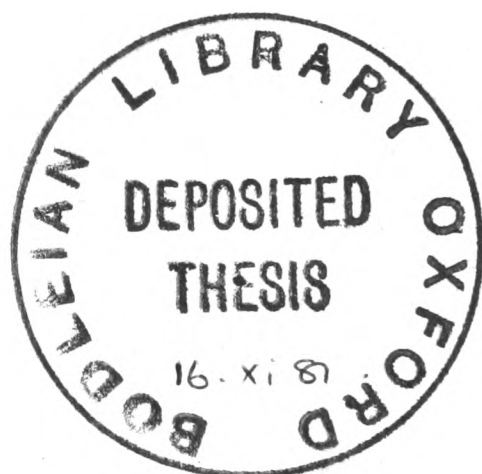


The lattice of normal
subgroups of an
infinite group

by

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The Queen's College,
Oxford



Thesis submitted for the degree of Doctor of Philosophy
Trinity Term, 1981

Abstract

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This thesis deals with various problems about the normal and subnormal structure of infinite groups. We first consider the relationship between the number of normal subgroups of a group G and of a subgroup H of finite index in G . We prove

Theorem 1.5 There exists a finitely generated group G which has a subgroup H of index 2 such that H has continuously many normal subgroups and G has only countably many normal subgroups.

Proposition 1.7 Let κ be an infinite cardinal. Then there exists a group G of cardinality κ that has only 12 normal subgroups but which contains a subgroup H of index 2 having κ normal subgroups.

We then consider partially ordered sets and investigate the subnormal structure of generalized wreath products. We deal with the question whether the number of subnormal subgroups of an infinite group is determined by the number of its n -step subnormal subgroups for an integer n . We prove

Theorem 5.3 Let G be a group. Then G has finitely many subnormal subgroups if and only if it has finitely many 2-step subnormal subgroups.

Theorem 5.5 Let $\underline{m}$ and $\underline{n}$ be infinite cardinals such that $\underline{m} \leq \underline{n}$. Then there exists a group G with the following properties:

- (1) The cardinality of G is $\underline{n}$.
- (2) The number of normal subgroups of G is $\aleph_0$.
- (3) The number of 2-step subnormal subgroups of G is $\underline{m}$.
- (4) The number of 3-step subnormal subgroups of G is $2^{\underline{n}}$.

Finally we consider characteristically simple groups with countably many normal subgroups. We construct a new type of characteristically simple groups:

Corollary 6.15 Let Λ be a partially ordered set such that for $\lambda, \mu \in \Lambda$ there exists an automorphism a of Λ such that $\mu \leq \lambda a$. Let $\mathcal{J}(\Lambda)$ be the distributive lattice of semi-ideals of Λ . Then there exists a group G with the following properties:

- (1) $|G| \leq \max(\aleph_0, |\mathcal{J}(\Lambda)|)$.
- (2) All subnormal subgroups of G are normal in G .
- (3) The lattice of normal subgroups of G is isomorphic to $\mathcal{J}(\Lambda)$.
- (4) The group G is characteristically simple.

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[Signature removed from online version of thesis]

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Chapter 0

Preliminaries

A. Definitions and notations

For basic concepts we shall use the standard definitions of group, lattice and set theory. We shall use the notations $\subseteq$, $\leq$ and $\trianglelefteq$ for the subset, subgroup and normal subgroup relations respectively, and $\subset$, $<$ and $\triangleleft$ will denote proper inclusion.

We shall assume the axiom of choice throughout this thesis. We shall, however, in general not assume the continuum hypothesis or generalized continuum hypothesis except where specifically stated.

If G is a group, H a subgroup of G , and S a subset of G we shall denote by H^S the subgroup of G generated by the conjugates of elements of H by elements of S , that is, $H^S = \langle s^{-1}hs \mid h \in H, s \in S \rangle$. We note that H^G is equal to the intersection of all normal subgroups of G which contain H . If N is a normal subgroup of G , and $\underline{n}$ a cardinal number, we say that N can be generated, qua normal subgroup, by at most $\underline{n}$ elements if there exists a subset S of G of cardinality at most $\underline{n}$ such that $N = \langle S \rangle^G$. Similarly, we say that N is, qua normal subgroup, finitely generated if there is a finite set S such that $N = \langle S \rangle^G$.

If G_i (for $i \in I$) is a family of groups we shall denote by $\prod_{i \in I} G_i$ the cartesian product of the groups G_i . The (restricted) direct product $\text{Dr}_{i \in I} G_i$ is the subgroup of $\prod_{i \in I} G_i$ consisting of those elements $(g_i)_{i \in I}$ for

which $g_i = 1$ for all but finitely many i .

If Ω is a set then $\text{Sym}(\Omega)$ will be the group of all permutations of Ω (the full symmetric group on Ω), $\text{SF}(\Omega)$ will be the subgroup of permutations of Ω with finite support (the restricted symmetric group on Ω), and $\text{AF}(\Omega)$ will be the subgroup of $\text{SF}(\Omega)$ consisting of those permutations which are even permutations on their support (the finitary alternating group on Ω).

B. Introduction

In this thesis we shall consider various problems about the lattice of normal subgroups of an infinite group, and also about the partially ordered set of subnormal subgroups of the group.

The first chapter of the thesis provides an answer to a problem of A.M. Brunner, R.G. Burns and James Wiegold. In [2] they considered the question of whether the modular group $\langle x, y \mid x^2 = y^3 = 1 \rangle$ has $2^{\aleph_0}$ non-isomorphic quotients with certain properties. Noting the fact that a finitely generated group has $2^{\aleph_0}$ non-isomorphic quotients if and only if it has $2^{\aleph_0}$ normal subgroups they state "Problem 1. Let H be a subgroup of finite index in a finitely generated group G . It is not hard to show that if G has $2^{\aleph_0}$ normal subgroups then so does H . Is the converse true?" We show that the answer to this question is no. We put this problem into the general context of an investigation of the relationship between the number of normal subgroups of any group G and the number of normal

subgroups of a subgroup H of finite index in G . We note that our solution to Problem 1 does not provide a solution to Problem 2 in [2]. There the question is that with G and H as in Problem 1, is it true that if H has $2^{\aleph_0}$ non-isomorphic residually finite quotients, then so does G ? This problem however seems to be more technical and less of general group theoretic interest than Problem 1.

If we take H to be normal of finite index in G then any statement about the relation between the numbers of normal subgroups of G and H will be a statement about the subnormal structure of G , as normal subgroups of H are 2-step subnormal in G . So a natural generalization of the question above is to consider the relationship between the number of subnormal subgroups of a group and the number of its n -step subnormal subgroups.

Whereas the concept of a normal subgroup was one of the first to be introduced in group theory, subnormal subgroups as such were not considered before the 1930s (see [21]). Robert Remak was the first to give them a special name ("nachinvariante Untergruppen"), and in a seminar in 1936 he asked, given A, B subnormal subgroups of a group G , if their join $\langle A, B \rangle$ was also subnormal. Helmut Wielandt [20] proved that the answer was yes if G satisfied the maximal condition on subnormal subgroups, and Hans J. Zassenhaus ([26], page 235, Ex. 23) showed that the answer was no in general. Problems concerning the join of subnormal subgroups still receive great attention [12].

Hence while the normal subgroups of a group form a complete modular lattice the partially ordered set of subnormal subgroups does not form a lattice in general. However the intersection of finitely many subnormal subgroups is, as can be easily seen, again subnormal (although the intersection of infinitely many need not be, see [16]).

Chapter 5 of this thesis deals with the relationship between the number of subnormal subgroups of a group and the number of its n -step subnormal subgroups. In a discussion of the topics presented in Chapter 1, Avinoam Mann conjectured that a countable group with at most countably many 2-step subnormal subgroups would have at most countably many subnormal subgroups. To construct groups which are counterexamples to this conjecture we have to use generalized wreath products over partially ordered sets. Therefore Chapter 2 deals with partially ordered sets and Chapter 3 gives the definition of generalized wreath products and some of their properties. In Chapter 4 we consider the subnormal structure of generalized wreath products and prove that some of their normal subgroups are again generalized wreath products.

Finally in Chapter 6 we deal with characteristically simple groups. If M and N are normal subgroups of a group G with $M < N$ we say that N/M is a chief factor of G if there does not exist a normal subgroup K of G with $M < K < N$. All chief factors of a group are characteristically simple groups.

If a group has at most countably many 2-step subnormal subgroups then all of its chief factors will be characteristically simple groups with at most countably many normal subgroups. Therefore for the study of the subnormal structure of countable groups (for example, for the question of whether a countable group with at most countably many 3-step subnormal subgroups will have at most countably many subnormal subgroups) it would be desirable to "know" all countable characteristically simple groups with countably many normal subgroups.

It is well known that all finite characteristically simple groups are direct powers of finite simple groups. Also the abelian characteristically simple groups can be characterized as being the additive groups of fields, that is, direct powers of either a group of order p for some prime p or the additive group of rational numbers. However it seems to be much more difficult (and it might not even be possible) to describe all countable characteristically simple groups with countably many normal subgroups in terms of certain constructions and simple groups.

We consider this problem and put certain additional conditions on the characteristically simple groups in question (e.g. the existence of a minimal normal subgroup). We also construct a type of characteristically simple group which has properties that are quite different from those of the formerly known characteristically simple groups.

Chapter 1

The normal subgroup lattice of a subgroup of finite index

In the study of properties of infinite groups it is one of the obvious questions to ask whether or not a certain property of a group is inherited by all its subgroups of finite index, or normal subgroups of finite index. In the case of chain conditions on the lattice of normal subgroups, a first result was obtained by John S. Wilson [22] in 1970.

Theorem 1.1 (John S. Wilson) Let G be a group which satisfies the maximal (minimal) condition on normal subgroups, and let H be a normal subgroup of finite index in G . Then H satisfies the maximal (minimal) condition on normal subgroups.

If G is a group and H a subgroup of finite index in G let K be maximal among the normal subgroups of G that are contained in H . If $\{g_1, g_2, \dots, g_n\}$ is a transversal of H in G then K is the intersection of all conjugates of H under $g_1, g_2, \dots, g_n$, and hence is also of finite index in G . The group G/K acts on the lattice of normal subgroups of K in the following way. If $gK \in G/K$ and $N \trianglelefteq K$ then the action of gK is given by $N \longmapsto N^g$

Furthermore N is fixed under all $gK \in G/K$ if and only if N is normal in G . Therefore we can apply to this situation the more general results of Joe W. Fisher [5] on the action of finite groups on modular lattices.

Theorem 1.2 (Joe W. Fisher) Let M be a modular lattice and let G be a finite subgroup of the automorphism group of M . If the sublattice of (common) fixed points (under G) satisfies any of a large class of chain conditions then M satisfies the same chain condition. Some chain conditions in this class are the following: the ascending chain condition; the descending chain condition; the property of having no chains order-isomorphic to the rational numbers, having no chains of a given infinite cardinal etc.

The theorem is proved by constructing a finite number of order-preserving maps $f_1, f_2, \dots, f_r$ from the lattice M to the sublattice of fixed points under G with the property: if x and y are elements of M with $x \leq y$ and $xf_i = yf_i$ for all $i \in \{1, 2, \dots, r\}$ then $x = y$. Now obviously Theorem 1.1 is a special case of the application of Theorem 1.2 to the lattice of normal subgroups of the normal subgroup H which is of finite index in the group G , and where the finite group and its action on the lattice are given as above.

In the following we shall study the relationship of the number of normal subgroups of a group G and the number of normal subgroups of a subgroup H of finite index in G . It is not difficult to give a lower bound for the number of normal subgroups of H in terms of the number of normal subgroups of G .

Proposition 1.3 Let G be a group and H a subgroup of finite index in G . Suppose that G and H have $\underline{n}$ and $\underline{m}$ normal subgroups respectively. If $\underline{n}$ is infinite then $\underline{m} \geq \underline{n}$.

Proof: Let K be the maximal normal subgroup of G contained in H . Then G/K is a finite group. It is sufficient to prove that the $\underline{n}$ normal subgroups of G have $\underline{n}$ different intersections with K (then they also have $\underline{n}$ different intersections with H). We can assume that G/K is simple (otherwise we proceed by induction on the length of a composition series of G/K).

First suppose that $\underline{n} = \aleph_\alpha$ where α is not an infinite limit ordinal, and suppose that the normal subgroups of G have less than $\underline{n}$ distinct intersections with K . Then there are $\underline{n}$ normal subgroups of G that have the same intersection with K , which can without loss of generality be assumed to be the identity subgroup. Then these $\underline{n}$ normal subgroups of G are all isomorphic to the finite simple group G/K . Let N be one of these and let $\{N_i \mid i \in I\}$ with $|I| = \underline{n}$ be the set of the others. Then $G = K \rtimes N$. Let H_i be the projection of N_i to K , that is $H_i = NN_i \cap K$. If N is non-abelian then all H_i are distinct. If N is cyclic of order p then to every H_i there can only correspond p different N_i . So in both cases we have $|\{H_i \mid i \in I\}| = \underline{n}$ contradicting the assumption that the normal subgroups of G have less than $\underline{n}$ distinct intersections with K .

Now suppose that $\underline{n} = \aleph_\alpha$ where α is an infinite limit ordinal, and suppose that the normal subgroups of G have less than $\underline{n}$ distinct intersections with K . Then for every $\beta < \alpha$ we can find $\aleph_\beta$ normal subgroups of G having the same intersection with K . As above it follows that for every $\beta < \alpha$ the normal subgroups of G have at least $\aleph_\beta$ distinct intersections with K , and hence have $\aleph_\alpha = \underline{n}$ such intersections contradicting the assumption. This proves the proposition.

Trivially this lower bound for the number of normal subgroups of H is the best possible as the case $H = G$ shows. We shall now show that upper bounds for the number of normal subgroups of H are very big although they will be the best possible. Consider the following construction. Let p be an odd prime. Let $A := B := Z_{p^\infty}$. Then A and B each have only countably many (normal) subgroups. For every endomorphism η of Z_{p^∞} we define a subgroup S_η of $Z_{p^\infty} \times Z_{p^\infty}$ by $S_\eta := \{(a, a\eta) \mid a \in Z_{p^\infty}\}$. Note that if $\eta \neq \eta'$ then $S_\eta \neq S_{\eta'}$. There is a bijection (see, for example, [6], p.181) between the endomorphisms of Z_{p^∞} and the p -adic integers. As the number of p -adic integers is $2^{\aleph_0}$, we have $2^{\aleph_0}$ (normal) subgroups of $A \times B$. If we take $D := \langle B, c \mid c^{-1}bc = b^{-1} \text{ for all } b \in B, c^2 = 1 \rangle$, a generalized dihedral group, then $H := A \times B$, which is of index 2 in $G := A \times D$, has continuously many normal subgroups, but G has only countably many. For, let N be a normal subgroup of G . Then obviously

$(N \cap A) \times (N \cap B) \leq N \cap H$. Let (a, b) be an element of $N \cap H$. Then $(a, b)^{(1, c)} = (a, b^{-1}) \in N$, and hence $(a^2, 1) = (a, b)(a, b^{-1}) \in N$, and also $(1, b^2) = (a, b)(a, b^{-1})^{-1} \in N$. But as p is odd it follows that $(a, 1) \in N$ and $(1, b) \in N$. Therefore $N \cap H = (N \cap A) \times (N \cap B)$. As A and B have only countably many subgroups, there are only countably many possibilities for $N \cap H$. As every normal subgroup of G/H is cyclic, every normal subgroup N of G is, qua normal subgroup, generated by $N \cap H$ and at most one further element of N not contained in H . Hence there are only countably many normal subgroups of G . Therefore we have now proved

Proposition 1.4 There exists a countable group G with a subgroup H of index 2 in G such that G has countably many normal subgroups, but H has continuously many.

Note that if H is normal and of finite index in a countable group G and has $2^{\aleph_0}$ normal subgroups then G has to have at least $\aleph_0$ normal subgroups. For if G had only a finite number of normal subgroups then by Theorem 1.1 the subgroup H would satisfy the maximal condition on normal subgroups. Hence every normal subgroup of H would be generated, qua normal subgroup, by finitely many elements of H , and therefore there could only exist countably many of them.

A.M. Brunner, R.G. Burns and James Wiegold asked in [2] the following question. Let G be a finitely generated group and H a subgroup of finite index in G . We have

seen in Proposition 1.3 that if G has $2^{\aleph_0}$ normal subgroups then H has $2^{\aleph_0}$ normal subgroups. Is the converse true? The group G in Proposition 1.4 does not provide an answer to this question as it is not finitely generated. However we shall construct a finitely generated group G in a similar way.

Theorem 1.5 There exists a finitely generated group G which has a subgroup H of index 2 such that H has $2^{\aleph_0}$ normal subgroups and G has only $\aleph_0$ normal subgroups.

Proof: (1) Construction of the group G .

Let p be a prime. For all $i \in \mathbb{N}$ define

$$x_i := u_i := \begin{pmatrix} 1 & p^{-i} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y_i := v_i := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & p^{-i} \\ 0 & 0 & 1 \end{pmatrix},$$

$$z_i := w_i := \begin{pmatrix} 1 & 0 & p^{-i} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad c := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$s := t := \begin{pmatrix} p^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^{-1} \end{pmatrix}.$$

Consider these as elements of $GL(3, \mathbb{Q})$. They satisfy the following relations: $x_{i+1}^p = x_i$; $y_{i+1}^p = y_i$; $[x_i, y_j] = z_{i+j}$; $[x_i, z_j] = [y_i, z_j] = 1$ for all $i, j \in \mathbb{N}$. Let

$$A_0 := \langle x_i, y_i, z_i \mid i \in \mathbb{N} \rangle;$$

$$B_0 := \langle u_i, v_i, w_i \mid i \in \mathbb{N} \rangle;$$

$Z := \langle z_i \mid i \in \mathbb{N} \rangle$ and $W := \langle w_i \mid i \in \mathbb{N} \rangle$.

The element s acts on A_0 by conjugation and we have

$$x_i^s = x_i^p; y_i^s = y_{i+1}; z_i^s = z_i \quad \text{for all } i \in \mathbb{N}.$$

Therefore the group A generated by A_0 and s is the semidirect product of A_0 and $\langle s \rangle$, and analogously $B = B_0 \langle t \rangle$. The element c acts on A_0 by conjugation and we have

$$x_i^c = x_i^{-1}; y_i^c = y_i; z_i^c = z_i^{-1} \quad \text{for all } i \in \mathbb{N}.$$

Furthermore c commutes with s , and therefore the group A_1 generated by A_0 , s and c is the semidirect product of A_0 with the direct product of $\langle s \rangle$ and $\langle c \rangle$. We take G to be the direct product of A_1 and B , and H its subgroup generated by A and B .

(2) Proof that G has the properties claimed in the statement of the theorem.

First note that A_1 can be generated by x_1, y_1, s, c , and that B can be generated by u_1, v_1, t . Therefore G is finitely generated. It should be clear, moreover, that H has index 2 in G .

The relations given in the construction show that z_i and w_i are centralised by x_1, y_1, s and u_1, v_1, t respectively (which are the generators of A and B) for all $i \in \mathbb{N}$. Therefore $Z \times W$ is contained in the centre of H . The quotient of $Z \times W$ by its subgroup $\langle z_1 \rangle \times \langle w_1 \rangle$ is isomorphic to $Z_{p^\infty} \times Z_{p^\infty}$. From the proof of Proposition 1.4 we know that $Z_{p^\infty} \times Z_{p^\infty}$ has $2^{\aleph_0}$ subgroups. Hence $Z \times W$ also has $2^{\aleph_0}$ subgroups, and as these all lie in the centre

of H , we have $2^{\aleph_0}$ normal subgroups of H .

Now it remains to prove that G has no more than $\aleph_0$ normal subgroups. Let N be a normal subgroup of G . Then obviously $(N \cap Z) \times (N \cap W) \leq N \cap (Z \times W)$. Let (z, w) be an element of $N \cap (Z \times W)$. Then $(z, w)^{(c, 1)} = (z^{-1}, w) \in N$, and hence $(1, w^2) = (z, w)(z^{-1}, w) \in N$, and also $(z^2, 1) = (z, w)(z^{-1}, w)^{-1} \in N$. Therefore $(N \cap (Z \times W))^2 \leq (N \cap Z) \times (N \cap W)$. As Z and W are locally cyclic, we have $|Z:Z^2| \leq 2$ and $|W:W^2| \leq 2$. Hence $|(Z \times W):(Z \times W)^2| \leq 4$ and $|N \cap (Z \times W):(N \cap (Z \times W))^2| \leq 4$. Therefore we get $|N \cap (Z \times W) : (N \cap Z) \times (N \cap W)| \leq 4$. So $N \cap (Z \times W)$ can be generated by its intersections with Z and W and at most two further elements of $Z \times W$. As Z and W have only countably many subgroups and are themselves countable, there are only countably many possibilities for $N \cap (Z \times W)$. As $(A_0 \times B_0)/(Z \times W)$ and $G/(A_0 \times B_0)$ are both abelian, $G/(Z \times W)$ is a finitely generated metabelian group. By a theorem of P. Hall (Theorem 3 in [7]), it therefore satisfies the maximal condition on normal subgroups, and so every normal subgroup of $G/(Z \times W)$ is, qua normal subgroup, finitely generated. Hence a normal subgroup N of G is, qua normal subgroup, generated by $N \cap (Z \times W)$ and a finite number of elements of N not contained in $Z \times W$. As there are only countably many possibilities for $N \cap (Z \times W)$ and only countably many finite subsets of G , there is only a countable number of normal subgroups of G .

We have now seen that for a countable (or even finitely generated) group G with countably many normal subgroups there is no upper bound (smaller than the obvious one) on the number of normal subgroups of a subgroup H of finite index in G . But in the case where G has only finitely many normal subgroups or where G is of higher cardinality, we can do better.

Proposition 1.6 Let H be a subgroup of finite index in a group G of infinite cardinality $\underline{k}$. Suppose that G and H have $\underline{n}$ and $\underline{m}$ normal subgroups respectively. If $\underline{n}$ is finite then $\underline{m} \leq \underline{k}$. If $\underline{n}$ is infinite then, assuming the generalized continuum hypothesis, if $\underline{n} < \underline{k}$ it follows that $\underline{m} \leq \underline{k}$.

Proof: If $\underline{n}$ is finite then by Theorem 1.1 the group H can have no infinite ascending chain, and hence every normal subgroup of H is, qua normal subgroup, finitely generated. Therefore $\underline{m} \leq |H| = |G| = \underline{k}$.

If $\underline{n}$ is infinite then by Theorem 1.2 the group H can have no chain of more than $\underline{n}$ normal subgroups*. Hence every normal subgroup of H can be generated, qua normal subgroup, by at most $\underline{n}$ elements. Therefore $\underline{m} \leq \underline{k}^{\underline{n}}$, and $\underline{k}^{\underline{n}} \leq \underline{k}$, assuming the generalized continuum hypothesis (see, for example, [11], IX, §6).

Again this bound is the best possible:

*We can assume that H is normal in G (otherwise replace H by a normal subgroup K of finite index in G with $K \leq H$, and note that going up from K to H multiplies the number of normal subgroups by a cardinal less or equal to $\underline{k}$).

Proposition 1.7 Let $\underline{k}$ be an infinite cardinal. Then there exists a group G of cardinality $\underline{k}$ that has only twelve normal subgroups, but which contains a subgroup H of index 2 having $\underline{k}$ normal subgroups.

Proof: Let K be a field of characteristic $p \neq 2$ and of cardinality $\underline{k}$. The special linear group $S := \text{SL}(2, K)$ acts naturally on K^2 . Let T be a cyclic group of order 2, and let $X_0 := X_1 := K^2$. Let $U := S \ltimes T$ act on X_0 and X_1 , where S acts naturally and T acts on X_0 trivially and on X_1 by multiplication with -1 . Let X be the direct sum of X_0 and X_1 . We take G to be the semidirect product of X with U , and H the subgroup generated by X and S . Now H is of index 2 in G and has $\underline{k}$ normal subgroups, as for every $q \in K$ the subgroup $N_q := \{(x, qx) \mid x \in X_0\}$ of X is normal in H . It remains to prove that G has only twelve normal subgroups. For this we shall use

Lemma 1.8 Let K be a field, $S := \text{SL}(2, K)$ acting naturally on $Y := K^2$. Then Y is a simple S -module.

Proof: Let $Z \neq \{0\}$ be an S -submodule of Y . Let $0 \neq x = (x_1, x_2) \in Z$. If $x_1 \neq 0$ then

$$s_1 = \begin{pmatrix} x_1^{-1} & -x_2 \\ 0 & x_1 \end{pmatrix} \in S, \text{ and } x^{s_1} = (1, 0) \in Z.$$

$$\text{If } x_2 \neq 0 \text{ then } s_2 = \begin{pmatrix} 0 & -x_2 \\ x_2^{-1} & x_1 \end{pmatrix} \in S, \text{ and } x^{s_2} = (1, 0) \in Z.$$

$$\text{For all } y \neq 0 \text{ we have } t_1 = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \in S \text{ and}$$

$t_2 = \begin{pmatrix} 0 & y \\ -y^{-1} & 0 \end{pmatrix} \in S$, and hence $(y,0) = (1,0)^{t_1} \in Z$

and $(0,y) = (1,0)^{t_2} \in Z$. Therefore $Z = Y$ and Y is simple.

We now finish the proof of Proposition 1.7.

Let $z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, then $Z = \langle z \rangle$ is the centre of S .

Let $D := \langle (z,t) \rangle \leq U$. We identify S and T with their corresponding subgroups of U . We want to show that the normal subgroups of G are precisely $\{1\}$, X_0 , X_1 , X_0^D , X , X_1^T , XD , XZ , XT , XZT , $H = XS$ and G , and that the lattice is given in figure 1.1. It is easy to see that these subgroups are, in fact, normal. So we have to show that these are all ^{the} normal subgroups of G .

Let $\{1\} \neq N \trianglelefteq G$. Suppose $N \cap X_0 \neq \{0\}$ or $N \cap X_1 \neq \{0\}$. Then by Lemma 1.8 we have $X_0 \leq N$ or $X_1 \leq N$ respectively.

Suppose that $(x_0, x_1) \in N$ where $x_0 \neq 0 \neq x_1$. Then $(x_0, -x_1) = (x_0, x_1)^{(1,t)} \in N$, hence $(2x_0, 0) \in N$ and $(0, 2x_1) \in N$. As $\text{char } K \neq 2$, these elements are non-zero. Hence $X_0 \leq N$ and $X_1 \leq N$. Therefore $X \leq N$.

Suppose $(z, 1) \in N$. Then $(z, 1)^{((1,0), (1,0))} = ((-2, 0), (-2, 0))(z, 1) \in N$. Hence $((-2, 0), (-2, 0)) \in N$, and $X \leq N$.

Suppose $(x_0, x_1)(s, 1) \in N$ where $1 \neq s \in S$. Then there exists $\bar{x} = (\bar{x}_0, \bar{x}_1)$ such that $\bar{x}(z, 1) \in N$ (As $N \trianglelefteq H$ we have $XN/X \trianglelefteq H/X \cong S$). We have

$(\bar{x}(z,1))^{(z,1)}\bar{x}(z,1) = (-2\bar{x}_0, 2\bar{x}_1) \in N$. If $\bar{x}_0 \neq 0 \neq \bar{x}_1$ it follows that $X \leq N$. If $\bar{x}_0 \neq 0$ or $\bar{x}_1 \neq 0$ it follows that $X_0 \leq N$ or $X_1 \leq N$, hence

$$X \leq N.$$

As H/X has only XZ/X as non-trivial normal subgroup, it follows that all normal subgroups of G contained in H are $\{1\}$, X_0 , X_1 , X , XZ , H .

Now suppose $(1,t) \in N$. Then

$$(1,t)^{(0,(1,0))} = (0,(-2,0))(1,t) \in N, \text{ and hence } X_1T \leq N.$$

Suppose $(0,x_1)(1,t) \in N$, where $0 \neq x_1 \in X_1$. Then

$$(0,x_1)(1,t)((0,x_1)(1,t))^{(z,1)} = (0,2x_1) \in N. \text{ Hence } X_1 \leq N \text{ and } X_1T \leq N.$$

Suppose $(x_0,x_1)(1,t) \in N$ where $x_0 \neq 0$. Then

$$((x_0,x_1)(1,t))^2 = (2x_0,0) \in N. \text{ Hence } X_0 \leq N, \\ (0,x_1)(1,t) \in N, X_1 \leq N, \text{ and } XT \leq N.$$

Suppose $(z,t) \in N$. Then $(z,t)^{((1,0),0)} = ((-2,0),0)(z,t) \in N$, and $X_0 \leq N$. Therefore $X_0D \leq N$.

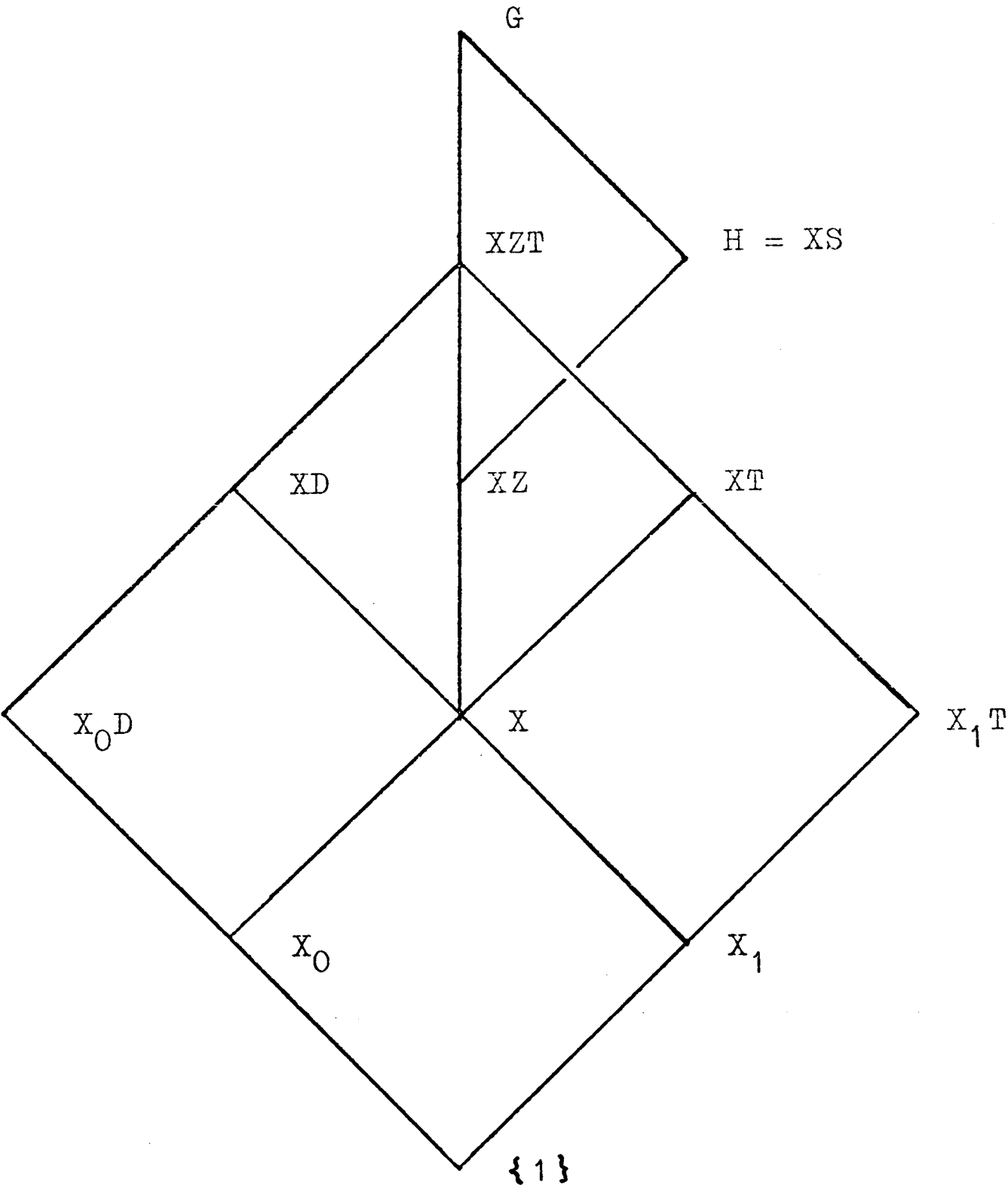
Suppose $(x_0,x_1)(s,t) \in N$, where $s \in S - Z$. Then

$$((x_0,x_1)(s,t))^2 = (x_0+x_0^{s^{-1}}, x_1-x_1^{s^{-1}})(s^2,1) \in N \text{ where } s^2 \neq 1. \text{ Hence } X \leq N.$$

Suppose $(x_0,x_1)(z,t) \in N$. Then $((x_0,x_1)(z,t))^2 = (0,2x_1) \in N$, and $(x_0,x_1)(z,t)((x_0,x_1)(z,t))^{(z,1)} = (2x_0,0) \in N$. It follows that $(z,t) \in N$, hence $X_0 \leq N$. Therefore $x_1 \neq 0$ implies $X \leq N$.

Therefore the rest of the normal subgroups of G are X_0D , X_1T , XD , XT , XZT , G , which concludes the proof of Proposition 1.7.

Figure 1.1



This leaves open the question whether or not there exists a group of cardinality $\underline{k}$ that has only $\underline{k}$ normal subgroups, but in which there is a subgroup of finite index having $2^{\underline{k}}$ normal subgroups. We have been unable to generalize the method used in the construction of the group G in Theorem 1.5 for $\underline{k} = \aleph_0$ to give examples of higher cardinality.

The main results of this chapter have been published in a joint paper by Peter M. Neumann and the author [1].

Chapter 2

Partially ordered sets

Let Λ be a partially ordered set. An automorphism a of Λ is a bijection $a: \Lambda \rightarrow \Lambda$ which preserves the order on Λ . Let Σ be a subset of Λ . We call Σ a semi-ideal of Λ if for $\sigma \in \Sigma$ and $\lambda \in \Lambda$ with $\lambda \leq \sigma$ it follows that $\lambda \in \Sigma$. The semi-ideals of a partially ordered set form a complete distributive lattice, the lattice operations being set-theoretic union and intersection. We can give the following test for the question whether there exists a non-trivial semi-ideal Σ which is invariant under all elements of a group A of automorphisms of Λ .

Theorem 2.1 Let Λ be a partially ordered set, and let A be a group of automorphisms of Λ . Then the following are equivalent:

- (1) For $\lambda, \mu \in \Lambda$ there exists $a \in A$ such that $\mu \leq \lambda a$.
- (2) For every semi-ideal Σ of Λ with $\emptyset \neq \Sigma \neq \Lambda$ there exists $a \in A$ with $\Sigma a \neq \Sigma$.

Proof: Assume (1). Let Σ be a semi-ideal of Λ with $\emptyset \neq \Sigma \neq \Lambda$. Choose $\lambda \in \Sigma$ and $\mu \in \Lambda - \Sigma$. Then there exists an automorphism $a \in A$ such that $\mu \leq \lambda a$. But then we have $\lambda a \in \Lambda - \Sigma$, and $\Sigma a \neq \Sigma$. Hence we have (2).

Assume (2). Let $\lambda, \mu \in \Lambda$. We define a semi-ideal Σ of Λ by $\Sigma = \{\nu \in \Lambda \mid \nu \leq \lambda\}$. Then for every $b \in A$ we have that Σb is a semi-ideal of Λ . As the lattice of semi-ideals

is complete, the set $\Phi = \bigcup_{b \in A} \Sigma b$ is a semi-ideal of Λ . Furthermore Φ is invariant under every element of A . As Φ is not empty, we therefore have $\Phi = \Lambda$. Hence there exists $a \in A$ such that $\mu \in \Sigma a$. So there exists $v \in \Sigma$ such that $\mu = va$. But $v \leq \lambda$, hence $\mu = va \leq \lambda a$. Therefore (1) holds.

For a partially ordered set Λ let the lattice of semi-ideals be denoted by $\mathcal{J}(\Lambda)$, and let $S(\Lambda) = |\mathcal{J}(\Lambda)|$. A totally ordered subset of Λ is called a chain in Λ . Let $L(\Lambda) := \sup \{S(\Phi) \mid \Phi \text{ is a chain in } \Lambda\}$, and let $D(\Lambda)$ be the minimal cardinal number such that every subset of Λ of cardinality $D(\Lambda)$ contains elements a, b such that $a < b$. We then have

Proposition 2.2 $S(\Lambda) \geq \max(|\Lambda|, D(\Lambda), L(\Lambda))$.

Proof: (i) For every $\lambda \in \Lambda$ we define a semi-ideal Σ_λ of Λ by $\Sigma_\lambda := \{\mu \in \Lambda \mid \mu \leq \lambda\}$. Note that $\lambda_1 \neq \lambda_2$ implies $\Sigma_{\lambda_1} \neq \Sigma_{\lambda_2}$. Hence $S(\Lambda) \geq |\Lambda|$.

(ii) First suppose that $D(\Lambda)$ is finite. Then there exists a subset Γ of Λ which consists of pairwise incomparable elements such that $|\Gamma| = D(\Lambda) - 1$ (if $D(\Lambda) = 0$ then $\Lambda = \emptyset$ and $S(\Lambda) = 1 > D(\Lambda)$). For every subset Γ_1 of Γ we define a semi-ideal Σ_{Γ_1} of Λ by $\Sigma_{\Gamma_1} = \{\mu \in \Lambda \mid \text{there exists } \lambda \in \Gamma_1 \text{ such that } \mu \leq \lambda\}$. If Γ_1, Γ_2 are different subsets of Γ then $\Sigma_{\Gamma_1} \neq \Sigma_{\Gamma_2}$. Hence $S(\Lambda) \geq 2^{D(\Lambda)-1} \geq D(\Lambda)$.

Now suppose that $D(\Lambda) = \aleph_{\alpha+1}$. Then there exists a subset Γ of pairwise incomparable elements of Λ such that $|\Gamma| = \aleph_{\alpha}$. As above, every subset Γ_1 of Γ gives rise to a distinct semi-ideal Σ_{Γ_1} of Λ . Hence $S(\Lambda) \geq 2^{\aleph_{\alpha}} \geq \aleph_{\alpha+1} = D(\Lambda)$.

Finally suppose that $D(\Lambda) = \aleph_{\alpha}$ where α is a limit ordinal. Then for every $\beta < \alpha$ there exists a set Γ of pairwise incomparable elements of Λ such that $|\Gamma| = \aleph_{\beta}$. Then, as above, we get $S(\Lambda) \geq 2^{\aleph_{\beta}} > \aleph_{\beta}$. Hence $S(\Lambda) \geq \aleph_{\alpha}$.

(iii) Let Φ be a totally ordered subset of Λ . For every semi-ideal Σ of Φ we define a semi-ideal Σ' of Λ by $\Sigma' = \{\lambda \in \Lambda \mid \text{there exists } \sigma \in \Sigma \text{ such that } \lambda \leq \sigma\}$. Again, if Σ_1 and Σ_2 are semi-ideals of Φ and $\Sigma_1 \neq \Sigma_2$, then $\Sigma'_1 \neq \Sigma'_2$. Hence $S(\Lambda) \geq S(\Phi)$, and $S(\Lambda) \geq L(\Lambda)$.

In a special case we can prove that equality holds:

Proposition 2.3 Let Λ be an infinite partially ordered set with $D(\Lambda)$ finite. Then $S(\Lambda) = L(\Lambda)$.

Proof: We know that $S(\Lambda) \geq L(\Lambda)$ from Proposition 2.2. If $D(\Lambda) = k$ then by Dilworth's Theorem [3] there exist disjoint chains $\Lambda_1, \dots, \Lambda_{k-1}$ in Λ such that $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_{k-1}$. Then a map $\varphi: \mathcal{J}(\Lambda) \rightarrow \mathcal{J}(\Lambda_1) \times \dots \times \mathcal{J}(\Lambda_{k-1})$ is given by $\Sigma \mapsto (\Sigma \cap \Lambda_1, \dots, \Sigma \cap \Lambda_{k-1})$. As Σ is the disjoint union of $\Sigma \cap \Lambda_1, \dots, \Sigma \cap \Lambda_{k-1}$, the map φ is injective. Hence $S(\Lambda) \leq |\mathcal{J}(\Lambda_1) \times \dots \times \mathcal{J}(\Lambda_{k-1})| \leq L(\Lambda)^{k-1} = L(\Lambda)$.

Note that if Λ is infinite and $D(\Lambda)$ finite then $L(\Lambda) \geq |\Lambda|$, because one of the chains Λ_i in Dilworth's Theorem must be of cardinality $|\Lambda|$, and hence $L(\Lambda) \geq S(\Lambda_i) \geq |\Lambda_i| = |\Lambda|$.

If $D^*(\Lambda)$ denotes the minimal cardinal such that Λ is a (disjoint) union of $D^*(\Lambda)$ chains then the proof of Proposition 2.3 shows that $L(\Lambda)^{D^*(\Lambda)}$ is an upper bound for $S(\Lambda)$. However this bound is usually not very good. Also we obviously have $\underline{n} \leq D^*(\Lambda)$ where $\underline{n}$ is the predecessor cardinal of $D(\Lambda)$ or $D(\Lambda) \leq D^*(\Lambda)$ where $D(\Lambda)$ is a limit cardinal, but equality does not hold in general. E.S. Wolk gives an example of a partially ordered set Λ of any infinite cardinality $\underline{k}$ such that $D(\Lambda) = \aleph_0$ but $D^*(\Lambda) = \underline{k}$ ([25], Thm. 2). Also note that there is another relationship between $|\Lambda|$, $D(\Lambda)$ and $L(\Lambda)$. Suppose that $|\Lambda|$ is infinite. If $D(\Lambda) \leq |\Lambda|$ then $L(\Lambda) \geq \aleph_0$. If $D(\Lambda) \leq \aleph_0$ then $L(\Lambda) \geq |\Lambda|$. This follows trivially from Theorems 5.24 and 5.25 in [4].

We can also give a characterisation of those countable totally ordered sets that have uncountably many semi-ideals. To do this we first show

Lemma 2.4 Let Λ be a countable chain with at least $\aleph_1$ semi-ideals. Then there exists $\gamma \in \Lambda$ such that both $\{\lambda \in \Lambda \mid \lambda < \gamma\}$ and $\{\lambda \in \Lambda \mid \lambda > \gamma\}$ have at least $\aleph_1$ semi-ideals.

Proof: Let Λ be a countable chain with (at least) $\aleph_1$ semi-ideals. For $\gamma \in \Lambda$ we define $L(\gamma) = \{\lambda \in \Lambda \mid \lambda < \gamma\}$ and $U(\gamma) = \{\lambda \in \Lambda \mid \lambda > \gamma\}$. We first show that there exist γ_U and γ_L such that $L(\gamma_U)$ and $U(\gamma_L)$ have $\aleph_1$ semi-ideals.

Let $\gamma_0 \leq \gamma_1 \leq \dots$ be a subchain of Λ which is co-final with Λ . For every semi-ideal $\Sigma \neq \Lambda$ of Λ there exist $n \in \mathbb{N}$ such that Σ is a semi-ideal of $L(\gamma_n)$. So if $L(\gamma_i)$ has only countably many semi-ideals for all $i \in \mathbb{N}$ then Λ has only countably many semi-ideals. Therefore there exists γ_U such that $L(\gamma_U)$ has at least $\aleph_1$ semi-ideals.

Let $\gamma_0 \geq \gamma_1 \geq \dots$ be a subchain of Λ which is co-initial with Λ . Then for every semi-ideal $\Sigma \neq \emptyset$ of Λ there exists $n \in \mathbb{N}$ such that Σ is the union of $\{\lambda \in \Lambda \mid \lambda \leq \gamma_n\}$ and a semi-ideal of $U(\gamma_n)$. So if $U(\gamma_i)$ has only countably many semi-ideals for all $i \in \mathbb{N}$ then Λ has only countably many semi-ideals. Therefore there exists γ_L such that $U(\gamma_L)$ has at least $\aleph_1$ semi-ideals.

We note that $\Lambda_L = \{\gamma \in \Lambda \mid U(\gamma) \text{ has at least } \aleph_1 \text{ semi-ideals}\}$ is a semi-ideal of Λ and $\Lambda_U = \{\gamma \in \Lambda \mid L(\gamma) \text{ has at least } \aleph_1 \text{ semi-ideals}\}$ is a dual semi-ideal of Λ . As for every $\gamma \in \Lambda$ every semi-ideal of Λ is either a semi-ideal of $L(\gamma)$ or it is $L_0(\gamma) = \{\lambda \in \Lambda \mid \lambda \leq \gamma\}$ or it is the union of $L_0(\gamma)$ and a semi-ideal of $U(\gamma)$ we must have $\Lambda = \Lambda_L \cup \Lambda_U$. All we now have to prove is that $\Lambda_L \cap \Lambda_U$ is not empty.

Suppose $\Lambda_L \cap \Lambda_U = \emptyset$. First note that in this case $\Lambda_L = \{\gamma \in \Lambda \mid L(\gamma) \text{ has at most } \aleph_0 \text{ semi-ideals}\}$ and $\Lambda_U = \{\gamma \in \Lambda \mid U(\gamma) \text{ has at most } \aleph_0 \text{ semi-ideals}\}$. Let $\alpha_0 \leq \alpha_1 \leq \dots$ be a subchain of Λ_L cofinal with Λ_L , and let $\beta_0 \geq \beta_1 \geq \dots$ be a subchain of Λ_U coinitial with Λ_U . Now let Σ be a semi-ideal of Λ . Then one of the following holds for Σ . Either there exists $n \in \mathbb{N}$ such that Σ is a semi-ideal of $L(\alpha_n)$, or $\Sigma = \Lambda_L$, or there exists $m \in \mathbb{N}$ such that Σ is the union of $\{\lambda \in \Lambda \mid \lambda \leq \beta_m\}$ and a semi-ideal of $U(\beta_m)$. But then there are only countably many possibilities for Σ which gives a contradiction. Hence $\Lambda_L \cap \Lambda_U \neq \emptyset$, and this set contains the required element.

Now we note that a countable chain Λ is order-isomorphic to the rational numbers if and only if it is dense in itself, that is, for $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 < \lambda_2$ there exists $\mu \in \Lambda$ such that $\lambda_1 < \mu < \lambda_2$. We then have

Theorem 2.5 Let Λ be a countable chain. Then the following are equivalent:

- (i) The chain Λ has at least $\aleph_1$ semi-ideals.
- (ii) The chain Λ has $2^{\aleph_0}$ semi-ideals.
- (iii) The chain Λ contains a subchain isomorphic to the rational numbers.

Proof: The implication (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is obvious, so we only have to prove (i) $\Rightarrow$ (iii). Let Λ be a countable chain with at least $\aleph_1$ semi-ideals, and set $\Lambda_0 := \Lambda$. By Lemma 2.4 there exists $\gamma_{0,0}$ such that both $\Lambda_{1,0} := \{\lambda \in \Lambda \mid \lambda < \gamma_{0,0}\}$ and $\Lambda_{1,1} := \{\lambda \in \Lambda \mid \lambda > \gamma_{0,0}\}$ have at least $\aleph_1$ semi-ideals. Continue this process to construct $\gamma_{n,i} \in \Lambda_{n,i}$ for $i \in \{0, \dots, 2^{n-1}-1\}$ and $n \in \mathbb{N}$ such that $\Lambda_{n,i} = \Lambda_{n+1,2i} \cup \Lambda_{n+1,2i+1} \cup \{\gamma_{n,i}\}$ and both $\Lambda_{n+1,2i}$ and $\Lambda_{n+1,2i+1}$ have at least $\aleph_1$ semi-ideals. Let $\Gamma = \{\gamma_{n,i} \mid i \in \{0, \dots, 2^{n-1}-1\}, n \in \mathbb{N}\}$. Then it is clear that Γ is dense in itself, and hence is order-isomorphic to the rational numbers.

Chapter 3

Generalized wreath products

In this chapter we shall define generalized wreath products and exhibit some of their properties. Generalized wreath products over totally ordered sets were introduced by P. Hall [8], but the definition easily generalizes to partially ordered sets, in the way described by Howard L. Silcock [19]. We shall give the definition for partially ordered sets.

Let Λ be a partially ordered set. Let $H_\lambda \neq \{1\}$ be a transitive permutation group on a set Ω_λ for $\lambda \in \Lambda$. For each λ we choose an element 0_λ of Ω_λ . For any subset Γ of Λ we define $\text{Dr}_{\lambda \in \Gamma} \Omega_\lambda$ to be the set of functions from Γ into $\bigcup_{\lambda \in \Gamma} \Omega_\lambda$ such that the image of λ lies in Ω_λ for all λ , and such that it is equal to 0_λ for all but finitely many λ . Let $\Omega = \text{Dr}_{\lambda \in \Lambda} \Omega_\lambda$. We denote elements of Ω by $\omega = (\omega_\lambda)_{\lambda \in \Lambda}$ where ω_λ is the image of λ under ω , and let $0 := (0_\lambda)_{\lambda \in \Lambda}$. For any $\lambda \in \Lambda$ we define an equivalence relation E_λ by $\omega \equiv \omega' \pmod{E_\lambda}$ if and only if $\omega_\mu = \omega'_\mu$ for all $\mu > \lambda$. We define a map φ_λ from H_λ to the symmetric group on Ω as follows. Let $h \in H_\lambda$ and $\omega \in \Omega$. If $\omega \equiv 0 \pmod{E_\lambda}$ then $(\omega(h\varphi_\lambda))_\lambda = \omega_\lambda h$ and $(\omega(h\varphi_\lambda))_\mu = \omega_\mu$ for all $\mu \neq \lambda$. If $\omega \not\equiv 0 \pmod{E_\lambda}$ then $\omega(h\varphi_\lambda) = \omega$. It is easy to see that $\varphi_\lambda: H_\lambda \longrightarrow \text{Sym}(\Omega)$ is a monomorphism. Let the image of H_λ under φ_λ be denoted by J_λ . We define the wreath

product W to be the subgroup of $\text{Sym}(\Omega)$ generated by the J_λ ($\lambda \in \Lambda$):

$$W = \text{wr}_{\lambda \in \Lambda} H_\lambda = \langle J_\lambda \mid \lambda \in \Lambda \rangle.$$

Using the transitivity of H_λ , it is easy to see that a different choice 0_λ^* for the distinguished element of Ω_λ leads to a wreath product W^* which is permutationally isomorphic to W . If the H_λ are all (permutationally) isomorphic to a given transitive permutation group H we call W the Λ th wreath power of H and write $W = \text{wr}^\Lambda H$. We remark that W is transitive on Ω . Furthermore, if $\Lambda = \{1, 2\}$ with $1 < 2$ then the wreath product W is just the ordinary (restricted) wreath product, and it will also be denoted by $W = H_1 \text{ wr } H_2$.

For a subset Γ of Λ we write $J_\Gamma := \langle J_\lambda \mid \lambda \in \Gamma \rangle$. Note that $J_\Gamma \cong \text{wr}_{\lambda \in \Gamma} H_\lambda$ ([19], Lemma 1). If J_Γ is regarded as a permutation group on its orbit containing 0 then this is a permutational isomorphism. In particular, if $\Gamma = \{\lambda, \mu\}$ then $J_\Gamma = H_\lambda \text{ wr } H_\mu$ if $\lambda < \mu$ and $J_\Gamma = H_\lambda \times H_\mu$ if λ and μ are incomparable.

If $\Gamma \subseteq \Lambda$ write $\omega \equiv \omega' \pmod{\Gamma}$ if $\omega_\lambda = \omega'_\lambda$ for all $\lambda \notin \Gamma$. We denote by D_Γ the subgroup of W consisting of those elements of W that leave each equivalence class mod Γ invariant. Also let $D_\lambda := D_{\{\lambda\}}$. If Σ is a semi-ideal of Λ then we have ([19], Lemma 2) that D_Σ is the normal closure of J_Σ in W , and $D_{\Lambda-\Sigma} = J_{\Lambda-\Sigma}$ is a complement to D_Σ in W . We shall now give some further properties of the generalized wreath product.

Lemma 3.1 For $\lambda, \mu \in \Lambda$ we have $D_\lambda D_\mu = D_\mu D_\lambda$.

Proof: If $\lambda \leq \mu$ let $\Gamma = \{\gamma \in \Lambda \mid \gamma \geq \lambda\}$. We then have $D_\lambda \trianglelefteq J_\Gamma$ as $\{\lambda\}$ is a semi-ideal of Γ , and we also have $D_\mu \leq J_\Gamma$. If λ and μ are incomparable let $\Gamma = \{\gamma \in \Lambda \mid \gamma \geq \lambda \text{ or } \gamma \geq \mu\}$. Then D_λ and D_μ both are normal in J_Γ , as $\{\lambda\}$ and $\{\mu\}$ are both semi-ideals of Γ .

The following lemma is a generalization of Lemma 2 in [8].

Lemma 3.2 Let Σ be a semi-ideal of Λ .

(a) Let $g \in D_\Sigma$. Then there exist $n \geq 0$ and $\lambda_1, \dots, \lambda_n \in \Sigma$ with $\lambda_i \not\leq \lambda_j$ for $i < j$ and $1 \neq g_{\lambda_i} \in D_{\lambda_i}$ for $1 \leq i \leq n$

such that $g = g_{\lambda_1} \cdots g_{\lambda_n}$ (Note that $g = 1$ if and only if $n = 0$).

(b) Let $g_1 = g_{\lambda_1} \cdots g_{\lambda_n}$, $g_2 = g_{\mu_1} \cdots g_{\mu_k} \in D_\Sigma$ with $\lambda_i \not\leq \lambda_j$ and $\mu_i \not\leq \mu_j$ for $i < j$, and $1 \neq g_{\lambda_i} \in D_{\lambda_i}$, $1 \neq g_{\mu_j} \in D_{\mu_j}$ for $1 \leq i \leq n$ and $1 \leq j \leq k$ with $n, k \geq 0$.

Then $g_1 = g_2$ if and only if $\{g_{\lambda_1}, \dots, g_{\lambda_n}\} = \{g_{\mu_1}, \dots, g_{\mu_k}\}$ (then, in particular, we have $k = n$).

Proof: (a) follows immediately from Lemma 3.1.

(b) We first assume that $g_1 = g_2$. We proceed by induction on $k+n$. The statement is clearly true for $k = 0$ or $n = 0$. So let us assume that $k, n > 0$. We choose a minimal element of $\{\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_k\}$,

say λ_i (for μ_j the proof is similar). Let $\Sigma' = \{\sigma \in \Sigma \mid \sigma \leq \lambda_i\}$. Then $D_\Sigma = D_{\Sigma'} D_{\Sigma - \Sigma'}$ with $D_{\Sigma'} \trianglelefteq D_\Sigma$ and $D_{\Sigma'} \cap D_{\Sigma - \Sigma'} = \{1\}$. As λ_i is minimal, it is incomparable with $\lambda_1, \dots, \lambda_{i-1}$, and we have $g_{\lambda_1} \dots g_{\lambda_n} = g_{\lambda_i} g_{\lambda_1} \dots g_{\lambda_{i-1}} g_{\lambda_{i+1}} \dots g_{\lambda_n}$. If $\lambda_i \notin \{\mu_1, \dots, \mu_k\}$ then $g_{\mu_1} \dots g_{\mu_k} \in D_{\Sigma - \Sigma'}$ in contradiction to $g_{\lambda_i} \in D_{\Sigma'}$. So there exists j such that $\lambda_i = \mu_j$. As $\lambda_i = \mu_j$ is minimal in $\{\mu_1, \dots, \mu_k\}$, we have, as above, $g_{\mu_1} \dots g_{\mu_k} = g_{\mu_j} g_{\mu_1} \dots g_{\mu_{j-1}} g_{\mu_{j+1}} \dots g_{\mu_k}$, and $g_{\lambda_i} = g_{\mu_j}$ by the decomposition $D_\Sigma = D_{\Sigma'} D_{\Sigma - \Sigma'}$. The rest follows using the induction hypothesis.

Conversely, let $1 \neq g_{\lambda_i} \in D_{\lambda_i}$, $1 \neq g_{\mu_j} \in D_{\mu_j}$ for $i, j \in \{1, \dots, n\}$ and $\{g_{\lambda_1}, \dots, g_{\lambda_n}\} = \{g_{\mu_1}, \dots, g_{\mu_n}\}$. We have to show that $g_1 = g_2$. We prove this by induction on n . It is clear for $n = 0, 1$. So let $n > 1$. Let j be such that $g_{\mu_j} = g_{\lambda_1}$. As $\lambda_1 = \mu_j$ is minimal in $\{\lambda_1, \dots, \lambda_n\} = \{\mu_1, \dots, \mu_n\}$ it follows that

$g_{\mu_1} \dots g_{\mu_n} = g_{\mu_j} g_{\mu_1} \dots g_{\mu_{j-1}} g_{\mu_{j+1}} \dots g_{\mu_n}$. We apply the induction hypothesis to $g_{\lambda_2} \dots g_{\lambda_n}$ and

$g_{\mu_1} \dots g_{\mu_{j-1}} g_{\mu_{j+1}} \dots g_{\mu_n}$ to show that they are equal.

Hence $g_1 = g_2$.

If Γ is a subset of Λ let $\Omega_\Gamma := \text{Dr}_{\lambda \in \Gamma} \Omega_\lambda$, and let π_Γ be the projection from $\Omega = \Omega_\Lambda$ onto Ω_Γ . Suppose that Φ is a partially ordered set and $\{\wedge_\varphi \mid \varphi \in \Phi\}$ is a

partition of Λ with the property that $\varphi_1 < \varphi_2$ if and only if $\lambda_1 < \lambda_2$ for all $\lambda_1 \in \Lambda_{\varphi_1}$ and $\lambda_2 \in \Lambda_{\varphi_2}$. It is not difficult to see that $\text{wr}_{\lambda \in \Lambda} H_\lambda \cong \text{wr}_{\varphi \in \Phi} (\text{wr}_{\lambda \in \Lambda_\varphi} H_\lambda)$. This is a straightforward generalization of the associative law given in [8]. We now can prove

Lemma 3.3 Let Σ be a semi-ideal in Λ , and let $\Gamma = \{\lambda \in \Lambda - \Sigma \mid \text{there exists } \sigma \in \Sigma \text{ such that } \lambda > \sigma\}$. Then $D_\Sigma = J_\Sigma^{J_\Gamma}$. In particular, as $\{\lambda\}$ is a semi-ideal in $\{\mu \in \Lambda \mid \mu \geq \lambda\}$, we have $D_\lambda = J_\lambda^{J_{U(\lambda)}}$ where $U(\lambda) = \{\mu \in \Lambda \mid \mu > \lambda\}$. Furthermore $D_\lambda \cong \text{Dr}_{\omega \in \Omega_{U(\lambda)}} J_\lambda(\omega)$ with $J_\lambda(\omega) = J_\lambda^g$ for $0g \equiv \omega \pmod{E_\lambda}$.

Proof: It is clear that J_Σ is normal in D_Σ , and hence we have $D_\Sigma = J_\Sigma^W = J_\Sigma^{D_\Sigma D_{\Lambda-\Gamma-\Sigma} J_\Gamma} = J_\Sigma^{D_{\Lambda-\Gamma-\Sigma} J_\Gamma}$. Also $J_\Sigma^{D_{\Lambda-\Gamma-\Sigma} J_\Gamma} = J_\Sigma^{J_\Gamma}$ as elements of D_Σ , and in particular of J_Σ , commute with elements of $D_{\Lambda-\Gamma-\Sigma}$. The isomorphism $D_\lambda \cong \text{Dr}_{\omega \in \Omega_{U(\lambda)}} J_\lambda(\omega)$ follows from the fact that $J_{\{\lambda\} \cup U(\lambda)} \cong J_\lambda \text{ wr } J_{U(\lambda)}$, and the fact that the base group of the ordinary wreath product is just the restricted direct product given above.

Now let all the groups H_λ ($\lambda \in \Lambda$) be permutationally isomorphic to a group H (that is, $\Omega_\lambda = \Omega_\mu$ for $\lambda, \mu \in \Lambda$ and the action of H_λ and H_μ is the same). If Λ is a

group of automorphisms of Λ , we can regard A as a subgroup of $\text{Sym}(\Omega)$ if we set $(\omega a)_\lambda = \omega_{\lambda a^{-1}}$ for $\omega \in \Omega$, $\lambda \in \Lambda$, $a \in A$. It is easy to see that $a^{-1} J_\lambda a = J_{\lambda a}$ for all $\lambda \in \Lambda$, $a \in A$. Hence A normalizes W , and the subgroup of $\text{Sym}(\Omega)$ generated by W and A is the semidirect product.

Finally we quote two results of Hall and Silcock on generalized wreath products. The first is Theorem D of [8], the second comprises Theorem 3 and Lemma 5 of [19].

Theorem 3.4 (P. Hall) Let Λ be a non-empty totally ordered set, $H \neq \{1\}$ a transitive permutation group, and A a group of automorphisms of Λ such that for any $\lambda, \mu \in \Lambda$ there exists $a \in A$ such that $\lambda \leq \mu a$. Let $W = \text{wr}^\Lambda H$, and let $G = WA$ be the semidirect product. Then the commutator subgroup W' of W is a minimal normal subgroup of G (in particular, it is characteristically simple).

Theorem 3.5 (Howard L. Silcock) Let Λ be a non-empty partially ordered set, and let $H_\lambda \neq \{1\}$ be a transitive permutation group for $\lambda \in \Lambda$. Let $W = \text{wr}_{\lambda \in \Lambda} H_\lambda$. Then the map $\Sigma \longmapsto D_\Sigma$ from the lattice of semi-ideals of Λ to the lattice of normal subgroups of W is a lattice monomorphism. If all H_λ are non-abelian simple groups, this map is a lattice isomorphism.

Chapter 4

The subnormal structure of generalized wreath products

If $W = \text{wr}_{\lambda \in \Lambda} H_\lambda$ is a generalized wreath product (where Λ is a partially ordered set), then by Theorem 3.5 the subgroups of the form D_Σ where Σ is a semi-ideal of Λ form a sublattice of the lattice of normal subgroups of W . Moreover if all H_λ are non-abelian simple, then this is the whole normal subgroup lattice. Therefore the subnormal structure of W is essentially determined by the structure of the groups D_Σ . We shall show that these groups are themselves generalized wreath products (over a different partially ordered set). In order to do this we need the following

Lemma 4.1 Let $W = \text{wr}_{\lambda \in \Lambda} H_\lambda$, and for $\lambda \in \Lambda$ let $U(\lambda) = \{\nu \in \Lambda \mid \nu > \lambda\}$. Let $\lambda < \mu$, $g \in J_\lambda(\omega)$, $h \in J_\mu(\omega')$ where $\omega \in \Omega_{U(\lambda)}$, $\omega' \in \Omega_{U(\mu)}$. Let $h_0 \in H_\mu$ such that $h = (h_0 \varphi_\mu)^{k_{\omega'}}$ with $Ok_{\omega'} = \omega'$ (and $k_{\omega'} \in J_{U(\mu)}$). If there exists $\nu \in U(\mu)$ such that $\omega_\nu \neq \omega'_\nu$ then $g^h = g \in J_\lambda(\omega)$. If $\omega_\nu = \omega'_\nu$ for all $\nu \in U(\mu)$ then $g^h \in J_\lambda(\bar{\omega})$ where $\bar{\omega}_\nu = \omega_\nu$ for $\nu \neq \mu$ and $\bar{\omega}_\mu = \omega_\mu h_0$. Furthermore if $g = (g_0 \varphi_\lambda)^{k_\omega}$ with $Ok_\omega = \omega$ then $g^h = (g_0 \varphi_\lambda)^{k_{\bar{\omega}}}$ with $Ok_{\bar{\omega}} = \bar{\omega}$.

Proof: The element h operates on $\Omega_{U(\lambda)}$ in the following way:

$$(\omega h)_\nu = \omega_\nu \text{ for } \nu \neq \mu \text{ and}$$

$$(\omega h)_\mu = \begin{cases} \omega_\mu h_0 & \text{if } \omega_\nu = \omega'_\nu \text{ for all } \nu > \mu \\ \omega_\mu & \text{otherwise.} \end{cases}$$

From this the lemma easily follows.

We can now state the result on the structure of D_Σ .

Theorem 4.2 Let Λ be a non-empty partially ordered set, H_λ a transitive permutation group on Ω_λ for $\lambda \in \Lambda$. Let Σ be a semi-ideal in Λ . Let $W = \text{wr}_{\lambda \in \Lambda} H_\lambda$, and let D_Σ be the subgroup of W consisting of those elements that leave each equivalence class modulo Σ invariant. For $\sigma \in \Sigma$ let $F(\sigma) = \{\lambda \in \Lambda \mid \lambda > \sigma \text{ and } \lambda \notin \Sigma\}$, and let $\bar{\Sigma} = \{(\sigma, \omega) \mid \sigma \in \Sigma, \omega \in \text{Dr}_{\lambda \in F(\sigma)} \Omega_\lambda\}$. We define a partial order on $\bar{\Sigma}$ by $(\sigma_1, \omega_1) < (\sigma_2, \omega_2)$ if and only if $\sigma_1 < \sigma_2$ and $(\omega_1)_\lambda = (\omega_2)_\lambda$ for all $\lambda \in F(\sigma_2)$. Let $H_{(\sigma, \omega)} = H_\sigma$ and $\Omega_{(\sigma, \omega)} = \Omega_\sigma$ for all $\omega \in \text{Dr}_{\lambda \in F(\sigma)} \Omega_\lambda$ and $\sigma \in \Sigma$. Then $D_\Sigma \cong \text{wr}_{(\sigma, \omega) \in \bar{\Sigma}} H_{(\sigma, \omega)}$.

Proof: Let $V := \text{wr}_{(\sigma, \omega) \in \bar{\Sigma}} H_{(\sigma, \omega)}$. Our goal is to define an isomorphism $\psi: V \longrightarrow D_\Sigma$.

We first note that $D_\sigma \cong \text{Dr}_{\omega \in \Omega_{U(\sigma)}} J_\sigma(\omega) \leq D_\Sigma$.

Also $D_{(\sigma, \omega)} \cong \text{Dr}_{\omega' \in \Omega_{U(\sigma, \omega)}} J_{(\sigma, \omega)}(\omega') \leq V$.

We define $\bar{D}_\sigma := \langle D_{(\sigma, \omega)} \mid \omega \in \Omega_{F(\sigma)} \rangle \leq V$.

By definition of the partial order on $\bar{\Sigma}$ we have that (σ, ω_1) and (σ, ω_2) are incomparable for $\omega_1 \neq \omega_2$. Therefore $\bar{D}_\sigma = \text{Dr}_{\omega \in \Omega_{F(\sigma)}} D_{(\sigma, \omega)}$

$$= \text{Dr}_{\omega \in \Omega_{F(\sigma)}} \text{Dr}_{\omega' \in \Omega_{U(\sigma, \omega)}} J_{(\sigma, \omega)}(\omega') .$$

We note that there is a bijection $U(\sigma, \omega) \longrightarrow U(\sigma) \cap \Sigma$ given by $(\nu, \xi) \longmapsto \nu$. For, if $(\nu, \xi) > (\sigma, \omega)$ then $\nu \in \Sigma$ and $\xi \in \Omega_{F(\nu)}$ such that $\xi_\gamma = \omega_\gamma$ for all $\gamma \in F(\nu)$, and hence ξ is then uniquely determined by σ, ω, ν . This gives rise to a natural bijection $\Omega_{U(\sigma, \omega)} \longrightarrow \Omega_{U(\sigma) \cap \Sigma}$ where $\omega' \longmapsto \omega'_*$ and $(\omega'_*)_\nu = \omega'_{(\nu, \xi)}$. We also note that given $\omega \in \Omega_{F(\sigma)}$ and $\omega'_* \in \Omega_{U(\sigma) \cap \Sigma}$ we get an element $(\omega'_*, \omega) \in \Omega_{U(\sigma)}$ in the obvious way.

Now let $\psi: J_{(\sigma, \omega)}(\omega') \longrightarrow J_\sigma(\omega'_*, \omega)$ in the natural way (i.e. if $g \in H_{(\sigma, \omega)} = H_\sigma$ and k, k' such that $Ok' = \omega'$ and $Ok = (\omega'_*, \omega)$ then $((g\psi_{(\sigma, \omega)})^{k'})_\psi = (g\psi_\sigma)^k$). We can linearly extend this mapping to the direct products to give an isomorphism $\psi: \bar{D}_\sigma \longrightarrow D_\sigma$.

By Lemma 3.2, we know that every element of D_Σ can be expressed as a product $g_{\lambda_1} \dots g_{\lambda_n}$ where $1 \neq g_{\lambda_i} \in D_{\lambda_i}$ and $\lambda_i \neq \lambda_j$ for $i < j$ and that the set $\{g_{\lambda_1}, \dots, g_{\lambda_n}\}$ is unique for every element of D_Σ . We also can write every element of V in the form $g(\lambda_1, \omega_1) \dots g(\lambda_k, \omega_k)$ where $1 \neq g(\lambda_i, \omega_i) \in D_{(\lambda_i, \omega_i)}$ and $(\lambda_i, \omega_i) \neq (\lambda_j, \omega_j)$

if $i < j$. It is easy to see that every element of V can be written as $g_{\lambda_1} \dots g_{\lambda_m}$ where $1 \neq g_{\lambda_i} \in \bar{D}_{\lambda_i}$ and $\lambda_i \neq \lambda_j$ for $i < j$, with the same uniqueness condition as in D_{Σ} .

Let $g = g_{\lambda_1} \dots g_{\lambda_m}$ be an element of V with $1 \neq g_{\lambda_i} \in \bar{D}_{\lambda_i}$ and $\lambda_i \neq \lambda_j$ if $i < j$. We define $g\psi := (g_{\lambda_1}\psi) \dots (g_{\lambda_m}\psi)$. To show that ψ is well-defined and a bijection from V to D_{Σ} we have to show that the definition is independent of the expression $g_{\lambda_1} \dots g_{\lambda_m}$. But this is obvious from Lemma 3.2(b).

What is left to prove now is that ψ is a homomorphism, i.e. given any $g, h \in V$ that $(gh)\psi = (g\psi)(h\psi)$. We write $g = g_{\lambda_1} \dots g_{\lambda_k}$ and $h = h_{\mu_1} \dots h_{\mu_m}$ with $1 \neq g_{\lambda_i} \in \bar{D}_{\lambda_i}$ and $1 \neq h_{\mu_j} \in \bar{D}_{\mu_j}$. Let $n(i) = \min \{n \mid \mu_n < \lambda_i\}$. Then

$$\begin{aligned} & g_{\lambda_1} \dots g_{\lambda_k} h_{\mu_1} \dots h_{\mu_m} \\ &= h_{\mu_1} \dots h_{\mu_{n(1)}} g_{\lambda_1} h_{\mu_{n(1)+1}} \dots h_{\mu_{n(2)}} \\ & \quad \cdot h_{\mu_1} \dots h_{\mu_{n(2)}} g_{\lambda_2} h_{\mu_{n(2)+1}} \dots \dots \dots \quad \text{and we have} \end{aligned}$$

$$(gh)\psi = (h_{\mu_1}\psi) \dots (h_{\mu_{n(1)}}\psi) (g_{\lambda_1} h_{\mu_{n(1)+1}} \dots h_{\mu_{n(2)}}\psi) \dots \dots \dots$$

We also get in the same way that

$$\begin{aligned} (g\psi)(h\psi) &= (g_{\lambda_1}\psi) \dots (g_{\lambda_k}\psi) (h_{\mu_1}\psi) \dots (h_{\mu_m}\psi) \\ &= (h_{\mu_1}\psi) \dots (h_{\mu_{n(1)}}\psi) (g_{\lambda_1} h_{\mu_{n(1)+1}} \dots h_{\mu_{n(2)}}\psi) \dots \dots \dots \end{aligned}$$

So it will be sufficient to show that $(g^h)_\psi = (g_\psi)^{(h_\psi)}$ for $g \in \bar{D}_\lambda$, $h = h_{\mu_1} \dots h_{\mu_n} \in \bar{D}_{\mu_1} \dots \bar{D}_{\mu_n}$ with $\mu_i \neq \mu_j$ if $i < j$ and $\lambda \neq \mu_1$.

Now for $g \in \bar{D}_\lambda$ we can write $g = g_1 \dots g_p$ where $g_j \in \bar{D}_{(\lambda, \omega_j)}$. And for g_j we can write $g_j = g_{j1} \dots g_{jq(j)}$ where $g_{ji} \in J_{(\lambda, \omega_j)}(\omega'_{ji})$. As $\bar{D}_\lambda$ is the direct product of the $J_{(\lambda, \omega_j)}(\omega'_{ji})$ and similarly $\bar{D}_\mu$ the direct product of the $J_{(\mu, \bar{\omega})}(\bar{\omega}'_i)$, we can without loss of generality assume that $g \in J_{(\lambda, \omega)}(\omega')$ for some ω, ω' .

Furthermore, as $h = h_{\mu_1} \dots h_{\mu_n} \in \bar{D}_{\mu_1} \dots \bar{D}_{\mu_n}$, we can assume that $h \in \bar{D}_\mu$ for some $\mu > \lambda$ (otherwise we can proceed by induction on n). Finally, decomposing $\bar{D}_\mu$ in the same way as above $\bar{D}_\lambda$, we can also assume that $h \in J_{(\mu, \bar{\omega})}(\bar{\omega}')$ for some $\bar{\omega}, \bar{\omega}'$.

So we have $\lambda < \mu$, $g \in J_{(\lambda, \omega)}(\omega')$, $h \in J_{(\mu, \bar{\omega})}(\bar{\omega}')$ where $\omega \in \Omega_{F(\lambda)}$, $\bar{\omega} \in \Omega_{F(\mu)}$. Now there are two cases.

(i) $(\lambda, \omega) \not< (\mu, \bar{\omega})$. Then there exists $v \in F(\mu)$ such that $\bar{\omega}_v \neq \omega_v$. It is clear that $g^h = g$ and $(g_\psi)^{(h_\psi)} = g_\psi$, hence $(g^h)_\psi = (g_\psi)^{(h_\psi)}$.

(ii) $(\lambda, \omega) < (\mu, \bar{\omega})$. Let $g_0 \in H_\lambda$, $h_0 \in H_\mu$ such that $g = (g_0 \varphi_\lambda)^k$ and $h = (h_0 \varphi_\mu)^{\bar{k}}$ where $0k = \omega'$ and $0\bar{k} = \bar{\omega}'$. If there exists $(v, \eta) \in U(\mu, \bar{\omega})$ such that $\bar{\omega}'_{(v, \eta)} \neq \omega'_{(v, \eta)}$ then $g^h = g$ by Lemma 4.1. If

$\bar{\omega}'_{(v,\eta)} = \omega'_{(v,\eta)}$ for all $(v,\eta) \in U(\mu, \bar{\omega})$ then
 $g^h \in J_{(\lambda, \omega)}(\xi)$ where $\xi_{(v,\eta)} = \omega'_{(v,\eta)}$ for $(v,\eta) \neq (\mu, \bar{\omega})$
 and $\xi_{(\mu, \bar{\omega})} = \omega'_{(\mu, \bar{\omega})} h_0$. Then $g^h \psi \in J_\lambda(\xi_*, \omega)$ such
 that $g^h \psi = (g_0 \psi_\lambda)^f$ where $Of = (\xi_*, \omega)$.

As $(\lambda, \omega) < (\mu, \bar{\omega})$ we have $\bar{\omega}_v = \omega_v$ for all $v \in F(\mu)$.
 If there exists $(v, \eta) \in U(\mu, \bar{\omega})$ such that $\bar{\omega}'_{(v,\eta)} \neq \omega'_{(v,\eta)}$
 we have $(\omega'_*)_{\eta} \neq (\bar{\omega}'_*)_{\eta}$ for this element $v \in U(\mu) \cap \Sigma$.
 Hence $(g\psi)^{(h\psi)} = g\psi$ by Lemma 4.1, and we have
 $(g^h)\psi = g\psi = (g\psi)^{(h\psi)}$ in this case.

Now suppose $\omega'_{(v,\eta)} = \bar{\omega}'_{(v,\eta)}$ for all $(v,\eta) \in U(\mu, \bar{\omega})$.
 Then $(\omega'_*)_{\eta} = (\bar{\omega}'_*)_{\eta}$ for all $v \in U(\mu) \cap \Sigma$. We then have
 $(\omega'_*, \omega)_v = (\bar{\omega}'_*, \bar{\omega})_v$ for all $v \in (U(\mu) \cap \Sigma) \cup F(\mu) = U(\mu)$.
 By Lemma 4.1 we therefore get $(g\psi)^{(h\psi)} \in J_\lambda(\xi)$ where
 $(g\psi)^{(h\psi)} = (g_0 \psi_\lambda)^{f'}$ with $Of' = \xi$, and where
 $\xi_v = (\omega'_*, \omega)_v$ for $v \neq \mu$ and $\xi_\mu = (\omega'_*)_{\mu} h_0 = \omega'_{(\mu, \bar{\omega})} h_0$.
 It is obvious that $\xi = (\xi_*, \omega)$, and hence
 $(g^h)\psi = (g\psi)^{(h\psi)}$ in this case too. This completes the
 proof of the theorem.

Note: If Λ is a totally ordered set then for any
 $\sigma \in \Sigma$ we have $F(\sigma) = \Lambda - \Sigma$ and $(\sigma_1, \omega_1) < (\sigma_2, \omega_2)$ if
 and only if $\sigma_1 < \sigma_2$ and $\omega_1 = \omega_2$. We therefore get

$$D_{\Sigma} \cong \text{wr}_{(\mathfrak{G}, \omega)} \in \bar{\Sigma}^H(\mathfrak{G}, \omega)$$

$= \text{wr}_{\omega} \in \text{Dr}_{\lambda \in \Lambda - \Sigma \Omega_{\lambda}} \text{wr}_{\mathfrak{G} \in \Sigma}^H(\mathfrak{G}, \omega)$, and this is isomorphic to a direct power of copies of J_{Σ} indexed by $\text{Dr}_{\lambda \in \Lambda - \Sigma \Omega_{\lambda}}$ as would be expected.

Chapter 5

The number of subnormal subgroups of infinite groups

If G is a group and H is a subgroup of G , and if n is a natural number, we say that H is n -step subnormal in G if there exist subgroups $H_1, H_2, \dots, H_{n-1}$ of G such that H_i is a normal subgroup of H_{i+1} for $i \in \{0, 1, \dots, n-1\}$ where $H_0 := H$ and $H_n := G$. We call H subnormal in G if there exists a number n such that H is n -step subnormal in G . If n is the least natural number such that H is n -step subnormal in G we shall call it the subnormal defect of H in G .

In this chapter we shall investigate the relation between the number of subnormal subgroups of a group G and the number of its k -step subnormal subgroups for a given k . It is well known that the minimal condition on subnormal subgroups is equivalent to the minimal condition on 2-step subnormal subgroups (see, for example, [17], Theorem 5.49.1). It was conjectured by Avinoam Mann that a countable group with countably many 2-step subnormal subgroups would have only countably many subnormal subgroups altogether. We shall give a counterexample to this conjecture.

We first show that the number of normal subgroups (i.e. the 1-step subnormal subgroups) of a group does not bound the number of subnormal subgroups. Note that

Theorem 1.5 gives an example of a finitely generated group with countably many normal subgroups but continuously many two-step subnormal subgroups. But if we do not restrict ourselves to take normal subgroups of normal subgroups of finite index as 2-step subnormal subgroups we get an even more striking example.

Proposition 5.1 Let $\underline{n}$ be an infinite cardinal number. Then there exists a group G of cardinality $\underline{n}$ such that G has 3 normal subgroups but $2^{\underline{n}}$ two-step subnormal subgroups.

Proof: Let H be a simple transitive permutation group of cardinality $\underline{n}$ on a set Ω of cardinality $\underline{n}$. Let $G = H_0 \text{ wr } H_1$ with $H_0 \cong H \cong H_1$. By Theorem 3.5 the only normal subgroups of G are $\{1\}$, $D_{\{0\}}$ and G . As $D_{\{0\}}$ is a direct power of $\underline{n}$ copies of H , it has $2^{\underline{n}}$ normal subgroups, which are all 2-step subnormal in G .

Note that the groups constructed in Proposition 1.7 give examples of groups of cardinality $\underline{n}$ with finitely many normal subgroups and $2^{\underline{n}}$ two-step subnormal subgroups. However the groups constructed in Proposition 5.1 achieve the smallest possible number of normal subgroups in this situation, namely three.

In order to deal with groups that have finitely many two-step subnormal subgroups we need the following lemma stated without proof by Wielandt in the footnote on p. 219 in [20]. The proof we shall give is different from the one given in [17] (part 1, p.184, Corollary to Thm. 5.48).

Lemma 5.2 Let G be a group with subnormal composition series of finite length. Then G has finitely many subnormal subgroups.

Proof: We shall prove the lemma by induction on the length of a composition series using the well known fact that all composition series have the same length. Lengths 0 and 1 are trivial. Let G be a counterexample of minimal length. Let $\mathfrak{M}$ be the set of maximal normal subgroups of G . As every $M \in \mathfrak{M}$ has finitely many subnormal subgroups by minimality of G , the set $\mathfrak{M}$ must be infinite. Let K be an element of $\mathfrak{M}$. Because K has only finitely many subnormal subgroups, there is an infinite subset $\mathfrak{N} \subseteq \mathfrak{M}$ and a normal subgroup L of G contained in K such that $K \cap M = L$ for all $M \in \mathfrak{N}$. But then G/L has infinitely many normal subgroups, hence by minimality of G we have $L = \{1\}$. Now we have $G = K \times M$ for every $M \in \mathfrak{N}$. Since K and M are maximal normal subgroups and $\mathfrak{N}$ has more than one element it follows that they are isomorphic simple groups. If they are non-abelian then G can only have 4 subnormal subgroups. If they

are abelian then G is finite and has only finitely many subnormal subgroups. Hence in both cases we have a contradiction to the assumption that G has infinitely many subnormal subgroups. This proves the lemma.

We then get

Theorem 5.3 Let G be a group. Then G has finitely many subnormal subgroups if and only if G has finitely many 2-step subnormal subgroups.

Proof: The one implication is trivial. So let us suppose that G has finitely many 2-step subnormal subgroups. Then G has a finite chief series

$\{1\} = N_0 \triangleleft N_1 \triangleleft \dots \triangleleft N_k = G$. For each $i \in \{0, \dots, k-1\}$ we have that $\bar{N}_{i+1} = N_{i+1}/N_i$ is a minimal normal subgroup of $\bar{G} = G/N_i$, and $\bar{G}$ has only finitely many 2-step subnormal subgroups.

Let $\bar{M}$ be a minimal normal subgroup of $\bar{N}_{i+1}$ and let $\mathfrak{M} = \{\bar{M}_0 = \bar{M}, \bar{M}_1, \dots, \bar{M}_1\}$ be the set of different conjugates of $\bar{M}$ in $\bar{G}$ (this is finite, as all $\bar{M}_j$ are 2-step subnormal in $\bar{G}$). Clearly the members of $\mathfrak{M}$ generate $\bar{N}_{i+1}$. We choose a subset $\mathfrak{M}'$ of $\mathfrak{M}$ which is minimal with respect to its members generating $\bar{N}_{i+1}$. Now $\bar{N}_{i+1}$ is the direct product of the subgroups contained in $\mathfrak{M}'$. For, let $\bar{K} \in \mathfrak{M}'$, and let $\bar{L}$ be the subgroup generated by the other elements of $\mathfrak{M}'$, we have $\bar{K} \cdot \bar{L} = \bar{N}_{i+1}$, and $\bar{K} \cap \bar{L}$ is a normal subgroup of $\bar{N}_{i+1}$. As $\bar{K}$ is a

minimal normal subgroup, it must be $\bar{K}$ or $\overline{\{1\}}$. If $\bar{K} \cap \bar{L} = \bar{K}$ then $\bar{K} \leq \bar{L}$ and $\bar{K}.\bar{L} = \bar{L} \neq \bar{N}_{i+1}$ which is a contradiction. Hence $\bar{K} \cap \bar{L} = \overline{\{1\}}$ and we have proved that $\bar{N}_{i+1}$ is the direct product of the elements of $\mathfrak{m}'$. Furthermore these groups are isomorphic simple groups, because if $\bar{K}_1 \trianglelefteq \bar{K} \in \mathfrak{m}'$ then $\bar{K}_1$ is normal in $\bar{N}_{i+1}$ as $\bar{K}$ is a direct factor and hence $\bar{K}_1 = \overline{\{1\}}$ or $\bar{K}_1 = \bar{K}$ by the minimality of $\bar{K}$. Therefore we can refine the chief series given above to a subnormal composition series of finite length. From Lemma 5.2 it then follows that G has finitely many subnormal subgroups.

We can also give another class of groups where the number of subnormal subgroups is equal to the number of 2-step subnormal subgroups:

Proposition 5.4 Let G be an uncountable soluble group of cardinality $\underline{n}$. Then G has $2^{\underline{n}}$ two-step subnormal subgroups.

Proof: The group G has a finite derived series $G = G^{(0)} \triangleright G^{(1)} \triangleright \dots \triangleright G^{(n)} = \{1\}$. One of the factors $G^{(i)}/G^{(i+1)}$ must have the same cardinality as G . Therefore it is an abelian group of cardinality $\underline{n}$. By [18], Theorem 9, it has $2^{\underline{n}}$ subgroups. Their inverse images in G are 2-step subnormal subgroups of G .

As uncountable soluble groups already have the maximal possible number of 2-step subnormal subgroups, their number of all subnormal subgroups cannot be greater. However we shall now construct non-soluble groups in which the number of 3-step subnormal subgroups is greater than the number of 2-step subnormal subgroups.

Theorem 5.5 Let $\underline{m}$ and $\underline{n}$ be infinite cardinals such that $\underline{m} \leq \underline{n}$. Then there exists a group G with the following properties:

- (1) The cardinality of G is $\underline{n}$.
- (2) The number of normal subgroups of G is $\aleph_0$.
- (3) The number of 2-step subnormal subgroups of G is $\underline{m}$.
- (4) The number of 3-step subnormal subgroups of G is $2^{\underline{n}}$.

Proof: Let Γ be an ordinal number of cardinality $\underline{m}$. Let $\Lambda = \Gamma \times \mathbb{Z}$ where $\mathbb{Z}$ denotes the integers. We define a total order on Λ by $(\gamma_1, n_1) < (\gamma_2, n_2)$ if and only if (i) $n_1 < n_2$ or (ii) $n_1 = n_2$ and $\gamma_1 < \gamma_2$. The mapping $t: (\gamma, n) \mapsto (\gamma, n+1)$ is an automorphism of Λ . Let A be the infinite cyclic group generated by t . If (γ_1, n_1) and (γ_2, n_2) are elements of Λ , then $(\gamma_1, n_1)t^{n_2-n_1+1} = (\gamma_1, n_2+1) > (\gamma_2, n_2)$. Let S be a simple transitive permutation group of cardinality $\underline{n}$ on a set Ω of cardinality $\underline{n}$. Let $W = \text{wr}^\Lambda S$, and let G be the semidirect product of W by A (in the way described

in chapter 3). We shall now prove that G has the properties (1) - (4) claimed in the statement of the theorem.

(1) We have $|S| = \underline{n}$ and $|\Lambda| = \underline{m} \leq \underline{n}$. The wreath power W is generated by $|\Lambda|$ copies of the group S , hence $|G| = |W| = \underline{n}$.

(2) By a slightly stronger version of Theorem 3.4 (given in [17], Theorem 6.25) the group W is the monolith (i.e. the unique minimal normal subgroup) of G . Therefore all non-trivial normal subgroups of G are of the form WB where B is a subgroup of A . Hence there are only countably many normal subgroups of G .

(3) If $\{1\} \neq B \leq A$ then B has also the property that for $(\gamma_1, n_1), (\gamma_2, n_2) \in \Lambda$ there exists $b \in B$ such that $(\gamma_1, n_1)b > (\gamma_2, n_2)$. Hence W is also the monolith of WB , and all normal subgroups of WB are already normal in G . So the subnormal subgroups of defect 2 in G are just the non-trivial normal subgroups of W .

Let Σ be a semi-ideal of Λ . Then either $\Sigma = \emptyset$ or $\Sigma = \Lambda$ or there exists an integer n_0 which is minimal with respect to the property that there exists $\gamma \in \Gamma$ such that $(\gamma, n_0) \notin \Sigma$. But as Γ is well-ordered there exists $\gamma_0 \in \Gamma$ which is minimal with respect to $(\gamma_0, n_0) \notin \Sigma$. But then clearly $\Sigma = \{(\gamma, n) \in \Lambda \mid (\gamma, n) < (\gamma_0, n_0)\}$, and hence $S(\Lambda) = |\Lambda| = \underline{m}$. By Theorem 3.5 the group W has $S(\Lambda) = \underline{m}$ normal subgroups.

(4) Let D_Σ for $\emptyset \neq \Sigma \neq \Lambda$ be one of the normal subgroups of W . Let $\Delta = \text{Dr}_{\lambda \in \Lambda - \Sigma} \Omega_\lambda$ with $\Omega_\lambda = \Omega$. By the note after Theorem 4.2 the group D_Σ is a direct product of copies of $\text{wr}^\Sigma S$ indexed by Δ . For every subset Δ' of Δ the group generated by those copies with indices in Δ' is normal in D_Σ , and hence is subnormal of defect 3 in G . As $|\Delta| = |\Omega| = \underline{n}$, there are $2^{\underline{n}}$ such subgroups.

Theorem 5.6 Let $\underline{n}$ be an infinite cardinal number. Then there exists a group G of cardinality $\underline{n}$ such that

- (1) the number of normal subgroups of G is 3,
- (2) the number of 2-step subnormal subgroups of G is $2^{\aleph_0}$,
- (3) the number of 3-step subnormal subgroups of G is $2^{\underline{n}}$.

Proof: Let S be a simple transitive permutation group of order $\underline{n}$ on a set Ω of cardinality $\underline{n}$. Let $\mathbb{Q}$ be the set of rational numbers in their natural order. Let A be the group consisting of the order automorphisms f of $\mathbb{Q}$ of the following form: for f there exist $x_0 < \dots < x_n \in \mathbb{Q}$ and $y_0 < \dots < y_n \in \mathbb{Q}$ (n finite) such that $xf = x$ for $x \leq x_0$ and $x \geq x_n$, and

$$xf = y_{i-1} + \frac{x - x_{i-1}}{x_i - x_{i-1}}(y_i - y_{i-1})$$

for $x_{i-1} \leq x \leq x_i$ and $1 \leq i \leq n$. Let B be the derived subgroup of A . Let $W = \text{wr}^{\mathbb{Q}} S$, and let $G = WB$ be the semi-direct product as before.

We first note that A (and hence B) is a countable group, as every element of A is determined by finitely many rational numbers $x_0 = y_0$, x_1 , $\dots$, x_{n-1} , $x_n = y_n$, y_1 , $\dots$, y_{n-1} . Hence G is a group of cardinality $\underline{n}$. We shall now prove that G has the properties (1) - (3) claimed above.

(1) In [9], Graham Higman has proved the following theorem: let A be a permutation group on a set Δ . For $f \in A$ let $D(f) = \{\delta \in \Delta \mid \delta f \neq \delta\}$. A subset Δ' of Δ is called totally unstable with respect to $f \in A$ if $\Delta' \cap (\Delta'f)$ is empty. If for every triple f_1 , f_2 , g ($g \neq 1$) of elements of A there exists an element h such that $(D(f_1) \cup D(f_2))h$ is totally unstable with respect to g then the derived group of A is a simple group.

Let f_1 , f_2 , g ($g \neq 1$) be elements of the group A defined above. As $D(f)$ is bounded for every $f \in A$ there exist elements $l < u$ of $\mathbb{Q}$ such that $D(f_1) \cup D(f_2) \subseteq \{q \in \mathbb{Q} \mid l < q < u\} =: I(l, u)$. As $g \neq 1$ there exist $q_1 < q_2$ in $\mathbb{Q}$ such that $(I(q_1, q_2))g \cap I(q_1, q_2)$ is empty (first it follows that there exists q_0 in $\mathbb{Q}$ such that $q_0 g \neq q_0$, but it is clear that there is a whole neighbourhood of q_0 that is mapped by g onto a disjoint subset). Let $L := \min(l, q_1)$ and $M := \max(u, q_2)$. We define the mapping $h: \mathbb{Q} \longrightarrow \mathbb{Q}$ by

$$xh = \begin{cases} x & \text{for } x \leq L-1 \text{ and } x \geq M+1 \\ L-1 + \frac{x-L+1}{1-L+1}(q_1-L+1) & \text{for } L-1 \leq x \leq 1 \\ q_1 + \frac{x-1}{u-1}(q_2-q_1) & \text{for } 1 \leq x \leq u \\ q_2 + \frac{x-u}{M+1-u}(M+1-q_2) & \text{for } u \leq x \leq M+1 \end{cases}$$

Then clearly h is an element of A , and we have $(D(f_1) \cup D(f_2))h \subseteq I(q_1, q_2)$. Hence $(D(f_1) \cup D(f_2))h$ is totally unstable with respect to g . By Higman's Theorem therefore B is simple.

For every $q \in \mathbb{Q}$ we define $f_q \in A$ by

$$xf_q = \begin{cases} x & \text{for } x \leq q-1 \text{ and } x \geq q+2 \\ q-1 + 2(x-q+1) & \text{for } q-1 \leq x \leq q \\ q+1 + \frac{1}{2}(x-q) & \text{for } q \leq x \leq q+2 \end{cases}$$

Let $h_q = f_q f_{q+1} f_q^{-1} f_{q+1}^{-1}$. Then $h_q \in B$ and $qh_q = q+1$.

Therefore B has the property that for $q_1, q_2 \in \mathbb{Q}$ there exists $b \in B$ such that $q_1 b > q_2$. As before W is the monolith of G , and hence $\{1\}$, W and G are the only normal subgroups of G .

We remark that even a stronger property than the existence of $b \in B$ such that $q_1 b > q_2$ holds for the group B . Namely, for every pair of ordered n -tuples $(x_1, \dots, x_n), (y_1, \dots, y_n)$ of elements of $\mathbb{Q}$ (where $x_i < x_{i+1}$ and $y_i < y_{i+1}$) there exists an element $h \in B$ such that $x_i h = y_i$ for $1 \leq i \leq n$. To prove this, we

first observe that there exists an element $f \in A$ such that $x_i f = y_i$ for $1 \leq i \leq n$. Now let $g \in A$ be chosen such that $D(f)$ is totally unstable with respect to g (such permutations f and g can be constructed explicitly by using similar methods as above). Then we define $h = f g f^{-1} g^{-1}$. Then x_i is mapped under f to y_i . This is mapped under g to $y_i g$. But as $D(f)$ is totally unstable with respect to g , the element $y_i g$ is left invariant under f and hence also under f^{-1} . Then $y_i g$ is mapped back to y_i under g^{-1} . Therefore h has the required property.

(2) Clearly the 2-step subnormal subgroups of G are just G and the normal subgroups of W . By Theorem 3.5 the normal subgroups of W correspond to the semi-ideals of $\mathcal{Q}$, and hence (Theorem 2.5) there are $2^{\aleph_0}$ of them.

(3) The proof that there are 2^n three-step subnormal subgroups of G is the same as the proof in the preceding theorem.

We note that the type of construction used in Theorems 5.5 and 5.6 can not be used to construct groups which have countably many 3-step subnormal subgroups but uncountably many subnormal subgroups:

Proposition 5.7 Let $H \neq \{1\}$ be a transitive permutation group on the set Ω , and let $\Lambda \neq \emptyset$ be a partially ordered set with a group A of automorphisms satisfying conditions (1) and (2) of Theorem 2.1.

Let $\underline{n}$ be the cardinality of the set $\{\lambda \in \Lambda \mid \lambda > \mu\}$ for some $\mu \in \Lambda$ (this is the same for all μ , because of Theorem 2.1), and assume $\underline{n} \neq 0$. Let $W = \text{wr}^\Lambda H$, and $G = WA$. Then $|G| = \max(|H|, |\Lambda|, |A|)$, and G has at least $2^{\max(|\Omega|, \underline{n})}$ three-step subnormal subgroups.

Proof: From $\underline{n} \neq 0$ it follows by Theorem 2.1 that $\underline{n}$ (and hence $|\Lambda|$) is infinite. The group W is generated by copies of H indexed by Λ , hence $|W| = \max(|H|, |\Lambda|)$, and as $G = WA$, we have $|G| = \max(|W|, |A|) = \max(|H|, |\Lambda|, |A|)$. Furthermore we have $W \triangleleft G$. Let μ be an element of Λ . Then $\Sigma = \{\lambda \in \Lambda \mid \lambda \leq \mu\}$ is a semi-ideal of Λ and $D_\Sigma \triangleleft W$. By Theorem 4.2 there is an injection from the set of semi-ideals of $\bar{\Sigma} = \{(\sigma, \omega) \mid \sigma \in \Sigma, \omega \in \text{Dr}_{\lambda \in F(\sigma)} \Omega_\lambda\}$ (with $\Omega_\lambda = \Omega$) into the set of normal subgroups of D_Σ . Let $\Gamma = \{(\mu, \omega) \mid \omega \in \text{Dr}_{\lambda \in F(\mu)} \Omega_\lambda\}$, then elements of Γ are incomparable, and hence every subset of Γ gives rise to a distinct semi-ideal of $\bar{\Sigma}$. Therefore the number of semi-ideals of $\bar{\Sigma}$ is at least $2^{|\Gamma|}$. But $F(\mu) = \{\lambda \in \Lambda \mid \lambda > \mu\}$, and hence $|\Gamma| = \max(|\Omega|, \underline{n})$, and therefore D_Σ has at least $2^{\max(|\Omega|, \underline{n})}$ normal subgroups, which are all three-step subnormal in G .

Corollary 5.8 Let G be as in Proposition 5.7, and suppose that G is countable. Then G has continuously many 3-step subnormal subgroups.

Proof: As G is countable, the set Λ has to be countable. Also $\underline{n} = \aleph_0$, and A , H and Ω can be at most countable. The rest follows from Proposition 5.7.

From Lemma 5.2 we know that for a group G it is equivalent to have finitely many subnormal subgroups and to have a finite composition series. We now ask the corresponding question for normal subgroups instead of subnormal subgroups: What is the relation between the property of having finitely many normal subgroups and the property of having a finite chief series?

Lemma 5.9 Let $(L, \wedge, \vee, 0, 1)$ be a distributive lattice of finite length n . Then L has at most 2^n elements.

Proof: We prove this by induction on n . The cases $n = 0$ and $n = 1$ are trivial. So suppose that $n > 1$. Let m be an atom (i.e. a minimal non-zero element) of L . Then the set $L_m = \{x \in L \mid x \geq m\}$ forms a distributive lattice of length $n-1$, and L_m has at most 2^{n-1} elements by induction hypothesis. We define a map $\varphi: (L - L_m) \longrightarrow L_m$ by $a\varphi = m \vee a$. Let $a, b \in L - L_m$, and suppose $a\varphi = b\varphi$, that is, $m \vee a = m \vee b$. Hence we have $a \wedge (m \vee a) = a \wedge (m \vee b)$ and, by distributivity, $(a \wedge m) \vee (a \wedge a) = (a \wedge m) \vee (a \wedge b)$. But $a \wedge m = 0$, and hence $a \leq b$. Similarly $b \leq a$, and hence $a = b$,

and φ is injective. Therefore $|L - L_m| \leq |L_m|$, and $|L| \leq 2|L_m| \leq 2 \cdot 2^{n-1} = 2^n$ as required.

Theorem 5.10 (A) Let G be a group with distributive lattice of normal subgroups and a chief series of finite length n . Then G has at most 2^n normal subgroups.

(B) Let $\underline{n}$ be an infinite cardinal number. Then there exists a group G of cardinality $\underline{n}$ that has a chief series of length 4 and that has $\underline{n}$ normal subgroups.

Proof: (A) follows immediately from Lemma 5.9.

(B) Let K be a field of cardinality $\underline{n}$ and prime characteristic $p \neq 2$. Let $X_1 \cong X_2 \cong K^2$, and let $S := \text{SL}(2, K)$ act naturally on X_1 and X_2 . Let $Z = Z(S)$ be the centre of S . Let X be the direct sum of X_1 and X_2 and let $G = XS$ be the semidirect product where S acts on X componentwise.

For $q \in K$ let $T_q = \{(x, qx) \mid x \in X_1\}$. Using Lemma 1.8 and the same methods as in Proposition 1.7 it is not difficult to see that the lattice of normal subgroups of G has the form given in figure 5.1, which proves the theorem.

We note that Theorem 5.10 (B) is the worst possible example: a group G of cardinality $\underline{n}$ with chief series of finite length m can have at most $\underline{n}$ normal subgroups, as every normal subgroup of G can be, qua normal sub-

Figure 5.1

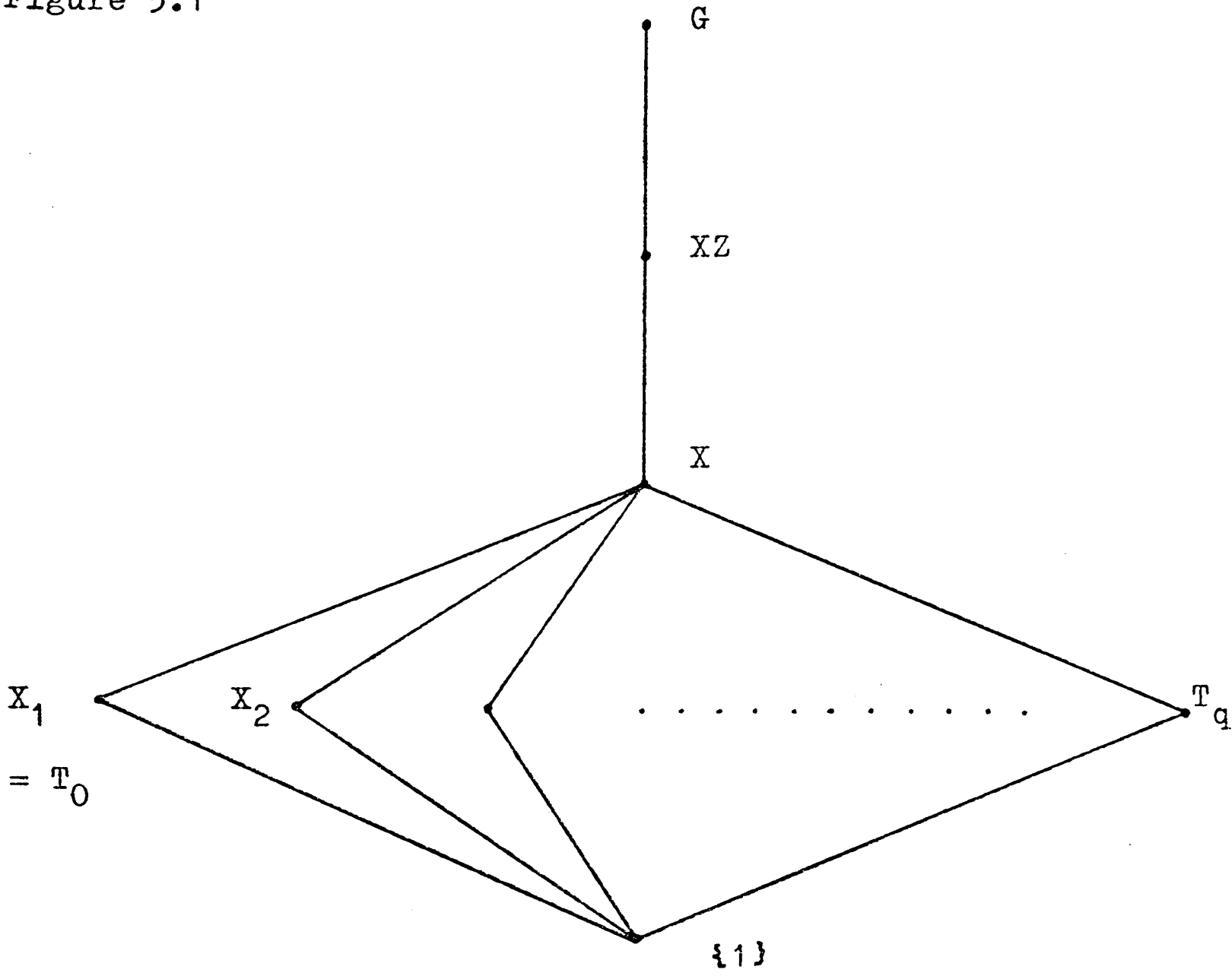
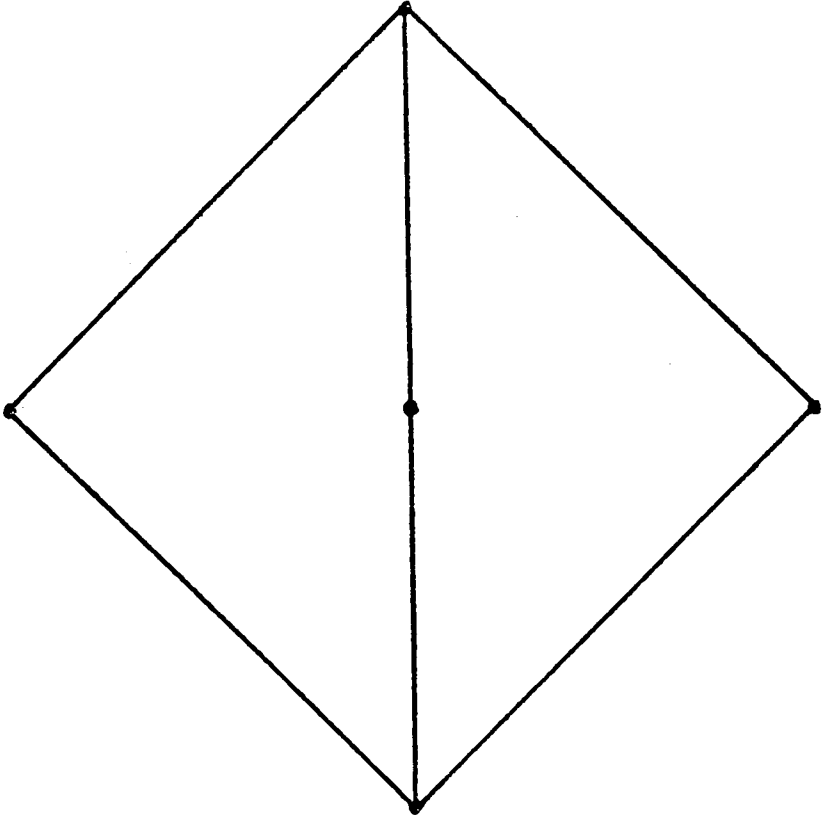


Figure 5.2



group, generated by at most m elements.

It is well known that for a modular lattice to be distributive it is necessary and sufficient that it does not contain a sublattice isomorphic to the lattice given in figure 5.2. From this it follows that any group G whose lattice of normal subgroups is not distributive will contain normal subgroups $N_1 \neq N_2$ such that $N_1/N_1 \cap N_2 \cong N_2/N_1 \cap N_2 \cong A$ where A is abelian. Hence if G is a group with no abelian chief factor then the lattice of normal subgroups of G is distributive.

Chapter 6

Characteristically simple groups

Characteristically simple groups with countably many normal subgroups, namely certain wreath powers, played an important role in the construction of a group with countably many 2-step but continuously many 3-step subnormal subgroups. We shall now look at these groups in general, and we shall construct a further type of characteristically simple group.

We recall that if $W = \text{wr}_{\lambda \in \Lambda} S_\lambda$ is a generalized wreath product where S_λ is a ^{non-abelian} simple group for $\lambda \in \Lambda$ then the lattice of normal subgroups of W is isomorphic to the lattice of semi-ideals of Λ . Furthermore the chief factors of W , that is, the factors M/N where M and N with $N < M$ are both normal subgroups of W , and where no normal subgroup of W lies properly between N and M , are isomorphic to certain direct powers of one of the simple groups S_λ . Note that we always have some chief factor being a proper power of some S_λ if there exist $\lambda, \mu \in \Lambda$ with $\lambda < \mu$. We shall now construct a group with a similar lattice of normal subgroups, but where all chief factors are simple groups.

A partially ordered set Λ is called a tree if for every pair of incomparable elements λ and μ of Λ there does not exist $\eta \in \Lambda$ such that $\lambda \leq \eta$ and $\mu \leq \eta$ (or

equivalently, if for every $\lambda \in \Lambda$ the set $\{\mu \in \Lambda \mid \mu \leq \lambda\}$ is a chain).

Let Λ be a partially ordered set. For $\lambda \in \Lambda$ let M_λ be the set of maximal chains in $\{\mu \in \Lambda \mid \mu \leq \lambda\}$. Note that λ is the maximal element of each member of M_λ . Let $T(\Lambda) = \bigcup_{\lambda \in \Lambda} M_\lambda$. The set $T(\Lambda)$ is partially ordered by set-theoretic inclusion. The set $T(\Lambda)$ is, in fact, a tree, for let $c_1, c_2 \in T(\Lambda)$ with c_1, c_2 incomparable, and suppose that $c_1 \subset c_3$ and $c_2 \subset c_3$. Let μ_i be the maximal element of c_i for $i \in \{1, 2, 3\}$. Then $c_j = \{\lambda \in c_3 \mid \lambda \leq \mu_j\}$ for $j \in \{1, 2, 3\}$. But then $c_1 \subset c_2$ or $c_2 \subset c_1$, which is a contradiction. We call $T(\Lambda)$ the tree associated to Λ . Note that if Λ is already a tree then Λ is order-isomorphic to $T(\Lambda)$, as M_λ consists of only one element for each $\lambda \in \Lambda$. Let $C_\lambda = T(\{\mu \in \Lambda \mid \mu \geq \lambda\})$, and let $c \in M_\lambda$. Then $\varphi_c: d \mapsto d \cup c$ is an order-isomorphism from C_λ to $\{e \in T(\Lambda) \mid e \supseteq c\}$.

Let $\mathbb{Z}$ denote the set of integers. Let $\Omega = \mathbb{Z}^{T(\Lambda)}$. For any subset Γ of $T(\Lambda)$, and $z = (z_c)_{c \in T(\Lambda)} \in \Omega$ let $\pi_\Gamma(z) = (z_c)_{c \in \Gamma}$. If $T(\Lambda) = \bigcup_{i \in I} \Gamma_i$ is any partition of $T(\Lambda)$, we identify z with $(\pi_{\Gamma_i}(z))_{i \in I}$ in the obvious way. For $c \in T(\Lambda)$ let $U(c) = \{d \in T(\Lambda) \mid d \supseteq c\}$. For $\lambda \in \Lambda$ let $\Delta_\lambda = \mathbb{Z}^{C_\lambda}$. Note that φ_c induces a bijection $\psi_c: \Delta_\lambda \longrightarrow \pi_{U(c)}(\Omega)$ by

$(z\psi_c)_d = z_{d\psi_c^{-1}} = z_{d-c\cup\{\lambda\}}$. Let $W(\lambda) = T(\Lambda) - \bigcup_{c \in M_\lambda} U(c)$.

We define an embedding $j_\lambda: AF(\Delta_\lambda) \longrightarrow \text{Sym}(\Omega)$ in the following way. Let $g \in AF(\Delta_\lambda)$, $z \in \Omega$. Then

$$z \cdot j_\lambda(g) = (\pi_{W(\lambda)}(z), (\pi_{U(c)}(z) \cdot \psi_c^{-1} g \psi_c)_{c \in M_\lambda}).$$

Denote the image of $AF(\Delta_\lambda)$ under j_λ by H_λ . For every semi-ideal Σ of Λ let $K_\Sigma = \langle H_\mu \mid \mu \in \Sigma \rangle$, and let $G(\Lambda) = \langle H_\mu \mid \mu \in \Lambda \rangle$.

Lemma 6.1 If μ, λ are incomparable then $[H_\mu, H_\lambda] = \{1\}$.

Proof: If $z \in \Omega$ then elements of H_λ and H_μ only affect those components of z which are indexed by elements of $\bigcup_{c \in M_\lambda} U(c)$ and $\bigcup_{d \in M_\mu} U(d)$ respectively.

As λ and μ are incomparable, we have

$(\bigcup_{c \in M_\lambda} U(c)) \cap (\bigcup_{d \in M_\mu} U(d)) = \emptyset$, and hence all elements of H_λ commute with all elements of H_μ .

Lemma 6.2 If $\mu \leq \lambda$, then H_λ normalizes H_μ .

Proof: Let $g \in AF(\Delta_\lambda)$, and let $y = (y_d)_{d \in C_\mu} \in \Delta_\mu$.

Let $N_\lambda = \{e \in C_\mu \mid \lambda \text{ is the maximal element of } e\}$.

If $e \in N_\lambda$ then there is a natural bijection

$\bar{\psi}_e: \Delta_\lambda \longrightarrow \pi_{S_e}(\Delta_\mu)$ induced by $\bar{\psi}_e$ where $c\bar{\psi}_e = e \cup c$

for $c \in C_\lambda$, and where $S_e = \{f \in C_\mu \mid f \supseteq e\}$. Let

$R = C_\mu - \bigcup_{e \in N_\lambda} S_e$. We define $h \in \text{Sym}(\Delta_\mu)$ by

$yh = (\pi_R(y), (\pi_{S_e}(y) \bar{\psi}_e^{-1} g \bar{\psi}_e)_{e \in N_\lambda})$. Then there exists

$g_1 \in \text{Sym}(\pi_{W(\mu)}(\Omega))$ such that

$$z \cdot j_\lambda(g) = (\pi_{W(\mu)}(z) \cdot g_1, (\pi_{U(c)}(z) \cdot \psi_c^{-1} h \psi_c)_{c \in M_\mu}) \text{ for}$$

all $z \in \Omega$. Hence if $k \in \text{AF}(\Delta_\mu)$ then

$$j_\lambda(g)^{-1} j_\mu(k) j_\lambda(g) = j_\mu(h^{-1} k h), \text{ and as } h^{-1} k h \in \text{AF}(\Delta_\mu),$$

we have $j_\lambda(g)^{-1} j_\mu(k) j_\lambda(g) \in H_\mu$.

From Lemma 6.1 and Lemma 6.2 we immediately get

Corollary 6.3 If Σ is a semi-ideal of Λ then K_Σ is a normal subgroup of $G(\Lambda)$. If Σ_1 and Σ_2 are semi-ideals with $\Sigma_2 = \Sigma_1 \cup \{\mu\}$ then $K_{\Sigma_2} = K_{\Sigma_1} H_\mu$, the semi-direct product.

Proposition 6.4 (a) Every element of $G(\Lambda)$ can be written in the form $h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_r}$ where $\lambda_i \neq \lambda_j$ whenever $i < j$, and $1 \neq h_{\lambda_i} \in H_{\lambda_i}$ for $i, j \in \{1, 2, \dots, r\}$ and $r \geq 0$.

(b) Let $g_1 = g_{\lambda_1} \dots g_{\lambda_r}$, $g_2 = g_{\mu_1} \dots g_{\mu_s} \in G(\Lambda)$ with $\lambda_i \neq \lambda_j$ and $\mu_i \neq \mu_j$ for $i < j$ and $1 \neq g_{\lambda_i} \in H_{\lambda_i}$, $1 \neq g_{\mu_j} \in H_{\mu_j}$ for $1 \leq i \leq r$ and $1 \leq j \leq s$ with $r, s \geq 0$. Then $g_1 = g_2$ if and only if

$$\{g_{\lambda_1}, \dots, g_{\lambda_r}\} = \{g_{\mu_1}, \dots, g_{\mu_s}\}.$$

This follows as in Lemma 3.2

Lemma 6.5 Let $1 \neq h = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_r} \in G(\Lambda)$, $\lambda_i \neq \lambda_j$ for $i < j$, $1 \neq h_{\lambda_i} \in H_{\lambda_i}$. Let Σ be the minimal semi-ideal containing $\{\lambda_1, \dots, \lambda_r\}$, that is,
 $\Sigma = \{\mu \in \Lambda \mid \text{there exists } i \in \{1, 2, \dots, r\} \text{ such that } \mu \leq \lambda_i\}$.
Then the normal closure N of h in K_Σ contains H_{λ_1} .

Proof: If $r = 1$ then $h \in H_{\lambda_1}$, and as H_{λ_1} is simple N must contain H_{λ_1} . So let $r > 1$. Now $h = h_{\lambda_1} k$ with $k = h_{\lambda_2} \dots h_{\lambda_r} \neq 1$. We have
 $[k, h_{\lambda_1}^{-1}] = h_{\lambda_1}^k h_{\lambda_1}^{-1} = (h_{\lambda_1} k)^k (h_{\lambda_1} k)^{-1} \in N$, and also
 $[k, h_{\lambda_1}^{-1}] \in H_{\lambda_1}$. Now if $[h_{\lambda_1}^{-1}, k] \neq 1$ we have $H_{\lambda_1} \leq N$ as above. So we have only to consider the case that
 $[h_{\lambda_1}^{-1}, k] = 1$, and also $[h_{\lambda_1}, k] = 1$.

First suppose that $\lambda_2, \dots, \lambda_r$ are all incomparable with λ_1 . Then choose $h'_{\lambda_1} \in H_{\lambda_1}$ such that $[h'_{\lambda_1}, h_{\lambda_1}] \neq 1$.
Then $(h_{\lambda_1} k)^{h'_{\lambda_1}} (h_{\lambda_1} k)^{-1} = [h'_{\lambda_1}, h_{\lambda_1}^{-1}] \in N \cap H_{\lambda_1}$, and hence $H_{\lambda_1} \leq N$ as above.

So we are left with the case that there exists $j \in \{2, \dots, r\}$ such that $\lambda_1 < \lambda_j$. Now we can choose $y \in \Delta_{\lambda_1}$ such that $(z, (y\psi_c)_{c \in M_{\lambda_1}})$ is fixed under h_{λ_1} for all $z \in \pi_{W(\lambda_1)}(\Omega)$ but moved by k to $(zt_k, (y'\psi_c)_{c \in M_{\lambda_1}})$ with $y' \in \Delta_{\lambda_1}$, $y' \neq y$ and $t_k: \pi_{W(\lambda_1)}(\Omega) \longrightarrow \pi_{W(\lambda_1)}(\Omega)$.

Also choose $y_1, y_2, y_3 \in \Delta_{\lambda_1}$ such that $(z, (y_i \gamma_c)_{c \in M_{\lambda_1}})$ is fixed under h_{λ_1} for all $z \in \pi_{W(\lambda_1)}(\Omega)$ and moved by k to $(zt_{i,k}, (y_i \gamma_c)_{c \in M_{\lambda_1}})$ with some $t_{i,k}: \pi_{W(\lambda_1)}(\Omega) \longrightarrow \pi_{W(\lambda_1)}(\Omega)$ for $i \in \{1, 2, 3\}$. Now let $g = (yy_1)(y_2y_3) \in \text{AF}(\Delta_{\lambda_1})$. We obviously have $[j_{\lambda_1}(g), h_{\lambda_1}] = 1$. Furthermore

$$(h_{\lambda_1} k)^{j_{\lambda_1}(g)} = h_{\lambda_1} k^{j_{\lambda_1}(g)} \in N, \text{ and } [k, j_{\lambda_1}(g)] \neq 1 \text{ as}$$

$$(z, (y_1 \gamma_c)_{c \in M_{\lambda_1}})^{j_{\lambda_1}(g)k} = (z, (y \gamma_c)_{c \in M_{\lambda_1}})^k$$

$$= (zt_k, (y' \gamma_c)_{c \in M_{\lambda_1}}), \text{ and}$$

$$(z, (y_1 \gamma_c)_{c \in M_{\lambda_1}})^{kj_{\lambda_1}(g)} = (zt_{1,k}, (y_1 \gamma_c)_{c \in M_{\lambda_1}})^{j_{\lambda_1}(g)}$$

$$= (zt_{1,k}, (y \gamma_c)_{c \in M_{\lambda_1}}) \text{ and } y \neq y'. \text{ Then, as above, we}$$

have $H_{\lambda_1} \leq N$.

Corollary 6.6 Let $h = h_{\lambda_1} \dots h_{\lambda_r} \in G(\Lambda)$, $\lambda_i \neq \lambda_j$ for $i < j$, $1 \neq h_{\lambda_i} \in H_{\lambda_i}$. Let $\Sigma = \{\mu \in \Lambda \mid \text{there exists } j \text{ such that } \mu \leq \lambda_j\}$. Then the normal closure of h in K_Σ is K_Σ .

Proof: Let N be the normal closure of h in K_Σ . Using Lemma 6.5, we can show by induction that $H_{\lambda_j} \leq N$ for $j \in \{1, 2, \dots, r\}$. Let $\mu < \lambda_j$. We can find elements $h_\mu \in H_\mu$ and $h_{\lambda_j} \in H_{\lambda_j}$ such that $[h_\mu, h_{\lambda_j}] \neq 1$. But

$[h_\mu, h_{\lambda_j}] \in H_\mu \cap N$, and hence as in Lemma 6.5 it follows that $H_\mu \leq N$. Therefore $K_\Sigma \leq N$.

Proposition 6.7 Every subnormal subgroup of $G(\Lambda)$ is normal in $G(\Lambda)$, and the lattice of normal subgroups of $G(\Lambda)$ is isomorphic to the lattice of semi-ideals of Λ .

Proof: For every semi-ideal Σ of Λ we have seen in Corollary 6.3 that K_Σ is a normal subgroup of $G(\Lambda)$.

Let N be a normal subgroup of $G(\Lambda)$. For $1 \neq h \in N$ let

$\Gamma(h) = \{\lambda_j \mid \text{there occurs } h_{\lambda_j} \neq 1 \text{ in the expression}$

$h = h_{\lambda_1} \dots h_{\lambda_r} \text{ with } 1 \neq h_{\lambda_j} \in H_{\lambda_j}\}$. If

$\Sigma(h) = \{\mu \in \Lambda \mid \text{there exists } \lambda_j \in \Gamma(h) \text{ with } \mu \leq \lambda_j\}$

then by Corollary 6.6 we have $K_{\Sigma(h)} \leq N$. We therefore

also get $K_\Sigma \leq N$ where $\Sigma = \bigcup \{\Sigma(h) \mid 1 \neq h \in N\}$. But

clearly also $N \leq K_\Sigma$, hence $N = K_\Sigma$.

Now let N be a 2-step subnormal subgroup of G . Then N is a normal subgroup of some K_Σ . The same argument

as above applies with K_Σ instead of $G(\Lambda)$. Therefore

there exists a semi-ideal $\Sigma' \subseteq \Sigma$ such that $N = K_{\Sigma'}$.

So N is normal, and hence all subnormal subgroups of $G(\Lambda)$ are normal.

Proposition 6.8 The following are equivalent:

(i) $G(\Lambda)$ is characteristically simple.

(ii) For any $\lambda, \mu \in \Lambda$ there exists an automorphism a of Λ such that $\mu \leq \lambda a$.

Proof: By Theorem 2.1 the condition (ii) is equivalent to the following: for any semi-ideal Σ of Λ there exists an automorphism a of Λ such that $\Sigma a \neq \Sigma$.

Let a be an automorphism of Λ . If $c \in T(\Lambda)$ let $ca = \{\mu a \mid \mu \in c\}$. Then $c \in M_\lambda$ implies $ca \in M_{\lambda a}$. Define a permutation p_a of Ω by $(zp_a)_c = z_{ca}$ for $c \in T(\Lambda)$. For $c \in T(\Lambda)$ we get a bijection $\tilde{p}_a: \pi_{U(c)}(\Omega) \longrightarrow \pi_{U(ca)}(\Omega)$ by $\pi_{U(c)}(z)\tilde{p}_a = \pi_{U(ca)}(zp_a)$, and a similar bijection $\bar{p}_a: \pi_{W(\lambda)}(\Omega) \longrightarrow \pi_{W(\lambda a)}(\Omega)$.

Also define a bijection $q_a: \Delta_\lambda \longrightarrow \Delta_{\lambda a}$ by

$(yq_a)_d = y_{da}$ for $d \in C_\lambda$. Note that $\gamma_c \tilde{p}_a = q_a \gamma_{ca}$ for

$c \in T(\Lambda)$. Now let $g \in AF(\Delta_\lambda)$. Consider the maps

$p_a^{-1} j_\lambda(g) p_a$ and $j_{\lambda a}(q_a^{-1} g q_a)$. For $z \in \Omega$ we have

$\pi_{W(\lambda a)}(z \cdot j_{\lambda a}(q_a^{-1} g q_a)) = \pi_{W(\lambda a)}(z)$ and

$\pi_{U(ca)}(z \cdot j_{\lambda a}(q_a^{-1} g q_a)) = \pi_{U(ca)}(z) \cdot \gamma_{ca}^{-1} q_a^{-1} g q_a \gamma_{ca}$

for $ca \in M_{\lambda a}$. But we also have

$\pi_{W(\lambda a)}(z \cdot p_a^{-1} j_\lambda(g) p_a) = \pi_{W(\lambda)}(z \cdot p_a^{-1} j_\lambda(g)) \cdot \bar{p}_a$

$= \pi_{W(\lambda)}(zp_a^{-1}) \cdot \bar{p}_a = \pi_{W(\lambda a)}(z) \cdot \bar{p}_a^{-1} \bar{p}_a = \pi_{W(\lambda a)}(z)$ and

$\pi_{U(ca)}(z \cdot p_a^{-1} j_\lambda(g) p_a) = \pi_{U(c)}(z \cdot p_a^{-1} j_\lambda(g)) \cdot \tilde{p}_a$

$= \pi_{U(c)}(z \cdot p_a^{-1}) \cdot \gamma_c^{-1} g \gamma_c \tilde{p}_a = \pi_{U(ca)}(z) \cdot \tilde{p}_a^{-1} \gamma_c^{-1} g \gamma_c \tilde{p}_a$

$= \pi_{U(ca)}(z) \cdot \gamma_{ca}^{-1} q_a^{-1} g q_a \gamma_{ca}$. Hence we have

$p_a^{-1} j_\lambda(g) p_a = j_{\lambda a}(q_a^{-1} g q_a)$, and $p_a^{-1} H_\lambda p_a = H_{\lambda a}$.

Hence if Σ is a semi-ideal of Λ we get $p_a^{-1}K_\Sigma p_a = K_{\Sigma a}$, and as the mapping $g \mapsto p_a^{-1}gp_a$ is an automorphism of $G(\Lambda)$, the result follows from Proposition 6.7.

We can now sum up the results we get from the construction of the groups $G(\Lambda)$ and show some consequences. In [19], Howard L. Silcock used the construction of the wreath powers together with the fact that every finite distributive lattice is isomorphic to the lattice of semi-ideals of some partially ordered set to prove that every finite distributive lattice is isomorphic to the lattice of normal subgroups of a finite group. As the group $G(\Lambda)$ is countable if Λ is finite we get a weaker form of this representation theorem from Proposition 6.7, namely that every finite distributive lattice is isomorphic to the lattice of normal subgroups of a countable group. But Proposition 6.7 also gives a different representation theorem:

Theorem 6.9 For every finite distributive lattice L there exists a countable group G such that the subnormal subgroups of G form a lattice isomorphic to L .

We finally sum up our results on $G(\Lambda)$ in the following

Theorem 6.10 Let Λ be a partially ordered set, and let $\mathcal{J}(\Lambda)$ be the distributive lattice of semi-ideals of Λ . Then there exists a group $G(\Lambda)$ with the following properties:

- (1) $|G(\Lambda)| \leq \max(\aleph_0, 2^{2^{|\Lambda|}})$.
- (2) All subnormal subgroups of $G(\Lambda)$ are normal in $G(\Lambda)$.
- (3) The lattice of normal subgroups of $G(\Lambda)$ is isomorphic to $\mathcal{J}(\Lambda)$.
- (4) The group $G(\Lambda)$ is characteristically simple if and only if for $\lambda, \mu \in \Lambda$ there exists an automorphism α of Λ such that $\mu \leq \lambda\alpha$.

Proof: We let the group $G(\Lambda)$ be constructed as above. Then (2), (3) and (4) follow from Propositions 6.7 and 6.8. To show (1) first suppose that Λ is finite. Then C_λ is finite for each $\lambda \in \Lambda$. It follows that $|\Delta_\lambda| = \aleph_0$ for $\lambda \in \Lambda$ and $|H_\lambda| = \aleph_0$ for $\lambda \in \Lambda$. Then $|G(\Lambda)| = \aleph_0 = \max(\aleph_0, 2^{2^{|\Lambda|}})$. Now suppose that Λ is infinite. Then

$$|C_\lambda| \leq 2^{|\Lambda|}, \text{ and } |\Delta_\lambda| \leq \aleph_0^{|C_\lambda|} \leq \aleph_0^{2^{|\Lambda|}} = 2^{2^{|\Lambda|}}$$

for $\lambda \in \Lambda$. Then also $|H_\lambda| = |\Delta_\lambda| \leq 2^{2^{|\Lambda|}}$, and

$$|G(\Lambda)| \leq 2^{2^{|\Lambda|}} = \max(\aleph_0, 2^{2^{|\Lambda|}}).$$

We note that, in general, the cardinality of $G(\Lambda)$ will be strictly greater than the cardinality of $\mathcal{J}(\Lambda)$. But there are two ways of getting around this.

The first way only works for certain partially ordered sets Λ , and it involves a slight change in the definition of the group $G(\Lambda)$. We suppose that Λ has the property that M_λ is finite for each $\lambda \in \Lambda$ (as $|M_\lambda| = 1$ for all $\lambda \in \Lambda$ if and only if Λ is a tree, it follows that all trees and, in particular, all totally ordered sets have this property). We then let $\Omega = \{z = (z_c)_{c \in T(\Lambda)} \in \mathbb{Z}^{T(\Lambda)} \mid z_c = 0 \text{ for all but finitely many } c\}$ and also $\Delta_\lambda = \{y = (y_d)_{d \in C_\lambda} \in \mathbb{Z}^{C_\lambda} \mid y_d = 0 \text{ for all but finitely many } d\}$. We can go through the construction with Ω and Δ_λ substituted wherever they occur. The group we get will have the same properties as the original one but it usually will be smaller as in this group $|\Delta_\lambda| = |C_\lambda|$ if Λ is infinite.

The second way to arrive at a smaller group will require the development of some theory of lattices of normal subgroups in general. We shall start with

Lemma 6.11 Let L_i be the lattice of normal subgroups of G_i for $i \in \{1, 2\}$. Let $M_i = \{N \in L_i \mid \text{there exists a finite subset } \{g_1, \dots, g_n\} \text{ of } G_i \text{ such that } N = \langle g_1, \dots, g_n \rangle^{G_i}\}$. This is partially ordered by the order in L_i , and suprema of finitely many elements of M_i taken in L_i are in M_i . Suppose there exists an order-isomorphism $\varphi: M_1 \longrightarrow M_2$. Then there exists a lattice isomorphism $\psi: L_1 \longrightarrow L_2$.

Proof: For $N \in L_1$ we define $N\psi = \langle K\psi \mid K \in M_1, K \leq N \rangle$, and for $M \in L_2$ we define $M\chi = \langle L\varphi^{-1} \mid L \in M_2, L \leq M \rangle$. The maps $\psi: L_1 \longrightarrow L_2$ and $\chi: L_2 \longrightarrow L_1$ are clearly both well-defined and order-preserving. To show that ψ is a lattice isomorphism it is therefore enough to show that $\psi\chi$ and $\chi\psi$ are the identities on L_1 and L_2 respectively. Because of the symmetry of the situation it is sufficient to show that $\psi\chi$ is the identity on L_1 .

We note that for $N \in L_1$ we have $N = \langle K \in M_1 \mid K \leq N \rangle$. Let $M = N\psi = \langle K\psi \mid K \in M_1, K \leq N \rangle$, and let $R = M\chi = \langle L\varphi^{-1} \mid L \in M_2, L \leq M \rangle$. We have to show that $N = R$. For $K \in M_1$ with $K \leq N$ it follows that $K\psi \in M_2$ with $K\psi \leq M$. Hence $R = \langle L\varphi^{-1} \mid L \in M_2, L \leq M \rangle \supseteq \langle (K\psi)\varphi^{-1} \mid K \in M_1, K \leq N \rangle = \langle K \mid K \in M_1, K \leq N \rangle = N$.

So it remains to prove that $R \leq N$. For this it is enough to show that $L\varphi^{-1} \leq N$ for all $L \in M_2$ with $L \leq M$. So let $L \in M_2$ and $L \leq M$. Then there exist elements $g_1, \dots, g_n \in M$ such that $L = \langle g_1, \dots, g_n \rangle^{G_2}$. And, as $M = N\psi$ for each $i \in \{1, \dots, n\}$ there exist $K_{i1}, \dots, K_{im_i} \in M_1$ contained in N such that $g_i \in \langle K_{i1}\varphi, \dots, K_{im_i}\varphi \rangle = \langle K_{i1}, \dots, K_{im_i} \rangle\varphi$. But then $S = \langle K_{ij} \mid j \in \{1, \dots, m_i\}, i \in \{1, \dots, n\} \rangle \in M_1$ and $S \leq N$. Furthermore $L \leq S\varphi$, and hence $L\varphi^{-1} \leq S\varphi\varphi^{-1} = S \leq N$. This concludes the proof of the lemma.

Using this lemma we can prove

Theorem 6.12 (a) Let G be a group with normal subgroup lattice L such that $\aleph_0 \leq |L| < |G|$. Let G_0 be a subgroup of G such that $|G_0| \leq |L|$. Then there exists a subgroup H of G with $G_0 \leq H$ and $|H| \leq |L|$ such that the lattice of normal subgroups of H is isomorphic to L . The lattice isomorphism is given by the natural map $N \mapsto N \cap H$ for $N \in L$.

(b) Let G, L, G_0 be as in (a). Also assume that G is characteristically simple. Then we can construct H with the properties as in (a) such that, in addition, the group H is characteristically simple.

Proof: (a) Let $\underline{N} = \{N \trianglelefteq G \mid \text{there exists a finite subset } \{g_1, \dots, g_n\} \text{ of } G \text{ such that } N = \langle g_1, \dots, g_n \rangle^G\}$. Let $F(G)$ be the set of finite subsets of G . We get a partition $F(G) = \bigcup_{N \in \underline{N}} F_N(G)$ where $F_N(G) = \{S \in F(G) \mid \langle S \rangle^G = N\}$. For each $N \in \underline{N}$ choose $S_N \in F_N(G)$. Let $H_0 := \langle G_0, S_N \mid N \in \underline{N} \rangle$. Then $|H_0| \leq |L|$. Suppose we have H_n constructed with $|H_n| \leq |L|$. For $N \in \underline{N}$ consider $N \cap H_n$. Let $F(N \cap H_n)$ be the set of finite subsets of $N \cap H_n$. Then $|F(N \cap H_n)| \leq |L|$. For every $S \in F(N \cap H_n) \cap F_N(G)$ there exists a subset $T(S)$ of G with $|T(S)| \leq |L|$ such that $N \cap H_n \leq \langle S \rangle^{T(S)}$. Let $T_{n,N} := \bigcup \{T(S) \mid S \in F(N \cap H_n) \cap F_N(G)\}$. Then $|T_{n,N}| \leq |L|$. Let $H_{n+1} := \langle H_n, T_{n,N} \mid N \in \underline{N} \rangle$.

Then $|H_{n+1}| \leq |L|$, and we have $G_0 \leq H_0 \leq H_1 \leq \dots$.

Let $H := \bigcup_{n \in \mathbb{N}} H_n$. Then $|H| \leq |L|$.

We claim that $N \mapsto N \cap H$ is an order-preserving bijection between the set $\underline{N}$ and the set $\underline{M} = \{N \trianglelefteq H \mid \text{there exists a finite subset } S \text{ of } H \text{ such that } N = \langle S \rangle^H\}$.

We first have to show that $N \cap H$ lies in fact in $\underline{M}$ for every $N \in \underline{N}$. Note that $N \cap H = N \cap \bigcup_{n \in \mathbb{N}} H_n = \bigcup_{n \in \mathbb{N}} (N \cap H_n)$. Let S_N be the set chosen above, we have $S_N \leq N$ and $S_N \leq H_n$ for all $n \in \mathbb{N}$. Hence $S_N \leq N \cap H$ and as $N \cap H \trianglelefteq H$ we have $\langle S_N \rangle^H \leq N \cap H$. Let $n \in \mathbb{N}$ then we have $N \cap H_n \leq \langle S_N \rangle^{T(S_N)} \leq \langle S_N \rangle^H$. Hence $N \cap H \leq \langle S_N \rangle^H$. Therefore $N \cap H = \langle S_N \rangle^H$, and hence $N \cap H \in \underline{M}$. Also the map $N \mapsto N \cap H$ is clearly order-preserving.

We next show it is surjective. Let U be a finite subset of H . We have to find $N \in \underline{N}$ such that $\langle U \rangle^H = N \cap H$. Let $N = \langle U \rangle^G$, then there exists $n \in \mathbb{N}$ such that $U \leq H_n$. We clearly have $\langle U \rangle^H \leq N \cap H$. But as $U \in F(N \cap H_m) \cap F_N(G)$ we have $N \cap H_m \leq \langle U \rangle^{T(U)} \leq \langle U \rangle^H$ for all $m \geq n$. Therefore we get $N \cap H \leq \langle U \rangle^H$.

We finally show the map is injective. Let $N_1 \neq N_2$. Then there exists $g \in S_{N_2}$ such that $g \notin N_1$ or there exists $h \in S_{N_1}$ such that $h \notin N_2$. But then $g \in N_2 \cap H$ and $g \notin N_1 \cap H$ or $h \in N_1 \cap H$ and $h \notin N_2 \cap H$. Hence $N_1 \cap H \neq N_2 \cap H$.

As we have an order-preserving bijection between $\underline{N}$ and $\underline{M}$ it follows from Lemma 6.11 that the lattices of normal subgroups of G and H are isomorphic, and that the isomorphism is given by $N \longmapsto N \cap H$.

(b) Let $\underline{N}$, $F(G)$, $F_N(G)$, S_N , H_0 be as in the proof of (a). Again we have $|H_0| \leq |L|$ and $|G_0| \leq |H_0|$. Let $A_0 := \{1\} \leq \text{Aut}(G)$. Suppose we have $H_n \leq G$ and $A_n \leq \text{Aut}(G)$ constructed with $|H_n| \leq |L|$ and $|A_n| \leq |L|$. For $N \in \underline{N}$ consider $N \cap H_n$. Let $F(N \cap H_n)$ be the set of finite subsets of $N \cap H_n$, then $|F(N \cap H_n)| \leq |L|$. For every $S \in F(N \cap H_n) \cap F_N(G)$ there exists a subset $T(S)$ of G with $|T(S)| \leq |L|$ such that $N \cap H_n \leq \langle S \rangle^{T(S)}$. Let $T_{n,N} := \bigcup \{T(S) \mid S \in F(N \cap H_n) \cap F_N(G)\}$. Then $|T_{n,N}| \leq |L|$. As G is characteristically simple, for each $h \in H_n$ there exists a subset $B_n(h)$ of $\text{Aut}(G)$ such that $H_n \leq \langle h\alpha \mid \alpha \in B_n(h) \rangle$ and $|B_n(h)| \leq |L|$. Let $A_{n+1} := \langle A_n, B_n(h) \mid h \in H_n \rangle$. Then $|A_{n+1}| \leq |L|$. Let $H_{n+1} := \langle T_{n,N} \ (N \in \underline{N}), h\alpha \ (h \in H_n, \alpha \in A_{n+1}) \rangle$. Then $|H_{n+1}| \leq |L|$ and $H_n \leq H_{n+1}$ (as $1 \in A_{n+1}$). So $G_0 \leq H_0 \leq H_1 \leq \dots$ and $A_0 \leq A_1 \leq A_2 \leq \dots$. Let $H = \bigcup_{n \in \mathbb{N}} H_n$ and $A = \bigcup_{n \in \mathbb{N}} A_n$.

We first note that the assertion that the lattice of normal subgroups of H is isomorphic to L follows exactly as in (a).

We note that the group H is invariant under A . For, let $h \in H, \alpha \in A$, and choose n such that $h \in H_n$ and $\alpha \in A_n$. Then $h\alpha \in H_{n+1} \leq H$. Therefore A acts as a group of automorphisms of H . To show that H is characteristically simple, it is therefore enough to show that $H \leq \langle h\alpha \mid \alpha \in A \rangle$ for all $h \in H$. Let $h \in H$. Choose m such that $h \in H_m$. Then for all $n \geq m$ we have $H_n \leq \langle h\alpha \mid \alpha \in B_n(h) \rangle \leq \langle h\alpha \mid \alpha \in A \rangle$. Therefore $H \leq \langle h\alpha \mid \alpha \in A \rangle$.

Note that we could also construct H such that, in addition to the properties given in the theorem above, the subgroup H is invariant under a given set B of automorphisms of G with $|B| \leq |L|$ (to do this, we start in the proof with $A_0 = \langle B \rangle$ instead of $A_0 = \{1\}$).

We note that with the same construction as in Theorem 6.12 we can prove that if G is an infinite group with finite lattice L of normal subgroups, and if G_0 is a countable subgroup of G then there exists a countable subgroup H of G with $G_0 \leq H$ such that the lattice of normal subgroups of H is isomorphic to L . As a special case this says that if G is simple then we can get H simple. This case is "folklore" which has been known for a very long time. A proof by Richard E. Phillips can be found in [15], and further references in [24].

We shall now consider the case where G is not only characteristically simple but also has all of its subnormal subgroups being normal, to show that we can construct a subgroup H as in Theorem 6.12 which has all of these properties. For this we need

Lemma 6.13 Let G be a group. Then the following are equivalent:

- (i) every subnormal subgroup of G is normal in G ,
- (ii) if K is normal in G and $g \in K$ then $\langle g \rangle^K = \langle g \rangle^G$,
- (iii) for all $g \in G$ we have $\langle g \rangle(\langle g \rangle^G) = \langle g \rangle^G$.

Proof: (i) $\Rightarrow$ (ii): Let $K \trianglelefteq G$, and let $g \in K$. We clearly have $\langle g \rangle^K \leq \langle g \rangle^G$. As $\langle g \rangle^K$ is normal in K , it is subnormal in G . By (i) it is therefore normal in G , and hence $\langle g \rangle^G \leq \langle g \rangle^K$.

(ii) $\Rightarrow$ (i): It is enough to show that every two-step subnormal subgroup of G is normal in G . So let N be two-step subnormal in G . Then there exists K such that $N \trianglelefteq K \trianglelefteq G$. We have $N = \langle \langle g \rangle^K \mid g \in N \rangle$
 $= \langle \langle g \rangle^G \mid g \in N \rangle \trianglelefteq G$.

(ii) $\Rightarrow$ (iii): Trivial, as $\langle g \rangle^G$ is normal in G .

(iii) $\Rightarrow$ (ii): Let K be normal in G , and let $g \in K$. We always have $\langle g \rangle(\langle g \rangle^G) \leq \langle g \rangle^K \leq \langle g \rangle^G$, and as $\langle g \rangle(\langle g \rangle^G) = \langle g \rangle^G$ we also have $\langle g \rangle^K = \langle g \rangle^G$.

We can now state the theorem.

Theorem 6.14 Let G be a characteristically simple group with normal subgroup lattice L such that $\kappa_0 \leq |L| < |G|$, and suppose that every subnormal subgroup of G is normal in G . Let G_0 be a subgroup of G with $|G_0| \leq |L|$. Then there exists a characteristically simple subgroup H of G with $G_0 \leq H$ and $|H| \leq |L|$ such that every subnormal subgroup of H is normal in H , and such that the lattice of normal subgroups of H is isomorphic to L .

Proof: Let $\underline{N} = \{N \trianglelefteq G \mid \text{there exists a finite subset } \{g_1, \dots, g_n\} \text{ of } G \text{ such that } N = \langle g_1, \dots, g_n \rangle^G\}$. Let $F(G)$ be the set of finite subsets of G . We get a partition $F(G) = \bigcup_{N \in \underline{N}} F_N(G)$ where $F_N(G) = \{S \in F(G) \mid \langle S \rangle^G = N\}$. For each $N \in \underline{N}$ choose $S_N \in F_N(G)$. Let $H_0 := \langle G_0, S_N \mid N \in \underline{N} \rangle$. Then $|H_0| \leq |L|$ and $G_0 \leq H_0$. Let $A_0 := \{1\} \leq \text{Aut}(G)$. Suppose we have $H_n \leq G$ and $A_n \leq \text{Aut}(G)$ constructed with $|H_n| \leq |L|$ and $|A_n| \leq |L|$. For $N \in \underline{N}$ consider $N \cap H_n$. Let $F(N \cap H_n)$ be the set of finite subsets of $N \cap H_n$, then $|F(N \cap H_n)| \leq |L|$. For every $S \in F(N \cap H_n) \cap F_N(G)$ there exists a subset $T(S)$ of G with $|T(S)| \leq |L|$ such that $N \cap H_n \leq \langle S \rangle^{T(S)}$. Let $T_{n,N} := \bigcup \{T(S) \mid S \in F(N \cap H_n) \cap F_N(G)\}$. Then

$|T_{n,N}| \leq |L|$. As G is characteristically simple, for each $h \in H_n$ there exists a subset $B_n(h)$ of $\text{Aut}(G)$ such that $H_n \leq \langle h\alpha \mid \alpha \in B_n(h) \rangle$ and $|B_n(h)| \leq |L|$. Let $A_{n+1} := \langle A_n, B_n(h) \mid (h \in H_n) \rangle$. Then $|A_{n+1}| \leq |L|$. For each $g \in H_n$ the group $\langle g \rangle^{H_n}$ is of cardinality at most $|L|$. Hence by Lemma 6.13 (as $\langle g \rangle^{(\langle g \rangle^G)} = \langle g \rangle^G$ for all $g \in G$, and $\langle g \rangle^{H_n} \leq \langle g \rangle^G$) there exists a subset $V(n,g)$ of $\langle g \rangle^G$ with $|V(n,g)| \leq |L|$ such that $\langle g \rangle^{H_n} \leq \langle g \rangle^{V(n,g)}$. Now let $H_{n+1} := \langle T_{n,N} \ (N \in \mathbb{N}), h\alpha \ (h \in H_n, \alpha \in A_{n+1}), V(n,g) \ (g \in H_n) \rangle$. Then $|H_{n+1}| \leq |L|$, and $H_n \leq H_{n+1}$ (as $1 \in A_{n+1}$). So $G_0 \leq H_0 \leq H_1 \leq \dots$ and $A_0 \leq A_1 \leq A_2 \leq \dots$. Let $H = \bigcup_{n \in \mathbb{N}} H_n$ and $A = \bigcup_{n \in \mathbb{N}} A_n$.

We note that the assertions that the lattice of normal subgroups of H is isomorphic to L and that H is characteristically simple follow as in Theorem 6.12.

What is left to prove now is that every subnormal subgroup of H is normal in H . By Lemma 6.13 it is enough to show that for every $g \in H$ we have

$\langle g \rangle^H \leq \langle g \rangle^{(\langle g \rangle^H)}$. We note that $\langle g \rangle^G \cap H = \langle g \rangle^H$ for all $g \in H$. Let m be such that $g \in H_n$ for $n \geq m$.

Then $V(n, g) \leq \langle g \rangle^G \cap H_{n+1} \leq \langle g \rangle^G \cap H = \langle g \rangle^H$.

Hence $\langle g \rangle^{H_n} \leq \langle g \rangle^{V(n, g)} \leq \langle g \rangle^{\langle g \rangle^H}$. But then it also follows that $\langle g \rangle^H \leq \langle g \rangle^{\langle g \rangle^H}$.

From this theorem and Theorem 6.10 we immediately get

Corollary 6.15 Let Λ be a partially ordered set such that for $\lambda, \mu \in \Lambda$ there exists an automorphism a of Λ such that $\mu \leq \lambda a$. Let $\mathcal{J}(\Lambda)$ be the distributive lattice of semi-ideals of Λ . Then there exists a group G with the following properties:

- (1) $|G| \leq \max(\aleph_0, |\mathcal{J}(\Lambda)|)$.
- (2) All subnormal subgroups of G are normal in G .
- (3) The lattice of normal subgroups of G is isomorphic to $\mathcal{J}(\Lambda)$.
- (4) The group G is characteristically simple.

We note that the characteristically simple groups we have constructed have quite distinct features from the well-known characteristically simple groups, which apart from direct powers of (characteristically) simple groups are McLain's groups [13], certain wreath powers [8], certain Boolean powers [14], and projective linear groups over certain rings [23]. To conclude this chapter we shall now look at the structure of certain characteristically simple groups with only countably many normal subgroups.

Proposition 6.16 Let G be a characteristically simple group with at most countably many normal subgroups. If G has a minimal normal subgroup then G is a direct product of finitely many isomorphic simple groups.

Proof: Let N be a minimal normal subgroup of G , and let $\underline{M}$ be the set of images of N under automorphisms of G . Then G is the join of all the members of $\underline{M}$ (as this is a characteristic subgroup of G). It follows (see, for example, [17], Lemma 5.23) that G is the direct product of the members of a subset $\underline{N}$ of $\underline{M}$. As normal subgroups of direct factors are normal in the whole group, these groups must be simple, and they are of course isomorphic. Suppose $\underline{N}$ is infinite. Then for every subset of $\underline{N}$ the subgroup generated by these groups is a different normal subgroup of G . Hence there are at least $2^{\aleph_0}$ of them in contradiction to the assumption. Therefore $\underline{N}$ is finite.

We note that this is a generalization of a well known theorem on finite groups. If we replace the existence of a minimal normal subgroup by the existence of a maximal normal subgroup of G , this does not seem to lead to an equally strong restriction on the structure of G . If M is a maximal normal subgroup of the characteristically simple group G then we get an embedding of G into a cartesian power of the simple

group G/M , but cartesian powers still have a very complicated structure. For example the groups constructed by J.S. Wilson [23] have maximal normal subgroups. And indeed, defined over suitable rings, they have only countably many normal subgroups. However Wilson remarks that if, in addition, we suppose that the group G/M is finite then G is a Boolean power of G/M . Therefore we shall now consider the structure of Boolean powers of simple groups which have at most countably many normal subgroups.

Proposition 6.17 Let G be a characteristically simple group which has no more than countably many normal subgroups. If G is the Boolean power of a simple group S then G is a finite direct power of S .

Proof: We first note that by B.H. Neumann and Sadayuki Yamamuro [14] the lattice of normal subgroups of G is isomorphic to the lattice L of ideals of the Boolean algebra A which is used in the construction of G .

Case 1: The algebra A has an atom a . Then $\{0, a\}$ is a minimal ideal of A and it corresponds to a minimal normal subgroup of G . Then the rest follows from Proposition 6.16.

Case 2: The algebra A has no atom. Then A can not be finite. As every element of A gives rise to a different element of L (namely the principal ideal

it generates), we have $|L| \geq |A|$. As L is at most countable we have both A and L countable. As a consequence of a theorem of Vaught (see Theorem 0.1. and Proposition 0.2. in [10]) it follows that there is exactly one isomorphism class of countable atomless Boolean algebras, and the dual space is isomorphic to the Cantor set. But this Boolean algebra has uncountably many ideals, and consequently the Boolean power constructed with it has uncountably many normal subgroups. So this case can not occur, which proves the proposition.

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