The Linguistic Expressions of Arithmetical Operations in the Italian *Libri d'Abbaco*

Abstract

The present paper is the first ever investigation of the formation of vernacular mathematical terminology in Italian Renaissance practical arithmetic manuals called *abbaci* (where *abbaco* means 'computation'). These are an excellent source of information about the lexicon and grammar of the Renaissance merchant and artisan, and reveal many of the regional and dialectal variations of the spoken language. The distinctive feature of *abbaco* literature is its emphasis on the practical aspect of learning: these works aimed to put mathematical knowledge into everyday use by providing practice in a great variety of mathematical problems, and these were mainly couched in situations familiar in the market-place of Renaissance towns. We focus on the lexicon (mainly verbs) used for the basic operations of addition, subtraction, multiplication, and division, because it reveals much about the formation of the mathematical lexicon, where vernacular terms having everyday meanings also acquire a specialized meaning within the mathematical register. Our analysis shows that in the *abbaci* some common everyday terms were employed to denote mathematical concepts and procedures; these were widely shared and established within the tradition despite geographical differences and local phonological differences, undergoing a process of meaning specialization. We investigate how vernacular mathematical vocabulary differs from the corresponding established Latin vocabulary in several respects: it seeks to explain new arithmetical procedures by using terms from the spoken language, thus embodying a process of new knowledge production addressing a new audience of common people. Terms belonging to the sphere of everyday language acquire new technical meanings in the context of computation.

Keywords:

Language of Mathematics; Vernacular Italian; Abbaco Manuscripts; Practical Mathematics; Indo-Arabic Numerals; Operations Verbs; Arithmetical Operations

1. Introduction

Before the introduction of Indo-Arabic figures, the numeration system in the West was based on Roman numerals, represented in writing by letters of the Roman alphabet.[1] The transition from Roman to Indo-Arabic numerals represented a complex paradigm shift (in the Kuhnian sense of a fundamental change in the basic concepts and experimental practices of a scientific discipline) with broad cultural, social, and linguistic implications. In Italy, the transition stimulated a specific vernacular corpus in the form of the *abbaco* literature, which played a major role in the dissemination of the new numerical notation. *Abbaco* texts were the primary vehicles for spreading knowledge of Indo-Arabic numerals and new procedures of calculation to the ordinary people (artisans, merchants, builders, shop-keepers), although they were not only aimed at them. Written in Italo-Romance vernaculars (Florentine and other early dialects of Italy)[2] *abbaci* focus on commercial uses of arithmetic; this textual genre became increasingly popular in the most commercially oriented cities of the Italian peninsula. The distinctive feature of this type of text is its emphasis on the practical aspect of learning: these works aimed to put mathematical knowledge into everyday use by providing practice in a great variety of mathematical problems, and these were mainly couched in situations familiar in the market-place of Renaissance towns.

The *abbaci* provide excellent material for a close study of the language of fourteenth and fifteenth century vernacular mathematics. They are an excellent source of information about the lexicon and grammar of the Renaissance merchant and artisan, and reveal many of the regional and dialectal variations of the spoken language. Technical registers in medieval Italian varieties have largely been neglected by linguists,[3] and discussions of technical texts by historians of mathematics tend to overlook their linguistic aspects. The complex and multi-faceted lexicon and phraseology found in the *abbaci* have far-reaching interdisciplinary implications. There are several aspects of the mathematical register of the time which make it particularly suitable for linguistic research, such as the role of language contact, Latinisms, and the relationship between specialized vocabulary and everyday spoken language. Here we aim to investigate the formation of vernacular mathematical terminology; in particular, we will focus on the linguistic expression of the four basic arithmetical operations. We demonstrate that, as a written product of their urban milieu, *abbaci* reveal much about the language of the ordinary people; these works were often not composed by men of letters, but by elementary-school teachers and merchants (sometime amateurs and land surveyors employed by the local council)—people who had probably themselves received only a minimal education in reading and writing, and were interested in

[1] On medieval Latin mathematics, see Folkerts and Hughes (2016: 4-223). Two classics on the history of Indo-Arabic numerals are by Smith and Karpinski (1911) and Cajori (1929). As explained by Chrisomalis (2010), in Europe Roman numerals were for a long time used side by side with the newly introduced Indo-Arabic figures. See also Burnett (2010).

[2] Iimproperly called 'Italian dialects', they are more correctly regarded as languages distinct from, but related to, Italian.

[3] Exceptions are Bocchi (2017), Feola (2008), and Manni (2001).

teaching the practical skills needed in the marketplace. Our analysis shows that in the *abbaci* some common everyday terms were employed to denote mathematical concepts and procedures. These were widely shared and established within the tradition despite geographical differences and local phonological differences, undergoing a process of meaning specialization while also departing from corresponding Latin forms such as those found in the Latin *Liber abaci* ("Book of Calculation", first written in 1202 CE by Leonardo Pisano (more widely known as 'Fibonacci'), which represents a seminal work in the mathematics of the Middle Ages (see more in section 7). There is certainly a high degree of specialized technical expression in the *Liber abaci* just as in the *abbaco* manuals, but there is not always an obvious continuity between the exponents of the mathematical expressions in the former and those of the latter. We argue that *abbaco* writers were actively seeking vernacular equivalents accessible to pupils and readers who did not know Latin. For instance,[4] while Latin term *multiplicatio* is continued in *moltiplicazione* to denote 'multiplication', the *abbaci* use *partizione* rather than derivatives of Lat. *divisio* to denote 'division'; Latin has *extrahere/subtrahere* and Italian has the related *trarre/sottrarre* but *also abbattere, cavare* to mean 'subtract'; and Latin has *addere* while the *abbaci* have for example *raccogliere* to mean 'add'. In the *Liber abaci*, a 'fraction' is denoted by *ruptus*, continued in the *abbaci* as *rotto,* and the dividing line of a written fraction is *virga* or *virgula,* which in *abbaci* is *verga* and *virgola;*[5] but other terms depart from the Latin tradition, for instance Lat. *denominatus* vs It. *partitore* to denote the denominator of a fraction.[6] We will see how vernacular mathematical vocabulary differs from the Latin tradition in several respects: it seeks to explain new arithmetical procedures by using terms from spoken language, thus embodying a process of new knowledge production and addressing a new audience of common people. Terms belonging to the sphere of everyday language acquire new technical meanings in the context of computation. We focus on the lexicon (mainly verbs) of the basic operations of addition, subtraction, multiplication, and division, because they reveal much about the formation of the mathematical lexicon, where vernacular terms having everyday meanings acquire a specialized meaning within the mathematical register. [7]

The main questions that guide our investigation are: what are the core features of the vernacular mathematical lexicon for arithmetical operations as found in *abbaci*? And what do the lexical items found therein tell us about the way the mathematical lexicon was shaped? What are the lexical tendencies in arithmetical operations and mathematical concept-naming in Italo-romance?[8] The

[4] The Latin terms are found in the *Liber abaci.*
[5] In the *Livero de l'abbecho* (the first *vernacular abbaco* to survive) the two operands of a fraction are expressed by referring to their position: *i nu(mer)e de sopre da le v(er)ge* (the numerator) and *gle num(er)e che sonno de sotta a le v(er)ge* (the denominator). Here the term *verga* can denote: 1) a fraction dividing line; 2) the whole fraction; 3) the denominator in the expression *moltiplicare (dividere) per la sua verga.* See more in Bocchi (p. 107). Such a usage of *verga* is wider than what is found in the *Liber abaci.*
[6] The Latin expression *verbi gratia* introducing a sample problem (used in the early thirteenth-century Latin work by Leonardo Pisano *Liber abaci* as well) is widely found in *abbaci*.
[7] We have drawn on the OVI database (http://gattoweb.ovi.cnr.it/).
[8] It is beyond the scope of this analysis to investigate individual mathematical authors' preferences for certain key terms.

analysis which follows is based on a set of twenty *abbaci* which have been selected according to the following criteria: 1) the subject treated (only *abbaci* that treat all four arithmetical operations); 2) geographical origin of the author/compiler (to reflect a range of vernacular varieties); 3) existence in in manuscript forms and in published scholarly transcriptions; 4) variety of focus, drawing on terminology from *abbaci* that treat arithmetic but also from those focusing on algebra and geometry, and 5) the timeframe, which is 1290-1500 CE, from the first surviving *abbaco* written in vernacular up to the sixteenth century, thus covering the golden age of *abbaco* literature.[9]

2. The Linguistic Expression of Arithmetical Operations in the Libri d'Abbaco

As stressed by Van Egmond (1980),[10] the *abbaci* can be defined as collections of practical mathematical problems drawn from many areas of everyday life, but primarily from situations related to the affairs of business and commerce. Their mathematical content varies, and includes business problems and recreational mathematics dealing with arithmetic, algebra, and geometry, as well as miscellaneous material such as calendars and astrology. It is this concern for the practice of mathematics that establishes the *abbaco* tradition as a distinct field of mathematics. The fact that they are written in the vernacular rather than in Latin indicates that *abbaci* belong to everyday life and particularly the world of the market place. As a literary product of their surrounding urban milieu, these practical mathematical texts reflect the native language of the ordinary people and the complex dynamics of the commercial, banking, and artisan sectors in the major Italian cities of the time. In their intention to instruct, vernacular mathematical works constitute a distinct branch of didactic literature. In the Renaissance period, Italy was the most urbanized area of Europe with a highly developed commercial and civic life resulting in widespread diffusion of education and literacy.[11] The church, dominant in education up to the thirteenth century, became far less important in education thereafter, and pre-university teaching passed overwhelmingly into secular hands. There were three main types of school: elementary school, to teach reading; grammar school, to teach Latin; *abbaco* school, to teach practical arithmetic. In elementary and grammar schools Latin was the exclusive medium of instruction, so the *abbaco* syllabus was always taught in the vernacular. *Abbaco* literature developed in tandem with the spread and popularity of so-called *botteghe d'abbaco*, commercially oriented schools attended by ten-to-twelve-year-old boys (mainly future bankers, merchants, artisans, builders, and shopkeepers) to learn practical

[9] The anonymous *Livero de l'abbecho* was written in the dialect of Perugia sometime around 1290 and the first decade of the 1300s. Danna (2021) shows that during the sixteenth century, the production of *abbaci* decreased considerably.
[10] See Van Egmond (1976 and 1980).
[11] Education in Renaissance Italy is investigated by Grendler (1991) and Black (2001).

mathematics. Some communal public schools, especially in Tuscany, Lombardy, and Venice, offered a commerce-directed curriculum instructing students in banking, commerce, and craftsmanship.[12] Tuscany was home to some of the most important and prestigious *abbaco* schools and masters. [13] *Botteghe d'abbaco* appeared in the mid-thirteenth century and are documented up to the early sixteenth, especially in cities with a dynamic economy.[14]

Commented [MM(1)]: NB Trinita, in note!

In *abbaci*, both numeral-signs and numeral-words were accompanied by lengthy explanations of mathematical procedures. Pre-modern mathematical writing employed discursive strategies to teach and expound knowledge and was mainly rhetorical in nature. By contrast, modern expressions of arithmetical operations are usually formulaic and (at least in writing) highly symbolic and independent of speech sounds.[15] In written formulations such as '4 + 2 = 6', '4 - 2 = 2', '4 × 2 = 8', '4 ÷ 2 = 2', the nature of the operation is expressed by the distinctive operator symbols '+, ×, -, ÷' which in modern Italian correspond, in oral formulation, to *più, meno, moltiplicato per, diviso* (*per*), respectively. The result of an operation is introduced by the symbol '=' (orally in Italian, '*uguale*'). The use of symbols to express arithmetical operations appears to have fully emerged no earlier than the late sixteenth century.[16] In the language of the *libri d'abbaco* mathematical expressions are grounded in extremely concrete vocabulary and their meaning is relatively transparent. The operation itself is typically expressed as an instruction (often in the imperative or future forms of the verb) of the type '(you will) join 4 and 2', '(you will) draw / pull / dig out 2 from 4', '(you will) multiply 4 by 2', '(you will) share 4 into/by 2' and the result of the operation is expressed by a separate (usually following) clause of the type 'it makes 6', 'it makes 8' (with additions or multiplications), '2 remains' (with subtractions), and '2 comes from it' (with divisions).[17] These English renderings of the typical vernacular structures in our texts are intended also to capture the fairly transparent and fundamentally 'popular' nature of the lexical items used ('join together', 'draw/pull', 'dig out', 'share into'). Distinctive lexical expression of the operators (the equivalents of modern English 'plus', 'minus', etc.) is absent, since the meaning of such expressions is conveyed by the operating verbs and their accompanying prepositions or conjunctions: 'and', 'with' (addition), 'from/out of' (subtraction), 'into, by' (division). Matters are slightly different with multiplications, since both the operator verb and the expression of the operator

[12] Carlsmith (2010) has investigated the schooling in Bergamo and the Venetian republic in the sixteenth century.
[13] See Goldthwaite (1972) and Black (2007).
[14] The Italian word *bottega* means 'workshop', and in Renaissance society it denotes the shops where manual jobs such as weaving and dying were practised. Among the best-known *abbaco* schools is that which belonged to Giovanni Di Bartolo, Piazza Santa Trinita in Florence. Schools and masters of *abbaco* in Tuscany are investigated in Ulivi (2004) and Ulivi (2013).
[15] Like the numerals themselves, they are non-linguistic, in the sense that they are interpretable independently of the native language of the reader. This is not, of course true, of the spoken formulations.
[16] See Cajori (1929): 200-356.
[17] While results typically follow operations, subtrahends typically follow minuends, and divisors typically follow dividends, there is no requirement for these orders in our texts and other orders are perfectly possible, their roles and relations being made explicit usually via the lexical verbs used.

turn out to be in certain respects specialized to the context of arithmetical operations: we discuss the reasons for this differential behaviour of multiplication later.

Some preliminary general observations about the lexicon of the four arithmetical operations are in order. Some *abbaco* writers seem to make a distinction between the terminology initially used to introduce and denote the definitions of arithmetical procedures, on the one hand, and the terminology used to carry out the algorithm, on the other. For instance, in Jacopo da Firenze's *Tractatus algorismi* one finds in the incipit of the text, where the author introduces the topics subdivided into chapters: "Lo primo capitolo si è moltiplicare. Lo secondo capitolo si è dividere. [...] Sonno moltiplicare, dividere, giungere, sobtrare" (p. 195). But then, in actually performing a mathematical procedure step by step and manipulating quantities, the verbs used throughout the text are *partire* for 'divide' and *cavare* for 'subtract' (see, for instance, p. 305 and 306).[18] Similarly, the anonymous fourteenth-century *abbaco* from Cremona, *Tractado algorismo*, employs in the introductory section (which represents a sort of table of contents) Latin nouns to indicate the arithmetical operations: "la p(ri)ma è adimandata numeratio, la s(e)co(n)da addicio, la 3ª subtractio, [...] la 6a multiplicacio, la 7a divisio, [etc...] (1v). But then, while explaining the operation of addition, which is introduced by the heading incipit addicio, the author says: Addicione non è altro se n(on) arecoliere molti nu(mer)i i(n)seme [...] e questo rezonzere è ditto asomare arecoliere vulgarme(n)te [...] E questo basti quanto a lo rezo(n)zere over somare. Seguita lo subtrare [...] Subtracione non e altro se no(n) asotrare [...] e questo sobtrare vulgarmente è adimanadato far resti [...]" (2v-3v). Here it is interesting to observe that the author uses the term *vulgarmente* to refer to the popular usage of the terms (verbs) for 'add up' and 'subtract', which implies that a distinction between abstract, general forms and popular, practical forms was already established. Similarly, at the beginning of a sample problem (and not in the introductory section) the anonymous author (most likely from Brescia, certainly from Lombardy) of an anonymous early fifteenth century *abbaco* says: "sotrae overo cava",[19] the *abbaco* by Dionigi Gori also has: "sotrare overo cavare".[20] The OVI corpus shows that *sottrarre* (*sotrare*, *sottrare*) is found in literary works, communal statutes, and letters with the meaning 'take away' but also 'deprive, remove'; *cavare* is also found in the OVI but only used, in non-mathematical texts, to mean literally 'dig, remove'.[21] There are 56 instances of *sottrarre/sotrare* and 413 instances of *cavare*, thereby demonstrating the prominent use of the latter and the appearance of the former in particular language registers requiring a more elaborate vocabulary. In the cases mentioned above, the word *overo/ovvero* 'that is, in other words' serves to clarify the meaning of abstract verbs (*sottrarre, dividere*) by referring to a context-based set of terms

[18] On the particular way *dividere/partire* is used in other *abbaci* see more on the subsection on division.
[19] See in Cattelani Degani and Mantovani (2000: 101).
[20] See in Franci and Toti Rigatelli (1982: 39).
[21] Compare: *ti fece sottrarre al tempo la stella* (Giovanni dalle Celle, Lettere, 1347/94) *and non lo poteano sotrare dalla voglia* (Jacopo della Lana, Inf. (M2), 1324-28) versus *cavare la terra* (Anonymous, Vita di S. Petronio, 1287-1330) , *cavare et estirpare e distrúgiare in tutto* (Statuto senese ca. 1303). Seven instances of *sottrazione* (mainly in Boccaccio and Statutes of Siena), one of *sotrazione* in Jacopo della Lana, and one of *sotraimento* (Esopo toscano).

with a more practical meaning. Such a usage affirms not the practical emphasis of the *abbaci*, their use not of abstract mathematical terminology but of concrete, everyday verbs.[22]

In the *abbaci*, expressions denoting arithmetical operations are mainly verb-forms rather than (abstract) nouns. Thus in the abovementioned expressions found in the *Tractado algorismo* from Cremona, the author explains the noun denoting the definition of the arithmetical operation by means of everyday verbs: *"*Addicione non è altro se n(on) arecoliere [...]; Subtracione non e altro se no(n) asotrare". Similarly, in the *abbaco* on algebra by Gratia de' Castellani one finds "Nota che quando ti venisse alcuna di queste partigione overo a partire in R meno numero" (p. 60), in his arithmetic manual Bastiano da Pisa states "la prova de ogni sotrare" (p. 20), the work by Giovanni de' Danti has *Ora avemo decto del multipricare e del partire* (p. 52), and some pages earlier states: "Questa ene la regola ad amestrare de multipricare a schacchieri" (p. 15) This usage is a common feature of several other *abbaci*, where one finds substantivized infinitives, which highlight the action rather than the abstract concepts. Further evidence is found in the *abbaco* of Piero Calandri: "Et questo basti quanto al partire de' rotti (p. 44) [...] Et questo basti quanto al trarre" (p. 45) and also in the work by Francesco Cortisi of Brescia: "La regula de fare una suma" (p. 36), "La regula del multiplicar segondo lo modo lombardo" p. (38), "A multiplicare segondo lo modo viniciano la regula sie questa" (p. 39), "La regula del partire "(p. 42), and "La regulla de l'azonzere" (p. 45). But nouns are also used; see for instance Maestro Dardi: "Perché nell'antiposto trattato abbiamo ditto sopra le multiplicatione, le adgiuntione, le sottratione e lle divisione delle R" (p. 64) and Jacopo da Firenze in: *"*abiamo dicto dele multiplicationi et dele divisioni" (p. 230) [...] "abiamo dicto del giongimento et del sobtraemento de numeri rotti" (p. 234).

3. Operation verbs

Basic arithmetical operations are commonly expressed by a limited group verbs, all of them rooted in basic vocabulary and none of them originally specialized for mathematics, which are clearly established terms shared across the lexicon of the voluminous corpus of more than 350 *abbaci*. They are commonly found in virtually all *abbaci*.

The analysis which follows is based upon a selection of twenty *abbaci*:[23]

[22] The conjunction *overo/ovvero* (which can serve as a disjunctive or as an explicative conjunctive) is common in our corpus also in sentence introducing topics or containing definitions of arithmetical operations. See, for instance, "El duodecimo capitolo è de reg(o)le de merto overo d'usura" (*Lo Livero de l'abbecho* p. 222); "e pa(r)tire en tante qua(n)te sonno le lib., overo en mese che noie vuoglessamo sap(er)e" (p. 223); "Et se fusse prestato overo guadagnasse l'anno el centinaro" (in Jacopo da Firenze, p. 246) and "Una sala, overo piazza, è lungha br 120, et largha br 36, né più né meno. Et io la voglio lastricare de lastre overo de pietre" (p. 276).

[23] We have only used abbreviations for the *abbaco* works which we have most frequently quoted. When we quote passages and add page references (p.), we refer to the published transcription; we add folio references (f.) only for the unpublished manuscript the anonymous *Tractado Algorismo* from Cremona. It has not been possible to

Maestro Dardi's *Aliabraa argibra* (henceforth AA);[24] the anonymous *Trattato dell'alcibra amuchabile*;[25] the *abbaco* by Altovito da Firenze (henceforth AF);[26] the anonymous *Arte giamata aresmetica*;[27] an anonymous *abbaco* from Brescia (henceforth BA);[28] Bastiano da Pisa's *Tratato d'arismeticha praticha*;[29] the anonymous *Tractado Algorismo* from Cremona (henceforth CTA); Dionigi Gori's collection of *abbaci* (henceforth DGA);[30] the *abbaco* by Francesco Cortisi;[31] Gratia de Castellani's *Chasi sopra chonpagnie* (henceforth CSC);[32] *Tractatus Algorismi* by Jacopo da Firenze (henceforth JF);[33] the anonymous *Livero de l'abbecho* (henceforth LA);[34] *Larte de labbacho,* an anonymous treatise from Treviso;[35] the *abbaco* by Paolo dell'Abbaco (henceforth PA);[36] Filippo Calandri's *Aritmetica*;[37] Piero della Francesca's *Trattato d'abaco*;[38] *Tractato d'abbacho* by Piero Maria Calandri;[39] *Regole di geometria pratica* by Obertano da Montepulciano;[40] *Trattato dei Fioretti* by Antonio de' Mazzinghi;[41] *Tractato de l'algorisimo* by Giovanni de' Danti (henceforth GD).[42]

3.1 Addition

The most frequent verbs used to mean 'add' are *giungere* or (*r*)*aggiungere* (or cognates thereof), whose basic meaning is 'join (together), add to'. One also sometimes encounters *raccogliere* 'gather (together)' (or cognate forms) and *addunare*/*radunare* 'assemble, bring together'. Some evidence of the use of these verbs in non-mathematical works is (from the OVI database):[43] i) nella casa la quale

inspect the manuscripts of the published transcriptions here employed, so that we rely on the texts of *abbaci* as found in the available publications. Given the significant amount of quotation, it is implicit that when we use the author or *abbaco* title followed by page number, we refer to the published edition, which for obvious space reason we cannot very time specify. In the footnotes below, we provide the references of the scholarly transcriptions we have used to quote passages from *abbaci*.

[24] Franci (2001).
[25] Simi (1994).
[26] Cattelani Degani and Mantovani (2000).
[27] Rivolo (1983).
[28] Cattelani Degani and Mantovani (2000).
[29] Barbieri and Lancellotti (1986).
[30] Franci and Toti Rigatelli (1982).
[31] Cattelani Degani and Mantovani (2000).
[32] Pancanti (1984).
[33] Høyrup (2007).
[34] Bocchi (2017).
[35] Swetz (1987).
[36] Arrighi (1964).
[37] Arrighi (1969).
[38] Dalai Emiliani (2012).
[39] Arrighi (1974).
[40] Simi (1991).
[41] Arrighi (1967).
[42] Arrighi (1985).
[43] Evidence from the OVI database shows that *giugnere* meant both 'arrive at' (Italian *raggiungere*, as in the second example) as well as 'add', as in Brunetto Latini's *Rettorica* (c. 1260-61, in Florentine, 76.2): *quella scienzia per la quale noi sapemo giungere ornamento di parole e di sentenze.*

sirà ordenata che se debia **radunare** la decta compagnia, et conme se debino **radunare**; [44]ii) cavalca poi sì forte, che, per che ti vogliano **giugnere**, che non possano;[45] iii) all'opera di colui nulla si possa **aggiungere**.[46]

Evidence from *abbaci*:

i) Ora **giungi** inseme tucti questi charrati, che sonno in tucto charrati 1140 e 5/12. Ora **giongi** inseme 7 oncie et 9 oncie et 16 oncie et 20 oncie, che sonno in tucto oncie 52 (JF 2.3) [...] **Agiongni** inseme 4 e 3, che sonno 7, et poi parti 7 in 12, che ne vene 7 . Et cotanto fa **gionto** inseme 1 (JF p. 339)

ii) **Rachoglj** prima quello ch'egli anno messo in fra tutti e.3, cioè: 19 F. che misse it primo e 17 F. che misse it sechondo et 14 F. che misse it terço, fanno 50 (CSC p. 1)

iii) **Raduna** ensieme quaranta e trenta e doie, fanno 72 (LA p. 176)

iv) **Arecolierli** tutti inseme [...] asomando i(n)seme [...] conviene azonzere (CTA 2v)

v) **Ragugni** 2/3 con ¾ [...] racoglj insieme 9 et 8 (*Aritmetica* by Filippo Calandri p. 11)[47]

vi) **Adduna** 49 e 49, fa 98 (p.28) [...] **adgiogni** inseme 6 e 6, fa 12 (*Regole di geometria pratica* by Orbetano da Montepulciano p.29)

Another verb expressing addition is the etymologically problematic *arrogere* 'add, integrate' but also 'complete, fill'; it usually implies the idea of an add-on, something added to a greater extent than is normal or proper (cf. GDLI s.v.). While the verb does occur in some of our texts (example (i) occurs precisely in the context of adulterating coins), only one *abbachista*, Paolo dell'Abbaco, uses it, occasionally, in the formulation of arithmetical operations:

> Fa' cosi, poni mente quanto quello che vi si vole **arogiare,** che vi se vole **arogiare** rame perciò che tu voli fare pigiore moneta che tu nonn ai e sempre quando vi si vole **arogiare** rame si dia sapere quanto ariento si vole nella quantita de la moneta che tu vuoli fare (GD p.48)

but

i) e **arrogi** uno a 24 e àj 25, e senpre multjpricha 12 via 25: f.a 300 (PA p. 36)

ii) Fa' choxj. **Arrogi** 6 a 24 fae 30, e' qualj 6 eglj gligli diè oltre al terzo (PA p. 71)

iii) E però di': **arrogiendovj** l rimarrà adunque uno a ciaschuno. E però **arrogi** uno a questo numero, cioè 7560, fa 7561 (PA p. 115).

[44] *Capitoli della Compagnia dei Disciplinati di Cortona*, 11 cap. 2, 128.7.
[45] Paolo da Certaldo, XIV sm. (tosc.), *Libro di buoni costumi.*
[46] Anonymous, vernacularization, Valerio Massimo, red. V1, 1336, in Florentine.
[47] *Ragugnare* is a phonological variant of *radunare* via **raunare*. Like *addunare,* this form contains the word for 'one' (Italian *uno*).

The *abbaco* by Piero Maria Calandri has four instances of *acchozzare* 'join, put togther', among which are:

- Dipoi acchozzato insieme queste somme fanno 1258 (p. 139)
- Dobbiamo prima acchozzare insieme 3 1/3 et 2 2/3, fanno 6 (p. 190)
- Et dopoi dirai che acchozzino insieme tutte queste multiprichationi coè 10, 234,1160, fanno 1404 (p. 141)

3.2 Subtraction

The operation of subtraction is predominantly expressed in our texts by one of two verbs, namely the local phonological or morphological cognates[48] of Italian *trarre* 'pull, draw' or *cavare* 'dig (out)'—or, more rarely, *abbattere* 'knock down'.[49] None of these is unique to mathematical usage but, as mentioned before for the verb meaning 'add', they have become established and specialized within the *abbaco* mathematical lexicon as the expressions for subtraction.[50] In the anonymous fourteenth-century *Trattato dell'alcibra amuchabile* and in Jacopo, for example, we find *c(h)avare* used both as a verb meaning 'subtract' and as a concrete verb meaning 'dig' (specifically, dig a well).[51] The *Livero de l'abbecho* employs the verbs *abattere, chavare, menovare,* and *soperchiare*; the result of a subtraction is there called *armanente*, *diferentia, menovamento, soperchio*, and *svario. Diferentia* is found in the *abbaco* by Giovanni de Danti (p. 23), Francesco Cortisi uses *resto* (p. 44). As we have previously observed, some *abbaci* distinguish between *sottrarre* and *trarre*, the former being the abstract form indicating the procedure (the theoretical, abstract aspect of the operation) of finding the subtraction between two numbers, whereas the latter denotes the act of subtracting itself (the practical procedure), that is to say the process of subtracting two numbers. For instance, in the arithmetic work by Bastiano da Pisa one

[48] The reflexes, respectively, of Latin TRAHERE, CAUARE, and AD+BATUERE.
[49] *LA* uses *abbattere*, but almost exclusively in the formula *devemo abbattere*: "Quista è la diritta reg(o)la chomo noie devemo fare, che noie devemo abatt(er)e l'uno de lo rotto, e rema(r)rà-ne 1/23, e devemo abatt(er)e 7 de 24 e rema(r)rà 17" (LA p. 370). *Abbattere* is also found in the fifteenth century abbaco from Lombardy *Arte giamata aresmetica*: "[...] fa 16; abate 9, romane 7" (p.2). *Togliere* 'remove', or 'take', appears quite often in our texts, but we have found no case in which it is obviously a verb of subtraction. Examples of *togliere* as 'remove, take' are, for instance, found in: *Trattato dell'alcibra amuchabile* p. 28; AF p. 70; and in the *abbaco* by Francesco Cortisi p. 36 and 42.
[50] As for other contemporary sources, in the OVI there are 188 instances of *abbattere* used to mean 'knock down, destroy'.
[51] "Uno toglie a **chavare** uno pozo a dentro 20 braccia e deve avere 20 lire" (*Trattato dell'alcibra amuchabile* p. 57); "Ora dice colui de chi è el puzzo, **cavalo** tanto più largho che sia 3 ˜br e 1 , et io te pagarò ala simile 2 ragione". (JF p. 359).

finds *modo di sotrare lire soldi e denari* (p. 18) and *la prova de ogni sotrare* (p. 20) as the heading of the sections explaining the operation of subtraction and the method for cross checking answers after subtracting, but then the verb used to indicate the subtracting itself is *cavare*. Similarly, in *Arte giamata aresmetica* (a fifteenth-century *abbaco* from Lombardy) the author clearly employs *sotrare* to denote the concept, whereas *abatere*/*cavare* are the verbs denoting the action: for instance, on p. 8 one finds *magistramento de sotrare de roto, zoe' cavare uno roto de uno altro roto.* The manuscript *Tractado algorismo* from Cremona specifies that *questo sobtrare vulgarmente è adimandato far resti* (folio 3v).

Evidence from non-mathematical texts: i) riempiere le valli [...] vedremo l'acqua **cavare** e encupare e fare le valli;[52] ii) de le qua' cose si possa **trarre** utilità;[53] iii) e in **abbattere** il nimico usa solamente il volere per sua forza.[54]

Evidence from *abbaci*:

i) **Sotrae** overo **cava** 2 1/2 de 3 1/4 (BA *p.* 101)
ii) **Tray** l'uno meritto da 15 l'altro, resta duchati 0 soldi 14 denari [8]1/15 (BA p. 96)
iii) A provare **togli** li 3/4 di 6 18/77, che sonno 4 52/77 (BA p. 113)
iv) Ora **trai** tante volte 5 di 24 che tu '1 possa partire in 3, che ne '1 **cavi** 3 volte e rimante 9 e questo 9 parti per 3 vienne 3 (GD p. 77)
v) **Abacti** 1 resta 7 e 7 n'a it secondo; ora di': 7 e 7 fa 14. **Abatti** 1 r[e]sta 13 e 13 n'avea it 1/3. (GD p.39)
vi) **Cava** L.1 di L.3 [...] **traendo** 4 di 5 (DGA p. 39)

3.3 Division

Evidence from non-mathematical works:[55] i) del meo tormento non posso **partire**;[56] ii) 'l guadagno se di' **pa(r)tire** p(er) mitade;[57] iii) col'affamato lo pane tuo **dividere**.[58]

With overwhelming consistency, across all our texts, mathematical division is expressed by the verb *partire* in the operation clause. For example:

[52] Restoro d'Arezzo, 1282 (aret.), *Composizione del mondo colle sue cascioni*.
[53] *Trattati morali di Albertano da Brescia volgarizzati*, Andrea da Grosseto (ed. Selmi), 1268 (tosc.).
[54] Anonymous, *Cassiano volgarizzato* (ed. Bini), XIII ex. (tosc., Collazioni (I-X) dei SS. Padri del venerabile Giovanni Cassiano (volgarizzate).
[55] In the OVI, *partire* is mainly found to mean 'leave, depart', which must have been its predominant meaning; the database also provides some instances of *spartire* meaning 'separate'.
[56] Guglielmo Beroardi, Rime, a. 1282 (fior.)
[57] Carta di Nicoluzzo de Mengo Cingilanza, Doc. ravenn., 1353, 448.8.
[58] *Trattato della Dilezione* di Albertano da Brescia volgarizzato (Il), 1275, Florentine, L1, cap. 6, 72.16.

i) **Parti** ora li diti duchati 36 soldi 14 per 12, che ne viene duchati 3 soldi 1 denari 2 e tanto viene el mese (BA p.95)
ii) Et se lo carcho fosse 108 libre, sì **parteresti** 65 in 27, che ne verrebbe dui denari et 11/27. (JF p. 251)
iii) Pone che la m(ultip)lichatione de la prima pa(r)te p(er) 4; conciosiachosaché la pa(r)te de rieto m(ultip)licha[ta] p(er) 6 façia 21 21, çioè 3, el quale viene del **pa(r)tire** 21 p(er) 6 (LA p.477)
iv) Premamente tu de' vedire in quello numero che tu **parti** quanti volti g'e lo numero che tu vo' **partire** (*abbaco* by Francesco Cortisi p. 44)
v) E se tu volessi **partire** numero per R (AA p. 58)
vi) 4 chonpagni anno a **partire** 60 ducati (AF p. 80)

The verb used in the *abbaci* to denote 'divide' is *partire*; *dividere* also appears but in different contexts. In *OVI* there are 2904 instances of *partire*: those with the meaning of 'depart' are by far the most numerous (so that 'depart' is the predominant meaning), the instances of *partire* as 'share, separate' are less than 5% of all instances in non-mathematical works. There are 315 instances of *dividere*. *Partire* is a popular term also meaning 'share, share out', and one that is transparently related to the noun *parte* 'part, share'. In the modern language, transitive *partire* 'share, divide' is partially distinguished from intransitive *partire* 'leave, depart', by the fact (see, e.g., Maiden 2004) that the former takes the morphological 'augment' *-isc-* in the singular and third person forms of the present tense (e.g., *partisce* 'he shares' vs *parte* 'he leaves'). In our texts the two lexemes are morphologically identical,[59] so that *parte* is both 'divides/shares' and (potentially) 'departs'. This fact means that the relationship of *partire* share, divide' to the noun *parte* 'share' may have been even more transparent then than it is today. Modern Italian uses almost exclusively the verb *dividere*, of learnèd origin,[60] not *partire*, to express division.[61] In medieval mathematical usage, in contrast, *dividere* was only rarely attested, and is greatly surpassed in our texts by *partire*. For example, in the work by Paolo dell'Abbaco the lexeme *partire* outnumbers *dividere* by 11 to 2, in Jacopo da Firenze by 60 to 3, or in Giovanni de' Danti by 90 to 4. Overall, there is no obvious difference in meaning between the two verbs. Høyrup (2007:127) even suggests that, in Jacopo, proximity within the text to the Latin word *divisio* may induce use of *dividere*: 'The term *divixio* in the Latin incipit seems to cause the appearance *dividere* instead of the regular *partire*. This happens thrice in 1.3 and in no other place in the manuscript. Elsewhere *dividere* appears only as an expression of bisection, namely in the locution *dividere per mezzo*.' In Gratia de' Castellani,

[59] Giovanni de Danti Aretino (9r) has one example of *partisici*: "E così simigliantemente partisci ongni numero che te fosse decto". In the formulation of arithmetical operations, however, only *parti* is ever found (e.g., "Parti 5 per 6, vienne 5/6", p. 65).
[60] Latin DIUIDERE would have developed by regular sound change as **divedere or **devedere. We do find *dovidere* and *ridovidono* in the *Trattato dell'alcibra amuchabile*, showing backing of unstressed front vowels when followed by a labial (cf. Rohlfs 2021a:269).
[61] This seems true of most other Romance languages, although Romanian regularly uses *a împărți* 'share, divide up', alongside *a divide.*

dividere in fact appears frequently, but almost never in the formulation of the arithmetical operation of division; rather, it is used in expressing situations in which capital or cattle are in some way being 'shared', 'dealt out', 'divvied up', 'split up', 'apportioned',[62] whereas the corresponding arithmetical calculation is expressed by *partire*:

i) Adunque abbiamo fatto che dovendo **dividere** .5 F., al primo toccherebbe 3 F. e al sechondo 2 F. E noj vogliamo **dividere** 100 L.: dove per lo primo moltiplicha .3. vie 100 L., fanno 300 L. e qualj **partiraj** in 5. vienne 60 L. E per lo sechondo moltiplicheraj 2. vie 100 L. e **partiraj** in 5. vienne 40 L. e tanto à il sechondo. (CSC p. 4)
ii) E sono .2. che l'uno da 40 pechore a soccio e choluj che lle guarda ne mette .10, e in chapo di .4. annj debbono **partire** per metà ['divide by half'], **dividono** ['they split up / separate'] in chapo di .2. annj 100 pechore [….] [….] Moltiplicheraj .2. vie 30. pechore e **partiraj** in .4 ['divide by 4'], vienne .15 pechore (CSC p. 47)
iii) E sono alquanti huomini ch'anno a **dividere** in tra lloro fiori 60 [...] dunque ti conviene **partire** 60 per una cosa (*Trattato dell'alcibra amuchabile* p. 59)
iv) 4 chonpangni anno a **dividere** 375 ducati [...] e multiplicha chontro a 375 e quello ne vene **partilo** per 2147 (AF p. 84)

As for the noun, the *Livero de l'abbecho* uses *divisione* (p. 195) and *partegione* (p. 326); in the abbaco by Antonio de' Mazzinghi the noun *partimento* denotes the result of the action of *partire*: *Fammi, di io, due parti che partito l'una per l'altra et l'altra per l'una et multiplichato ciaschuno partimento*. As already stated above, some authors seems to prefer a substantivized verb:

i) E questo ti basta per inte(n)dere **el partire** p(er) colona (*L'arte dell'abbaco* p. 27)
ii) Volendo mostrare **lo partire** a galea (CTA 17v)
iii) A lo m(ultip)licha(r)e e a **lo partire** (LA p. 405)
iv) Qui finisce il capitolo **del partire** (TA p.22)
v) **Al partire** a danda (DGA p. 47)

'To halve' is *dimezare* (p. 25, where one also finds the noun *dimezamento*); forms of *dimeçare* are found in *Trattato dei Fioretti* by Antonio de' Mazzinghi (p. 24), *piglia la metà* (*pigliare* is literally 'take') also occurs in Piero Maria Calandri's *Tractato d'abbacho* (p. 188), *partire per metà* is found in the work by Gratia de' Castellani (p. 43), *dimezzare* in the text by Jacopo da Firenze (p. 313 and 365, where one also widely finds the noun *dimezzamento*), and *amezare* is found in the *abbaco* by Francesco Cortisi (p. 36).

[62] See, for instance, in CSC p. 4 and p.7.

3.4 Multiplication

Finally, and in notable contrast to the expressions of the other three arithmetical operations, the operation of multiplication is conveyed exclusively by phonologically learnèd (or semi-learnèd) forms derived from Latin *multiplicare*.[63] The predictable 'popular' phonological development of this word[64] would have been something like *moltepjeˈkare, showing the regular change of [l] to [j] after an obstruent (cf. popular PLICAT > *piega*, IMPLET > *empie*, EXPLICAT > *spiega*), and opening of short u and i to [e] and [o] respectively (cf. UIDET > *vede*, PLICAT > *piega* and even the semi-learnèd adjective *molteplice* < MULTIPLICEM). The predicted type *moltepjeˈkare is, to the best of our knowledge, not found anywhere. The range of root forms that which we actually encounter in our sources is: *multiplic*(*h*)-, *multipric*(*h*), *moltipric*(*h*)-, *moltiplic*-, and *montiplic*(*h*)-. The apparently[65] wholly learnèd type *multiplic*(*h*)- is widely predominant in the Vatican manuscript of Jacopo da Firenze (e.g., 278 examples of the form *multipric(h)a* vs 57 of *multiplica*) as well and in the three *libri d'abbaco* of the *Documenti campori* (1420, 1456?, 1433-37). In the text by Giovanni de'Danti we have overwhelmingly *multipric*- (for example, 43 examples of infinitive *multipricare* vs 3 of *multiplicare*), and *multripric*- is consistently found in Paolo dell'Abbaco. In Maestro Dardi, forms in *multiplic*- are greatly outnumbered by *moltiplic*- (the form which survives into modern standard Italian), while Piero della Francesca consistently has the distinctive type *montiplic*- (occasionally *muntiplic-)*. The forms in *-pric*- show a type of rhotacism which is typical of popular pronunciations of words originally introduced with the cluster /Cl/. Rohlfs (2021a:255;356) identifies [pr] as a reflex of (word-initial) [pl] in 'parole prese in prestito dal latino in epoca più recente' for Sicilian and Neapolitan, while words such as *applicare* and *duplicare* (indeed partly cognate with *multiplicare*) are identified as 'imprestiti, talvolta con passaggio di *l* a *r*; cfr. in Guittone *risprende, esempro*.'[66] Given that this lexeme is everywhere of learnèd origin, it would seem reasonable to suppose that that the *o* of *moltiplic*- is not a direct phonological development at all, but rather a matter of 'contamination' from the semantically closely related *molto* (< MULTUM). The distinctive *montiplicare*[67] used by some authors of *abbaci*, including by Piero della Francesca, (e.g., imperative *montiplica, montiplicalo,* participle *montiplicato*) seems[68] to have a similar

[63] Extensive exemplification from *abbaci* would therefore be otiose. The verb is everywhere a derivative of *multiplicare*.
[64] See, Rohlfs (2021a:5f.;162f.;165;355).
[65] In principle, the occurrence of *u* and *i* in pretonic position could (at least in Tuscan) reflect a general tendency towards raising of pretonic mid vowels, rather than a learnèd development (Rohlfs 2021a:162f.;165). However, the stressed vowel of present tense *multíplica*(*no*) etc. is always *i* and never the etymologically predicted *e*.
[66] As Castellani (2000:26) shows, EXEMPLUM is an example of a 'voce semidotta' which often yields *as*(*s*)*empro* (also attested in Jacopo da Firenze and Giovanni de' Danti).
[67] Note also *monteplicano* in the Sienese *Egidio Romano volgarizzato* (1288), see furhter.
[68] A purely phonological change [lt] > [nt] is not unknown (cf. Rohlfs 2021a:346, notably pistoiese and romanesco *antro* for *altro*, and *minza* for *milza* in various parts of Tuscany), but we find no other evidence for such a change in Piero's text.

origin, the influence of *montare* 'amount (to)', a verb often used in expressing the resulting of calculations, apparently being at work.[69] *Montare* is sometimes used to denote the result of a multiplication, sometimes to mean 'amount (to)'. For instance, in the abbaco by Cortisi, one finds: *primo dey tore quarti 14 de mellio per denari 6 quarti, che monta soldi 7* (p. 58) but there is no use of this verb to denote the amount of a multiplication, whereas in the *Livero* from Perugia one finds both usages: compare "devemo noie m(ultip)licare 360 via 100, 27 che montano 36000" (p. 228) and "adoma(n)do quanto monta lo cha(r)rato e quanto monta l'una ma(r)cha e quanto monta l'una onçia e qua(n)to mo(n)ta lo qua(r)tiere de l'onçia e qua(n)to mo(n)ta lo ca(r)rato piçiolo e quanto mo(n) ta tutto em so(m)ma" (p. 403).[70] Cortelazzo and Zolli (s.v. *moltiplicare*) describe the word as a 'v[o]c[e] dotta che un passo biblico ha contribuito a diffondere ("che dia lor potenza, come dice il prete all'altare, di crescere e multiplicare": 1566 A. F. Grazzini. *Pinzochera* V 10'.') The earliest attestation dates from the mid thirteenth century (Giacomo da Lentini), and the term is by no means restricted to mathematical usage, being used in the wider sense of 'increase (in quantity, volume, weight, etc.)'. The more general and more popular term for 'multiply' seems, however, to be *accrescere,* and there are examples where the two terms are used side by side or even presented as apparent synonyms.[71] Note that the operation of doubling, which was a separate operation, in the *Livero de l'abbecho* is indicated by the nouns *adopiagione* and *doppio* and by the verbs *adopiare, aradoppia, endopicha, radopiare* (for instance p. 317); *adoppiare* also occurs in the text by Maestro Dardi (for instance p. 53); *radoppiare/raddoppiare* is found in the abbaco by Piero Calandri (for instance p. 109), whereas Francesco Cortisi uses *indoplare* (for instance p. 35). The noun *duplicazione* is used by Giovanni de' Danti (p. 6).

Evidence from non-mathematical texts: i) le cose della natura generano l'una l'altra, e **monteplicano** fra loro medesime;[72] ii) e poi **multiplicare** la so(m)ma del detto di tutti; iii) che puoi li meriti **multiplicare** et così vita eterna.[73]

Evidence from *abbaci*:

i) E poi **multiplicha** 22 mesi chontro a (AF p. 79)
ii) Dobiamo **multiplichare** per si medessmo (BA p. 123)
iii) Dibiamo **m(ultip)licare** 5 via 40, ke fa 200 (LA p. 409)
iv) **Multiprica** prima ei rocti (GD p. 23)
v) **Montiplica** 2 via 4 fa 8 (Piero della Francesca's *Trattato d'abaco* p. 5)

[69] It is used twice in Jacopo da Firenze, see Høyrup (2007:267 and 269).
[70] In two arithmetic fifteenth-century manuscripts in Catalan transcribed by Arrighi (1982: 16), one finds: "e de montiplique 3 vegades 1 march e fan 3 marchs, [...] e moltiplique-lo per 6 e fan 42 onzes".
[71] "E moltiplicherà il vostro seme, e accrescerà il compimento e l' accrescimento delle biade della vostra giustizia (1. 2 Cor 9)". "E anchora se ttu il volexxj acresciere 4 chotantj, multjpricha 4 via 49" (PA).
[…] li tre sergenti a cui lo ricco huomo diede li bisanti ad acrescere et a moltipricare" (Inchiesta San Gradale).
[72] Anonymous, *Reggimento de' principi di Egidio Romano* (1238).
[73] Giordano da Pisa, *Prediche inedite* (1309).

vi) **Montiplicalo** in sé fa 81 (*Trattato d'abaco* p. 402)

vii) **Montiplicato** la seconda nella quarta (*Trattato d'abaco* p. 574)

4. Expressing the result

The result of an arithmetical operation is characteristically expressed in a separate clause, which usually follows the clause expressing the operation. The clauses are frequently asyndetic (not explicitly connected to the preceding clause which formulates the operation), but may be linked by means of the conjunctions *e* or *che*. Some typical examples illustrating all four operations are:

i) Regol'è che devemo **trare** de libre 100 quiste libre 20, e **resta** libre 80 necto; or devemo dire: "se 'l centonaio d(e) la d(i)c(t)a polve vale lib. 10, que ne va(r)rà le libre 20?". Sì devemo **m(ultip)licare** 20 v(i)a 10, che **fa** 200 libre, e **pa(r)tire** p(er) 100, che **ne viene** lib. 2; or devemo **trare**, de libre 25, libre 2, e **rema(r)rà** lib. 23. (LA p.167)

ii) **Rachoglj** prima quello ch'egli anno messo in fra tutti e.3, cioè: 19 F. che misse it primo e 17 F. che misse it sechondo et 14 F. che misse it terço, **fanno** 50 e questo si dice chorpo di chopagnia. E di poj per llo primo che misse 19 F., **moltiplicha** 19. vie 300 L., **fanno** 5700 L., le qualj **parti** in 50, cioè per lo chorpo, **vienne** 114 L. e tanto tocha al prima di guadagnio. (CSC p.2)

iii) Prima **agugneraj** le 12 L. meno cho' lle 8 L. meno, **fanno** .20 L. meno, che **fanno** chon.100 L., L.120, della quale somma **traj** 10 L. più del sechondo, **rimane** 110 L. (CSC p.11)

iv) Ora fa cosi: **multiplicha** 8 via 8, zoe per crose 1'uno 8 chontro a l'altro, e poi **multiplicha** 7 via 9 che **fa** 63, **trai** 63 di 64, **rimane** 1, **partilo** per la figura di soto, zoe 8 via 9,[74] che **fa** 72, e **partito** questo 1 per 72 **ne vene** 7 2 ed è provata. (AF p. 72)

Both the lexicon and the syntax of the product clause are extremely formulaic. There is a high degree of mutual correlation between expressions of multiplication/addition, subtraction, and division and, respectively, the selection of *fare, rimanere/restare,* and *venirne*. For example, in Jacopo da Firenze *moltiplicare* 'multiply' in the operation clause selects only *fare* in the product clause (135 examples),[75] *trarre* 'subtract' in the operation clause selects only *restare* or (occasionally) *rimanere* in the product clause (45 examples), and *partire* 'divide' in the operation clause selects only *venirne* in the product clause (130 examples). As this implies, there is also a strict lexical specialization: *venirne* is exclusively used with divisions, *rimanere/restare* exclusively with subtractions, and *fare* exclusively with multiplications and additions. Thus, despite the fact that *fare* could conceivably be used for the result

[74] This is an implicit multiplication, as indicated by *via*.

[75] This is also true of *(ag)giungere* and *fare*, but there is actually rather little discussion in the texts of the operation of addition, and only four relevant examples.

of any kind of arithmetical operation, we do not find formulations such as 'Trai 2 di 6, fa / ne viene 4' or 'Parti 6 per 2, fa / rimane 3' or 'Moltiplica 6 via 2, ne viene 12'.

The operation of division is characteristically expressed, in all our texts, by the verb *partire* and the corresponding verb in the result clause, in all our texts, is almost always[76] a third person singular form of the verb *venire* combined with the clitic pronoun *ne* (typically *ne viene* or encliticized *vienne*).[77] Even in the presence of an explicit plural subject within the clause, the verb practically never[78] agrees for number.

i) Adunque se partiremo .1800 F. per .6, **ne verrà** quello che a messo el terço, e partiremo 1800 F. per .6., **ne viene** 300 F. (CSC p.37)
ii) Ora voglio dire cosi: Jo voglio partire 40 per radicie di 30; vo' sapere quanto **ne viene**. Anche simigliantemente fa come faciesti dinanzi, cioe di rechare 40 a radicie che fanno 1600. Ora parte radicie di 1600 per radicie di 10 in verita **vienne** radicie di 53 1/3 e dirai che, partendo 40 per radicie di 40 che li **ne vengha** radicie di 53 1/3 ed e fatta (AA p.19)
iii) Fà-ne secondo ch'ò mostro de sotto e m(ultip)lica 7 fiade lib. 34 e s. 12 e d. 6 e p(re)nde p(er) lo 1/2 la mitade de le lib. 34 e s. 12 e d. 6 e giogne om(n)e causa ensieme e fa', degle d., s. e, degle s., fa' lib. ed averaie lib. 259 e s. 13 e d. 9, a pa(r)tire en 100, che de le lib. 200 **vegnerà-ne** lib. 2 e de le lib. 55 **vegnerà-ne** s. 11, e remangono lib. 4, che sonno s. 80; giogne s. 13, averaie 93, che de s.91 e d. 8 gle viene d. 11, puoie remane d. 25 che **ne viene** 1/4. (LA p.166)
iv) Ora sapia che viene il mese parti 90 per 12 **vienne** lr. 7 e s. 10, ora sapia che viene il di parti 7 lr. e 1/2 per 30 **vienne** s. 5 el di. (GD p. 44)
v) Ora parti 20 anni per 7 e si è facta che vedi che **ne viene** anni 2 e 6/7 d'anno e in due anni e 6/7 d'anno sarano radopiate le 2000 lr. ed è facta. (GD p. 30)

[76] The *Livero de l'abbecho* has just three examples of *venire* without *ne* in this context:
i) de quista t(er)ra pa(r)teraie 20 p(er) meço, che viene 10, e pa(r)te(r)aie 30 p(er) meço, che viene 15 (LA 175v)
ii) Multipricha 6 via 24, fa 144, parti in j, viene 144. (LA p. 354)
[77] As examples of *venirne* used to express the result of a division outside the didactic environment, cf. "però che l'oste della Fede Cristiana non è sì poca come suole, ma per la vittoria c'ha avuta sopra la Fede dell'idoli è sì multiplicata e cresciuta che son più che non sogliono ben mille cotanti, e vienne più che cento per uno de la tua gente" (*Libro de' vizî e delle virtù* 1292) and "Fumo in achordo che intra nnoi si dividesono in questo modo: che Charoccio dovesse avere di detti danari piùe che Duccio lbr. cinquecento a ffior., e Duccio dovesse avere di detti danari piùe ch' Alberto lbr. mille a ffior. Vienne dunque a Charoccio lbr. undicimilia settecento novantacinque a ffior., e a Duccio di Lapo lbr. undicimilia dugento novantacinque a ffior., e ad Alberto lbr. diecimilia dugento novantacinque a ffior." (*Libro delle possessioni di Duccio e d'Alberto di Lapo degli Alberti del Giudice,* 1334)
[78] The type *ne vengono / vengo(n)ne* with plural agreement is attested in the *Livero de l'abbecho* but it represents under 5% of tokens of *venirne* in product clauses (13 out of approximately 400). For example:
i) egl quagle parte p(er) 17, ch'è de sotto da la v(er)ga: vengone 6/17 e chotante parte tiene el primo de la pa(r)te del se(con)do (LA p.339)
ii) egl quagle parte p(er) 25: vengo(n)ne hore 19/25 5 p(er) lo voietare de la tina. (LA p.346)
iii) E sse p(er) regola retta la vogle av(er)e, sì m(ultip)licha10 p(er) 11, che fonno 110, e pa(r)te p(er) 19: vengone 5, egl quagle tratte de 11 remangono 5, s(econd)o de sopre trova(m)mo. (LA p.476)
iv) agiongne 12 e 20 e traie 15 de 16, e rema(r)rà p(er) essa radiçie 1 e 32, egl quagle pa(r)te p(er) 5: vengo(n)ne 6 e, egl quagle quadrate fonno 41. (LA p.480).

vi) Parti ora li diti duchati 36 soldi 14 per 12, che **ne viene** duchati 3 soldi 1 denari 2 e tanto viene el mese. Poy partii diti fiorini 3 soldi 1 denari 2 in 30, che **ne viene** soldi 2 denari 0 7/15 e tanto viene per dì. (BA p.95)
vii) Et se lo carcho fosse 108 libre, sì parteresti 65 in 27, che **ne verrebbe** dui denari et 11/27. (JF p. 251)

Occasionally, *venirne* may occur even where the operation of division is only implicit, as in the example below which involves calculating an individual's share of the profits:

> Una chollegança è una conpag(n)ia che à lib. 345 e s. 10, çioè 1/2 345; aguadagnano lib. 123 e s. 12, e vuogle sap(er)e que **ne viene** de guadagno a chuluie che nçe a(v)ve lib. 17. (LA p.262)

And also in Filippo Calandri's *Aritmetica*:

In the passage below, the reference of the *ne* that we see in other expressions of the products of division is made explicit, as *viene del partire* (i.e., 'comes from the division'):

> Pone che la m(ultip)lichatione de la prima pa(r)te p(er) 4; conciosiachosaché la pa(r)te de rieto m(ultip)licha[ta] p(er) 6 façia 21 21, çioè 3, el quale viene del pa(r)tire 21 p(er) 6 (LA p.477)

In subtraction, the result clause, virtually without exception, contains a verb of 'remaining', *restare* or *rimanere/remanere*. As with expressions of division, there is a tendency for the verb to be singular, even in the context of an explicit plural subject noun accompanying the numeral quantifier, although this syntactic characteristic is less consistently present than for *venirne*. For example, in *Trattato dell'Alcibra amuchabile* we have just two cases of agreement vs 41 of non-agreement with plural subjects, and in the work by Gratia de' Castellani one case of plural agreement vs 12 of non-agreement. There is no obvious motivation for the selection of the agreeing third person plural form over the singular.

i) Multipricha 3 per se medesimo, fae 9, tralo di 40, che debono fare le parti, **rimane** 31. (AA p. 40)
ii) Ora multipricha 1'altra schisa, cioe 60 vie una cosa, fanno 60 cose, chavalo di 60 cose e di 60 numeri, **rimanghono** 60 numeri. (AA p.61)
iii) Onde el 1/5 di .100. pechore è .20. pechore, che llo traj di .50. pechore, **rimane** .30 pechore (CSC p.47)

iv) E pero trarraj .6 L, et.4 L.di.100, **rimanghono**.90 L. (CSC p.7)

v) Pone sopre la ema(r)gene el 159, degle quale traie 119: **resta** 40, egl quagle è-ne el dopio degl d. degle 3 huomene; emp(er)çiò pa(r)te 40 p(er) 2, che ne viene 20, egl quagle so(n)no p(er) la so(m)ma degle d. de quiste 3 huomene; egl quagle tratte de 119, **resta** 99 p(er) la somma de le 3 borscie, degle quagle traie 10 e de. 23 che trovate fuoro e· lla s(econd)a ê· lla t(er)ça b[o]rscia più ch'e· lla prima: **remane** d. 66, egl q(ua)gle pa(r)te p(er) lo nu(mer)o de le borscie, çioè p(er) 3, che ne viene 22, e chotanto trovaro e· lla s(econd)a borscia; chom gle quagle agiogne d. 13 ch'e· lla maiure borscia trova(m)mo più ch'e· lla s(econd)a e siro(n)-no d. 45, egl quagle sonno egl d. de la maiure b{o}rscia. E traie de 28, çioè de la maiure bo(r)scia, d. 22: **restano** 6 e chotante d. a(v)ve el p(r)imo. Anchora traie de la s(econd)a borscia, çioè de 32, de la so(m)ma degl d. del s(econd)o huomo e de la s(econd)a bo(r)scia, çioè de d. 39: **restano** d. 7, e chota(n)te n'avve el s(econd)o; semeglantem(en)te tra-ne de la maiu(r)e borscia, çioè de 45, de la so(m)ma d'essa borscia [e] del t(er)ço huomo, çioè de 52: **rema(n)gono** dr. 7, 6 e chotante n'a(v)ve el t(er)ço. (LA p.308)

vi) Adunqua trai 40 cose meno 60 de 100, **rimane** 160 meno 40 cose (JF 319)

vii) Dimezza le cose, **remanghono** 15. Multipricha per se medesemo, fanno 225. Traine li numeri, che sonno 144, **resta** 81. Trova la sua radice, che è 9. Trailo del dimezzamento dele cose, cioè de 15. **Resta** 6, et cotanto vale la chosa. (JF p.315)

While subtraction and division each show a distinctive lexical verb in the product clause, no lexical distinction is made between multiplication and addition in this respect, both verbs most commonly selecting *fare.*[79]

i) Ragugneraj .41 L. 3/7 et .61 L. 3/7, **fanno** .102 L. 6/7 e di poj lo sechondo moltiplicha 41 3/7 vie 90 L., **fanno** .3728 L. 4/7 (CSC p.15)

ii) 4 via 4, che **fa(n)no** 16, e 3 via 3, [c]he **fanno** 9, e mette ensieme 16 e 9, che **fo(n)no** 25, ed è fatta. (LA 161v)

iii) Se ne fosse ditto che 3 via 3 **faciono** 9, e 4 via 4 **faciono** 16, egl quagle emsieme agionte **fo(n)no** 25 (LA p.458)

iv) Multiplica quelle doi de prima che diciva vintiquattro per doi [sic] **faranno** ducento quaranta […]~~e,~~ che dece via 248 **fa** dumilia quactrocento ottanta. (JF p.202)

v) Ancora diremo, se 3 via 4 **facesse** 13, quanto **farebbe** 7 via 9 a quella medesema ragione. Fa così et di', 3 via 4 **fa** 12, et io dico che **fa** 13. (JF p.238)

[79] One sometimes also encounters *montare* 'amount to', with additions and multiplications: '25 via 25, che **monta** 625; anchora devemo m(ultip)lichare la lina da lato, çioè 15, em sé medesma, e di-ne: « 15 via 15 », che fa 225; quiste 2 m(ultip)lichat(i)o(n)e devemo agiongn(er)e emsieme, e **monta** 8<0>50,' (*LA* 177v).

vi) Ridu' ogni cossa a ssano e fa 15 via 176, che **fa** 2640: agiugi 1 però che dice 1/[1]5 e **farà** 2641. Poy multiplica 15 via 9 1/6 e **fa** 15 via 1/6: **fanno** 15/6, che sono do' sani ½: poy fa 15 via 9 **fa** 135, agiungi 2 ½ per lo multiplicare fato 1/5, **farono** 137 ½. Ridu' anche a sano, perché el v'è ½, e **fa** denari 2 via 137; agiugi uno che **fanno** 275. (BA p.97)

One sign of the 'functionalization' of *fare* in the result clause of calculations is its use as an invariant singular, even with plural subjects, and even where the relevant texts otherwise show systematic agreement for number of third person verbs with their subjects. Our texts show some significant differences in this respect. In *Chasi sopra chonpagnie* by Gratia de' Castellani, the result clause always contains a plural form of *fare*. Altovito da Firenze, in contrast, only ever has singular *fa*,[80] and the same is predominantly true of Cortisi.[81] In the anonymous *abbaco* from Brescia, about 70% of tokens show singular agreement; in the *Livero de l'abbecho* approximately 55% do so. It needs to be stressed that in these texts *fare*, when otherwise used as a lexical verb, consistently agrees with its subject. Thus:

i) Et dovete sapere che tucto ciò che li homini **fanno** naturalmente et accidentalmente, sì è che el nostro padre à conceduto a sapere per la sua santissima virtù et grazia et misericordia. (JF p.194)
ii) Che per intellecto et per bono ingegno et sottile si **fanno** li homini molte sperientie et congielationi de tractati (JF p.195)

In parts 1-15 of Jacopo's text, with overwhelming consistency, [82] the verb *fare* in the product clause shows singular agreement if the multiplicands are expressed simply as numerals, be they integers or fractions:

i) Ancora diremo, se 3 via 4 **facesse** 13, quanto **farebbe** 7 via 9 a quella medesema ragione. Fa così et di', 3 via 4 **fa** 12, et io dico che **fa** 13. Et 7 via 9 **fa** 63. Et però dobiame multiplicare 13 via 63, che **fa** 819, et parti in 12 che è la terza cosa, che ne vene 68 et 1 . Et tanto **farebbe** 7 via 9 se 3 via 4 **facesse** 13. (JF p.238)
ii) E però moltiplica 3 via 92 grani, **fa** 276 grani. Et tanto vale la libra. (JF p.251)
iii) Et però multiplica 8 via 4320 pietre, che **fa** 34560 pietre. (JF p.276)

Where Jacopo does use plural agreement, this tends to be in the presence of plural *nouns* rather than simple numerals:

[80] In the future he has some examples of plural *faranno*, in addition to singular *farà.*
[81] The sole exception is 'Se la de' esser bona, tu de' fare: 2 via 87, che **fano** denari 2' (*Cortisi* 11v).
[82] The only exception is the following: "Et poi multiplica 3 via 4, che fa 12. Et poi di' così, 10 tornesi vagliono 12 parigini, che varranno 25 tornesi. Multiplica 12 via 25, che fanno 300, et parti in 10, che ne vene 30 parigini. Et cotanto varranno 25 tornesi". (JF p.239)

Et però multiplica 6 via tornesi 3 et ½ , **fanno** tornesi 21. Et di', 6 via parigini 4 1/3 **fanno** 26 parigini. Et noi vogliamo sapere che varrano 20 parigini. Et però dirai così, 21 tornesi vagliono 26 parigini, que varranno 20 parigini. Multiplica 21 via 20, **fa** 420, et parti in 26, che ne vene 16 2/(1)3 (JF p.241)

5. Specialized lexical expression of the operator

For subtraction and addition, our texts lack lexically specialized expressions of the operator. That is to say that there is nothing corresponding to English 'minus', 'plus', or to modern Italian *meno* and *più* in the relevant contexts. Of course in our texts, as in modern Italian, the words *meno* and *più* are abundantly attested with the general meanings 'less' and 'more', but we do not find formulations such as modern 'sei meno due fa quattro', or 'sei più due fa otto'. Rather the operation is expressed via the verb of the operation clause, and the relation between the operands is expressed by appropriate prepositions and conjunctions: *di* or *da* 'from' in subtractions introduced by *trarre* 'draw, remove' or *cavare* 'dig'; in additions, the addends are sometimes linked by the conjunction *e* 'and' and *con* 'with'. In divisions, the verb *partire* 'divide, share' takes the dividend as its direct object while the divisor in introduced by the basic prepositions *per* 'by, through, for' or *in* 'in, into'. *Per* and *in* seem[83] to be generally indistinguishable but in some of our texts (e.g. Jacopo) *in* clearly predominates, whereas in others (e.g the *Livero de le'abbecho*) *per* is normal. Some authors use *per* when the verb *partire* is followed by a noun, while *in* precedes a numeral; in others this distinction is either not systematic or absent. It seems that in Jacopo da Firenze *in* is predominantly used when dividing numerals by other numerals, although *per* is possible there too: for instance, "partire per le figure de quelli rotti" (p. 242), "partire per 3 e 1/7" (p. 290), "partire in 1/4 censo" (p. 312), "se vole partire nelli chubi" (p. 321), "a partire 60*f* in 15" (p. 326).

i) Et parti per 7 £ de tornesi, cioè, parti 1080 in 7 (JF p.237)
ii) Et parti 1356 per la terza cosa, che è 7 che fa 819 (JF p.238)
iii) Et parti in 12 che è la terza cosa, che ne vene 68 et 1/4. (JF p.238)
iv) Sì dobiamo partire 20 anni per tante parti quanti guadagnia la £ el mese. (JF p.244)
v) A volere sapere quanto è el suo diametro, si vole partire per 3 e 1/7 (JF p.290)

[83] On this topic, see the observations by Høyrup (2007:16, 17, 161fn312, 68fn324).

vi) Et però devi partire li numeri nelle cose (JF p.306)

vii) Si vole partire el numero per li censi (JF p.307)

viii)E parti per duchati 367 in 10, ne vene duchati 36 soldi 14 e tanto vene 1'anno. Parti ora li diti duchati 36 soldi 14 per 12, che ne viene duchati 3 soldi 1 denari 2 e tanto viene el mese (BA p.95)

viii) Ora multiplica 29 via 47, 1363 e parti per 76 (BA p.99)

ix) E poy multiplicha 20 via 30, fa 600 e parti in 60 (BA p.99)

x) E noi dovemo partire 1 s. in 12 d. (GD p.17)

xi) Se noi avesimo a partire 1 lr. per 3 s. (GD p.17)

xiii)Partire uno numero per un altro [...] a partire 2 in 36 che nne viene [...] (*Tractato d'abbacho* by Piero Maria Calandri, p. 43)

xiv) [...] partire 32 3/4 in 6 2/3 [...] partire 66 2/3 per 15 (*Tractato d'abbacho* p. 47)

xv) Dove multipricherai 75 via 42, fanno 315 o e' quali dobbiamo partire per 5, vienne 630 (*Tractato d'abbacho* p. 118)

Matters are different when it comes to multiplication. We have already seen that the verb expressing multiplication stands out by its (semi-)learnèd origin: unlike *trarre, cavare, partire,* or *(agg)iungere*, *multiplicare* it is not originally part of the everyday lexicon. It also stands out from the other operations because its operator (the equivalent of English 'times') does have a specialized lexical expression. Unlike *multiplicare*, this term is not 'learnèd': it is the word *via* or its plural *vie* and it is entirely 'popular', being an everyday term meaning also 'way', 'road', 'route' and, by extension 'by means of', 'via', 'away'.[84] In old Tuscan the word had the further sense of '(a) time', 'occasion'.[85] From an onomasiological perspective, however, it is peculiarly specialized as the term used for the multiplier, as noted for example in GRADIT: 'Prep. Disus. Per, volta (introduce il secondo termine di una moltiplicazione). Dante, *Vita nuova*, 29-3 (124): Tre via tre fa nove. *Paolo dell'Abbaco*, 2-24: Multiprica 8 via 3, che fae 24. *S. Agostino volgar.*, 1-3-19: Quattro vie sei fanno ventiquattro. *Piero della Francesca*, 83: Radici relata de 32 è 2; cioè 2 via 2 fa 4, e 4 via 4 fa 16, e 2 via 16 fa 32. *P. Cattaneo*, 7-39: Multiplichisi 45 via 180 e quel che fa partisi per io e ne verrà 810. *Scaruffi*, 17: Si dèe considerare che 12 via 12 fanno la somma di 144. *Algarotti*, 1-II-184: Due via due dà quattro. *Gazzetta urbana veneta*, XL- 589: Tre via tre fanno nove, e non 100.' What is noteworthy about *via* in our texts is that it is strictly specialized as a connector of multiplicands and multipliers expressed as numerals. When at least one, but usually both, of the factors is a numeral, overwhelmingly it is *via*[86] that is

Commented [AP2]: Martin: Page numbers for quotations from Paolo dell'Abbaco in footnote 69?

[84] See also Meyer-Lübke (1972:134) for confirmation that multiplicative *via* originates in the word for 'road'.
[85] Note also the temporal expressions old Sienese *issa via*, old Umbrian *essa via* 'immediately', Tuscan *via via* 'right now' (Rohlfs 2021b:269).
[86] Usually singular *via*; much more rarely plural *vie* is encountered, e.g., Paolo dell'Abbaco (p 28) "multjprichare 5 vie 8 3/5" but also "multjpricha 5 ¼ via 8 3/5", etc. We cannot detect any difference in meaning or function between the singular and plural.

employed.[87] This tends also to be the case with nouns specifically denoting a numeral (such as *cosa*, *censo*, *radice*). When the factors being multiplies are not expressed as numerals but as nouns (e.g., 'multiply length by breadth'), alternative expressions, such as *per* 'by', *con* 'with', or (*di*) *contro* (*a*) 'against', are generally used to express the operation.

i) Multipricha la lunghezza contra ala larghezza, cioè 16 via 25, che fa 400. (JF p. 355)
ii) Ora multiplica contra ala chupezza, cioè 4 via 50, che fa 200. (JF p. 296)
iii) Et multiplica 4 via 4, cioè la lunghezza contra la larghezza, che fa 16. Et poi multiprica per l'altezza, cioè 4 via 16, che fa 64. (JF p. 269)
iv) Ora moltiplica la figura delo rotto l'una contra all'altra. (JF p. 242)
v) Sì'tti convene multipricare una chosa via quello che egli ànno guadagniato, et partire in tucto el corpo dela conpagnia. Et però abbiamo a multiprichare 30 via una cosa. (JF p. 306)
vi) Ora qui appresso insegnaremo moltiplicare l'una figura contra all'altra, et queste se chiamano librectine minori. Et simile insegniaremo moltiplicare una figura contra a doi, et doi contra altro du, et ancora contra a tre, et queste se chiamano librettine magiori. (JF p. 203)
vii) Multiplicha sano e rotto cum sano e rotto, cioè 21/2. Fa' cossì: ridu' ciaschaduna parte al so roto, cioè 5/2 e 5/2; poy multiplicha 5 via 5, fa 25, e 2 via 2, fa 4. (BA p.100)

Lo *Livero de l'abbecho* employs *fiade*/*fiada* 'times' as well as *via*, although the latter is the multiplication operator by far most used in the text.[88] There are 160 occurrences of *fiade,* 43 instances of *fiada.* No differences in terms of semantics and context-based usage is found. Compare:

i) 3 fiade 17 denare, che fanno 51 denaio (p. 163)
ii) 13 via 47, che fanno 611 (p. 171)
iii) M(ultip)lica 1 fiada 60, che fa s. 60, (p. 180)

Fiada/fiade is equivalent to *via*, and can also appear in contexts such as *tuttavia /tutta fiada*. Bocchi (p. 315 f82v) underlines that *tutta fiada* in *tutta fiada faciea el doppio* translates *semper* of the *Liber abaci.* It means 'however' in: i) Ma tutta fiada agio(n)gniamo el numero del p(r)imo e del s(econd)o cho(n) lu nu(mer)o del [...] (p.283); and in ii): Che tutta fiada faciea el doppio (p. 315). But *fiada* by itself, like *via*, means '(a) time': i) La quale chade una fiada la prima co(n)solatione e la t(er)ça una fiada (p. 331); ii) Che noie lo podemo sap(er)e per apositio(n)e 30 falsa la prima fiada (p. 383).

[87] Examples of (*di*) *contro* (*a*) or *per* with purely numeral multiplicands are rare but not unknown: Jacopo da Firenze "che diciva vintiquattro per doi faranno ducento quaranta" (p.202). Cases of *via* with non-numeral multiplicands are extremely rare. An example from Jacopo is 'Appresso si vole multiprichare la minore, che è una cosa, *via* la magiore, che è 10 meno una cosa'.
[88] Notably, the *Larte de labbacho* of Treviso uses *fia* as the arithmetic operator: *fia* is a local phonological development showing deletion of intervocalic[d].

Aritmetica by Filippo Calandri uses *vie* and *per*. *Vie* is always used with numbers larger than 1 (there appear to be no instances with the multiplicand 1). The connector *per* is used when the two multiplication operands are not numbers:

i) Multipricha 18 vie 32 (p. 181)
ii) Multipricha 12 fiorini vie 15 (p.73)
iii) Multiprica el diamitro per sè (p. 149)
iv) Multpricha la largheça per sè (p. 148)
v) Multiprica tutte le figure che sono sopra alle linee, l'una per l'autra (p.5)

Via and *per* are the multiplication operators used by Bastiano da Pisa:

i) Multiprica 5 via 21 (p. 33)
ii) Multiprica la quantità delle lire per 5 (p. 25)
iii) Multiprica il 9 per 4 (p. 31)

There appears to be no semantic distinction between *via* and *per* as both are used when the multiplier is a numeral, while *in* is used quantities expressed by noun, a case in which some *abbaci* use *contra*, but some authors such Orbetano da Montepulciano use *per*:

i) Fa cosi: moltiplica una delle faccie per se medesma e dì, 10 via 10, fa 100 (p. 31)

This author uses both *via* with verbal expressions and numbers:

ii) Moltiplica la mità de 14 che è 7 via la mità de 44 che è 22 (p. 36)
iii) Moltiplica 12 per 31.7 farà 37 5/7 (p. 62)
iv) Moltiplica el mezo de 14 per lo mezo de 44 (p. 27)
v) Moltiplica 4 via 12.4/7, farà 50 (p. 63)

In the *Trattato dell'alcibra amuchabile* one finds *via* and *vie* used indifferently both for singular and for plural; in this work *contra* is used thirty times for multiplication: five times with figures and the remainder with quantities expressed by nouns (for instance, *contra cienso, contra la radicie*).

i) Multipricha radicie di 25 via radicie di 20 (p. 17)
ii) Multipricha 15 via 20 (p. 17)
iii) Multiprichi 10 e più radicie di 20 contra lo risiduo suo, cioè contra 10 (p. 21)
iv) Multiprichati l'uno contra l'altro (p. 24)
v) Ti conviene multiprichare la minore, ch'e una cosa, vie la magiore (p. 26)

Altovito da Firenze uses the singular *via* and *chontro* interchangeably, *chontro* is always used when quantities are nominal and not numerical (and it is never replaced by *via* in such cases). There are 61 instances of *via,* 48 of *chontro.*

i) Multiplicha 25 via 3/4. Fa chosì: multiplicha 25 chontro allo 3 (p. 67)
ii) Multiplicha lo medesimo 2 chontro allo 3 (p. 68)
iii) Debbi multiplichare 36 e 5/9 chontro a 23 e 2/3 (p. 73)

In the anonymous *abbaco* from Brescia,[89] the operators used are *contra, per*, and *via:* there is a clear distinction between the three. There are 7 instances of *contra*, 11 of *per*, and 93 of *via.* The former is used only with verbal expressions of quantities, *per* only when a number or quantity is multiplied "by itself" (see "*si medessmo*"), *via* is used only with numerical quantities:

i) Dobiamo multiplicare l'uno contra l'altro, cioe 10 via 10 (p. 123)
ii) Multiplicha la mytà di braza 10, ch'è 5, contra lo diamitro (p. 123)
iii) Dobiamo multiplichare 9 per si medessmo, fa 81, e poi la facia delle 15 braza per si medesma (p. 129)

In the *abbaco* by Francesco Cortisi, there are 213 instances of *via*, and two of *contra,* which is only used with verbal expressions of quantities. Via is used with numerical quantities:

i) Multiplicare questo 4 contra l'altro 4 (p. 40)
ii) 7 via 4 fa 28 (p. 40)
iii) Soldi 8 via 35 fa 280 (p. 57)
iv) Multiplichare la cosa che noy voliamo sapire in contra quella (p. 60)

Gratia de' Castellani employs the plural *vie* extensively. No instances of *contra* are found, some instances of *per* (mainly with units of money, but the use is not consistent as it is often replaced by *vie*) and three instances of *in* all followed by the term and unit of money *denari* (underlying that the number is multiplied by the number of parts contained in the given quantity of *denari* according to the conversion ratio between currencies):

i) Moltiplichata ne' denari (p. 36)
ii) Moltiplicha .12000 L. per .3 d. (p. 38)
iii) Moltiplica 150 vie 600 L. (p. 41)

[89] See *Una raccolta di tre libri d'abbaco* (pp. 91-123)

Maestro Dardi uses *in, per, contra* (rarely), and by far the most frequently the singular *via.* The operator *in* is used when a quantity is multiplied by itself, *per* with both verbal explanation of quantities and occasionally with figures, very few instances of *contra* (used with quantities verbally defined). The operator mostly used is the singular *via*:.

i) Multiplicato in se medesimo (p. 39)
ii) Multiplicata per lo primo numero (p. 39)
iii) Multiplicato lo ditto 4 per lo 2 fa 8 (p.39)
iv) Dej multiplicare 4 via 9 che fa 36 (p. 40)
v) Multiplicare R dj 4 via R dj 9 (p. 40)
vi) Multiplicare 2 R contra 2 altre R (p. 50)

The specialization of *via* to numeral contexts must be to some extent a consequence of its origin. This origin we believe to be the medieval use of the term *via*, to mean 'time, occasion, occurrence', and this is the view also taken by Rohlfs (2021b:262n1): cf. GDLI s.v. *via*$_{32}$: *Anonimo*, I-504: Poi che fallì una via / a lo suo ben vogliente, / non l'adorna neiente / gioia né cortesia. *Guittone*, VII-69: E' veggio spesse via / per orgoglio atutare / ciò che merzé chiamare / non averea di far mai segnoraggio. *S. Girolamo volgar*. [Tommaseo]: Quando lo nostro Signore andava una via al tempio, sì vi trovò venditori e compratori'. An expression such as 'tre via tre' would therefore have meant 'three occurrences of three, three threes', 'three times three'. The meaning 'times, occurrences' readily explains the observed compatibility of *via* with numerals and its incompatibility with non-numeral expressions. 'Times' may be modified by quantifier expressions (of which the numerals are a subset): thus one may say '120 times 36' but not by non-quantifiers such as 'length times breadth' or 'what it costs you times what you want to earn'.[90] An objection to the claim that the specialization of *via* presupposes an original meaning of 'time(s), occurrence(s)' might be that *via* is a morphological singular, whilst one would expect to find plural *vie* whenever the multiplier is greater than the multiplicand. Plural *vie* does occur, but it is significantly rarer that *via*. Thus, in Paolo dell'Abbaco's *Trattato* there are over 300 cases of *via* vs just 13 of *vie*, without any discernible functional distinction between the two: e.g., "Dej fare per una reghola la quale è gienerale per tutte e dire senpre che fae 21 via tutta la poxexxione, cioè vie 107 4/5" (p. 118). That *via* is almost never plural is not the objection that it might seem, for the simple reason that in the sense 'times, occurrences', as well, the word was also frequently *via*. An example is Guittone's *spesse via* 'often times', given above, also 'Morte, per pietate, sia guerenz'a me di sì cocente foco, che m'aucide vivendo mille via!', 'Per che seguo ragion, non lecciaria, und'ho già *mante via* portato in

[90] These utterances are perfectly possible in modern English, but only because of the prior specialization of 'times' as a multiplication operator. In the original sense of 'occasions, occurrences' they are infelicitous: compare the fact that one might say '120 occurrences of 36', '10 occurrences of 5' but never 'length occurrences of breadth' or 'what it costs you occurrences of what you want to earn'.

loco di gran ver menzogna ed in loco d'onor propia vergogna. Further examples are: Arrigo da Castiglia (1267) 'ond'io audito agio dir *molte via* di tal morte', *Anonimo, Arte d'Amare di Ovidi* "l'amore è fiero e *molte via* mi contrasta […]", *Anonimo, Commento all'Arte d'A* "Cioè che le pulcelle vi perdéno *molte via* la virginità". However one seeks to explain the morphological invariance of *via* in its more general meaning, its prevalent appearance in multiplications is compatible with an origin in the noun *via*. We may also note the expression *vie più* '(many) times more', the source of modern *vie*(*p*)*più*, which in the thirteenth and fourteenth centuries also occurred as *via più*:[91] e.g., Guittone "se om da pare ingiurie porta magne in pace, è manto; ma *via* più troppo, se da suo menore; e molto avante, ben de mal rendendo"; Sacchetti "Or questo risparmio fece questo valente uomo, ch' e' porci valeano forse dieci fiorini, ed egli ne spese poi forse altrettanti, senza le beffe che furono *via* più" (but also "Morte è bisogno e necessario fine; s'ella non fosse, *vie* più mal sería"); Fazio degli Uberti "Solin, diss'io, di vedere avea brama questo animale e parmi scontrafatto assai *via* più che non porta la fama" (but also "Otto anni e trenta al mio onore intese e vo' che sappi che, per adornarmi, assai *vie* più ch'alcun de' primi spese"). That the operator is so consistently *via* in our texts (far more often, it seems, than plural *via* in the sense 'occurrence'), may reflect its functional specialization as an operator. This specialization is further apparent in that, when the multiplier is 'one', it never agrees in gender with *via* ('one times..'[92] is always *uno via*, never **una via*) and in that verbs rarely agree for number with *via*: e.g., Paolo dell'Abbaco '4 *via* 6 fae 24' (not **fanno*). The same development[93] is occasionally observable in *fiata* (or *fiada*) 'time, occasion'. While plural *fiate* (*fiade*) is vastly predominant, the OVI database offers several examples of plural *fiata* (*fiada*). For example, Guido Faba 'Lo Signore Deo spesse fiata monstra la via p(er) la quale d(e) andare l'omo'; 'multe fiata fano tale overe ch'enduxeno l'omini a pecare'; "Dua fiada u tree dé l'omo rechedere lo soe amigo"; "li n(ost)ri subditi spesse fiada ènno robbati i(n) lo v(ost)ro tereno"; Andrea da Grosseto "lo studio aiuta lo 'ngieno e vince spesse fiata la natura"; Accurso di Cremona "ca eciandeu quisti chosi spissi fiata assemblaru curaiusi skeri di cavaleri"; Maramauro "spesse fiata facea novi tormenti e delectavasi in essi"; Matteo dei Libri "lo homo de questo mundo spese fiata convene sustinere briga"; Laudi della Scuola Urbinate "La bocca dé aver clusa e la lengua affrenata, e stare in parlamento molte poke fiata"; Anonimo, Volgarizzamento REF "d(e) lu q(u)ale ung(u)ento se unga dui fiata in dì"; Volgarizzamento della Regola "et sia dicto quisto v(er)su dintra la clesia da tutti li fratri tre fiata"; Anonimo, Elucidario "perzò ke in questo nomero de nove se conten lo nomero de tri trea fiada , ké trea fiada tri fa nove"; "Quanta fiada aparite Criste pox la resurectione? M. El aparite xij fiada"; "sì com quatro fiada x leva xl"; "e imperzò nuy fimo somerzuy trea fiada soto l'aqua"; Anonimo REF "quaranta dì in gran beleza plusor fiada con dolzeza ay so amis' sì se demostrà". The

[91] The OVI database offers 120 attestations of *via più* in this sense, against 220 of *vie più.*
[92] It is interesting in this connexion to note how the English 'times', specialized as a mathematical operator, can only occur as a plural form. One says 'one times two', never **'one time two'.
[93] See also see also Meyer-Lübke (1890:202); Rohlfs (2021b:22). For a similar phenomenon with the noun *ora* 'time', see Ascoli (1884:324n2).

Livero dell'abbecho systematically uses *fiade* in multiplication, but also *via*. Compare: "m(ultip)lichare 7 fiade 5 (p. 163) and 5 via 24, mo(n)tano 120" (p. 463); and compare: "p(er) chotante fiade te co(n)viene radopiar (p. 462) and a m(ultip)lichandole 5 fiade, 5 montano 25" (p. 463).

6. Lexicon, Computation Techniques, and Spatial Visualization

The *abbaci* show that a great deal of importance was given to visual modes of teaching and visual reasoning. While the use of drawings and illustrations therein, found to facilitate understanding and memorization of mathematical techniques, has been examined by Botana (2020),[94] we focus here on the use of a metaphorical, analogical, everyday lexicon which supported memorization of computation techniques while also embodying the algorithm process involved. It is worth noting that, unlike in the *abbaci*, in the Latin *Liber abaci* by Leonardo Pisano multiplication and division techniques are not given any particular name. Several factors came into play that markedly contributed to determining this situation. First, Pisano's treatise is an encyclopaedic work (combining lengthy theoretical explanations with practical examples) aimed at the broad audience of all the people who shared the Latin culture. In the introduction, Pisano dedicates his work to his contemporary Michael Scot (1175-1232 CE), an astrologer under the patronage of Frederick II. By contrast, vernacular *abbaci* addressed a popular audience of both young and adult learners and focused on the practice of commercial mathematics, no theoretical observations are found. Also, by the time these training manuals had established themselves as a new textual genre, new multiplication and division method emerged, as the decimal system of numerical notation and numeration was fully adopted, adapted, and new computation techniques developed.

Authors of *abbaci* had to explain how to perform an arithmetical operation by employing a reasoning style which makes use of spatial lexicon and spatial images: in doing so, they describe how quantities should be arranged in a numerical layout while being computed. For instance, the action verbs *pigliare, porre*, and *mettere,* and the prepositions *sopra, sotto,* and *appresso* guide readers through every step of calculating an algorithm. Numerical tables sometime accompany verbal explanations but they seem to play a marginal role and are not numerous. Another aspect illustrating the relationship between spatial lexicon and visual reasoning regards computation methods, which are often curiously named; and most names reflect the way digits are arranged in the layout to be computed, for instance:

- *partire a galera*[95]
- *partire disteso*[96]

[94] See Botana (2020: 156-189).
[95] DGA p. 46, called *a galea* in CTA 17v.
[96] TA p. 18.

- *partire a danda*[97]
- *multiplicare a scacchieri*[98]
- *multiplicare a casella*[99]
- *multiplichare a baricocolo*[100]

The term *galea/galera* (English 'galley') indicates a type of ship that was propelled mainly by oars, and Franci and Rigatelli (1982: 46) underline that the way digits were arranged to perform this division method must have reminded the mathematicians of a ship; they (p. 42) also emphasize that the multiplication method a *baricocolo* was so called because it reminded them of a multilayered apricot tart, and specify that it was used only in Tuscany and was sometimes also called *a scachieri* (the chessboard method, *scacho* was a 'chess piece') in Venice, implying that these two expressions indicate the same multiplication method. The name implies that the numerical table used in this multiplication method had a chessboard form. Smith (vol II, p. 109) explains and shows how the digits were arranged to be computed in a chessboard-like mathematical layout where squares were marked off, and that this name and method is also found in later practical arithmetic manuals in Germany, Spain, and England. For instance, Petzensteiner (1483) says: "Also ich multipliciren in Scachir". Recorde (1542) speaks of "one way that is wrought by a checker table", and Texada (1546) describes multiplication "escaqr o berricolo". Luca Pacioli in *Summa de arithmetica, geometria, proportioni et proportionalita* (first published in 1494), while explaining various multiplication methods clarifies that their names derive from the form of the numerical layout arranged where to insert and compute digits: "Così a simili hano facto li pratici ragionieri dando el nome a l'operatione secondo che a qualche cosa materiale sa simeglia la sua dispositione. Bericuocolo: perch'pare la figura d qsti bricuocoli [...] che si vendono a le feste. E scachiero lo chiamano perchè capo piedi recogiola la sua operatione fanne una figura che sasimiglia al scachieri da scachi (f. 36r)".

Commented [MM(3]: Please check that this is correct

Commented [MM(4]: 'where'? I don't quite understand this

In his *abbaco,* Giovanni de' Danti distinguishes multiplication a *barichocolo/biricuocolo* and multiplication *a schachieri* (p. 14-15):

> [...] quale se po multipricare ongni grande multipricasgione senca avere bisongnio di tenere troppo a le mani in questa forma, cioe a bricuocolo e a schacchieri come vedarete figurato seguente la materia. [...] Questa ene la regola ad amestrare de multipricare a schacchieri ongni quantita de figure [...] tenere quella medesima regola che si tiene a multipricare a brichuocolo salvo che non si tiene a la mano nulla. Ancho se mecte in ciascheuno quadro de lo scacchieri.

[97] TA p. 19, CTA 18v, DGA p. 47.
[98] TA p. 15, *multiplichare lo schachiero* in the LA p. 399.
[99] TA p. 16.
[100] Called *brichuocolo* in TA p. 14, *berichochulo* in Bastiano da Pisa's *Tratato* p. 15, *birichocolo* in DGA p. 42.

The verb *combactere* ('fight against')[101] reflects the way the multiplicand and multiplier were multiplied by each other and the digits were arranged in the mathematical layout, which reminds a strategic pen and paper game for two players, a battleship game played on ruled grids.[102] He explains that le "figure vanno combattute. E poni 7 nella prima casella del brichuocolo da capo, poi combacti: 1 via 2 fa 2 e poni 2 [...] E 1 che tu tieni fa 5 e cosi combacti a figura a figura tucte quelle di socto [...] Donqua vedi che sono 7 figure e combactendo a una a una quelle di socto faranno 7 verghe nel brichuocolo" (p. 14). And to explain the multiplication method called *a scacchieri,* he says: "Ancho se mecte in ciascheuno quadro de lo scacchieri quello che fa 1'una figura combactuta coll'altra, bene che ciasceuno quadro de lo schachieri [...] Ora ai combactuto da 1 a tucte le figure di sopto, ora lascia 1 e piglia il 2 che gli e di lato e combacti con tucte quelle di socto a una a una e di' "(p. 15). [103]

The verb *abbattere* ('knock down') is found in some *abbaci* to indicate 'subtract' (we have discussed this verb in section 3.2) and seems to reflect the top-down direction of the mathematical procedure of subtraction and the way digits were arranged and computed in the mathematical layout one below the other according to their power of ten.[104]

References to everyday objects, metaphorical associations, and most importantly the choice of terms able to reflect the computation techniques, and thus the way digits were arranged and computed during the algorithm calculation, were effective linguistic devices able to facilitate mathematics learning and memorization.

7. The lexicon of mathematical operations in the *Liber abaci*

Although there is no general agreement on the role played by it in the West in stimulating the transition from Roman to Indo-Arabic numerals, Pisano's *Liber abaci* and the innovations introduced had an enormous impact for well over three centuries.[105] In Italy, several vernacularizations of sections of this monumental work appeared towards the end of the thirteenth and beginning of the fourteenth century.[106] Pisano seems to have been the first to use the word *abacus* to mean "computation, calculation" based on Indo-Arabic numerals and decimal arithmetic rather than to denote the reckoning device; the *abbaco*

[101] The OVI database shows that, in non-mathematical texts, the verb *combattere* is used to mean 'fight against, combat'.
[102] See the table drawn by de' Danti to demonstrate how to compute digits according to the operation explained (p. 16).
[103] We have not found other instances of the verb *combattere* in our corpus.
[104] The mathematical tables drawn in the *abbaci* demonstrate that the two quantities to be subtracted were arranged as we would do nowadays: the larger is above, the smaller is below, and then units below units, tens below tens, and so forth.
[105] Edited by Boncompagni (1857: 1–459). The *Liber abaci* has been translated into English by Sigler (2002). See also Giusti and Petti (2002), and Franci (2003).
[106] See, for instance, Arrighi (1996) and Feola (2008).

mathematical literature of Renaissance Italy teaching business arithmetic in the vernacular developed to some extent upon the tradition of business arithmetic initiated by Pisano, and used the word *abacus* (Italian *abbaco*) with the same meaning.[107] This is the earliest surviving work, written in Latin, which teaches methods of computation making use of positional decimal numeration and expounding business arithmetic in great detail, while also providing a wide variety of sample problems. This is the earliest Latin work to apply mathematics to daily and commercial life. In the dedicatory letter, Pisano explains that he became acquainted with the new science of numbers (Indo-Arabic figures and method of computation based on them) during his training in Northern Africa and travels in the Mediterranean basin, when his father was the scribe responsible for the customs post in Béjaïa (in modern-day Algeria).

To what extent were the patterns and formulae that we have identified in the vernacular texts already present also in the *Liber abaci*? There is certainly a high degree of formulaicity in the *Liber abaci* just as there is in the *libri d'abbaco*, but there is not always an obvious continuity between the mathematical expressions of the former and those of the latter. The *libri d'abbaco* repeatedly have (*ag*)*giungere, trarre/cavare, moltiplicare*, and *partire* in the operator clause, and the *Liber abaci* also has recurrent verbs in these respective functions, although they are partly different, both semasiologically and onomasiologically, from what we find in the vernacular. In Leonardo's text we find, typically, *addere*, *extrahere*, *multiplicare*, and *dividere*. Of these, as we have seen, only *multiplicare* is continued in the vernacular texts, while *dividere* is almost entirely supplanted by *partire*, and *addere* by *(ag)giungere*.[108] *Trarre* 'draw, pull' derives from Latin TRAHERE, of the same meaning, which is the derivational base in Latin of *extrahere*, used in the *Liber abaci*, but *extrahere* means specifically 'extract, pull out', a meaning more faithfully reflected in the etymologically distinct *cavare* 'dig out, excavate' (< Latin CAUARE). Notably, for the operation of addition Pisano also employs *additio/ addictio, additatio*, *iunctio*, and *collectio*, and 'add' is also *additare, iungere,* and *colligere.*

There is nothing in the *Liber abaci* corresponding to the lexical specialization of the multiplication operator *via/vie* found in the *libri d'abbaco*. Multipliers and multiplicands, just like divisors and dividends, are generally linked by *per* 'by, through'. There are a few examples of the word *vices* 'times, occasions' used in expressing the operation of multiplication, such as (p. 402) "item 4 uices 4 faciunt 16, et 5 uices 5 faciunt 25". But most of the examples of *uices* occur in the arithmetical 'times tables' at the beginning of the *Liber abaci* (whose status as part of the text is problematic) [109] where the multiplication formula is of the kind '6 uices 6 fiunt 36'. In fact, where *uices* does occur in

[107] In the preface, Leonardo states how he has found the Indian number system and its methods (*modus indorum*) of calculation to be superior to all other methods.
[108] *Iungere* 'join' is occasionally encountered in the *Liber abaci* (e.g., et habebis 9 et 10; que insimul iunge, erunt 19), but with nothing like the frequency with which its vernacular counterpart appears in the *libri d'abbaco.*
[109] There are various respects in which the table headed 'Introductiones in acditione et multiprichatione numerorum' is unrepresentative of the text. This is the only place in which we encounter the phonologically popular form *multiprichatione*, or the spelling of the word with 'h'. And it is also the only place in which the result of an operation is introduced by *fiunt* (rather than *sunt, erunt*, *faciunt*, etc.).

the text, it is almost always in its ablative plural form *uicibus*, expressing precisely the number of *times by which* a number is being multiplied.[110] E.g (p. 288).:

> Rursus accipe 2/5, et extrahe 2 de 5, remanent 3; que multiplica 1429 inuenta, erunt 4287; super que adde multiplicationem de 1, quod est super 4, in 5, que sunt super 11 **uicibus** 9, **uicibus** 7, **uicibus** 5, que sunt sub uirgis, scilicet 1575, erunt 5862; et tot habuit primus homo: additis ergo inuentis quantitatibus .iiu.M hominum, reddent pro tota summa eorum 13860; que summa inuenitur ex multiplicatione etiam omnium numerorum, qui sunt sub uirgis, uidelicet de 4 in 11 **uicibus** 9, **uicibus** 7, **uicibus** 5.

Pisano uses mainly the operators *per* but also *in* and *contra*; but they seem to have different semantic values, compare:

i) adhuc nultiplicetur 1 de superiori numero per unum de subteriori (p. 8)
ii) multiplicauimus 7 per 4 (p. 10)
iii) item multiplicatio de 1 in 37 (p. 8)
iv) quem ostendam in multiplicatione de 567 in 4321 (p. 19)
v) item si queratur multiplicatio unius figure contra duas (p. 10)
vi) ut sint in multiplicatione sex figure contra sex (p. 17)

Our inspection of the *Liber Abaci* shows that the operator *per* is the most frequently used, and occurs when forms of the verb *multiplicō, -āre* are used to denote that the multiplication should be performed between two numbers expressed in figures or by verbal explanations emphasising their characteristics or position in the numerical layout .[111] The operator *in* is used in association with the noun *multiplicatio* governed by the preposition *de* followed by figures. The operator *contra* is found when the noun *multiplicatio* is used with verbal expressions denoting quantities to multiply. Like the *libri d'abbaco,* in the *Liber abaci* the result of multiplications and additions may be introduced sometimes by the verb 'make' (*facere*) (see pp. 53, 54) but more frequently we find the plural third person present or future form (*est, sunt; erit, erunt*) of the verb *esse* 'be'. The result of a subtraction, however, is expressed, as in the vernacular texts, by the verb meaning 'remain' (*remanere*) (pp. 55, 56, 57, 58), the etymological and semantic counterpart of Italian *rimanere*:

[110] But see also: *39 uices miliaria 60* (p. 169) and *tres uices 7* (p. 240).
[111] It is also found with the noun *multiplicatio*: *et divides ipsas multiplicationes per 14* (p. 152), *et divides utramque multiplicationem per 2 posita super 3* (p. 153).

i) Et multiplica 7 per 6**, erunt** 42, que pone super 7; et multiplica 19 per 2, **erunt** 38, que pone super 19: post hec extrahe 38 de 42, **remanent** 4; multiplica 38 per 21, **erunt** 789; que divide per 4, **exibunt** denarii 1/2 199; et tot investivit (p. 170)

ii) Et multiplicabitur tantum per 7, et per 9, idest per 63, **erunt** 877733; de quibus extrahe 815360, **remanent** 72373 (p. 179)

iii) Unde colligas 1, et 2, et 4, **faciunt** 7 (p. 181) […] Quare multiplica 12 per 12, **faciunt** 144 (p. 184)

iv) Multiplica 12 per 30, **erunt** 360; que divide per 13, **exibunt** 9/13 27: pro quibus iterum dices: est numerus, super quem si addideris 1/4 1/3 et 12, **facient** 9/13 27: quare extrahe 12 de 9/13 27, **remanebunt** 9/13 15: deinde pone, ut ipse numerus sit 12, super quem adde 1/41/3 ipsius, **erunt** 19 (p. 187)

For division, the correlation between expression of the result as *venirne* and of the operation of division as *partire* in the *libri d'abbaco* mirrors a similar lexical specialization in the *Liber abaci*, where there is a very strong correlation between *dividere* (or the noun *divisio*) in the operation clause and the verb *exire*, literally 'go out',[112] in the result clause. There are between 350 and 400 tokens of *exire* in in the *Liber abaci*, virtually all of them expressing the outcome of a division,[113] and virtually all correlated with the verb *dividere* or the noun *divisio* in the operation clause (example 4 is an exception, of the 'take a fraction of' kind). By far the most common form of the verb is 3PL.FUT.IND *exibunt* (or 3SG.FUT.IND *exibit*):

i) Multiplicabis itaque 1 per 10; que **divides** per 7, **exibit** pro quantitate prime partis 3/7 1. Item multiplicabis eademque ratione secundam partem, scilicet 2 per 10, erunt 20: que **divides** iterum per 7, **exibunt** 6/7 2; et tantum est secunda pars. (p. 181)

ii) Quare multiplica horas 12 diei, scilicet 144 per 1; et **divide** per secundum numerum, scilicet per 1/3 26, **exibunt** hore 37/79 5; et in tot implebitur tina illa. (p. 183)

iii) Quare **accipe** ¼ de 459, cum cadat in integrum, **exibunt** 153 pro summa denariorum hominum (pp. 440-441)

The verb 'come' (*uenire*) and its derivatives *prouenire* 'come forth' and *euenire* 'come out' are used quite extensively by Leonardo to express the result of an arithmetical operation, but they do not appear to have any particular specialization according to *type* of operation. E.g.:

[112] Examples of this verb used in the literal sense of 'come/get out' can be observed where we meet the problem of the lion trying to get out of a well: 'Quidam leo est in quodam puteo, cuius profunditas est palmis 50; et ascendit cotidie 1/7 unius palmi, et descendit 1/9. Queritur in quot diebus **exierit** de puteo. Pone, ut **exiret** extra puteum in diebus 63; ideo quia in 63 invenitur et 1/9 1/7.'

[113] We have detected 8 examples where *exire* expresses the result of a multiplication. See, for instance, p. 228: Sunt enim 2 de 3 due tercie: multiplicatis quidem 2 per aliquem numerum, numerus qui **exierit** ex multiplicatione, erit 2/3 ex numero, qui procreabatur ex multiplicatione de 3 in ipso numero, in quo multiplicata fuerint 2.

Verbi gratia: quoniam ex diuisore ducto in exeuntem numerum **prouenit** diuisus numerus: ergo si multiplicamus probam diuisoris per probam exeuntis, **ueniet** proba diuisi numeri: sed ex diuiso numero per 23, remanserunt 15, quibus extractis de 13976, remanent 13961, quibus diuisis per 23, **ueniunt** 607. Ergo ex multiplicatione de 23 in 607 **proueniunt** 13961. Quare si multiplicatur proba de 607 que est 4 per probam de 23 que est 5, **ueniunt** 20, quorum proba , scilicet 2, est proba de 13961 , quibus additur proba de 15 que super sunt que est 6, faciunt 8, scilicet proba de 13976, et hoc uolui demonstrare. (p. 34)

8. Conclusion

Our linguistic analysis demonstrates the neglected fact that vernacular mathematical teaching involved extensive, often metaphorical, use of everyday words for everyday concepts.[114] Paradoxically, however, the everyday words may become specialized as part of a vernacular mathematical vocabulary, widespread across the *abbaci*. The reuse of ordinary terms in technical registers is obviously a teaching method based on repetition and recognizability. We could speak of a process of "concretization of abstract mathematical concepts": there is a transference of abstract concepts and terms into practical, everyday language and techniques. This semantic path appears to be one of metaphorization. For example, 'dig' and 'collect', one the one hand, and 'subtract' and 'add', on the other, illustrate how the abstraction/concretization relationship is resolved in *abbaci*: the two sets of concrete terms have been metaphorized, and the practical everyday sense of 'digging' and 'collecting' (activities in the lives of people generally, not just of students of *abbaco*) has been exploited metaphorically to express the arithmetical concepts.

The vernacular expression of arithmetical operations was to a large degree *motivated* for learners, through the deployment of terms from everyday language. The use of the verbs *(ag)giungere* 'join', *trarre* 'pull (out)', *cavare* 'dig (out)', *partire* 'share' seems to root the relevant operations in concrete, real-world, operations or experiences. An 'addition' clearly involves 'joining' entities together; in a subtraction one 'pulls out' a set on entities from a larger set; a division involves 'sharing out' amounts from a larger amount. Verbs expressing a result seem similarly rooted. The semantically generic, basic, transitive verb *fare* is used indifferently for both addition and multiplication and its argument structure is such that the operation clause acts as the subject or agent of the action, and the product is the result: the frequent absence of agreement of this verb with its subject suggests a degree of reanalysis as a specialized term introducing arithmetical results. The unaccusative verbs

[114] A conceivable alternative would be that the mathematical lexicon should consist only of specialized, 'learnèd', terms whose meanings would have to be wholly learned. This seems to be far more the case in modern Italian mathematical usage.

rimanere/restare 'remain, be left' express what is 'left behind' once one amount has been extracted from another. In contrast, in the result clause of divisions the unaccusative *venirne* is a *motion* verb expressing not simply 'come' but, via the clitic pronoun, also the source, so that the subject (the result of the division) is that which 'comes out of, emerges from' the division. The underlying image seems to be one of separation out of a greater original whole, of emergence of a new, distinct, set of entities, and *venirne* seems to be consistent with our cognitive experience of 'sharing', the focus being on the fractions, or 'shares', that come out of the operation.[115]

Far less 'motivated' and familiar are the means used to express 'multiplication'. We have seen that the way in which it is expressed rather stands apart from the other three operations: it involves a phonologically learnèd, or at least semi-learnèd, form, it is not part of 'everyday' vocabulary, and it does not have a dedicated 'result' verb, since it shares *fare* with the results of additions. We surmise that these linguistic facts may reflect in some degree the cognitive reality of multiplication. Multiplication is not, we submit, cognitively distinct from addition: the outcome for either is perceived simply as 'more'. In concrete situations, the only way to achieve '3 times 4 eggs' is to *add* another 4 eggs, and then another 4 eggs, to the 4 eggs we already have. The operation is not essentially different from '3 plus 4 eggs', in which 4 eggs may be added to 3. One operation *makes* a collection of 12 eggs, the other *makes* a collection of 7 eggs. In sum, 'multiplication' is simply a special form of addition: people cannot achieve multiples by any means other than addition. This is quite distinct from division, for example, where the action of sharing out, and the emergence of a set of distinct entities as a result, is a matter of common experience. We suggest, also, that it is the fundamentally more abstract nature of multiplication, in comparison with the other operations, which motivates both the selection of a (semi-)learnèd verb and the existence of a specialized operator in the form of *via* or *vie*. In sum, the language of the *libri d'abbaco* draws on, for pedagogic and practical ends, the lexicon of everyday experience-insofar as the relevant concepts can easily be related to everyday experience. But in so doing it begins to create a specialized vernacular lexicon and brings new, mathematical, meanings to basic vocabulary.

References

[115] Note how, in Italian as well as English, division is the only operation whose outcome is distinctively lexicalized, as *frazione* or *fraction*. There is no ordinary equivalent for subtraction, while for both addition and multiplication we have the *somma* or *sum*, without further lexical distinction. It is hard to think of a distinctive term for the result of a multiplication beyond the purely technical 'product'.

Manuscripts

Ash 1168 [1097). Medicea Library, Florence: *Tractado algorismo*, 15th century.

Primary Sources

Arrighi, G. 1964. *Trattato d'aritmetica* of Paolo Dell'Abaco. Pisa: Domus Galilæana.

—1967. *Trattato di Fioretti nella trascelta a cura di Mo Benedetto* of Antonio de' Mazzinghi, secondo la lezione del Codice L.IV.21 (sec. XV) della Biblioteca degl'Intronati di Siena. Siena: Domus Galilaeana.

—1969. *Aritmetica* of Filippo Calandri, Secondo la lexione del codice 2669 (sec. XV) della Biblioteca Riccardiana di Firenze. 2 vols. Firenze: Cassa di Risparmio di Firenze.

—1974. *Tractato d'abbacho* of *Pier Maria Calandri*. Dal codice Acq. e doni 154 (sec. XV) della Biblioteca Medicea Laurenziana di Firenze. Pisa: Domus Galiaeana.

—1985. "Tractato de l'algorisimo" of Giovanni de' Danti Aretino. Dal Cod. Plut. 30. 26 (sec. XIV) della Biblioteca Medicea Laurenziana di Firenze. In *Atti e Memorie della Accademia Petrarca di Lettere*, Arti e Scienze, nuova serie 47, 3-91.

— 1996. *La pratica di geometria: volgarizzata di Cristofano di Gherardo di Dino cittadino pisano dal codice 2186 della Biblioteca Riccardiana di Firenze*. Pisa: Domus Galilaeana.

Barbieri, F. e P. Lancellotti. 1986. *Tratato d'arismeticha praticha,* Bastiano da Pisa detto il Bevilacqua: dal Codice ital. 1110 della Biblioteca estense di Modena. Quaderni del Centro Studi della Matematica Medioevale, 17. Siena: Università degli Studi di Siena.

Bocchi, A. 2017. *Lo livero de l'abbacho*. Pisa: Edizioni ETS.

Boncompagni, B. 1857. "Il Liber Abbaci di Leonardo Pisano pubblicato secondo la lezione del codice Magliabechiano C. I. 2616". In *Scritti di Leonardo Pisano, matematicodel secolo decimoterzo*, vol. 1, Rome: Tipografia delle Scienze matematiche e fisiche.

Cattelani Degani, F. And A.Mantovani. 2000. *Una raccolta di tre libri d'abbaco*. Quaderni del Centro Studi della Matematica Medioevale, 25. Siena: Università degli Studi di Siena.

Dalai Emiliani, M. 2012. *Trattato d'abaco di Piero della Francesca. Roma*: Istituto Poligrafico dello Stato.

Franci, R. 2001. *Aliabraa argibra* of Maestro Dardi, dal manoscritto I.VII.17 della Biblioteca Comunale di Siena. Quaderni del Centro Studi della Matematica Medioevale, 26. Siena: Università degli Studi di Siena.

Høyrup, H. 2007. *Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture*. Basel, Boston: Birkhäuser.

Pancanti, M. 1984. *Chasi sopra chonpagnie* of Gratia de' Castellani: dal codice Palatino 573 della Biblioteca nazionale di Firenze. Quaderni del Centro Studi della Matematica Medioevale, 11. Siena:Servizio Editoriale dell' Università di Siena.

Rivolo, M.Teresa. 1983. *Arte giamata aresmetica*, anonimo maestro lombardo. Un'antologia dal codice N.III.53 della Biblioteca nazionale di Torino. Quaderni del Centro Studi della Matematica Medioevale, 8. Siena: Servizio Editoriale dell' Università di Siena.

Simi, A. 1991. *Regole di geometria pratica* of Orbetano da Montepulciano, dal manoscritto Moreni 130 (sec. XV) della Biblioteca Riccardiana di Firenze. Quaderni del Centro Studi della Matematica Medioevale, 19. Siena: Servizio Editoriale dell' Università di Siena.

—1994. *Trattato dell'alcibra amuchabile,* Anonimo (sec. XIV), dal Codice Ricc. 2263 della Biblioteca Riccardiana di Firenze. Quaderni del Centro Studi della Matematica Medioevale, 22. Siena: Servizio Editoriale dell' Università di Siena.

Secondary Sources

Allard, A. 1990. "La formation du vocabulaire latin de l'arithmétique médiévale." In *Méthodes et instruments du travail intellectuel au moyen âge:études sur le vocabulair*e, edited by Olga Weijers, 137-181. Turnhout: Brepols.

Ascoli, G. I. 1884. Varia. *Archivio glottologico italiano* 15-17:323-336.

Black, R. 2001. *Humanism and Education in Medieval and Renaissance Italy.* Cambridge: Cambridge University Press.

—2007. *Education and society in Florentine Tuscany: teachers, pupils and schools, c. 1250-1500*. Leiden, Netherlands; Boston Mass.: Brill.

Burnett, C. 2010. *Numerals and Arithmetic in the Middle Ages*. Farnum, Surrey: Ashgate.

Cajori. F.1929. *A History of Mathematical Notations.* Chicago, Ill.: The Open court publishing company.

Carlsmith, C. 2010. *A Renaissance Education: Schooling in Bergamo and the Venetian Republic*, *1500-1650.* Toronto: University of Toronto Press.

Castellani, A. 2000. *Grammatica storica della lingua italiana. 1. Introduzione.* Bologna: il Mulino.

Chrisomalis, S. 2010. *Numerical notation: a comparative history*. Cambridge: Cambridge University Press.

Danna, R. 2021. "The Spread of Hindu-Arabic Numerals in the European Tradition of Practical Mathematics (13th-16th Centuries)." *Nuncius* 36(1): 1–44

Folkerts, M. and B. Hughes. 2016. "The Latin mathematics of medieval Europe." In *Sourcebook in the mathematics of medieval Europe and North Africa*, edited by Victor J. Katz, 4-223. NJ: Princeton University Press.

Feola, F. 2008. *Gli esordi della geometria in volgare: un volgarizzamento trecentesco della Practica geometriae di Leonardo Pisano*. Firenze: Accademia della Crusca.

Franci, R. and L. Toti Rigatelli.1982. *Introduzione all'aritmetica mercantile del Medioevo e del Rinascimento. Realizzata attraverso un'antologia degli scritti di Dionigi Gori (sec. XVI).* Urbino: Quattro Venti.

Franci. R. 2003. "Leonardo Pisano e la trattatistica dell'abaco in Italia nei secoli XIV e XV." In *Bollettino di storia delle scienze matematiche*: XXIII, 2, Fabrizio Serra Editore, pp. 1-22.

Giusti, E.and R. Petti, eds.2002. *Un ponte sul Mediterraneo. Leonardo Pisano, la scienza araba e la rinascita della matematica in Occidente*. Firenze: Edizioni Polistampa.

Goldthwaite, R.A. 1972. "Schools and Teachers of Commercial Arithmetic in Renaissance Florence." In *Journal of European Economic History* 1(2):418-433.

— 2009. *The Economy of Renaissance Florence*. Baltimore: Johns Hopkins University Press.

—2015. "The Practice of Accounting in Renaissance Florence." In *Enterprise and Society*, 16(3): 611–647.

Grendler. P. 1991. *Schooling in Renaissance Italy: Literacy and Learning, 1300-1600*. Baltimore: Johns Hopkins University Press.

Scienza.

—2013. *Gli abacisti fiorentini delle famiglie del maestro Luca, Calandri e Micceri e le loro scuole d'abaco* (secc. XIV–XVI). Florence: Olschki.

Pellegrini, G.B. 1972. *Gli arabismi nelle lingue neolatine con speciale riguardo all'Italia*. Brescia: Peideia.

Mancini, M. 1993. "Voci orientali ed esotiche nella lingua italiana." In *Storia della lingua italiana,* edited by L. erianni and P.Trifone, 3 vols Torino:Einaudi, pp. 825-879.

Manni, P. 2001. "La matematica in volgare nel medioevo (con particolare riguardo al linguaggio algebrico)." In *Le parole della scienza. scritture tecniche e scientifiche in volgare (secc. XIII-XV)*, edited by R. Gualdo, pp. 127-152Atti del convegno di Lecce, 16-18 aprile 1999, Galatina: Congedo.

Meyer-Lübke, W. 1890. *Italienische Grammatik.* Leipzig: Reisland.

Meyer-Lübke, W. 1972. *Grammatica storica della lingua italiana e dei dialetti toscani.* Turin: Loescher.

Petrocchi, A. 2019. "Medieval Literature in Comparative Perspective: Language and Number in Sanskrit and Latin." *Journal of Medieval Worlds*, 1 (2): 57–76

Rohlfs, G. 2021a. *Grammatica storica della lingua italiana e dei suoi dialetti. Fonetica.* Bologna: il Mulino.

Rohlfs, G. 2021b. *Grammatica storica della lingua italiana e dei suoi dialetti. Sintassi e formazione delle parole.* Bologna: il Mulino.

Sigler, L. *Fibonacci's Liber Abaci. A Translation into Modern English of Leonardo Pisano's Book of Calculation.* New York, NY: Springer New York: Imprint: Springer.

Smith D.E. and L.C. Karpinski. 1911. *The Hindu-Arabic Numerals.* Boston, Mass.; London: Ginn and Company.

Swetz, F.J. 1987. *Capitalism and Arithmetic: The New Math of the 15th Century*. Including the Full Text of the Treviso Arithmetic of 1478, translated by David Eugene Smith. La Salle: Open Court Publishing.

Ulivi, E.2004. ”Maestri e scuole d’Abaco a Firenze: la Bottega di Santa Trinita”, *Bollettino di storia delle scienze matematiche* 24(1): 43-91.

Van Egmond, W. 1976. *The Commercial Revolution and the Beginnings of Western Mathematics in Renaissance Florence, 1300 –1500* (Ph.D. diss., Indiana University).

—. 1980. *Practical mathematics in the Italian Renaissance. A Catalog of Italian Abbacus Manuscripts and printed Books to 1600.* Firenze: Istituto e Museo di Storia della Scienza.