Stability for nonlinear diffusive PDEs

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Abstract

This thesis focuses on well-posedness and stability estimates (i.e. continuous dependence with respect to the nonlinearities) for a family of cross-diffusion systems that includes several models used to describe cell diffusion in Mathematical Biology for one or more species. We discuss rigorous quantitative results for a class of (possibly degenerate) parabolic PDEs, including scalar equations as well as systems. The approach is typical of Analysis of PDEs, hence the concepts of weak solutions in Sobolev spaces, a priori estimates and well-posedness are crucial. Considering scalar, nonlinear diffusion, we extend the existing solvability, uniqueness, boundedness and stability results. Regarding non-degenerate models for multiple species, we work in the framework of systems with an entropy structure and limited set of other assumptions that allow us to prove stability estimates, improving some of the existing results. Finally, we have been working on the multi-species, degenerate case, providing a continuous dependence estimate under suitable (but rather restrictive) assumptions. Two different types of numerical simulations are presented in order to complement the abstract results.

Contents

Notation

d	number of spatial dimensions.
N	number of equations.
Ω	open, connected, bounded and sufficiently smooth domain in $\mathbb{R}^d$.
$\lvert\Omega\rvert$	Lebesgue measure of Ω.
$\bar{\Omega}$	closure of Ω.
Q_T	parabolic cylinder $\Omega \times (0, T)$.
$\lvert x\rvert$	Euclidean norm of the vector x.
$x \cdot y$	Euclidean scalar product of the vectors x, y.
$B_r(x)$	the ball with center x and radius r in $\mathbb{R}^d$.
X^*	dual of a Banach space X.
$\langle \cdot, \cdot \rangle$	duality pairing of a Banach space and its dual.
$C^k(\Omega)$	set of real functions with continuous k-th derivative on Ω.
$BV(\Omega)$	real functions of bounded variation in Ω.
$L^p(\Omega)$	functions with integrable p-th power in Ω.
$W^{k,p}(\Omega)$	functions with weak k-th derivative in $L^p(\Omega)$.
$H^k(\Omega)$	$W^{k,2}(\Omega)$.
$L^p(0, T; X)$	L^p functions on $[0, T]$ with values in X.
$L^p(Q_T)$	$L^p(0, T; L^p(\Omega))$.
$\lVert u\rVert_p$, $\lVert u\rVert_{L^p}$	L^p norm of u.
u_+, u_-	$\max(u, 0)$, $\max(-u, 0)$ respectively.
χ_E	indicator function of the set E.
Df	Jacobian matrix of $f : \mathbb{R}^N \to \mathbb{R}^M$ (or gradient if $M = 1$).
$\mathrm{Hess}(f)$	Hessian of a function $f : \mathbb{R}^N \to \mathbb{R}$.

Chapter 1

Introduction

Contents

1.1 Preliminaries

1.1.1 Opening remarks

This thesis aims to present advances in the study of continuous dependence and stability estimates for nonlinear parabolic problems arising principally in Mathematical Biology. Throughout this work we study the behaviour of the solutions of partial differential equations (PDEs) that represent the evolution and interplay of particle or cell densities in a given domain. We consider the case of a single species, as well as of multiple interacting species. We are explicitly interested in the comparison between different models. In particular, we focus on estimates for the difference of solutions corresponding to PDEs with distinct nonlinear terms of first and second order.

From the point of view of modelling, systems of many interacting particles (such as biological cells) can be described in multiple ways, employing individual-based models at a microscopic level. For example, one can study the dynamics of particles interacting stochastically on a lattice, or consider them as Brownian particles evolving in a continuous domain.

Microscopic models determine the evolution of macroscopic quantities, the most important of which is the density of particles in the domain. Passing from microscopic to macroscopic descriptions constitutes a vast field by itself. Here we do not discuss this topic, as we focus on macroscopic PDE models.

Intuitively, there is no unique way to construct a model for a fundamental biological mechanism like cell diffusion and migration. Indeed, different assumptions at the microscopic level can lead to different models at the macroscopic level, as well as to dissimilar predictions. The differences between microscopic models are usually reflected into the properties of the corresponding macroscopic models. Therefore, it becomes natural to ask how to compare the discrepancies that arise. This thesis provides mathematical tools allowing the comparison of different models in the form of stability (or continuous dependence) estimates.

Our general framework will be made more precise in section 1.1.3. For an overview of problems arising in Mathematical Biology we refer to [Per15, Fri18].

1.1.2 Motivation

In recent years the interest of mathematicians and biologists towards a better understanding of cell diffusion and migration has increased significantly. Cell diffusion in a tissue or other medium is a basic mechanism in Biology and its comprehension becomes extremely relevant in processes like tumour growth. Nevertheless, modelling these phenomena can be arduous because of their complex and multi-scale nature. Several models have been recently derived in the attempt to reproduce different behaviours that are observed experimentally, and a large fraction of these involves cross-diffusion systems, both strictly parabolic and degenerate parabolic. We think this category of models is particularly challenging because it displays strong interactions between (densities of) particles, resulting in non-trivial properties of the solutions. Many of the techniques that have recently been used derive from the study of hyperbolic systems. In fact, the solutions of degenerate problems show properties that are typical of systems of conservation laws, like finite speed of propagation for the support of the solutions.

A variety of models describing cell behaviour and interactions has been produced recently, and the range of different assumptions, strategies, levels of approximation and typical scales that are considered is extremely diversified. Nevertheless, there are a few elements that many models have in common, and the most interesting one for us is that, at the macroscopic level, cell densities can be described as the solutions of (systems of) PDEs with diffusion, drift and reaction terms (see e.g. [MB14]).

Nonlinear diffusion equations are often used to model phenomena like tumour growth, cell motility and population dynamics. Let us mention some works in this direction.

- Mathematical Oncology has been growing fast over the last twenty years, and recent trends are discussed, for example, in [GM03, SBMA03, Fri07, AQ08, Byr10, CL10, ALM15, AM18] in general terms, and in [MB14, FMM15, JB02] for some specific examples of mathematical modelling of cancer.

- Cell motility has been studied with several approaches, building more and more sophisticated models that consider particles as hard spheres, to better capture the interplay between cell behaviour and volume exclusion effects. Moreover, on- and off-lattice descriptions, chemotaxis and multiple-species interactions have been considered. In this perspective we refer to [WBB17, SLH09, BC12a, BC15, BCR17, VMG+18] and, for a more analytical point of view, to [BDP06, BDFPS10, BBRW17].

- Concerning models of population dynamics, two of the most studied phenomena are pattern formation and swarming behaviour. The models in this field can be very different since they try and capture the underlying complex behaviour of the individuals and of the crowd. Among those including nonlinear diffusion we cite [SKT79a, FKK03, TBL06a, MEK99, PS12].

Given the great diversity of models describing cell diffusion and migration, it seems natural to compare the various predictions, seeking a deeper understanding of the modelling assumptions and of the underlying biological mechanisms. The most popular way of comparison involves numerical simulation, which has the advantage of producing explicit results that can be processed and analysed in multiple ways, but also has many limitations. Each numerical simulation is obviously performed for a given set of parameters, making it necessary to run computationally expensive algorithms multiple times. Moreover, each model might require the development a specific numerical method of simulation, making the analysis longer and more complicated. Our interest lies in different ways of comparing models, and we use theoretical tools that are typical of the field of Analysis of PDEs. Stability and continuous dependence estimates for parabolic problems can be effectively used in the context of Mathematical Biology to quantify (in a suitable norm) the difference between solutions of two equations or systems. This category of estimates, when applicable, provides almost immediate answers since it only requires to compute norms of the difference of the nonlinearities in the models (or other possible parameters).

1.1.3 General framework

In this thesis we are going to work both with scalar problems and systems of equations, therefore we introduce different indices to refer to the ambient space variables and to the component or species number. The indices $1 \leq \alpha, \beta \leq d$ refer to directions in the ambient space, $\mathbb{R}^d$, for $d \in \{1, 2, 3\}$. The indices $1 \leq i, j \leq N$ are used to refer to the component number. The domain Ω in $\mathbb{R}^d$, where the problem is formulated, is supposed to be bounded, connected and sufficiently smooth. The outward normal on $\partial\Omega$ is indicated by ν. Consider the following general system:

$$\begin{aligned} \partial_t u^i - \partial_\alpha \left[\mathcal{A}^{\alpha\beta}_{ij}(t,x,u)\partial_\beta u^j + \mathcal{B}^{\alpha}_{ij}(t,x,u)u^j \right] &= 0 \qquad \text{in } \Omega, \\ \left[\mathcal{A}^{\alpha\beta}_{ij}(t,x,u)\partial_\beta u^j + \mathcal{B}^{\alpha}_{ij}(t,x,u)u^j \right] \cdot \nu_\alpha &= 0 \qquad \text{on } \partial\Omega, \\ u^i(0,\cdot) &= u^i_0(\cdot) \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where we adopt the summation convention for repeated indices. We will make the assumptions on each term precise later on, for the moment we just consider the following set of general hypotheses:

1. the diffusion tensor $\mathcal{A}$ is continuous in all its arguments, bounded and non-negative. In particular, there exist two non-negative functions $\lambda, \Lambda : \mathbb{R}^N \to \mathbb{R}_+$ such that

$$\lambda(u)|X|^2 \leq \mathcal{A}^{\alpha\beta}_{ij}(t,x,u)X^i_\alpha X^j_\beta \leq \Lambda(u)|X|^2, \tag{1.2}$$

   for all $t \geq 0,\ x \in \Omega,\ u \in \mathbb{R}^N,\ X \in \mathbb{R}^{N\times d}$;

2. concerning the drift $\mathcal{B}$, we assume that there exists a constant $B > 0$ such that

$$|\mathcal{B}^{\alpha}_{ij}(t,x,u)| \leq B|u|, \quad \text{for all} \quad t \geq 0,\ x \in \mathbb{R}^d\ u \in \mathbb{R}^N; \tag{1.3}$$

3. We assume that the initial data are non-negative and bounded, i.e., there exists $M_0 > 0$ such that

$$0 \leq u^i_0(x) \leq M_0, \quad \text{for all } \ x \in \Omega,\ i \in \{1 \dots N\}. \tag{1.4}$$

We will focus on the study of the following map:

$$\begin{aligned} \left\{ \mathcal{A} \in \mathbb{R}^{N\times N} \,\middle|\, \mathcal{A} \text{ satisfies } (1.2) \right\} &\longrightarrow \{ \text{ Solution of } (1.1) \ \} \\ \mathcal{A} \quad &\mapsto \quad u_{\mathcal{A}}. \end{aligned} \tag{1.5}$$

As an example, let us consider a linear PDE. Let $u_{\mathcal{A}_i}$, $i = 1, 2$, denote the solution of problem (1.1) with $N = 1$, $\mathcal{B} = 0$ and the simple diffusion coefficients $\mathcal{A}_1 = a_1(x)$ and $\mathcal{A}_2 = a_2(x)$ such that, for some constant $\rho > 0$ and a.e. x, $\rho^{-1} \leq a_i(x) \leq \rho$.

We call *stability estimate* an inequality that quantifies the differences between $u_{\mathcal{A}_1}$ and $u_{\mathcal{A}_2}$ in terms of the differences between $\mathcal{A}_1$ and $\mathcal{A}_2$. More precisely, in the linear example we obtain

$$\| u_{\mathcal{A}_1} - u_{\mathcal{A}_2} \|_{L^2(0,T,H^1(\Omega))} \leq C \, \| \mathcal{A}_1 - \mathcal{A}_2 \|_{L^\infty(\Omega)}, \tag{1.6}$$

where C is a constant depending only on ρ and u_0.

Such notion of stability is also known as well-posedness in the sense of Hadamard. As we shall see, stability estimates become more complicated for nonlinear systems, they can involve lower order terms and other types of norms.

1.1.4 Thesis structure and statement of originality

The main body of the thesis is organised as follows:

- Chapter 2 concerns nonlinear scalar diffusion equations that generalise the porous medium equation, which we denote by *filtration equations*. In the first section, we prove existence, boundedness and comparison properties of the solutions. The second section completes and extends the results of the first section, including the case of an autonomous, anisotropic diffusion matrix. We prove a stability estimate with respect to the L^1 norm. The third section is dedicated to the study of a nonlinear, nonlocal equation of Fokker–Plank type. In the presence of an infinite confinement potential in a portion of space, we prove convergence of the weak formulation of the equation in full space to a limit problem on a bounded domain and no-flux boundary condition.
  Regarding the first two sections, most of the techniques that are used are not new, but they have been reorganised and adapted in an original way in order to reach our goals (see Remarks 2.4 and 2.7). The third section is original work and part of a research paper co-authored by M. Bruna and J. A. Carrillo, see [ABC18b] (the analysis and main theorems are L. A.'s contribution).

- In Chapter 3, we present the analysis of a class of systems which is "close" (in a sense defined later on) to a set of uncoupled equations. After explaining the origin of this type of models, we study well-posedess of the problem and we prove stability estimates in a strong parabolic norm. The key tool is a fixed point theorem applied in a function space that is "tailored" for this type of systems. In a second step, we show that, under an additional assumption, our results become uniform in time. In the second section we discuss some numerical simulations and we explore the limits of the model in a one-dimensional setting.
  This chapter is original work and the first section is part of a research paper co-authored by M. Bruna and Y. Capdeboscq, see [ABC18a] (L. A. contributed to the analysis, with particular attention to Theorem 3.5). The numerical simulations we present were done independently from the above-mentioned paper.

- Chapter 4 is dedicated to the study of more general cross-diffusion systems. In the first section we introduce strictly parabolic systems that have an entropy structure, and can therefore be interpreted as gradient flows. This assumption is substantial as it ensures existence and boundedness of solutions. Subsequently we discuss higher integrability of the gradient of the solution, a propriety that is particularly useful in one and two dimensions. Moreover, we prove a stability result subject to a regularity assumption (which is satisfied in two dimensions thanks to the higher integrability result). The second section concerns degenerate parabolic systems. We prove a positivity result that enables us to construct a simple and efficient regularised version of the original problem. We use this regularisation method to prove a stability result in the special case in which the diffusion matrix can be written as the Jacobian of a certain vector field. Finally, we turn to the problem of simulating the solution numerically, especially in the presence of a free-boundary in one spatial dimension. We construct a front-tracking algorithm and we test it on a specific example.

The first section of this chapter combines different techniques available in the literature, the novelty consists in a substantial rearrangement and partial extension of the previous results, which were developed in different frameworks. The second section, including the numerical simulations, is original work, with the exception of the H^{-1} technique used to prove uniqueness of solutions.

1.2 Overview and state of the art

In this section we present an overview of the Thesis, summarise our main contributions and refer to the results in the literature that have been more relevant for our work.

1.2.1 Scalar diffusive equations

The starting point of our analysis is given by scalar quasilinear diffusion PDEs which are referred to as *generalised porous medium equation* or *filtration equation*. Such equations typically have the form

$$\partial_t u - \Delta\phi(u) = 0, \tag{1.7}$$

where u is an unknown function to be determined and the nonlinearity ϕ is called *filtration* function and typically it satisfies $\phi(0) = \phi'(0) = 0$ (standard references for this equation include [WYLZ01, Váz07, DK07, DiB12]). We are mostly interested in the autonomous case, which means that the diffusion coefficient $\phi'(\cdot)$ only depends on the solution itself. This situation already captures the major features and difficulties of the analysis and it is not too restrictive since many multi-species models for cell diffusion, ion transport, drift-diffusion in semiconductors, etc. are autonomous (see examples 1, 2 and 4).

We will usually consider purely diffusive equations of second order, without other terms, since the nonlinearity in the highest order is responsible for most of the technical difficulties that we will encounter. First-order drift terms and zero-order reaction terms are common in biological applications. Despite not being treated here, they could be included in this work without serious issues (see e.g. [BH86,CG99,FT81], or references for the system case given below). Additionally, results for problems involving drift (or confinement) terms can be recovered "freezing" one of the components in the multi-species models that we will introduce later in this section.

From an analytical point of view, it is crucial to distinguish between strongly parabolic and degenerate parabolic equations. Indeed, non-degenerate problems can be treated with classical techniques and the issues of well-posedness, boundedness and stability with respect to the nonlinearities have been widely studied in the literature (see table 1.1 for some key references, as well as [MPSP00, AT88]). On the other hand, the theory of degenerate parabolic equations has been developed more recently and still presents some challenges. For example, we could not find references that cover the study of boundary value problems for a filtration equation that can degenerate (having zero diffusion) at multiple values of the unknown. This type of degeneracy seems rather natural when considering the phenomenon of cell diffusion. In fact, the absence of cells in a certain portion of the domain clearly determines zero diffusion. Also, in the opposite scenario, a fully packed region of the domain prevents the cells from moving and diffusing.

Table 1.1 gives a minimal list of references in which the theory of degenerate diffusion equations is treated and points the reader to the results present in this thesis. It has to be mentioned that the techniques we use are similar to those available in the literature (see e.g. [BH86, CG99, Váz07]). However, we could not simply apply previous results since the notions of solution, the type of boundary value problem and the assumptions on the nonlinearities were not the same. One of the main difficulties consists in proving boundedness of the solutions *a priori* when the equation does degenerate for a value of the solution other than zero (in other words, the filtration function ϕ is not strictly monotone), see section 2.1 for all the details.

	Non-degenerate	Degenerate
Nonlinear diffusion	Existence, [LSU88], [Váz07]	Existence, Theorem 2.3 and [Váz07]
	Boundedness, [LSU88], [Váz07]	Boundedness, Lemma 2.7 and [DK07]
	Uniqueness, [Váz07], [CG99]	Uniqueness, Theorem 2.4
	Stability, [CH05], [CG99]	Stability, Theroem 2.10 and [CK05]

Table 1.1: This table summarises the key results for nonlinear, scalar diffusion equations of degenerate and non-degenerate type. For each category of results (existence, boundedness, uniqueness and stability) we either refer to a paper, or to a statement in this document.

The most complete reference for scalar diffusion equations that generalise the porous medium equation is the book by J. L. L. Vázquez [Váz07]. Such book has been very useful for our purposes since most of the standard techniques for this type of equations are presented therein. In addition to the references already mentioned, we also recall the rather extensive work of [Kal87] on degenerate second-order PDEs.

A characteristic property of degenerate parabolic equations is to generate free boundaries, especially if the initial condition is compactly supported. For further details in this sense we refer to [Váz07, Bar52, DH98, DHL01].

An essential tool in the theory of scalar nonlinear diffusion is the so-called *comparison principle*, which ensures that two solutions that are initially ordered will remain ordered almost everywhere in time and space. This property was first proved in [Pie82], and proposed again in [Váz07, DK07, WYLZ01]. The comparison principle is deeply connected to the notion of L^1 contraction, or, in other words, continuous dependence with respect to the initial data in the L^1 norm in space (see [CT80]). This allows to prove uniqueness of solutions, but it is also very useful in the proofs of existence, boundedness and stability.

We prove stability estimates adapting the approach of [CG99, CK05], which is, in turn, a generalisation of the L^1 contraction property, also studied in [BC81, Ott96, EKR01, JK02, CMR09]. The difference of two solutions corresponding to distinct filtration functions in the L^1 norm is proportional to the difference of the filtration functions themselves (more precisely, it is bounded by the supremum norm of the difference of the square roots of the filtration functions). Our proof of the stability estimate is similar to those available in the literature but the previous results were formulated in full space for a different notion of solution (see table 1.1 and above references).

1.2.2 Nonlinear, nonlocal Fokker-Planck equation

We[1] are now going to discuss a different type of results, which are, in a sense, complementary to those presented above. Unlike in the case of stability estimates for the purely diffusive equations, we are now going to focus on the behaviour of the solutions of a more general PDE in which one of the lower order terms becomes unbounded in a portion of the domain. The techniques that we will use in this case are different and they are based on particular *a priori* estimates.

We consider a nonlinear, nonlocal Fokker–Planck equation of the form

$$\partial_t u = \operatorname{div}\left[\nabla\phi(u) + u\nabla V + u\nabla(W * u)\right], \tag{1.8}$$

where ϕ is a filtration function, V is a confinement potential and W is a nonlocal interaction potential. The problem is initially set in the whole $\mathbb{R}^d$, see section 2.3 for all the details. We are particularly interested in the behaviour of the solution when V becomes infinite outside a given domain Ω. We point out that, under suitable hypothesis, the filtration equation (1.7) can be seen as a special case of nonlinear Fokker–Planck equation. Problems of type (1.8) have been studied in the literature with diferent approaches. For example, for the equation in full space we cite [CMV03, CMV06], where entropy methods are employed. The problem of well-posedness of a general nonlinear diffusion equation in L^1, on one-dimansional bounded domains is treated in [BT95], in the context of semigroup theory and entropy solutions. We also mention the work in [Gil89], concerning the Cauchy-Dirichlet problem for a nonlinear equation of Fokker–Planck type; in this case generalised solutions are obtained as limits of sequences of classical solutions.

Equation (1.8) is often used to describe a system of interacting particles at the macroscopic level and explain how individual-level mechanisms give rise to population-level or collective behaviour. Among the physiscal and biological applications we mention granular materials [BCP97, BCCP98], self-assembly of nanoparticles [HP06], colloidal systems [GWL16], ionic transport [HLLE12], cell motility [HP08], animal swarms [CDP09],

[1] The results described here are part of a joint work with M. Bruna and J. A. Carrillo, [ABC18b].

pedestrian dynamics [BMi11], and social sciences [Tos06, PT13]. For example, choosing $\phi(u) = u$ and $W = 0$ can be used to describe a system of non-interacting Brownian particles under the influence of an external potential V (representing a chemical concentration in the case of chemotaxis).

Interactions between particles may affect the macroscopic model (1.8) in two forms (at least): either as a modification of the diffusion term ϕ, for example $\phi(u) = u + \beta u^2$, or as a nonlocal convolution. The former typically arises from short-range repulsive interactions between particles (such as excluded-volume interactions, see [CC06, CCH17, BC12b]), whereas the latter is used to model long-range attractive-repulsive interactions (such as electrostatic or chemoattractive interactions, see [CCH17, TBL06b, BCR17]).

Our first aim is to provide a suitable mathematical framework for the rigorous study of confinement potentials that diverge to infinity. Our second aim is to extend our previous work, passing from a purely diffusive filtration equation to a more general type of model. We also show the connections between infinite confinement and in an unbounded domain and a new boundary value problem in the bounded domain Ω. This can be useful in the application whenever one wants to transfer results in $\mathbb{R}^d$ to a bounded domain (as it is often the case in Mathematical Biology).

In practical terms, we introduce a sequence of potentials, depending on a parameter $k \geq 0$, which will tend to infinity as $k \to \infty$ on the complement of Ω. Regardless of the approach we choose, the key step to prove any convergence result for $k \to \infty$ will involve uniform estimates with respect to k. More specifically, we prove that the sequence of weak formulations of problem (1.8), with $V = V_k$ (initially posed in $\mathbb{R}^d$), will converge to a limit problem posed the bounded domain Ω with no-flux boundary conditions. This is entirely consistent with the intuition about the action of a large confinement potential, which will indeed confine all the mass in a smaller domain.

The table below shows some important references, as well as our own results, for the two distinct approaches that we explore. We denote them as L^2 setting and entropy setting.

	L^2 setting	Entropy setting
Fokker–Planck	Well-posedness, [BS09] Uniform bounds, Lemma 2.27 and Lemma 2.30 Behaviour for large confinement, Theorem 2.23	Well-posedness, [AGS08] Uniform bounds, Corollary 2.36 Behaviour for large confinement, Theorem 2.31

Table 1.2: This table summarises the key results for a nonlinear, nonlocal Fokker–Planck equation. For each issue (well-posedness, uniform bounds and behaviour of the solution for large confinement potential), we refer to a paper or to a statement in this document.

In the L^2 case, we rely on a relatively standard approach based on energy estimates, whereas, in the entropy case, we see the equation as a gradient flow in a suitable Wasserstein space. Each approach has advantages and disadvantages. For example, the L^2 setting produces better estimates but it only works when either the diffusion term is linear or the nonlocal potential W is chosen to be zero. In contrast, in the entropy approach weaker norms have to be considered, but the results we obtain hold for the full equation and even for degenerate diffusion coefficients.

As mentioned before, the main difficulty lies in the derivation of estimates for the solution that are uniform with respect to k. In the L^2 case, this is obtained thanks to a change of variables that transforms the confinement potentials V_k into a *weight*. In the gradient flow formulation, the entropy functional is defined so that the large potential automatically appears on the right-hand side of an entropy decay estimate. In both cases, it is crucial to assume the initial datum is supported inside the domain Ω. This hypothesis seem to be necessary to identify the limit problem uniquely, otherwise the mass outside Ω will accumulate on the lateral boundary of Ω as $k \to \infty$, generating a singular measure which, a priori, is not uniquely determined in any dimension greater than one.

In terms of well-posedness of equation (1.8) we refer to [BS09, AL83] for the L^2 setting and to [AGS08]. For the study of a nonlocal system see [DFEF18]. Many results concerning convergence to equilibrium can be found in [CJM$^+$01, AMTU01, CT98, BNCGP16]. For the related results concerning drift-diffusion equations for semiconductors, see [MS89, Mar13].

1.2.3 Cross diffusion systems

Our study of diffusive systems of PDEs is divided into three categories of problems, namely non-degenerate systems with small cross-diffusion, non-degenerate systems with general (possibly large) cross-diffusion, and degenerate systems of PDEs. These cases are ordered in terms of increasing difficulty in the analysis since the system becomes more strongly coupled in the diffusion tensor. This implies that our hypotheses on the systems we study will become more and more restrictive in terms of the structure of the diffusion, in order to deal with the possibly degenerate behaviour of the solutions.

I. Systems of equations with small cross-diffusion terms.[2]

We consider systems of PDEs of the form

$$\partial_t u - \operatorname{div}\left[\mathcal{A}(t,x,u)\nabla u - \mathcal{B}(t,x,u)u\right] = 0, \quad \text{in} \quad \Omega, t > 0, \tag{1.9}$$

where $u \in \mathbb{R}^N$ and the diffusion and drift matrices $(N \times N)$ have the form

$$\mathcal{A}(t,x,u) = D(t,x) + \epsilon a(t,x)\phi(u), \quad \mathcal{B}(t,x,u) = F(t,x) + \epsilon b(t,x)\psi(u), \tag{1.10}$$

for suitable diagonal matrices $D, F \in \mathbb{R}^{N\times N}$, and sufficiently regular tensors a, b, ϕ, ψ specified in chapter 3. Here $\epsilon \geq 0$ is a small parameter that represents the strength of the

[2] The results described here are part of a joint work with M. Bruna and Y. Capdeboscq, [ABC18a].

interactions between particles.

System (1.9) is "close" to a linear, decoupled system which is obtained setting $\epsilon = 0$. Unsurprisingly, this is a very useful property since we show that many properties of the underlying linear system are inherited by (1.9). Models of this type are quite common in the applications, as explained in section 3.1.1 and a similar type of equations is studied in [BBRW17].

We develop the existence and uniqueness theory for problem (1.9) with no-flux boundary conditions in a strong Sobolev space by means of Banach's fixed point theorem. As a consequence, we also obtain boundedness of the solution in L^∞ with respect to time and space. In addition, we prove a strong stability estimate with respect to the nonlinearities in $\mathcal{A}$ and $\mathcal{B}$, which involves the norm of the same space in which well-posedness is proven.

Under slightly more restrictive assumptions, we improve the above-mentioned stability result, making it uniform in time. This is relevant in many applications since it is often important to understand the long time behaviour of a system and our new result gives a uniform way to compare solutions at all times.

We verify our results numerically and, more importantly, we investigate the range of values of ϵ for which stability holds, providing a sharp estimate in certain cases. We develop a second order accurate numerical scheme and we present simulations in one dimension for a system of two interacting species. The numerical scheme is based on a finite volume discretisation with upwinding and second order reconstruction, we refer to [BCF12,CCH15] and the references therein for further details on this type of approach. This numerical strategy has been successfully used in many gradient flow type equations and systems [CHS18], and it has been recently generalized to high order DG-approximations in [SCS18]. See section 3.2 for more details.

II. Systems of equations with large cross-diffusion terms.

Quasilinear, non-degenerate systems with large cross diffusion terms are more difficult to study than those with small cross-diffusion. In a way, they already present many of the difficulties that are typical of degenerate problems. One reason is that the components are strongly coupled and the cross-diffusion terms are not necessarily small compared to the self-diffusive ones. Regularity results for these systems are available, for example, in [LSU88, Ark95, BF13]. Interestingly, solutions of cross diffusion systems may present spiking behaviour in certain circumstances, for more details about phenomenon we cite [Ni98,KW11]. Regarding of the study of pattern formation and convergence to equilibrium, we refer to [VE09, MNB15] and [LN96, BDFPS10, BBRW17] respectively.

Existence of weak solutions for this type of problems has been studied following different approaches, for example using *a priori* estimates, in [LSU88] and [CFL76], or abstract parabolic theory, in [Ama89]. A particularly interesting perspective has been recently presented in [Jün15], where it was noticed that a large class of parabolic systems possesses a so-called *entropy structure*. This means that there exists a Lyapunov-type functional, commonly referred to as the entropy functional, that decreases along solutions over time. Furthermore, the first variation of this functional defines, under suitable conditions, a useful change of variables that allows to reformulate the problem in full space rather than on a bounded domain. Such change of variables defines a one-to-one correspondence between the domain $U \subset \mathbb{R}^N$ where the solutions of the original problem take values, and $\mathbb{R}^N$ itself, where the new unknown takes values. Exploiting the properties of this map, it is possible to prove a weaker version of a maximum principle that ensures that if the initial datum belongs to the domain U, then the solution will remain in U for almost all times. The method introduced in [Jün15] is particularly powerful because it can be applied to non-symmetric, degenerate systems, provided they have an entropy structure. It is legitimate to wonder how common it is for a system of PDEs to have such structure. A partial answer was given in [DGJ97], where it was proven that having an entropy is equivalent to the existence of a change of variables that makes the system symmetric. Other works based on entropy methods include [CJM+01], treating large-time asymptotics, [BDFPS10], studying cross-diffusion with volume exclusion, [DLM14, DLMT15, DT15], considering generalisations of the SKT model and triangular cross-diffusion, and [DFEF18], studying a system with nonlocal interactions.

This table shows some important references and our results for cross-diffusion systems.

	Small cross-diffusion	Large cross-diffusion / degenerate
System of PDEs	Existence, Proposition 3.1 Boundedness, Proposition 3.1 Uniqueness, Proposition 3.1 Stability, Theorem 3.5	Existence, [Jün15], [CFL76], [Ama89] Boundedness, [Jün15] Uniqueness, Theorem 4.3 and [CFL76] Minimum principle, Proposition 4.13 Stability, Th. 4.11, Prop. 4.19, [CFL76]

Table 1.3: This table summarises the key results for cross-diffusion systems. For each category of results (existence, boundedness, uniqueness, stability) we either refer to relevant papers or to a statement in this document.

When we are working with systems with large cross-diffusion, we are usually assuming that they have an entropy structure in the sense of [Jün15], so that existence and boundedness of solutions is guaranteed, as discussed in section 4.1.1.

We prove a stability estimate for non-degenerate systems under a regularity assumption. In particular we control the difference of the solutions corresponding to two distinct diffusion tensors in the H^1 norm in terms of the supremum norm of the difference of the diffusion tensors themselves. Our proof combines the approaches presented in [CFL76] and [CH05] in order to obtain a shorter proof and to simplify the assumptions. The regularity assumption (also present in [CFL76]) requires that the gradient of the solution satisfies an integrability condition, namely $\nabla u \in L^p$, for some $p > d$ (the typical values of d are 1, 2 and 3). In dimension one and two, we are able to show that this requirement is satisfied, as we prove a higher integrability result for the gradient using the so-called Gehring's Lemma. See sections 4.1.2 and 4.1.3 for all the details, as well as [GS82, GM79, DFT10, Par07]. For other results on higher integrability we refer to [Cam80, Sof13].

The study of degenerate systems is more delicate. We would like to study their properties as limit problems of certain non-degenerate systems, therefore we have to choose a way to approximate them with strongly parabolic systems. Inspired by the strategy used in the scalar case, it seems natural to construct a regularised problem by "lifting" the initial datum by a constant vector with small, non-negative entries. This allows us to avoid the presence of free boundaries in the solution, a phenomenon that is typical for degenerate systems with compactly supported initial data. We have to ensure that the above-mentioned lifted problem effectively provides a form of regularisation. In particular, we have to verify that solutions of degenerate problems that are strictly positive at the initial time remain positive during the evolution. It is indeed the case, as shown in section 4.2.1 (a weaker non-negativity result was proven in [CHL03]).

The next step consists in proving a stability result. To do so, we have to make an additional structural assumption and require that the diffusion matrix can be written as the Jacobian of a suitable monotone vector field. It is well-known that in this case the solutions of the system of PDEs are unique despite the possible degeneracy (it can be shown by means of the so-called H^{-1} technique, i.e., testing the weak formulation against the inverse Laplacian of the solution). With this extra assumption, we are able to show stability in the L^2 norm, as discussed in section 4.2.2. Some regularity results for degenerate systems are available in [LN06, Le10, DT15].

Section 4.2.3 is dedicated to the formal derivation of a different formulation of a degenerate system of PDEs. In particular, we focus on the possible free boundaries in one space dimension. We derive a new PDE in a re-mapped domain and a number of ODEs equal to the number of points on the free-boundary (for example, two components of the solution having two free-boundaries each). We use this reformulation in the final section 4.2.4 in order to construct a numerical method that keeps track of the points of the free boundary (for $d = 1$). We test our *front-tracking algorithm* on a specific example and we compare it to the method introduced in section 3.2.2. Another algorithm was derived in [DH84].

Chapter 2

Nonlinear diffusion equations

Contents

In this chapter we analyse scalar diffusion equations in divergence form with diffusion coefficients depending on the solution itself. We will first consider the case of filtration equations with a quite general type of nonlinearity. We then extend our analysis to a family of anisotropic diffusion equations proving existence of solutions and adapting tools available in the literature - such as comparison principle and stability estimates - to the case at hand. Finally, we consider an aggregation-drift-diffusion problem that can not be treated with the previous techniques since the drift term tends to infinity outside a given domain. The novelties consist in

1. extension of the comparison principle in [WYLZ01] and of Alikakos estimates to the case of filtration equations which may degenerate at two different values of the solution;

2. reformulation of the stability estimates in [CG99,CK05] for weak solutions on bounded domains with no-flux boundary conditions;

3. extension of the proof of existence of weak solutions in [Váz07] tho the case of anisotropic diffusion;
4. study of the effects of a "strong" confining potential on scalar diffusion equations.

2.1 Filtration equation

This section is dedicated to a family of scalar problems known as filtration equations, or generalised porous medium equations. These are nonlinear, possibly degenerate PDEs that generalise the diffusion equation. We will discuss existence and boundedness of solutions, as well as a the comparison principle, which implies uniqueness and the contraction property in L^1.

Our main references for this section are the books of J. L. Vázquez, *The Porous Medium Equation*, [Váz07], and of Wu, Yin and Li, *Nonlinear Diffusion equations*, [WYLZ01]. Another relevant reference in this context is the work of Bertsh and Hilhorst [BH86].

The main difference with respect to the above-mentioned references is that we keep the *filtration function* (see (2.1) below) general and, in particular, we allow its derivative (the diffusion coefficient) to vanish both at zero and at another point. Consequently we have to deal with some difficulties concerning, for example, the proof of boundedness of solutions, which is carried out with a generalisation of Alikakos estimates.

2.1.1 Well-posedness results

We consider the following scalar problem:

$$\begin{aligned} \partial_t u - \Delta\phi(u) &= 0 \text{ on } (0,T)\times\Omega, \\ \nabla\phi(u)\cdot n &= 0 \text{ on } (0,T)\times\partial\Omega, \\ u(t=0) &= u_0 \text{ in } \Omega, \end{aligned} \tag{2.1}$$

which can be obtained from problem (1.1) assuming that, in addition to the assumptions of Section 1.1.3, we have $N=1$, $\mathcal{B}=0$ and that $\mathcal{A}$ is autonomous, i.e. $\mathcal{A}(u) = D_u(\phi(u))$ for a suitable function ϕ (see Definition 2.1).

Depending on the properties of ϕ, such model can describe different phenomena such as fast or slow diffusion in different contexts. The derivative ϕ' determines the ellipticity (or parabolicity) of the equation. We say that problem (2.1) becomes degenerate at all values of $s \in \mathbb{R}_+$ where $\phi'(s) = 0$. One example to keep in mind is the porous medium equation, corresponding to $\phi(u) = u^m$, $m > 1$. Interestingly, in this case it is possible to write a family of solutions explicitly; they are known as *Barenblatt solutions* and they are obtained choosing a Dirac delta centred at the origin as initial condition (see [Bar52]).

They have the form:

$$u(x,t) = \max\left\{0, t^{-J}\left[H - J\left(\frac{|x|}{t^{\frac{J}{d}}}\right)^2\right]^{\frac{1}{m-1}}\right\},$$

where H is the mass the Dirac delta for $t = 0$ and $J = \frac{m-1}{2dm(m-1+2/d)}$. These functions are *a priori* defined in $\mathbb{R}^d$, but they can be redefined in a bounded domain for a fixed T.

The choice of homogeneous Neumann conditions ensures that the initial mass is conserved for all $t > 0$. We always assume that the initial condition is bounded and non negative and we shall see that, under suitable conditions, the solution will remain bounded and non-negative at all times. Let us state precisely the hypotheses that we make on the functions ϕ and u_0 appearing in problem (2.1).

Definition 2.1. We call *filtration function* a function $\phi \in C^1(\mathbb{R}^+) \cap C(\mathbb{R})$, such that

- (monotonicity) either $\phi' > 0$ on $\mathbb{R}^+$, or $\phi' > 0$ on $(0, s_0)$ and $\phi' = 0$ on $[s_0, \infty)$;
- (positivity) $\phi \geq 0$ on $\mathbb{R}^+$and $\phi = 0$ on $\mathbb{R}^-$;
- (polynomial growth) there exists $\aleph, \beth \geq 1$ such that $\phi(s) \leq |s|^{\aleph} + \beth$ for all $x \in \mathbb{R}$.

The typical behaviour of ϕ is sketched in figure 2.1.

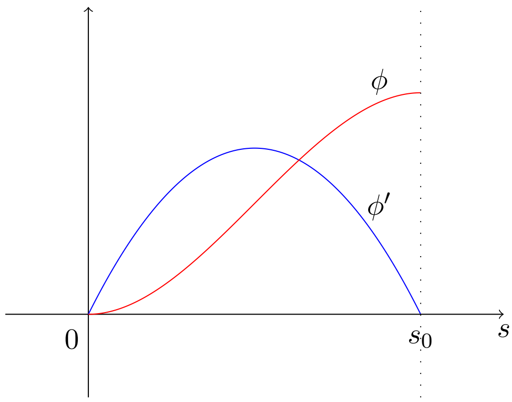

Figure 2.1: The typical behaviour of a filtration function and its derivative.

Remark 2.1 (The role of s_0)**.** According to Definition 2.1, problem (2.1) may be degenerate at either $s = 0$, or $s = s_0$, or both (we exclude the case of more than two degeneracies). Note that problem (2.1) is unchanged if a constant is added to the filtration function. To fix ideas, we have assumed that $\phi(0) = 0$. The constant s_0 plays the role of a threshold for the values of a solution u and for the properties of our model which may be interpreted as a model for the steric hindrance. When particles (or cells) are fully packed, we expect the diffusion coefficient to vanish. When s_0 is finite, we shall say we are in presence of a steric hindrance constraint.

Definition 2.2. We say that u_0 is an admissible initial datum for problem (2.1) if it is non negative and bounded in $L^\infty(\Omega)$. Additionally, if the model contains a steric hindrance constraint s_0, then an admissible initial datum satisfies

$$0 \le u_0 \le s_0 \text{ a.e. in } \Omega.$$

We now introduce the two notions of weak solutions that we are going to use (see also Definition 2.5 for a more general PDE).

Definition 2.3 (L^2-weak solution)**.** Suppose that u_0 and ϕ are admissible (see Definitions 2.1, 2.2). We say that u is an L^2 weak solution of (2.1) if

$$u \in C^0(0,T;L^2(\mathbb{R}^d)), \quad \nabla\phi(u) \in L^2(Q_T), \quad \partial_t u \in L^2(0,T;(H^1(\mathbb{R}^d))'),$$

and, for all test functions $\eta \in H^1(Q_T)$,

$$\int_\Omega u(t)\eta(t)\,dx\Big|_{t=0}^{t=T} - \int_{Q_T} \left[\nabla\phi(u)\cdot\nabla\eta - u\partial_t\eta\right]\,dx\,dt = 0. \tag{2.2}$$

The initial datum is satisfied in the L^2 sense.

This notion of weak solution will be used in the proof of existence results. We will also invoke an a-priori weaker notion of solution, in the proof of the so-called *comparison principle* (see Section 2.1.2). In particular,

Definition 2.4 (L^1-weak solutions)**.** Given $T > 0$, a function $u : [0,T] \to L^1(\Omega)$ is an L^1-weak solution of problem (2.1) if $u \in L^\infty(0,T;L^1(\Omega))$, $\phi(u) \in L^1(Q_T)$, and there holds

$$\int_\Omega u(t)\eta(t)\,dx\Big|_{t=0}^{t=T} - \int_{Q_T} [u\partial_t v + \phi(u)\Delta v]\,dx\,d\tau = 0$$

for all non-negative $v \in H^2\left([0,\infty)\times\bar{\Omega};\mathbb{R}^+\right)$ such that $\frac{\partial v}{\partial \nu} = 0$ on $[0,\infty)\times\partial\Omega$.

Remark 2.2. A L^2-weak solution in the sense of Definition 2.3 is also a L^1-weak solution according to Definition 2.4.

The following lemma shows that $\phi(u)$ is an admissible test function in (2.2) and it will be useful later.

Lemma 2.1. *If u is a L^2-weak solution of problem (2.1), then $\phi(u) \in L^2\left(0,T;H^1(\Omega)\right)$.*

Proof of Lemma 2.1. The definition of L^2-weak function does not specify the integrability of $\phi(u)$, but only that of u and that of $\nabla\phi(u)$. Let us verify that it is the case. Note that $\phi(u)$ is integrable. By definition, there exists $\aleph, \beth \ge 1$ such that

$$1 \le \phi(u) + 1 \le |u|^\aleph + \beth + 1,$$

therefore $(\phi(u)+1)^{\frac{2}{\aleph}} \in L^1(\Omega)$. If $\aleph = 0$ or $\aleph = 1$, the proof is complete. If $\aleph \geq 2$ then $(1+\phi(u))^{\frac{2}{\aleph}-1} \leq 1$, and

$$\left|\nabla\left((1+\phi(u))^{\frac{2}{\aleph}}\right)\right| = \frac{2}{\aleph}\left|(1+\phi(u))^{\frac{2}{\aleph}-1}\nabla\phi(u)\right| \leq |\nabla\phi(u)|,$$

thus $(1+\phi(u))^{\frac{2}{\aleph}} \in W^{1,1}(\Omega)$. Thus if $d=1$, the proof if complete by Sobolev embedding. If $d \geq 2$, then $(1+\phi(u))^{\frac{2}{\beth}}(u) \in L^1(\Omega)$, with $\beth = \aleph\frac{d-1}{d} < \aleph$. Repeating the same argument finitely many times, we obtain our result. □

For future reference, we remind the reader of an existence result for linear parabolic equations with bounded coefficients depending on time and space (we refer to Dautray–Lions, [DL93], chap. XVIII).

Proposition 2.2 (Existence for linear, time dependent problems, [DL93]). *Let L be a linear operator of the form*

$$L := -\partial_i(a_{ij}(t,x)\partial_j) + b_i(t,x)\partial_i + c(t,x),$$

where $a \in L^\infty\left(Q_T;\mathbb{R}^{d\times d}\right)$, $b \in L^\infty\left(Q_T;\mathbb{R}^d\right)$, $c \in L^\infty\left(Q_T;\mathbb{R}^+\right)$, and a is uniformly elliptic, that is, there exist two positive constants λ and Λ such that

$$\lambda|\xi|^2 \leq a_{ij}(t,x)\xi_i\xi_j \leq \Lambda|\xi|^2,$$

for any $\xi \in \mathbb{R}^d$ and for all $(t,x) \in Q_T$. Then for any $u_0 \in L^2(0,T;L^2(\Omega))$ there exists a unique weak solution u to

$$\begin{aligned} \partial_t u + Lu &= 0 \text{ on } Q_T, \\ \partial_\nu u &= 0 \text{ on } (0,T)\times\partial\Omega, \\ u(t=0) &= u_0, \end{aligned}$$

such that $u \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$, $\partial_t v \in L^2(0,T;(H^1(\Omega))')$.

Remark 2.3. Our aim is to prove an existence and uniqueness result for weak solutions. As far as we know these results are available in the literature as long as ϕ' is degenerate only at the origin, while the study of two degeneracies was not treated in any of the references we used.

We now state two (new) theorems, concerning existence and uniqueness of weak solutions to problem (2.1), which extend the analogous results available in the literature.

Theorem 2.3 (Existence). *Suppose that ϕ is a filtration function and u_0 is an admissible initial datum as introduced in Definitions 2.1 and 2.2. Then there exists an L^2-weak solution to problem (2.1) in the sense of Definition 2.3.*

The existence proof will follow from a fixed point approach in section 2.2.3. The main steps of such proof are the same as in [Váz07], but our result is more general (as we consider less restrictive assumptions for ϕ).

In terms of uniqueness of solutions, we will prove what follows:

Theorem 2.4 (Uniqueness). *Under the assumptions of Theorem 2.3,*

- *if $s_0 = \infty$, there exists at most one L^1-weak solution to problem (2.1) in the sense of Definition 2.4;*
- *in all cases, there is at most one L^2-weak solution to problem (2.1) in the sense of Definition 2.3.*

Remark 2.4. The uniqueness proof is based on a new formulation of the comparison principle in Proposition 2.5. In turn, the proof of the comparison principle is based on the techniques presented in [WYLZ01], which we adapt to our framework. A key estimate is given in Lemma 2.7 (we later found out that a similar technique is presented in [DK07]).

2.1.2 Comparison principle and boundedness

The comparison principle constitutes a very powerful tool since, as the name says, it ensures that solutions that are initially ordered remain ordered over time.

The following statement is general in the sense that it involves only weak solutions (not bounded a priori). A more classical but less general version of the comparison principle is presented afterwards.

Proposition 2.5 (Comparison principle for weak solutions). *Let $u_1, u_2 \in L^2(Q_T)$ be L^2-weak solution of problem (2.1), with admissible initial data u_{01} and u_{02} respectively, such that $u_{01} \leq u_{02}$. Then $u_1 \leq u_2$ a.e. in Q_T.*

First of all, we need some preliminary results that will contribute to the proof of proposition 2.5 and theorem 2.4. We have the following weaker form of comparison principle:

Lemma 2.6 (Comparison principle for bounded solutions). *Let $u_1, u_2 \in L^1(Q_T) \cap L^\infty(Q_T)$ be L^1-weak solutions of problem (2.1) with bounded initial data u_{01} and u_{02} respectively. In addition, suppose that*

- *either $s_0 = \infty$ and $0 \leq u_{01} \leq u_{02}$ in Ω,*
- *or $s_0 < \infty$, $0 \leq u_{01} \leq u_{02} \leq s_0$ in Ω and $0 \leq u_1, u_2 \leq s_0$ in Q_T.*

Then $u_1 \leq u_2$ in Q_T.

The proof of this lemma, done in Section 2.1.3, is an adaptation to bounded domains and more general ϕ of the one presented in [WYLZ01], Section 1.3.1.

Remark 2.5. Due to the extra assumption concerning boundedness of solutions, the Lemma above is somewhat unsatisfactory, as it assumes that solutions satisfy an a priori bounds for all times. A full comparison principle should not include such assumption (as in Proposition 2.5).

Remark 2.6. Uniqueness in the class of L^1-weak solutions does not follow immediately since the we can only control weak solutions by the initial datum (smallness result).

We now establish that weak solutions are bounded by their initial data in L^∞. The technique we use is similar to that in [DK07].

Lemma 2.7 (L^∞ estimates)**.** *Let u be a L^2-weak solution of problem (2.1). Suppose u_0 is an admissible initial datum according to Definition 2.2. Then*

$$u \geq 0 \text{ a.e. in } (0,T) \times \Omega,$$

and there holds

$$\|u\|_{L^\infty((0,T)\times\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} .$$

For the proof of this Lemma we need some technical results, we will proceed in several steps. Given $p \geq 1$, we define F_p as an anti-derivative of ϕ^p, that is,

$$F_p(s) = \int_0^s \phi(\sigma)^p \, d\sigma.$$

Lemma 2.8. *For any $s < s_0$, $a \leq s_0$ and $b > a$, there holds*

$$\chi_{\{s>b\}} \int_0^a \phi(\sigma)^p \, d\sigma \leq F_p(s) \leq \phi\,(s)^p\, s. \tag{2.3}$$

Proof of Lemma 2.8. The lower bound follows from the positivity of ϕ, as

$$\begin{aligned} F_p(s) &= \chi_{\{s\leq b\}} \int_0^s \phi(\sigma)^p \, d\sigma + \chi_{\{s>b\}} \int_0^s \phi(\sigma)^p \, d\sigma \\ &\geq \chi_{\{s>b\}} \int_0^a \phi(\sigma)^p \, d\sigma. \end{aligned}$$

As ϕ is monotonously increasing until s_0,

$$F_p(s) \leq \phi\,(s)^p\, s \text{ for all } s < s_0.$$

□

Lemma 2.9. *Under the hypothesis of Lemma (2.7), we have*

$$u \geq 0 \text{ and, when } s_0 \neq \infty, u \leq s_0 \text{ a.e. in } \Omega \times (0,T).$$

Furthermore,

$$\int_\Omega F_p(u)\,dx \le \int_\Omega F_p\left(u_0\right)\,dx \text{ for a.e. } t\in(0,T).$$

Proof of Lemma 2.9. Note that for $u\le 0$, $\phi(u)=0$, defining $u_*=-\min(u,0)$, we find $\nabla\phi(u)\cdot\nabla u_*\equiv 0$ a.e. in Q_T. Therefore, testing the equation against u_*, we have

$$\partial_t\int_\Omega u_*^2=0 \text{ a.e. } t\in(0,T)$$

and, since $u_0\ge 0$, we obtain $u\ge 0$ almost everywhere. Now recall that ϕ is constant on (s_0,∞) by Definition 2.1In particular, for $u^*=\max(u,s_0)$, we have $\phi(u^*)=\phi(s_0)$. We deduce that

$$\partial_t\int_\Omega (u^*)^2=0 \text{ a.e. } t\in(0,T),$$

leading to the result as $u_0\le s_0$. In order to prove the other inequalities, we have to identify a suitable test function. Given $k>0$, we introduce the following truncation:

$$\begin{aligned} T_k:\mathbb{R} &\to \mathbb{R}\\ s &\mapsto \min\{k,s\}.\end{aligned}$$

Now consider the function given by $\psi_k(s)=\phi\left(T_k\left(s\right)\right)^{p-1}\phi(s)$, and its anti-derivative $\Psi_k(s)=\int_0^s\psi_k(r)\,dr$. Note that $\Psi_k\to F_p$ as $k\to\infty$. Thanks to Proposition 2.1, $\phi(u)\in L^2\left(0,T;H^1(\Omega)\right)$ and, consequently, we also have $\psi_k(u)\in L^2\left(0,T;H^1(\Omega)\right)$. We notice that

$$\nabla\psi_k(u)=\begin{cases} p\phi(u)^{p-1}\nabla\phi(u) & \text{when } u\le k,\\ \phi\left(k\right)^{p-1}\nabla\phi(u) & \text{otherwise.}\end{cases}$$

Integrating equation (2.1) against $\psi_k(u)$ we find, for a.e. $t\in(0,T)$,

$$\int_\Omega \Psi_k(u\left(t,x\right))\,dx+\int_{Q_T}\min\{p\phi(u)^{p-1},\phi\left(k\right)^{p-1}\}\left|\nabla\phi(u)\right|^2\,dx\,dt\le\int_\Omega\Psi_k(u_0\left(x\right))\,dx,$$

which gives

$$\int_\Omega \Psi_k(u\left(t,x\right))\,dx\le\int_\Omega\Psi_k(u_0(x))\,dx. \tag{2.4}$$

Passing to the limit for $k\to\infty$ in (2.4), we obtain the result:

$$\int_\Omega F_p(u\left(t,x\right))\,dx\le\int_\Omega F_p(u_0(x))\,dx.$$

□

Proof of Lemma 2.7. The upper bound of F_p given by Lemma 2.8, we find

$$\left(\int_\Omega F_p\left(u_0\right)\,dx\right)^{\frac{1}{p}}\le\left(\int_\Omega\phi(u_0)^p u_0\,dx\right)^{\frac{1}{p}}\le\left(|\Omega|\left\|u_0\right\|_{L^\infty(\Omega)}\right)^{\frac{1}{p}}\phi\left(\left\|u_0\right\|_{L^\infty(\Omega)}\right).$$

Thanks to Lemma 2.9, we only need to consider the case when $\|u_0\|_{L^\infty(\Omega)} < s_0$. Given $a > 0$ such that $a \le s_0$, the lower bound on F_p given by Proposition 2.8 shows that for *a.e.* $t \in (0, T)$ there holds

$$|\{x \in \Omega \,:\, u(t,x) > a\}| \left(\int_0^a \phi(\sigma)^p \, d\sigma \right) \le \int_\Omega F_p \left(u \left(t, x \right) \right) \, dx$$

Proposition 2.9 then shows that

$$|\{x \in \Omega \,:\, u(t,x) > a\}|^{\frac{1}{p}} \left(\int_0^a \phi(\sigma)^p \, d\sigma \right)^{\frac{1}{p}} \le \left(|\Omega| \, \|u_0\|_{L^\infty(\Omega)} \right)^{\frac{1}{p}} \phi \left(\|u_0\|_{L^\infty(\Omega)} \right). \tag{2.5}$$

Note that

$$\lim_{p \to \infty} \left(|\Omega| \, \|u_0\|_{L^\infty(\Omega)} \right)^{\frac{1}{p}} . = 1,$$

Furthermore if $|\{x \in \Omega \,:\, u(t,x) > a\}| \neq 0$,

$$\lim_{p \to \infty} |\{x \in \Omega \,:\, u(t,x) > a\}|^{\frac{1}{p}} \left(\int_0^a \phi(\sigma)^p \, d\sigma \right)^{\frac{1}{p}} = \phi \left(a \right),$$

thus $\phi\left(a\right) \le \phi\left(\|u_0\|_{L^\infty(\Omega)}\right)$, and, as ϕ is strictly increasing in $(0, s_0)$,

$$a \le \|u_0\|_{L^\infty(\Omega)} .$$

Thus $\left| \left\{ x \in \Omega \,:\, u(t,x) > \|u_0\|_{L^\infty(\Omega)} + \epsilon \right\} \right| = 0$ for all $0 < \epsilon \le s_0 - \|u_0\|_{L^\infty(\Omega)}$ and *a.e.* $t \in (0, T)$. In other words, $\|u\|_{L^\infty(Q_T)} \le \|u_0\|_{L^\infty(\Omega)}$, as announced. □

We are now in the position to prove the comparison principle stated in Proposition 2.5.

Proof of Proposition 2.5. Thanks to Lemma 2.7, any L^2-weak solution with admissible initial datum given by Definition 2.2 is bounded and satisfies $\|u\|_{L^\infty((0,T)\times\Omega)} < s_0$. We may therefore apply Lemma 2.6 to conclude. □

We can now proceed with the proof of our uniqueness result.

Proof of Theorem 2.4. Uniqueness in the class of L^2-weak solutions introduced in Definition 2.3 follows from the comparison principle (Proposition 2.5). In particular, given two weak solutions u_1 and u_2 to problem (2.1) satisfying the same initial datum u_0, it follows that $u_1(x,t) \le u_2(x,t)$ as well as $u_2(x,t) \le u_1(x,t)$ for a.e. $(x,t) \in Q_T$, which implies $u_1(x,t) = u_2(x,t)$ a.e. in Q_T. □

2.1.3 Proof of the comparison principle for bounded solutions

Proof of Lemma 2.6. The structure of this proof is an adaptation of an analogous result proven in [WYLZ01]. Unlike in that proof, we work in a bounded domain and our assumptions are less restrictive.

Integrating equation (2.1) against a test function $v \in C_c^\infty(Q_T)$ we find for any L^1-weak solution u of Problem (2.1),

$$\int_\Omega [u(T,x)v(T,x) - u(0,x)v(0,x)]\, dx = \int_0^T \int_\Omega [u\partial_t v + \phi(u)\Delta v]\, dx\, dt. \tag{2.6}$$

Now we define $z = u_1 - u_2$ and $z_0 = u_{01} - u_{02}$. As (2.6) is satisfied by u_1 and u_2, there holds

$$\begin{aligned} \int_\Omega [z(T,x)v(T,x) - z(0,x)v(0,x)]\, dx &= \int_0^T \int_\Omega [z\partial_t v + (\phi(u_1) - \phi(u_2))\, \Delta v]\, dx\, dt \\ &= \int_0^T \int_\Omega z\, [\partial_t v + a(t,x)\Delta v\, dx\, dt]\, . \end{aligned}$$

The coefficient a takes the form of a difference quotient and it is defined by

$$a(t,x) = \begin{cases} \frac{\phi(u_1)-\phi(u_2)}{u_1-u_2} & \text{if } u_1 \neq u_2 \\ \phi'(u_1) & \text{if } u_1 = u_2 \end{cases}$$

We want to find a suitable function $v \geq 0$ such that the above relation gives

$$\int_\Omega [z(T,x)v(T,x) - z(0,x)v(0,x)]\, dx \leq 0,$$

where we also require $v \geq 0$. Provided $v(T,x) \geq 0$ can be chosen in a sufficiently large set, the condition $z(0,x) \leq 0$ will imply $u_1(T,x) \leq u_2(T,x)$ a.e. in Ω, which is our thesis (the final time T can be arbitrarily chosen). Thus we would like to solve the following linear problem

$$\begin{array}{ll} \partial_t v + a\Delta v = 0 & \text{in } Q_T, \\ v(t,x) = 0 & \text{on } \partial\Omega \times (0,T), \\ v(T,x) = g(x) & \text{in } \Omega, \end{array}$$

with $g \in L^2(Q_T)$, $g \geq 0$. This problem needs not be regular enough since we have limited information about the coefficient a. Therefore we consider the following mollified problem:

$$\begin{array}{ll} \partial_t v_n + a_n\Delta v_n = 0 & \text{in } Q_T, \\ v_n(t,x) = 0 & \text{on } \partial\Omega \times (0,T), \\ v_n(T,x) = g(x) & \text{in } \Omega, \end{array} \tag{2.7}$$

where a_n is a mollified and translated version of a defined as follows:

$$a_n = \int_\Omega \rho_n(x-y)a(y)\, dy + \frac{1}{n}, \qquad \text{for } \frac{1}{n} < \operatorname{dist}(x, \partial\Omega),$$

and $\rho_n, g \in C_c^\infty(B_R)$. Furthermore we choose ρ_n such that the following inequality is satisfied:

$$\int_0^T \left\| a - \int_\Omega \rho_n(x-y)a(y)\,dy \right\|^2_{L^2(\Omega)} dt \le \frac{1}{n^2}. \tag{2.8}$$

Note that standard mollifiers provide precisely such a sequence, up to the extraction of a suitable subsequence. Thanks to the linear theory of parabolic equations (see Proposition 2.2) we know that problem (2.7) has a unique weak solution.

We can now choose our original test function to be $v = v_n$ and we obtain:

$$\begin{aligned}
\int_\Omega z(T,x)v_n(T,x) - z_0(x)v_n(0,x)\,dx &= \int_0^T \int_\Omega z\left(\partial_t v_n + a\Delta v_n\right)\,dx\,dt \\
&= \int_0^T \int_\Omega z\left(-a_n\Delta v_n + a\Delta v_n\right)\,dx\,dt \\
&= I_n.
\end{aligned}$$

Let us now show that I_n tends to 0 as $n \to \infty$, we need two preliminary estimates. We test equation (2.7) against Δv_n over $B_R \times [t,T]$:

$$\int_\Omega \frac{1}{2}|\nabla v_n|^2\,dx + \int_t^T \int_\Omega a_n|\Delta v_n|^2\,dx\,dt = \int_\Omega \frac{1}{2}|\nabla g|^2\,dx, \tag{2.9}$$

which implies the two following bounds (independent of n):

$$\int_\Omega |\nabla v_n|^2\,dx \le C(g), \qquad \int_t^T \int_\Omega a_n|\Delta v_n|^2\,dx\,d\tau \le C(g). \tag{2.10}$$

Now we use these two results to estimate I_n.

$$\begin{aligned}
I_n &= \int_0^T \int_\Omega z(a-a_n)\Delta v_n\,dx\,dt \\
&\le \|z\|_\infty \left(\int_0^T \int_\Omega \frac{(a-a_n)^2}{a_n}\,dx\,dt\right)^{\frac{1}{2}} \left(\int_0^T \int_\Omega a_n(\Delta v_n)^2\,dx\,dt\right)^{\frac{1}{2}} \\
&\le C(g)\,\|z\|_\infty \left(\int_0^T \int_\Omega \frac{(a-a_n)^2}{a_n}\,dx\,dt\right)^{\frac{1}{2}}.
\end{aligned}$$

In addition, from the definition of a_n we get

$$\int_0^T \int_\Omega \frac{(a-a_n)^2}{a_n}\,dx\,dt \le n\int_0^T \int_\Omega \left(a - \rho_n * a - \frac{1}{n}\right)^2 dx\,dt \le \frac{C}{n},$$

hence we conclude that $|I_n| \to 0$ as $n \to \infty$. This implies, as announced, that

$$\int_\Omega z(x,T)g(x)\,dx \le \int_\Omega z_0(x)v(0,x)\,dx \le 0,$$

and since $g, \phi(x,0) \ge 0$, this concludes the proof. □

2.2 Anisotropic diffusion

In this section we prove well-posedness and stability of scalar equations with anisotropic diffusion. We use Krushkov doubling variable technique (see [Kru70]) in order to prove a stability estimate for nonlinear diffusion equations with spatial anisotropy. An essential ingredient in this approach is the L^1 contraction (see for example [Ott96, Váz07]). Similar results are also available in [CK05] and [Pan17]. This result is crucial since it holds true both for degenerate and non-degenerate equations, and even for some general conservation laws. The extension of this result to systems presents many difficulties and it remains mostly open. Similar results (in full space) can be found in [CK05].

In addition to the assumptions stated in section 1.1.3, we are going to assume that $N = 1$, $\mathcal{B} = 0$, $\mathcal{A} = A^{\alpha\beta}(u)$, where A is a continuous function with values in the set of symmetric, positive semi-definite $d \times d$ matrices, written $M_{s+}^{d\times d}(\mathbb{R})$. In the particular case where $A(u) = \phi'(u)I$ we recover the filtration equation.

The problem we consider takes the form:

$$\begin{aligned} \partial_t v - \operatorname{div}\left(A\left(u\right)\nabla u\right) = 0 \qquad &\text{on } Q_T, \\ A\left(u\right)\nabla u \cdot n = 0 \qquad &\text{on } Q_T, \\ u\left(t=0\right) = u_0 \qquad &\text{on } \Omega. \end{aligned} \tag{2.11}$$

The aim of this section is to show a continuous dependence estimate with respect to the nonlinearity A .

If A is positive definite and bounded (not necessarily symmetric), existence of an L^2-weak solution is given by Corollary (2.20).

2.2.1 Continuous dependence in L^1

In the spirit of Cockburn and Griepenberg in [CG99], we prove a continuous dependence result with respect to the nonlinearity $A(\cdot)$.

Theorem 2.10. *Given $A, B \in C([0,T], M_{s+}^{d\times d}(\mathbb{R}))$, in particular*

$$\min_{s\in\mathbb{R}} \left(A(s)\xi\cdot\xi, B(s)\xi\cdot\xi\right) \geq 0 \textit{ for all } \xi \in \mathbb{R}^d,$$

and there exists $M > 0$ such that

$$\max_{s\in\mathbb{R}} \left(A(s)\xi\cdot\xi, B(s)\xi\cdot\xi\right) \leq M\left|\xi\right|^2 \textit{ for all } \xi \in \mathbb{R}^d.$$

We write their principal square roots $a(\cdot)$ and $b(\cdot)$. Let u and v be bounded, non-negative solutions of (2.11) corresponding to (A, u_0) and (B, v_0). Assume further that u and u_0

are in $BV(\Omega)$. Then for almost every t, the following inequality holds:

$$\|u(t)-v(t)\|_{L^1(\Omega)} \leq C\,\|a-b\|_{L^\infty(I_0)}\sqrt{t}\left(|u|_{BV(\Omega)}+|u_0|_{BV(\Omega)}\right) \tag{2.12}$$

where C is a universal constant, and

$$I_0 := (-\max\{\|u\|_\infty, \|v\|_\infty\}, \max\{\|u\|_\infty, \|v\|_\infty\}).$$

Remark 2.7. This result is proven in Section 2.2.2. The proof is similar to the original one in [CG99], and it was already extended in [CK05]. We have simplified it where possible and adapted it to the case of a bounded domain.

Let us focus on the consequences of Theorem 2.10.

Corollary 2.11 (Comparison principle, non degenerate case.)**.** *Suppose $u_0, v_0 \in L^\infty(\Omega)$ satisfy $m \leq u_0 \leq v_0 \leq M$ a.e. in Ω, for two constants $m, M \in \mathbb{R}$. Suppose that $A \in C\left([m,M]; M^{d\times d}_{s+}(\mathbb{R})\right)$ is symmetric, and that there exists $\lambda > 0$ such that, for all $x \in [m,M]$ and $\xi \in \mathbb{R}^d$,*

$$\sup_{[m,M]}|A| \leq \lambda^{-1} \quad \textit{and} \quad \lambda\xi\cdot\xi \leq A(x)\xi\cdot\xi.$$

Then the L^2-weak solution $u \in C([0,T), H^1(\Omega))$, $\nabla u \in L^2(Q_T)$, of (2.11) is unique, and

$$m \leq u \leq v \leq M \ \textit{a.e. in } Q_T.$$

We postpone the proof of this Corollary to section 2.2.3.

Notice that Theorem 2.10 concerns stability (or continuous dependence) in L^1 with respect to the diffusion matrix. The result can easily be extended to include a term corresponding to the the difference of initial data. In this sense, it is actually useful to consider the following L^1 contraction estimate by [Ott96], which will allow us to control the BV norms in (2.12) a priori.

Lemma 2.12 (L^1 contraction, [Ott96])**.** *Let u and v be solutions of the problem (2.11) corresponding to initial data u_0 and v_0 in $L^1(\Omega)$ respectively. Then*

$$\|u(t)-v(t)\|_{L^1(\Omega)} \leq \|u_0-v_0\|_{L^1(\Omega)}, \tag{2.13}$$

for a.e. $t > 0$.

Remark 2.8. In the case where the equation (2.11) reduces to the filtration equation, the L^1 contraction principle follows immediately from the comparison principle we proved in the previous section.

We would like to highlight that, using the L^1 contraction, we can easily prove a BV estimate:

Proposition 2.13 (BV estimate)**.** *Let u be a solution of the problem* (2.11) *with initial datum u_0 in $BV(\Omega)$. Then $u(t) \in BV(\Omega)$ for a.e. $t > 0$.*

Proof of Proposition 2.13. Consider the functions $u(t,x)$ and $u(t,x+h)$ and apply the L^1 contraction estimate:

$$\|u(t,x+h) - u(t,x)\|_{L^1(\Omega)} \le \|u_0(x+h) - u_0(x)\|_{L^1(\Omega)} \le |h| \, \|u_0\|_{BV} .$$

The conclusion follows dividing by $|h|$ (recall that $\|u\|_{BV} = \|u\|_{L^1} + |\nabla u|_{TV}$). □

2.2.2 Proof of the stability estimate

This subsection is dedicated to the proof of Theorem 2.10. The fundamental idea consists in testing the equation for the difference $u - v$ against an approximation of $\operatorname{sign}(u-v)$. A key step consists in doubling the variables (in the sense of Kruzkov).

Let ρ be a smooth and convex approximation of the absolute value, namely, for $\varepsilon \ge 0$,

$$\rho(s) = \sqrt{\varepsilon + s^2}.$$

Note that $\rho'(s) = \frac{s}{\rho(s)}$ gives an approximation of $\operatorname{sign}(s)$ for $\varepsilon \to 0$. Furthermore $\rho''(s) = \frac{\varepsilon}{\rho(s)^3}$ converges to a Dirac delta in the sense of distributions (clearly this is not the only function with such properties).

For the rest of this section we adopt the summation convention for repeated indices. Moreover, given $x, y \in \Omega$ and $t > 0$ we write

$$u = u(t,x), \qquad v = v(t,y).$$

Notice that we have *doubled the variables*, meaning that we consider u as a function of x and v as a function of y, with x and y being independent variables. It is clear that they will need to be identified again eventually.

Proof of Theorem 2.10. Using the equations for u and v, we obtain

$$\partial_t \rho(u-v) = \rho'(u-v) \left(\partial_{x^i} \left(A_{ij}(u) \, \partial_{x^j} u \right) - \partial_{y^i} \left(B_{ij}(v) \, \partial_{y^j} v \right) \right). \tag{2.14}$$

We are now going to rewrite this equation in a different form which will allow us to identify a signed quantity and subsequently to integrate by parts more freely. Note that, as v is independent of x,

$$\rho'(u-v) \, \partial_{x^i} \left(A_{ij}(u) \, \partial_{x^j} u \right) = \partial_{x^i} \left(\rho'(u-v) \, A_{ij}(u) \, \partial_{x^j} u \right) - \rho''(u-v) \, A_{ij}(u) \, \partial_{x^j} u \partial_{x^i} u,$$

and in turn,

$$\rho'(u-v) \, A_{ij}(u) \, \partial_{x^j} u = \partial_{x^j} \left(Q^1_{ij}(u,v) \right),$$

with

$$Q^1_{ij}(s,r) = \int_r^s A_{ij}(\sigma)\rho'(\sigma - r)\,d\sigma, \text{ for all } s, r \in \mathbb{R}. \tag{2.15}$$

Similarly,

$$\rho'\left(u(x) - v(y)\right)\partial_{y^i}\left(B_{ij}\left(u\right)\partial_{y^j}u\right) = -\partial^2_{y^iy^j}\left(Q^2_{ij}(u,v)\right) + \rho''\left(u - v\right)B_{ij}\left(v\right)\partial_{y^j}v\partial_{y^i}v,$$

with

$$Q^2_{ij}(s,r) = \int_r^s B_{ij}(\sigma)\rho'(\sigma - r)\,d\sigma, \qquad \text{for } s, r \in \mathbb{R}. \tag{2.16}$$

Altogether we rewrite (2.14) as

$$\begin{aligned} \partial_t\rho(u-v) &= \partial_{x^ix^j}Q^1_{ij}(u,v) + \partial_{y^iy^j}Q^2_{ij}(u,v) \\ &\quad -\rho''(u(x) - v(y))\left(A_{ij}(u)\partial_{x^i}u\partial_{x^j}u\right. && (2.17) \\ &\qquad \left.+B_{ij}(v)\partial_{y^i}v\partial_{y^j}v\right). && (2.18) \end{aligned}$$

We wish to write the last term of the right-hand side as the square of a difference, plus a remainder. Namely, we write (keeping in mind that a and b are symmetric)

$$\begin{aligned} A_{ij}(u)\partial_{x^i}u\partial_{x^j}u &= a_{ik}(u)\partial_{x^i}u\; a_{jk}(u)\partial_{x^j}u, \\ B_{ij}(v)\partial_{y^i}v\partial_{y^j}v &= b_{ik}(v)\partial_{x^i}v\; b_{jk}(v)\partial_{x^j}v, \\ A_{ij}(u)\partial_{x^i}u\partial_{x^j}u + B_{ij}(v)\partial_{y^i}v\partial_{y^j}v &= \left(a_{ik}(u)\partial_{x^i}u - b_{ik}(v)\partial_{y^i}v\right)\left(a_{jk}(u)\partial_{x^j}u - b_{jk}(v)\partial_{y^j}v\right) \\ &\quad +a_{ik}(u)\partial_{x^i}u\; b_{jk}(v)\partial_{y^j}v \\ &\quad +a_{jk}(u)\partial_{x^j}u\; b_{ik}(v)\partial_{y^i}v. \end{aligned}$$

Note that the two reminder terms derive from potentials, namely, incorporating the multiplying factor appearing in (2.18), there holds

$$\begin{aligned} \rho''(u-v)\; a_{ik}(u)\partial_{x^i}u\, b_{jk}(v)\partial_{y^j}v &= \partial_{x^i}\partial_{y^j}Q^{1,2}_{ij}(u,v), \\ \text{and } \rho''(u-v)\, a_{jk}(u)\partial_{x^j}u\, b_{ik}(v)\partial_{y^i}v &= \partial_{y^i}\partial_{x^j}Q^{2,1}_{ij}(u,v), \end{aligned}$$

where

$$\begin{aligned} Q^{1,2}_{ij}(s,r) &= \int_r^s a_{ik}(\sigma)\int_\sigma^r b_{jk}(\tau)\rho''(\tau-\sigma)\,d\tau\,d\sigma && (2.19) \\ \text{and } Q^{2,1}_{ij}(s,r) &= \int_r^s b_{ik}(\sigma)\int_\sigma^r a_{jk}(\tau)\rho''(\tau-\sigma)\,d\tau\,d\sigma. \end{aligned}$$

We have obtained the equivalent formulation of (2.18) :

$$\begin{aligned}\partial_t\rho(u-v) &= \partial_{x^i}\left(\partial_{x^j}Q^1_{ij}(u,v)-\partial_{y^j}Q^{1,2}_{ij}(u,v)\right)\\ &\quad+\partial_{y^i}\left(\partial_{y^j}Q^2_{ij}(v,u)-\partial_{x^j}Q^{2,1}_{ij}(v,u)\right)\\ &\quad-\rho''(u-v)\left|(a(u)\partial_x u-b(v)\partial_y v)\right|^2.\end{aligned}$$

Notice that we have isolated a negative square on the right-hand side. In particular, we have the inequality

$$\begin{aligned}\partial_t\rho(u-v) &\leq \partial_{x^i}\left(\partial_{x^j}Q^1_{ij}(u,v)-\partial_{y^j}Q^{1,2}_{ij}(u,v)\right)\\ &\quad+\partial_{y^i}\left(\partial_{y^j}Q^2_{ij}(v,u)-\partial_{x^j}Q^{2,1}_{ij}(v,u)\right)\end{aligned} \tag{2.20}$$

We now proceed to unify (or de-double) the variables. Let η be the standard mollifier, and write $\eta_\delta(x)=\frac{1}{\delta^d}\eta\left(\frac{x}{\delta}\right)$. Let

$$\theta_m(x)=\min\left(1,\frac{d(x,\partial\Omega)}{m}\right)$$

and notice that $\theta_m\to 1$ as $m\to 0$. We write

$$\psi_\delta(x,y)=\eta_\delta(x-y)\theta_m(x)\theta_m(y).$$

Note that

$$\begin{aligned}&\int_{\Omega\times\Omega}\partial_{x^i}\left(\partial_{x^j}Q^1_{ij}(u,v)-\partial_{y^j}Q^{1,2}_{ij}(u,v)\right)\psi_\delta(x,y)\,dx\,dy\\
=&-\int_{\Omega\times\Omega}\left(\partial_{x^j}Q^1_{ij}(u,v)-\partial_{y^j}Q^{1,2}_{ij}(u,v)\right)\partial_i\eta_\delta(x-y)\,\theta_m(x)\,\theta_m(y)\,dx\,dy\\
&-\int_{\Omega\times\Omega}\left(\partial_{x^j}Q^1_{ij}(u,v)-\partial_{y^j}Q^{1,2}_{ij}(u,v)\right)\eta_\delta(x-y)\,\partial_i\theta_m(x)\,\theta_m(y)\,dx\,dy,\\
=&-\int_{\Omega\times\Omega}\partial_{x^j}\left(Q^1_{ij}(u,v)+Q^{1,2}_{ij}(u,v)\right)\partial_i\eta_\delta(x-y)\,\theta_m(x)\,\theta_m(y)\,dx\,dy\\
&-\int_{\Omega\times\Omega}\left(\partial_{x^j}Q^1_{ij}(u,v)-\partial_{y^j}Q^{1,2}_{ij}(u,v)\right)\eta_\delta(x-y)\,\partial_i\theta_m(x)\,\theta_m(y)\,dx\,dy\\
&-\int_{\Omega\times\Omega}Q^{1,2}_{ij}(u,v)\,\partial_i\eta_\delta(x-y)\left(\theta_m(x)\,\partial_j\theta_m(y)+\partial_j\theta_m(x)\,\theta_m(y)\right)dx\,dy,\\
=&-\int_{\Omega\times\Omega}\partial_{x^j}\left(Q^1_{ij}(u,v)+Q^{1,2}_{ij}(u,v)\right)\partial_i\eta_\delta(x-y)\,\theta_m(x)\,\theta_m(y)\,dx\,dy\\
&-\int_{\Omega\times\Omega}E^1_m\,dx\,dy,\end{aligned}$$

where E^1_m contains all remainder terms namely

$$
\begin{aligned}
E^1_m &= \left(\partial_{x^j} Q^1_{ij}(u,v) - \partial_{y^j} Q^{1,2}_{ij}(u,v)\right) \eta_\delta(x-y)\, \partial_i \theta_m(x)\, \theta_m(y) \\
& \quad Q^{1,2}_{ij}(u,v)\, \partial_i \eta_\delta(x-y) \left(\theta_m(x)\, \partial_j \theta_m(y) + \partial_j \theta_m(x)\, \partial \theta_m(y)\right)
\end{aligned}
$$

Note that

$$E^1_m \to 0 \text{ a.e. in } \Omega,$$

and from (2.15) and (2.19) we have

$$
\begin{aligned}
\int_{\Omega\times\Omega} \left|E^1_m\right| dx\, dy &\leq C(\lambda,,\delta,\epsilon) \int_{\Omega\times\Omega} \left|\partial_{x^j} Q^1_{ij}(u,v)\, \partial_j \theta_m(y)\right| dx\, dy, \\
&\leq C(\lambda,\delta,\epsilon) \left|u\right|_{BV(\Omega)} \left(\int_\Omega \left|\nabla \theta_m\right| dy\right) \\
&\leq C(\lambda,\delta,\epsilon) \left|u\right|_{BV(\Omega)} \frac{1}{m} \left|\{x : d(x,\partial\Omega)\} < m\right| \\
&\leq \tilde{C}
\end{aligned}
$$

where the constant $\tilde{C}$ does not depend on m. Thus by the Dominated Convergence Theorem,

$$\lim_{m\to 0} \|E^1_m\|_{L^1(\Omega\times\Omega)} \to 0.$$

Similarly,

$$
\begin{aligned}
& \int_{\Omega\times\Omega} \partial_{y^i} \left(\partial_{y^j} Q^2_{ij}(v,u) - \partial_{x^j} Q^{2,1}_{ij}(v,u)\right) \psi_\delta(x,y)\, dx\, dy \\
= & \int_{\Omega\times\Omega} \partial_{y^j} \left(Q^2_{ij}(v,u) + Q^{2,1}_{ij}(v,u)\right) \partial_i \eta_\delta(x-y) \theta_m(x) \theta_m(y)\, dx\, dy \\
& + \int_{\Omega\times\Omega} E^2_m\, dx\, dy,
\end{aligned}
$$

and

$$\lim_{m\to 0} \|E^2_m\|_{L^1(\Omega\times\Omega)} = 0.$$

Two more integration by parts show that

$$
\begin{aligned}
& \int_{\Omega\times\Omega} \partial_{y^j} \left(Q^2_{ij}(v,u) + Q^{2,1}_{ij}(v,u)\right) \partial_i \eta_\delta(x-y) \theta_m(x) \theta_m(y)\, dx\, dy \\
= & -\int_{\Omega\times\Omega} \partial_{x^j} \left(Q^2_{ij}(v,u) + Q^{2,1}_{ij}(v,u)\right) \partial_i \eta_\delta(x-y) \theta_m(x) \theta_m(y)\, dx\, dy \\
& + \int_{\Omega\times\Omega} E^3_m\, dx\, dy,
\end{aligned}
$$

with

$$E^3_m = \left(Q^2_{ij}(v,u) + Q^{2,1}_{ij}(v,u)\right) \partial_i \eta_\delta(x-y) \theta_m(x) \left(\partial_j \theta_m(y) \theta_m(y) - \partial_j \theta_m(x) \theta_m(y)\right),$$

and again

$$\lim_{m\to 0}\|E_m^2\|_{L^1(\Omega\times\Omega)}=0.$$

Taking the limit in (2.20) tested against ψ_δ as $m\to 0$, and noticing that $\theta_m\to 1$, we have shown

$$\begin{aligned}&\iint \partial_t\rho(u-v)\eta_\delta(x-y)\,dx\,dy\\&\le -\iint \partial_{x^j}\left(Q^1_{ij}(u,v)+Q^{1,2}_{ij}(u,v)+Q^2_{ij}(v,u)+Q^{2,1}_{ij}(v,u)\right)\partial_i\eta_\delta(x-y)\,dx\,dy.\end{aligned}\tag{2.21}$$

By definition,

$$\begin{aligned}&\partial_{x^j}\left(Q^1_{ij}(u,v)+Q^2_{ij}(v,u)+Q^{1,2}_{ij}(u,v)+Q^{2,1}_{ij}(v,u)\right)\\&=\left(\rho'(u-v)A_{ij}(u)-\int_u^v B_{ij}(\sigma)\rho''(\sigma-u)\,d\sigma\right)\partial_{x^j}u\\&\quad+\left(a_{ik}(u)\int_u^v b_{jk}(\sigma)\rho''(\sigma-u)\,d\sigma+\int_u^v b_{ik}(\sigma)a_{jk}(u)\rho''(u-\sigma)\,d\sigma\right)\partial_{x^j}u\end{aligned}$$

Passing to the limits as $\epsilon\to 0$ we find

$$\rho'(u-v)A_{ij}(u)\to \operatorname{sign}(u-v)A_{ij}(u)\ \text{a.e. in}\ \Omega\times\Omega\tag{2.22}$$

$$\rho'(v-u)B_{ij}(u)\to \operatorname{sign}(v-u)B_{ij}(u)\ \text{a.e. in}\ \Omega\times\Omega\tag{2.23}$$

$$a_{ik}(u)\int_u^v b_{jk}(\sigma)\rho''(\sigma-u)\,d\sigma\to a_{ik}(u)b_{jk}(u)\operatorname{sign}(v-u)\ \text{a.e. in}\ \Omega\times\Omega\tag{2.24}$$

$$\int_u^v b_{ik}(\sigma)a_{jk}(u)\rho''(u-\sigma)\,d\sigma\to b_{ik}(u)a_{jk}(u)\operatorname{sign}(v-u)\ \text{a.e. in}\ \Omega\times\Omega\tag{2.25}$$

Let us verify the convergence in (2.23).

$$\begin{aligned}&\int_u^v B_{ij}(\sigma)\rho''(\sigma-u)\,d\sigma=\int_0^{v-u}B_{ij}(z+u)\rho''(z)\,dz\\&=\int_0^{v-u}(B_{ij}(z+u)-B_{ij}(u))\rho''(z)\,dz+\int_0^{v-u}B_{ij}(u)\rho''(z)\,dz\\&=\int_0^{v-u}(B_{ij}(z+u)-B_{ij}(u))\frac{\varepsilon}{(\varepsilon+z^2)^{3/2}}\,dz+\int_0^{v-u}B_{ij}(u)\rho''(z)\,dz\\&=\int_0^{(v-u)/\sqrt{\varepsilon}}(B_{ij}(\sqrt{\varepsilon}w+u)-B_{ij}(u))\frac{1}{(1+w^2)^{3/2}}\frac{\varepsilon\sqrt{\varepsilon}}{\varepsilon^{3/2}}\,dw+B_{ij}(u)\int_0^{v-u}\rho''(z)\,dz,\end{aligned}$$

so the first term in the last line converges to 0 provided that B is continuous. For the second term we have

$$B_{ij}(u)\int_0^{v-u}\frac{\varepsilon}{(\varepsilon+z^2)^{3/2}}\,dz=B_{ij}(u)\frac{v-u}{\sqrt{\varepsilon+(u-v)^2}}\to B_{ij}(u)\operatorname{sign}(v-u)\qquad\text{as }\varepsilon\to 0.$$

Relations (2.24) and (2.25) are obtained in a very similar way. Overall we have obtained:

$$\begin{aligned}&\partial_{x^i}\left(Q^1_{ij}(u,v)+Q^2_{ij}(v,u)+Q^{1,2}_{ij}(u,v)+Q^{2,1}_{ij}(v,u)\right)\\ \to\quad &\operatorname{sign}(u-v)(a_{ik}(u)-b_{ik}(u))(a_{jk}(u)-b_{jk}(u))\partial_{x^i}(u)\quad \text{a.e. in } \Omega, \text{ as } \varepsilon\to 0.\end{aligned}\tag{2.26}$$

It follows that

$$\lim_{\epsilon\to 0}\left|\int_{\Omega\times\Omega}\partial_{x^i}\left(Q^1_{ij}(u,v)+Q^2_{ij}(v,u)+Q^{1,2}_{ij}(u,v)+Q^{2,1}_{ij}(v,u)\right)\partial_j\eta_\delta\,dx\,dy\right|=L,$$

$$L=\left|\int_{\Omega\times\Omega}\operatorname{sign}(u-v)(a_{ik}(u)-b_{ik}(u))(a_{jk}(u)-b_{jk}(u))(\partial_{x^i}u)\partial_j\eta_\delta\,dx\,dy\right|.$$

We can estimate the limit L by

$$L\le\frac{1}{\delta}\,|u|_{BV(\Omega)}\,\|a-b\|^2_{L^\infty(I_0)}\,C,\tag{2.27}$$

where $C=\int_{\mathbb{R}^d\times\mathbb{R}^d}|\nabla\eta|\,(x-y)\,dx\,dy$. Integrating in time, for $\varepsilon\to 0$, (2.21) becomes

$$\left[\iint|u(x)-v(y)|\eta_\delta(x-y)\,dx\,dy\right]_0^t\le\frac{C}{\delta}\,\|a-b\|^2_{L^\infty(I_0)}\,t\,|u|_{BV(\Omega)}\tag{2.28}$$

Finally, we wish to replace the left-hand side by an L^1 norm. Adding and subtracting $u(x)$ in $|u(y)-v(y)|$ and using the fact that $\iint g_\delta(x-y)\,dx\,dy=1$, we obtain

$$\begin{aligned}\|u(t)-v(t)\|_{L^1(\Omega)}&=\int_{\Omega\times\Omega}|u(y)-v(y)|g_\delta(x-y)\,dx\,dy\\&\le\int_{\Omega\times\Omega}(|u(y)-u(x)|+|u(x)-v(y)|)g_\delta(x-y)\,dx\,dy\\&\le 2\delta\,|u|_{BV(\Omega)}+\int_{\Omega\times\Omega}|u(x)-v(y)|g_\delta(x-y)\,dx\,dy,\end{aligned}\tag{2.29}$$

Notice that the factor 2δ appears because $x-y\in\operatorname{supp}g_\delta\subseteq B_\delta(0)$. Arguing in the opposite direction,

$$\int_{\Omega\times\Omega}|u_0(x)-v_0(y)|g_\delta(x-y)\,dx\,dy\le\|u_0-v_0\|_{L^1(\Omega)}+2\delta\,|u_0|_{BV(\Omega)}.$$

Altogether, (2.28) becomes

$$\|u(t)-v(t)\|\le 2\delta\,|u|_{BV(\Omega)}+\frac{C}{\delta}\,\|a-b\|^2_{L^\infty(I_0)}\,t\,|u|_{BV(\Omega)}+\|u_0-v_0\|_{L^1(\Omega)}+2\delta\,|u_0|_{BV(\Omega)}.$$

Choosing $\delta=\|a-b\|_{L^\infty(I_0)}$ yields

$$\|u(t)-v(t)\|_{L^1(\Omega)}\le 5C\,\|a-b\|_{L^\infty(I_0)}\,\sqrt{t}\left(|u|_{BV(\Omega)}+|u_0|_{BV(\Omega)}\right),$$

which is our thesis. □

2.2.3 Existence via fixed point approach

In this section we will present two proofs of existence of solutions, one for filtration equations and one for anisotropic diffusion. The structure of the proofs is similar but some of the details differ since the identification of the limit for filtration equations is simpler.

The main idea consists in constructing an ordered sequence of solutions of the same problem with different initial data converging monotonically to the desired limit. Such idea is introduced, for example, in [Váz07], who uses it in the filtration case (with more restrictive assumptions). A similar fixed point technique was used in [BDFDS06] for an equation with nonlinear diffusion coefficient given by $u(1-u)$.

Before getting started, let us recall two classical results that we will use in this section.

Lemma 2.14 (Aubin–Lions–Simon, [BF12], Th. II.5.16)**.** *Let $X_1 \hookrightarrow X_2 \hookrightarrow X_3$ be Banach spaces, where the first embedding is compact and the second one is continuous. For any $T > 0$, the set $W = \{u \in L^2((0,T), X_1) \,|\, \partial_t u \in L^2((0,T), X_3)\}$ is compactly embedded in $L^2([0,T], X_2)$.*

Theorem 2.15 (Schauder's Fixed Point, [GT98], Th. 11.1)**.** *Let S be a Banach space and let $X \subset S$ be non-empty, convex and bounded. If $\mathcal{F}$ is a continuous, compact map from X into itself, then it has a fixed point in X.*

The main existence result of this section is the following:

Theorem 2.16 (Existence via fixed point)**.**

- *Given a filtration function ϕ defined according to definition (2.1) and an admissible initial datum defined according to definition (2.2), suppose that $\phi'(0) > 0$. Then there exists a L^2-weak solution to problem (2.1) in the sense of Definition 2.3.*
- *Given a non-negative diffusion matrix $A \in C\left([0,M]; \mathbb{R}^{d\times d}\right)$ and a non-negative initial datum u_0, then there exists a (unique) function u such that*

$$\int_\Omega u(t,\cdot)\partial_t\psi(t,\cdot)\,dx = \int_{Q_T} A(u)\nabla u \nabla\psi\,dx\,dt$$

for all $\psi \in C^1([0,T]; L^2(\Omega)) \cap L^2([0,T]; H^1(\Omega))$.

The proof will follow from Propositions 2.21 and 2.22. We are going to define a suitable solution map F form a space $X \subseteq L^2([0,T], L^2(\Omega))$ into itself. We will show that F is continuous on X in the sense that if $w_n \to w$ in X, then $F(w_n) \to F(w)$ in X (with respect to the strong topology induced by $L^2([0,T], L^2(\Omega))$.

If we prove the existence of a fixed point $\bar{u}$, i.e. $\bar{u} = F(\bar{u})$, a compactness argument and continuity of F will provide that

$$\tilde{u} = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} F(u_n) = F(\tilde{u}).$$

Our approach is to linearise the diffusion term, namely to consider the sequence of heat-type equations.

Lemma 2.17. *Suppose $u_0 \in L^2(\Omega) \cap L^\infty(\Omega)$ satisfies $m \le u_0 \le M$ a.e. in Ω, for two constants $m, M \in \mathbb{R}$. Suppose that $A \in C\left([m,M];\mathbb{R}^{d\times d}\right)$ and that there exists $\lambda > 0$ such that for all $s \in [m,M]$ and $\xi \in \mathbb{R}^d$*

$$\sup_{[m,M]} |A(s)| \le \lambda^{-1} \text{ and } \lambda\xi\cdot\xi \le A(s)\xi\cdot\xi$$

Let $X_{m,M} := \left\{u \in L^2\left(Q_T\right) : m \le u \le M \text{ a.e. in } Q_T\right\}$. The map

$$\begin{aligned} F : X_{m,M} &\to X_{m,M} \\ u &\mapsto v, \end{aligned} \tag{2.30}$$

the unique solution in $L^2((0,T);H^1(\Omega)) \cap C([0,T);L^2(\Omega))$ of

$$\begin{aligned} \partial_t v - \operatorname{div}\left(A\left(u\right)\nabla v\right) = 0 \quad & \text{in } (0,\infty)\times\Omega, \\ A\left(u\right)\nabla v\cdot n = 0 \quad & \text{on } (0,\infty)\times\partial\Omega, \\ v\left(t=0\right) = u_0 \quad & \text{on } \Omega. \end{aligned}$$

is continuous and compact with respect to the $L^2(Q_T)$ induced topology.

Proof of Lemma 2.17. As $u \in X_{m,M}$ is admissible, it is bounded, therefore

$$\lambda\xi\cdot\xi \le A(u)\xi\cdot\xi \text{ and } |A(u)| \le \lambda^{-1}\xi\cdot\xi \text{ for a.e. } x\in\Omega.$$

Therefore Problem 2.30 is uniformly parabolic and non degenerate, and classical theory (Proposition 2.2) shows that v is indeed as described. The parabolic maximum principle applies providing the announced bounds upper and lower bounds for v, thus $F\left(X_{m,M}\right) \subset X_{m,M}$. Using v as a test function, and integrating we find for any $T > 0$,

$$\frac{1}{2}\int_\Omega v^2(T,\cdot)\,dx + \int_{Q_T} A\left(u\right)\nabla v\cdot\nabla v\,dx\,dt = \frac{1}{2}\int_\Omega u_0^2(x)\,dx,$$

which shows that

$$\lambda\int_{Q_T} |\nabla v|^2\,dx\,dt \le \frac{1}{2}\int_\Omega u_0^2(x)\,dx. \tag{2.31}$$

Finally, testing (2.30) against $w \in L^2(0,T;H^1(\Omega))$, we find

$$\begin{aligned}\int_{Q_T} \partial_t v w &= \int_{Q_T} A(u) \nabla v \cdot \nabla w \\ &\leq \sqrt{\frac{1}{2}\int_\Omega u_0^2(x)\,dx}\sqrt{\int_{Q_T} A(u)\nabla w \cdot \nabla w \,dx\,dt} \\ &\leq \sqrt{\frac{1}{2\lambda}\int_\Omega u_0^2(x)\,dx},\end{aligned}$$

for any w such that $\|w\|_{L^2(0,T;H^1(\Omega))} \leq 1$. Thus

$$\|\partial_t u_n\|^2_{L^2(0,T;H^1(\Omega)^*)} \leq \frac{1}{2\lambda}\int_\Omega u_0^2\,dx.$$

This implies that

$$F(X_{m,M}) \subset \left\{ u \in L^2((0,T), H^1(\Omega)) \,\middle|\, \partial_t u \in L^2((0,T), \left(H^1(\Omega)\right)^*)\right\}$$

and is bounded within that set. Thanks to the Aubin–Lions Lemma, $F(X_{m,M})$ is therefore compactly embedded into $L^2(Q_T)$, and thus in $X_{m,M}$.

To conclude, let us show that F is indeed continuous. Given a sequence $(u_n)_{n\in\mathbb{N}} \in X_{m,M}$ assume that $u_n \to u$ in $L^2(Q_T)$ with $u \in X_{m,M}$. Writing $v_n = F(u_n)$, $v = F(u)$ and $w_n = v_n - v$, we have

$$\partial_t w_n = \operatorname{div}(A(v)\nabla w_n) + \operatorname{div}((A(w_n+v) - A(v))\nabla v_n). \tag{2.32}$$

We have, thanks to (2.31),

$$\|(A(w_n+v) - A(v))\nabla v_n\|_{L^\infty(0,T;L^2(\Omega))} \leq 2\lambda^{-2}\|u_0\|_{L^2(\Omega)}. \tag{2.33}$$

In particular, integrating by parts (2.32) against $z := v_1 - v_2$ we find for any $t \in (0,T)$,

$$\begin{aligned}&\frac{1}{2}\int_\Omega w_n^2(T,\cdot)\,dx + \int_{Q_T} A\left(\frac{1}{2}v_n + \frac{1}{2}v\right)\nabla w_n \cdot \nabla w_n\,dx\,dt \\ &= \int_{Q_T} (A(w_n+v) - A(v))\nabla v_n \cdot \nabla w_n\,dx,\end{aligned}$$

which in turn thanks to (2.33) and (2.31) implies

$$\|\partial_t w_n\|_{L^2(0,T;H^1(\Omega)')} + \|w_n\|_{L^\infty(0,T;H^1(\Omega))} \leq 4\lambda^{-2}\|u_0\|_{L^2(\Omega)}.$$

Applying Aubin-Lions' Lemma again, we deduce that we may extract a strongly converging subsequence $w_n \to w$ in $L^2(Q_T)$. As A is continuous, $A(w_n+v) \to A(v)$ in $L^2(Q_T)$, and $\int_{(0,T)\times\Omega} (A(w_n+v) - A(v))\nabla v_n \cdot \nabla\phi \to 0$ for any test function ϕ. Thus, w is a weak solution of (2.32) with a null source term, that is, $w \equiv 0$, and our claim is established. □

Corollary 2.18. *Suppose that u_0 is an admissible initial datum. Suppose additionally that $s_0 > u_0 > \alpha$ a.e. in Ω, where $\alpha > 0$ is a given constant.*

Let $X_\alpha := \left\{u \in L^2(Q_T) : \|u_0\|_{L^\infty(\Omega)} > u > \alpha \text{ a.e. in } Q_T\right\}$. The map

$$\begin{aligned} F : X_\alpha &\to X_\alpha \\ u &\mapsto v, \end{aligned} \tag{2.34}$$

the unique solution in $L^2((0,T); H^1(\Omega)) \cap C([0,T); L^2(\Omega))$ of

$$\begin{aligned} \partial_t v - \operatorname{div}\left(\phi'(u)\nabla v\right) = 0 \quad & \text{in } (0,\infty)\times\Omega, \\ \phi'(u)\nabla v \cdot n = 0 \quad & \text{on } (0,\infty)\times\partial\Omega, \\ v(t=0) = u_0 \quad & \text{on } \Omega. \end{aligned}$$

is continuous and compact with respect to the $L^2(Q_T)$ induced topology.

Proof of Corollary 2.18. As $u \in X_\alpha$ is admissible, it is bounded, therefore

$$\phi'(\|u_0\|_{L^\infty(\Omega)}) > \phi'(u) \geq \phi'(\alpha) > 0 \text{ for a.e. } x \in \Omega.$$

We conclude by Lemma (2.17) with $\lambda = \min(\phi'(\alpha), \phi'(\|u_0\|_{L^\infty(\Omega)})^{-1})$ and $A = \phi' I_d$. □

Corollary 2.19. *Suppose that u_0 is an admissible initial datum. Suppose additionally that $u_0 > \alpha$ a.e. in Ω, where $\alpha > 0$ is a given constant. Then there exists an L^2-weak solution for Problem (2.1).*

Proof of Corollary 2.19. We apply Schauder's Fixed Point Theorem to F given by (2.34) on X_α, as Corollary 2.18 shows that F satisfies all the requirements. □

Corollary 2.20 (Existence, non-degenerate case)**.** *Suppose $u_0 \in L^\infty(\Omega)$ satisfies $m \leq u_0 \leq M$ a.e. in Ω, for two constants $m, M \in \mathbb{R}$. Suppose that $A \in C\left([m,M]; \mathbb{R}^{d\times d}\right)$ and that there exists $\lambda > 0$ such that such that for all $x \in [m,M]$ and $\xi \in \mathbb{R}^d$*

$$\sup_{[m,M]} |A| \leq \lambda^{-1} \text{ and } \lambda\xi\cdot\xi \leq A(x)\xi\cdot\xi$$

Then problem (2.11) has an L^2-weak solution, $u \in C([0,T], H^1(\Omega))$, $\nabla u \in L^2(Q_T)$.

Proof. It's a direct consequence of Lemma 2.17. □

Proposition 2.21 (Existence, filtration equation)**.** *Suppose that u_0 is an admissible initial datum. Then there exists a L^2-weak solution for Problem (2.1), and it is unique.*

Proof of Proposition 2.21. Consider first the case when $\|u_0\|_{L^\infty(\Omega)} < s_0$, Let n_0 be so large that

$$\|u_0\|_{L^\infty(\Omega)} + \frac{1}{n_0} < s_0.$$

Then, for every $n > n_0$, consider the sequence of L^2-weak solutions of Problem (2.1) with initial data $u_{0,n} = u_0 + \frac{1}{n}$. Thanks to Corollary 2.19, there is a unique solution to this problem. Furthermore, the comparison principle given by Proposition 2.5 shows that $0 \leq u_n \leq u_m$ when $n > m$. Thus the sequence u_n converges pointwise to u. Note furthermore that ϕ is monotonously increasing on $(0, s_0)$ therefore we also have that

$$\phi(u_n) \leq \phi(u_m)$$

when $n > m$. And therefore $\phi(u_n)$ also converges to a limit, which is $\phi(u)$ as ϕ is continuous. If we can test equation (2.1) against $\phi(u_n)$, we obtain an energy identity of the form

$$\int_\Omega \Phi(u_n)\, dx + \int_0^T \int_\Omega |\nabla \phi(u_n)|^2\, dx\, dt = \int_\Omega \Phi(u_{0,n})\, dx,$$

where Φ is a primitive of ϕ. This implies in particular that $\phi(u_k)$ is uniformly bounded in $L^2([0,T], H^1(\Omega))$. Thus a subsequence converges weakly $L^2([0,T], H^1(\Omega))$, to $\phi(u)$, with the same bound. Passing to the limit in the weak formulation, we deduce that u is a L^2-weak solution of (2.1).

Suppose now that $s_0 < \infty$, and that $u_0 \in L^2(\Omega) \cap L^\infty(\Omega)$ satisfies $0 \leq u_0 \leq s_0$. Then there exists a L^2-weak solution for problem (2.1). We construct a sequence $u_{0,n} = \min(u_0, s_0 - \frac{1}{n+\frac{1}{s_0}})$, $n \geq 0$. This is a decreasing sequence and for each such problem, Theorem (2.21) applies, and delivers a unique solution u_n. Thanks to the comparison principle (2.5), we have $s_0 \geq u_{n+1} \geq u_n$ for all $n \geq 0$, and $\phi(s_0) \geq \phi(u_{n+1}) \geq \phi(u_n)$, and both sequences are bounded in $L^\infty(Q_T)$ and $L^2\left(0, T; H^1(\Omega)\right)$ respectively, independently of n. Therefore $u_n \to u$ in $L^2(Q_T) \cap L^\infty(Q_T)$, by continuity, $\phi(u_n) \to \phi(u)$ (weakly) in $H^1(\Omega)$ and passing to the limit in the weak formulation we conclude that u is an L^2-weak solution.

Uniqueness is given by the comparison principle, Proposition 2.5. □

Proposition 2.22 (Existence - degenerate, anisotropic case.)**.** *Suppose $u_0 \in L^\infty(\Omega)$ satisfies $0 \leq u_0 \leq M$ a.e. in Ω, for a constant $M \geq 0$. Suppose that $A \in C\left([0,M]; \mathbb{R}^{d\times d}\right)$ is such that $A(s)\xi \cdot \xi > 0$ for all $s \in (0, M]$ and let $a(\cdot)$ be the principal square root of A.*

Then there exists a unique L^2-weak solution $u_n \in C([0,T), L^2(\Omega))$ of (2.11) corresponding to the initial data $u_{0,n} = \max(u_0, \frac{1}{n})$.

Furthermore there exists $u \in L^\infty(Q_T; \mathbb{R})$ such that

$$\begin{aligned} u_n &\downarrow u \text{ in } L^2(Q_T), \\ a(u_n)\nabla u_n &\rightharpoonup a(u)\nabla u \text{ in } L^2(Q_T), \\ A(u_n)\nabla u_n &\rightharpoonup A(u)\nabla u \text{ in } L^1(Q_T), \end{aligned}$$

with $0 \leq u \leq M$*. Moreover,* u *is unique and satisfies the weak formulation:*

$$\int_\Omega u(t,\cdot)\partial_t\psi(t,\cdot)\,dx = \int_{Q_T} A(u)\nabla u\nabla\psi\,dx\,dt$$

for all $\psi \in C^1([0,T];L^2(\Omega)) \cap L^2([0,T];H^1(\Omega))$*, and we have the estimate:*

$$\frac{1}{2}\int_\Omega u(t,\cdot)^2\,dx + \int_{Q_T} A(u)\nabla u\nabla u\,dx\,dt \leq \frac{1}{2}\int_\Omega u_0^2\,dx.$$

Proof of Proposition 2.22. Uniqueness follows from Theorem 2.10.

Consider the sequence $u_0^n = \max(u_0, \frac{1}{n})$. As $u_0^{n,} \in [\frac{1}{n}, M]$, and $\min_{[\frac{1}{n},M]} A(x)\xi\cdot\xi > 0$ for all $\xi \in \mathbb{R}^d$, there exists a unique u^n solution of 2.11 for this initial data. Furthermore, Corollary 2.11 shows that

$$0 \leq u^n \leq u^p \text{ when } p < n.$$

Integrating by parts against u^n, we find

$$\frac{1}{2}\int_\Omega (u^n(t,\cdot))^2\,dx + \int_{Q_T} A(u^n)\nabla u^n\cdot\nabla u^n\,dx\,dt = \frac{1}{2}\int_\Omega (u_0^n)^2\,dx$$

thus for all n,

$$\|a(u^n)\nabla u^n\|_{L^2(Q_T)} \leq \frac{1}{2}\|u_0\|_{L^2(\Omega)}.$$

As a consequence, $\lim_{n\to\infty} u^n = u$ exists almost everywhere due to the combination of monotone convergence theorem and comparison principle, indeed $0 \leq u \leq u^n \leq M$ for any n. Since a is continuous, we have $a(u^n) \to a(u)$ in $L^2(Q_T)$ by Dominated Convergence Theorem. Further, up to a subsequence, $a(u^n)\nabla u^n \rightharpoonup \zeta \in L^2(Q_T)$. By sequential weak lower semi-continuity, it holds

$$\int_{Q_T} \zeta^2\,dx\,dt \leq \int_{Q_T} A(u^{n,m})\nabla u^{n,m}\cdot\nabla u^{n,m}\,dx\,dt.$$

Alternatively, testing 2.11 against $\psi \in C_c^\infty(Q_T)$ we obtain

$$\int_{Q_T} u^{n,m}\partial_t\psi\,dx\,dt = \int_{Q_T} a(u^{n,m})(a(u^{n,m})\nabla u^{n,m})\nabla\psi\,dx\,dt.$$

Passing to the limit in this identity, we obtain our result. In addition, since $a(u^n) \to a(u)$ in $L^2(Q_T)$ and we just saw that $a(u_n)\nabla u_n \rightharpoonup \zeta$, we conclude that $A(u_n)\nabla u_n \rightharpoonup \Xi$ in L^1.

We now proceed to identify the limits. Let Θ_{ij} be a (non negative) primitive satisfying $\frac{d}{ds}\Theta_{ij}(s) = A_{ij}(s)$, and observe that

$$A_{ij}(u_n)\partial_{x^j}u_n = \partial_{x^j}\Theta_{ij}(u_n).$$

Let $\eta \in C_c^\infty(Q_T)$ be a test function and consider

$$\begin{aligned}\int_{Q_T} A_{ij}(u_n)\partial_{x^j} u_n \eta \, dx\, dt &= \int_{Q_T} \partial_{x^j}\Theta_{ij}(u_n)\eta \, dx\, dt \\ &= -\int_{Q_T} \Theta_{ij}(u_n)\partial_{x^j}\eta \, dx\, dt.\end{aligned}$$

Notice that, up to a subsequence, $\Theta_{ij}(u_n) \to \Theta_{ij}(u)$ a.e. since $u_n \downarrow u$ in $L^2(Q_T)$ and because Θ_{ij} is continuous. Hence

$$\int_{Q_T} \Theta_{ij}(u_n)\partial_{x^j}\eta \, dx\, dt \to \int_{Q_T} \Theta_{ij}(u)\partial_{x^j}\eta \, dx\, dt,$$

and, integrating back,

$$\int_{Q_T} A_{ij}(u_n)\partial_{x^j} u_n \eta \, dx\, dt \to \int_{Q_T} A_{ij}(u)\partial_{x^j} u \, \eta \, dx\, dt,$$

which is the definition of weak convergence. By uniqueness of the limit we have $\Xi = A_{ij}(u)\partial_{x^j} u$. □

The following proof is now straightforward:

Proof of Corollary 2.11. Theorem 2.10 applied to $B = A$ shows uniqueness of L^1-weak solution, which implies uniqueness of L^2-weak solutions. Corollary 2.20 ensures that such solutions exist. Furthermore the proof relies on a fixed point argument for a linear system. The classical comparison principle applies for this system, therefore it also applies, passing to the limit, for the fixed point solution. □

2.3 The role of confinement and interaction potentials

This section is based on a paper co-authored by M. Bruna and J. A. Carrillo, see [ABC18b].

2.3.1 Equations of Fokker–Planck type: discussion

In the previous sections we have considered purely diffusive equations, as this is the main focus of this work. Nevertheless, additional terms occur very frequently in the models that are relevant for mathematical biology. For example, we might be interested in drift terms of the form $\operatorname{div}(Fu)$, where $u : \mathbb{R}^d \to \mathbb{R}$ is the unknown function and $F : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field representing a force (in the conservative case we have $F = \nabla V$, for a suitable potential V). The simplest example of drift-diffusion equation takes the form

$$\partial_t u - \operatorname{div}\left[\nabla u + Fu\right] = 0,$$

however nonlinear and nonlocal versions of the above equations have become more and more relevant recently. As an example, we consider the following equation

$$\partial_t u = \operatorname{div}\left[\nabla\phi(u) + u\nabla V + u\nabla(W * u)\right]. \tag{2.35}$$

Here ϕ determines the nonlinear diffusive behaviour, V can be interpreted as a confinement potential and W is a nonlocal interaction potential which may be attractive, repulsive or a combination of the two.

Equation (2.35) can be posed in full space $\mathbb{R}^d$ or on a bounded domain Ω. In the first case, suitable decay conditions at infinity can be imposed, whereas, in the case of a bounded domain, no-flux boundary conditions constitute the most common and natural choice. In either case, it is easy to see that, since the equation is in conservative (divergence) form, the initial mass is preserved over time. Furthermore, if the initial condition is non-negative, then solutions will remain non-negative for all positive times.

This facts are consistent with the intuition behind the model: we can interpret the solution u as a density of cells in space; in the absence of proliferation or death terms the total number of cells (mass) will remain constant.

There are at least two points of view in the study of well-posedness of equation (2.35), i.e. the L^2 approach and the entropy approach. In the first case a key role is played by estimates for a quadratic energy and relative generalisations. Such approach has been developed, for example in [BS09].

The second case makes use of the gradient flow structure of the equation in a suitable space of probability measures (hence the solutions are only measure a priori). In fact, it is well known that there exists a convex entropy functional that decreases along the solutions of equation (2.35). Precise well-posedness results can be obtained as a consequence of the general theory of gradient flows exposed in [AGS08]. Entropy methods have been employed in the study of nonlinear, nonlocal Fokker–Planck equations in [CMV06,CMV03] and [JKO98]. A comparison of entropy and energy methods can be found in [Ott01].

The two approaches we mentioned are not mutually exclusive but they rely on different assumptions that depend on the specific properties of V and W (boundedness, integrability, convexity etc.).

We would like to point out that there is a link between scalar equations with drift and cross diffusion systems. In particular, we can consider the following example:

$$\partial_t u^i = \partial_\alpha (A_{ij}(u) \partial_\alpha u^i)),$$

which is a special case of the general system introduced in 1.1.3 with $\mathcal{B} = 0$. Let us focus on the equation for the first component:

$$\partial_t u^1 = \operatorname{div}(A_{11}(u^1, u^2) \nabla u^i + A_{12}(u^1, u^2) \nabla u^2,$$

we can see that, if the second component u^2 was fixed, we would have an equation of drift-diffusion type provided that $A_{12}(u^1, u^2) = u^1 f(u^2)$.

In section 2.2.1 discussed a stability result (or continuous dependence) with respect to the nonlinearity for purely diffusive systems. Such result can be extended to include drift terms, as shown in [CK05]. It has to be noted that the above-mentioned result holds on full space, whereas we are dealing with bounded domains. As shown in section 2.2.1, all we have to do is to consider a modified test function that takes the distance from the boundary into account.

In the next section we are going to consider a different type of result, namely we would like to study the case in which the drift term becomes infinite outside a bounded region. This will provide a further connection between problems posed in bounded and unbounded domains and we would like to remark that a problem on full space with infinite confinement potential and a problem on a bounded domain cannot be compared with the stability estimates treated in the previous sections since the regularity assumptions are violated and, clearly, the domains of definition of the solutions are a priori different. Therefore the next section can be viewed as a necessary step towards more complicated stability results concerning different domains and unbounded potentials.

2.3.2 Analysis in the L^2 setting

As announced, we consider an equation of the form (2.3.1) in Ω, where $u(x,t) \geq 0$ satisfies no-flux boundary conditions and a suitable initial condition that we will specify later. The function $\phi(\cdot)$ represents nonlinear diffusion (used to model short-range interactions between particles), W is a symmetric interaction potential (used to model long-range interactions) and the external potential V_0 describes the interactions with the environment (for example, distribution of food). In Eq. (2.3.1), Ω is a bounded and connected domain

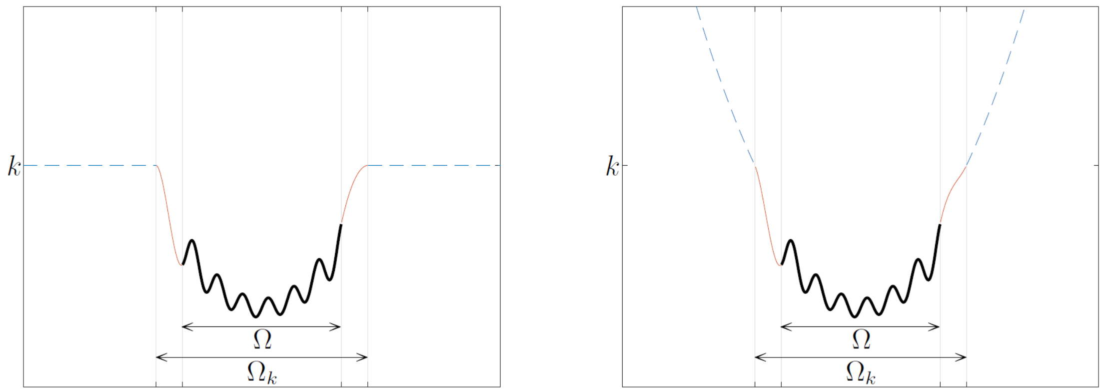

Figure 2.2: Sketch of the global potential $V_k(x)$ defined in (2.40).

of class C^2 in $\mathbb{R}^d$ (typically $d = 1, 2, 3$).

The goal of this section is to understand how the solutions of (2.3.1) in the bounded domain Ω relate to the solutions of the following equation in the whole space, as $k \to \infty$:

$$\partial_t u_k = \operatorname{div}\left[\nabla\phi(u_k) + u_k \nabla V_k + u_k \nabla(W * u_k)\right], \quad x \in \mathbb{R}^d, t > 0, \tag{2.36}$$

where the confinement potential is fixed in the bounded domain, i.e. $V_k(x) = V_0(x)$ for $x \in \Omega$ and it becomes stronger outside Ω as $k \to \infty$.

We want to understand the behaviour of the solution u_k to (2.36) when V_k becomes a strong confinement potential outside Ω (where the initial condition is supported). In particular, we consider a potential V_k of the form depicted in Figure 2.2 (see Definition 2.6). The parameter k determines the level of confinement and, as $k \to \infty$, we obtain infinite confinement. We will show that this is equivalent to a reformulation of the problem on a bounded domain with no-flux boundary conditions.

More precisely, consider the problem

$$\begin{aligned} \partial_t u &= \operatorname{div}\left[\nabla\phi(u) + u\nabla V + u\nabla(W * u)\right], \quad x \in \mathbb{R}^d, t > 0, \\ u(x, 0) &= u_0(x). \end{aligned} \tag{2.37}$$

We denote by Q_T the space $\mathbb{R}^d \times [0, T]$ and by Ω_T the space $\Omega \times [0, T]$.

Assumption 1 (L^2 setting)**.** The first set of assumptions that we consider is the following:

1. $0 \le u_0(x) \le M_0$, $\operatorname{supp}(u_0) \subseteq \Omega$.

2. $W \in W^{1,\infty}(\mathbb{R}^d)$ is symmetric, $\nabla W \in L^1(\mathbb{R}^d)$ and, without loss of generality, $W \ge 0$.

3. $V \in W^{1,\infty}(\mathbb{R}^d)$ and, without loss of generality, we also assume $V \ge 0$.

4. $\phi \in C^1(\mathbb{R})$ has the form $\phi(s) = s + \sigma(s)$, and $\phi(0) = 0$. We also suppose that there

exist constants $\mu > 0$ and $b \geq a \geq 1$ such that

$$\mu s^a \leq \sigma'(s) \leq \frac{1}{\mu} s^b. \tag{2.38}$$

Definition 2.5 (L^2 (weak) solution)**.** Suppose that Assumption 1 holds. We say that u is an L^2 weak solution of (2.1) if

$$u \in L^2(0,T;H^1(\mathbb{R}^d)) \cap C^0(0,T;L^2(\mathbb{R}^d)) \cap L^\infty(0,T,L^1(\mathbb{R}^d)),$$

$$\nabla\phi(u) \in L^2(Q_T), \quad \partial_t u \in L^2(0,T;(H^1(\mathbb{R}^d))'),$$

and, for all test functions $\eta \in H^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} u(t)\eta(t)\,dx\Big|_{t=0}^{t=T} - \int_0^T \int_{\mathbb{R}^d} \left[(\nabla\phi(u) + u\nabla V_0 + u\nabla W * u)\cdot\nabla\eta - u\partial_t\eta\right]\,dx\,dt = 0. \tag{2.39}$$

The initial datum is satisfied in the L^2 sense (i.e. $\lim_{t\to 0} \|u(t,\cdot) - u_0(\cdot)\|_{L^2(\mathbb{R}^d)} = 0$).

We now define the sequences of confinement potentials that we are will use in this section.

Definition 2.6 (Sequence of potentials, L^2 setting)**.** We define the following sequence of potentials $V_k \in W^{1,\infty}(\mathbb{R}^d)$:

$$V_k(x) = \begin{cases} V_0(x) & x \in \Omega, \\ \psi_k(x) & x \in \Omega_k \setminus \Omega, \\ k & x \in \mathbb{R}^d \setminus \Omega_k, \end{cases} \tag{2.40}$$

where Ω_k is an extended domain around Ω,

$$\Omega_k = \left\{ x + \frac{1}{k}e \,\middle|\, x \in \Omega, e \in S^d \right\},$$

so that $\Omega_k \searrow \Omega$ as $k \to \infty$, and ψ_k is a suitable C^1 extension of V.

Our main result in the L^2 setting concerns the convergence of the sequence of solutions u_k, which are defined in $\mathbb{R}^d$, to a limit function u that solves a problem in the bounded domain Ω.

Theorem 2.23 (L^2 setting)**.** *Suppose that one of the following conditions is satisfied:*

$$(1) \quad \phi(s) = s, \qquad\qquad (2) \quad W = 0.$$

Consider a solution u_k of problem (2.1) in the sense of Definition 2.5 and let $V = V_k$ satisfy the conditions in Definition 2.6. Then u_k converges for $k \to \infty$ to a function u

satisfying the following weak formulation in Ω:

$$\int_\Omega u(t)\eta(t)\,dx\Big|_{t=0}^{t=T} - \int_0^T\int_\Omega \left[(\nabla\phi(u) + u\nabla V_0 + u\nabla W * u)\cdot\nabla\eta - u\partial_t\eta\right]\,dx\,dt = 0. \quad (2.41)$$

for all $\eta \in H^1(\Omega)$.

Remark 2.9 (No-flux and confinement)**.** Notice that (2.41) is the weak formulation of a problem with *no-flux boundary conditions.* This justifies rigorously the correspondence between infinite confinement and no-flux conditions for the family of PDEs at hand.

Remark 2.10 (Convolution on a bounded domain)**.** Given $f : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}$, we use the following convention:

$$(W * f)(x) = \int_\Omega W(x-y)f(y)\,dy.$$

We are not going to address the issue of well-posedness of problem (2.1) in the L^2 setting, however we refer, for example, to the existence and uniqueness results in [BS09].

Lemma 2.24 (Non-negativity and conservation of mass)**.** *Any weak solution u of problem* (2.1) *in the sense of Definition 2.5 is non-negative and furthermore it preserves mass, that is,*

$$\int_{\mathbb{R}^d} u\,dx = \int_{\mathbb{R}^d} u_0\,dx.$$

Proof. To obtain conservation of mass, it is sufficient to test with the function $\eta = 1$ and integrate by parts to obtain $0 = \langle \partial_t u, 1\rangle = \partial_t \int_{\mathbb{R}^d} u\,dx$.

Let $M > 0$. In order to obtain positivity, we introduce the following truncated function

$$\phi_M'(s) = \min(M, \phi'(s))$$

and the corresponding C^1 primitive ϕ_M.

We now consider a solution u^M of problem (2.1) (in the sense of Definition 2.5) with ϕ replaced by ϕ_M and we test the equation against $\theta = (u^M)_-$ (which is non-negative and supported in the set $\{u^M \geq 0\}$). In particular we have

$$\partial_t \int_{\mathbb{R}^d} \frac{1}{2}\theta^2\,dx + \int_{\mathbb{R}^d}\left[\phi_M'(u^M)|\nabla\theta|^2 + \theta\nabla(V + W * u^M)\cdot\nabla\theta\right]\,dx = 0,$$

which implies, for $\lambda = \min_{s\geq 0}\phi_M'(s)$ and any $\varepsilon > 0$,

$$\int_{\mathbb{R}^d}\frac{1}{2}\theta^2\,dx + \int_{Q_T}\lambda|\nabla\theta|^2\,dx \leq \left\|\nabla(V + W * u^M)\right\|_{L^\infty(Q_T)}\int_{Q_T}\left[\frac{\varepsilon}{2}|\nabla\theta|^2 + \frac{1}{2\varepsilon}|\theta|^2\right]\,dx,$$

Notice that $\int_{\mathbb{R}^d} u^M = \int_{\mathbb{R}^d} u_0$, therefore $\left\|\nabla(V + W * u^M)\right\|_{L^\infty(Q_T)}$ is finite thanks to Assumption 1 and Young's inequality for convolutions. Hence we choose

$$\varepsilon = \lambda\left\|\nabla(V + W * u^M)\right\|_{L^\infty(Q_T)}^{-1}.$$

It follows that

$$\int_{\mathbb{R}^d} \theta^2 \, dx + \int_{Q_T} \lambda |\nabla \theta|^2 \, dx \leq \|\nabla (V + W * u)\|^2_{L^\infty(Q_T)} \frac{1}{\lambda} \int_{Q_T} |\theta|^2 \, dx.$$

Using Gronwall's inequality we obtain $\theta = (u^M)_- = 0$ a.e. $(x,t) \in Q_T$.

Finally, we claim that, up to a subsequence, u^M converges to u as $M \to \infty$ in the following sense

$$\begin{aligned} & u^M \to u \text{ in } L^2(Q_T), \\ & \nabla u^M \rightharpoonup \nabla u \text{ in } L^2(Q_T), \\ & \phi_M(u^M) \to \phi(u) \text{ a.e. in } Q_T, \\ & \nabla \phi_M(u^M) \rightharpoonup \nabla \phi(u) \text{ in } L^2(Q_T), \\ & \partial_t u^M \rightharpoonup \partial_t u \text{ in } L^2(0,T; H^1(\mathbb{R}^d)'). \end{aligned} \tag{2.42}$$

It is indeed the case since all the standard energy estimates for u^M are uniform with respect to M (recall that the diffusion is non-degenerate at this stage). So $u^M \in L^2(0,T;H^1(\Omega))$ and, by compactness, claim (2.42) holds. This ensures that the limit function u is a solution of (2.1) in the sense of Definition 2.5. □

Linear Fokker–Planck equation

We begin with the simplest case with non-interacting particles ($W = 0$, $\phi(s) = s$); in this setting it is possible to work in an L^2 setting.

Lemma 2.25 (Energy identity and boundedness)**.** *Consider the scalar equation*

$$\begin{aligned} \partial_t u &= \Delta u + \operatorname{div}(u \nabla V), \quad x \in \mathbb{R}^d, t > 0, \\ u(x,0) &= u_0(x). \end{aligned} \tag{2.43}$$

For every weak solution of problem (2.43) *(in the sense of Definition* 2.5*) the following identity holds:*

$$\int_{\mathbb{R}^d} u(T)^2 e^V \, dx + 2 \int_0^T \int_{\mathbb{R}^d} e^V |\nabla u + u \nabla V|^2 \, dx \, dt = \int_{\mathbb{R}^d} {u_0}^2 e^V \, dx, \tag{2.44}$$

for a.e. $T > 0$. In addition, u is bounded in $L^\infty(Q_T)$ in the following way:

$$u_0 \leq m e^{-V} \Rightarrow u \leq m e^{-V} \text{ a.e. } (t,x) \in Q_T, \tag{2.45}$$

for a fixed $m \geq 0$.

Remark 2.11. Lemma 2.25 is a particular case of Lemma 3.7, which we will state and prove in Chapter 3. We present it here in order to make this section self-contained.

Proof. We test the equation against ue^V. We obtain

$$\int_{\mathbb{R}^d} \partial_t u \left(ue^V\right) \, dx = \int_{\mathbb{R}^d} \left[\Delta u + \operatorname{div}(u\nabla V)\right] ue^V \, dx. \tag{2.46}$$

Integrating the left-hand side of (2.46) in time, we obtain

$$\int_0^T \int_{\mathbb{R}^d} \partial_t u \left(ue^V\right) \, dx \, dt = \int_0^T \int_{\mathbb{R}^d} \partial_t \left(\frac{1}{2}u^2 e^V\right) \, dx \, dt = \int_{\mathbb{R}^d} \frac{1}{2} u(T)^2 e^V \, dx - \int_{\mathbb{R}^d} \frac{1}{2} {u_0}^2 e^V \, dx.$$

Using integration by parts in the right-hand side of (2.46) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \left[\Delta u - \operatorname{div}(u\nabla V)\right] \cdot \left(ue^V\right) \, dx &= -\int_{\mathbb{R}^d} (\nabla u + u\nabla V) \cdot \nabla \left(ue^V\right) \, dx \\ &= -\int_{\mathbb{R}^d} e^V |\nabla u + u\nabla V|^2 \, dx \leq 0.\end{aligned}$$

Altogether we have obtained

$$\int_{\mathbb{R}^d} \frac{1}{2} e^V u(T)^2 \, dx = \int_{\mathbb{R}^d} \frac{1}{2} {u_0}^2 e^V \, dx - \int_0^T \int_{\mathbb{R}^d} e^V |\nabla u + u\nabla V|^2 \, dx \, dt,$$

as required. In order to prove boundedness, we consider the function $\tilde{u} = ue^V$. Let us rewrite equation (2.43) in terms of $\tilde{u}$:

$$e^{-V} \partial_t \tilde{u} = \operatorname{div}\left(e^{-V} \nabla \tilde{u}\right).$$

We integrate the equation above against the test function $(\tilde{u} - m)_+$:

$$\int_{\mathbb{R}^d} \frac{1}{2} e^{-V} (\tilde{u}(t) - m)_+^2 \, dx + \int_0^t \int_{\mathbb{R}^d} e^{-V} |\nabla(\tilde{u} - m)_+|^2 \, dx \, d\tau = \int_{\mathbb{R}^d} \frac{1}{2} e^{-V} (\tilde{u}(0) - m)_+^2 \, dx.$$

Notice that $\tilde{u}(0) = u_0 e^V$, thus the right-hand side in the equality above vanishes. This means that $\tilde{u}(t) \leq m$ for a.e. $(t, x) \in Q_T$ and the proof is complete. □

Corollary 2.26. *Consider the assumptions of Lemma 2.25 and recall that* $\operatorname{supp}(u_0) \subseteq \Omega$*. Let* $V = V_k$ *as in Definition 2.6 and consider* $u = u_k$ *as in Lemma 2.25. Then* $u_k(x, t) \to 0$ *in* $L^\infty([0, T] \times \Omega_k^c)$ *for* $k \to \infty$*.*

Proof. It is a direct consequence of (2.45) (recalling that $V_k \to \infty$ on Ω_k^c). □

Nonlocal Fokker–Planck equation

Here we consider the extend the linear case of the previous subsection to include a nonlinear interaction potential W.

Lemma 2.27 (L^2 energy estimate, case $\phi(s) = s$)**.** *Let* $u = u_k$ *be a weak solution of problem (2.1) with* $\phi(s) = s$ *and* $V = V_k$ *given by Definition 2.6. Let* $\bar{u}_0$ *be the (constant)* *mass of u. In addition, recall that* $\mathrm{supp}(u_0) \subset \Omega$. *Then we have*

$$\int_{\mathbb{R}^d} u(T)^2 e^V \, dx + \int_0^T \int_{\mathbb{R}^d} e^{-V} |\nabla(e^V u)|^2 \, dx \, dt \leq C(\nabla W, \bar{u}_0, T) \int_\Omega {u_0}^2 e^V \, dx. \tag{2.47}$$

Proof. We are going to use $u \exp(V)$ as our test function.

$$\int_{\mathbb{R}^d} u(T)^2 e^V \, dx + 2 \int_0^T \int_{\mathbb{R}^d} \left[e^V |\nabla u + u \nabla V|^2 + u(\nabla W * u) \cdot \nabla(e^V u) \right] \, dx \, dt = \int_\Omega {u_0}^2 e^V \, dx,$$

and

$$\int_0^T \int_{\mathbb{R}^d} u e^{V/2} (\nabla W * u) \cdot e^{-V/2} \nabla(e^V u) \, dx \, dt \leq \|u e^{V/2}\|_{L^2} \|e^{-V/2} \nabla(e^V u)\|_{L^2} \|\nabla W\|_{L^\infty} \|u\|_{L^1},$$

resulting into

$$\begin{aligned} \int_{\mathbb{R}^d} u(T)^2 e^V \, dx &+ \int_0^T \int_{\mathbb{R}^d} e^{-V} |\nabla(e^V u)|^2 \, dx \, dt \\ &\leq \int_\Omega {u_0}^2 e^V \, dx + \|\nabla W\|_{L^\infty}^2 \|u\|_{L^1}^2 \int_0^T \int_{\mathbb{R}^d} u^2 e^V \, dx \, dt. \end{aligned}$$

Using Gronwall's lemma we obtain

$$\int_{\mathbb{R}^d} u(T)^2 e^V \, dx + \int_0^T \int_{\mathbb{R}^d} e^{-V} |\nabla(e^V u)|^2 \, dx \, dt \leq \exp\left(T \|\nabla W\|_{L^\infty}^2 \|u\|_{L^1}^2\right) \int_\Omega {u_0}^2 e^V \, dx,$$

as announced. □

Remark 2.12. Unlike in the linear case, boundedness of (weak) solutions is not so straightforward. It can be shown arguing as in [LSU88], Chapter 3, Theorem 7.1 or in Stampacchia, [Sta65]. Since we do not need boundedness in our analysis, we do not discuss it further.

Nonlinear (local) Fokker–Planck equation

In the case of nonlinear diffusion, we generalise the familiar procedure often used to obtain energy estates and we introduce a new quantity, indicated by $P(u)$, which coincides with u in the linear case.

Lemma 2.28 (Energy inequality, $W = 0$)**.** *Let u be a weak solution the following equation*

$$\begin{aligned} \partial_t u &= \Delta \phi(u) + \mathrm{div}(u \nabla V), \quad x \in \mathbb{R}^d, t > 0, \\ u(x, 0) &= u_0(x), \end{aligned} \tag{2.48}$$

where $\phi(s) = s + \sigma(s)$ satisfies Assumptions 1. Then:

$$\int_{\mathbb{R}^d} e^V Q(u(T)) \, dx + \int_{\mathbb{R}^d} e^V \frac{P(u)}{u} |\nabla \phi(u) + u \nabla V(x)|^2 \, dx \leq \int_\Omega e^V Q(u_0) \, dx, \tag{2.49}$$

for a.e. $T > 0$, *with*

$$P(u) = \exp\left(\int_1^u \frac{\phi'(s)}{s}\,ds\right), \qquad \textit{and} \qquad Q(u) = \int_0^u P(s)\,ds.$$

Proof. First of all, we notice that

$$\begin{aligned}\nabla(e^V P(u)) &= e^V P(u) \nabla \left(V(x) + \int^u \frac{\phi'(s)}{s}\,ds\right) \\ &= e^V P(u)\left(\nabla V(x) + \frac{\phi'(u)}{u}\nabla u\right) \\ &= e^V \frac{P(u)}{u}\left(\nabla\phi(u) + u\nabla V(x)\right).\end{aligned}$$

We test equation (2.63) against $e^V P(u)$ and we obtain

$$\int_{\mathbb{R}^d} \partial_t u e^V P(u)\,dx = \int_{\mathbb{R}^d} \operatorname{div}\left[\nabla\phi(u) + u\nabla V\right] e^V P(u)\,dx. \tag{2.50}$$

Considering the left-hand side of (2.50), we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} \partial_t u \left[e^V P(u)\right]\,dx\,dt &= \int_0^T \int_{\mathbb{R}^d} \partial_t \left[Q(u)e^V\right]\,dx\,dt \\ &= \int_{\mathbb{R}^d} Q(u(T))e^V\,dx - \int_{\mathbb{R}^d} Q(u_0)e^V\,dx,\end{aligned}$$

where Q is a primitive of P.

Notice that, since $P(u) > 0$, $Q(u)$ is increasing and we can choose $Q(0) = 0$. Using integration by parts in the right-hand side of (2.50), we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \operatorname{div}\left[\nabla\phi(u) + u\nabla V\right] e^V P(u)\,dx &= -\int_{\mathbb{R}^d} \left[\nabla\phi(u) + u\nabla V\right]\cdot\nabla\left[e^V P(u)\right]\,dx \\ &= -\int_{\mathbb{R}^d} e^V \frac{P(u)}{u}\left|\nabla\phi(u) + u\nabla V(x)\right|^2\,dx,\end{aligned}$$

Altogether we have obtained

$$\int_{\mathbb{R}^d} e^V Q(u(T))\,dx + \int_{\mathbb{R}^d} e^V \frac{P(u)}{u}\left|\nabla\phi(u) + u\nabla V(x)\right|^2\,dx \le \int_\Omega e^V Q(u_0)\,dx. \tag{2.51}$$

□

Remark 2.13. Notice that, thanks to Assumption 1 (point 4), we have

$$\frac{\phi'(s)}{s} = \frac{1+\sigma'(s)}{s} \ge \frac{1}{s} + \mu s^{a-1},$$

for any $s \geq 0$ and some given $\mu > 0$, $a \geq 1$. From the definition of P, for $u \geq 0$, we obtain

$$P(u) = \exp\left(\int_1^u \frac{1+\sigma'(s)}{s}\,ds\right) \geq \exp\left(\log(u) + \mu \int_1^u s^{a-1}\,ds\right) = u\exp\left(\frac{\mu}{a}(u^a - 1)\right),$$

consequently, $P(u)$ is well defined and, using the lower bound just obtained, we observe that it satisfies

$$P(u) \geq c(\mu, a) u \exp(u^a), \tag{2.52}$$

for $c(\mu, a) = \exp(-\frac{\mu}{a})$. It follows that, for any $u \geq 0$,

$$\frac{P(u)}{u} \geq c(\mu, a) \quad \text{and} \quad \lim_{s \to 0^+} \frac{P(s)}{s} = c(\mu, a).$$

In addition, notice that since $P(u) \geq c(\mu, a)u$, we also have $Q(u) \geq \frac{1}{2}c(\mu, a)u^2$.

These facts will be useful in the proof of Theorem 2.23, providing useful lower bounds for the left-hand side of inequality (2.49).

We are now going to exploit the properties of the function Q in order to obtain a bound in L^p. First, we need the following simple result.

Lemma 2.29. *Let $p > 1$, $a > 1$. The following inequality holds for any $s > 0$:*

$$s^p \leq \theta_{p,a} s \exp(s^a), \qquad \textit{where} \qquad \theta_{p,a} = \left(\frac{p-1}{ea}\right)^{\frac{p-1}{a}}. \tag{2.53}$$

Proof. Maximizing the function $s^{p-1}\exp(-s)$ we obtain the optimal value for $\theta_{p,a}$, which is attained at $s = p - 1$ and hence $\theta_{p,a} = \left(\frac{p-1}{ea}\right)^{\frac{p-1}{a}}$. □

Corollary 2.30 (L^p estimate)**.** *Under the hypotheses of Lemma 2.28, for any $1 \leq p < \infty$ and $\theta_{p,a}$ as in Lemma 2.29, it holds*

$$\int_{\mathbb{R}^d} e^V u^p\,dx \leq \theta_{p,a} \int_\Omega e^V Q(u_0)\,dx. \tag{2.54}$$

Proof. The result follows combining the inequalities (2.49), (2.52) and (2.53). □

Remark 2.14. Notice that the term $\operatorname{div}(u\nabla V)$ can be generalised to the form $\operatorname{div}(\alpha(u)\nabla V)$, where α is such that $\alpha(s) = s(1 + \rho(s))$ and there exists a constant $\mu \in [0, 1)$ such that

$$\mu\sigma'(s) \leq \rho(s) \leq \sigma'(s). \tag{2.55}$$

In this case we can define P as follows:

$$P(u) = \exp\left(\int_1^u \frac{\phi'(s)}{\alpha(s)}\,ds\right),$$

and we can obtain a result analogous to Lemma 2.28.

Proof of Theorem 2.23

We are going to presents the two cases separately (i.e. $W = 0$ or $\phi(s) = s$).

Remark 2.15. Suppose that the function $ue^{V/2}$ belongs to $L^2(0,T;H^1(\Omega))$, then u belongs to the same space. Indeed $\nabla(ue^{V/2}) = e^{V/2}(\nabla u + u\nabla V/2)$ and we know that $u \in L^\infty(0,T;L^1(\Omega))$ and that $V \in W^{1,\infty}(\Omega)$.

Proof of Theorem 2.23, case $\phi(s) = s$. The weak formulation introduced in Definition 2.5 gives us

$$\int_{\mathbb{R}^d} u_k(t)\eta(t)\,dx\Big|_{t=0}^{t=T} - \int_0^T \int_{\mathbb{R}^d} \left[(\nabla u_k + u_k \nabla(V_k + W * u_k)) \cdot \nabla\eta - u_k \partial_t \eta\right]\,dx\,dt = 0. \tag{2.56}$$

for all test functions $\eta \in H^1(\mathbb{R}^d)$.

We notice that, from the energy inequality (2.47), the function $u_k e^{\frac{V_k}{2}}$ is bounded in $L^2(0,T;H^1(\mathbb{R}^d)) \cap L^\infty(0,T;L^2(\mathbb{R}^d))$. Since the term $e^{\frac{V_k}{2}}$ is bounded from below, we also have that u_k belongs to the same space (see Remark 2.15).

We divide $\mathbb{R}^d$ into three parts, namely Ω, Ω_k^c and $\Omega_k \setminus \Omega$. Consequently, we split (2.56) as $I_\Omega + I_{\Omega_k^c} + I_{\Omega_k\setminus\Omega} + J_\Omega + J_{\Omega_k^c} + J_{\Omega_k\setminus\Omega}$, where

$$\begin{aligned} I_A &= \int_0^T \int_A \left[(\nabla\phi(u_k) + u_k\nabla V_k + u_k \nabla W * u_k) \cdot \nabla\eta\right]\,dx\,dt, \\ J_A &= \int_A [u_k(T)\eta(T) - u_0\eta(0)]\,dx - \int_0^T \int_A u_k \partial_t \eta\,dx\,dt \end{aligned}$$

We want to show that all terms but I_Ω and J_Ω vanish in the limit $k \to \infty$. Then $I_\Omega + J_\Omega$ will characterise the limit problem defined in Ω.

- I_Ω: Restricting our attention to I_Ω, from (2.47) we obtain

$$\int_\Omega u_k(T)^2 e^{V_k}\,dx + \int_0^T \int_\Omega e^{-V_k} |\nabla(e_k^V u_k)|^2\,dx\,dt \le C \int_\Omega {u_0}^2 e^V\,dx. \tag{2.57}$$

  This implies that

$$u_k \in L^\infty(0,T;L^2(\Omega)), \qquad \nabla(e^{V_0} u_k) \in L^2(0,T;L^2(\Omega)),$$

  uniformly in k (notice that V is bounded and sufficiently smooth). Hence we can extract a subsequence u_{k_n} that converges weakly in $L^2(0,T;H^1(\Omega))$ to a limit denoted by u. By compactness, u_{k_n} converges strongly in $L^2(\Omega_T)$ as well. To simplify the notation, in what follows we write u_k instead of u_{k_n}. In particular, thanks to the strong convergence of u_k in L^2, we have

$$u_k \nabla V_K \to u\nabla V \text{ in } L^2(\Omega_T)$$

and

$$u_k\nabla(W*u_k) \to u\nabla(W*u),$$

in fact, for $k\to\infty$,

$$\|\nabla W*(u_k-u)\|_{L^2(Q_T)} \le \|\nabla W\|_{L^1(\Omega)}\|(u_k-u)\|_{L^2(Q_T)} \to 0$$

Thus we have obtained

$$I_\Omega \to \int_\Omega [(\nabla u + u\nabla V + u\nabla W*u)\cdot\nabla\eta]\,dx,$$

as $k\to\infty$.

- $I_{\Omega_k^c}$: Considering $I_{\Omega_k^c}$ and using again (2.47), since $V_k\ge k$ on Ω_k^c, we have

$$\int_{\Omega_k^c} u_k(T)^2 e^k\,dx + \int_0^T\int_{\Omega_k^c} e^k|\nabla u_k|^2\,dx\,dt \le C\int_\Omega {u_0}^2 e^{V_0}\,dx,$$

and therefore, for $k\to\infty$, we obtain $I_{\Omega_k^c}\to 0$ because we have that

$$\|u_k\|_{L^2(0,T;H^1(\Omega_k^c))\cap L^\infty(0,T;L^2(\Omega_k^c))} \to 0. \tag{2.58}$$

- $I_{\Omega_k\setminus\Omega}$: It remains to be checked that $I_{\Omega_k\setminus\Omega}$ also vanishes. Once more, from the energy identity (2.47), we obtain

$$\int_0^T\int_{\Omega_k\setminus\Omega} e^{\psi_k}|\nabla u_k + u_k\nabla(\psi_k+W*u_k)|^2\,dx\,dt \le C\int_\Omega {u_0}^2e^{V_0}\,dx.$$

Since $\exp(V_k)\ge 1$ and $|\Omega_k\setminus\Omega|\to 0$, we deduce that

$$\begin{aligned}
&\int_0^T\int_{\Omega_k\setminus\Omega}(\nabla u_k + u_k\nabla(\psi_k+W*u_k))\cdot\nabla\eta\,dx\,dt\\
&\le\left(\int_0^T\int_{\Omega_k\setminus\Omega}|\nabla u_k+u_k\nabla(\psi_k+W*u_k)|^2\,dx\,dt\right)^{1/2}\left(\int_0^T\int_{\Omega_k\setminus\Omega}|\nabla\eta|^2\,dx\,dt\right)^{1/2}\\
&\le\left(C\int_\Omega {u_0}^2e^{V_0}\,dx\right)^{1/2}\left(\int_0^T\int_{\Omega_k\setminus\Omega}|\nabla\eta|^2\,dx\,dt\right)^{1/2}\to 0.
\end{aligned}$$

- J_Ω: The sequence u_k converges weakly in in $H^1(\Omega)$ to a limit u, hence

$$J_\Omega \to \int_\Omega [u(T)\eta(T)-u_0\eta(0)]\,dx - \int_0^T\int_\Omega u\partial_t\eta\,dx\,dt \quad \text{as } k\to\infty, \tag{2.59}$$

- $J_{\Omega_k^c}$: Notice that $u_0=0$ in Ω_k^c. The remaining terms in $J_{\Omega_k^c}$ vanish thanks to (2.58).

- $J_{\Omega_k\setminus\Omega}$: The integral $J_{\Omega_k\setminus\Omega}$ goes to zero because the integrand is uniformly bounded in L^1 (thanks to the conservation of mass) and $|\Omega_k \setminus \Omega| \to 0$.

The weak formulation we obtain in the limit is the following:

$$\int_\Omega [u(T)\eta(T) - u(0)\eta(0)]\,dx - \int_0^T \int_\Omega [(\nabla\phi(u) + u\nabla V_0 + u\nabla W * u)\cdot\nabla\eta - u\partial_t\eta]\;dx\,dt = 0.$$

Notice that the initial datum is still satisfied in the L^2 sense and that since the test function can be any element of $H^1(\Omega)$ this implies that no-flux conditions on $\partial\Omega$ are implicitly enforced. It is easy to see that the initial datum is satisfied in the L^2 sense. □

Sketch of the proof of Theorem 2.23, case $W = 0$. We now consider the case $W = 0$ (and ϕ generic). We can repeat all the steps above using the energy inequality (2.49) instead of (2.47). This is natural once we have observed that the quotient $\frac{P(u)}{u} \geq c(\mu, a)$ for any $u \geq 0$, see Remark 2.13. In particular, from (2.49), Remark 2.13 and recalling that $Q(0) = 0$, it follows that

$$\int_{\mathbb{R}^d} e^{V_k}\frac{1}{2}u_k(T)^2\,dx + \int_{\mathbb{R}^d} e^{V_k}|\nabla\phi(u_k) + u_k\nabla V_k(x)|^2\,dx \leq \int_\Omega e^{V_0}Q(u_0)\,dx. \tag{2.60}$$

In the case $\phi(s) = s$, thanks to inequality (2.60) we can prove that, up to a subsequence, u_k converges to 0 outside of Ω and it converges strongly in $L^2(Q_T)$ (hence almost everywhere) and weakly in $L^2(0, T; H^1(\Omega))$ to a function u satisfying the limit weak formulation (2.41). □

Remark 2.16. We do not treat the most general case in the present section since it is not clear how to obtain a suitable energy estimate that is uniform with respect to k. However, the general equation can be studied using entropy techniques as shown in the next section.

2.3.3 Analysis in the entropy setting

Under a slightly different set of assumptions, it is possible to obtain a convergence result analogous to Theorem 2.23. The key ingredient this time is the entropy structure of equations (2.3.1) and (2.36).

Assumption 2 (Entropy setting)**.** Our second set of assumptions is the following:

1. $u_0(x) \geq 0$, $\int_{\mathbb{R}^d} u_0\,dx = 1$ and $\mathrm{supp}(u_0) \subseteq \Omega$.
2. $W \in W^{1,\infty}_{loc}(\mathbb{R}^d)$ is symmetric and, without loss of generality, $W \geq 0$.
3. $V \in W^{1,\infty}_{loc}(\mathbb{R}^d)$ and we assume that $V \geq c|x|^2$ for $|x| \to \infty$ and for some $c > 0$.
4. $\phi \in C^1(\mathbb{R}_+)$ has the form $\phi(s) = s + \sigma(s)$, $\phi(0) = 0$ and it is monotone. We suppose that there exist constants $\mu > 0$ and $b \geq a \geq 1$ such that

$$\mu s^a \leq \sigma'(s) \leq \frac{1}{\mu}s^b. \tag{2.61}$$

Notice that we have not assumed boundedness or decay of V and W at infinity and, for our purposes, convexity is not required. Nevertheless, in other to apply the existence theory (see Theorem 2.32), we will have to assume convexity (see [AGS08] and [CMV03]).

We denote the 2-Wasserstein space of probability measures by $\mathcal{P}_2(\mathbb{R}^d)$ and the relative distance by d_{W_2}.

Definition 2.7 (Entropy (weak) solution)**.** Suppose that Assumption 2 is satisfied. We say that $u \in L^\infty(0,T;L^1(\mathbb{R}^d))$ is an entropy solution of (2.1) if u is a distributional solution in $\mathcal{D}'(\mathbb{R}^d)$ and it satisfies

$$u(t,\cdot) \to u_0 \text{ in } \mathcal{P}_2(\mathbb{R}^d) \quad \text{as } t \to 0,$$

for all $T > 0$,

$$\int_{Q_T} u \left| \frac{1}{u} \nabla \phi(u) + \nabla V + \nabla W * u \right|^2 dx\, dt < \infty,$$

and, for $\Theta(u) = \int_0^u \int_1^s \frac{\phi'(r)}{r} \, dr\, ds$, it is a gradient flow in $\mathcal{P}_2(\mathbb{R}^d)$ for the entropy functional:

$$E[u(t)] = \int_{\mathbb{R}^d} \Theta(u) + uV + \frac{1}{2} u(W * u)\, dx.$$

We now consider the following sequence of potentials:

Definition 2.8 (Sequence of potentials, entropy setting)**.** We define the following sequence of potentials $V_k \in W^{1,\infty}_{loc}(\mathbb{R}^d)$:

$$V_k(x) = \begin{cases} V_0(x) & x \in \Omega, \\ \psi_k(x) & x \in \Omega_k \setminus \Omega, \\ \zeta_k(x) & x \in \mathbb{R}^d \setminus \Omega_k, \end{cases} \tag{2.62}$$

where Ω_k is an extended domain around Ω,

$$\Omega_k = \left\{ x + \frac{1}{k} e \,\middle|\, x \in \Omega, e \in S^d \right\},$$

so that $\Omega_k \searrow \Omega$ as $k \to \infty$. Here $\zeta_k(x) \geq k$ is such that $V_k(x) \geq c|x|^2$ for $|x|$ sufficiently large (in particular $\int_{\mathbb{R}^d} e^{-\zeta_k(x)} \mathrm{d}x < \infty$), and $\psi_k(x)$ is a C^1 interpolant between the values of V_0 on $\partial\Omega$ and ζ_k outside Ω_k.

Theorem 2.31 (Entropy setting)**.** *Consider a solution u_k of problem (2.1) in $\mathbb{R}^d$ in the sense of Definition 2.7 and let $V = V_k$ satisfy the conditions in Definition 2.8. Then u_k converges for $k \to \infty$ to a function u satisfying the following weak formulation in Ω:*

$$\int_0^T \int_\Omega \left[(\nabla \phi(u) + u \nabla V_0 + u \nabla W * u) \cdot \nabla \eta - u \partial_t \eta \right] dx\, dt = 0,$$

for any $\eta \in C^\infty(\Omega_T)$. The initial datum is satisfied in $\mathcal{P}_2(\Omega)$.

Let us consider the full problem with the nonlocal term

$$\begin{aligned} \partial_t u &= \operatorname{div}\left[\nabla\phi(u) + u\nabla V + u\nabla(W * u)\right], \quad x \in \mathbb{R}^d, t > 0, \\ u(x,0) &= u_0(x). \end{aligned} \tag{2.63}$$

We consider a solution u_k to (2.63) when $V = V_k$ given in Definition 2.8. We have to prove that u_k exists and is unique and that the sequence u_k converges to u solving problem (2.1) in Ω. The steps involved are:

1. finding bounds independent of k,
2. showing that $u_k \to 0$ outside Ω.
3. passing to the limit in the weak formulation.

The following well-posedness result has been proven in [AGS08], Theorem 11.2.8.

Theorem 2.32 (Existence and uniqueness of solutions)**.** *Suppose that V and W are strictly convex. For every $u_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exists a unique distributional solution $u \in \mathcal{P}_2(\mathbb{R}^d)$ of (2.1) in $\mathbb{R}^d$ among those satisfying*

$$u(t,\cdot) \to u_0 \text{ in } \mathcal{P}_2(\mathbb{R}^d) \text{ as } t \to 0,$$

$\phi(u) \in L^1_{loc}(0,\infty; W^{1,1}_{loc}(\mathbb{R}^d))$, and, for all $T > 0$,

$$\int_{Q_T} \left| \frac{1}{u}\nabla\phi(u) + \nabla V + \nabla W * u \right|^2 u \, dx \, dt < \infty. \tag{2.64}$$

Remark 2.17 (Existence theory)**.** Since $u \in \mathcal{P}_2(\mathbb{R}^d)$, the solution could a priori be a probability measure without density. In this case, in order to avoid the abuse of notation in (2.64), we shall rewrite the integration with respect to u itself (instead of the Lebesgue measure). For further results concerning existence and uniqueness of entropy solutions, we refer to [CMV03], Proposition 2.1. For a different approach based on hyperbolic techniques, we refer to [CK05], Section 6.

Step 1: bounds for u_k

Lemma 2.33. *Let $u = u_k$ be a weak solution of problem (2.63) and let Assumption 2 hold. Problem (2.63) is a gradient flow for the following associated entropy functional:*

$$E[u(t)] = \int_{\mathbb{R}^d} u \log u + \Xi(u) + uV + \frac{1}{2}u(W * u)\, dx,$$

with

$$\Xi(u) = \int_0^u \xi(s)\, ds, \qquad \xi(s) = \int_1^s \frac{\sigma'(r)}{r}\, dr.$$

More specifically we have

$$E[u(t)] + \int_0^t D[u(\tau)]\mathrm{d}\tau = E[u_0], \tag{2.65}$$

where

$$D[u(t)] = \int_{\mathbb{R}^d} u\,|\nabla(\log(u) + \xi(u) + V + W * u)|^2\; dx \tag{2.66}$$

Moreover, we have that, for a.e. $t \geq 0$, $u_k \geq 0$ *and*

$$\int_{\mathbb{R}^d} u_k\,dx = \int_{\mathbb{R}^d} u_0\,dx.$$

Proof. Identity (2.65) is obtained differentiating $E(u(t))$ with respect to time and noticing that equation (2.1) can be rewritten as follows:

$$\partial_t u = \operatorname{div}\left[u\nabla(\log(u) + \xi(u) + V + W * u)\right].$$

See for example Chapter 11 in [AGS08] for further details. □

Remark 2.18. Note that the estimates for the L^1 norm and for the entropy are uniform with respect to k and t.

We now state a useful technical Lemma, for its proof we refer the reader to [BCC12].

Lemma 2.34 (Carleman estimate [BCC12])**.** *Consider two functions* $\rho \in L^1_+(\mathbb{R}^d)$ *and* $\gamma : \mathbb{R}^d \to \mathbb{R}$ *with* $\gamma(x) \geq 0$, $e^{-\gamma} \in L^1(\mathbb{R}^d)$ *and such that the moment* $\int_{\mathbb{R}^d} \gamma(x)\rho(x)\,dx$ *is bounded. Then*

$$\int_{\mathbb{R}^d} \rho(x)(\log\rho(x))_-\,dx \leq \int_{\mathbb{R}^d} \gamma(x)\rho(x)\,dx + \frac{1}{e}\int_{\mathbb{R}^d} e^{-\gamma(x)}\,dx. \tag{2.67}$$

Thanks to Lemma 2.34, we obtain a bound that takes into account the negative part of $u_k \log u_k$.

Lemma 2.35. *Let* u_k *be the solution to* (2.63) *when* $V = V_k$ *is given by* (2.40)*. Then the following inequality holds*

$$\int_{\mathbb{R}^d} \left[u_k|\log u_k| + u_k V_k + u_k(W * u_k)\right]\,dx \leq C_0, \tag{2.68}$$

where C_0 *is a constant depending on* Ω, u_0 *and* V_0 *only and it is given by* $C_0 = 2E(u_0) + \frac{2}{e}\left(\int_\Omega e^{-\frac{1}{2}V_0}\,dx + \varepsilon_k\right)$ *and* $\varepsilon_k = \int_{\mathbb{R}^d\setminus\Omega} e^{-\frac{1}{2}V_k}\,dx \to 0$ *as* $k \to \infty$.

Additionally, we have that

$$\int_{\mathbb{R}^d\setminus\Omega_k} u_k\,dx \leq \frac{2C_0}{k}, \tag{2.69}$$

and $u_k \to 0$ *in* $L^1(\mathbb{R}^d \setminus \Omega_k)$ *as* $k \to \infty$*, uniformly in* $t \in [0, T]$.

Proof. First we consider the case $\phi(s) = s$.

In order to use the entropy inequality of Lemma 2.33, we have to ensure that the term involving $u_k \log(u_k)$ is non-negative. We have

$$\int_{\mathbb{R}^d} \left[(u_k \log u_k) + u_k V_k + \frac{1}{2} u_k (W * u_k) \right] dx \leq E(u_0).$$

In order to estimate the negative part of $u_k \log u_k$ we will use Lemma 2.34 with $\rho = u_k$ and $\gamma = \frac{1}{2} V_k$ (notice that $\int_{\mathbb{R}^d} V_k u_k \, dx$ is bounded but we do not know that the bound is uniform in k at this stage). More specifically we have

$$\int_{\mathbb{R}^d} u_k (\log u_k)_- \, dx \leq \frac{1}{2} \int_{\mathbb{R}^d} V_k u_k \, dx + \frac{1}{e} \int_{\mathbb{R}^d} e^{-\frac{1}{2} V_k} \, dx.$$

This implies that

$$\int_{\mathbb{R}^d} \left[u_k |\log u_k| + \frac{1}{2} u_k V_k + \frac{1}{2} u_k ((W) * u_k) \right] dx \leq E(u_0) + \frac{1}{e} \int_{\mathbb{R}^d} e^{-\frac{1}{2} V_k} \, dx,$$

and, in turn,

$$\frac{1}{2} \int_{\mathbb{R}^d \setminus \Omega_k} u_k V_k \, dx \leq E(u_0) + \frac{1}{e} \int_{\mathbb{R}^d} e^{-\frac{1}{2} V_k} \, dx.$$

This means

$$0 \leq \int_{B \setminus \Omega_k} u_k \, dx \leq 2 \frac{C_0}{k} \to 0 \quad \text{as } k \to \infty.$$

The general case with a nonlinear diffusion $\phi(s)$ can be treated in an analogous way, after observing that, from (2.38), $\frac{\phi'(s)}{s} = \frac{1}{s} + \frac{\sigma'(s)}{s} \geq \frac{1}{s} + \mu s^{a-1} > 0$, for $s > 0$. Therefore $\phi(u) = u \log u + \Xi(u)$ is convex with respect to u and defines a suitable entropy. In addition, all the terms involving $\frac{\sigma'(s)}{s}$ that appear in the generalisation of the previous computations are automatically non-negative. □

Remark 2.19. Notice that the assumption $W \geq 0$ is not restrictive and the same argument applies if W has a lower bound, in particular

$$\int_B u(W * u) = \int_B u((W + q) * u) - q \left(\int_B u \right)^2$$

The following estimate will be used extensively in the next subsection. Its proof is a direct consequence of (2.65) and (2.68).

Corollary 2.36. *Let u_k be the solution to (2.63) when $V = V_k$ is given by (2.62). Then, given the constant C_0 from Lemma 2.35, the following estimate (uniform in k) holds:*

$$\begin{aligned} &\int_{\mathbb{R}^d} \left[u_k |\log u_k| + \Xi(u_k) + u_k V_k + u_k ((W + q) * u_k) \right] dx \\ &+ \int_0^T \int_{\mathbb{R}^d} u_k \left| \nabla (\log u_k + \xi(u_k) + V_k + W * u_k) \right|^2 dx \, dt \leq C_0. \end{aligned} \tag{2.70}$$

Step 2: passage to the limit and proof of Theorem 2.31.

Similarly to what we did in the L^2 case, we consider the weak formulation (2.39) and we divide $\mathbb{R}^d$ into three parts, namely Ω, Ω_k^c and $\Omega_k \setminus \Omega$. Consequently, we split the weak formulation in the following way:

$$\begin{aligned} &\int_{\mathbb{R}^d} u_k\eta\Big|_0^T dx + \int_0^T \int_{\mathbb{R}^d} [(\nabla\phi(u_k) + u_k\nabla V_k + u_k\nabla W * u_k)\cdot\nabla\eta - u_k\partial_t\eta]\, dx\, dt \\ &= I_\Omega + I_{\Omega_k\setminus\Omega} + I_{\Omega_k^c} + J_\Omega + J_{\Omega_k\setminus\Omega} + J_{\Omega_k^c}, \end{aligned} \tag{2.71}$$

for any test function $\eta \in C^\infty(Q_T)$. Here we have defined

$$\begin{aligned} I_A &= \int_0^T \int_A [(\nabla\phi(u_k) + u_k\nabla V_k + u_k\nabla W * u_k)\cdot\nabla\eta]\, dx\, dt, \\ J_A &= -\int_0^T \int_A u_k\partial_t\eta\, dx\, dt. \end{aligned}$$

Below we show that all terms except I_Ω and J_Ω vanish in the limit $k \to \infty$. Then $I_\Omega + J_\Omega$ will characterise the limit problem defined in Ω.

- I_Ω: First, we focus on I_Ω. Notice that, restricting (2.70) in Ω, we have

$$\int_0^T \int_\Omega u_k \left|\nabla(\log u_k + \xi(u_k) + V_0 + W * u_k)\right|^2 dx\, dt \le C_0. \tag{2.72}$$

Hence, we have that $\sqrt{u_k}\,\nabla(\log u_k + \xi(u_k) + V_0 + W * u_k) \in L^2(\Omega_T)$. We now proceed to show that the last two terms in this expression are bounded in $L^2(\Omega_T)$. We have

$$\int_0^T \int_\Omega u_k|\nabla V_0|^2\, dx\, dt \le T\, \|\nabla V_0\|^2_{L^\infty(\Omega)} \int_\Omega u_0\, dx, \tag{2.73}$$

and

$$\int_0^T \int_\Omega u_k|\nabla W * u_k|^2\, dx\, dt \le T\, \|\nabla W\|_{L^\infty(\Omega)} \left(\int_\Omega u_0\, dx\right)^2. \tag{2.74}$$

Now we rewrite the diffusion terms as

$$\int_0^T \int_\Omega u_k \left|\nabla(\log u_k + \xi(u_k))\right|^2 dx\, dt = \int_0^T \int_\Omega (1 + \sigma'(u_k))|\nabla(\sqrt{u_k})|^2\, dx\, dt. \tag{2.75}$$

Recalling that $\sigma' \ge 0$ and combining (2.72), (2.73) and (2.74), from (2.75) it follows that that $\nabla(\sqrt{u_k}) \in L^2(\Omega_T)$ uniformly in k. We deduce that the sequence $\sqrt{u_k}$ is compact in $L^2(\Omega_T)$. Therefore, we can extract a subsequence (still denoted by u_k) that converges strongly in the same space:

$$\sqrt{u_k} \to \sqrt{u} \quad \text{in } L^2(\Omega_T). \tag{2.76}$$

We now rewrite I_Ω as follows

$$I_\Omega = \int_0^T \int_\Omega \sqrt{u_k}\sqrt{u_k}\,\nabla(\log u_k + \xi(u_k) + V_k + W * u_k)\cdot\nabla\eta\; dx\, dt, \qquad (2.77)$$

and we notice that the integrand is the product of a strongly converging sequence and a weakly converging sequence in $L^2(\Omega_T)$, in particular $\sqrt{u_k} \to \sqrt{u}$ and $F_k \rightharpoonup F \in L^2(\Omega_T)$, where $F_k = \sqrt{u_k}\nabla(\log u_k + \xi(u_k) + V_k + W * u_k)$. This means that

$$\int_0^T \int_\Omega \sqrt{u_k} F_k \cdot \nabla\eta\; dx\, dt \to \int_0^T \int_\Omega \sqrt{u} F \cdot \nabla\eta\; dx\, dt.$$

Furthermore, again combining (2.72), (2.73) and (2.74), it follows that $F = \nabla\phi(u) + u\nabla V_0 + u\nabla W * u$ and

$$I_\Omega \to \int_0^T \int_\Omega [(\nabla\phi(u) + u\nabla V_0 + u\nabla W * u)\cdot\nabla\eta]\; dx\, dt, \qquad \text{as } k\to\infty. \qquad (2.78)$$

- $I_{\Omega_k^c}$: The term $I_{\Omega_k^c}$ is dealt with in an analogous way to I_Ω since equation (2.77) holds replacing Ω by Ω_k^c. In particular, this term then vanishes since $u_k \to 0$ strongly in $L^1(\Omega_k^c)$ (using Lemma 2.35).

- $I_{\Omega_k\setminus\Omega}$: We now check that $I_{\Omega_k\setminus\Omega}$ also vanishes. Indeed we have

$$I_{\Omega_k\setminus\Omega} \le \|u_k\|^{1/2}_{L^\infty(0,T;L^1(\Omega_k\setminus\Omega))}\left(\int_0^T D_{\Omega_k\setminus\Omega}[u(t)]\, dt\right)^{\frac{1}{2}}.$$

for $D_{\Omega_k\setminus\Omega}[u(t)] = \int_{\Omega_k\setminus\Omega} u_k\,|\nabla(\log u_k + \xi(u_k) + V_k + W * u_k)|^2\; dx$

The second factor in the right hand side is bounded by the Lemma 2.36. The first factor is bounded and converges to zero as $k\to\infty$ using the following argument. By Jensen's inequality we obtain, for a region R such that $|R| < 1$,

$$\int_R g(x)\, dx \le (-\log|R|))^{-1}\left(\int_R g(x)\log g(x)\, dx + e^{-1}\right). \qquad (2.79)$$

We use this inequality with $R = \Omega_k \setminus \Omega$ and $g = u_k$. Recalling that $|\Omega_k \setminus \Omega| \to 0$ as $k\to\infty$ we obtain the desired result, $I_{\Omega_k\setminus\Omega} \to 0$.

- J_Ω: Up to a subsequence, u_k converges weakly in $L^1(\Omega)$ to some function u, hence

$$J_\Omega \to -\int_0^T \int_A u\partial_t\eta\; dx\, dt, \qquad \text{as } k\to\infty. \qquad (2.80)$$

- $J_{\Omega_k^c}$: Notice that $u_0 = 0$ in Ω_k^c. The remaining terms in $J_{\Omega_k^c}$ vanish by Lemma 2.35.

- $J_{\Omega_k\setminus\Omega}$: The integral $J_{\Omega_k\setminus\Omega}$ goes to zero because the integrand is uniformly bounded in L^1 (thanks to the conservation of mass, see Lemma 2.24) and $|\Omega_k \setminus \Omega| \to 0$.

Thanks to (2.78) and (2.80), the weak formulation we obtain in the limit is the following:

$$\int_0^T \int_\Omega [(\nabla\phi(u) + u\nabla V_0 + u\nabla W * u)\cdot\nabla\eta - u\partial_t\eta]\, dx\, dt = 0.$$

Notice that since the test function can have arbitrary values on $\partial\Omega$, the no-flux conditions on $\partial\Omega$ are implicitly enforced.

We now show that initial datum is satisfied in $\mathcal{P}_2(\mathbb{R}^d)$. To do so, we use the characterisation of $\mathcal{P}_2(\mathbb{R}^d)$ convergence given in Proposition 7.1.5, p. 154 in [AGS08]. In particular, the the second moment $\int_{\mathbb{R}^d} u_k|x|^2\, dx$ is bounded uniformly ($V_k(x) \geq c|x|^2$ at infinity) and the so-called narrow convergence is implied by the L^1 bounds obtained in (2.69), (2.76) and (2.79). We deduce that u_k converges to u in $\mathcal{P}_2(\mathbb{R}^d)$ as well. As a consequence, we obtain that the initial datum is satisfied in the $\mathcal{P}_2(\mathbb{R}^d)$ sense, indeed, since our L^1 bounds are uniform in time, we conclude that

$$\lim_{t\to 0}\lim_{k\to\infty} d_{W_2}(u(t,\cdot), u_0) \leq \lim_{t\to 0}\lim_{k\to\infty} (d_{W_2}(u_k(t,\cdot), u(t,\cdot)) + d_{W_2}(u_k(t,\cdot), u_0)) = 0.$$

Corollary 2.37 (Degenerate case)**.** *Theorem 2.32 still holds if we drop the linear diffusion term, i.e. $\phi = \sigma$, leading to an equation with degenerate diffusion.*

Proof. Almost all the results above remain unchanged, in particular it is straightforward to see that u_k converges to 0 in the complement of Ω. Therefore, the only steps that are not obvious concern the passage to the limit in the term I_Ω in the proof of Theorem 2.32. More specifically, we have to obtain strong L^1 convergence in Ω finding an alternative to (2.75). Since, by assumption, $\sigma(s) \geq s^a$ for some $a > 1$, the entropy satisfies the inequality

$$\int_\Omega u^{a+2}\, dx \leq C(a)\int_\Omega \Xi(u)\, dx < \infty. \tag{2.81}$$

We know that $\nabla\xi(u_k)$ is bounded in $L^2(\Omega)$, hence, by compactness, the sequence $\xi(u_k)$ converges to $\bar{\xi}$ a.e. (t, x). Notice that ξ is monotone and sufficiently regular, thus u_k converges a.e. as well. Combining this fact, inequality (2.81) and uniqueness of weak limits we obtain strong L^1 convergence of u_k to a weak solution u for $k \to \infty$. □

Remark 2.20 (Moments)**.** We have used the hypothesis of quadratic growth of V at infinity only to ensure that the second moment $\int_{\mathbb{R}^d} u_k|x|^2\, dx$ is bounded and therefore that the initial datum is satisfied in $\mathcal{P}_2(\mathbb{R}^d)$. It is possible to make less restrictive assumptions, for example, if V grows linearly at infinity we get control over the first moment and the initial datum is satisfied in $\mathcal{P}_1(\mathbb{R}^d)$.

Chapter 3

Systems with small cross diffusion

Contents

In this chapter we focus on the study of non-degenerate systems of PDEs of type (1.1). In addition to the assumptions in section 1.1.3, we suppose there exists $\lambda > 0$ such that

$$\mathcal{A}_{ij}^{\alpha\beta}(t,x,u)X_\alpha^i X_\beta^j \geq \lambda |X|^2, \quad \text{for all} \quad t \geq 0,\ x \in \mathbb{R}^d,\ u \in \mathbb{R}^N,\ X \in \mathbb{R}^{d \times N}.$$

We study of systems that are nearly decoupled, in particular all the nonlinearities are supposed to be multiplied by an arbitrarily small parameter ϵ for $d \in \{1,2,3\}$. The results of this chapter have been obtained jointly with Y. Capdeboscq and M. Bruna and they have been published in [ABC18a]. The "smallness" assumption for the cross-diffusion terms may seem quite restrictive, however it is relatively common in the biological applications since it is often related to the fact that pairwise interactions between individuals in a microscopic model are dominant (whereas the interaction of three or more individuals is considered to be uncommon, at least in a low-density regime). As one might expect, the fact that the system is close to a linear, decoupled one can be used to prove regularity and stability of solutions in strong Sobolev spaces. In this case we include in the analysis first order terms in divergence form. Chapter 4 will be dedicated to the study of non-degenerate systems with "large" cross diffusion. For a different approach to perturbations of (possibly degenerate) decoupled systems, we refer to [ZM15].

3.1 Analysis and stability

3.1.1 Background and motivation

We are going to analyse a class of nonlinear cross-diffusion systems of PDEs which model multi-species populations in presence of short-range interactions between individuals. We assume that these systems are close, in a suitable sense, to decoupled sets of linear parabolic evolution problems. Such problems arise in many applications in mathematical biology, such as chemotactic cell migration, ion transport through cell membranes, and spatial segregation in interacting species. The strength of the interactions (and therefore of the nonlinear terms) is quantified with a small parameter ϵ, so that when $\epsilon = 0$ the system becomes diagonal and linear. The biological justification for these models comes from weakly-interacting species, whereby interactions between populations (such as excluded-volume or chemotactic interactions) are present but are not dominant over the isolated species behaviour.

The cross-diffusion systems we are interested in have the form

$$\partial_t u - \operatorname{div}\left[\mathcal{A}(t,x,u)\nabla u - \mathcal{B}(t,x,u)u\right] = 0, \quad \text{in} \quad \Omega, t > 0, \tag{3.1a}$$

with boundary and initial conditions

$$\left[\mathcal{A}(t,x,u)\nabla u - \mathcal{B}(t,x,u)u\right]\cdot \nu = 0, \qquad \text{on} \quad \partial\Omega, t > 0, \tag{3.1b}$$

$$u(0,\cdot) = u^0, \quad \text{in} \quad \Omega, \tag{3.1c}$$

where Ω is a smooth, bounded, and connected domain in $\mathbb{R}^d$ ($d = 1,2,3$), ν denotes the outward normal on $\partial\Omega$, and $u = (u_1, \dots, u_m)$ is the vector of densities of each species. The divergence div and gradient ∇ represent derivatives with respect to the d spatial variables. Here $\mathcal{A}(t,x,u)$ and $\mathcal{B}(t,x,u)$ are $m \times m$ matrices of diffusion tensors and drift vectors, respectively (see (3.12) for further details). In particular, the entries of the diffusion tensor $\mathcal{A}$ may be scalars in the case of isotropic diffusion, or $d \times d$ tensors in the case of anisotropic diffusion. The drift matrix elements $\mathcal{B}_{ij}$ are $d-$dimensional vectors. In our class of cross-diffusion systems, the matrices $\mathcal{A}$ and $\mathcal{B}$ are close to matrices that are diagonal and independent of u, that is, they can be written in the form

$$\begin{aligned} \mathcal{A}(t,x,u) &= \mathcal{A}^{(0)}(t,x) + \epsilon\mathcal{A}^{(1)}(t,x,u) + O(\epsilon^2), \\ \mathcal{B}(t,x,u) &= \mathcal{B}^{(0)}(t,x) + \epsilon\mathcal{B}^{(1)}(t,x,u) + O(\epsilon^2), \end{aligned} \tag{3.2}$$

where ϵ is a small parameter.

The focus of this section is to study the stability of the solutions to (3.1) under perturbations of order ϵ. We establish that the solutions depend continuously on the nonlinearities $\mathcal{A}^{(1)}$ and $\mathcal{B}^{(1)}$ for ϵ small enough. The cross-diffusion model (3.1) is a non-linear system, and this combines two types of difficulties, namely the non-linearity and the fact that fully coupled parabolic systems of equations do not enjoy, in general, the same smoothness properties as parabolic equations (see, for example, [GM13, Chap. 9] , and [GS82]). Our results are detailed in Proposition 3.1 and Theorem 3.5. They are quantitative, in the sense that we provide a bound on ϵ below which our perturbation result applies. The novelty of our analysis consists in the unified approach to the study of regularity and stability properties in "strong" Sobolev norms for a relatively wide class of nonlinear cross-diffusion systems.

Our stability estimate uses the underlying regularity of the system, which, as we will see, it inherits from the leading order model, consisting of decoupled linear evolution equations. We show that for small perturbations at least some of the regularity is preserved and, using a fixed point argument, we deduce a stability estimate with respect to the nonlinearities of the model.

Similar results concerning nonlinear systems where interactions between species (or components) are limited to lower order term (so-called weakly coupled systems) are available in the work of Camilli and Marchi [CM12]. They extend the results available for scalar equations in terms of continuous dependence estimates in the sup norm using the doubling variable method [Kru70] and viscosity solutions. Their results do not apply to fully coupled systems with cross diffusion present such as the ones we are considering. Continuous dependence for fully coupled quasilinear systems was studied by Cannon, Ford and Lair [CFL76]. They established existence and uniqueness, following arguments of Ladyzhenskaya, Solonnikov, and Ural'tseva [LSU88] in larger Sobolev spaces (weaker norms). They derive stability estimates under additional integrability properties assumptions for the gradients. We establish existence and uniqueness in stronger norms, removing the need of additional regularity assumptions.

There are several models, especially in mathematical biology, that fit into the class of systems (3.1) and (3.2). This is the case for models describing the transport of cells or ions while accounting for the finite-size of particles [BC12a, BDFPS10, SLH09, Per15]. These models were derived from stochastic agent-based models assuming that the concentration of cells or ions is not too large, so that the transport dominates over the finite-size interactions between cells or ions. The diffusion and drift matrices become density-dependent due to the interactions, but this correction is small since it scales with the excluded volume in the system. Below we present three of such models, and show how they fit into our framework.

Example 1 (Random walk on a lattice with size exclusion)**.** A cross-diffusion model for two interacting species was employed to describe the motility of biological cells by Simpson et al. [SLH09] or ion transport by Burger et al. [BDFPS10]. The models were derived assuming that particles are restricted to a regular square lattice and undergo a simple exclusion random walk, in which a particle can only jump to a site if it is presently unoccupied. In order to obtain a continuum model such as (3.1) from these so-called lattice-based models, it is generally assumed that the occupancies of adjacent sides are independent, so that the jumping probabilities take a simple form and do not require correlation functions [BDFPS10,SLH09]. Clearly, such an approximation is poor when the overall occupancy of the lattice is high. As a result, these models are generally considered valid for low-lattice occupancies.

The models in [BDFPS10,SLH09] consider two species of equal size, whose diameter is given by the lattice spacing ε, that undergo a random walk with isotropic diffusion D_i and external potential $V_i(x)$, for $i = 1, 2$ (the jumping rates increase with D_i and the jumps are biased in the direction of $-\nabla V_i(x)$). There are N_1 particles of the first species, and N_2 of the second species. Under these assumptions, a cross-diffusion model of the form (3.1) is obtained, where the population densities $u_1(t,x)$ and $u_2(t,x)$ represent the probability that a particle from first or second species respectively is at $x \in \Omega$ at time t. The diffusion and drift matrices are given by [BDFPS10]

$$\mathcal{A}(u) = \begin{pmatrix} D_1(1-\epsilon\bar{N}_2 u_2) & \epsilon D_1 \bar{N}_2 u_1 \\ \epsilon D_2 \bar{N}_1 u_2 & D_2(1-\epsilon\bar{N}_1 u_1) \end{pmatrix}, \tag{3.3a}$$

$$\mathcal{B}(u) = \begin{pmatrix} -\nabla V_1(1-\epsilon\bar{N}_1 u_1) & \epsilon \bar{N}_2 u_1 \nabla V_1 \\ \epsilon \bar{N}_1 u_2 \nabla V_2 & -\nabla V_2(1-\epsilon\bar{N}_2 u_2) \end{pmatrix}, \tag{3.3b}$$

where $\epsilon = (N_1+N_2)\varepsilon^d/|\Omega| \ll 1$ represents the total volume fraction of the lattice occupied by particles and $\bar{N}_i = N_i/(N_1+N_2)$. Like in most works using lattice-based models, the continuum model is written in terms of the volume concentrations $\hat{u}_i$, so that the mass of $\hat{u}_i$ equals the total volume occupied by species i (that is, $\int_\Omega \hat{u}_i(t,x)\mathrm{d}x = N_i\varepsilon^d/|\Omega|$). We write (3.3) in terms of probability densities u_i, which implies that $\int_\Omega u_i \mathrm{d}x = 1$. The two quantities are related by the identity $\hat{u}_i = \bar{N}_i\epsilon u_i$. The potentials appearing in (3.3b), V_i, are not rescaled by the diffusion coefficient as it is done in [BDFPS10]. The number of species can take any values provided that ϵ, is small. The matrices in (3.3) are of the form (3.2) that we consider in this chapter. We have written (3.3) in a form consistent with our notations, which differ slightly from those used in [BDFPS10]. In particular, we have the following correspondence: $[r,b,V,W,D_r,D_b] = [\varepsilon\bar{N}_1u_1, \varepsilon\bar{N}_2u_2, V_1, V_2, D_1, D_2]$. For the first flux (and similarly for the second one) we have:

$$\begin{aligned} &D_r((1-b)\nabla r + r\nabla b + r(1-r-b)\nabla V) \\ &= \varepsilon\bar{N}_1\big[(1-\varepsilon\bar{N}_2u_2)\nabla u_1 + u_1\nabla(\varepsilon\bar{N}_2u_2) + u_1(1-\varepsilon\bar{N}_1u_1 - \varepsilon\bar{N}_2u_2)\nabla V_1\big] \\ &= \varepsilon\bar{N}_1\big[\mathcal{A}(u)_{11}\nabla u_1 + \mathcal{A}(u)_{12}\nabla u_2 + \mathcal{B}(u)_{11}u_1 + \mathcal{B}(u)_{12}u_2\big] \end{aligned}$$

Global existence for such model was shown in [CJ06]. There are also other lattice-based models that fit well into such framework, such as that derived by Shigesada et al. [SKT79b] to describe spatial segregation of interacting animal populations.

Example 2 (Brownian motion with size exclusion)**.** A cross-diffusion model for two interacting species of diffusive particles was obtained by Bruna and Chapman for $d = 2, 3$

in [BC12a], starting from a system with two types of Brownian hard spheres. The population densities $u_i(t,x)$, $i = 1,2$, represent the probability that a particle of species i is at $x \in \Omega$ at time t, and so $\int_\Omega u_i(t,x)\mathrm{d}x = 1$. The model assumes there are N_i particles of species i, of diameter ε_i and isotropic diffusion constant D_i. The position X_i of each particle in species i evolves in time according to the stochastic differential equation

$$\mathrm{d}X_i(t) = \sqrt{2D_i}\mathrm{d}W(t) - \nabla V_i(X_i(t))\mathrm{d}t, \tag{3.4}$$

where $i = 1$ or 2, and W are independent, d-dimensional standard Brownian motions. Reflective boundary conditions are imposed whenever two particles are in contact ($\|X_i - X_j\| = (\varepsilon_i + \varepsilon_j)/2$, when X_i and X_j are of type i and j, respectively), as well as on the boundary of the domain $\partial\Omega$.

The cross-diffusion model is derived using the method of matched asymptotic expansions under the assumption that the volume fraction of the system is small, or equivalently, that $(N_1\varepsilon_1^d + N_2\varepsilon_2^d)/|\Omega| \sim \epsilon \ll 1$, where ϵ is defined as in Example 1 with $\varepsilon = (\varepsilon_1+\varepsilon_2)/2$. When the number of particles in each species is large, the cross-diffusion model in [BC12a] is of the form (3.1), with diffusion matrix

$$\mathcal{A}(u) = \begin{pmatrix} D_1(1+\epsilon a_1 u_1 - \epsilon c_1 u_2) & \epsilon D_1 b_1 u_1 \\ \epsilon D_2 b_2 u_2 & D_2(1+\epsilon a_2 u_2 - \epsilon c_2 u_1) \end{pmatrix}, \tag{3.5a}$$

and drift matrix

$$\mathcal{B}(u) = \begin{pmatrix} -\nabla V_1 & \epsilon c_1 \nabla(V_1 - V_2) u_1 \\ \epsilon c_2 \nabla(V_2 - V_1) u_2 & -\nabla V_2 \end{pmatrix}. \tag{3.5b}$$

The parameters a_i, b_i, c_i ($i = 1,2$) are all positive numbers that depend on the problem dimension, particle sizes, numbers, and relative diffusion coefficients (see specific values in Section 3.2). Model (3.5) also fits into the form (3.2), with $\epsilon = 0$ when particles are non-interactive (point particles) and evolve according to two decoupled linear drift-diffusion equations.

Example 3 (Asymptotic gradient-flow structures)**.** Certain cross-diffusion systems possess a formal gradient-flow structure, that is, they can be formulated as

$$\partial_t u - \nabla \cdot \left(M \nabla \frac{\delta E}{\delta u} \right) = 0, \tag{3.6}$$

where $M \in \mathbb{R}^{m\times m}$ is known as mobility matrix and $\delta E/\delta u$ is the variational derivative of the entropy (or free energy) function $E[u]$. While the underlying microscopic model (3.4) of Example 2 has a natural entropy, in [BBRW17] it was noted that model (3.5) does not have an obvious gradient-flow structure, but that it is close to one that does have such convenient structure. More specifically, consider the following entropy

$$E_\epsilon[u] = \int_\Omega \left[u_1 \log u_1 + u_2 \log u_2 + u_1 \frac{V_1}{D_1} + u_2 \frac{V_2}{D_2} + \frac{\epsilon}{2}\left(a_1 u_1^2 + 2a_{12}u_1u_2 + a_2 u_2^2\right) \right] \mathrm{d}x, \tag{3.7a}$$

with $a_{12} = (d-1)(c_1+c_2)$, and the mobility matrix

$$M_\epsilon(u) = \begin{pmatrix} D_1 u_1(1-\epsilon c_1 u_2) & D_1 c_2 \epsilon u_1 u_2 \\ D_2 c_1 \epsilon u_1 u_2 & D_2 u_2(1-c_2\epsilon u_1) \end{pmatrix}. \tag{3.7b}$$

The cross-diffusion system (3.1) with diffusion and drift matrices (3.5) and $N_1 = N_2$[1], can be rewritten as

$$\partial_t u = \nabla \cdot \left(M_\epsilon \nabla \frac{\delta E_\epsilon}{\delta u} - \epsilon^2 G \right), \tag{3.8}$$

where $G = G(u, \nabla u)$ (see more details in Section 3.2). In particular, the discrepancy between the system in Example 2 and the gradient-flow induced by (3.7) is of order ϵ^2, an order higher than that of the model.[2] Does this legitimise the use of (3.7) as a gradient-flow structure of the system? Having a formal gradient-flow structure can facilitate the analysis of cross-diffusion models [Jün15]. The gradient-flow model (3.6)-(3.7) was studied in [BBRW17]; stability, uniqueness of the stationary solutions, and a global-in-time existence result was shown.

It is natural to ask whether the approximation argument in Example 3 can be made rigorous, and, more generally if minor changes in the models can be safely ignored. For instance, given a two-species biological system, does it matter if we choose a lattice-based model (like in Example 1), or an off-lattice model (like in Example 2 with equal particle number, size, diffusivity, etc.)? If so, can we quantify the differences? Lattice-based approaches have become very common, as they offer a simple way to derive continuum PDE models. They can be unrealistic since most biological transport processes modelled by these are not constrained on a lattice [PS12]. Nevertheless, if one is solely interested in the population-level behaviour of the system, is it worth using a more realistic off-lattice model? When is the even simpler model (linear advection-diffusion) sufficiently accurate? The aim of this chapter is to answer these questions and quantify the differences between models of the form (3.1).

3.1.2 Well-posedness

As we are working with systems of equations, we use different indices to refer to the ambient space variables and the component or species number. Greek indices $1 \le \alpha, \beta \le d$ refer to directions in the ambient space, $\mathbb{R}^d$, for $d = 1, 2, 3$. Latin indices $1 \le i, j \le m$ are used to refer to the species number. The domain Ω where the problem is formulated is bounded, connected and of class C^2 in $\mathbb{R}^d$. The outward normal on $\partial\Omega$ is written ν.

[1] In [BBRW17] the more general case when $N_1 \neq N_2$ was also considered, by writing the system in terms of number densities $N_i u_i$.

[2] Systems (3.1)-(3.5) and (3.6)-(3.7) are in fact identical when both species have the same particle sizes, $\varepsilon_1 = \varepsilon_2$, and diffusivities, $D_1 = D_2$, since G vanishes in that particular case.

The parabolic models we consider are weak formulations of problems of the form

$$\begin{aligned} \partial_t u_i - \partial_\alpha \left[\mathcal{A}_{ij}^{\alpha\beta}(t,x,u)\partial_\beta u_j - \mathcal{B}_{ij}^{\alpha}(t,x,u)u_j\right] &= 0 \quad \text{in } \Omega, \\ \left[\mathcal{A}_{ij}^{\alpha\beta}(t,x,u)\partial_\beta u_j - \mathcal{B}_{ij}^{\alpha}(t,x,u)u_j\right] \cdot \nu_\alpha &= 0 \quad \text{on } \partial\Omega, \\ u(0,\cdot) &= u^0 \quad \text{in } \Omega, \end{aligned} \tag{3.9}$$

for $1 \le i \le m$,. The Einstein summation convention is used, that is, repeated indices are implicitly summed.

Our main result is a stability estimate for cross-diffusion systems that are close to diagonal, decoupled, linear diffusion problems. Our reference problem will be the weak formulation of

$$\begin{aligned} \partial_t u_i - \partial_\alpha \left[D_i^{\alpha\beta}(t,x)\partial_\beta u_i - F_i^{\alpha}(t,x)u_i\right] &= 0 \quad \text{in } \Omega, \\ \left[D_i^{\alpha\beta}(t,x)\partial_\beta u_i - F_i^{\alpha}(t,x)u_i\right] \cdot \nu_\alpha &= 0 \quad \text{on } \partial\Omega, \\ u(0,\cdot) &= u^0 \quad \text{in } \Omega, \end{aligned} \tag{3.10}$$

The initial datum u^0 in (3.9) and (3.10) belongs to $H^2(\Omega)$. Note that throughout this chapter we write $H^2(\Omega)$ for $H^2(\Omega;\mathbb{R}^m)$, and similarly for other spaces.

Compared to the general system (3.9), in (3.10) we have specified that $\mathcal{A}_{ij} = \mathcal{B}_{ij} = 0$ if $i \neq j$, and $\mathcal{A}$ and $\mathcal{B}$ do not depend on u. In Examples 1, 2, and 3, the reference problem corresponds to the case $\epsilon = 0$, with $D_i^{\alpha\beta}(x,t) = \delta_{\alpha\beta} D_i$ and $F_i = -\nabla V_i$. We allow time and space variations of the diffusion coefficients as it does not affect the analysis. We could also have safely included lower-order terms, but it would have resulted in somewhat longer and relatively routine developments. Additionally such terms do not appear in the three examples of interest.

System (3.10) is strongly parabolic, that is, there exist a positive constant λ such that for every $t \in [0,\infty)$, $x \in \Omega$ and $\xi \in \mathbb{R}^d$, there holds

$$D_i^{\alpha\beta}(t,x)\xi^\alpha\xi^\beta \ge \lambda\,|\xi|^2, \quad i = 1,\ldots,m. \tag{3.11}$$

Furthermore, we shall assume that D is symmetric in the space indices α and β.

We allow perturbations of system (3.10) scaled by a small parameter ϵ. Namely we consider (3.9) with

$$\begin{aligned} \mathcal{A}_{ij}^{\alpha\beta}(t,x,u) &= D_i^{\alpha\beta}(t,x) + \epsilon a_{ij}^{\alpha\beta}(t,x)\phi_{ij}^{\alpha\beta}(u), \\ \mathcal{B}_{ij}^{\alpha}(t,x,u) &= F_i^{\alpha}(t,x) + \epsilon b_{ij}^{\alpha}(t,x)\psi_{ij}^{\alpha}(u). \end{aligned} \tag{3.12}$$

The variations of the coefficients a and b are of class C^2 in time and space, that is,

$$\|(a,b)\|_{C^2\left([0,\infty)\times\mathbb{R}^d\right)} \le M, \tag{3.13}$$

and the dependence on u of the perturbations is also of class C^2,

$$\phi, \psi \in C^2\left(\mathbb{R}^m\right)^{m\times m}, \qquad \phi(0) = \psi(0) = 0. \tag{3.14}$$

Furthermore, we assume that D and F satisfy the bound

$$\sum_{\alpha,\beta,i} \|D_i^{\alpha\beta}\|_{C^1\left([0,\infty)\times\mathbb{R}^d\right)} + \sum_{\alpha,i} \|F_i^{\alpha}\|_{C^1\left([0,\infty)\times\mathbb{R}^d\right)} \leq M. \tag{3.15}$$

In the context of biological models, one is often interested in arbitrarily long behaviour and, in turn, convergence to a steady state. Along this line, we prove sharper estimates when the coefficients D and F of the reference problem (3.10) do not depend on time and F is derived from a potential (as in Examples 1, 2, and 3). In particular, consider the following additional assumption:

(H) for each $i \in \{1, \dots, m\}$, D_i is independent of time and there exists V_i such that $F_i = -D_i \nabla V_i$.

Our estimates will be expressed in terms of the constants appearing in assumptions (3.11), (3.13), (3.14) and (3.15). More specifically, the following positive-valued functions will appear:

$$L_i \;:\; R \to \|(\phi,\psi)\|_{C^i\left(\overline{B_R(0)}\right)} \qquad i = 0, 1, 2, \tag{3.16}$$

$$K_0 \;:\; R \to M\left(5L_0\left(C_S^\infty R\right) + 2C_S^2 L_1\left(C_S^\infty R\right) R\right), \tag{3.17}$$

$$K_1 \;:\; R \to C_S M\left(L_1(R)R + L_2(R)R^2\right), \tag{3.18}$$

$$K_2 \;:\; R \to 6RC_{T/\infty} C_S \max\left(\left(L_0(R) + L_1(R)R\right), M(1+R)\right), \tag{3.19}$$

where C_S^2, C_S, and C_S^∞ depend on Ω and d and are given by (3.30), (3.31), and (3.69) respectively. The constant $C_{T/\infty}$ determines the dependence on a final time $T > 0$ of our estimates and is given by

$$C_{T/\infty} = \begin{cases} C_T & \text{when } \textbf{(H)} \text{ does not apply} \\ C_\infty & \text{when } \textbf{(H)} \text{ applies,} \end{cases} \tag{3.20}$$

where C_T is given by (3.64) and depends on M, Ω, L_0, L_1 and T only, and C_∞ is specified in (3.65) and it depends on M, Ω, L_0 and L_1 only – not T. The upper bound ϵ_0 on the range of values ϵ allowed will be determined by means of the following function

$$\epsilon_0 : R \to \min\left(\frac{1}{2 + 2K_0(R)}, \frac{1}{1 + K_1(R)}\right). \tag{3.21}$$

Our first result, which is instrumental to our main theorem, provides an existence result and a regularity estimate for solutions of system (3.9). Given $T > 0$, we denote the parabolic cylinder by $Q_T = (0,T) \times \Omega$.

Definition 3.1. We name $W(Q_T)$ the Banach space of functions with two weak derivatives in space in $L^2(\Omega)$ continuously in time, and one time derivative in $H^1(Q_T)$, that is,

$$W(Q_T) = \left\{u \in C\left([0,T]\,; H^2(\Omega)\right), \partial_t u \in H^1(Q_T)\right\}.$$

We are now ready to state our first result, concerning existence and uniqueness of solutions of (3.9).

Proposition 3.1. *Assume that hypothesis (3.11), (3.12), (3.13), (3.14) and (3.15) hold. Consider $u^0 \in W(Q_T)$ satisfying the compatibility condition*

$$\left[\mathcal{A}_{ij}^{\alpha\beta}(t,x,u^0)\partial_\beta u_j^0 - \mathcal{B}_{ij}^{\alpha}(t,x,u^0)u_j^0\right]\cdot\nu = 0 \quad \text{on } \partial\Omega, \quad i = 1,\ldots,m. \tag{3.22}$$

Let

$$Y_0 = C_{T/\infty}\|u^0\|_{H^2(\Omega)}. \tag{3.23}$$

If $\epsilon < \epsilon_0(Y_0)$, then system (3.9) admits a unique solution $u \in W(Q_T)$ and there holds

$$\|u\|_{W(Q_T)} \leq Y_0.$$

Remark 3.1. Any compatible initial data in $H^2(\Omega)$ is allowed, provided ϵ is small enough. Note that the compatibility condition (3.22) holds for any initial data compactly supported in Ω. All ϵ within the range $[0, \epsilon_0(Y_0))$ are allowed, and the solution u is bounded linearly by its initial condition. When assumption **(H)** holds, the solution is bounded for all times.

Our result holds for space dimension $d = 1, 2$ and 3, but not above. Two embeddings are used in our proofs: $L^4(\Omega) \subset H^1(\Omega)$, which does not hold when $d \geq 5$, and $L^\infty(\Omega) \subset H^2(\Omega)$, which does not hold when $d \geq 4$.

In Lemma 3.2, we derive an estimate for a linearisation of system (3.9).

Lemma 3.2. *Assume that $\mathcal{A}$ and $\mathcal{B}$ are given by (3.12), and that a, b and ϕ, ψ satisfy (3.13) and (3.14) respectively. Suppose that $h \in W(Q_T)$ satisfies*

$$\epsilon K_0\left(\|h\|_{W(Q_T)}\right) < 1, \tag{3.24}$$

where K_0 is given by (3.17).

For all $u^0 \in H^2(\Omega)$ and $f \in C([0,T]; H^1(Q_T)) \cap H^1(0,T; L^2(\Omega))$ such that

$$\left[\mathcal{A}_{ij}^{\alpha\beta}(t,x,h)\partial_\beta u_j^0 - \mathcal{B}_{ij}^{\alpha}(t,x,h)u_j^0 + f_i^\alpha(t=0)\right]\cdot\nu = 0 \quad \text{on } \partial\Omega, \quad i = 1,\ldots,m \tag{3.25}$$

there exists a unique weak solution $u \in W(Q_T)$ to the linearised system

$$\begin{aligned} \partial_t u_i - \partial_\alpha \left[\mathcal{A}_{ij}^{\alpha\beta}(t,x,h)\partial_\beta u_j - \mathcal{B}_{ij}^{\alpha}(t,x,h)u_j + f_i^\alpha\right] &= 0 \quad \text{in } \mathcal{D}'(\Omega), \\ \left[\mathcal{A}_{ij}^{\alpha\beta}(t,x,h)\partial_\beta u_j - \mathcal{B}_{ij}^{\alpha}(t,x,h)u_j + f_i^\alpha\right]\nu_\alpha &= 0 \quad \text{on } \partial\Omega, \quad i = 1,\dots,m \\ u(0,x) &= u^0 \quad \text{in } \Omega. \end{aligned} \tag{3.26}$$

Furthermore, the solution map

$$S : \big(h, u^0, f\big) \to u, \text{ where } u \text{ is the solution of (3.26)}, \tag{3.27}$$

satisfies

$$\begin{aligned} &\big\|S(h,u^0,f)\big\|_{W(Q_T)}\left[1 - \epsilon K_0(\|h\|_{W(Q_T)})\right] \\ &\le \frac{1}{2}C_{T/\infty}\Big(\|u^0\|_{H^2(\Omega)} + \|f\|_{C([0,T];H^1(Q_T))\cap H^1(0,T;L^2(\Omega))}\Big), \end{aligned}$$

*where $C_{T/\infty} > 0$ is given by (3.20) and does not depend on T if **(H)** holds.*

The proof of Lemma 3.2 is in section 3.1.4. This first result has an immediate corollary.

Corollary 3.3. *For any u^0 and h in $W(Q_T)$, suppose that*

$$\left[\mathcal{A}_{ij}^{\alpha\beta}(t,x,h)\partial_\beta u_j^0 - \mathcal{B}_{ij}^{\alpha}(t,x,h)u_j^0\right]\cdot\nu = 0 \quad \text{on } \partial\Omega,$$

and

$$\epsilon \le \frac{1}{2 + 2K_0\left(C_{T/\infty}\|u^0\|_{H^2(\Omega)}\right)}, \qquad \|h\|_{W(Q_T)} \le Y_0,$$

where K_0, $C_{T/\infty}$, and Y_0 are defined in (3.17), (3.20), and (3.23) respectively. Then

$$\big\|S(h,u^0,0)\big\|_{W(Q_T)} < Y_0.$$

Proof. Since K_0 is a non decreasing function, we obtain

$$\epsilon K_0\left(\|h\|_{W(Q_T)}\right) \le \frac{K_0\left(Y_0\right)}{2 + 2K_0\left(Y_0\right)} < \frac{1}{2},$$

hence (3.24) is satisfied. Applying Lemma 3.2 with $f = 0$, we obtain the announced estimate. □

In a second step, we establish a contraction property.

Lemma 3.4. *Given $\epsilon > 0$, $u^0 \in H^2(\Omega)$, and $h, \tilde{h} \in W(Q_T)$, suppose that on $\partial\Omega$*

$$\left[\mathcal{A}_{ij}^{\alpha\beta}(t,x,h)\partial_\beta u_j^0 - \mathcal{B}_{ij}^{\alpha}(t,x,h)u_j^0\right]\cdot\nu = 0, \qquad \left[\mathcal{A}_{ij}^{\alpha\beta}(t,x,\tilde{h})\partial_\beta u_j^0 - \mathcal{B}_{ij}^{\alpha}(t,x,\tilde{h})u_j^0\right]\cdot\nu = 0.$$

Suppose also that

$$\epsilon \le \frac{1}{2[1+K_0(Y_0)]}, \qquad \max(\|h\|_{W(Q_T)}, \|\tilde{h}\|_{W(Q_T)}) \le Y_0,$$

where K_0*,* $C_{T/\infty}$*, and* Y_0 *are defined in* (3.17)*,* (3.20)*, and* (3.23) *respectively. Then we have*

$$\left\|S(h,u^0,0) - S(\tilde{h},u^0,0)\right\|_{W(Q_T)} \le \epsilon K_1(Y_0)\|h-\tilde{h}\|_{W(Q_T)},$$

with K_1 *given by* (3.18)*.*

Proof. Write $u = S(h,u^0,0)$ and $\tilde{u} = S(\tilde{h},u^0,0)$. We have

$$u - \tilde{u} = \epsilon S\left(h, 0, g\right)$$

where

$$g_i^\alpha = a_{ij}^{\alpha\beta}(t,x)\left[\phi_{ij}^{\alpha\beta}(h) - \phi_{ij}^{\alpha\beta}(\tilde{h})\right]\partial_\beta \tilde{u}_j + b_{ij}^\alpha(t,x)\left[\psi_{ij}^{\alpha\beta}(h) - \psi_{ij}^{\alpha\beta}(\tilde{h})\right]\tilde{u}_j. \tag{3.28}$$

Noting that

$$\left|\phi_{ij}^{\alpha\beta}(h) - \phi_{ij}^{\alpha\beta}(\tilde{h})\right| \le L_1(Y_0)\left|h-\tilde{h}\right|,$$

we find

$$\max_{[0,T]} \|g\|_{L^2(\Omega)} \le ML_1(Y_0)\|h-\tilde{h}\|_{W(Q_T)}\,\|\tilde{u}\|_{W(Q_T)} \le ML_1(Y_0)M_0\|h-\tilde{h}\|_{W(Q_T)}.$$

Similarly, we can estimate the gradient as follows

$$\begin{aligned}|\nabla g| \le M\left(L_1(Y_0)\left|h-\tilde{h}\right| + L_2(Y_0)\left|h-\tilde{h}\right|\,|\nabla h| + L_1(Y_0)\left|\nabla h - \nabla\tilde{h}\right|\right)\left(|\nabla\tilde{u}| + |\tilde{u}|\right)\\ + ML_1(Y_0)\left|h-\tilde{h}\right|\left(\left|\nabla^2\tilde{u}\right| + |\nabla\tilde{u}|\right).\end{aligned}$$

Therefore

$$\begin{aligned}\|\nabla g\|_{L^2(\Omega)} \le ML_1(Y_0)\left[2\|h-\tilde{h}\|_{L^\infty(Q_T)}\|\tilde{u}\|_{H^2(\Omega)} + \|h-\tilde{h}\|_{L^4(\Omega)}\left(\|\nabla\tilde{u}\|_{L^4(\Omega)} + \|\tilde{u}\|_{L^4(\Omega)}\right)\right]\\ + ML_2(Y_0)\|h-\tilde{h}\|_{L^\infty(Q_T)}\,\|\nabla h\|_{L^4(\Omega)}\left(\|\nabla\tilde{u}\|_{L^4(\Omega)} + \|\tilde{u}\|_{L^4(\Omega)}\right).\end{aligned}$$

Thanks to the Ladyzhenskaya (or Gagliardo–Nirenberg) inequality, we obtain

$$\max_{[0,T]} \|\nabla g\|_{L^2(\Omega)} \le C_S M\left(L_1(Y_0)Y_0 + L_2(Y_0)Y_0^2\right)\|h-\tilde{h}\|_{W(Q_T)},$$

where C_S^1 is a product of Sobolev embedding constants, depending on Ω and d, namely

$$C_S^1 = \max\left(1, C\left(H^2(\Omega)\hookrightarrow L^\infty(\Omega)\right)^3, C\left(H^2(\Omega)\hookrightarrow W^{1,4}(\Omega)\right)^3\right). \tag{3.29}$$

We now turn to the time derivative

$$|\partial_t g| \leq M \left(L_1(Y_0)\big|h-\tilde{h}\big| + L_2(Y_0)\big|h-\tilde{h}\big|\,|\partial_t h| + L_1(Y_0)\big|\partial_t h - \partial_t \tilde{h}\big|\right)(|\nabla \tilde{u}| + |\tilde{u}|) \\ + M L_1(Y_0)\big|h-\tilde{h}\big|\,(|\nabla \partial_t \tilde{u}| + |\partial_t \tilde{u}|)\,.$$

Thus, using that $\partial_t h, \partial_t \tilde{h} \in L^4(Q_T)$ and $\partial_t \nabla \tilde{u} \in L^2(Q_T)$, we have

$$\|\partial_t g\|_{L^2(Q_T)} \leq C_S^2 M \left(L_1(Y_0)Y_0 + L_2(Y_0)Y_0^2\right) \left\|h-\tilde{h}\right\|_{W(Q_T)},$$

where C_S^2 is also a product of Sobolev embedding constants, depending on Ω and d, namely

$$C_S^2 = \max\left(C\left(H^1(\Omega) \hookrightarrow L^4(\Omega)\right)^2, 1\right). \tag{3.30}$$

Finally, we apply Lemma 3.2 to obtain

$$\|u-\tilde{u}\|_{W(Q_T)} \leq \epsilon Y_0 C_S M \left[L_1(Y_0)Y_0 + L_2(Y_0)Y_0^2\right] \|h-\tilde{h}\|_{W(Q_T)},$$

with

$$C_S = C_S^1 + C_S^2. \tag{3.31}$$

□

We now turn to the proof of Proposition 3.1.

Proof of Proposition 3.1. Recall that

$$\epsilon_0 : R \to \min\left(\frac{1}{2+2K_0(R)}, \frac{1}{1+K_1(R)}\right).$$

where K_0 and K_1 are defined in (3.17) and (3.18), respectively.

Given $u^0 \in W(Q_T)$ we introduce the sequence v_n given by $v_0 = u^0$ and, for all $n \geq 0$,

$$v_{n+1} = S(v_n, u^0, 0),$$

where S is the solution map defined in (3.27). Note that the compatibility condition (3.22) is satisfied at every step. Corollary 3.3 shows that $\|v_n\|_{W(Q_T)} \leq Y_0$ for each n. Furthermore, thanks to Lemma 3.4,

$$\|v_{n+2} - v_{n+1}\|_{W(Q_T)} \leq \epsilon_0 K_1(Y_0)\|v_{n+1} - v_n\|_{W(Q_T)} \leq \frac{K_1(Y_0)}{1+K_1(Y_0)}\|v_{n+1} - v_n\|_{W(Q_T)}.$$

The sequence thus converges to a solution of (3.9), thanks to the contraction mapping theorem. □

3.1.3 Stability estimate

Our purpose is to establish a stability result under perturbations, therefore we consider a second problem with $\mathcal{A}$ and $\mathcal{B}$ replaced by

$$\begin{aligned} \widetilde{\mathcal{A}}_{ij}^{\alpha\beta}(t,x,u) &= D_i^{\alpha\beta}(t,x) + \epsilon\tilde{a}_{ij}^{\alpha\beta}(t,x)\tilde{\phi}_{ij}^{\alpha\beta}(u), \\ \widetilde{\mathcal{B}}_{ij}^{\alpha}(t,x,u) &= F_i^{\alpha}(t,x) + \epsilon\tilde{b}_{ij}^{\alpha}(t,x)\tilde{\psi}_{ij}^{\alpha}(u), \end{aligned} \tag{3.32}$$

where $\tilde{a}$, $\tilde{b}$, $\tilde{\phi}$ and $\tilde{\psi}$ satisfy hypothesis (3.13), (3.14) and, without loss of generality,

$$\left\|(\tilde{\phi},\tilde{\psi})\right\|_{C^i(\overline{B_R(0)})} \leq L_i(R) \quad \text{for all} \quad 0 \leq R, \quad i = 0,1,2.$$

for L_i defined in (3.16). Our main result is as follows.

Theorem 3.5. *Given $u^0, \tilde{u}^0 \in H^2(\Omega)$ compactly supported in Ω, write*

$$Y_1 = C_{T/\infty} \max\left(\|u^0\|_{H^2(\Omega)}, \|\tilde{u}^0\|_{H^2(\Omega)}\right),$$

and assume $\epsilon < \epsilon_0(Y_1)$ so that Proposition 3.1 applies for both sets of parameters. Let $u \in W(Q_T)$ be the solution of (3.9) and $\tilde{u} \in W(Q_T)$ be the solution of (3.9) with $\mathcal{A}$, $\mathcal{B}$ and u^0 are replaced by $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ and $\tilde{u}^0$, respectively. Then the following stability estimate holds:

$$\begin{aligned} \|\tilde{u} - u\|_{W(Q_T)} \leq{}& \Gamma_1 \|\tilde{u}^0 - u^0\|_{H^2(\Omega)} \\ &+ \epsilon\Gamma_2 \left(\|(\tilde{a},\tilde{b}) - (a,b)\|_{C^1([0,\infty)\times\mathbb{R}^d)} + \|(\tilde{\phi},\tilde{\psi}) - (\phi,\psi)\|_{C^1\left(\overline{B_{Y_1}(0)}\right)}\right), \end{aligned} \tag{3.33}$$

where $\Gamma_1 = (1 + K_1(Y_1))C_{T/\infty}$, $\Gamma_2 = (1 + K_1(Y_1))K_2(Y_1)$ and K_1, K_2 are non decreasing functions given by (3.18), (3.19) respectively. They depend on Ω, M, λ, L_0, L_1 and L_2 and $C_{T/\infty}$ only.

Theorem 3.5 implies, for example, that we can control the differences between the solutions of the models in Examples 1 and 2, by considering the differences in their respective diffusion and drift matrices, which appear at order ϵ. Similarly, we can also use this result to predict the error we will make by approximating model (3.5) in Example 2 as the gradient flow in Example 3. Since the differences between models appear at order ϵ^2 in this case, provided the initial data are equal, the error will be bounded and of order ϵ^2 for all times (see Section 3.2).

Remark 3.2. In Proposition 3.1 the compatibility condition (3.22) appears, which is automatically satisfied by compactly supported initial data as we have assumed in Theorem 3.5. However, Theorem 3.5 also holds (with the same proof) provided that u^0 and $\tilde{u}^0$

satisfy the following four conditions:

$$\left[\mathcal{A}_{ij}^{\alpha\beta}(t,x,u^0)\partial_\beta u_j^0 - \mathcal{B}_{ij}^{\alpha}(t,x,u^0)u_j^0\right]\cdot\nu = 0 \quad \text{on } \partial\Omega, \quad i=1,\dots,m,$$
$$\left[\mathcal{A}_{ij}^{\alpha\beta}(t,x,\tilde u^0)\partial_\beta \tilde u_j^0 - \mathcal{B}_{ij}^{\alpha}(t,x,\tilde u^0)\tilde u_j^0\right]\cdot\nu = 0 \quad \text{on } \partial\Omega, \quad i=1,\dots,m,$$
$$\left[\widetilde{\mathcal{A}}_{ij}^{\alpha\beta}(t,x,u^0)\partial_\beta u_j^0 - \widetilde{\mathcal{B}}_{ij}^{\alpha}(t,x,u^0)u_j^0\right]\cdot\nu = 0 \quad \text{on } \partial\Omega, \quad i=1,\dots,m,$$
$$\left[\widetilde{\mathcal{A}}_{ij}^{\alpha\beta}(t,x,\tilde u^0)\partial_\beta \tilde u_j^0 - \widetilde{\mathcal{B}}_{ij}^{\alpha}(t,x,\tilde u^0)\tilde u_j^0\right]\cdot\nu = 0 \quad \text{on } \partial\Omega, \quad i=1,\dots,m.$$

We choose to write the result for compactly supported initial data to improve readability.

We now focus on the proof of Theorem 3.5. Consider the linearised system given by

$$\begin{aligned}
\partial_t\tilde u_i - \partial_\alpha\left[\widetilde{\mathcal{A}}_{ij}^{\alpha\beta}(t,x,\tilde h)\partial_\beta\tilde u_j - \widetilde{\mathcal{B}}_{ij}^{\alpha}(t,x,\tilde h)\tilde u_j\right] &= \tilde f_i \quad \text{in} \quad \mathcal{D}'(\Omega),\\
\left[\widetilde{\mathcal{A}}_{ij}^{\alpha\beta}(t,x,\tilde h)\partial_\beta\tilde u_j - \widetilde{\mathcal{B}}_{ij}^{\alpha}(t,x,\tilde h)\tilde u_j\right]\nu_\alpha &= 0 \quad \text{on} \quad \partial\Omega,\\
\tilde u(0,\cdot) &= \tilde u^0 \quad \text{in} \quad \Omega,
\end{aligned} \tag{3.34}$$

Following the notation of Lemma 3.2 (see (3.27)), the solution operator associated to (3.34) is denoted by $\tilde S(\tilde h, \tilde u^0, \tilde f)$.

Proposition 3.6. *Let $h, \tilde h \in W(Q_T)$ be compactly supported in Ω for $t=0$ and write*

$$Y_1 = C_{T/\infty}\max\left(\|\tilde h\|_{W(Q_T)}, \|h\|_{W(Q_T)}\right).$$

Assume $\epsilon < \epsilon_0(Y_1)$, so that the solution operators S and $\tilde S$ corresponding to (3.26) and (3.34) respectively are well defined. For any $u^0, \tilde u^0 \in H^2(\Omega)$ with compact support in Ω there holds

$$\begin{aligned}
&\left\|\tilde S(\tilde h,\tilde u^0,0) - S(h,u^0,0)\right\|_{W(Q_T)}\\
&\qquad \le C_{T/\infty}\|\tilde u^0 - u^0\|_{H^2(\Omega)} + \epsilon K_1(Y_1)\|\tilde h - h\|_{W(Q_T)}\\
&\qquad\quad + \epsilon K_2(Y_1)\left(\|(\tilde a,\tilde b)-(a,b)\|_{C^1([0,\infty)\times\mathbb{R}^d)} + \|(\tilde\phi,\tilde\psi)-(\phi,\psi)\|_{C^1\left(\overline{B_{Y_1}(0)}\right)}\right).
\end{aligned}$$

where K_2 depends on L_0, L_1, Ω, M and $C_{T/\infty}$ and is given by (3.19).

Proof. We write

$$\begin{aligned}
\tilde S(\tilde h,\tilde u^0,0) - S(h,u^0,0) &= \tilde S(\tilde h,\tilde u^0,0) - S(\tilde h,\tilde u^0,0)\\
&\quad + S(\tilde h,\tilde u^0,0) - S(h,\tilde u^0,0) + S(h,\tilde u^0,0) - S(h,u^0,0).
\end{aligned}$$

Thanks to Lemma 3.2 and to the linearity of S with respect to the initial data, we have

$$\left\|S\left(h,\tilde u^0,0\right) - S\left(h,u^0,0\right)\right\|_{W(Q_T)} \le C_{T/\infty}\|u^0-\tilde u^0\|_{H^2(\Omega)}.$$

Note that the compatibility condition is satisfied due to the compact support of u^0, $\tilde u^0$

and $h(t=0)$ in Ω. On the other hand, Lemma 3.4 shows that

$$\left\| S(\tilde{h}, \tilde{u}^0, 0) - S(h, \tilde{u}^0, 0) \right\|_{W(Q_T)} \le \epsilon K_1(Y_1) \|\tilde{h} - h\|_{W(Q_T)}.$$

We write

$$\tilde{S}(\tilde{h}, \tilde{u}^0, 0) - S(\tilde{h}, \tilde{u}^0, 0) = \epsilon \tilde{S}(\tilde{h}, 0, \tilde{g}),$$

where $\tilde{g}$ is given by

$$\begin{aligned}\tilde{g}_i^\alpha &= \epsilon^{-1} \left[\left(\tilde{\mathcal{A}}_{ij}^{\alpha\beta} - \mathcal{A}_{ij}^{\alpha\beta} \right) (t, x, \tilde{h}) \partial_\beta \tilde{u}_j - \left(\tilde{\mathcal{B}}_{ij}^{\alpha} - \mathcal{B}_{ij}^{\alpha} \right) (t, x, \tilde{h}) \tilde{u}^j \right] \\ &= \left(\tilde{a}_{ij}^{\alpha\beta} \tilde{\phi}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta} \phi_{ij}^{\alpha\beta} \right) (t, x, \tilde{h}) \partial_\beta \tilde{u}_j + \left(\tilde{b}_i^{\alpha\beta} \tilde{\psi}_{ij}^{\alpha} - b_{ij}^{\alpha} \psi_{ij}^{\alpha} \right) (t, x, \tilde{h}) \tilde{u}^j,\end{aligned}$$

and $\tilde{u} = S(\tilde{h}, \tilde{u}^0, 0)$. In other words, $\tilde{g}$ is of the form

$$\tilde{g} = \left[(\tilde{a} - a)\tilde{\phi} + a(\tilde{\phi} - \phi) \right] \nabla \tilde{u} + \left[(\tilde{b} - b)\tilde{\psi} + b(\tilde{\psi} - \psi) \right] \tilde{u}.$$

Thus we are in a setting similar to that of the proof of Lemma 3.4. In particular we have

$$\begin{aligned}&(|\nabla g| + |g|) \\ &\le \left(\left\| (\tilde{a}, \tilde{b}) - (a, b) \right\|_{C^1([0,\infty) \times \mathbb{R}^d)} L_0(Y_1) + M \max_{B_{Y_1}(0)} \left| (\tilde{\phi}, \tilde{\psi}) - (\phi, \psi) \right| \right) (|\nabla^2 \tilde{u}| + 2|\nabla \tilde{u}| + |\tilde{u}|) \\ &\quad + \left(\left| (\tilde{a}, \tilde{b}) - (a, b) \right| L_1(Y_1) + M \max_{B_{Y_1}(0)} \left| (\tilde{D\phi}, \tilde{D\psi}) - (D\phi, D\psi) \right| \right) |\nabla \tilde{h}| (|\nabla \tilde{u}| + |\tilde{u}|)\end{aligned}$$

As in the proof of Lemma 3.4, using Gagliardo–Nirenberg's inequality to bound the last term, we find

$$\begin{aligned}(1/C_S) \max_{[0,T]} \|\tilde{g}\|_{H^1(\Omega)} &\le \left\| (\tilde{a}, \tilde{b}) - (a, b) \right\|_{C^1([0,\infty) \times \mathbb{R}^d)} \left[2L_0(Y_1) Y_1 + L_1(Y_1) Y_1^2 \right] \\ &\quad + M \left\| (\tilde{\phi}, \tilde{\psi}) - (\phi, \psi) \right\|_{C^1(\overline{B_{Y_1}(0)})} (2Y_1 + Y_1^2),\end{aligned}$$

where C_S is given by (3.31). Finally, we bound $\partial_t g$ to show that $\tilde{g} \in C([0,T]; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$ in the same way, namely

$$\begin{aligned}(1/C_S) \|\partial_t \tilde{g}\|_{L^2(Q_T)} &\le \left\| (\tilde{a}, \tilde{b}) - (a, b) \right\|_{C^1([0,\infty) \times \mathbb{R}^d)} \left[L_0(Y_1) Y_1 + L_1(Y_1) Y_1^2 \right] \\ &\quad + M \left\| (\tilde{\phi}, \tilde{\psi}) - (\phi, \psi) \right\|_{C^1(\overline{B_{Y_1}(0)})} (Y_1 + Y_1^2).\end{aligned}$$

Because of the compact support of u^0, $\tilde{u}^0$, $\tilde{h}(t=0)$ and $h(t=0)$ in Ω, we can conclude thanks to Lemma 3.2 that

$$\begin{aligned}\left\| \tilde{S}(\tilde{h}, \tilde{u}^0, 0) - S(\tilde{h}, \tilde{u}^0, 0) \right\|_{W(Q_T)} &\le \epsilon K_2(Y_1) \big(\left\| (\tilde{a}, \tilde{b}) - (a, b) \right\|_{C^1([0,\infty) \times \mathbb{R}^d)} \\ &\quad + \left\| (\tilde{\phi}, \tilde{\psi}) - (\phi, \psi) \right\|_{C^1(\overline{B_{Y_1}(0)})} \big).\end{aligned}$$

□

Proof of Theorem 3.5. As in the proof of Proposition 3.1, the sequences $v_{n+1} = S\left(v_n, u^0, 0\right)$ and $\tilde{v}_{n+1} = \tilde{S}\left(\tilde{v}_n, \tilde{u}^0, 0\right)$ for all $n \geq 1$, with $v_0 = u^0$ and $\tilde{v}_0 = \tilde{u}^0$, converge to u and $\tilde{u}$, respectively as $n \to \infty$. Thanks to Proposition 3.6 we have

$$\begin{aligned}\|\tilde{v}_{n+1} - v_{n+1}\|_{W(Q_T)} \leq C_{T/\infty}\|\tilde{u}^0 - u^0\|_{H^2(\Omega)} + \epsilon K_1(Y_1)\|\tilde{v}_n - v_n\|_{W(Q_T)} \\ + \epsilon K_2(Y_1)\left(\|(\tilde{a},\tilde{b}) - (a,b)\|_{C^1([0,\infty)\times\mathbb{R}^d)} + \|(\tilde{\phi},\tilde{\psi}) - (\phi,\psi)\|_{C^1(\overline{B_{Y_1}(0)})}\right).\end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain the required inequality:

$$\begin{aligned}\|\tilde{u} - u\|_{W(Q_T)} \leq [1 + K_1(Y_1)]C_{T/\infty}\|\tilde{u}^0 - u^0\|_{H^2(\Omega)} \\ + \epsilon[1 + K_1(Y_1)]K_2(Y_1)\left(\|(\tilde{a},\tilde{b}) - (a,b)\|_{C^1([0,\infty)\times\mathbb{R}^d)} + \|(\tilde{\phi},\tilde{\psi}) - (\phi,\psi)\|_{C^1(\overline{B_{Y_1}(0)})}\right).\end{aligned}$$

□

3.1.4 Proof of the estimates for the linearised problem

In this section we prove Lemma 3.2. Our approach is classical and the parabolic estimate mostly follows from an elliptic regularity estimate. Yet, for general cross-diffusion systems, it is well known that such elliptic results do not always hold, including for quasilinear systems with analytic dependence on u (see for example [GM13] and [SJ95]). Therefore this result needs to be proved in the case at hand. Some of the more technical arguments are detailed in well known references (for example [Gri11, Tro13] concerning elliptic regularity and [DL93, Eva98, LSU88, LM68] for the parabolic case), so we safely skip a certain number of intermediate steps, and we give the relevant references.

The following lemma provides the key regularity result.

Lemma 3.7. *Given $\omega \in C^1\left(\overline{\Omega}; \mathbb{R}^+\right)$, for any $u^0 \in H^2\left(\Omega; \mathbb{R}^m\right)$ and*

$$g_i \in C\left([0,T]; H^1\left(\Omega; \mathbb{R}^d\right)\right) \cap H^1\left(0,T; L^2\left(\Omega; \mathbb{R}^d\right)\right), \quad i = 1,\ldots,m$$

the weak solution u of

$$\begin{aligned}\omega\partial_t u_i - \partial_\alpha\left[D_i^{\alpha\beta}(x,t)\partial_\beta u_i + F_{ij}^\alpha(x,t)u_j + g_i^\alpha\right] = 0 \quad &\text{in} \quad \Omega, \\ \left[D_i^{\alpha\beta}(x,t)\partial_\beta u_i + F_{ij}^\alpha\left(x,t\right]u_j + g_i^\alpha\right] \cdot \nu^\alpha = 0 \quad &\text{on} \quad \partial\Omega, \\ u_i(0) = u_i^0 \quad &\text{in} \quad \Omega,\end{aligned} \tag{3.35}$$

for $i = 1,\ldots,m$, is unique in the space

$$L^2\left(0,T; H^1\left(\Omega\right)\right) \cap C\left([0,T]; L^2\left(\Omega\right)\right) \text{ with } \partial_t u \in L^2(0,T; (H^1(\Omega))').$$

If the compatibility condition

$$\left[D_i^{\alpha\beta}(x,t)\partial_\beta u_i^0 + F_{ij}^\alpha u_j^0 + g_i^\alpha\right]\cdot\nu^\alpha = 0 \quad \text{on } \partial\Omega, \quad i=1,\dots,m \tag{3.36}$$

holds, then u satisfies

$$\|u\|_{W(Q_T)} \le \frac{1}{2}C_T\left(\left\|u^0\right\|_{H^2(\Omega)} + \|g\|_{C([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))}\right), \tag{3.37}$$

where the constant C_T is given by (3.64) and depends on m, λ, T, the C^1 norms of ω, D, and F, and the domain Ω only.

Furthermore, if $F_{ij}^\alpha = D_i^{\alpha\beta}\partial_\beta V_i$ with $V_i \in C^1\left(\overline{\Omega};\mathbb{R}\right)$, and for each i, D_i and V_i do not depend on time, then

$$\|u\|_{W(Q_T)} \le \frac{1}{2}C_\infty\left(\left\|u^0\right\|_{H^2(\Omega)} + \|g\|_{C([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))}\right), \tag{3.38}$$

where C_∞, given by (3.65), depends on m, λ, the C^1norms of ω, D, F and the domain Ω only. In particular, C_∞ is independent of T.

Proof. Note that no coupling appears in (3.35), therefore the index i can be dropped, as the result relates to equations, and not systems. For the purpose of this proof, it is convenient to modify the formulation of the problem to simplify the computations. We will write $D = A^2$, with $A \in C^1\left(\overline{Q_T};\mathbb{R}^{d\times d}\right)$ symmetric, positive definite and A satisfies

$$\left\|A^{-1}(x,t)\right\|_\infty \le \lambda^{-1/2} \text{ in } Q_T. \tag{3.39}$$

We write $F = AB$, and $g = Af$, so that the evolution problem under consideration can be written under the form

$$\omega\partial_t u - \operatorname{div}\left(A^2\nabla u + ABu + Af\right) = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{3.40}$$

The *a priori* bounds we will use are

$$\|A\|_{L^\infty(Q_T)} + \|\nabla A\|_{L^\infty(Q_T)} \le M_A, \tag{3.41}$$

$$\|B\|_{L^\infty(Q_T)} + \|B\|_{L^\infty(Q_T)} + \|\nabla u_2\|_{L^\infty(Q_T)} \le M_B, \tag{3.42}$$

$$\left\|\omega^{-1}\right\|_{L^\infty(Q_T)} + \|\omega\|_{L^\infty(Q_T)} + \|\nabla\omega\|_{L^\infty(Q_T)} \le M_\omega, \tag{3.43}$$

and

$$\|\partial_t A\|_{C(\overline{Q_T})} + \left\|A^{-1}\partial_t A\right\|_{C(\overline{Q_T})} + \left\|\partial_t A^{-1}\right\|_{C(\overline{Q_T})} + \left\|\partial_t\left(A^{-1}B\right)\right\|_{C(\overline{Q_T})} \le M_T. \tag{3.44}$$

For *a.e.* $t \in [0,T]$, we define $\mathcal{A}(t,u,v) : H^1(\Omega;\mathbb{R}) \times H^1(\Omega;\mathbb{R}) \to \mathbb{R}$ by

$$\mathcal{A}(t,u,v) = \int_\Omega \left(A^2\right)^{\alpha\beta}(t,x)\,\partial_\beta u \partial_\alpha v\,\mathrm{d}x + \int_\Omega (AB)^\alpha (t,x)\, u\partial_\alpha v\,\mathrm{d}x. \tag{3.45}$$

Using the *a priori* bounds (3.41) and (3.42), we find the upper bound

$$\mathcal{A}(t,u,v) \le M_A\left(M_A + M_B\right)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

Furthermore, using (3.39) as well, we have the lower bound

$$\mathcal{A}(t,u,v) \ge \lambda\|u\|^2_{H^1(\Omega)} - M_A M_B\|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)} \ge \frac{1}{2}\lambda\|u\|^2_{H^1(\Omega)} - \frac{1}{2\lambda}M_A^2 M_B^2\|u\|^2_{L^2(\Omega)}. \tag{3.46}$$

We may therefore apply the parabolic version of the Lax–Milgram Theorem of Lions [Bre10,LM68] to deduce that there exists a unique solution of (3.35) $u \in L^2(0,T;H^1(\Omega;\mathbb{R}))\cap C([0,T];L^2(\Omega;\mathbb{R}))$ with $\partial_t u \in L^2(0,T;H^1(\Omega;\mathbb{R})')$.

We now derive an explicit bound. Integrating (3.40) by parts against u we find

$$\partial_t\frac{1}{2}\int_\Omega \omega u^2\,\mathrm{d}x + \int_\Omega A^2(x,t)\nabla u\cdot\nabla u\,\mathrm{d}x + \int_\Omega uAB\cdot\nabla u\,\mathrm{d}x + \int_\Omega Af\cdot\nabla u\,\mathrm{d}x = 0.$$

Thus, using (3.46) and Cauchy–Schwarz

$$\partial_t\left(\frac{1}{2}\left\|\sqrt{\omega}u\right\|^2_{L^2(\Omega)}\right) + \frac{1}{2}\|A\nabla u\|^2_{H^1(\Omega)} \le \|f\|^2_{L^2(\Omega)} + \left\|\omega^{-\frac{1}{2}}B\right\|^2_{L^\infty(\Omega)}\left\|\sqrt{\omega}u\right\|^2_{L^2(\Omega)},$$

which leads to two bounds

$$\|u\|_{C([0,T],L^2(\Omega))} \le M_\omega^{\frac{1}{2}}\left(\exp\left(\sqrt{2}M_\omega M_B T\right)\|f\|_{L^2(Q_T)} + M_\omega^{\frac{1}{2}}\left\|u^0\right\|_{L^2(\Omega)}\right), \tag{3.47}$$

and

$$\sqrt{\lambda/2}\|\nabla u\|_{L^2(0,T;L^2(\Omega))} \le \sqrt{1/2}\|A\nabla u\|_{L^2(0,T;L^2(\Omega))} \le \sqrt{\frac{M_\omega}{2}}\left\|u^0\right\|_{L^2(\Omega)} + \|f\|_{L^2(Q_T)}. \tag{3.48}$$

Note that $\int_\Omega u\,\mathrm{d}x = \int_\Omega u^0\,\mathrm{d}x$ for all times. As a result,

$$\begin{aligned}
\|u\|_{L^2(0,T;H^1(\Omega))} &\le \sqrt{T}\left|\frac{1}{|\Omega|}\int_\Omega u^0\,\mathrm{d}x\right| + \left\|u - \frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}x\right\|_{L^2(0,T;L^2(\Omega))} + \|\nabla u\|_{L^2(0,T;L^2(\Omega))} \\
&\le \sqrt{T}\left|\frac{1}{|\Omega|}\int_\Omega u^0\,\mathrm{d}x\right| + (C_P(\Omega)+1)\lambda^{-1/2}\|A\nabla u\|_{L^2(0,T;L^2(\Omega))} \\
&\le C_1\left(\left\|u^0\right\|_{L^2(\Omega)} + \|f\|_{L^2(Q_T)}\right),
\end{aligned} \tag{3.49}$$

where $C_P(\Omega)$ is the Poincaré–Wirtinger constant, and

$$C_1 = \sqrt{T}\,|\Omega|^{-1/2} + (C_P(\Omega)+1)\sqrt{\frac{M_\omega}{2\lambda}}. \tag{3.50}$$

Let us now focus on higher regularity. We are going to show that

$$u \in C\left([0,T]; H^2(\Omega)\right) \cap H^1\left(0,T; H^1(\Omega)\right).$$

We write

$$\Phi = A\nabla u + Bu + f. \tag{3.51}$$

Thanks to (3.48) and (3.49), we have

$$\|\Phi\|_{L^2(0,T;L^2(\Omega))} \leq C_2 \left(\left\|u^0\right\|_{L^2(\Omega)} + \|f\|_{L^2(Q_T)} \right),$$

with

$$C_2 = 1 + \sqrt{M_\omega} + M_B C_1. \tag{3.52}$$

Next, we are going to test (3.35) against $\eta = \partial_t u - \omega^{-1}\mathrm{div}\,(A\Phi)$. Notice that we have to ensure that η is a valid test function. We just sketch the procedure, namely we consider $\eta_{\tau,h} = \Delta_\tau u - \omega^{-1}\Delta_h^\alpha \left(A^{\alpha\beta}\Phi^\beta\right)$, where the difference quotient time derivative is given by $\Delta_\tau u = \left(u\left(\cdot + \tau\right) - u\left(\cdot\right)\right)\tau^{-1}$ and difference quotient space derivatives in direction i is given by $\Delta_{-h}^\alpha \psi = \left(\psi\left(\cdot + he_\alpha\right) - \psi\left(\cdot\right)\right)h^{-1}$. We have to test (3.35) against $\eta_{\tau,h}$ and subsequently pass to the limit for $\tau, h \to 0$, paying attention to the direction normal to the boundary near $\partial\Omega$. This step is somewhat technical but straightforward and it justifies the following calculations rigorously. To simplify the exposition, we use directly $\partial_t u - \omega^{-1}\mathrm{div}\,(A\Phi)$ as the test function in the following steps, and obtain

$$\int_\Omega \omega\left(\partial_t u\right)^2 \,\mathrm{d}x + \int_\Omega (A\Phi)\cdot\nabla\left(-\omega^{-1}\mathrm{div}\,(A\Phi)\right)\,\mathrm{d}x - 2\int_\Omega \partial_t u \mathrm{div}(A\Phi)\,\mathrm{d}x = 0 \tag{3.53}$$

As $A\Phi\cdot\nu = 0$, we find that

$$\int_\Omega (A\Phi)\cdot\nabla\left(-\omega^{-1}\mathrm{div}\,(A\Phi)\right)\,\mathrm{d}x = \int_\Omega \omega^{-1}\left(\mathrm{div}\,(A\Phi)\right)^2\,\mathrm{d}x. \tag{3.54}$$

Let us now turn to the mixed term. We have

$$\begin{aligned} -2\int_\Omega \partial_t u \mathrm{div}(A\Phi)\,\mathrm{d}x &= 2\int_\Omega \partial_t\left(\left(A^{-1}A\right)\nabla u\right)\cdot(A\Phi)\,\mathrm{d}x && (3.55)\\ &= 2\int_\Omega \left[\partial_t(A\nabla u) + A\partial_t(A^{-1})A\nabla u\right]\cdot\Phi\,\mathrm{d}x && (3.56)\\ &= 2\int_\Omega \left[\partial_t(\Phi) + A\partial_t(A^{-1})\Phi\right]\cdot\Phi\,\mathrm{d}x && (3.57)\\ &\quad - 2\int_\Omega \left[\partial_t(Bu+f) + A\partial_t(A^{-1})(Bu+f)\right]\cdot\Phi\,\mathrm{d}x. \end{aligned}$$

Inserting (3.54) and (3.55) into (3.53) and using Cauchy–Schwarz, we obtain

$$\begin{aligned}\left\|\sqrt{\omega}\partial_t u\right\|_{L^2(\Omega)}^2 + \int_\Omega \omega^{-1}\mathrm{div}\,(A\Phi)^2 \,\mathrm{d}x + \partial_t \left\|\Phi\right\|_{L^2(\Omega)}^2 \\ \le 2M_T M_A \left(\left\|\Phi\right\|_{L^2(\Omega)}^2 + \left\|f\right\|_{L^2(\Omega)}\left\|\Phi\right\|_{L^2(\Omega)}\right) + 2\left\|\partial_t f\right\|_{L^2(\Omega)}\left\|\Phi\right\|_{L^2(\Omega)} \\ + 2M_T M_A M_B \left\|u\right\|_{L^2(\Omega)}\left\|\Phi\right\|_{L^2(\Omega)} + 2M_B M_\omega^{\frac{1}{2}} \left\|\sqrt{\omega}\partial_t u\right\|_{L^2(\Omega)}\left\|\Phi\right\|_{L^2(\Omega)}.\end{aligned} \tag{3.58}$$

Using Young's inequality, we recombine inequality (3.58) to find

$$\begin{aligned}&\frac{1}{2}\left\|\sqrt{\omega}\partial_t u\right\|_{L^2(\Omega)}^2 + \left\|\sqrt{\omega}\mathrm{div}\,(A\Phi)\right\|_{L^2(\Omega)}^2 + \partial_t \left\|\Phi\right\|_{L^2(\Omega)}^2 \\ &\le (2C_3+1)\left\|\Phi\right\|_{L^2(\Omega)}^2 + M_B^2 \left\|u\right\|_{L^2(\Omega)}^2 + \left\|\partial_t f\right\|_{L^2(\Omega)}^2 + \left\|f\right\|_{L^2(\Omega)}\end{aligned}$$

with

$$C_3 = 2M_T M_A (1 + 2M_T M_A) + 2M_B^2 M_\omega.$$

Integrating in time, we find

$$\begin{aligned}\left\|\Phi\right\|_{C([0,T],L^2(\Omega))}^2 + \frac{1}{2}\left\|\sqrt{\omega}\partial_t u\right\|_{L^2(Q_T)}^2 + \left\|\sqrt{\omega}\mathrm{div}\,(A\Phi)\right\|_{L^2(Q_T)}^2 \\ \le C_4\left(\left\|u^0\right\|_{L^2(\Omega)} + \left\|f\right\|_{L^2(Q_T)}\right)^2 \\ + \left\|A\nabla u^0 + Bu^0 + f\,(t=0)\right\|_{L^2(\Omega)}^2 + \left\|\partial_t f\right\|_{L^2(Q_T)}^2 + \left\|f\right\|_{L^2(Q_T)}^2,\end{aligned}$$

with

$$C_4 = (2C_3+1)\,C_2^2 + M_B^2 C_1^2. \tag{3.59}$$

Let us now check that this allows us to define $\partial_t u|_{t=0}$ in an appropriate sense. Since

$$\left\|\nabla u\right\|_{C([0,T],L^2(\Omega))} \le \lambda^{-\frac{1}{2}}\left(\left\|\Phi\right\|_{C([0,T],L^2(\Omega))} + M_B\left\|u\right\|_{C([0,T],L^2(\Omega))} + \left\|f\right\|_{C([0,T],L^2(\Omega))}\right),$$

for any $v \in H^1(\Omega)$, the map

$$t \to \int_\Omega \left[A(x,t)\nabla u\cdot\nabla v + Bu\cdot\nabla v + fu\cdot\nabla v\right]\mathrm{d}x$$

is continuous on $[0,T]$. In other words, we define $\partial_t u|_{t=0} \in \left(H^1(\Omega)\right)'$ as follows

$$\begin{aligned}\int_\Omega \partial_t u|_{t=0}\, v\,\mathrm{d}x &= \lim_{t\downarrow 0}\int_\Omega \left[A(x,t)\nabla u\cdot\nabla v + B(x,t)u.\nabla v + f(x,0)\cdot\nabla v\right]\mathrm{d}x \\ &= \int_\Omega \left[A(x,0)\nabla u^0\cdot\nabla v + B(x,0)u^0.\nabla v + f(x,0)\cdot\nabla v\right]\mathrm{d}x,\end{aligned}$$

provided that the compatibility condition (3.36) holds, that is,

$$\left[A(x,0)\nabla u^0 - B(x,0)u^0 - f(x,0)\right]\cdot\nu = 0.$$

An integration by parts then shows that

$$\int_\Omega \partial_t u|_{t=0}\, v \,\mathrm{d}x = \int_\Omega \operatorname{div}\left[A(x,0)\nabla u^0 + B(x,0)u^0 + f(x,0)\right] v \,\mathrm{d}x,$$

which, in turn, shows that $\partial_t u|_{t=0} \in L^2(\Omega)$ and

$$\| \partial_t u|_{t=0}\|_{L^2(\Omega)} \leq (M_A + M_B)\left(\left\|u^0\right\|_{H^2(\Omega)} + \|f\|_{C([0,T];H^1(\Omega))}\right). \tag{3.60}$$

We now notice that $\partial_t u$ is a weak solution of (3.35), where f is replaced by $\partial_t f + \partial_t A\nabla u + \partial_t Bu$ and u^0 is replaced by $\partial_t u|_{t=0}$. From (3.49) we obtain

$$\|\partial_t f + \partial_t A\nabla u + \partial_t Bu\|_{L^2(Q_T)} \leq \max(M_T C_1, 1)\left(\left\|u^0\right\|_{L^2(\Omega)} + \|f\|_{H^1(0,T;L^2(\Omega))}\right) \tag{3.61}$$

Thus (3.48) becomes

$$\begin{aligned}\sqrt{\frac{\lambda}{2}}\,\|\partial_t \nabla u\|_{L^2((0,T);L^2(\Omega))} &\leq \sqrt{\frac{M_\omega}{2}}\,\| \partial_t u|_{t=0}\|_{L^2(\Omega)} + \|\partial_t f + \partial_t A\nabla u + \partial_t Bu\|_{L^2(Q_T)} \\ &\leq \sqrt{\frac{M_\omega}{2}}(M_A + M_B)\left(\left\|u^0\right\|_{H^2(\Omega)} + \|f\|_{C([0,T];H^1(\Omega))}\right) \\ &\quad + \max(M_T C_1, 1)\left(\left\|u^0\right\|_{L^2(\Omega)} + \|f\|_{H^1(0,T;L^2(\Omega))}\right),\end{aligned}$$

and (3.47) gives

$$\begin{aligned}\|\partial_t u\|_{C([0,T];L^2(\Omega))} &\leq M_\omega^{\frac{1}{2}}\left[\exp\left(\sqrt{2}M_\omega M_B T\right)\|\partial_t f + \partial_t A\nabla u + \partial_t Bu\|_{L^2(Q_T)} + M_\omega^{\frac{1}{2}}\,\| \partial_t u|_{t=0}\|_{L^2(\Omega)}\right] \\ &\leq C_5\left(\left\|u^0\right\|_{H^2(\Omega)} + \|f\|_{C([0,T];H^1(\Omega))} + \|f\|_{H^1(0,T;L^2(\Omega))}\right),\end{aligned}$$

with

$$C_5 = M_\omega\,(M_A + M_B) + M_\omega^{\frac{1}{2}} \max(M_T C_1, 1)\exp\left(\sqrt{2}M_\omega M_B T\right). \tag{3.62}$$

Finally, we observe that the right-hand side of the identity

$$\operatorname{div}(A\nabla u) = \partial_t u - \operatorname{div}(Bu + f),$$

belongs to $C\left([0,T];L^2(\Omega)\right)$, and therefore the left-hand side belongs to the same space. This in turn shows that $u \in H^2(\Omega)$ for any t, in fact $u \in C\left([0,T];H^2(\Omega)\right)$, see, for example, [Plu92], with

$$\|u\|_{C([0,T];H^2(\Omega))} \leq C(\Omega, M_A, \lambda)\,(C_5 + M_B C_1)\left(\left\|u^0\right\|_{H^2(\Omega)} + \|f\|_{C([0,T];H^1(\Omega))} + \|f\|_{H^1(0,T;L^2(\Omega))}\right).$$

Altogether, we have shown

$$\|u\|_{C([0,T];H^2(\Omega))} + \|u\|_{H^1(0,T;H^1(\Omega))} \leq \frac{1}{2}C_T\left(\left\|u^0\right\|_{H^2(\Omega)} + \|f\|_{C([0,T];H^1(\Omega))} + \|f\|_{H^1(0,T;L^2(\Omega))}\right), \tag{3.63}$$

where

$$C_T = 2\left(C(\Omega, M_A, \lambda)\left(C_5 + M_B C_1\right) + \sqrt{\frac{M_\omega}{\lambda}}\left(M_A + M_B\right) + \sqrt{\frac{2}{\lambda}}\max(M_T C_1, 1)\right), \tag{3.64}$$

and C_4 and C_5 are given by (3.59) and (3.62), respectively, as announced.

Let us now turn to the particular case when $B = A\nabla V$, with $V \in C^2\left(\overline{\Omega}\right)$, and A and V are independent of time. We perform the change of unknown $w = u \exp V$ and, thanks to Lemma 3.8, we can study the problem satisfied by w. We have

$$\begin{aligned} \exp\left(-V\right)\partial_t w - \operatorname{div}\left[A\exp\left(-V\right)\nabla w + f\right] &= 0 && \text{in} \quad \Omega, \\ \left[A\exp(-V)\nabla w + f\right]\cdot \nu &= 0 && \text{on} \quad \partial\Omega, \\ w(0) &= u^0\exp(V) && \text{in} \quad \Omega, \end{aligned}$$

that is, the same system as (3.35) above, with $\omega = \exp(-V)$, $M_B = 0$ and $M_T = 0$. In this case,

$$C_2 = 1 + \sqrt{M_\omega}, \quad C_3 = 0, \quad C_4 = C_2^2, \quad C_5 = M_\omega M_A + M_\omega^{\frac{1}{2}},$$

and the constant C_T in (3.64) becomes

$$\tilde{C}' = 2\left(C(\Omega, M_A, \lambda)C_5 + \sqrt{\frac{M_\omega}{\lambda}}M_A + \sqrt{\frac{2}{\lambda}}\right),$$

and it does not depend on T. Thanks to Lemma 3.8, we find that in terms of u the bound (3.63) holds with the following constant

$$C_\infty = \tilde{C}\left[\left(1 + M_V'\right)^2 + M_V''\right]\exp M_V, \tag{3.65}$$

which, again, is independent of T. M_V, M_V' and M_V'' are defined in Lemma 3.8. □

Remark 3.3 (Ellipticity bound for ϵ)**.** Suppose that an *a priori* bound for u on Q_T is known, say $u^* = \sup_{Q_T}|u|$. For any $\xi_i^\alpha \in \mathbb{R}^{d\times m}, \zeta_j \in \mathbb{R}^m$, we have the lower bound

$$\mathcal{A}_{ij}^{\alpha\beta}(t,x,y)\xi_i^\alpha\xi_j^\beta = D_i^{\alpha\beta}(t,x)\xi_i^\alpha\xi_i^\beta + \epsilon a_{ij}^{\alpha\beta}(t,x)\phi_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta \geq (\lambda - \epsilon L_0(u^*)\|a\|_\infty)|\xi|^2, \tag{3.66}$$

where $\|a\|_\infty = \max_{i,j,\alpha,\beta,x}|a_{ij}^{\alpha\beta}(x)|$ and L_0 is given in (3.16). Therefore, choosing

$$\epsilon < \min\left(\frac{\lambda}{1 + \|a\|_\infty L_0(u^*)}, 1\right) \tag{3.67}$$

guarantees coercivity, and this is sufficient to ensure existence and uniqueness of weak solutions of (3.26), and consequently of (3.9) via Lax–Milgram lemma. We use relation (3.67) to derive an *a priori* upper bound for ϵ in a specific case, see Lemma 3.11.

Lemma 3.8. *Given $V \in C^2\left(\overline{\Omega}\right)$, the map $u \to u\exp(V)$ is a bi-continuous isomorphism in $C\left([0,T];H^2(\Omega)\right) \cap H^1\left(0,T;H^1(\Omega)\right)$. The following inequalities hold*

$$\|u\exp(V)\|_{W(0,T,\Omega)} \leq \left[\left(1+M_V'\right)^2 + M_V''\right] \exp M_V \|u\|_{W(0,T,\Omega)},$$
$$\|u\|_{W(0,T,\Omega)} \leq \left[\left(1+M_V'\right)^2 + M_V''\right] \exp M_V \|u\exp(V)\|_{W(0,T,\Omega)},$$

where $M_V = \sup_\Omega |V|$, $M_V' = \sup_\Omega |\nabla V|$ and $M_V'' = \sup_\Omega \left|\nabla^2 V\right|$.

Proof. Note that it is sufficient to prove one inequality, as replacing V by $-V$ changes the map to its inverse. Indeed, we have

$$\|u\exp(V)\|_{L^2(\Omega)} \leq \exp M_V \|u\|_{L^2(\Omega)},$$
$$\|u\exp(V)\|_{H^1(\Omega)} \leq \left(1+M_V'\right)\exp M_V \|u\|_{H^1(\Omega)},$$
$$\|u\exp(V)\|_{H^2(\Omega)} \leq \left[\left(1+M_V'\right)^2 + M_V''\right]\exp M_V \|u\|_{H^2(\Omega)}.$$

□

The second step in the proof of Lemma 3.2 concerns the regularity of the forcing term f, which coincides with the regularity of the cross-diffusion term, provided that h and u are in $W(Q_T)$.

Lemma 3.9. *The map*

$$\begin{aligned} P : Q_T \times C^\infty\left(Q_T;\mathbb{R}^m\right)^2 &\to C^2\left(Q_T : \mathbb{R}^{m\times d}\right) \\ (t,x,h,u) &\to a_{ij}^{\alpha\beta}(t,x)\phi_{ij}^{\alpha\beta}(h)\,\partial_\beta u_j + b_{ij}^{\alpha}(t,x)\psi_{ij}^{\alpha}(h)\,u_j, \end{aligned} \tag{3.68}$$

has the following property

$$P\left(Q_T \times W(Q_T) \times W(Q_T)\right) \subset C\left([0,T];H^1(\Omega;\mathbb{R}^m)\right) \cap H^1\left(0,T;L^2(\Omega;\mathbb{R}^m)\right).$$

Furthermore, there holds

$$\sup_{[0,T]}\left(\|\nabla P_i(t,x,h,u)\|_{L^2(\Omega)} + \|P_i(t,x,h,u)\|_{L^2(\Omega)}\right) + \|\partial_t P_i(t,x,h,u)\|_{L^2(Q_T)}$$
$$\leq K_0\left(\|h\|_{W(Q_T)}\right)\|u\|_{W(Q_T)},$$

where K_0 is given by (3.17).

Proof. Note that $L^\infty(Q_T) \subset C\left([0,T];H^2(\Omega;\mathbb{R}^m)\right)$. Therefore

$$\sup_{Q_T}|h| \leq C_S^\infty \|h\|_{W(Q_T)},$$

where

$$C_S^\infty = C(H^2(\Omega) \hookrightarrow L^\infty(\Omega)) \tag{3.69}$$

is the Sobolev constant associated to the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$, and depends on Ω and d. We compute the following bounds for P

$$|P_i(t,x,h,u)| \leq \sup_{\Omega\times[0,\infty)} (|a|,|b|) L_0 \left(\sup_{Q_T} |h| \right) (|\nabla u| + |u|) ,$$
$$\|P_i(t,x,h,u)\|_{L^2(\Omega)} \leq M \|u\|_{H^1(\Omega)} \leq M L_0 \left(C_S^\infty \|h\|_{W(Q_T)} \right) \|u\|_{W(Q_T)} ,$$

for all $t \in [0,T]$. Similarly, for the spatial derivatives of P we have

$$\begin{aligned} |\partial_\alpha P_i(t,x,h,u)| \leq M &\left(L_0 \left(C_S^\infty \|h\|_{W(Q_T)} \right) + L_1 \left(C_S^\infty \|h\|_{W(Q_T)} \right) |\nabla h| \right) (|\nabla u| + |u|) \\ &+ M L_0 \left(C_S^\infty \|h\|_{W(Q_T)} \right) \left(|\nabla^2 u| + |\nabla u| \right) . \end{aligned}$$

Therefore, using Cauchy–Schwarz and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ we find

$$\begin{aligned} \|\nabla P_i(t,x,h,u)\|_{L^2(\Omega)} &\leq 2ML_0 \left(C_S^\infty \|h\|_{W(Q_T)} \right) \|u\|_{W(Q_T)} \\ &\quad + ML_1 \left(C_S^\infty \|h\|_{W(Q_T)} \right) \|\nabla h\|_{L^4} \left(\|\nabla u\|_{L^4} + \|u\|_{L^4} \right) \\ &\leq M \Bigg[2L_0 \left(C_S^\infty \|h\|_{W(Q_T)} \right) \\ &\qquad + C_S^2 L_1 \left(C_S^\infty \|h\|_{W(Q_T)} \right) \|h\|_{W(Q_T)} \Bigg] \|u\|_{W(Q_T)} , \end{aligned}$$

where C_S^2 is defined by (3.30). This shows that $P_i(t,x,h,u) \in C\left([0,T];H^1(\Omega)\right)$. Finally, for the time derivative we obtain

$$\begin{aligned} |\partial_t P_i(t,x,h,u)| \leq M &\left[L_0 \left(C_S^\infty \|h\|_{W(Q_T)} \right) + L_1 \left(C_S^\infty \|h\|_{W(Q_T)} \right) |\partial_t h| \right] (|\nabla u| + |u|) \\ &+ M L_0 \left(C_S^\infty \|h\|_{W(Q_T)} \right) (|\nabla \partial_t u| + |\partial_t u|) , \end{aligned}$$

and

$$\begin{aligned} &\|\partial_t P_i(t,x,h,u)\|_{L^2(Q_T)} \\ &\leq M \left[2L_0 \left(C_S^\infty \|h\|_{W(Q_T)} \right) + L_1 \left(C_S^\infty \|h\|_{W(Q_T)} \right) \|\partial_t h\|_{L^2(Q_T)} \right] \|u\|_{W(Q_T)} . \end{aligned}$$

Altogether we have shown that

$$\begin{aligned} \sup_{[0,T]} \left(\|\nabla P_i(t,x,h,u)\|_{L^2(\Omega)} + \|P_i(t,x,h,u)\|_{L^2(\Omega)} \right) + \|\partial_t P_i(t,x,h,u)\|_{L^2(Q_T)} \\ \leq K_0 \left(\|h\|_{W(Q_T)} \right) \|u\|_{W(Q_T)} , \end{aligned}$$

where K_0 is defined by (3.17), as announced. □

Proof of Lemma 3.2. We write

$$\partial_\alpha \left[\mathcal{A}_{ij}^{\alpha\beta}(t,x,h)\partial_\beta u_j - \mathcal{B}_{ij}^{\alpha}(t,x,h)u_j + f_i^\alpha\right] = \partial_\alpha \left[D_i^{\alpha\beta}(t,x)\partial_\beta u_j - F_i^{\alpha}(t,x)u_j + g_i^\alpha\right],$$

with $g_i^\alpha = f_i^\alpha + \epsilon P_i^\alpha(t,x,h,u)$, and P given by (3.68). Lemma 3.7 shows that

$$\|u\|_{W(Q_T)} \leq \frac{1}{2}C_T\left(\left\|u^0\right\|_{H^2(\Omega)} + \|g\|_{C([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))}\right),$$

and

$$\begin{aligned}\|g\|_{C([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))} \leq{}& \|f\|_{C([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))}\\ &+ \epsilon\,\|P(t,x,h,u)\|_{C([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))}.\end{aligned}$$

Thanks to Lemma 3.9, there holds

$$\sup_{[0,T]}\left(\|\nabla P_i\,(t,x,h,u)\|_{L^2(\Omega)}\right) + \|\partial_t P_i\,(t,x,h,u)\|_{L^2(Q_T)} \leq K_0\left(\|h\|_{W(Q_T)}\right)\|u\|_{W(Q_T)},$$

and therefore

$$\|u\|_{W(Q_T)}\left[1 - K_0\big(\|h\|_{W(Q_T)}\big)\right] \leq \frac{1}{2}C_T\left(\left\|u^0\right\|_{H^2(\Omega)} + \|f\|_{C([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))}\right),$$

which is our thesis since, thanks to the Fredholm Alternative, boundedness implies existence and uniqueness. If **(H)** holds, the proof is analogous with C_T replaced by C_∞. □

3.2 Numerical simulations

We are going to study the equations described in (2) and (1) and we will verify the stability results numerically.

In this section we choose the constants in (3.5) as follows:

$$a_i = \frac{2\pi}{d}(d-1),\ b_i = \frac{2\pi}{d}\frac{[(d-1)D_i + dD_j]}{D_i + D_j},\ c_i = \frac{2\pi}{d}\frac{D_i}{D_i+D_j},\ i = 1,2,\ d = 2,3. \quad (3.70)$$

3.2.1 Preliminary computations

Before focusing on the numerical simulations, we will check that models (2) and (1) are related through equation (3.8).

Fact 3.10. *System (3.6) can be written in the form of (3.8).*

Proof. Let u_1 and u_2 be are probability densities (non-negative, with mass equal to 1).

Notice that

$$\begin{aligned}
a + c_i &= \frac{2\pi}{d}\left((d-1) + \frac{D_i}{D_i + D_j}\right)\\
&= \frac{2\pi}{d}\frac{(d-1)(D_i + D_j) + D_i}{D_i + D_j}\\
&= \frac{2\pi}{d}\frac{(d-1)D_J + dD_i}{D_i + D_j}\\
&= b_j.
\end{aligned}$$

Using the expression of (3.7a), we obtain

$$\begin{aligned}
\partial_{u_1} E_\epsilon &= \log u_1 + 1 + V_1 + a\epsilon u_1 + a\epsilon u_2,\\
\partial_{u_2} E_\epsilon &= \log u_2 + 1 + V_2 + a\epsilon u_2 + a\epsilon u_1,\\
\nabla(\partial_{u_1} E_\epsilon) &= \left(\frac{1}{u_1} + a\epsilon\right)\nabla u_1 + a\epsilon\nabla u_2 + \nabla V_1,\\
\nabla(\partial_{u_2} E_\epsilon) &= \left(\frac{1}{u_2} + a\epsilon\right)\nabla u_2 + a\epsilon\nabla u_1 + \nabla V_2.
\end{aligned}$$

We now compute each term of the flux:

$$\begin{aligned}
M_\epsilon^{11}\nabla(\partial_{u_1} E_\epsilon) &= D_1(1 - c_1\epsilon u_2)u_1\left(\left(\frac{1}{u_1} + a\epsilon\right)\nabla u_1 + a\epsilon\nabla u_2 + \nabla V_1\right)\\
&= D_1[(1 - c_1\epsilon u_2 + a\epsilon(1 - c_1\epsilon u_2)u_1)\,\nabla u_1\\
&\qquad + a\epsilon(1 - c_1\epsilon u_2)u_1\nabla u_2 + (1 - c_1\epsilon u_2)u_1\nabla V_1]\\
&= D_1[\big(1 - c_1\epsilon u_2 + a\epsilon u_1 - c_1\epsilon^2 u_2 u_1\big)\,\nabla u_1\\
&\qquad + (a\epsilon u_1 - ac_1\epsilon^2 u_2 u_1)\nabla u_2 + (1 - c_1\epsilon u_2)u_1\nabla V_1]\\
&= D_1\left[(1 - c_1\epsilon u_2 + a\epsilon u_1)\,\nabla u_1 + a\epsilon u_1\nabla u_2 + (1 - c_1\epsilon u_2)u_1\nabla V_1\right]\\
&\quad - D_1 ac_1\epsilon^2 u_2 u_1\nabla u_1 - D_1 ac_1\epsilon^2 u_2 u_1\nabla u_2,
\end{aligned}$$

$$\begin{aligned}
M_\epsilon^{12}\nabla(\partial_{u_2} E_\epsilon) &= D_1 c_2\epsilon u_1 u_2\left(\left(\frac{1}{u_2} + a\epsilon\right)\nabla u_2 + a\epsilon\nabla u_1 + \nabla V_2\right)\\
&= D_1\left(\big(c_2\epsilon u_1 + ac_2\epsilon^2 u_1 u_2\big)\,\nabla u_2 + ac_2\epsilon^2 u_1 u_2\nabla u_1 + c_2\epsilon u_1 u_2\nabla V_2\right)\\
&= D_1\left(c_2\epsilon u_1\nabla u_2 + c_2\epsilon u_1 u_2\nabla V_2\right)\\
&\quad + D_1 ac_2\epsilon^2 u_1 u_2\nabla u_2 + D_1 ac_2\epsilon^2 u_1 u_2\nabla u_1.
\end{aligned}$$

Summing up the fluxes we have:

$$\begin{aligned}
M_\epsilon^{11}\nabla(\partial_{u_1}E_\epsilon)+M_\epsilon^{12}\nabla(\partial_{u_2}E_\epsilon) &= D_1[(1-c_1\epsilon u_2+a\epsilon u_1)\nabla u_1 \\
&\qquad +a\epsilon u_1\nabla u_2+(1-c_1\epsilon u_2)u_1\nabla V_1] \\
&\quad +D_1\left(c_2\epsilon u_1\nabla u_2+c_2\epsilon u_1 u_2\nabla V_2\right) \\
&\quad -D_1ac_1\epsilon^2u_2u_1\nabla u_1-D_1ac_1\epsilon^2u_2u_1\nabla u_2 \\
&\quad +D_1ac_2\epsilon^2u_1u_2\nabla u_2+D_1ac_2\epsilon^2u_1u_2\nabla u_1 \\
(\bigcirc) \qquad &= D_1(1-c_1\epsilon u_2+a\epsilon u_1)\nabla u_1 \\
(\triangle) \qquad &\quad +D_1(a+c_2)\epsilon u_1\nabla u_2 \\
(\square) \qquad &\quad +D_1c_2\epsilon u_1u_2\nabla V_2+D_1(1-c_1\epsilon u_2)u_1\nabla V_1 \\
&\quad -D_1ac_1\epsilon^2u_2u_1\nabla u_1-D_1ac_1\epsilon^2u_2u_1\nabla u_2 \\
&\quad +D_1ac_2\epsilon^2u_1u_2\nabla u_2+D_1ac_2\epsilon^2u_1u_2\nabla u_1.
\end{aligned}$$

The last two addenda above will determine the first component of G. We also have:

$$\begin{aligned}
D^{11}\nabla u_1+D^{12}\nabla u_2+F^{11}u_1+F^{12}u_2 &= D_1(1+\epsilon a u_1-\epsilon c_1 u_2)\nabla u_1 && (\bigcirc) \\
&\quad +\epsilon D_1 b_1 u_1\nabla u_2 && (\triangle) \\
&\quad +D_1u_1\nabla V_1+\epsilon D_1\nabla(c_2V_2-c_1V_1)u_1u_2, && (\square)
\end{aligned}$$

and we see that the terms denoted by $(\bigcirc)$, $(\triangle)$ and $(\square)$ coincide. We proceed analogously for the second component. Now we can rewrite the system as

$$\partial_t\begin{pmatrix}u_1\\u_2\end{pmatrix}=\nabla\cdot\left[M_\epsilon\nabla\begin{pmatrix}\partial_{u_1}E_\epsilon\\\partial_{u_2}E_\epsilon\end{pmatrix}-\epsilon^2G\right],$$

where $G(u_1,u_2)$ is the vector

$$\begin{aligned}
\epsilon^2G =&a\epsilon^2(D_2-D_1)u_1u_2(\nabla u_1+\nabla u_2)\begin{pmatrix}c_1\\-c_2\end{pmatrix} \\
=&\begin{pmatrix}c_1a\epsilon^2(D_2-D_1)u_1u_2(\nabla u_1+\nabla u_2)\\-c_2a\epsilon^2(D_2-D_1)u_1u_2(\nabla u_1+\nabla u_2)\end{pmatrix}.
\end{aligned}$$

Hence we have verified that model 2 is a gradient flow and that model 2 is an asymptotic gradient flow. We have also computed the difference between the two models explicitly. □

The expression of ϵ_0 found in the proof of Lemma 3.2 for a general system can be improved for the specific systems at hand. In this section we will use a new bound, which we denote by ϵ^*, that ensures ellipticity of the diffusion matrix (3.5a). This is in fact the practical bound required to obtain meaningful numerical results, and it is in general less restrictive than ϵ_0.

Fact 3.11 (Sharp ellipticity bound)**.** *We present a necessary to ensure coercivity of the diffusive term.*

Suppose that the two components of the solution of (3.1) *coincide at at least one point, i.e.*$u_1(\bar{t},\bar{x}) = u_2(\bar{t},\bar{x}) = u^* > 0$, *and let*

$$\theta = (D_1 - D_2)^2/4D_1D_2 \geq 0.$$

Then the symmetrisation of the diffusion matrix (3.5a) *is non-degenerate for* $u_i = u^*$ *if and only if*

$$\epsilon \leq \epsilon^* = \frac{1+\sqrt{9+4\theta}}{2+\theta}(\pi u^*)^{-1}.$$

Proof. Recall that the diffusion matrix of Example 2 is

$$\mathcal{A}(u) = \begin{pmatrix} D_1(1+\epsilon a_1 u_1 - \epsilon c_1 u_2) & \epsilon D_1 b_1 u_1 \\ \epsilon D_2 b_2 u_2 & D_2(1+\epsilon a_2 u_2 - \epsilon c_2 u_1) \end{pmatrix}.$$

From the numerical point of view, a realistic bound can be obtained imposing that the symmetrised diffusion matrix does not degenerate.

Concerning the trace of $\mathcal{A}$, we have

$$\mathrm{tr}(\mathcal{A}) = D_1 + D_2 + \epsilon((D_1 a - D_2 c_2)\bar{u_1} + (D_2 a - D_1 c_1)\bar{u_2}) > 0, \quad \forall \bar{u_i} \geq 0.$$

For the determinant we have

$$\det(\mathrm{Sym}(\mathcal{A})) = \det(\mathcal{A}) - \left(\frac{\mathcal{A}_{12}-\mathcal{A}_{21}}{2}\right)^2 = D_1 D_2 \left[1 + \frac{1}{2}\epsilon\pi u^* - \frac{1}{4}(\epsilon\pi u^*)^2(2+\theta)\right],$$

where $\theta = (D_1 - D_2)^2/(4D_1D_2) \geq 0$. Imposing that $\det(\mathrm{Sym}(D)) = 0$ leads to

$$\epsilon\pi u^* = \frac{1+\sqrt{9+4\theta}}{2+\theta},$$

as required. □

3.2.2 Simulations for initial data varying in one direction

The cross-diffusion systems we are interested in have the form

$$\partial_t u - \partial_x\left[\mathcal{A}(u)\partial_x u - \mathcal{B}(x,u)u\right] = 0, \quad \text{in} \quad \Omega, t > 0, \tag{3.71}$$

with boundary and initial conditions

$$\mathcal{A}(u)\partial_x u - \mathcal{B}(x,u)u = 0, \quad \text{for } x = \pm 1, t > 0, \tag{3.72}$$

$$u(0,\cdot) = u_0, \quad \text{in } \Omega, \tag{3.73}$$

where Ω is the interval $[-1,1]$ and $u=(u_1,u_2)$ is the vector of densities of each species.

We will consider the following cases:

- zero order model

$$\mathcal{A}_0(u)=\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \qquad \mathcal{B}_0(u)=\begin{pmatrix} -\nabla V_1 & 0 \\ 0 & -\nabla V_2 \end{pmatrix}, \tag{3.74}$$

- first order model, already introduced in example 2

$$\mathcal{A}_1(u)=\begin{pmatrix} D_1(1+\epsilon a_1 u_1-\epsilon c_1 u_2) & \epsilon D_1 b_1 u_1 \\ \epsilon D_2 b_2 u_2 & D_2(1+\epsilon a_2 u_2-\epsilon c_2 u_1) \end{pmatrix}, \tag{3.75a}$$

$$\mathcal{B}_1(u)=\begin{pmatrix} -\partial_x V_1 & \epsilon c_1 \partial_x (V_1-V_2) u_1 \\ \epsilon c_2 \partial_x (V_2-V_1) u_2 & -\partial_x V_2 \end{pmatrix}, \tag{3.75b}$$

- second order model, already introduced in example 1

$$\mathcal{A}_2(u)=\mathcal{A}_1(u)+\epsilon^2 u_1 u_2 \begin{pmatrix} -D_1\theta_1 & D_1\theta_2 \\ D_2\theta_1 & -D_2\theta_2 \end{pmatrix}, \tag{3.76a}$$

$$\mathcal{B}_2(u)=\mathcal{B}_1(u),$$

where $\theta_i=a_i c_i-(d-1)(c_i+c_j)c_j$. We recall that the coefficients in (3.75) are given by

$$a_i=\frac{2\pi}{d}(d-1), \quad b_i=\frac{2\pi}{d}\frac{[(d-1)D_i+dD_j]}{D_i+D_j}, \quad c_i=\frac{2\pi}{d}\frac{D_i}{D_i+D_j}, \tag{3.77}$$

for $i,j=1,2$ ($j\neq i$). The small parameter ϵ is chosen to be 0.1.

We present numerical simulations for the cross-diffusion systems above. We consider these examples when the physical dimension is $d=2$, but with initial data and potentials V_i varying in one direction such that the solutions can be represented as one-dimensional.

We solve (3.71) in the domain $\Omega=(-1,1)$ using a second-order accurate finite-difference scheme in space and the method of lines with the inbuilt Matlab ode solver `ode15s` in time. In particular, we use an equidistant mesh of size $\Delta x=|\Omega|/J$, with nodes $x_n=-1+n\Delta x$, $0\leq n\leq J$, $J=1500$. The fluxes are evaluated at the nodes x_n to ensure the no-flux conditions are imposed accurately, while the solutions u_i are computed at the midpoints $x_{n+1/2}=-1+(n+1/2)\Delta x$. The unknowns are $u_n(t)=(u_{1,n},u_{2,n})$, where $u_{i,n}(t)\approx u_i(x_{n+1/2},t)$, $i=1,2$. The discretisation of the spatial derivatives is done in the spirit of the positivity-preserving scheme proposed in [ZB00]. For example, the terms of the form $u_i\nabla u_j$ are discretised as

$$\left(u_i\frac{\partial u_j}{\partial x}\right)(x_{n+\frac{1}{2}})\approx u_{i,n+\frac{1}{2}}\left(\frac{u_{j,n+1}-u_{j,n}}{\Delta x}\right),$$

where

$$u_{i,n+\frac{1}{2}} = \frac{2u_{i,n+1}u_{i,n}}{u_{i,n+1} + u_{i,n}}.$$

We consider the following expression for the numerical fluxes:

$$F_{i,n+1} = \mathcal{A}_{ij}(u_{n+\frac{1}{2}})\frac{u_{j,n+1} - u_{j,n}}{\Delta x} - \mathcal{B}_{ij}(x_{n+\frac{1}{2}}, u_{n+\frac{1}{2}})u_{j,n+1},$$

and we obtain the system of ODEs

$$\frac{du_n}{dt} = \frac{F_{n+1} - F_n}{\Delta x},$$

The scheme in Figure 3.1 visualises the way in which we compute fluxes F and approximate solutions u (n particular, the first ones are computed at the nodes in red and the second ones at the half-nodes in black).

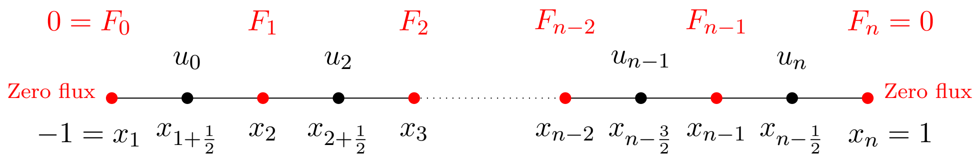

Figure 3.1: A schematic representing nodes and half-nodes, where the fluxes and the solutions are evaluated in the numerical algorithm used to solve problem (3.71).

We choose $D_1 = 1.3$, $D_2 = 1$ and $d = 2$. We set initial data $u_1(x, 0) = C\exp(-20(x - 0.5)^2)$, $u_2(x, 0) = C\exp(-20(x + 0.5)^2)$, where C is the constant that normalizes the mass to 1, and external potentials $V_1(x) = 5(x + 0.5)^2$, $V_2(x) = 5(x - 0.5)^2$. We run the time-dependent simulation until $T = 1$.

Figure 3.2 shows different profiles of the two components of u_{A_1} (first order model) for a sequence of time steps. Both diffusion and drift effects are clearly recognizable, in particular we see that the first component is moving to the left and the second one to the right, according to the gradient of the potentials V_i.

Figure 3.3 shows the deviation of each component of the mass of the solution (in absolute value) at different time steps. The deviation is initially zero for all cases, subsequently increases and reaches a plateau. The deviation is significantly smaller in the case of a linear system (zero order model) and it is comparable for first and second order models (approximately 10^{-9}).

Figure 3.4 shows the rate of convergence of the solutions towards equilibrium in terms of the maximum of the variation between subsequent time steps taken over all nodes. Such variations decrease exponentially fast in time in all three our examples. The first component (characterised by a larger diffusion coefficient) decreases faster in all cases as expected.

Figure 3.5 confirms our expectations in terms of stability of the models. In particular,

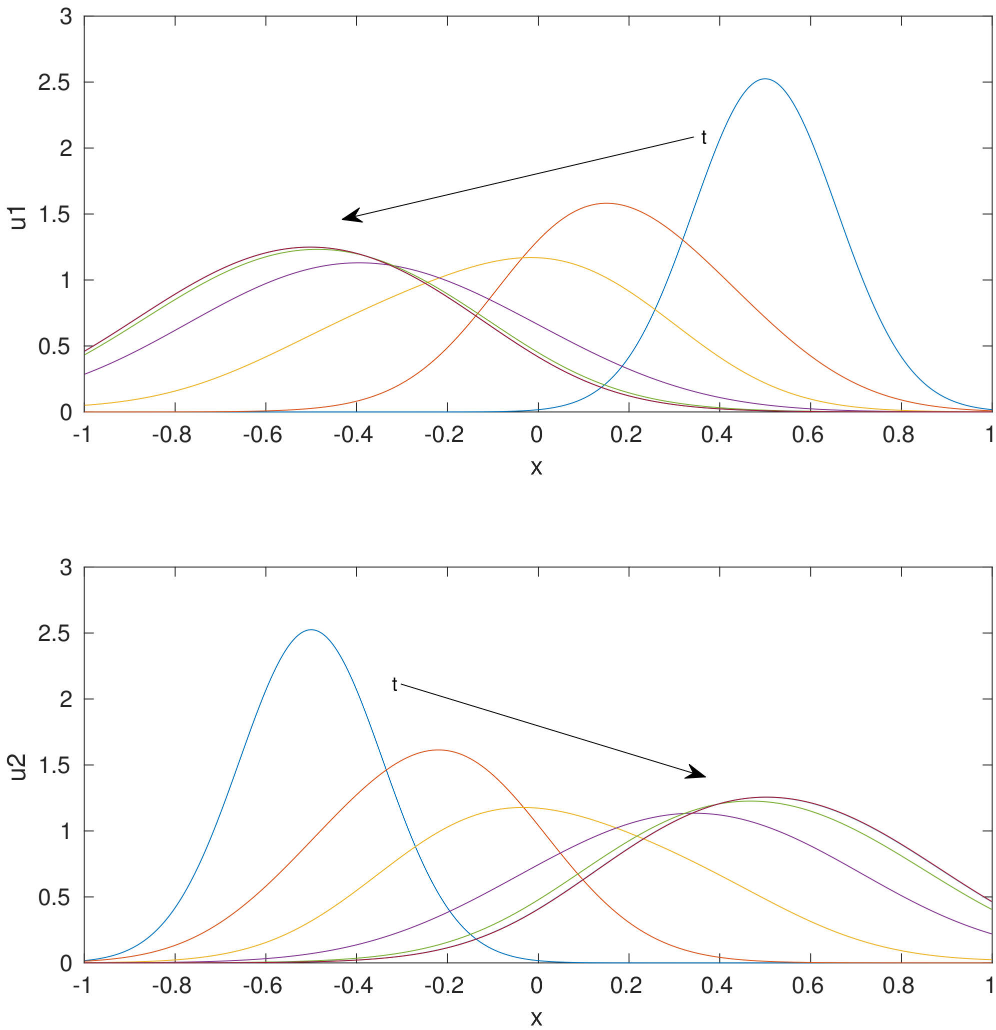

Figure 3.2: The two components of the solutions of problem (3.1) in the unperturbed case ($\mathcal{A} = \mathcal{A}_1$) at times $[0.0323, 0.0645, 0.1290, 0.2258, 0.3548, 0.6129, 1]$. The arrow indicates the direction of time (for example, in the upper part, the profile on the right corresponds to time $t = 0$).

we observe that the difference $u_{A_0} - u_{A_1}$, after an initial phase, stabilises at order 10^{-1}, which is consistent with our choice of $\epsilon = 0.1$. Moreover, the difference $u_{A_1} - u_{A_2}$ takes values of order 10^{-2} in the initial phase and afterwards even smaller ($\approx 10^{-3}$), which is again consistent with the choice of ϵ and the theoretical estimates.

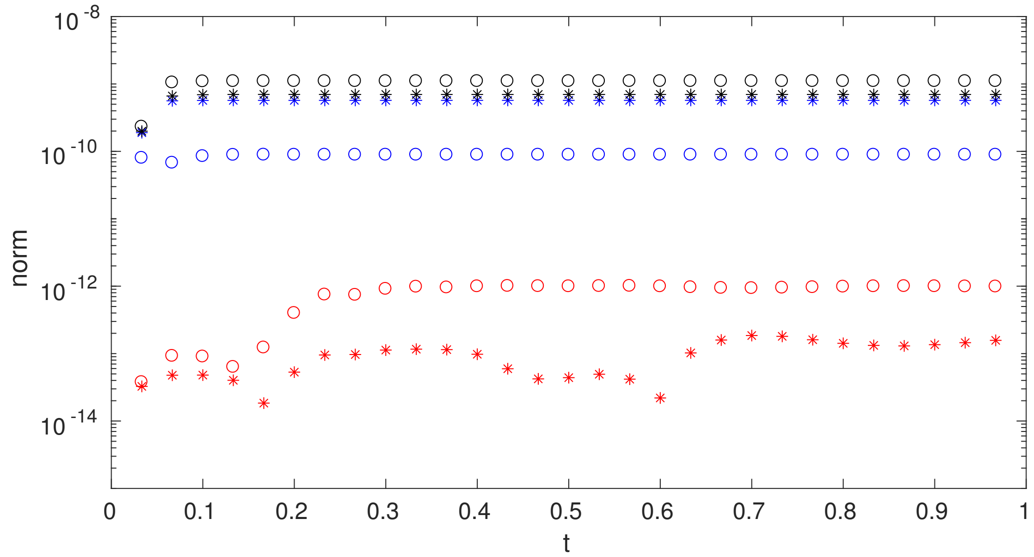

Figure 3.3: Evolution of the mass deviation given by (the discrete equivalent of) $\left|\int_\Omega (u^i_{A_k} - u^i_0)dx\right|$ over time. Values corresponding to the first and the second component are identified by a circle and an asterisk respectively. Different colours characterise each of the systems, linear in red, first order in blue, second order in black.

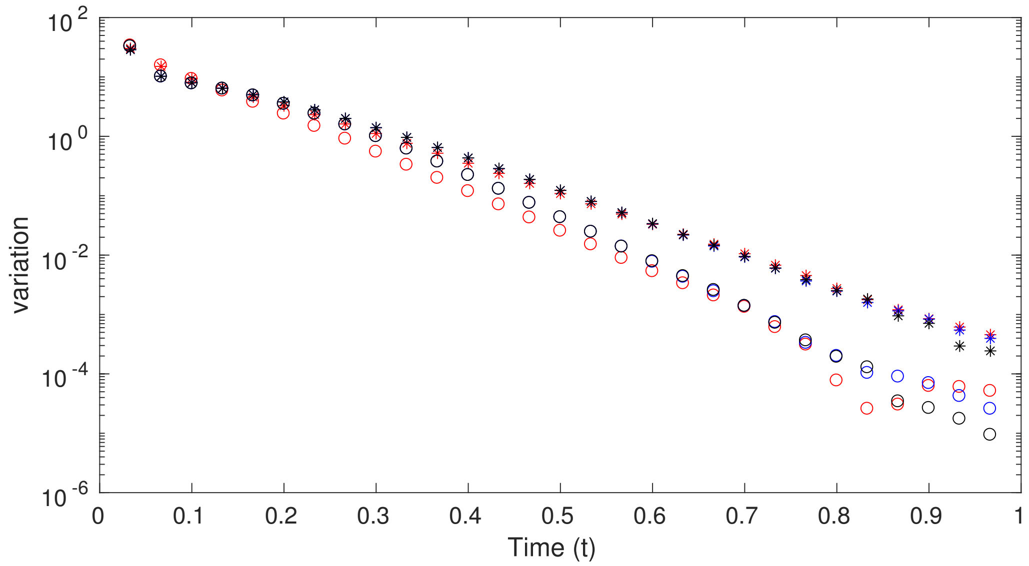

Figure 3.4: Convergence to equilibrium is evaluated computing $\max_{[-1,1]} \|u(t_n, \cdot) - u(t_{n-1,} .)\|$ at different time steps. Values corresponding to the first and the second component are identified by a circle and an asterisk respectively. Different colours characterise each of the systems, linear in red, first order in blue, second order in black.

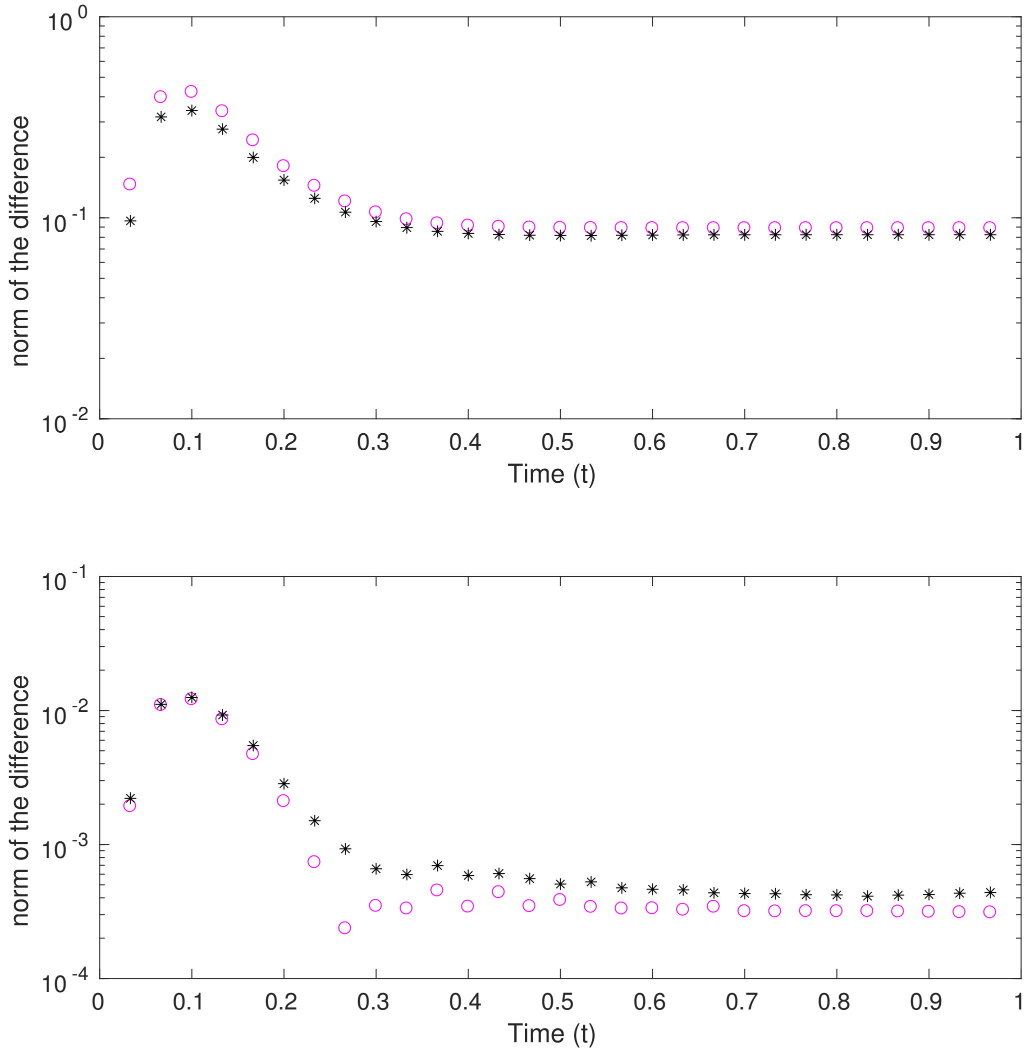

Figure 3.5: This picture shows the behaviour of the L^{∞} norm of the differences $u_{A_1} - u_{A_0}$ (upper part) and $u_{A_1} - u_{A_2}$ (lower part). The different colours and symbols correspond to the two different species.

Chapter 4

Systems with large cross diffusion

Contents

This chapter is dedicated to systems with cross-diffusion terms that are not necessarily small; in the first part we consider strongly parabolic problems, in the second one we discuss some properties of degenerate parabolic systems. An important tool in this chapter is the so-called *entropy structure* of the systems we consider. This is a powerful tool that allows us to study existence and boundedness of both degenerate and non-degenerate systems. In addition, it is not necessary to assume symmetry of the diffusion matrix and in fact non-symmetric examples are common in the applications. This approach is not the only possible one, but it has been very successful in recent times. We would like to cite the works of Amann, [Ama93, Ama89, Ama90] in which global existence for non-degenerate quasilinear problems is obtained via abstract parabolic theory.

4.1 Non-degenerate case

In section 4.1.1 we will introduce the concept of entropy structure. We shall see that, when present, such structure constitutes a key ingredient in the existence theory and, under suitable assumptions, it can guarantee boundedness of the solutions, as shown in [Jün15]. In section 4.1.2 we will show how to improve the integrability of the gradients of the solutions. This information will be useful to close our stability estimates in dimension $d = 2$. Sections 4.1.1 and 4.1.2 are based on results available in the literature which will be referenced therein. Section 4.1.3 is dedicated to a stability estimate for autonomous cross-diffusion systems. We refer to the technique used to prove this result as the "homotopic approach". In particular, we extend the results in [CH05] for classical solutions to the case of weak solutions in a vectorial setting.

4.1.1 Systems with entropy structure

In this section we consider systems of equations describing the behaviour of multiple species whose mutual interactions are not necessarily small. An important assumption in our analysis consists in the existence of a so-called *entropy structure*. As shown by Jüngel in [Jün15], this ensures existence and boundedness of solutions globally in time. As discussed in Section 1.2.3, many models having the form of a cross-diffusion system have been proposed. The different species can represent, for example, groups of mutated and non-mutated cells in a tumour growth model (see example 4).

Systems with large cross-diffusion can be rather complicated to analyse (the solutions may have spikes, as shown in [JB02]), it is therefore important to identify a class of problems for which solvability has been shown.

Let us introduce in more detail the notion of *entropy structure* in the sense of [JS12].

Definition 4.1 (Entropy structure)**.** Consider a system of equations of the type:

$$\partial_t u = \operatorname{div}\left(A(u)\nabla u\right), \tag{4.1}$$

where $u : \Omega \to \mathbb{R}^N$. We say that system (4.1) has an *entropy structure* if there exists a function $h : \mathbb{R}^N \to \mathbb{R}$ such that

- h is a convex function of class C^2 and it defines the following *entropy functional*

$$E[u] = \int_\Omega h(u)\, dx.$$

- the map $Dh(\cdot)$ defines a change of coordinates (Lipschitz diffeomorphism) from an open and connected domain U in $\mathbb{R}^N$ into the whole $\mathbb{R}^N$.

- system (4.1) can be written in the form:

$$\partial_t u = \operatorname{div}\left(B(u)\nabla\left(Dh(u)\right)\right), \tag{4.2}$$

  where $B(u) = A(u)(\operatorname{Hess}(h))^{-1}(u)$ is assumed to be a positive semi-definite matrix. Such matrix is typically called *motility matrix*.

Remark 4.1 (Entropy variables)**.** The new unknown $w \in \mathbb{R}^N$ obtained setting $w =$

$Dh(u)$, for $u \in U$, is commonly referred to as the *entropy variable*. The domain U is typically a Lipschitz subset of the region $\{u_i \geq 0, 1 \leq i \leq N\} \subset \mathbb{R}^N$.

Remark 4.2. It has been shown in [DGJ97] that a system has an entropy structure if and only if there exists a change of variable that provides a "symmetric" reformulation of the original system. Additionally, all the notions we introduced so far can easily be adapted in the presence of reaction terms.

Definition 4.2 (Weak formulation)**.** Consider the following reaction-diffusion system:

$$\partial_t u = \operatorname{div}(A(u)\nabla u) + f(u) \quad \text{in } \Omega, \tag{4.3}$$

$$A(u)\nabla u \cdot n = 0 \quad \text{on } \partial\Omega, \tag{4.4}$$

$$u(0) = u_0, \tag{4.5}$$

where $A : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is a matrix-valued function, $f : \mathbb{R}^N \to \mathbb{R}^N$ is a vector field of sub-linear growth and the second order term is interpreted as follows:

$$\operatorname{div}(A(u)\nabla u) = \sum_{j=1}^{d}\sum_{k=1}^{N} \partial_j(A_{ik}(u)\partial_j u^k).$$

We say that u is a weak solution of (4.3) if

$$u \in L^2(Q_T), \quad A(u)\nabla u \in L^2(Q_T) \quad \partial_t u \in L^2(0,T;(H^1(\Omega;\mathbb{R}^N))').$$

and, for any test function $\eta \in C^\infty(\Omega)$ and a.e. $t \geq 0$, it holds

$$\langle \partial_t u, \eta\rangle + \int_\Omega [A(u)\nabla u \cdot \nabla\eta - f(u)\eta]\, dx\, dt = 0.$$

Moreover, $u(t,\cdot) \to u_0(\cdot)$ in $H^1(\Omega)'$ as $t \to 0$.

Remark 4.3 (Entropy decay)**.** An important consequence of Definition 4.1 is the following estimate, that holds for weak solutions of (4.1) in the sense of Definition 4.2:

$$\frac{dE}{dt} = \int_\Omega \partial_t u Dh(u)\mathrm{d}x = -\int_\Omega \nabla w : B\nabla w \mathrm{d}x, \tag{4.6}$$

where $w = Dh(u)$ is the new unknown after the change of variables.

For this type of systems, global existence and boundedness of solutions under suitable assumptions was established in [Jün15]. We now state the existence result proved therein

Theorem 4.1 (Boundedness-by-entropy Principle)**.** *Consider problem* (4.3)*, let U be an open subset of $\mathbb{R}^n$ and suppose $u_0 \in U$. Consider the following hypotheses:*

1. *There exists a convex function $h \in C^2(U, [0,\infty))$ such that its derivative $Dh : U \to \mathbb{R}^n$ is invertible on $\mathbb{R}^n$.*

2. *There exist* $-\infty < a < b < \infty$ *such that* $U \subset (a,b)^n$*, and there exist* $\alpha_i^*, m^i \geq 0$ $(i = 1...N)$ *be such that for any vectors* $z \in \mathbb{R}^N$ *and any* $u \in U$

$$z^t D^2 h(u) A(u) z \geq \sum_{i=1}^{n} \alpha_i(u^i)^2 (z^i)^2,$$

where $\alpha_i(u^i)$ *coincides either with* $\alpha_i^*(u^i - a)^{m^i - 1}$ *or with* $\alpha_i^*(b - u_i)^{m_i - 1}$.

3. *There exists* $\Lambda > 0$ *such that for all* $u \in U$ *and all* $i.j = 1...n$ *for which* $m_j > 1$ *it holds that*

$$|A_{ij}(u)| \leq \Lambda |\alpha_j(u^j)|.$$

4. *It holds* $A \in C^0(U, \mathbb{R}^{N \times N})$ *and there exists a constant* C *such that for all* $u \in D$

$$f(u) \cdot Dh(u) \leq C(1 + h(u)).$$

5. *It holds* $u_0 \in L^1(\Omega, \mathbb{R}^N)$ *and* $u_0(x) \in U$ *for a.e.* $x \in \Omega$.

Then there exists a weak solution of problem (4.3) *in the sense of definition 4.2. Additionally,* $u(t,x) \in \bar{U}$ *for a.e.* $(x,t) \in Q_T$*. Furthermore, if* $m_i \geq 0$*, we have*

$$u \in L^2_{loc}(0, \infty; H^1(\Omega; \mathbb{R}^N)), \quad \partial_t u \in L^2_{loc}(0, \infty; (H^1(\Omega; \mathbb{R}^N))').$$

Let us now introduce an example of cross diffusion system that has many of the features we are interested in:

Example 4. The following cross-diffusion model has been introduced by Jackson and Byrne in [JB02] starting from mechanical considerations.

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} = \operatorname{div}\left(D(u,v) \begin{bmatrix} \nabla u \\ \nabla v \end{bmatrix} \right) + R(u,v),$$

where

$$D(u,v) = \begin{bmatrix} 2u(1-u) - \beta\theta u v^2 & -2\beta u v(1+\theta u) \\ -2uv + \beta\theta(1-v)v^2 & 2\beta v(1-v)(1+\theta u) \end{bmatrix}, \quad \beta > 0, \theta \geq 0$$

$$R(u,v) = \begin{bmatrix} \gamma u(1-u-v) - \delta u \\ \alpha u v(1-u-v) \end{bmatrix}, \quad \alpha, \gamma, \delta \geq 0,$$

to be complemented with suitable initial and boundary conditions.

Such model generalises the filtration equation to the case of multiple species in the sense that the nonlinear diffusion is degenerate (e.g. for u or v vanishing). Moreover, we shall see that such system has an entropy structure in the sense of definition 4.1 (see [JS12]). In particular, for this example we have $U = \{(u_1, u_2) : u_i > 0, u_1 + u_2 < 1\}$ and the entropy functional is given by

$$\int_\Omega (u \log(u) + v \log(v) + (1 - u - v) \log(1 - u - v))\, dx.$$

In section 4.2.4 we will consider a simplified version of this model with $\theta = 0$.

The entropy structure often plays an important role in the proof of existence of solutions, however it does not imply any uniqueness result unless further assumptions are made. On the other hand, uniqueness can be proven if the solutions are sufficiently regular, as we will see in the next Theorem.

We will use the following interpolation result (see, for example, [BM17]):

Proposition 4.2 (Gagliardo-Nirenberg interpolation inequality). *Let Ω be a Lipschitz, bounded domain in $\mathbb{R}^d$ and consider a function $f : \Omega \to \mathbb{R}^d$ such that all the norms appearing below are bounded. For any choice of exponents $1 \le p, q, r \le \infty$ and real number $\alpha \in (0, 1]$ such that*

$$\frac{1}{p} = \left(\frac{1}{r} - \frac{1}{d}\right)\alpha + \frac{1}{q}(1 - \alpha),$$

there exists constants γ_1 and γ_2, depending on d, p, q, α and Ω, such that

$$\|f\|_{L^p(\Omega)} \le \gamma_1 \|\nabla f\|^{\alpha}_{L^r(\Omega)} \|f\|^{1-\alpha}_{L^q(\Omega)} + \gamma_2 \|f\|_{L^s(\Omega)}, \tag{4.7}$$

for an arbitrary $s \ge 1$.

The following uniqueness result is classical, we present it for the sake of completeness (see, for example, [CFL76]).

Proposition 4.3 (Uniqueness). *Consider the (scalar or vectorial) equation*

$$\begin{aligned} &\partial_t u = \operatorname{div}(A(u)\nabla u) \qquad \text{in } \Omega,\\ &A(u)\nabla u \cdot n = 0 \qquad \text{on } \partial\Omega,\\ &u(0) = u_0, \end{aligned}$$

under the following assumptions:

- *the matrix A is elliptic with ellipticity constant $\lambda > 0$ and Lipschitz in u.*
- *any solution u is bounded a.e. by a constant M depending only on A, Ω and u_0,*
- *there exist a constant K depending only on A, Ω, $T > 0$ and u_0 and an exponent $q_0 > d \ge 2$ such that $\sup_{[0,T]} \|\nabla u\|_{L^{q_0}(\Omega)} \le K$,*

Then the problem above has a unique weak solution on the interval $[0, T]$.

Proof of proposition 4.3 . Given two solutions u and v corresponding to the same initial datum, consider the equation for their difference and test it against $(u - v)$:

$$\int_\Omega (u-v)\partial_t(u-v)\mathrm{d}x = -\int_\Omega (A(u)\nabla u - A(v)\nabla v)\cdot\nabla(u-v)\mathrm{d}x.$$

Adding and subtracting $A(u)\nabla v$ and $A(v)\nabla u$ on the right hand side we obtain:

$$\begin{aligned}\partial_t \int_\Omega (u-v)^2\mathrm{d}x = &-\int_\Omega (A(u)-A(v))\nabla(u+v)\cdot\nabla(u-v)\mathrm{d}x\\ &-\int_\Omega (A(u)+A(v))|\nabla(u-v)|^2\mathrm{d}x.\end{aligned} \tag{4.8}$$

Remember that we assumed $\sup_{[0,T]}\|\nabla(u+v)\|_q^q\,\mathrm{d}t < \infty$ (for some $q > d$) and consider the term.

$$I(u,v) = \int_\Omega |A(u)-A(v)||\nabla(u+v)||\nabla(u-v)|\mathrm{d}x.$$

Apply Hölder inequality with exponents $\frac{1}{p}+\frac{1}{q}+\frac{1}{2}=1$ to obtain

$$I(u,v) \le \|\nabla(u-v)\|_2\,\|\nabla(u+v)\|_q\,\|u-v\|_p\,.$$

Applying inequality (4.7) for $\frac{1}{p} = \left(\frac{1}{2}-\frac{1}{d}\right)\alpha + \frac{1-\alpha}{2} = \frac{1}{2}-\frac{\alpha}{d}$, with $\alpha \in (0,1)$, we have:

$$I(u,v) \le \|\nabla(u-v)\|_2\,\|\nabla(u+v)\|_q\left(\gamma_1\|\nabla(u-v)\|_2^\alpha\|u-v\|_2^{1-\alpha}+\gamma_2\|u-v\|_2\right)$$

$$\le \gamma_1\|\nabla(u-v)\|_2^{1+\alpha}\|\nabla(u+v)\|_q\|u-v\|_2^{1-\alpha}+\gamma_2\|\nabla(u-v)\|_2\|\nabla(u+v)\|_q\|u-v\|_2$$

Now we apply Young inequality twice with exponents $\frac{1+\alpha}{2}+\frac{1-\alpha}{2}=1$, $\frac{1}{2}+\frac{1}{2}=1$, $\nu,\mu>0$:

$$\begin{aligned}I(u,v) \le&\gamma_1\left[\frac{(1+\alpha)}{2}\nu^{\frac{2}{1+\alpha}}\|\nabla(u-v)\|_2^2+\frac{1-\alpha}{2}\nu^{-\frac{2}{1-\alpha}}\|\nabla(u+v)\|_q^{\frac{2}{1-\alpha}}\|u-v\|_2^2\right]+\\ &+\gamma_2\left[\frac{\mu}{2}\|\nabla(u-v)\|_2^2+\frac{1}{2\mu}\|\nabla(u+v)\|_q^2\|u-v\|_2^2\right].\end{aligned}$$

Rearranging we obtain

$$\begin{aligned}I(u,v) \le&\frac{1}{2}\left[\gamma_1(1+\alpha)\nu^{\frac{2}{1+\alpha}}+\gamma_2\mu\right]\|\nabla(u-v)\|_2^2+\\ &+\left[\frac{(1+\alpha)\gamma_1}{2\nu^{\frac{2}{1-\alpha}}}\|\nabla(u+v)\|_q^{\frac{2}{1-\alpha}}+\frac{\gamma_2}{2\mu}\|\nabla(u+v)\|_q^2\right]\|u-v\|_2^2.\end{aligned}$$

Recall that $\frac{1}{p}=\frac{1}{2}-\frac{\alpha}{d}$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. We get $q=\frac{d}{\alpha}>d$ and we choose α so that $q=q_0$. From ellipticity and the relation (4.8) we obtain

$$\frac{1}{2}\left[\gamma_1(1+\alpha)\nu^{\frac{2}{1+\alpha}}+\gamma_2\mu\right]\|\nabla(u-v)\|_2^2-\int_\Omega (A(u)+A(v))\nabla(u-v)\cdot\nabla(u-v)\mathrm{d}x \le 0,$$

and we choose ν and μ so that

$$\frac{1}{2}\left[\gamma_1(1+\alpha)\nu^{\frac{2}{1+\alpha}} + \gamma_2\mu\right] < \lambda. \tag{4.9}$$

Recalling that $u_0 - v_0 = 0$. Given the constant

$$C = \frac{(1+\alpha)\gamma_1}{2\nu^{\frac{2}{1-\alpha}}}|\Omega|^{1/q_0} + \frac{\gamma_2}{2\mu}|\Omega|^{1/2}, \tag{4.10}$$

we conclude that:

$$\begin{aligned}\|u-v\|_2^2 &\leq \|u_0 - v_0\|_2^2 + C\int_0^T \|\nabla(u+v)\|_{q_0}^{q_0}\,\|u-v\|_2^2\,\mathrm{d}t \\ &\leq \|u_0 - v_0\|_2^2 \exp\left(CK^qT\right) \leq 0,\end{aligned}$$

where K is defined in the statement and the initial data coincide by assumption. □

4.1.2 Higher integrability of the gradients

Now that existence of solutions is established, we proceed showing higher integrability of the gradient of the solutions. In the interior of the domain we use a rather standard parabolic version of Gehring's Lemma and we then extend the result to the initial and lateral boundary. We follow the approach of [GS82, Par07, AM07]. The higher integrability property will be useful in the next section, especially in the case of two spatial dimensions.

We are going to show that weak solutions have gradients that are integrable with an exponent greater than 2. Consider the system of equations

$$\begin{aligned}\partial_t u - \operatorname{div}[A(t,x,u)\nabla u] = 0,\\ \nabla u\cdot n = 0 \text{ on } \partial\Omega, \quad u(0) = 0,\end{aligned} \tag{4.11}$$

where A is uniformly continuous, bounded and elliptic with constant λ.

We will obtain the following result:

Theorem 4.4 (Higher integrability)**.** *Let $u \in L^2(0,T,H^1(\Omega);\mathbb{R}^N)$ be a weak solution of problem (4.11). Then there exists $p > 2$ such that $u \in L^p(0,T,W^{1p}(\Omega))$.*

The rest of this section is dedicated to the proof of the result above and it is divided in three main parts, namely Lemma 4.7 shows higher integrability of ∇u in the interior of the domain Ω and we refer to Giaquinta and Struwe [GS82], Lemma 4.8 provides the same result including the lateral boundary, and Lemma 4.9 for the initial boundary of the parabolic cylinder $(0,T)\times\Omega$. For the latter two extensions, we follow closely the strategy presented in [Par07, AM07].

In the rest of this section we will use the following notation:

$$Q = [0,T]\times\Omega, \qquad z=(t,x),$$
$$J_R = J(t_0,R) = (t_0-R^2,t_0),$$
$$Q_R = Q(z_0,R) = J(t_0,R)\times B_R(x_0).$$

Consider a smooth cut-off function χ supported in $B_{2R}(x_0)$ and identically equal to 1 on $B_R(x_0)$ with gradient bounded by 2. We define the following functions:

$$\chi_{2R}(x) = \chi(x/R),$$
$$\tilde{u}_{2R} = \tilde{u}_{x_0,2R}(t) = \frac{\int_{B_{2R}(x_0)} u(t,x)\chi_{2R}^2(x)dx}{\int_{B_{2R}(x_0)} \chi_{2R}^2(x)dx}.$$

As a preliminary result, we prove two fundamental inequalities following the approach of Giaquinta and Struwe (see [GS82]).

Lemma 4.5 (Caccioppoli and Poincaré inequalities)**.** *Let $u\in L^2(0,T,H^1(\Omega;\mathbb{R}^N))$. The following Caccioppoli-type estimate holds: for all $Q_R\subset Q_{2R}\subset Q$, there exists a constant c such that*

$$\int_{Q_R}|\nabla u|^2dz \le \frac{c_1}{R^2}\int_{Q_{2R}}|u-\tilde{u}_{2R}|^2dz. \tag{4.12}$$

Moreover we have the following Poincaré-type inequality holds: for all $Q_R\subset Q_{2R}\subset Q$,

$$\sup_{t\in J(t_0,R)}\int_{B_R}|u-\tilde{u}_R|^2dz \le c_2\int_{Q_{2R}}|\nabla u|^2dz. \tag{4.13}$$

The constants c_1 and c_2 are independent of R.

Proof of Lemma 4.5. Fix x_0 and t_0 and introduce a cut-off function in time, indicated with η, that is compactly supported in (t_0-2R^2,∞), being identically 1 on $[t_0-R^2,t_0]$. Consider the test function:

$$\phi = (u-\tilde{u}_{2R}(t))\chi_{2R}^2(x)\eta^2 1_{(-\infty,t_0)},$$

testing against equation (4.11) we obtain:

$$\int_{B_{2R}}\chi^2\eta^2|u-\tilde{u}_{2R}|^2dx + \int_{Q_{2R}}\chi^2\eta^2 A\nabla u\cdot\nabla u\,dz$$
$$\le 2\int_{Q_{2R}}|u-\tilde{u}_{2R}|^2\chi^2\eta\partial_t\eta dz - 2\int_{Q_{2R}}(u-\tilde{u}_{2R}(t))\chi\eta^2 A\nabla u\cdot\nabla\chi\,dz,$$

$$(\lambda-\varepsilon)\int_{Q_R}|\nabla u|^2\,dz \le C_\varepsilon\left[\int_{Q_{2R}}|u-\tilde{u}_{2R}|^2\chi^2\eta\frac{d}{dt}\eta dz + \int_{Q_{2R}}|u-\tilde{u}_{2R}|^2|\nabla\chi|^2\,dz\right].$$

Hence we obtain the following inequality:

$$\int_{Q_R} |\nabla u|^2 \, dz \le C_{\varepsilon,\lambda} \left(1 + \frac{1}{R^2}\right) \int_{Q_{2R}} |u - \tilde{u}_{2R}|^2 dz.$$

□

The well-known result that we will use to obtain higher integrability is the following:

Lemma 4.6 (Gehring's Lemma, [GM79]). *Consider $g \ge 0$ in Q satisfying the following relation for $q > 1$:*

$$⨍_{Q(z_0,R)} g^q \, dz \le b \left(⨍_{Q(z_0,4R)} g \, dz \right)^q + \theta ⨍_{Q(z_0,4R)} g^q \, dz,$$

for every $z_0 \in Q$ and every $Q(z_0, 4R) \subset Q$. Then there exists a constant θ_0 depending on q and d if $\theta < \theta_0$, such that $g \in L^p_{loc}(Q)$ for $q \in [q, q + \varepsilon)$ and

$$\left(⨍_{Q(z_0,R)} g^p \, dz \right)^{1/p} \le c \left(⨍_{Q(z_0,4R)} g^q \, dz \right)^{1/q},$$

for all $Q(z_0, 4R) \subset Q$. The constants $c, \varepsilon > 0$ depend on b, q, θ and d.

The following lemma by Giaquinta and Struwe guarantees higher integrability of ∇u in the interior of $\Omega \times (0, T)$:

Lemma 4.7 (Higher integrability, [GS82]). *Let $u \in L^2(0, T, H^1(\Omega); \mathbb{R}^N)$ be a weak solution of problem (4.11), then there exists an exponent $p > 2$ such that $\nabla u \in L^p_{loc}(Q)$. Moreover, for all $Q_R \subset Q_{4R} \subset Q$ it holds*

$$⨍_{Q_R} |\nabla u|^p \, dz \le C \left(⨍_{Q_{4R}} |\nabla u|^2 \, dz \right)^{\frac{p}{2}}.$$

Proof of Lemma 4.7. The proof is an application of Caccioppoli inequality, Gehring's lemma and other well known inequalities.

In the next computation we will use the following relation:

$$\int_{Q_{2R}} |u - \tilde{u}_{2R}|^2 dz \le \sup_{t \in J(t_0, 2R)} \int_{B_{2R}} |u - \tilde{u}_R|^2 dz.$$

Let $I = \int_{J_R}\int_{B_{2R}} |u - \tilde{u}_{2R}|^2 dx\, dt$. We proceed with the following estimates:

$$I \leq \int_{J_R} \left(\int_{B_{2R}} |u - \tilde{u}_{2R}|^2 dx\right)^{\frac{1}{2}} \cdot$$

$$\left[\left(\int_{B_{2R}} |u - \tilde{u}_{2R}|^{2_*} dx\right)^{\frac{1}{2_*}} \left(\int_{B_{2R}} |u - \tilde{u}_{2R}|^{2^*} dx\right)^{\frac{1}{2^*}}\right]^{\frac{1}{2}} dt$$

$$\leq \left(\sup_{t \in J(t_0, 2R)} \int_{B_R} |u - \tilde{u}_R|^2 dz\right)^{\frac{1}{2}} \cdot$$

$$\int_{J_R} \left[\left(\int_{B_{2R}} |u - \tilde{u}_{2R}|^{2_*} dx\right)^{\frac{1}{2_*}} \left(\int_{B_{2R}} |u - \tilde{u}_{2R}|^{2^*} dx\right)^{\frac{1}{2^*}}\right]^{\frac{1}{2}} dt,$$

from Poincaré's inequality (version above),

$$I \leq c \left(\int_{Q_{4R}} |\nabla u|^2 dz\right)^{\frac{1}{1}} \int_{J_R} \left[\left(\int_{B_{2R}} |u - \tilde{u}_{2R}|^{2_*} dx\right)^{\frac{1}{2_*}} \left(\int_{B_{2R}} |u - \tilde{u}_{2R}|^{2^*} dx\right)^{\frac{1}{2^*}}\right]^{\frac{1}{2}} dt$$

from usual Poincaré's and Sobolev inequalities,

$$I \leq c \left(\int_{Q_{4R}} |\nabla u|^2 dz\right)^{\frac{1}{2}} R^{1/2} \int_{J_R} \left(\int_{B_{2R}} |\nabla u|^{2_*} dx\right)^{\frac{1}{(2 \cdot 2_*)}} \left(\int_{B_{2R}} |\nabla u|^2 dx\right)^{\frac{1}{2}} dt,$$

from Hölder inequality for the time integral with $\frac{1}{2_*} + \frac{1}{s} = 1$,

$$I \leq cR^{1/2} \left(\int_{Q_{4R}} |\nabla u|^2 dz\right)^{\frac{1}{2}} \left(\int_{Q_{2R}} |\nabla u|^{2_*} dz\right)^{\frac{1}{2 \cdot 2_*}} \left[\int_{J_R} \left(\int_{Q_{2R}} |\nabla u|^2 dz\right)^{s/2} dt\right]^{\frac{1}{s}},$$

from Gagliardo-Nirenberg interpolation inequality,

$$I \leq cR^{\frac{3}{2} - \frac{1}{d}} \left(\int_{Q_{4R}} |\nabla u|^2 dz\right)^{\frac{3}{4}} \left(\int_{Q_{2R}} |\nabla u|^{2_*} dz\right)^{\frac{1}{2 \cdot 2_*}},$$

from Young inequality with exponents $\frac{4}{3}$ and 4,

$$I \leq \varepsilon R^2 \int_{Q_{4R}} |\nabla u|^2 dz + C_\varepsilon R^{-\frac{4}{d}} \left(\int_{Q_{2R}} |\nabla u|^{2_*} dz\right)^{\frac{1}{2_*}}.$$

Therefore, from Caccioppoli inequality,

$$\frac{R^2}{c} \int_{Q_R} |\nabla u|^2 dz \leq \int_{Q_{2R}} |u - \tilde{u}_{2R}|^2 dz \leq \varepsilon R^2 \int_{Q_{4R}} |\nabla u|^2 dz + C_\varepsilon R^{-\frac{4}{d}} \left(\int_{Q_{2R}} |\nabla u|^{2_*} dz\right)^{\frac{2}{2_*}}$$

and the conclusion follows from Gehring's Lemma with $g = |\nabla u|$ and $q = 2$. □

Let us now include the lateral boundary:

Lemma 4.8 (Extension to the lateral boundary)**.** *Let Ω be a domain with C^1 boundary, in particular suppose that for every point $x \in \partial\Omega$ there exists a neighbourhood U of x and a change of coordinates $\Phi \in C^1(U, \mathbb{R}^d)$ such that*

$$y_i = \Phi_i(x), \quad \Phi(\Omega \cap U) = \{y_d > 0\} \cap B_R(0) = B_R^+.$$

Then the higher integrability result 4.7 can be extended up to the lateral boundary boundary, i.e. $\nabla u \in L^p((\tau, T) \times \Omega)$ for some $p > 2$ and all $0 < \tau < T$.

Proof of Lemma 4.8. We will perform a reflection in the transformed domain in order to obtain a suitable extension of u. Consider a finite open covering U_j of the boundary of Ω with the corresponding maps $\Phi_j \in C^1(U_j, \mathbb{R}^d)$ and an associated partition of unity θ_j. Let us define

$$\tilde{u}_j(y) = \begin{cases} \theta_j u(\Phi^{-1}(y)) & y \in \{y_d > 0\} \cap B_R(0), \\ \theta_j u(\Phi^{-1}(y_1, ..., y_{d-1}, -y_d)) & y \in \{y_d \leq 0\} \cap B_R(0). \end{cases}$$

Notice that for $y = 0$ we have $\tilde{u} = u$. In addition, in the new domain $B_R(0)$, we have

$$\nabla_y \tilde{u} = (D\Phi)(\Phi^{-1}(y)) \begin{pmatrix} I_{d-1} & 0 \\ 0 & sign(y_d) \end{pmatrix} \nabla u = (D\Phi) J \nabla u.$$

We can now observe that $\tilde{u}$ satisfies the following equations (in weak form) in the two parts of B_R:

$$\begin{aligned} \text{in } B_R^+, \quad & \int_{B_R^+} \partial_t u \cdot \eta \, |\det(D\Phi)| dy = \int_{B_R^+} A(u)((D\Phi)\nabla u) \cdot (D\Phi \nabla \eta) \, |\det(D\Phi)| dy, \\ \text{in } B_R^-, \quad & \int_{B_R^-} \partial_t u(y', -y_d) \cdot \eta(y', -y_d) \, |\det(D\Phi)| dy \\ & = \int_{B_R^-} A(u(y', -y_d))((D\Phi) J \nabla_y u(y', -y_d)) \cdot (D\Phi \nabla \eta(y', -y_d)) \, |\det(D\Phi)| dy \\ & = \int_{B_R^-} A(u(y', -y_d))((D\Phi) \nabla_y u(y', -y_d)) \cdot (D\Phi J \nabla \eta(y', -y_d)) \, |\det(D\Phi)| dy, \end{aligned}$$

for a test functions η. Choosing Φ such that $|\det(D\Phi)| = 1$ and defining

$$A'(y, u) = (D\Phi)^t A(u \circ \Phi^{-1}) D\Phi,$$

we obtain the new weak formulation

$$\int_{B_R} \partial_t(u \circ \Phi^{-1}) \cdot \eta \, dy$$
$$= \int_{B_R^+} A'(y,u)\nabla(u\circ\Phi^{-1})\cdot\nabla\eta\, dy + \int_{B_R^-} A'(y,u)\nabla u\circ\Phi^{-1}(y',-y_d)\cdot J\nabla\eta(y',-y_d)\, dy.$$

The matrix $A'(x,\tilde{u})$ is elliptic and it allows us to repeat the same estimates that we had for A. □

Lemma 4.9 (Extension to the initial boundary)**.** *Let $d=2$. Consider u be a weak solution of system* (4.11) *and suppose that the initial datum u_0 belongs to the space $W^{1,2+\delta}(\Omega)$. Consider a parabolic cylinder in $\mathbb{R}\times\mathbb{R}^2$ such that*

$$Q_R(t_{0,}x_0) = \Lambda_{R^2} \times B_R(x_0)$$

where $B_{8R}(x_0)\subset\Omega$ and $0\in\Lambda_{(4R)^2}(t_0) = (t_0 - \frac{1}{2}(4R)^2, t_0 + \frac{1}{2}(4R)^2)$.

Then there exist two positive constants ε_0 and c such that, for any $\varepsilon<\varepsilon_0$,

$$\int_{Q_0} |\nabla u|^{2+\varepsilon} dz \le C \max\left\{1, \left(\fint_{Q_0} |\nabla u|^2 dz\right)^{1/2}\right\}^{2+\varepsilon} + \frac{CR^{\frac{\varepsilon}{1+\varepsilon}}\varepsilon}{1+\varepsilon}\left(\int_{B_{4R}} |\nabla u_0|^{1+\varepsilon} dx\right)^{\frac{2+\varepsilon}{1+\varepsilon}},$$

where $Q_0 = Q_{4R,(4R)^2}(t_0)$.

Proof of Lemma 4.9. We use the following convention in this proof:

$$\fint_{Q_r} f(z)dz := \frac{1}{|Q_r|}\int_{Q_r\cap Q} f(z)dz.$$

Let $Q_0 = Q_{4R,(4R)^2}(t_0)$. We divide Q_0 into Whitney-type cylinders $Q_i = Q_{r_i,r_i^2}(y_i,t_i)$ for $i=1,2,\dots$. We require that the radii r_i are proportional to the parabolic distance $dist(Q_i,\partial Q_0)$ and every z belongs to at most a fixed number of cylinders. Finally we assume $Q_{5r_i}\subset Q_0$.

Step 1. We choose $\lambda_0' = \left(\fint_{Q_0} |\nabla u|^2 dz\right)^{1/2}$ and $\lambda > \lambda_0 := \max\{1,\lambda_0'\}$.

For $(t,x)\in Q_0\cap Q$ we define

$$h(t,x) = \frac{1}{c_2}|\nabla u(t,x)|\frac{1}{|Q_0|^{1/2}}\min\{|Q_i|^{1/2} : (t,x)\in Q_i\},$$

where $c_2>1$ is a constant to be chosen later. Now choose $(\bar{t},\bar{x})\in Q$ such that $h(\bar{t},\bar{x})>\lambda$, i.e.,

$$\frac{1}{c_2}|\nabla u(\bar{t},\bar{x})|\frac{1}{|Q_0|^{1/2}}\min\{|Q_i|^{1/2}:(t,x)\in Q_i\}>\lambda>\max\left\{1,\left(\fint_{Q_0}|\nabla u|^2dz\right)^{1/2}\right\}.$$

Notice that if there exists no such $(\bar{t},\bar{x})$, then we have a pointwise bound for $\nabla u(\bar{t},\bar{x})$ and hence we have nothing to prove. Furthermore, fix Q_i such that $(\bar{t},\bar{x})\in Q_i\cap Q$ and define

$$\alpha=\alpha(\bar{t},\bar{x})=|Q_0|/|Q_i|.$$

Claim 1. *For almost all $(\bar{t},\bar{x})$ as above and for some cylinder $Q_\rho(\bar{t},\bar{x})\subset Q_i$, we obtain*

$$\frac{1}{C}\lambda^2\le\fint_{Q_\rho}h^2dz\le C\fint_{Q_{20\rho}}h^2dz\le C^2\lambda^2,$$

for some constant C independent of ρ and λ. Also note that $Q_{20\rho}\subset Q_0$.

Proof of Claim 1. For the upper bound, consider $Q_r=Q_{r,r^2}(\bar{t},\bar{x})$ with $\frac{r_i}{20}<r<r_i$. Using the fact that $Q_r\subset Q_i\subset Q_0$ and that $\frac{1}{|Q_i|}\min\{|Q_j|^{1/2}:(t,x)\in Q_j\cap Q_i\}^2\le 1$, we obtain

$$\begin{aligned}\fint_{Q_r}h^2dz &\le \frac{|Q_0|}{|Q_i|}\fint_{Q_0}h^2\chi_{Q_i}dz\\ &= \frac{1}{|Q_i|c_2^2}\fint_{Q_0}\Big(|\nabla u(t,x)|\min\{|Q_j|^{1/2}:(t,x)\in Q_j\cap Q_i\}\Big)^2dz\\ &\le \frac{1}{c_2^2}\fint_{Q_0}|\nabla u(t,x)|^2dz\\ &\le \frac{1}{c_2^2}\lambda^2\le\lambda^2.\end{aligned}$$

where we had already assumed $c_2\ge 1$.

For the lower bound, we recall Lebesgue's differentiation theorem: Given a point x_0 in $\mathbb{R}^d$ and a family of measurable, bounded sets U_γ such that $|U_\gamma|>c|B_\gamma(x_0)|$ and $U_\gamma\to\{x_0\},\quad\gamma\to 0$. Then, for an integrable function $f:\mathbb{R}^d\to\mathbb{R}$, and for a.e. $x_0\in\mathbb{R}^d$, we have

$$f(x_0)=\lim_{\gamma\to\infty}\frac{1}{|U_\gamma|}\int_{U_\gamma}f(x)dx.$$

We deduce that, for almost every $(\bar{t},\bar{x})\in Q_i\cap Q$ such that $h(\bar{t},\bar{x})>\lambda$, we have

$$\lim_{r'\to 0}\fint_{Q_{r'}}h^2dz\quad=|h(\bar{t},\bar{x})|^2>\lambda^2.$$

Now notice that the function

$$f(r,t,x)=\frac{1}{|Q_r(t,x)|}\int_{Q_r(t,x)}h^2(z)dz$$

is continuous on $[0,R]\times[0,T]\times\bar{\Omega}$, and hence uniformly continuous. Furthermore, for all $\frac{r_i}{20}<r<r_i$, we have $\fint_{Q_r} h^2dz\le\lambda^2$ and, for r' small enough, $\fint_{Q_{r'}} h^2dz\ge\lambda^2$. Therefore there exists a radius $\rho_1\in(0,r_i/20)$ depending only on λ and h such that, the following equality holds $\fint_{Q_{\rho_1}} h^2dz=\lambda^2$ and, by continuity

$$\frac{1}{C}\lambda^2\le\fint_{Q_\rho} h^2dz\le C\fint_{Q_{20\rho}} h^2dz\le C^2\lambda^2,$$

for some $0<\rho<\rho_1<\frac{r_i}{20}$. □

Step 2. Claim 1 allows us to apply Lemma 6.1 in [Par07]. In particular we obtain

$$\fint_{Q_{20\rho}} h^2dz\le c\left(\fint_{Q_{4\rho}} hdz\right)^2+c\left(\fint_{B_{4\rho}}|\nabla u_0|dx\right)^2$$

for $0\in\Lambda_{(4\rho)^2}$. Now we define the level sets

$$\begin{aligned}G(\lambda)&=\{(t,x)\in Q_0\cap Q:h(t,x)>\lambda\},\\ \Gamma(\lambda)&=\{x\in B_{4R}(x_0):|\nabla u_0|>\lambda\}.\end{aligned}$$

Claim 2. *We obtain*

$$\fint_{Q_{20\rho}} h^2dz\le\frac{c\lambda}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap G(\eta\lambda)} hdz+\left(\frac{c}{|B_{4\rho}|}\int_{B_{4\rho}\cap\Gamma(\eta\lambda)}|\nabla u_0|dx\right)^2.$$

Proof of Claim 2. From Step 1 we obtained

$$\fint_{Q_{20\rho}} h^2dz\le c\left(\fint_{Q_{4\rho}} hdz\right)^2+c\left(\fint_{B_{4\rho}}|\nabla u_0|dx\right)^2.$$

Let's split the domains of integration as follows: $Q_{4\rho}=(Q_{4\rho}\cap G(\lambda))\cup((Q_{4\rho}\cap G(\lambda)^c)$ and $B_{4\rho}=(B_{4\rho}\cap\Gamma(\lambda))\cup((B_{4\rho}\cap\Gamma(\lambda)^c)$. We obtain, for a suitable $0<\eta<1$,

$$\begin{aligned}c\left(\fint_{Q_{4\rho}} hdz\right)^2+c\left(\fint_{B_{4\rho}}|\nabla u_0|dx\right)^2\ &\le\ c\left(\fint_{Q_{4\rho}\cap G(\eta\lambda)} hdz\right)^2+c\eta^2\lambda^2\\ &\quad+c\left(\fint_{B_{4\rho}\cap\Gamma(\eta\lambda)}|\nabla u_0|dx\right)^2+c\eta^2\lambda^2.\end{aligned}$$

Now choose η small enough so that

$$2c\eta^2\lambda^2\le\fint_{Q_{20\rho}} h^2dz,$$

and notice that this is possible thanks to the lower bound in Claim 1. By Hölder inequality and the upper bound from Claim 1 we have $\fint_{Q_{4\rho}} h dz \leq c\lambda$ and, altogether

$$\fint_{Q_{20\rho}} h^2 dz \leq c\lambda \fint_{Q_{4\rho}\cap G(\eta\lambda)} h dz + \left(c \fint_{B_{4\rho}\cap\Gamma(\eta\lambda)} |\nabla u_0| dx \right)^2.$$

□

Step 3. Multiplying by $|Q_{20\rho}|$ we obtain

$$\int_{Q_{20\rho}} h^2 dz \leq c\lambda \int_{Q_{4\rho}\cap G(\eta\lambda)} h dz + c \left(\int_{B_{4\rho}\cap\Gamma(\eta\lambda)} |\nabla u_0| dx \right)^2. \tag{4.14}$$

In particular, we would like to extend the inequality above to the whole $G(\lambda)$. We proceed with a covering argument taking $G(\lambda) = \bigcup_{i=1}^{\infty} Q_{20\rho_i}(r_i) \subset Q_0$, which gives a Vitali covering. Summing up (4.14) for each cylinder in the covering and using additivity of the integral we get

$$\int_{G(\lambda)} h^2 dz \leq c\lambda \int_{G(\eta\lambda)} h dz + c \left(\int_{\Gamma(\eta\lambda)} |\nabla u_0| dx \right)^2. \tag{4.15}$$

Claim 3. *The following inequality holds*

$$\begin{aligned} \int_{G(\lambda_0)} h^{2+\varepsilon} dz \quad &\leq \quad c\frac{\varepsilon}{1+\varepsilon} \int_{G(\lambda_0)} h^{2+\varepsilon} dz + c\lambda_0^{\varepsilon} \int_{G(\lambda_0)} h^2 dz \\ &\quad + c\varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1} \left(\int_{\Gamma(\eta\lambda)} |\nabla u_0| dx \right)^2 d\lambda. \end{aligned}$$

Proof of claim 3. We are now going to use equation (4.15):

$$\begin{aligned} \int_{G(\lambda_0)} h^{2+\varepsilon} dz &= \int_{G(\lambda_0)} \left(\int_{\lambda_0}^{h} \varepsilon\lambda^{\varepsilon-1} d\lambda + \lambda_0^{\varepsilon} \right) h^2 dz \\ &= \varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1} \int_{G(\lambda)} h^2 dz d\lambda + \lambda_0^{\varepsilon} \int_{G(\lambda_0)} h^2 dz \\ &\leq c \int_{\lambda_0}^{\infty} \left[\varepsilon\lambda^{\varepsilon} \int_{G(\eta\lambda)} h dz + \left(\int_{\Gamma(\eta\lambda)} |\nabla u_0| dx \right)^2 \right] d\lambda + \lambda_0^{\varepsilon} \int_{G(\lambda_0)} h^2 dz. \end{aligned}$$

For the first term on the right-hand side above we have, using Fubini's Theorem,

$$
\begin{aligned}
\int_{\lambda_0}^{\infty} \varepsilon\lambda^{\varepsilon} \int_{G(\eta\lambda)} hdzd\lambda &= c\varepsilon \int_{G(\eta\lambda_0)} \int_{\lambda_0}^{h/\eta} \lambda^{\varepsilon} h \, d\lambda dz \\
&= c\varepsilon \int_{G(\lambda_0)} \left[\frac{\lambda}{1+\varepsilon}\right]_{\lambda_0}^{h/\eta} hdz \\
&= \frac{c\varepsilon}{1+\varepsilon} \int_{G(\lambda_0)} \left[\left(\frac{h}{\eta}\right)^{1+\varepsilon} - \lambda_0^{1+\varepsilon}\right] hdz.
\end{aligned}
$$

In particular we have obtained

$$
\begin{aligned}
\int_{G(\lambda_0)} h^{2+\varepsilon} dz &\leq \frac{c\varepsilon}{1+\varepsilon} \int_{G(\lambda_0)} \left[\frac{h^{2+\varepsilon}}{\eta} - \lambda_0^{1+\varepsilon} h\right] dz \\
&\quad + c \int_{\lambda_0}^{\infty} \left(\int_{\Gamma(\eta\lambda)} |\nabla u_0| dx\right)^2 d\lambda + \lambda_0^{\varepsilon} \int_{G(\lambda_0)} h^2 dz.
\end{aligned}
$$

Notice that the term $\int_{G(\lambda_0)} \left[-\lambda_0^{1+\varepsilon} h\right] dz$ is negative and we can discard it. □

We are now going to estimate the term $\varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1} \left(\int_{\Gamma(\eta\lambda)} \nabla u_0 dx\right)^2 d\lambda$, namely

$$
\begin{aligned}
&\varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1} \left(\int_{\Gamma(\eta\lambda)} |\nabla u_0| dx\right)^2 d\lambda \\
&\leq \left(\int_{\Gamma(\eta\lambda_0)} |\nabla u_0| dx\right) \int_{\lambda_0}^{\infty} \varepsilon\lambda^{\varepsilon-1} \left(\int_{\Gamma(\eta\lambda)} |\nabla u_0| dx\right) d\lambda \\
&\leq \left(\int_{\Gamma(\eta\lambda_0)} |\nabla u_0| dx\right) \int_{\Gamma(\eta\lambda_0)} \int_{\lambda_0}^{|\nabla u_0|/\eta} \varepsilon\lambda^{\varepsilon-1} |\nabla u_0| d\lambda dx \\
&\leq \left(\int_{\Gamma(\eta\lambda_0)} |\nabla u_0| dx\right) \int_{\Gamma(\eta\lambda_0)} \frac{\varepsilon}{1+\varepsilon} \left[\left(\frac{|\nabla u|}{\eta}\right)^{\varepsilon} - \lambda_0^{\varepsilon}\right] |\nabla u_0| dx \\
&\leq \frac{\varepsilon}{1+\varepsilon} |B_{4R}|^{\frac{\varepsilon}{1+\varepsilon}} C \left(\int_{\Gamma(\eta\lambda_0)} |\nabla u_0|^{1+\varepsilon} dx\right)^{1/(1+\varepsilon)} \int_{\Gamma(\eta\lambda_0)} |\nabla u_0|^{1+\varepsilon} dx \\
&= \frac{\varepsilon}{1+\varepsilon} |B_{4R}|^{\frac{\varepsilon}{1+\varepsilon}} C \left(\int_{\Gamma(\eta\lambda_0)} |\nabla u_0|^{1+\varepsilon} dx\right)^{\gamma_2}.
\end{aligned}
$$

Altogether we have

$$
\begin{aligned}
\int_{G(\lambda_0)} h^{2+\varepsilon} dz \;\; &\leq \;\; c\frac{\varepsilon}{1+\varepsilon}\int_{G(\lambda_0)} h^{2+\varepsilon}dz + c\lambda_0^{\varepsilon}\int_{G(\lambda_0)} h^2 dz \\
&\quad + \frac{\varepsilon}{1+\varepsilon}|B_{4R}|^{\varepsilon/(1+\varepsilon)} C \left(\int_{\Gamma(\eta\lambda_0)} |\nabla u_0|^{1+\varepsilon} dx\right)^{\frac{2+\varepsilon}{1+\varepsilon}},
\end{aligned}
$$

and, for ε small enough,

$$
\begin{aligned}
\int_{G(\lambda_0)} h^{2+\varepsilon} dz \;\; &\leq \;\; c\lambda_0^{\varepsilon}\int_{G(\lambda_0)} h^2 dz \\
&\quad + \frac{\varepsilon}{1+\varepsilon} C R^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_{\Gamma(\eta\lambda_0)} |\nabla u_0|^{1+\varepsilon} dx\right)^{\frac{2+\varepsilon}{1+\varepsilon}}.
\end{aligned}
$$

Notice that in $(Q_0 \cap Q)\backslash G(\lambda_0)$ by definition it holds $h^{2+\varepsilon} \leq \lambda_0^{\varepsilon} h^2$. Moreover, we can replace h by $|\nabla u|$ because of the way the Whitney-type covering is constructed. In particular, the cubes containing points of the form $(0, x)$, can't be too close to the boundary. In conclusion, we have obtained

$$
\begin{aligned}
&\int_{Q_0} |\nabla u|^{2+\varepsilon} dz \\
&\leq C \max\left\{1, \left(\fint_{Q_0} |\nabla u|^2 dz\right)^{\frac{1}{2}}\right\}^{\varepsilon} \int_{Q_0} |\nabla u|^2 dz + \frac{C R^{\frac{\varepsilon}{1+\varepsilon}} \varepsilon}{1+\varepsilon} \left(\int_{B_{4R}} |\nabla u_0|^{1+\varepsilon} dx\right)^{\frac{2+\varepsilon}{1+\varepsilon}} \\
&\leq C \max\left\{1, \left(\fint_{Q_0} |\nabla u|^2 dz\right)^{\frac{1}{2}}\right\}^{2+\varepsilon} + \frac{C R^{\frac{\varepsilon}{1+\varepsilon}} \varepsilon}{1+\varepsilon} \left(\int_{B_{4R}} |\nabla u_0|^{1+\varepsilon} dx\right)^{\frac{2+\varepsilon}{1+\varepsilon}}.
\end{aligned}
$$

This concludes the proof of Lemma 4.9. □

4.1.3 Stability estimates, homotopic approach

We now turn to stability problems for nonlinear diffusion systems without any smallness assumption. There are already some results available in the literature in this direction, in particular, the work of Cannon, Ford and Liar [CFL76] and that of of Coclite and Holden [CH05]. In the case of [CFL76], the authors derive a stability estimate with a direct approach under the assumption of sufficiently high integrability for the gradient of weak solutions. In the second one, the authors deal with classical solutions and they introduce a homotopic method that allows them to prove stability in a standard parabolic norm. Our strategy combines both these approaches, generalising the previous results. Specifically, we will work with weak solutions under the assumption of an entropy structure and integrability of the gradient to some power greater than the spatial dimension d.

We are going to obtain stability results based on an estimate of the length of a special "curve" with values in a suitable space of solutions. Similarly to [CFL76], a regularity assumption (higher integrability of the gradients) is required. However, such assumption this is satisfied in dimensions $d \leq 2$ thanks to Theorem 4.4 in the previous section.

Consider the following systems of equations:

$$\begin{aligned} \partial_t u &= \operatorname{div}\left(A(u)\nabla u\right) + f(u) && \text{in } \Omega, \\ A(u)\nabla u \cdot n &= 0 && \text{on } \partial\Omega, \\ u(0) &= u_0. && (4.16) \end{aligned} \qquad \begin{aligned} \partial_t v &= \operatorname{div}\left(B(v)\nabla v\right) + f(v) && \text{in } \Omega, \\ B(v)\nabla v \cdot n &= 0 && \text{on } \partial\Omega, \\ v(0) &= v_0. && (4.17) \end{aligned}$$

We assume that the matrices A and B are elliptic with ellipticity constants λ_A and λ_B respectively. A key object in the development of this approach is the following system, which is derived combining the previous two:

$$\begin{aligned} \partial_t u_\theta &= \operatorname{div}\left[(\theta A(u_\theta) + (1-\theta)B(u_\theta))\nabla u_\theta\right] + f(u_\theta) && \text{in } \Omega, \\ (\theta A(u_\theta) + (1-\theta)B(u_\theta))\nabla u_\theta \cdot n &= 0 && \text{on } \partial\Omega, \\ u_\theta(0) &= \theta u_0 + (1-\theta)v_0. \end{aligned} \tag{4.18}$$

Given the uniqueness result of Proposition 4.3, for $\theta = 0$, we obtain $u_\theta = v$, and for $\theta = 1$, we have $u_\theta = u$. Intuitively, the function $\theta \to u_\theta, \quad \theta \in [0,1]$, defines a curve connecting the solutions u and v corresponding to different nonlinearities.

The starting point of our strategy, like in [CH05], is given by the following chain of relations:

$$\|u - v\|_p \leq \mathcal{L}_p[u_\theta] = \int_0^1 \left\| \frac{\partial u_\theta}{\partial \theta} \right\|_{L^p(\Omega)} \mathrm{d}\theta, \tag{4.19}$$

where $\mathcal{L}_p$ indicates the length with respect to the p-distance in L^p. The expression of $\mathcal{L}_p$ is justified by Theorem 4.1.6 in [AT04] and we will see that the map $\theta \mapsto u_\theta$ is Lipschitz *a posteriori*. This allows us to transform a stability problem into L^p estimates for $z_\theta := \frac{\partial u_\theta}{\partial \theta}$.

The following technical Lemma guarantees that system (4.18) has an entropy structure, provided that (4.16) and (4.17) do.

Lemma 4.10 (Convex combination of entropies)**.** *Let D be a convex domain. Consider the equations (4.16) and (4.17) and their respective entropy densities, namely $h : U \to \mathbb{R}$ and $k : U \to \mathbb{R}$. Then the function $h_\theta = \theta h + (1-\theta)k$ defines an entropy for equation (4.18).*

Proof of Lemma 4.10. By definition, an entropy density η is a convex, $C^2(D;\mathbb{R})$ where U is a convex domain, and $Dh : U \to \mathbb{R}^N$ is a diffeomorphism.

Since h and k are convex functions, h_θ is convex as well:

$$\begin{aligned} h_\theta(\alpha u + (1-\alpha)v) &\leq \theta\alpha h(u) + (1-\theta)\alpha h(v) + \theta(1-\alpha)k(u) + (1-\theta)(1-\alpha)k(v) \\ &= \alpha\, h_\theta(u) + (1-\alpha)\, h_\theta(v). \end{aligned}$$

In particular, it follows that Dh_θ is a diffeomorphism, indeed it is C^1 and non-singular in U, and therefore it has C^1 inverse thanks to the Inverse Function Theorem. We still have to show that the entropy functional $H_\theta[u_\theta] = \int_\Omega h_\theta(u_\theta)\mathrm{d}x$ satisfies an entropy inequality of type (4.6). Let $M_\theta = \theta A(u_\theta) + (1-\theta)B(u_\theta)$

$$\begin{aligned} \frac{dH_\theta}{dt} &= \int_\Omega \partial_t u_\theta Dh_\theta(u_\theta)\mathrm{d}x \\ &= -\int_\Omega M_\theta(u_\theta)\nabla u_\theta \cdot \nabla Dh_\theta(u_\theta)\mathrm{d}x \\ &= -\int_\Omega M_\theta(u_\theta)\nabla[Dh_\theta^{-1}(Dh_\theta(u_\theta))] \cdot \nabla Dh(u_\theta)\mathrm{d}x \\ &= -\int_\Omega M_\theta(u_\theta)(D^2h_\theta)^{-1}(Dh_\theta(u_\theta))\nabla Dh(u_\theta))] \cdot \nabla Dh(u_\theta)\mathrm{d}x, \end{aligned}$$

where $(D^2h_\theta)^{-1}$ indicates the inverse of the Hessian of h_θ which is a positive definite matrix since h_θ is convex. Therefore,

$$\tilde{M_\theta} = M_\theta(u_\theta)(D^2h_\theta)^{-1}(Dh_\theta(u_\theta))$$

is symmetric and positive definite and we have obtained $\frac{dH_\theta}{dt} \leq 0$. □

Let us now state our main result concerning stability for cross diffusion systems:

Theorem 4.11 (Stability for sufficiently regular solutions)**.** *Consider the weak solutions u, v and u_θ of problems (4.16), (4.17) and (4.18) respectively. Assume that there exists two constants $K < \infty$ and $0 < \gamma < 1$ such that*

$$\begin{aligned} \max_{\theta\in[0,1]} \Big(\|\nabla u_\theta\|_{L^{d/\gamma}(Q_T)} &+ \|A(u_\theta)\|_{L^\infty(Q_T)} + \|DA(u_\theta)\|_{L^\infty(Q_T)} \\ &+ \|B(u_\theta)\|_{L^\infty(Q_T)} + \|DB(u_\theta)\|_{L^\infty(Q_T)} \\ &+ \|Dg(u_\theta)\|_{L^\infty(Q_T)} + \|g(u_\theta)\|_{L^\infty(Q_T)} \\ &+ \|Df(u_\theta)\|_{L^\infty(Q_T)} + \|f(u_\theta)\|_{L^\infty(Q_T)} \Big) \leq K, \end{aligned} \tag{4.20}$$

Then the following stability estimate holds for a.e $t \in [0,T]$:

$$
\begin{aligned}
&\|u-v\|^2_{L^\infty((0,T);L^2(\Omega))} + \|\nabla u - \nabla v\|^2_{L^2(Q_T)} \\
&\le C\left(\|u_0 - v_0\|^2_{L^2(\Omega)} + \|A-B\|_{L^\infty(\mathbb{R}^{N\times N})}\|\nabla u\|^2_{L^2(Q_T)}\right),
\end{aligned}
\tag{4.21}
$$

where the constant C depends on T, Ω, d, u_0, v_0 and K.

The proof will be presented later as it relies on estimates for the solution u_θ of equation (4.18) and its derivative with respect to θ. Let us derive the equation for $z_\theta = \frac{\partial u_\theta}{\partial \theta}$ via differentiation of the equation for u_θ, namely (4.18). We start by considering the equation:

$$
\frac{d}{dt}u_\theta = \operatorname{div}\left[(\theta A(u_\theta) + (1-\theta)B(u_\theta))\nabla u_\theta\right] + f(u_\theta) \qquad \text{in } \Omega, \tag{4.22}
$$

$$
(\theta A(u_\theta) + (1-\theta)B(u_\theta))\nabla u_\theta \cdot n = 0 \quad \text{ on } \partial\Omega, \tag{4.23}
$$

$$
u_\theta(0) = \theta u_0 + (1-\theta)v_0. \tag{4.24}
$$

We denote the ellipticity constants of A and B by λ_A and λ_B respectively.

Remark 4.4 (Equation for z_θ)**.** Given the system (4.22), the equation satisfied by $z_\theta = \frac{\partial u_\theta}{\partial \theta}$ (in the sense of distributions) has the following expression:

$$
\partial_t z_\theta = \operatorname{div}\left[M\nabla z_\theta\right] + \operatorname{div}\left(L z_\theta \nabla u_\theta\right) + F z_\theta + \operatorname{div} G, \tag{4.25}
$$

$$
\left[M\nabla z_\theta + L z_\theta \nabla u_\theta\right]\cdot n = 0 \quad \text{ on } \partial\Omega, \qquad z_\theta(0) = u_0 - v_0.
$$

where

$$
\begin{aligned}
M_{ij} &:= \theta A_{ij}(u_\theta) + (1-\theta)B_{ij}(u_\theta) \ge 0,\\
L &:= \left[\theta DA(u_\theta) + (1-\theta)DB(u_\theta)\right],\\
F &:= Df(u_\theta),\\
G &:= (A(u_\theta) - B(u_\theta))\nabla u_\theta.
\end{aligned}
$$

We derived the expression of each term in the equation for z_θ as follows:

$$
\begin{aligned}
\frac{d}{dt}z_\theta &= \frac{d}{dt}\partial_\theta u_\theta\\
&= \operatorname{div}\,\partial_\theta\left[(\theta A(u_\theta) + (1-\theta)B(u_\theta))\nabla u_\theta\right] + \partial_\theta f(u_\theta)\\
&= \operatorname{div}\left[(A(u_\theta) - B(u_\theta))\nabla u_\theta\right]\\
&\quad + \operatorname{div}\left[(\theta DA(u_\theta) + (1-\theta)DB(u_\theta))z_\theta \nabla u_\theta\right]\\
&\quad + \operatorname{div}\left[(\theta A(u_\theta) + (1-\theta)B(u_\theta))\nabla z_\theta\right]\\
&\quad + Df(u_\theta)z_\theta,
\end{aligned}
$$

and we also obtain the following boundary conditions:

$$\partial_\theta \left[(\theta A(u_\theta) + (1-\theta)B(u_\theta))\nabla u_\theta\right] \cdot n = 0 \quad \text{on } \partial\Omega, \qquad z_\theta(0) = u_0 - v_0.$$

Notice that the matrix M is elliptic for any θ and its ellipticity constant is bounded from below by $\lambda = \min\{\lambda_A, \lambda_B\}$. The tensor T and the field F are bounded in L^∞ provided that u_θ is bounded in in $L^\infty([0,T]\times\Omega)$. The vector field G is bounded in $L^2([0,T]\times\Omega)$ provided that u_θ is bounded in $L^\infty([0,T]\times\Omega)$.

Proposition 4.12 (Estimate for z_θ)**.** *Let $T > 0$ and consider the linear system of equations (4.25) derived in remark 4.4. Such system admits weak solutions*

$$z_\theta \in L^\infty([0,T], L^2(\Omega)) \cap L^2([0,T], H^1(\Omega)).$$

Furthermore, assume that there exists a number $0 < \sigma < 1$ and a constant $M = M(\Omega)$ such that:

$$\int_0^T \|\nabla u\|^{d/\sigma}_{L^{d/\sigma}}(\Omega)\, dt \le M.$$

Then the following estimate holds:

$$\sup_{[0,T]} \|z\|_2^2 + \int_0^t \|\nabla z\|_2^2 \, d\tau \le \left(\|z_0\|_2^2 + \frac{2}{\lambda}\|A-B\|_\infty^2 \|\nabla u\|_2^2\right) \exp(4C_1 t), \tag{4.26}$$

where $\lambda = \min\{\lambda_A, \lambda_B\} > 0$ and the constant C_1 (depending on λ is specified in the proof.

Proof of Proposition 4.12. The existence of a weak solution to the linear system for z_θ is guaranteed, for example, by Theorem 1.1 in [LSU88], Chapter VII or by [CFL76].

We now replace z_θ by z to simplify the notation and set $F = 0$. In the following it is understood that all repeated indexes are implicitly summed up.

Testing the equation against the solution itself gives

$$\int_\Omega \partial_t z^k \cdot z^k = -\int_\Omega M_{ij}\partial_h z^i \partial_h z^j \, dx - \int_\Omega T_{ij,m} z^m \partial_h u^i \partial_h z^j \, dx - \int_\Omega (A-B)_{ij}\partial_h u^i \partial_h z^j \, dx.$$

Fix γ such that $0 < \gamma < \sigma$. Let p be the exponent determined by the relation $\frac{1}{p} + \frac{\gamma}{d} + \frac{1}{2} = 1$. We estimate the "convection" term by Hölder's inequality according to the relation just mentioned and subsequently using Gagliardo-Nirenberg interpolation inequality with the exponents specified below:

$$\begin{aligned}\int_\Omega L_{ij,m} z^m \partial_h u^i \partial_h z^j \, dx \le& \|L\|_\infty \|z\|_p \|\nabla u\|_{d/\gamma} \|\nabla z\|_2 \\ \le& C\|L\|_\infty \left(\|z\|_q^{1-\alpha} \|\nabla u\|_{d/\gamma} \|\nabla z\|_2^{1+\alpha} + \|z\|_2 \|\nabla u\|_{d/\gamma} \|\nabla z\|_2\right).\end{aligned}$$

The following relation between the exponents must hold:

$$\frac{1}{p} = \left(\frac{1}{r} - \frac{1}{d}\right)\alpha + \frac{1-\alpha}{q},$$

and we choose $r = 2$, $q = 2$, which gives $\frac{1}{p} = \frac{1}{2} - \frac{\alpha}{d}$ in addition, from the relation of the Hölder exponents we have $\frac{1}{p} + \frac{\gamma}{d} = \frac{1}{2}$ therefore we obtain $\frac{1}{2} - \frac{\alpha}{d} = \frac{1}{2} - \frac{\gamma}{d}$, which implies $\alpha = \gamma$. Hence, in particular we have

$$p = \frac{2d}{d-2\gamma} < 2^*, \quad 0 < \gamma < 1.$$

Young's inequality implies

$$\|z\|_2^{1-\alpha}\|\nabla z\|_2^{1+\alpha} \le \frac{1-\alpha}{2\varepsilon^{\frac{1+\alpha}{1-\alpha}}}\|z\|_2^2 + \varepsilon\frac{1+\alpha}{2}\|\nabla z\|_2^2,$$

for some $\varepsilon, \mu > 0$ to be specified later. Now let

$$\mathcal{L} = \int_\Omega \partial_t z^k \cdot z^k + \int_\Omega M_{ij}\partial_h z^i \partial_h z^j dx,$$

then

$$\begin{aligned}
\mathcal{L} &\le C_{GN}\|L\|_\infty\|\nabla u\|_{d/\gamma}\Bigg(\frac{1-\alpha}{2\varepsilon^{\frac{1+\alpha}{1-\alpha}}}\|z\|_2^2 + \varepsilon\frac{1+\alpha}{2}\|\nabla z\|_2^2 + \\
&\quad + \frac{1}{2\mu}\|z\|_2^2 + \mu\frac{1}{2}\|\nabla z\|_2^2\Bigg) - \int (A-B)_{ij}\partial_h u^i \partial_h z^j dx \\
&\le C_{GN}\|L\|_\infty\|\nabla u\|_{d/\gamma}\Bigg[\left(\frac{1-\alpha}{2\varepsilon^{\frac{1+\alpha}{1-\alpha}}} + \frac{1}{2\mu}\right)\|z\|_2^2 + \\
&\quad + \left(\varepsilon\frac{1+\alpha}{2} + \frac{\mu}{2}\right)\|\nabla z\|_2^2\Bigg] - \int (A-B)_{ij}\partial_h u^i \partial_h z^j dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{L} &\le \frac{1}{2}Q_0(t)\left(\frac{1-\alpha}{\varepsilon^{\frac{1+\alpha}{1-\alpha}}} + \frac{1}{\mu}\right)\|z\|_2^2 \\
&\quad + \frac{1}{2}Q_0(t)\left((1+\alpha)\varepsilon + \mu\right)\|\nabla z\|_2^2 + \|A-B\|_\infty\|\nabla u\|_2\|\nabla z\|_2 \\
&\le \frac{1}{2}Q_0(t)\left(\frac{1-\alpha}{\varepsilon^{\frac{1+\alpha}{1-\alpha}}} + \frac{1}{\mu}\right)\|z\|_2^2 \\
&\quad + \frac{1}{2}Q_0(t)\left((1+\alpha)\varepsilon + \mu\right)\|\nabla z\|_2^2 + \frac{1}{\lambda}\|A-B\|_\infty^2\|\nabla u\|_2^2 + \frac{\lambda}{4}\|\nabla z\|_2^2,
\end{aligned}$$

and hence

$$\mathcal{L} \le C_1(t)\|z\|_2^2 + C_2(t)\|\nabla z\|_2^2 + C_3(t)\|A-B\|_\infty^2,$$

where $C_{GN} = \max(\gamma_1, \gamma_2)$, where γ_i are the constants from Proposition 4.2 and we define

$$Q_0(t) = C_{GN}\|L\|_\infty\|\nabla u\|_{d/\gamma},$$
$$C_1(t) = \frac{1}{2}Q_0(t)\left(\frac{1-\alpha}{\varepsilon^{\frac{1+\alpha}{1-\alpha}}} + \frac{1}{\mu}\right),$$
$$C_2(t) = \frac{1}{2}Q_0(t)\left((1+\alpha)\varepsilon + \mu\right) + \frac{\lambda}{4},$$
$$C_3(t) = \frac{\|\nabla u\|_2^2}{\lambda}.$$

Therefore, thanks to the ellipticity of M (whose ellipticity constant is controlled from below by a certain $\lambda > 0$ depending on λ_A and λ_B), we have obtained

$$\frac{1}{2}\partial_t\|z\|_2^2 + (\lambda - C_2(t))\,\|\nabla z\|_2^2 \leq C_1(t)\|z\|_2^2 + C_3(t)\|A-B\|_\infty^2.$$

Integrating in time, we have

$$\|z\|_2^2 + 2\int_0^t (\lambda - C_2(t))\,\|\nabla z\|_2^2\,d\tau \leq \|z_0\|_2^2 + 2\int_0^t C_1(t)\|z\|_2^2\,d\tau + 2\|A-B\|_\infty^2\int_0^t C_3(t)\,d\tau$$

Now choose $\mu = (1-\alpha)\varepsilon$ and $\varepsilon = \frac{\lambda}{4Q_0(t)}$ so that

$$2(\lambda - C_2(t)) = 2\lambda - (Q_0(t)\,(2\varepsilon) + \frac{\lambda}{2}) = \lambda,$$

and let

$$J(t) := \|z\|_2^2 + \int_0^t \lambda\,\|\nabla z\|_2^2\,d\tau.$$

Adding $\int_0^t C_1(\tau)\int_0^\tau \lambda\,\|\nabla z\|_2^2\,d\sigma\,d\tau$ on the right hand side, we obtain the inequality (notice that $J(0) = 0$) :

$$J(t) \leq \|z_0\|_2^2 + 2\int_0^t C_1(t)J(\tau)\,d\tau + 2\|A-B\|_\infty^2\int_0^t C_3(t)\,d\tau.$$

By Gronwall's Lemma we obtain:

$$\sup_{[0,T]}\|z\|_2^2 + \int_0^t \lambda\,\|\nabla z\|_2^2\,d\tau \leq \left(\|z_0\|_2^2 + 2\int_0^t C_3(t)\,d\tau\|A-B\|_\infty^2\right)\exp\left(2\int_0^t C_1(t)\,d\tau\right),$$

where

$$
\begin{aligned}
C_1(t) &= \frac{1}{2}Q_0(t)\left(\frac{1-\alpha}{\varepsilon^{\frac{1+\alpha}{1-\alpha}}}+\frac{1}{\mu}\right)\\
&= \frac{1}{2}Q_0(t)\left(\left(\frac{4Q_0(t)}{\lambda}\right)^{\frac{1+\alpha}{1-\alpha}}(1-\alpha)+\frac{1}{(1-\alpha)\varepsilon}\right)\\
&\leq \frac{1}{2}Q_0(t)\left(\left(\frac{4Q_0(t)}{\lambda}\right)^{\frac{1+\alpha}{1-\alpha}}+\frac{1}{(1-\alpha)}\frac{4Q_0(t)}{\lambda}\right)\\
&\leq \frac{1}{2}\left(\left(\frac{4}{\lambda}\right)^{\frac{1+\alpha}{1-\alpha}}Q_0(t)^{\frac{2}{1-\alpha}}+\frac{1}{(1-\alpha)}\frac{4Q_0(t)^2}{\lambda}\right).
\end{aligned}
$$

This provides the desired estimate for z. □

We conclude with the proof of the main theorem of this section:

Proof of Theorem 4.11 . Proposition 4.4 provides us the equation satisfied by z_θ and Theorem 4.12 shows that, under suitable assumptions on the integrability of ∇u_θ, an estimate of the type

$$\|z_\theta\|^2 \leq C\|z_0\|_2^2 + C'\|A-B\|_\infty\|\nabla u\|_2^2$$

holds, where

$$\|z\|^2 = \sup_{[0,T]}\|z\|_2^2 + \int_0^T \|\nabla z\|_2^2\, d\tau.$$

In order to conclude we have to integrate the inequality above with respect to θ and use the following relation:

$$\|u-v\| \leq \int_0^1 \left\|\frac{\partial u_\theta}{\partial \theta}\right\| d\theta. \tag{4.27}$$

In conclusion we have obtained the inequality $\|u-v\|_2^2 \leq C\|z_0\|_2^2 + C'\|A-B\|_\infty\|\nabla u\|_2^2$. □

4.2 Degenerate case

To the best of our knowledge uniqueness and stability estimates for a general degenerate cross-diffusion system is an open problem. Even if the system has an entropy structure (like in [Jün15]), uniqueness is not clear unless we assume a special structure of the system or extra regularity of the solution is required. For systems with a special structure, relevant references include [DLMT15, DT15]. Additionally, a class of degenerate systems has been studied as gradient flow of an entropy functional con an infinite-dimensional Riemannian manifold in [ZM15]. An interesting attempt to prove further regularity can be found in [Le10]. More recently, Chen and Jüngel presented in [CJ18] a new method to prove uniqueness which, when applicable, involves in two steps, i.e. proving that the sum of the component satisfies a scalar equation for which uniqueness of solutions holds, and consequently transferring this information to the full system via relative entropy methods.

In this section we will show that the problem of stability with respect to the nonlinearities can be reduced to the study of the convergence rate of a suitable vanishing viscosity approximation of the system. Let us describe the strategy we have adopted. Considering the stability estimate (4.21) we notice that the constants on the right-hand side depend on the ellipticity constant of the system and therefore the bounds blow up when we consider a degenerate diffusion system. Our idea to circumvent this problem consists in taking two solutions u_A and u_B of two different equations (with diffusion matrices A and B respectively and the same initial and boundary conditions for simplicity) and to consider the corresponding "regularised" solutions u_A^ε and u_B^ε obtained adding artificial viscosity to the system. Inequality (4.21) can be applied to the regularised problem and we have, schematically,

$$\|u_A^\varepsilon - u_B^\varepsilon\| \leq \frac{C}{\lambda(\varepsilon)} \|A - B\|_\infty ,$$

where $\lambda(\varepsilon)$ is determined by A and B and it tends to 0 as $\varepsilon \to 0$. Hence we have to show that there exists a suitable regularisation (of vanishing viscosity type) for which we can prove an explicit rate of convergence: $\|u_A^\varepsilon - u_A\| \leq K\varepsilon^\alpha$, for some $\alpha > 0$. This would imply that, for $\varepsilon > 0$,

$$\|u_A - u_B\| \leq \frac{C}{\lambda(\varepsilon)} \|A - B\|_\infty + 2K\varepsilon^\alpha. \tag{4.28}$$

Therefore, we can optimise in ε and obtain a new stability result. For example, if $\lambda(\varepsilon) = \varepsilon^\beta$ ($\beta > 0$), we choose $\varepsilon = \|A - B\|_\infty^{\frac{1}{\alpha+\beta}}$ and we obtain

$$\|u_A - u_B\| \leq C' \|A - B\|_\infty^{\frac{\alpha}{\alpha+\beta}} .$$

The first step in this direction is to construct a suitable regularisation of the system. We believe that the most natural way of doing so consists in "lifting" the initial datum and proving that the system obtained in this way is non-degenerate. In particular, we will see in section 4.2.1 that the systems we consider satisfy a weak form of a minimum principle. In section 4.2.2 we will obtain an improved stability result for the special case in which the diffusion matrix can be identified as the Jacobian of some monotone vector field.

Section 4.2.3 will be dedicated to the derivation of a free boundary problem associated with degenerate systems with compactly supported initial data. This reformulation of the problem will be used in section 4.2.4 to design a front-tracking numerical algorithm. Finally, we will compare numerically the predictions of two numerical simulations for a toy problem that does not present the special Jacobian structure.

4.2.1 Positivity results

Th following positivity result is very useful in the construction of approximations. Non-negativity of solutions was already proven in [CHL03].

Proposition 4.13 (Strict positivity). *Consider the system:*

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = 0 \qquad \text{in } \Omega, \tag{4.29}$$

$$A(u)\nabla u \cdot n = 0 \qquad \text{on } \partial\Omega, \tag{4.30}$$

$$u(0,x) = u_0(x) \qquad \text{in } \Omega, \tag{4.31}$$

where Ω is a fixed, bounded, Lipschitz domain and

$$A(u)\xi\xi \geq \lambda(u)|\xi|^2 \geq 0, . \tag{4.32}$$

In addition, suppose that $A(0) = 0$, $\lambda(0) = 0$ and u is the unique solution of problem (4.29). Furthermore, suppose that, if $u_i = 0$ for some i, then $A_{kh}(u) = 0$ for any $k \neq h$. Suppose that there exist constants δ_k and M such that $0 < \delta^k \leq (u_0)^k \leq M$ for all k. Then, we have

$$u_k(t,x) \geq \delta_k \text{ a.e. in } [0,T]\times\Omega, \text{ for } k \geq 1.$$

The proof of Proposition 4.13 consists of various steps, the strategy is the following:

1. we introduce a modified diffusion matrix, $\tilde{A}$, and we prove the result for the corresponding system,
2. we show that the new system has an entropy structure in the sense of Chapter 3,
3. we prove the existence of solutions for the modified system.
4. we conclude that, by uniqueness, the solutions of the modified problem and of the original one coincide, which give the result.

Step 1 (modified problem).

Lemma 4.14. *Consider following modified matrix*

$$\tilde{A}(v) = \begin{cases} A(v) & for\ v^k \geq \delta_*^k,\ k = 1 \dots N, \\ \sigma(v)A(v) + (1-\sigma(v))\lambda(v)I & for\ \delta^k \leq v^k \leq \delta_*^k,\ k = 1 \dots N, \\ \lambda(v)I & otherwise. \end{cases} \tag{4.33}$$

where $\delta_*^k > \delta^k$ *and* $\sigma : \mathbb{R}^N \mathbb{R}$ *is such that*

$$\sigma(v) = \begin{cases} 0 & for\ v^k \geq \delta_*^k,\ k = 1 \dots N, \\ \theta(v) & for\ \delta^k \leq v^k \leq \delta_*^k,\ k = 1 \dots N, \\ 1 & otherwise, \end{cases}$$

for a suitable cut off function θ *connecting the values* 0 *and* 1*. Suppose* v *satisfies the equation* (4.29) *with* $\tilde{A}$ *in place of* A *and under the assumption* $\delta_k \leq v_k(0) \leq 1$, $k = 1 \dots N$, *then* $v_k(t,x) \geq \delta_k$ *a.e. in* $[0,T] \times \Omega$, *for all* k, $A(v) = \tilde{A}(v)$ *and so* v *solves problem* (4.29)*.*

Proof. Given a suitable test function η, We have the following weak formulation of the equation for v:

$$\int_0^T \int_\Omega (\partial_t v^k)\eta^k \, dx \, dt = -\int_0^T \left(\int_\Omega \tilde{A}(v)^{kh} \nabla v^k \cdot \nabla \eta^h \, dx \right) dt. \tag{4.34}$$

We consider the following test function:

$$\eta_k = S^k(v) = -[v^k - \delta^k]_- = \begin{cases} 0 & v^k > \delta^k \\ v^k - \delta^k & v^k \leq \delta^k \end{cases}.$$

Notice that, given our assumptions on the initial datum, $S_k(u_0) = 0$ almost everywhere. Furthermore, from (4.33) and (4.34), we have

$$\int_0^T \left(\int_\Omega \tilde{A}^{kh}(v) \nabla v^k \cdot \nabla S^h(v) \, dx \right) dt = 0, \qquad \forall k \neq h.$$

We obtain

$$\begin{aligned} \int_0^T \int_\Omega (\partial_t v_k) S^k(v) \, dx \, dt &= -\int_0^T \left(\int_\Omega \tilde{A}^{kh}(v) \nabla v_k \cdot \nabla S_h(v_h) \, dx \right) dt \\ \int_0^T \int_\Omega (\partial_t S^k(v)) S^k(v) \, dx \, dt &= -\int_0^T \left(\int_\Omega \tilde{A}^{kh}(v) \nabla S^k(v) \cdot \nabla S_h(v_h) \, dx \right) dt \\ \frac{1}{2} \int_\Omega |S_(v(t,x))|^2 \, dx \, dt &= -\int_0^T \left(\int_\Omega \lambda(v) |\nabla S(v)|^2 \, dx \right) dt \end{aligned}$$

by definition of the initial datum and of $\tilde{A}$. It follows that, for every k,

$$\int_\Omega S^k(v(t,x))^2\,dx = 0,$$

for a.e. t and hence $v^k \geq \delta^k$ almost everywhere. □

Step 2 (entropy).

Lemma 4.15. *Suppose that system (4.29) admits an entropy structure in the sense of Definition 4.1. Then the modified system obtained replacing A with $\tilde{A}$ in (4.29) also admits an entropy structure with the same entropy functional.*

Proof. Let us denote the entropy function by $h : \mathbb{R}^N \to \mathbb{R}$. Recall that, by definition of entropy structure, h is a C^2 convex function and its gradient, $D_u h$, defines an invertible map between a fixed domain D to $\mathbb{R}^N$. By assumption, the matrix Hess(h) is symmetric and positive definite, and therefore invertible. Let us denote Hess(h) simply by H. Then we can rewrite problem (4.29) with $A = \tilde{A}$ as in (4.2), namely

$$\partial_t v = \operatorname{div}(\tilde{A}(v)\nabla v) = \operatorname{div}\left(\tilde{B}(v)\nabla\left(D_v h\right)\right), \tag{4.35}$$

where $\tilde{B} = \tilde{A}H^{-1}$. It is easy to check that B is positive semidefinite, in fact it is equivalent to verify it for the matrix $H\tilde{A}$. We have that $H(v)\tilde{A}(v)\xi\cdot\xi \geq 0$, where

$$H(v)\tilde{A}(v)\xi\cdot\xi = \begin{cases} H(v)A(v)\xi\cdot\xi & \text{for } v^k \geq \delta_*^k, \\ \sigma(v)H(v)A(v)\xi\cdot\xi + (1-\sigma(v))\lambda(v)H(v)\xi\cdot\xi & \text{for } \delta^k \leq v^k \leq \delta_*^k, \\ \lambda(v)H(v)\xi\cdot\xi & \text{otherwise.} \end{cases}$$

□

Step 3 (existence). Recall the following fixed point theorem (see, for example, Theorem 11.3 in [GT98]):

Theorem 4.16 (Schaefer's fixed point theorem)**.** *Let P be a continuous and compact map from a Banach space X into itself. If the set*

$$\{x \in X \;:\; x = \lambda P(x)\}$$

is bounded for any $0 \leq \lambda \leq 1$, then P has a fixed point.

Now we can proceed as in [Jün15] and obtain the existence result. We will only present a sketch of the proof since it is the same presented in the above mentioned reference.

Lemma 4.17. *Assume $v_0 \in dom(h)$. There exists a solution $w^k \in L^\infty(\Omega)$ for the following approximation scheme:*

$$\frac{1}{\tau}\int [v(w^k) - v(w^{k-1})]\eta \, dx + \int_\Omega B(w^k)\nabla w^k \cdot \nabla \eta \, dx + \varepsilon \int_\Omega \nabla w^k \cdot \nabla \eta^k \, dx = 0,$$
$$w^0 = Dh(v_0),$$

where $\varepsilon, \tau > 0$, $w = Dh(v)$ and $B(w) = A(v(w))H(v(w))^{-1}$.

Sketch of the proof.

Part 1. Let $y \in L^\infty(\Omega; \mathbb{R}^N)$ and $\delta \in [0,1]$. Define the linear problem

$$a(w,\eta) = F(\eta) \qquad \text{for all } \eta \text{ in } H^1(\Omega),$$

where

$$a(w,\eta) = \int_\Omega B(y)\nabla w \cdot \nabla \eta \, dx + \varepsilon \int_\Omega \nabla w \cdot \nabla \eta \, dx,$$

and

$$F(\eta) = -\frac{\rho}{\tau}\int v(y) - v(w^{k-1})\eta \, dx.$$

Such problem admits a solution w in $H^1(\Omega; \mathbb{R}^N)$ depending on y and ρ.

Part 2. We can now consider the following fixed point problem defined through solution operators S_ρ such that $S_\rho(y) = w$ and $S_1(w) = w$, meaning that we look for a fixed point y of the solution operator S of the elliptic problem above. In particular, the operator S acts between the spaces:

$$S_\rho : L^\infty(\Omega; \mathbb{R}^N) \times [0,1] \to L^\infty(\Omega; \mathbb{R}^N).$$

Notice that $S_\rho(y) = \rho S(y,1)$. The existence of a solution of the fixed point problem for S_ρ in $L^\infty(\Omega; \mathbb{R}^N)$ is shown in [Jün15], hence we omit the proof.

Part 3. The next step is to derive uniform bounds with respect to ε and τ. First of all, consider the following:

$$\int_\Omega B(w)\nabla w \cdot \nabla w \, dx = \int_\Omega H(v)A(v)\nabla v \cdot \nabla v \, dx \geq \sum_{i=1}^N \int_\Omega \alpha_i(v_i)|\nabla v_i|^2 \, dx,$$

for a suitable choice of the functions α_i. Let us identify such functions. Finally, we have the following bounds:

$$\|v\|_{L^2([0,T],H^1(\Omega))} + \sqrt{\varepsilon}\|w\|_{L^2([0,T],H^1(\Omega))} \leq C,$$

and this allows us to control the time derivative as well:

$$\left|\frac{1}{\tau}\int_\Omega [v(w) - v(w^{k-1})]\eta \, dx\right| \leq \int_\Omega H(v)A(v)\nabla v \cdot \nabla v \, dx \leq C.$$

In conclusion, we can pass to the limit for $\varepsilon, \tau \to 0$ and obtain a solution to

$$\int_\Omega \partial_t v \eta \, dx + \int_\Omega A(v) \nabla v \cdot \nabla \eta \, dx = 0,$$

that satisfies the initial datum in the $H^1(\Omega)'$ sense. □

Let us now conclude the proof of the positivity result.

Step 4.

We can apply Lemma 4.17 to system (4.29) and consequently pass to the limit for $\varepsilon \to 0$ and $\tau \to 0$, as shown in [Jün15]. Hence we deduce that solutions to the modified problem exist and have the desired positivity properties thanks to Lemma 4.14. By uniqueness, the solution of problem (4.32) coincides with the solution of the modified problem, hence it satisfies the same bounds.

4.2.2 Stability in the special case $A = D\phi$

In the special case in which the diffusion matrix is a Jacobian, it is possible to obtain a stability result directly, using the so called $(H^1)'$ approach. The key idea is, roughly speaking, to test the equation against the inverse Laplacian of the solution. As a preliminary result, we reformulate the uniqueness results in [Jün15], Section 6(ii), as follows:

Proposition 4.18 (Uniqueness in a special case)**.** *Suppose that the assumptions (1-5) of Theorem 4.1 are satisfied, and consider a weak solution u of the following problem:*

$$\begin{aligned} \partial_t u - \operatorname{div}(A(u)\nabla u) &= 0 \qquad \text{in } \Omega, \\ A(u)\nabla u \cdot n &= 0 \qquad \text{on } \partial\Omega, \\ u(0,x) &= u_0. \end{aligned} \tag{4.36}$$

In particular, system (4.36) is supposed to have an entropy structure in the sense of definition 4.1 and we denote the corresponding motility matrix by B. Suppose additionally that there exists a monotone vector field $\phi : \mathbb{R}^N \to \mathbb{R}^N$ such that either $A(u) = D_u\phi(u)$, or $B(u) = D_u\phi(u)$, where $D_u\phi \in \mathbb{R}^{N\times N}$ indicates the Jacobian matrix of ϕ. Then (4.36) admits a unique weak solution.

We extend the uniqueness result above in two steps. First, we establish a stability result for non-degenerate problems:

Proposition 4.19 (Stability in a special case)**.** *Consider two systems of the form:*

$$\partial_t u_i^k - \Delta\phi_i^k(u) = 0 \qquad \text{in } \Omega, \tag{4.37}$$

$$\nabla\phi_i^k(u_i) \cdot \nu = 0 \qquad \text{on } \partial\Omega, \tag{4.38}$$

$$u_i^k(0,x) = u_{i,0}^k(x), \tag{4.39}$$

for $k = 1, \dots, N$, $i = 1, 2$. *Suppose that the solutions* u_i *are bounded and non-negative, in particular* $0 \le u_i^k \le M$, *and we assume strict monotonicity of* $\phi_i \in C^1(\mathbb{R}^N)$, *i.e. there exist* $\lambda_i > 0$ *such that*

$$(\phi_i(z) - \phi_i(w)) \cdot (z - w) \ge \lambda_i |z - w|^2, \quad \forall z, w \in \mathbb{R}^N.$$

Then the following stability estimate holds

$$\begin{aligned} &\|u_1(t) - u_2(t) - (\bar{u_1} - \bar{u_2})\|^2_{(H^1(\Omega))'} + \lambda_1 \|u_1 - u_2 - (\bar{u_1} - \bar{u_2})\|^2_{L^2(Q_T)} \\ &\le \|u_{0,1} - u_{0,2} - (\bar{u_1} - \bar{u_2})\|^2_{(H^1(\Omega))'} + \frac{|\Omega|}{\lambda_1} \max_{E_M} |\phi_1(\bar{u_1} + \cdot) - \phi_2(\bar{u_2} + \cdot))|^2, \end{aligned} \tag{4.40}$$

where $E_M = \{\xi \in \mathbb{R}^N_+, |\xi| < M\}$ *and we have*

$$\bar{u}_i^k = \frac{1}{|\Omega|} \int_\Omega u_{i,0}^k \, dx.$$

Notice that the constant functions $\bar{u}_i^k$ *are stationary solutions of* (4.37).

Proof. We can rewrite (4.37) as follows:

$$\partial_t (u^k - \bar{u}^k) - \Delta(\tilde{\phi}^k(u - \bar{u})) = 0 \quad \text{in } \Omega, \tag{4.41}$$

$$\nabla \tilde{\phi}^k (u - \bar{u}) \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{4.42}$$

$$u^k(0, x) - \bar{u}^k = u_0^k(x) - \bar{u}^k, \tag{4.43}$$

where $\tilde{\phi}^k(w) = \phi^k(\bar{u} + w)$. Now we consider two solution $w_i = u_i^k - \bar{u_i}^k$ of problem (4.41) with $\tilde{\phi_i}^k(w) = \phi_i^k(\bar{u_i} + w_i)$ for $i = 1, 2$. The equation for the difference $w_1 - w_2$ is given by

$$\partial_t (w_1^k - w_2^k) - \Delta(\tilde{\phi}_1^k(w_1) - \tilde{\phi}_2^k(w_2)) = 0 \quad \text{in } \Omega, \tag{4.44}$$

$$\nabla(\tilde{\phi}^k(w_1) - \tilde{\phi}^k(w_2)) \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{4.45}$$

$$w_1^k(0, x) - w_2^k = u_{0,1}^k(x) - u_{0,2}^k(x) - (\bar{u}_{0,1}^k - \bar{u}_{0,2}^k). \tag{4.46}$$

Consider a test function η such that

$$\begin{aligned} -\Delta \eta^k &= w_1^k - w_2^k \quad \text{in } \Omega, \\ \nabla \eta^k \cdot \nu &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Testing equation 4.44 against η, after an integration by parts by parts we obtain

$$\begin{aligned} \partial_t \int_\Omega \frac{1}{2} |\nabla \eta|^2 \, dx &+ \int_\Omega (\tilde{\phi}_1^k(w_1) - \tilde{\phi}_1^k(w_2))(w_1^k - w_2^k) \, dx \\ &+ \int_\Omega (\tilde{\phi}_1^k(w_2) - \tilde{\phi}_2^k(w_2))(w_1^k - w_2^k) \, dx = 0. \end{aligned}$$

Using the strict monotonicity of ϕ_1 and Young's inequality we have

$$\partial_t \int_\Omega |\nabla \eta|^2 \, dx + \lambda_1 \int_\Omega |w_1^k - w_2^k)|^2 \, dx \leq \frac{1}{\lambda_1} \int_\Omega |\tilde{\phi}_1^k(w_2) - \tilde{\phi}_2^k(w_2))|^2 \, dx,$$

and the result follows integrating in time. □

We now combine the previous result to obtain our final stability estimate:

Theorem 4.20 (Stability). *Consider two systems of the form:*

$$\partial_t u_i^k - \Delta \phi_i^k(u) = 0 \quad \text{in } \Omega \tag{4.47}$$

$$\nabla \phi_i^k(u_i) \cdot \nu = 0 \quad \text{on } \partial\Omega \tag{4.48}$$

$$u_i^k(0,x) = u_0^k(x), \tag{4.49}$$

for $k = 1, \ldots, N$, $i = 1, 2$. Suppose that the solutions u_i are bounded and non-negative, in particular $0 \leq u_i^k \leq M_0$. Assume that $\phi_i(0) = 0$ and that $\phi_i \in C^1(\mathbb{R}^N)$ is monotone, i.e. there exist function $\lambda_i > 0$ such that

$$(\phi_i(z) - \phi_i(w)) \cdot (z - w) \geq 0, \quad \forall z, w \in \mathbb{R}^N,$$

and that, given $R^{kh} := \int_0^1 D_h \phi^k(\tau z + (1-\tau)w) d\tau$, there exists an exponent $\alpha \geq 1$ and a constant $c_0 > 0$ such that

$$(z - w)^t R(z - w) \geq c_0 \delta^\alpha, \quad \forall z, w \in B_\delta(0) \subset \mathbb{R}^N.$$

Then the following stability estimate holds

$$\|u_1 - u_2\|_{L^2(Q_T)}^2 \leq \left(1 + \frac{|\Omega|}{c_0}\right) \max_{E_M} |\phi_1(\cdot) - \phi_2(\cdot))|^{\frac{2}{1+\alpha}}, \tag{4.50}$$

where $E_M = \{\xi \in \mathbb{R}_+^N, |\xi| < M\}$.

Proof. The proof follows combining Propositions 4.13 and 4.19. In particular, thanks to the positivity result, we can construct the approximations u_i^δ by considering two systems of the form (4.47) where the initial condition is "lifted" by $\delta > 0$, i.e. replaced by $u_0^\delta = u_0 + \delta$. We can now apply 4.19 to the approximations, obtaining

$$\|u_1 - u_2\|_{L^2(Q_T)}^2 \leq \delta^2 + \frac{|\Omega|}{c_0 \delta^\alpha} \max_{E_M} |\phi_1(\cdot) - \phi_2(\cdot))|^2,$$

where we used the monotonicity assumption and the fact that the initial data are identical. We can conclude optimising with respect to delta, in particular we choose

$$\delta = \max_{E_M} |\phi_1(\cdot) - \phi_2(\cdot))|^{\frac{1}{1+\alpha}}.$$

□

4.2.3 Free boundary in one dimension, formal computations

We now present a heuristic argument that allows us to rewrite the degenerate problem specified below into a free boundary problem (all the computations are formal). This is in fact useful to construct a numerical scheme that takes into account the finite speed of propagation property of degenerate diffusions and keeps track of the evolution of the support of the solution. In particular we consider

$$\partial_t u^i - \partial_x \left[A(u)^{ij}\partial_x u^j\right] = 0, \quad \text{in} \quad \Omega, t > 0, \tag{4.51}$$

with boundary and initial conditions

$$A(u)^{ij}\partial_x u^j = 0, \qquad \text{for } x = \pm 1, t > 0, \tag{4.52}$$

$$u(0,\cdot) = u_0, \qquad \text{in } \Omega, \tag{4.53}$$

where Ω is the interval $[-1, 1]$.

Let us first focus on the scalar case, in order to illustrate the technique.

Fact 4.21 (Free boundary problem, scalar case)**.** *Consider the scalar equation:*

$$\begin{aligned}
\partial_t u - \partial_x(A(u)\partial_x u) &= 0 \textit{ in } \Omega_t := (P(t), Q(t)),\\
u(t, P(t)) &= 0,\\
u(t, Q(t)) &= 0,\\
u(0, x) &= u_0(x) \textit{ in } \Omega_0 := (P(0), Q(0)),
\end{aligned}$$

where P and Q are (continuous) real functions. A sufficiently regular solution of this problem can be rewritten as the following re-mapped free boundary problem:

$$\begin{aligned}
&\partial_t u(t,x) + \left(\frac{(1-y)P'(t) + (1+y)Q'(t)}{Q(t) - P(t)}\right)\partial_y u(t, x(y))\\
&\quad - \left(\frac{2}{Q(t) - P(t)}\right)^2 \partial_y(A(u(t, x(y))\partial_y u(t, x(y)) = 0 \textit{ in } (-1, 1),
\end{aligned}$$

with boundary conditions

$$u(t,-1) = 0, \qquad u(t, 1) = 0, \qquad u(0, x(y)) = u_0(x(y)) \textit{ in } (-1, 1),$$

where

$$P'(t) = -A'(0)\partial_x u(t, P(t)), \qquad Q'(t) = -A'(0)\partial_x u(t, Q(t)).$$

Proof of Fact (4.21). Let us consider the following change of variables:

$$\begin{aligned} x &= \frac{1-y}{2}P(s) + \frac{1+y}{2}Q(s), \\ t &= s, \end{aligned}$$

$$\frac{\partial x}{\partial y} = \frac{Q(s)-P(s)}{2}, \quad \frac{\partial x}{\partial s} = \frac{1-y}{2}P'(s) + \frac{1+y}{2}Q'(s).$$

The different terms in the equation transform as follows:

$$\begin{aligned} \partial_y u(t,x(y)) &= \frac{\partial x}{\partial y}\partial_x u(t,x) \\ &= \frac{Q(t)-P(t)}{2}\partial_x u(t,x), \\ \partial_s(u(t,x)) &= \partial_t u(t,x) + \partial_x u(t,x)\frac{\partial x}{\partial s} \\ &= \partial_t u(t,x) + \left(\frac{(1-y)P'(t)+(1+y)Q'(t)}{Q(t)-P(t)}\right)\partial_y u(t,x(y)). \end{aligned}$$

All together the equation becomes

$$\begin{aligned} \partial_t u(t,x) &+ \left(\frac{(1-y)P'(t)+(1+y)Q'(t)}{Q(t)-P(t)}\right)\partial_y u(t,x(y)) \\ &- \left(\frac{2}{Q(t)-P(t)}\right)^2 \partial_y(A(u(t,x(y))\partial_y u(t,x(y)) = 0 \ \textit{in} \ (-1,1) \end{aligned}$$

with boundary conditions

$$u(t,-1)=0, \qquad u(t,1)=0, \qquad u(0,x(y)) = u_0(x(y)) \ \textit{in} \ (-1,1).$$

Let us now derive the equations for P' and Q'.

$$u(t,P(t)) = 0 \Rightarrow \partial_t u(t,P(t)) = 0.$$

We differentiate the identity we just derived.

$$\begin{aligned} 0 = \partial_t u(t,P(t)) &= \partial_t u(t,P(t)) + \partial_x u(t,P(t))P'(t) \\ &= \partial_x(A(u(t,P(t)))\partial_x u(t,P(t))) + \partial_x u(t,P(t))P'(t). \end{aligned}$$

This implies that

$$\begin{aligned} -\partial_x u(t,P(t))P'(t) &= \partial_x(A(u(t,P(t)))\partial_x u(t,P(t))) \\ &= A'(u(t,P(t)))(\partial_x u(t,P(t)))^2 + A(u(t,P(t)))\partial_x^2 u(t,P(t)) \\ &= A'(0)(\partial_x u(t,P(t)))^2, \end{aligned}$$

and thus

$$P'(t) = -A'(0)\partial_x u(t, P(t)),$$

and, analogously, for Q'we have

$$Q'(t) = -A'(0)\partial_x u(t, Q(t)).$$

□

We now move to the system case:

Fact 4.22 (Free boundary problem, system case)**.** *Consider the system*

$$\begin{aligned}\partial_t u - \partial_x(A_{11}(u,v)\partial_x u + A_{12}(u,v)\partial_x v) = 0 \ \textit{in} \ \tilde{\Omega}_1 := (P_1(t), Q_1(t)),\\ \partial_t v - \partial_x(A_{21}(u,v)\partial_x u + A_{22}(u,v)\partial_x v) = 0 \ \textit{in} \ \tilde{\Omega}_2 := (P_2(t), Q_2(t)),\end{aligned}$$

with boundary conditions

$$\begin{aligned}&u(t, P_1(t)) = u(t, Q_1(t)) = 0,\\ &v(t, P_2(t)) = v(t, Q_2(t)) = 0,\\ &\qquad u(0,x) = u_0(x) \qquad \textit{in} \ \Omega_{1,0} := (P_1(0), Q_1(0)),\\ &\qquad v(0,x) = v_0(x) \qquad \textit{in} \ \Omega_{2,0} := (P_2(0), Q_2(0)),\end{aligned}$$

where P_i and Q_i are (continuous) real functions. Additionally, suppose that

$$A(u,v) = \begin{pmatrix} f(u) & g(u)h(v) \\ g(u)h(v) & l(v) \end{pmatrix},$$

for suitable real functions f, g, h, l that vanish at 0. A sufficiently regular solution of this problem can be rewritten as the following re-mapped free boundary problem:

$$\begin{aligned}\partial_t u(t, x(y_1)) + \left(\frac{(1-y_1)P_1'(t) + (1+y_1)Q_1'(t)}{Q_1(t) - P_1(t)}\right)\partial_y u(t, x(y_1)) +\\ -\partial_{y_1}(A_{11}(u(t,x(y_1)), v(t,x(y_1))\partial_{y_1} u(t,x(y_1)) +\\ -\partial_{y_1}(A_{12}(u(t,x(y_1)), v(t,x(y_1))\partial_{y_1} v(t,x(y_1))) = 0,\end{aligned}$$

$$\begin{aligned}\partial_t v(t, x(y_2)) + \left(\frac{(1-y_2)P_2'(t) + (1+y_2)Q_2'(t)}{Q_1(t) - P_1(t)}\right)\partial_y u(t, x(y_2)) +\\ -\partial_{y_1}(A_{21}(u(t,x(y_2)), v(t,x(y_2))\partial_{y_2} u(t,x(y_2)) +\\ -\partial_{y_1}(A_{22}(u(t,x(y_2)), v(t,x(y_2))\partial_{y_2} v(t,x(y_2))) = 0,\end{aligned}$$

where the first equation is set in $\tilde{\Omega}_1 := (P_1(t), Q_1(t))$ and the second one in $\tilde{\Omega}_2 :=$

$(P_2(t), Q_2(t))$, *with boundary conditions*

$$\begin{aligned} u(t, P_1(t)) &= 0, \\ u(t, Q_1(t)) &= 0, \\ v(t, P_2(t)) &= 0, \\ v(t, Q_2(t)) &= 0, \\ u(0,x) &= u_0(x) \quad \text{in } \tilde{\Omega}_{1,0} := (P_1(0), Q_1(0)), \\ v(0,x) &= v_0(x) \quad \text{in } \tilde{\Omega}_{2,0} := (P_2(0), Q_2(0)), \end{aligned}$$

and, finally,

$$\begin{aligned} P_1'(t) &= -D_u A_{11}(0, v(t, P_1(t)))\partial_x u(t, P_1(t)) - D_u A_{12}(0, v(t, P_1(t)))\partial_x v(t, P_1(t)), \\ P_2'(t) &= -D_v A_{21}(u(t, P_2(t)), 0)\partial_x u(t, P_2(t)) - D_v A_{22}(u(t, P_2(t)), 0)\partial_x v(t, P_2(t)), \\ Q_1'(t) &= -D_u A_{11}(0, v(t, Q_1(t)))\partial_x u(t, Q_1(t)) - D_u A_{12}(0, v(t, Q_1(t)))\partial_x v(t, Q_1(t)), \\ Q_2'(t) &= -D_v A_{21}(u(t, Q_2(t)), 0)\partial_x u(t, Q_2(t)) - D_v A_{22}(u(t, Q_2(t)), 0)\partial_x v(t, Q_2(t)). \end{aligned}$$

Proof of fact 4.22. Let us consider two changes of variables:

$$x = \frac{1-y_1}{2}P_1(t) + \frac{1+y_1}{2}Q_1(t),$$

$$x = \frac{1-y_2}{2}P_2(t) + \frac{1+y_2}{2}Q_2(t).$$

Therefore the relation between $y_1$and y_2 is given by

$$\frac{1-y_1}{2}P_1(t) + \frac{1+y_1}{2}Q_1(t) = \frac{1-y_1}{2}P_1(t) + \frac{1+y_1}{2}Q_1(t),$$

$$\frac{Q_1(t)+P_1(t)}{2} + y_1\frac{Q_1(t)-P_1(t)}{2} = \frac{Q_2(t)+P_2(t)}{2} + y_2\frac{Q_2(t)-P_2(t)}{2},$$

$$y_2 = \frac{1}{Q_2(t)-P_2(t)}\left(Q_1(t) - Q_2(t) + P_1(t) - P_2(t) + y_1(Q_1(t) - P_1(t))\right).$$

In addition we have

$$\frac{\partial x}{\partial y_i} = \frac{Q_i(t)-P_i(t)}{2},$$

$$\frac{\partial x}{\partial s} = \frac{1-y_1}{2}P_1'(s) + \frac{1+y_1}{2}Q_1'(s) = \frac{1-y_2}{2}P_2'(s) + \frac{1+y_2}{2}Q_2'(s).$$

The different terms in the equation transform as follows:

$$\begin{aligned}\partial_{y_i}u(t,x(y_i)) &= \frac{\partial x}{\partial y_i}\partial_x u(t,x)\\ &= \frac{Q_i(t)-P_i(t)}{2}\partial_x u(t,x),\\ \partial_s(u(t,x(y_i))) &= \partial_t u(t,x(y_i)) + \partial_x u(t,x(y_i))\frac{\partial x(y_i)}{\partial s}\\ &= \partial_t u(t,x(y_i)) + \left(\frac{(1-y_i)P_i'(t)+(1+y_i)Q_i'(t)}{Q_i(t)-P_i(t)}\right)\partial_y u(t,x(y_i)).\end{aligned}$$

All together the equations become:

$$\begin{aligned}\partial_t u(t,x(y_1)) + \left(\frac{(1-y_1)P_1'(t)+(1+y_1)Q_1'(t)}{Q_1(t)-P_1(t)}\right)&\partial_y u(t,x(y_1)) +\\ -\partial_{y_1}(A_{11}(u(t,x(y_1)),v(t,x(y_1))&\partial_{y_1}u(t,x(y_1))+\\ -\partial_{y_1}(A_{12}(u(t,x(y_1)),v(t,x(y_1))&\partial_{y_1}v(t,x(y_1))) = 0\\ u(t,P_1(t)) &= u(t,Q_1(t)) = 0,\\ u(0,x) &= u_0(x),\end{aligned}$$

in $\tilde{\Omega}_1 := (P_1(t),Q_1(t))$, $t \geq 0$, coupled with

$$\begin{aligned}\partial_t v(t,x(y_2)) + \left(\frac{(1-y_2)P_2'(t)+(1+y_2)Q_2'(t)}{Q_1(t)-P_1(t)}\right)&\partial_y u(t,x(y_2)) +\\ -\partial_{y_1}(A_{21}(u(t,x(y_2)),v(t,x(y_2))&\partial_{y_2}u(t,x(y_2))+\\ -\partial_{y_1}(A_{22}(u(t,x(y_2)),v(t,x(y_2))&\partial_{y_2}v(t,x(y_2))) = 0\\ v(t,P_2(t)) &= v(t,Q_2(t)) = 0\\ v(0,x) &= v_0(x),\end{aligned}$$

in $\tilde{\Omega}_2 := (P_2(t),Q_2(t))$. Let us now derive the equations for P_i' and Q_i'. To do so, we are going to (formally) differentiate the boundary conditions with respect to time:

$$u(t,P_1(t)) = 0 \Rightarrow \partial_t u(t,P_1(t)) = 0,$$

hence,

$$\begin{aligned}0 &= \partial_t u(t,P_1(t)) = \partial_t u(t,P_1(t)) + \partial_x u(t,P_1(t))P_1'(t)\\ &= \partial_x(A_{11}(u(t,P_1(t)),v(t,P_1(t)))\partial_x u + A_{12}(u(t,P_1(t)),v(t,P_1(t)))\partial_x v(t,P_1(t)))\\ &\quad + \partial_x u(t,P_1(t))P_1'(t).\end{aligned}$$

Notice that

$$\begin{aligned}
&\partial_x(A_{11}(u,v)\partial_x u + A_{12}(u,v)\partial_x v)\\
&= D_u A_{11}(u,v)(\partial_x u)^2 + D_v A_{11}(u,v)\partial_x v \partial_x u + A_{11}(u,v)\partial_x^2 u +\\
&\quad D_u A_{12}(u,v)\partial_x u \partial_x v + D_v A_{12}(u,v)(\partial_x v)^2 + A_{12}(u,v)\partial_x^2 v
\end{aligned}$$

We will now make use of the assumption on the that the dependence of A on the different components of the solutions, i.e.:

$$A(u,v) = \begin{pmatrix} f(u) & g(u)h(v) \\ g(u)h(v) & l(v) \end{pmatrix}, \quad A(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then

$$\begin{aligned}
-\partial_x u(t,P_1(t))P_1'(t) &= D_u A_{11}(u,v)(\partial_x u)^2 + D_u A_{12}(u,v)\partial_x u \partial_x v\\
&= D_u A_{11}(0,v(t,P_1(t)))(\partial_x u(t,P_1(t)))^2\\
&\quad + D_u A_{12}(0,v(t,P_1(t)))\partial_x u(t,P_1(t))\partial_x v(t,P_1(t)),
\end{aligned}$$

assuming $\partial_x u(t,P_1(t)) \neq 0$, we obtain

$$P_1'(t) = -D_u A_{11}(0,v(t,P_1(t)))\partial_x u(t,P_1(t)) - D_u A_{12}(0,v(t,P_1(t)))\partial_x v(t,P_1(t)).$$

Th remaining equations are obtained in an analogous way. □

4.2.4 Numerical simulations for a degenerate system

We are going to simulate a nonlinear system of the form:

$$\partial_t u^i - \partial_x \left[A(u)^{ij}\partial_x u^j\right] = 0, \quad \text{in} \quad \Omega, t > 0, \tag{4.54}$$

with boundary and initial conditions

$$A(u)^{ij}\partial_x u^j = 0, \quad \text{for } x = \pm 1, t > 0, \tag{4.55}$$

$$u(0,\cdot) = u_0, \quad \text{in } \Omega, \tag{4.56}$$

where Ω is the interval $[-1,1]$. We will consider the following case:

$$A(u) = \begin{pmatrix} u_1(1-u_1) & -u_1 u_2 \\ -u_1 u_2 & u_2(1-u_2) \end{pmatrix}. \tag{4.57}$$

We present two different numerical approaches to the system above.

In the first case, we solve (4.54) in the domain $\Omega = [-1,1]$ using the method presented in section 3.2.2. We briefly recall that this is a second-order accurate finite-difference

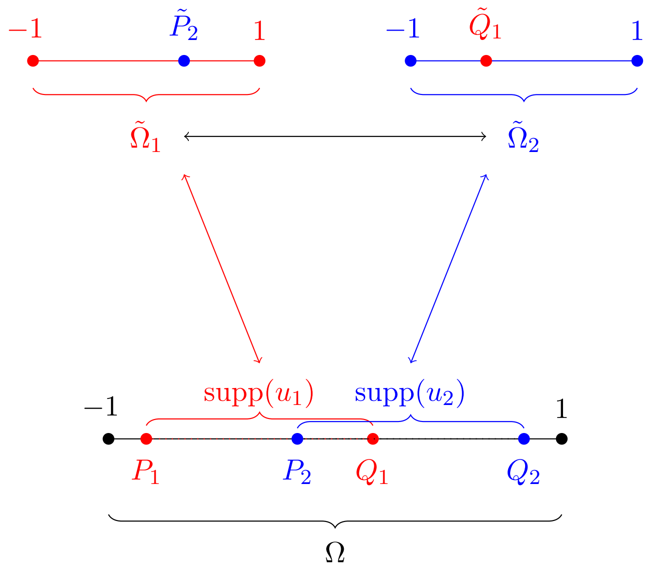

Figure 4.1: A schematic reperesentation of the remapping strategy adopted in the development of the front-tracking algorithm.

scheme in space and the method of lines with the inbuilt Matlab ode solver `ode15s` in time. We use an equidistant mesh of size $|\Omega|/J$, with nodes $x_n = -1 + n\Delta x$, $0 \leq n \leq J$, $J = 100$. The fluxes are evaluated at the nodes x_n to ensure the no-flux conditions are imposed accurately, while the solutions u_i are computed at the midpoints $x_{n+1/2}$.

It is well known that in the presence of degenerate diffusion, as in (4.57), when the initial datum is compactly supported the solution presents a free boundary. It is often important to be able to locate precisely the points of the free boundary, but the first algorithm we presented is not designed for this purpose. We now present one way of constructing a a *front-tracking algorithm.*

The key idea consists in mapping the evolving support of each of the two components of u to fixed domains, $\tilde{\Omega}_i$, and subsequently to obtain and solve an equation for each of the two components on the new domains, as summarised in figure 4.1.

In order to preserve the properties of the original system, each component of the solution u_i has to be re-mapped to the domain $\tilde{\Omega}_j$ $(i \neq j)$ at each time-step. In addition, the new system is coupled with as many ODEs as points on the boundary of the support of the components of u. For convenience, we choose $\tilde{\Omega}_i = [-1, 1]$. In order to ensure that all the equations are well posed, in our practical examples we assume that both components of u have compact and connected support, i.e. each component is supported on a closed interval that is strictly contained in $\Omega = [-1, 1]$. The extreme points of such interval are used to provide the initial conditions of the ODEs corresponding to each point of the free boundary. The new equations are specified in Fact 4.22. We would like to stress that the equations that we obtain in the new, remapped domain (image of the support) have to satisfy homogeneous Dirichlet boundary conditions.

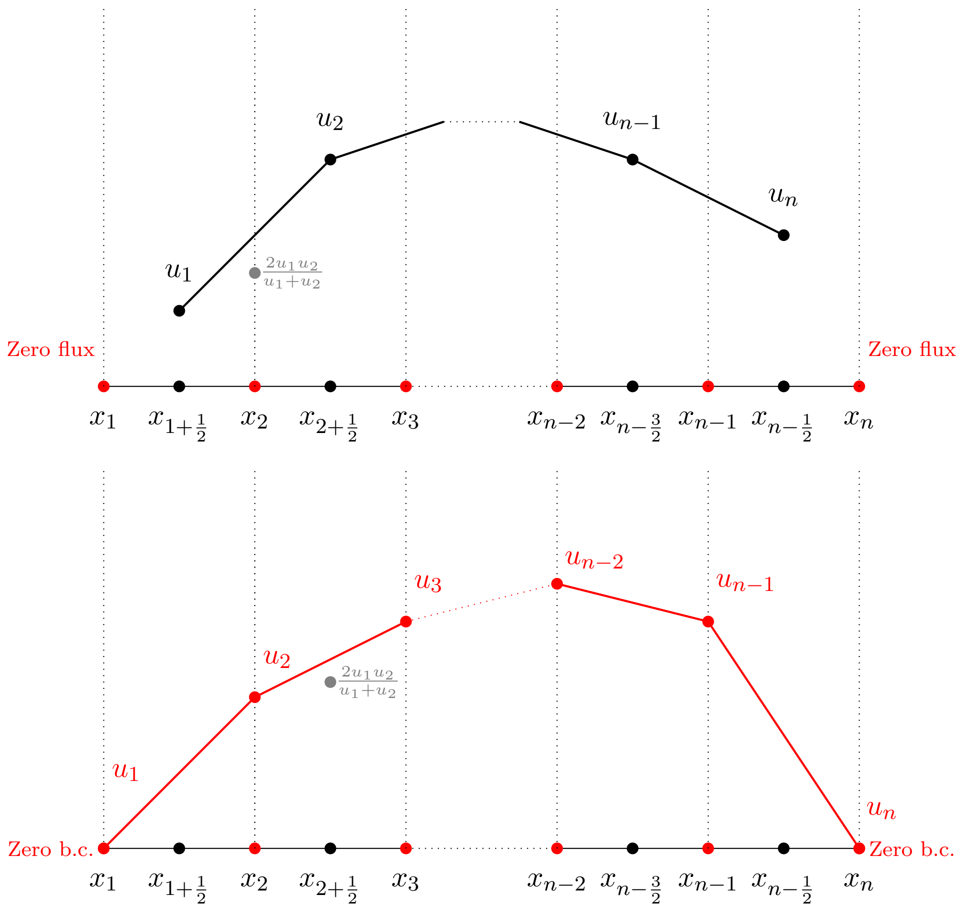

Figure 4.2: A schematic representation of the two methods of simulation for problem 4.54. In the first type of discretisation (see section 3.2.2), the solutions are computed at the black half-nodes, while the fluxes are computed at the red ones. In the second case, namely the front-tracking method, the solutions are computed at the red nodes, while the fluxes are computed at the black half-noes.

Once again, we simulations use a second-order accurate finite-difference scheme in space and the method of lines with the inbuilt Matlab ode solver `ode15s` in time. We consider the following expression for the numerical fluxes:

$$F_{i,n+1} = \mathcal{A}_{ij}(u_{n+\frac{1}{2}})\frac{u_{j,n+1} - u_{j,n}}{\Delta x}$$

with $u_{i,n+\frac{1}{2}} = \frac{2u_{i,n+1}u_{i,n}}{u_{i,n+1}+u_{i,n}}$. We obtain the following system of ODEs:

$$\frac{du_n}{dt} = \frac{F_{n+1} - F_n}{\Delta x}.$$

The approximate solutions is computed on the new domains Ω_i' and subsequently mapped back to Ω. We set as initial data $u_{i,0} = \max(0, \frac{1}{3}(1-(x/\sigma \pm \tau)^2))$, for $\sigma = 0.3$ and $\tau = 0.5$. We run the time-dependent simulation until $T = 1$. Figure 4.2 highlight the differences with the numerical scheme described in section 3.2.2.

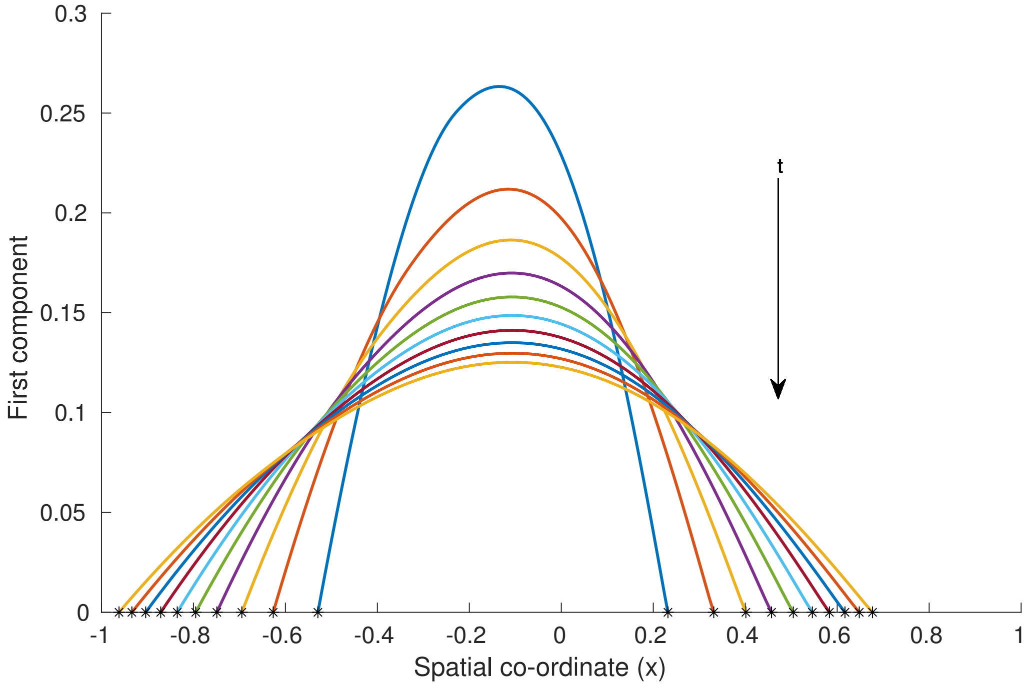

Figure 4.3: Profile of the first component of the solution of (4.54) obtained with the front-tracking algorithm. The initial condition is the function with the smallest support, as time passes the support slowly expands and the points on the free boundary are marked by black asterisks. The extreme points of the support itself are plotted as black asterisks.

Figure 4.3 shows different profiles of the first component of the solution of problem(4.54) for a sequence of equidistant time steps. It is apparent that the support of the solutions grows with finite speed.

Figures 4.4 and 4.5 clarify some of the strengths of the front-tracking method. The evolution in time of the points of the free boundary is shown in figure 4.4. Notice that when the external branches of the blue and the red curves reach the boundary of $\Omega = [-1, 1]$ the method is no longer applicable since there is no more a free-boundary. Figure 4.5 is obtained zooming in figure 4.3 and comparing the approximate solution obtained with the front-tracking method with the one obtain with the method of section 3.2.2. As expected, the new method captures the behaviour of the free boundary in greater detail (compare the red circles on the line $y = 0$ and the blue curves).

Figure 4.6 shows the evolution of the difference $\tilde{u} - u$ in different norms, namely $L^\infty(-1, 1)$, $L^2(-1, 1)$ and $L^1(-1, 1)$. Here $\tilde{u}$ indicates the solution obtained with the method in 3.2.2 and u corresponds to the front-tracking method. We observe that the differences are small, especially in the L^2 and L^1 norms. The maximum of the difference is actually attained near the free boundary points, whereas the methods give very similar predictions near the centre of the support of each component.

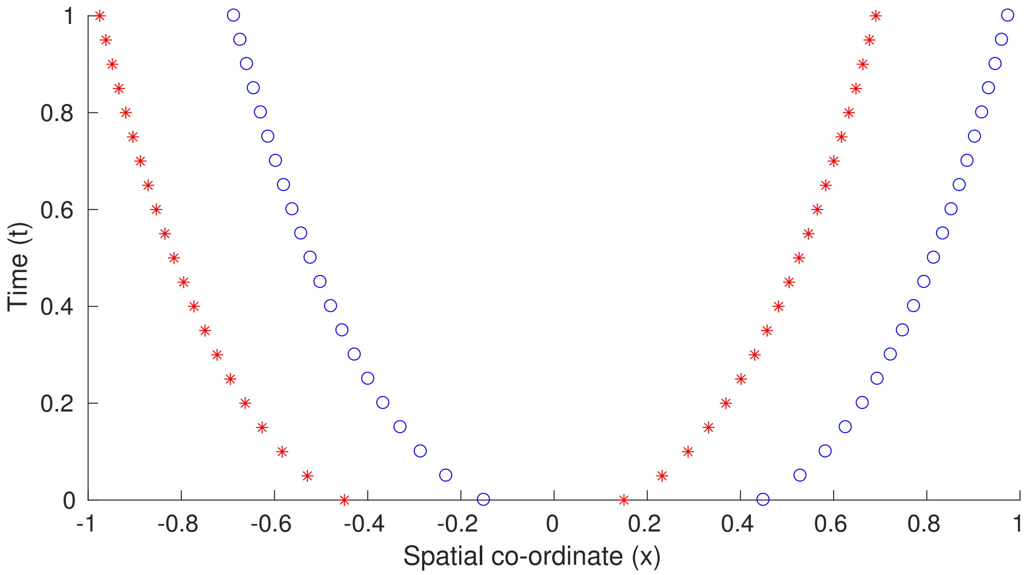

Figure 4.4: Evolution of the points on the free boundary of u_1 over time. The front-tracking algorithm stops working when such points reach the boundary of the domain.

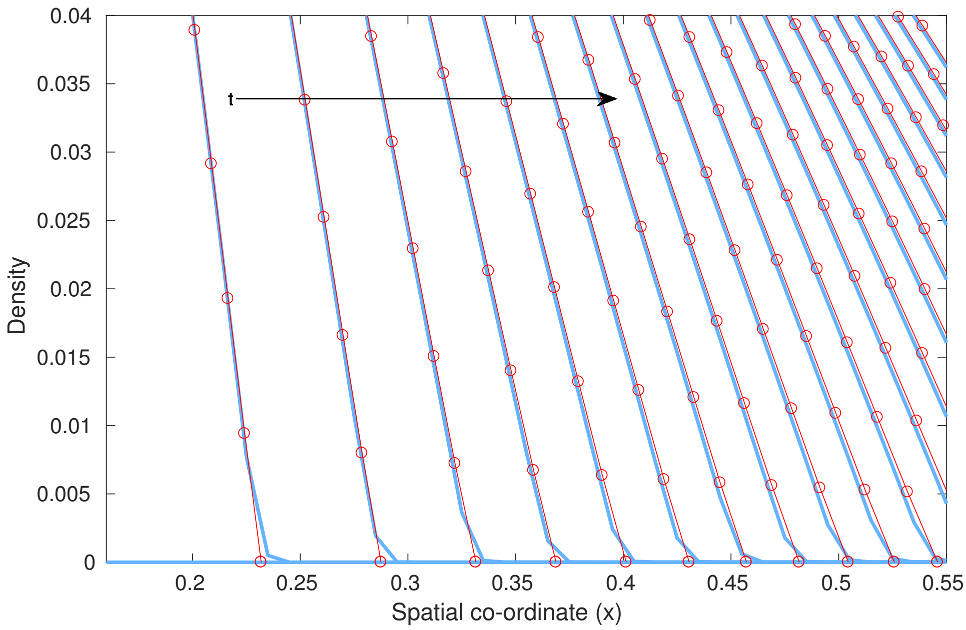

Figure 4.5: A comparison of the two algorithms. We plot the solution obtained with the first method in blue and the solution obtained with the front-tracking method in red. In this case we are just considering the first component of the solution and we focus on a subdomain of Ω. By construction, the second method is much more accurate at the boundary of the support of the solution.

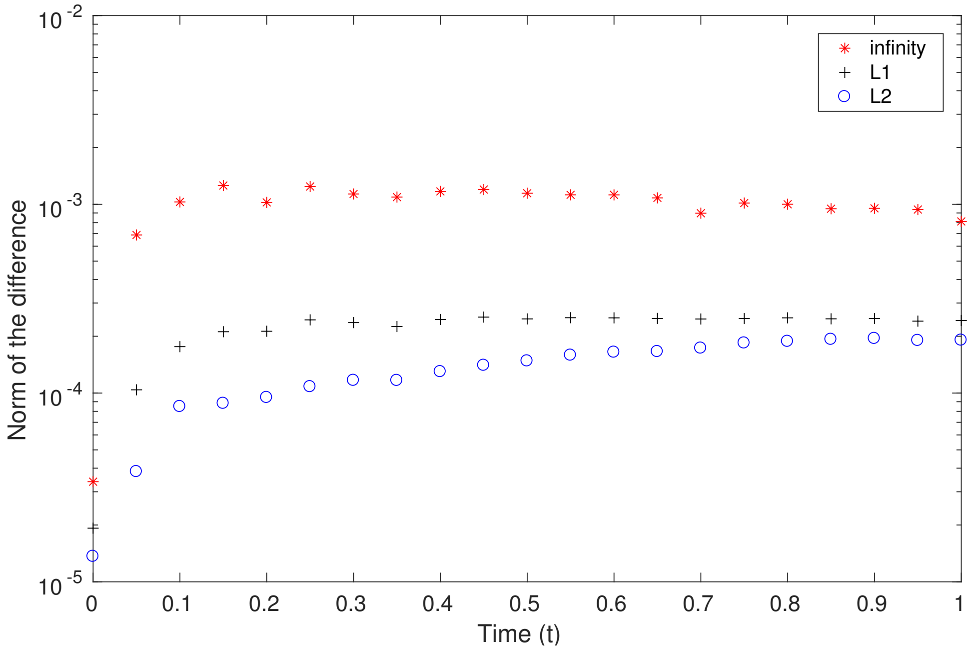

Figure 4.6: Difference of the solutions obtained with the two methods described above. We consider the difference of the first components in different norms. The difference, which is zero initially, is larger near the boundary of the support of the solution.

Final remarks

Let us mention some of the challenges and open problems in the analysis of nonlinear cross-diffusion systems that we would like to address in the future.

For *general* degenerate parabolic systems, many issues concerning well-posedness and stability of solutions are still open. Their study is interesting both from the theoretical and the applied point of view (a special case was considered in Section 4.2) and we illustrate some possible developments.

Existence results for a wide class of cross-diffusion problems of the form $u_t = \Delta A(u) + R(u)$ have recently been improved in [LM17]. For this family of problems they introduce the notion of *non-uniform entropy*, allowing for more general positive semidefinite motility matrices. Generalising the known results and techniques to such systems will probably require the development of new mathematical tools.

To the best of our knowledge, uniqueness and stability estimates for general degenerate cross-diffusion systems are an open problem. Even if the system has an entropy structure, uniqueness is not clear unless we assume either a special structure of the system, or extra regularity of the solution is required (see, for example, [CJ18]). The techniques introduced in Section 4.2 might be useful to proceed in this direction.

Systems with non-local terms, such as interaction potentials describing long-range attraction or repulsion, are particularly relevant in the modelisation of collective behaviour. New results for this type of problems can be found in [DFEF18]. A simple example of scalar equation involving a nonlocal term of this type was presented in Section 2.3.

It is also desirable to obtain better stability estimates for a class of cross-diffusion systems including a broader range of models used in Mathematical Biology. In addition, we think it is important to consider an heterogeneous environment (i.e. allowing the diffusion coefficients to depend on the space variable). In this sense, the theory of homogenisation can be relevant to study problems that involve a distribution of spatially heterogeneous obstacles affecting the evolution of the other components.

Finally, we recall that nonlinear diffusion equations play a role not only in Mathematical Biology, but also in image de-noising, [Wei98], in the study of geometric flows, [DdP95], and in drift-diffusion models for semiconductor devices, [CJ07].

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