname{Ass}_A(T^m(T^m(M)))$, and so $\operatorname{coht}\underline{p} = m$ by (12.10).

Suppose, conversely, that all the primes in $\operatorname{Ass}_A(M)$ have coheight m.

From the proof of (12.11), I need to show that, if $0 \neq x \in M$, $\lambda(M)(x) \neq 0$ in $\mathrm{Hom}_A(\mathrm{Hom}_A(M,T^m(A)),T^m(A))$, i.e. there is an A-homomorphism $f:M \longrightarrow T^m(A)$ s.t. $f(x) \neq 0$. Let N be the submodule of M generated by x. Recalling from (2.22) that the functors T^m and $\mathrm{Hom}_A(\ ,T^m(A))$ from $\underline{\mathcal{C}}_A^f$ to $\underline{\mathcal{C}}_A$ are naturally equivalent, it is thus sufficient to show that $T^m(j):T^m(M) \longrightarrow T^m(N)$ is non-zero, where $j:N \overset{\subseteq}{\longrightarrow} M$ is the natural inclusion, for then there will be an $f \in \mathrm{Hom}_A(M,T^m(A))$ s.t. the restriction of f to N is non-zero, i.e. $f(x) \neq 0$.

Now (E) $0 \longrightarrow N \xrightarrow{j} M \longrightarrow M/N \longrightarrow 0$ induces:-

(E) $T^m(M) \xrightarrow{T^m(j)} T^m(N) \longrightarrow T^{m-1}(M/N)$; suppose $T^m(j) = 0$. Then:-

(E) $0 \longrightarrow T^m(N) \longrightarrow T^{m-1}(M/N)$. Since N is a non-zero submodule of M, $\mathrm{Ass}_A(N) \subseteq \mathrm{Ass}_A(M)$, and so $\text{K-Dim}_A N = m$. By (12.10), $T^m(N) \neq 0$ and is a f.-g. A-module, and $\text{K-Dim}_A(T^m(N)) = m$. Thus $T^{m-1}(M/N) \neq 0$, and is f.-g. By (12.10) $\text{K-Dim}_A(T^{m-1}(M/N)) \leqslant m-1$, a contradiction.

Thus $T^m(j) \neq 0$, and the result follows. //

The next step is to show that, when the complete local ring A of Krull Dimension m is Cohen-Macaulay, then, if M is a non-zero, f.-g., A-module of Krull Dimension m which is Cohen-Macaulay, the A-homomorphism $\chi(M):M \longrightarrow T^mT^m(M)$ of (12.11) is, in fact, an isomorphism. However, I first need a Lemma.

(12.13)Lemma. Suppose the complete local ring A is Cohen-Macaulay. Then every A-endomorphism of the canonical Gorenstein A-module W can be realized as multiplication by exactly one element of A.

Proof. If M is an A-module, let $1_M:M \longrightarrow M$ denote the identity map on M. By (9.10), $(0:W)_A = 0$, so if a,b are two distinct elements of A,

$a.1_W : W \longrightarrow W$ and $b.1_W : W \longrightarrow W$ are distinct A-endomorphisms of W.

Now suppose $\phi \in \mathrm{Hom}_A(W,W)$. By (12.6), $R^m L_{\underline{m}}(W) \cong W^m$, the final non-zero term in the Cousin Complex for W. But W is a Gorenstein A-module, so the Cousin Complex for W provides a minimal injective resolution for W. Thus, by (10.4), $W^m \cong E_A(A/\underline{m})$, and so, by Theorem 3.7 of Matlis (17), $R^m L_{\underline{m}}(\phi) = c.1_{R^m L_{\underline{m}}(W)} : R^m L_{\underline{m}}(W) \longrightarrow R^m L_{\underline{m}}(W)$ for some element c of A.

Thus $T^m(\phi) = c.1_{T^m(W)} : T^m(W) \longrightarrow T^m(W)$, and so:-
$T^m T^m(\phi) = c.1_{T^m T^m(W)} : T^m T^m(W) \longrightarrow T^m T^m(W)$. Now W is a Cohen-Macaulay A-module of Krull Dimension m, so that the A-homomorphism of (12.11), $\chi(W) : W \longrightarrow T^m T^m(W)$, is a monomorphism (by (4.14) and (12.12)). Thus:-

$$\begin{array}{ccccc} (E)\ 0 & \longrightarrow & W & \xrightarrow{\chi(W)} & T^m T^m(W) \\ & & \downarrow \phi & & \downarrow T^m T^m(\phi) = c.1_{T^m T^m(W)} \\ (E)\ 0 & \longrightarrow & W & \xrightarrow{\chi(W)} & T^m T^m(W) \end{array} \qquad (C)\ .$$

It follows that $\phi = c.1_W$, and the Lemma is proved. //

Next comes the powerful Theorem for which I have been working.

(12.14)Theorem. Suppose the complete local ring A of Krull Dimension m is Cohen-Macaulay. If the non-zero, f.-g., A-module M is Cohen-Macaulay with $\text{K-Dim}_A M = m$, then the A-homomorphism $\chi(M) : M \longrightarrow T^m T^m(M)$ of (12.11) is an isomorphism.

Proof. I first show that $\chi(A) : A \longrightarrow T^m T^m(A)$ is an isomorphism. By (4.14) & (12.12), $\chi(A)$ is a monomorphism. Referring to the proof of (12.11), it is sufficient to show that $\lambda(A) : A \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(A, T^m(A)), T^m(A))$ is an isomorphism, and, for this, it is in turn sufficient to show that every A-endomorphism of $T^m(A)$ can be realized as multiplication by exactly one element of A. But this is immediate from (12.13), for $T^m(A) \cong W$ (see (12.9)).

Thus $\chi(A)$ is an isomorphism. Since χ:Identity $\rightsquigarrow T^m T^m$ is a natural transformation of functors from $\underline{\mathcal{C}}_A^f$ to $\underline{\mathcal{C}}_A$ and $T^m T^m$ is additive and A-linear, it follows that $\chi(F):F \longrightarrow T^m T^m(F)$ is an isomorphism whenever F is a non-zero, f.-g., free A-module.

Now consider the Cohen-Macaulay A-module M of Krull Dimension m. Let F be a f.-g. free A-module s.t. there is an exact sequence:-
(E) $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ (*). By (12.1), F is a Cohen-Macaulay A-module of Krull Dimension m. If K = 0, $F \cong M$, and so certainly $\chi(M):M \longrightarrow T^m T^m(M)$ is an isomorphism. Consequently, I suppose $K \neq 0$. K is certainly f.-g.

(*) induces:- (E) $0 \longrightarrow \mathrm{Hom}_A(A/\underline{m},K) \longrightarrow \mathrm{Hom}_A(A/\underline{m},F) \longrightarrow \ldots \longrightarrow \mathrm{Ext}_A^{m-2}(A/\underline{m},M) \longrightarrow \mathrm{Ext}_A^{m-1}(A/\underline{m},K) \longrightarrow \mathrm{Ext}_A^{m-1}(A/\underline{m},F) \longrightarrow \mathrm{Ext}_A^{m-1}(A/\underline{m},M) \longrightarrow \mathrm{Ext}_A^{m}(A/\underline{m},K) \longrightarrow \ldots$. From this it follows that $\mathrm{codh}_A K = m$, so that K, too, is Cohen-Macaulay of Krull Dimension m.

Now $R^{m+1}L_{\underline{m}}(X) = 0$ for all A-modules (by (11.7)). Also $R^{m-1}L_{\underline{m}}(M) \cong \varinjlim_k \mathrm{Ext}_A^{m-1}(A/\underline{m}^k,M)$ (by (11.3)). Since M is Cohen-Macaulay and $\mathrm{ht}_M \underline{m}^k = m$, $\mathrm{Ext}_A^{m-1}(A/\underline{m}^k,M) = 0$ for each $k > 0$. Thus $R^{m-1}L_{\underline{m}}(M) = 0$, and:-
(E) $0 \longrightarrow R^m L_{\underline{m}}(K) \longrightarrow R^m L_{\underline{m}}(F) \longrightarrow R^m L_{\underline{m}}(M) \longrightarrow 0$.
Consequently, (E) $0 \longrightarrow T^m(M) \longrightarrow T^m(F) \longrightarrow T^m(K) \longrightarrow 0$.

Consider $T^m(K)$: by (12.10), $T^m(K)$ is a non-zero f.-g. A-module of Krull Dimension m. Also, by (12.8), $T^m(K)$ is A-isomorphic to $\mathrm{Ext}_B^{n-m}(K,B)$ (endowed with its natural A-module structure). But K is a C.-M. B-module of Krull Dimension m, so by (10.2), $\mathrm{Ext}_B^{n-m}(K,B)$ is a C.-M. B-module of Krull Dimension m. It follows that $T^m(K)$ is a C.-M. A-module of Krull Dimension m. Similarly, $T^m(F)$ and $T^m(M)$ are C.-M. A-modules of Krull

Dimension m. So, applying T^m as before:-

(E) $0 \longrightarrow T^m T^m(K) \longrightarrow T^m T^m(F) \longrightarrow T^m T^m(M) \longrightarrow 0.$

Since χ:Identity $\rightsquigarrow T^m T^m$ is a natural transformation of functors from $\underline{\mathcal{C}}_A^f$ to $\underline{\mathcal{C}}_A$, it follows that:-

$$\begin{array}{ccccccccc} (E) & 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & M & \longrightarrow 0 \\ & & & \downarrow \chi(K) & & \downarrow \chi(F) & & \downarrow \chi(M) & \\ (E) & 0 & \longrightarrow & T^m T^m(K) & \longrightarrow & T^m T^m(F) & \longrightarrow & T^m T^m(M) & \longrightarrow 0 \quad (C). \end{array}$$

Now $\chi(F)$ is an isomorphism, and by (4.14) and (12.12), $\chi(M)$ and $\chi(K)$ are monomorphisms. Thus $\chi(M)$ is also an isomorphism, as was to be proved. //

I can now prove the result which has been the second of the two main aims of this Chapter.

(12.15)Theorem. (Structure Theorem for Gorenstein modules over a complete Cohen-Macaulay local ring.) Suppose the complete local ring A having maximal ideal $\underline{m}$ and Krull Dimension m is Cohen-Macaulay. Then the Gorenstein A-modules are precisely those A-modules isomorphic to a direct sum of a (strictly positive) finite number of copies of the canonical Gorenstein A-module W.

In fact, if M is an arbitrary Gorenstein A-module, $M \cong \bigoplus \mu^m(\underline{m}, M)W$.

Also, if h and k are positive integers s.t. $\bigoplus hW \cong \bigoplus kW$, then h = k.

Proof. It follows immediately from (10.4) and (12.1) that, if k is a strictly positive integer, $\bigoplus kW$ is a Gorenstein A-module and $\mu^m(\underline{m}, \bigoplus kW) = k$.

Suppose, conversely, that the (non-zero, f.-g.,) A-module M is Gorenstein: let $k = \mu^m(\underline{m}, M)$. By (6.27) (& (9.10)) k is a strictly positive integer. By (12.6), $R^m L_{\underline{m}}(M) \cong M^m$, the final non-zero term in the Cousin Complex for M. But, by (6.27), $M^m \cong \bigoplus kE_A(A/\underline{m})$, since the

Cousin Complex for M provides a minimal injective resolution for M.

Similarly, $R^m L_{\underline{m}}(\oplus kW) \cong \oplus kE_A(A/\underline{m})$, and so $R^m L_{\underline{m}}(\oplus kW) \cong R^m L_{\underline{m}}(M)$. Thus $T^m(\oplus kW) \cong T^m(M)$, and so $T^m T^m(\oplus kW) \cong T^m T^m(M)$. But $\oplus kW$ and M, being Gorenstein A-modules, are Cohen-Macaulay A-modules of Krull dimension m, so that, by (12.14), $M \cong T^m T^m(M)$ and $\oplus kW \cong T^m T^m(\oplus kW)$. Thus $M \cong \oplus kW = \oplus \mu^m(\underline{m}, M)W$.

Finally, suppose h and k are positive integers s.t. $\oplus hW \cong \oplus kW$. Then $R^m L_{\underline{m}}(\oplus kW) \cong R^m L_{\underline{m}}(\oplus hW)$, and so $\oplus kE_A(A/\underline{m}) \cong \oplus hE_A(A/\underline{m})$. It follows from (5.2)(iv) that h = k. //

Theorem (12.15) completely determines all Gorenstein modules over a complete Cohen-Macaulay local ring. There are two reasons why the methods employed to prove Theorem (12.15) cannot be expected to yield such agreeable results when applied to the study of Gorenstein modules over a non-complete Cohen-Macaulay local ring A. Firstly, it might not be possible to express A as a homomorphic image of a Gorenstein local ring, so the theory might not even "get off the ground". Secondly, in building up to the proof of Theorem (12.15), considerable use has been made of the "duality" developed by Matlis in (17) (see (12.3)) between f.-g. modules and Artinian modules over a complete local ring. The corresponding results are not necessarily true in the category of modules over a non-complete local ring, and so, in the case of a non-complete Cohen-Macaulay local ring A, even if the problems of existence of Gorenstein modules were solved, it could not be expected that the methods used in this Chapter would yield a complete classification of all Gorenstein A-modules comparable with that provided by Theorem (12.15) for Gorenstein modules over a complete Cohen-Macaulay local ring. However, in this

connection it is probably worth recalling Corollary (8.32), which states that, if M is a non-zero, f.-g., module over a local ring A, then M is a Gorenstein A-module if and only if $M \otimes_A \hat{A}$ is a Gorenstein $\hat{A}$-module (where, of course, $\hat{A}$ denotes the completion of A).

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