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Large-*n* asymptotics for Weil-Petersson volumes of moduli spaces of bordered hyperbolic surfaces

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Abstract: We study the geometry and spectral theory of Weil-Petersson random surfaces with genus-g and n cusps in the large-n limit. We show that for a random hyperbolic surface in $\mathcal{M}_{g,n}$ with n large, the number of small Laplacian eigenvalues is linear in n with high probability. By work of Otal and Rosas [42], this result is optimal up to a multiplicative constant. We also study the relative frequency of simple and non-simple closed geodesics, showing that on random surfaces with many cusps, most closed geodesics with lengths up to $\log(n)$ scales are non-simple. Our main technical contribution is a novel large-n asymptotic formula for the Weil-Petersson volume $V_{g,n}(\ell_1, \ldots, \ell_k)$ of the moduli space $\mathcal{M}_{g,n}(\ell_1, \ldots, \ell_k)$ of genus-g hyperbolic surfaces with k geodesic boundary components and $n-k$ cusps with k fixed, building on work of Manin and Zograf [31].

Contents

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1. Introduction

Let $\mathcal{M}_{g,n}$ denote the moduli space of genus-g hyperbolic surfaces with n cusps. One can equip $\mathcal{M}_{g,n}$ with a natutral probability measure $\mathbb{P}_{g,n}$ by normalising the Weil-Petersson volume form. Stemming from the works of Guth, Parlier and Young [18], and Mirzakhani [37] there has been significant interest in the geometry and spectral theory of Weil-Petersson random surfaces of large volume. By the Gauss-Bonnet theorem, sampling a large volume hyperbolic surface corresponds to taking $g + n \to \infty$.

So far, the vast majority of attention in the literature has focused on compact surfaces in the large-genus limit, $g \to \infty$, with great success in understanding their typical geometric and spectral features. On the other hand, the low genus setting has seen a reemergence in the physics literature in relation to the study of Jackiw-Teitelboim (JT) gravity (see the recent survey [33]), a simple solvable model of quantum gravity in two dimensions. Moreover, there have been recent connections [9] drawn between Weil-Petersson random surfaces in the $n \to \infty$ regime and random planar maps which have been fruitful in investigating local and global geometric properties of genus zero surfaces (see Sect. 1.4). These developments give strong impetus to study Weil-Petersson random surfaces with many cusps and is the focus of the current paper.

Prior work has shown that such surfaces exhibit very different and interesting spectral geometric behaviours [21,47,54] to their compact, large genus counterparts, but they are yet to be intensively studied. The results obtained here focus on eigenvalues of the Laplacian and structure of closed geodesics and are proven through novel asymptotics for Weil-Petersson volumes of moduli spaces of bordered hyperbolic surfaces with many cusps that will be applicable to other spectral geometric questions in this setting.

1.1. Small eigenvalues. Let X be a complete, connected and orientable finite-area hyperbolic surface. The spectrum of the Laplacian Δ_X is contained inside $[0, \infty)$ with 0 a simple eigenvalue. When X is non-compact, the spectrum in $(0, \frac{1}{4})$ consists of finitely many discrete eigenvalues called *exceptional eigenvalues* and absolutely continuous spectrum in $[\frac{1}{4}, \infty)$, with possibly infinitely many embedded eigenvalues. The first result of this paper is concerned with the number of exceptional eigenvalues, $N^{\text{exc}}(X)$, for random surfaces in $\mathcal{M}_{g,n}$ as $n \to \infty$.

A result of Zograf [53] states that for $X \in \mathcal{M}_{g,n}$, the infimum $\lambda_1(X)$ of the non-zero spectrum of the Laplacian satisfies

$$\lambda_1(X) \leqslant C\frac{g+1}{n}, \tag{1.1}$$

for a universal constant $C > 0$. On the other hand, by a result of Otal and Rosas [42], any surface in $\mathcal{M}_{g,n}$ has at most $2g + n - 3$ exceptional eigenvalues. Thus for fixed g and sufficiently large n,

$$1 \leqslant N^{\text{exc}}(X) \leqslant 2g + n - 3. \tag{1.2}$$

Our first result says that for fixed g, a random surface in $\mathcal{M}_{g,n}$ saturates the upper bound (1.2) up to a constant multiplicative factor, with probability tending to 1 as $n \to \infty$.

Theorem 1.1. *For any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that for $g \geqslant 0$ a Weil-Petersson random surface $X \in \mathcal{M}_{g,n}$ has at least $C(\varepsilon)n$ eigenvalues below ε with probability tending to 1 as $n \to \infty$.*

A weaker version of Theorem 1.1 with $C(\varepsilon)n$ replaced by any function $\nu : \mathbb{N} \to \mathbb{N}$ with $\nu(n) = o(n)$ was proven in [21, Theorem 1.10]. The proof of Theorem 1.1 and the prior result in [21] is to use a min-max theorem with orthogonal functions that localise around short closed geodesics each separating off two cusps from the surface. The strengthening of the lower bound obtained here is due to the strong Weil-Petersson volume asymptotics proven later that allow for simultaneous control over a linear (rather than the previous sub-linear) number of such curves that hold with high probability.

Remark 1.2. The constant $C(\varepsilon)$ can be given explicitly. If $\varepsilon < \frac{1}{4}$, the constant $C(\varepsilon)$ can be taken to be $\frac{1}{25\pi}\varepsilon J_1(j_0) I_1\left(\frac{j_0\varepsilon}{12\pi}\right)$. The functions J_1 and I_1 are the Bessel and modified Bessel functions of the first kind and j_0 is the first positive zero of the Bessel function $J_0(x)$ (see the Notation section below).

It is interesting to ask whether or not the eigenvalues guaranteed by Theorem 1.1 are cusp forms. We believe the answer is no although we do not have a proof. We note that when $g = 0$ or 1, any small eigenvalue is residual [22] (i.e. arises as a pole of the scattering matrix). It is conjectured by Otal and Rosas [42] that on any surface in $\mathcal{M}_{g,n}$, the number of small cuspidal eigenvalues is bounded above by $2g - 3$. It would then follow that the number of residual eigenvalues on a random surface in $\mathcal{M}_{g,n}$ is linear in n with probability tending to 1 as $n \to \infty$.

1.2. Relative frequencies of closed curves. Our next result is on the relative frequencies of simple and non-simple closed geodesics. For $X \in \mathcal{M}_{g,n}$, we define $N(X, L)$ to be the number of unoriented primitive closed geodesics on X of length less than L. Similarly we define $N^s(X, L)$ (resp. $N^{\text{ns}}(X, L)$) to be the number of such curves that are simple (resp. non-simple). By the results of [21], a random surface with many cusps has lots of geodesics with lengths at most 2arccosh(3), all of which must be simple. The prime geodesic theorem [14] says that for any X,

$$N(X, L) \sim \frac{e^L}{2L} \tag{1.3}$$

as $L \to \infty$, whilst by [36] (see also [45]),

$$N^s(X, L) \sim \eta(X) L^{6g-6+2n}, \tag{1.4}$$

where $\eta : \mathcal{M}_{g,n} \to \mathbb{R}_+$ is a continuous proper function. It then follows from (1.3) and (1.4) that for any fixed $X \in \mathcal{M}_{g,n}$ one has

$$N^s(X, L) \leqslant \varepsilon(L) N^{\text{ns}}(X, L),$$

for large enough L, where $\varepsilon(L) \to 0$ as $L \to \infty$. In other words, non-simple geodesics become more abundant at large enough length scales. It is natural to ask at what scale this transition occurs.

This problem has recently been studied for random large-genus compact surfaces. It was shown by Wu and Xue [52, Theorem 4] that for a Weil-Petersson random genus g compact surface, as $g \to \infty$, most geodesics of length much shorter than $\sqrt{g}$ are simple and non-separating whereas most geodesics of length much longer than $\sqrt{g}$ are non-simple. This confirmed a conjecture made by Lipnowski and Wright [26]. Subsequently, Dozier and Sapir [15, Theorem 1.1] gave an alternative proof of the fact that most geodesics much longer than $\sqrt{g}\log g$ are non-simple via dynamical methods which also apply to other random models.

We study the analogous problem for random surfaces with many cusps and prove the following.

Theorem 1.3. *Let $L = L(n) > 0$ be any function with $L \to \infty$ as $n \to \infty$ and $L = O(\log n)$. Then there exists a function $\varepsilon(n)$ with $\varepsilon \to 0$ as $n \to \infty$ such that*

$$\mathbb{P}_{g,n}\left(X \in \mathcal{M}_{g,n} : N^{\mathrm{s}}(X, L) \leq \varepsilon(n) N^{\mathrm{ns}}(X, L)\right) \to 1,$$

as $n \to \infty$.

In other words, for any $g \geqslant 0$ and large n, on a typical surface in $\mathcal{M}_{g,n}$, most long geodesics (which are not too long) are non-simple. This is in stark contrast to the large genus case.

Remark 1.4. We believe the conclusion of Theorem 1.3 is true for any $L \to \infty$ without the $L = O(\log n)$ condition. The condition $L = O(\log n)$ is essentially an artifact of our current error term $\frac{1}{n^{\frac{1}{4}}}\cosh\left(\frac{L}{2}\right)$ for Weil-Petersson volumes of moduli spaces in Theorem 1.5 which causes problems when L is too large. Removing the condition could be done by improving error terms in Theorem 1.6 or possibly employing a more involved method such as [15,51], which we do not pursue here.

We now highlight a related problem. Let $N^s_{\mathrm{sep}}(X, L)$ (resp. $N^s_{\mathrm{nonsep}}(X, L)$) count the number of separating (resp. non-separating) closed geodesics on X of length at most L. By the work of Mirzakhani [36], on any surface $X \in \mathcal{M}_{g,n}$,

$$\lim_{L\to\infty} \frac{N^s_{\mathrm{sep}}(X, L)}{N^s_{\mathrm{nonsep}}(X, L)} = \frac{c_{g,n,\mathrm{sep}}}{c_{g,n,\mathrm{nonsep}}},$$

where $\frac{c_{g,n,\mathrm{sep}}}{c_{g,n,\mathrm{nonsep}}} \in \mathbb{Q}$ depends only on g and n. The asymptotic behaviour of $\frac{c_{g,n,\mathrm{sep}}}{c_{g,n,\mathrm{nonsep}}}$ in g and n was recently studied by Delecroix, Goujard, Zograf and Zorich in [13] and by Ren [44]. Interestingly, the asymptotics of $\frac{c_{g,n,\mathrm{sep}}}{c_{g,n,\mathrm{nonsep}}}$ are not particularly sensitive to the ratio $\frac{n}{g}$, in contrast to many geometric quantities [21,47,54].

For a random surface in $\mathcal{M}_{g,n}$ with n large, there is an abundance of very short separating simple closed geodesics [21]. This indicates a transition in the ratio of $N^s_{\mathrm{sep}}(X, L)$ vs $N^s_{\mathrm{nonsep}}(X, L)$ for some $L = L(n)$. At what scale does this transition occur for a random surface?

1.3. Asymptotics of Weil-Petersson volumes. We consider the Weil-Petersson volume $V_{g,n}(\ell_1, \ldots, \ell_k)$ of the moduli space $\mathcal{M}_{g,n}(\ell_1, \ldots, \ell_k)$ of bordered hyperbolic surfaces of genus g with $n-k$ cusps and k labelled geodesic boundaries with lengths $\ell_1, \ldots, \ell_k \geqslant 0$.

After Mirzakhani's celebrated thesis works [34,35], the asymptotic behaviour of $V_{g,n}(\ell_1, \ldots, \ell_k)$ is crucial for computing the integrals of geometric functions with respect to the Weil-Petersson volume form (or from a probabilistic perspective, computing moments of random variables with respect to $\mathbb{P}_{\mathrm{WP}}$). A key insight in the large-genus case is the sinh expansion, appearing first in the work of Mirzakhani and Petri [38], which states

$$\frac{V_{g,n}(\ell_1, \ldots, \ell_k)}{V_{g,n}} = \prod_{i=1}^{k} \frac{\sinh\left(\frac{\ell_i}{2}\right)}{\left(\frac{\ell_i}{2}\right)} \left(1 + O_n\left(\frac{\sum \ell_i^2}{g}\right)\right), \tag{1.5}$$

as $g \to \infty$. The estimate (1.5), or variants of it (see the works [4,38,41,52]), have been paramount in understanding questions about the spectral gap, eigenvalue distribution and geometry of geodesics on random hyperbolic surfaces of large genus (Sect. 1.4).

The main technical contribution of this paper is to develop an analogue of (1.5) for the large-n case. We prove the following.

Theorem 1.5. *For any $g, k \geqslant 0$,*

$$\frac{V_{g,n}(\ell_1, \ldots, \ell_k)}{V_{g,n}} = \prod_{i=1}^{k} I_0\left(\frac{j_0}{2\pi}\ell_i\right) + O_g\left(\frac{1}{n^{\frac{1}{4}}} \prod_{i=1}^{k} \cosh\left(\frac{\ell_i}{2}\right)\right),$$

where I_0 is the modified Bessel function of the first kind and j_0 is the first positive zero of the Bessel function of the first kind J_0.

By a result of Mirzakhani [35] (c.f. Theorem 2.1), the intersection numbers $[\tau_0^{n-k}\tau_{d_1}\ldots\tau_{d_k}]_{g,n}$ of tautological classes on $\overline{\mathcal{M}_{g,n}}$ (see Sect. 2) play a central role in understanding $V_{g,n}(\ell_1, \ldots, \ell_k)$. For the large-genus case, the formula (1.5) can be deduced from an estimate, e.g. [37, Page 286], of the form

$$\frac{[\tau_{d_1}\ldots\tau_{d_n}]_{g,n}}{V_{g,n}} = 1 + O_n\left(\frac{\left(\sum_{i=1}^{n} d_i\right)^2}{g}\right). \tag{1.6}$$

To prove Theorem 1.5, we develop a novel asymptotic formula for $[\tau_0^{n-k}\tau_{d_1}\ldots\tau_{d_k}]_{g,n}$ as $n \to \infty$.

Theorem 1.6. *Suppose that $k : \mathbb{N} \to \mathbb{N}$ is such that $k(n) \leqslant \frac{1}{8}\log(n)$. Then for any $d_1, \ldots, d_k$ with $\sum_{i=1}^{k} d_i \leq 3g + n - 3$, as $n \to \infty$,*

$$\frac{[\tau_0^{n-k}\tau_{d_1}\cdots\tau_{d_k}]_{g,n}}{V_{g,n}} = \prod_{i=1}^{k}\left(\frac{j_0}{\pi}\right)^{2d_i} \frac{(2d_i+1)}{\sqrt{\pi}} \frac{\Gamma(d_i+\frac{1}{2})}{\Gamma(d_i+1)} + O_g\left(\frac{\sum_{i=1}^{k} d_i}{n^{\frac{1}{4}}}\right),$$

where j_0 is the first positive zero of the Bessel function of the first kind J_0.

To our knowledge, Theorem 1.6 provides the first result in the literature that studies the intersection numbers $[\tau_0^{n-k}\tau_{d_1}\cdots\tau_{d_k}]_{g,n}$ in the $n\to\infty$ regime. One difficulty with studying this quantity is that, as opposed to the $g\to\infty$ regime, the complexity of recursive formulae satisfied by the intersection numbers increases rapidly in n. In particular, the number of terms in such formulae that contribute to the asymptotic leading order grows rapidly. We explain more about the widespread interest of the asymptotics of intersection numbers in Sect. 1.4.

1.4. Related work. We now put our work into context with existing literature.

Large n regime The spectral geometric properties of surfaces in $\mathcal{M}_{g,n}$ in the $n\to\infty$ regime is not well studied. Early work by Zograf [54] and Manin and Zograf [31] focussed on the asymptotics of Weil-Petersson volumes of the moduli space for unbordered surfaces which we replicate in Theorem 2.4. The spectrum of the Laplacian has been studied by Zograf [53] where the deterministic bound (1.1) on the first non-zero eigenvalue was obtained, and we proved a sub-linear in n version of Theorem 1.1 in [21] for random surfaces.

Additionally, counting functions for the number of closed geodesics whose lengths are in a shrinking window were shown to be Poisson distributed in the $n\to\infty$ regime in [21] allowing for the expected systole size and distribution of short geodesics to be studied on Weil-Petersson random surfaces. The Bers' constant for deterministic punctured spheres (genus zero, n cusps) has also been studied. The length of a pants decomposition $\mathcal{P}$ of $X\in\mathcal{M}_{0,n}$ is the length of the longest geodesic in $\mathcal{P}$. The Bers' constant associated with X is the shortest pants length among all pants decompositions of X. The optimal Bers' constant $\mathcal{B}_{0,n}$ is the supremum of the Bers' constants associated to every surface in $\mathcal{M}_{0,n}$. Balacheff and Parlier [7] then proved the following.

Theorem 1.7. *If $X\in\mathcal{M}_{0,n}$ has Bers' constant equal to $\mathcal{B}_{0,n}$, that is, any pants decomposition contains a geodesic with length at least $\mathcal{B}_{0,n}$, then all simple closed geodesics of X have length strictly greater than* $2\operatorname{arcsinh}(1)$.

This contrasts the typical behaviour of Weil-Petersson random surfaces with $n\to\infty$, where there are many curves on length scales const $\cdot\frac{1}{\sqrt{n}}$. In comparison with Theorem 1.1 where the many short curves give rise to linear in n many exceptional eigenvalues, a surface as in Theorem 1.7 may be a candidate to exhibit few exceptional eigenvalues.

Finally, we mention forthcoming work of Budd and Curien [9] where a connection between Weil-Petersson random surfaces in $\mathcal{M}_{0,n}$ and the scaling limits of random planar maps has been established. Through this, the authors have investigated the local geometry of a Weil-Petersson random surface in $\mathcal{M}_{0,n}$ by proving pointed Benjamini–Schramm convergence to a random surface with genus zero and countably many punctures. In addition, they show that globally, after cutting the cusps at horocycles of fixed length and appropriately rescaling the hyperbolic distance, the compact core converges as a metric space in the Gromov–Hausdorff sense to a scaled Brownian sphere.

Remark 1.8. It is known that in many settings that Benjamini–Schramm convergence implies the convergence of spectral measures of the Laplacian. For example, in [1] it is shown that if Γ_n are a uniformly discrete sequence of lattices in a connected semi-simple Lie group G with trivial centre and maximal compact subgroup K, then Benjamini–Schramm convergence of the manifold $\Gamma_n\backslash G/K$ to G/K implies convergence of the spectral measure to the normalised Plancherel measure on $L^2(G)$. It would be interesting to investigate the spectral measure for Weil-Petersson random surfaces in the large-n regime and it is likely that precise asymptotics for Weil-Petersson volumes as in Corollary 1.5 will be helpful for this.

Large g regime The literature for studying random surfaces in the $g \to \infty$ is much more well established whereby the sinh approximation of (1.5) and its variants have played an important role in obtaining the current state of the art results.

The first application of (1.5) was in [38], where it was applied to prove Poisson statistics for the number of closed geodesics whose lengths are in a fixed window on closed surfaces. It was also used in investigating the size of the first non-zero eigenvalue (spectral gap) in [26,51] where the authors independently prove that a Weil-Petersson random closed surface of large genus has spectral gap at least $\frac{3}{16} - \varepsilon$. Recently, this result was improved to $\frac{2}{9} - \varepsilon$ by Anantharaman and Monk [5], where they applied the sharpest known version of (1.5), proven in [4]. See also the works [20,29,30] for spectral gap results for random covers. Other applications have been in proving delocalisation estimates on Laplacian eigenfunctions [16,25,48], connecting eigenvalue distribution to random matrix theory [46], and investigating the Cheeger constant and behaviour of the systole [41,43] for closed surfaces. The sinh-approximation has also been used to prove results of a similar flavour when n grows with the genus g, where the behaviour of geodesics and eigenvalues changes dependent on the n and g regime [19,47].

Theorem 1.5, the large-n analogue of (1.5), offers an explanation for the contrasting results seen for the spectral geometry in the large g and n regimes: the leading order asymptotics in n are subdominant to those in g leading to shorter scale geometry to permeate. As with (1.5), we expect that our result will be useful for investigating other spectral geometric questions for Weil-Petersson random surfaces in addition to the results that we prove here. These applications are of significant interest in there own right and because of their sharp contrast to the behaviour witnessed on large genus closed surfaces.

Asymptotics of intersection numbers Aside from appearing in Mirzakhani's formula for moduli space volumes (c.f. Theorem 2.1) the intersection numbers $\left[\tau_{d_1} \cdots \tau_{d_n}\right]_{g,n}$ (after a suitable normalisation—compare (2.1) to the definition in the introduction of [3] for example), play the role of correlation functions in Witten's model of two-dimensional quantum gravity [49]. In proving a conjecture of Witten regarding the nature of the intersection numbers, Kontesevich [24] related them to the combinatorics of trivalent ribbon graphs with genus g and n boundaries. Additionally, the intersection numbers appear in formulae for frequencies of geodesic multicurves on hyperbolic and random flat surfaces by Mirzakhani [36] and Delecroix, Goujard, Zograf and Zorich [11–13] respectively.

Computation of the intersection numbers can in principle be carried out exactly using recursive relations of Witten and Kontesevich [24,49] and their connection to Virasoro constraints or the recursive relations of Liu and Xu [27] (some of which we reproduce in Sect. 2.2). There are however, no explicit closed form expressions of the intersection numbers outside of some exceptional cases, and so it is of immense interest to obtain asymptotic formulae for them instead in various regimes of g and n. For $g \to \infty$ (and with control over the growth of n in terms of g), there has been significant progress on these asymptotics made by Mirzakhani and Zograf [39] and Liu and Xu [28]. When $\sum_{i=1}^{n} d_i = 3g + n - 3$, the exact form of the intersection numbers in the $g \to \infty$ limit were conjectured by Delecroix, Goujard, Zograf and Zorich [11, Conjecture E.6] and recently proven by Aggarwal [3] (see also a special case by Liu and Xu [28]). Our result, Theorem 1.6, marks significant progress in understanding the asymptotics of the intersection numbers $[\tau_0^{n-k}\tau_{d_1}\cdots\tau_{d_k}]_{g,n}$ in the $n\to\infty$ regime.

1.5. Overview of the proofs of Theorems 1.5 and 1.6. We first outline the proof of Theorem 1.6 from which Theorem 1.5 follows. We use the following recursive formula, Lemma 2.2, which follows from [27, Propositions 3.3 and 3.4], c.f. [39, eq. (Ib)].

$$\begin{aligned}\left[\tau_0^2\tau_{\ell+1}\prod_{i=1}^n\tau_{d_i}\right]_{g,n+3} &= \left[\tau_0^4\tau_\ell\prod_{i=1}^n\tau_{d_i}\right]_{g-1,n+5} + 8\sum_{\substack{I\sqcup J=\{1,\dots,n\}\\ g_1+g_2=g}}\left[\tau_0^2\tau_\ell\prod_{i\in I}\tau_{d_i}\right]_{g_1,|I|+3}\left[\tau_0^2\prod_{i\in J}\tau_{d_i}\right]_{g_2,|J|+2}\\ &\quad + 4\sum_{\substack{I\sqcup J=\{1,\dots,n\}\\ g_1+g_2=g}}\left[\tau_0\tau_\ell\prod_{i\in I}\tau_{d_i}\right]_{g_1,|I|+2}\left[\tau_0^3\prod_{i\in J}\tau_{d_i}\right]_{g_2,|J|+3}. \qquad (1.7)\end{aligned}$$

The benefit of studying (1.7), as opposed to e.g. Mirzakhani's recursion formula [35], is that the right hand side is solely in terms of intersection numbers with lower indices which makes it very well suited to induction. We explain the method for the simpler case where $d_i=0$ for $i=1,\dots,n$. We remind the reader that $\left[\tau_0^n\right]_{g,n}=V_{g,n}$. The starting point for our analysis is the large-n asymptotic for $V_{g,n}$ proved by Manin and Zograf, c.f. Theorem 2.4, which states that for $g\geqslant 0$ fixed,

$$V_{g,n}=\frac{\left(2\pi^2\right)^{3g+n-3}}{x_0^n}n!(n+1)^{\frac{5g-7}{2}}\left(B_g+O_g\left(\frac{1}{n}\right)\right), \qquad (1.8)$$

as $n\to\infty$, for some constant $B_g>0$ with $x_0=-\frac{1}{2}j_0J_0'(j_0)$. Using (1.8), we study (1.7) inductively. By comparing each term of (1.7) with (1.8), we are able to prove (e.g. Lemma 4.2) that

$$\frac{[\tau_{\ell+1}\tau_0^{n+2}]_{g,n+3}}{V_{g,n+3}}=C_{\ell+1}+O_g\left(\frac{\ell}{n^{\frac{1}{4}}}\right),$$

where

$$C_{\ell+1}\overset{\text{def}}{=}8C_\ell\sum_{i=0}^\infty\frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1}+4\sum_{i=0}^\infty\frac{[\tau_0^{i+2}\tau_\ell]_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1}. \qquad (1.9)$$

It is an important point that the constants C_ℓ involve only genus-0 intersection numbers, regardless of the value of g. Our goal is now to compute the constants C_ℓ explicitly. To do this we need to compute the constant

$$b_\ell\overset{\text{def}}{=}\sum_{i=0}^\infty\frac{[\tau_0^{i+2}\tau_\ell]_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1},$$

for each ℓ which then allows us to solve the recurrence relation (1.9) to compute C_ℓ. We consider the generating functions

$$\Phi_\ell(x) \stackrel{\text{def}}{=} \sum_{i=3}^{\infty} \frac{[\tau_0^{i-1}\tau_\ell]_{0,i}}{(2\pi^2)^{i-2} i!} x^i,$$

noting that $b_\ell = \Phi_\ell''(x_0)$. Crucially, for the case that $\ell = 0$, it was shown in [23, eq. (0.8)] that $y(x) \stackrel{\text{def}}{=} \Phi_0''(x)$ can be obtained by inverting the function $x(y) \stackrel{\text{def}}{=} -\sqrt{y} J_0'\left(2\sqrt{y}\right)$ which gives that $b_0 = \frac{j_0^2}{4}$. This follows from a recursive formula for genus-0 Weil-Petersson volumes due to Zograf [54]. We observe, in Lemma 4.3, that (1.7) implies that $\Phi_\ell(x)$ satisfies the following family of ODE's

$$\Phi_\ell'''(x) = 8\Phi_0''(x)\Phi_{\ell-1}'''(x) + 4\Phi_0'''(x)\Phi_{\ell-1}''(x). \tag{1.10}$$

We solve (1.10) for $\Phi_\ell''(x)$ to find that

$$\Phi_\ell''(x) = \frac{2^{3\ell+1}}{\sqrt{\pi}} \frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(\ell+2)} \Phi_0''(x)^{\ell+1}.$$

Using $b_0 = \frac{j_0^2}{4}$, we learn that

$$b_\ell = \frac{2^{3\ell+1}}{\sqrt{\pi}} \frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(\ell+2)} \left(\frac{j_0^2}{4}\right)^{\ell+1}.$$

Solving (1.9) then gives

$$C_\ell = \left(\frac{j_0}{\pi}\right)^{2\ell} \frac{(2\ell+1)}{\sqrt{\pi}} \frac{\Gamma(\ell+\frac{1}{2})}{\Gamma(\ell+1)}.$$

After some more technical computations, the general case of $[\tau_0^2\tau_{\ell+1}\prod_{i=1}^{n-3}\tau_{d_i}]_{g,n}$ for $d_1, \ldots, d_k > 0$ essentially reduces to the above considerations.

Finally, since by [35, Theorem 1.1],

$$V_{g,n}(2x_1, \ldots, 2x_n) = \sum_{\substack{d_1,\ldots,d_n \\ |d| \leqslant 3g+n-3}} \left[\prod_{i=1}^{n} \tau_{d_i}\right]_{g,n} \frac{x_1^{2d_1}}{(2d_1+1)!} \cdot \cdots \cdot \frac{x_n^{2d_n}}{(2d_n+1)!},$$

Theorem 1.5 follows from Theorem 1.6 by recognising the Taylor expansion for $\prod_{i=1}^{k} I_0(x_i)$, where we recall

$$I_0(x) = \sum_{d=0}^{\infty} \left(\frac{x}{2}\right)^{2d} \frac{1}{(d!)^2}.$$

1.6. Outline of the paper. In Sect. 2 we will gather together the definitions of the objects we are interested in studying along with some results from the literature that we shall use throughout the article. Section 3 proves some preliminary results involving the moduli space volumes that we will frequently apply when computing the asymptotics of the intersection numbers. Mostly these results are technical and their proofs can be skipped over without loss for reading the remainder of the paper. In Sect. 4 we prove Theorems 1.6 and 1.5 starting with some special cases that lead into an inductive argument later in the section. The remaining two sections are devoted to proving the respective applications outlined previously.

1.7. Notation. For real valued functions f, h depending on a parameter n we write $f \ll h$ or $f = O(h)$ if there exists $C, N > 0$ such that $|f(n)| \leqslant Ch(n)$ for all $g > N$. We add subscripts to the $\ll$ sign if the constants C and N depend on another variable. E.g. we write $f \ll_\epsilon h$ if there exists $C = C(\varepsilon)$, $N = N(\varepsilon)$ such that $|f(n)| \leqslant Ch(n)$ for all $g > N$. We write $f \sim h$ if $f \ll h$ and $h \ll f$.

We will frequently use the shorthand $\delta_{n\in 2\mathbb{Z}}$ to indicate

$$\delta_{n\in 2\mathbb{Z}} = \begin{cases} 1 & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$$

Finally, we recall that for $n \in \mathbb{Z}_{\geq 0}$,

$$J_n(x) = \left(\frac{1}{2}x\right)^n \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j} j!(n+j)!} x^{2j},$$

$$I_n(x) = i^{-n} J_n(ix) = \left(\frac{1}{2}x\right)^n \sum_{j=0}^{\infty} \frac{x^{2j}}{2^{2j} j!(n+j)!}.$$

2. Background

In this section we briefly introduce the necessary background and state some results we will use later in the paper.

2.1. Moduli space. Let $\Sigma_{g,n,b}$ denote a topological surface with genus g, n labelled punctures and b labelled boundary components where $2g+n+b \geqslant 3$. A marked surface of signature (g, n, b) is a pair (X, φ) where X is a hyperbolic surface and $\varphi : \Sigma_{g,n,b} \to X$ is a homeomorphism. Given $(\ell_1, \dots, \ell_b) \in \mathbb{R}^b_{>0}$, we define the Teichmüller space $\mathcal{T}_{g,n,b}(\ell_1, \dots, \ell_b)$ by

$$\mathcal{T}_{g,n,b}(\ell_1, \dots, \ell_b) \overset{\text{def}}{=} \left\{ \begin{array}{c} \text{Marked surfaces}(X,\varphi) \text{ of signature } (g,n,b) \\ \text{with labelled totally geodesic boundary components} \\ (\beta_1,\dots,\beta_b) \text{ with lengths } (\ell_1,\dots,\ell_b) \end{array} \right\} / \sim,$$

where $(X_1, \varphi_1) \sim (X_2, \varphi_2)$ if and only if there exists an isometry $m : X_1 \to X_2$ such that φ_2 and $m \circ \varphi_1$ are isotopic. Let $\text{Homeo}^+\left(\Sigma_{g,n,b}\right)$ denote the group of orientation preserving homeomorphisms of $\Sigma_{g,c,d}$ which leave every boundary component set-wise fixed and do not permute the punctures. Let $\mathrm{Homeo}_0^+\left(\Sigma_{g,n,b}\right)$ denote the subgroup of homeomorphisms isotopic to the identity. The mapping class group is defined as

$$\mathrm{MCG}_{g,n,b} \overset{\text{def}}{=} \mathrm{Homeo}^+\left(\Sigma_{g,n,b}\right)/\mathrm{Homeo}_0^+\left(\Sigma_{g,n,b}\right).$$

$\mathrm{Homeo}^+\left(\Sigma_{g,n,b}\right)$ acts on $\mathcal{T}_{g,n,b}(\ell_1,\ldots,\ell_b)$ by pre-composition of the marking and we define the moduli space $\mathcal{M}_{g,n,b}(\ell_1,\ldots,\ell_b)$ by

$$\mathcal{M}_{g,n,b}(\ell_1,\ldots,\ell_b) \overset{\text{def}}{=} \mathcal{T}_{g,n,b}(\ell_1,\ldots,\ell_b)/\mathrm{MCG}_{g,n,b}.$$

By convention, a geodesic of length 0 is a cusp and we suppress the distinction between cusps and boundary components in our notation by allowing $\ell_i \geqslant 0$. In particular we write

$$\mathcal{M}_{g,n+b} = \mathcal{M}_{g,n,b}(0,\ldots,0).$$

2.2. Intersection numbers and Weil-Petersson volumes. Let ω_{WP} denote the Weil-Petersson symplectic form on $\mathcal{T}_{g,n}(\ell_1,\ldots,\ell_n)$. ω_{WP} is invariant under the action of the mapping class group [17] and descends to a symplectic form on $\mathcal{M}_{g,n}(\ell_1,\ldots,\ell_n)$ where it induces the volume form

$$\mathrm{dVol}_{WP} \overset{\text{def}}{=} \frac{1}{(3g-3+n)!}\bigwedge_{i=1}^{3g-3+n}\omega_{WP}.$$

We write $V_{g,n}(\ell_1,\ldots,\ell_n)$ to denote $\mathrm{Vol}_{WP}\left(\mathcal{M}_{g,n}(\ell_1,\ldots,\ell_n)\right)$, which is finite. We define the Weil-Petersson probability measure $\mathbb{P}_{g,n}$ on $\mathcal{M}_{g,n}$ by normalising dVol_{WP}.

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$. There are n tautological line bundles $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g,n}$ whose fiber at $X \in \overline{\mathcal{M}}_{g,n}$ is the cotangent space at the ith marked point on X. We define the ψ-classes $\psi_i \overset{\text{def}}{=} c_1(\mathcal{L}_i)$ where c_1 denotes the first Chern class of the bundle $\mathcal{L}_i$. For $d = (d_1,\ldots,d_n) \in \mathbb{Z}^n_{\geqslant 0}$ with $|d| \overset{\text{def}}{=} \sum_{i=1}^n d_i \leqslant 3g+n-3$, we define

$$\left[\tau_{d_1}\ldots\tau_{d_n}\right]_{g,n} \overset{\text{def}}{=} \frac{2^{2|d|}\prod_{i=1}^n(2d_i+1)!!}{(3g+n-3-|d|)!}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\omega_{WP}^{3g+n-3-|d|}. \tag{2.1}$$

If $|d| > 3g+n-3$, $\left[\tau_{d_1}\ldots\tau_{d_n}\right]_{g,n}$ is taken to be identically 0. For further background, see [6]. Mirzakhani proved that $V_{g,n}(\ell_1,\ldots,\ell_n)$ is a polynomial in $\ell_1,\ldots,\ell_n$ with coefficients given by the intersection numbers $\left[\tau_{d_1}\ldots\tau_{d_n}\right]_{g,n}$.

Theorem 2.1 ([35, Theorem 1.1]). *For $n > 0$ and $\ell_1,\ldots,\ell_n > 0$,*

$$V_{g,n}(2\ell_1,\ldots,2\ell_n) = \sum_{\substack{d_1,\ldots,d_n \\ |d|\leqslant 3g+n-3}}\left[\prod_{i=1}^n\tau_{d_i}\right]_{g,n}\frac{\ell_1^{2d_1}}{(2d_1+1)!}\cdot\cdots\cdot\frac{\ell_n^{2d_n}}{(2d_n+1)!}.$$

By Theorem 2.1, we can study the asymptotics of $\frac{V_{g,n}(\ell_1,\ldots,\ell_n)}{V_{g,n}}$ by understanding the asymptotics of $\frac{\left[\prod_{i=1}^n\tau_{d_i}\right]_{g,n}}{V_{g,n}}$. To this end, we rely heavily on the following recursive formula which follows from [27, Propositions 3.3 and 3.4], c.f. [39, eq. (Ib)].

Lemma 2.2 ([27]).

$$\begin{aligned}&\left[\tau_0^2\tau_{\ell+1}\prod_{i=1}^n\tau_{d_i}\right]_{g,n+3}\\&\quad=\left[\tau_0^4\tau_\ell\prod_{i=1}^n\tau_{d_i}\right]_{g-1,n+5}+8\sum_{\substack{I\sqcup J=\{1,\dots,n\}\\g_1+g_2=g}}\left[\tau_0^2\tau_\ell\prod_{i\in I}\tau_{d_i}\right]_{g_1,|I|+3}\left[\tau_0^2\prod_{i\in J}\tau_{d_i}\right]_{g_2,|J|+2}\\&\qquad+4\sum_{\substack{I\sqcup J=\{1,\dots,n\}\\g_1+g_2=g}}\left[\tau_0\tau_\ell\prod_{i\in I}\tau_{d_i}\right]_{g_1,|I|+2}\left[\tau_0^3\prod_{i\in J}\tau_{d_i}\right]_{g_2,|J|+3}.\end{aligned}$$

We also record a basic estimate for $\left[\tau_{d_1},\dots,\tau_{d_n}\right]_{g,n}$ in terms of Weil-Petersson volumes.

Lemma 2.3 ([37, Lemma 3.2, Part (1)]). *For any $d_1,\dots d_n$,*

$$\left[\tau_{d_1},\dots,\tau_{d_n}\right]_{g,n}\leqslant V_{g,n}.$$

We shall make essential use of the following large-n asymptotic formula for $V_{g,n}$ due to Manin and Zograf [31].

Theorem 2.4 ([31, Theorem 6.1]). *For any $g\geqslant 0$, there exists a constant $B_g>0$ such that*

$$V_{g,n}=\frac{\left(2\pi^2\right)^{3g+n-3}}{x_0^n}n!(n+1)^{\frac{5g-7}{2}}\left(B_g+O_g\left(\frac{1}{n}\right)\right),$$

where $x_0\overset{\text{def}}{=}-\frac{1}{2}j_0J_0'(j_0)$ and J_0 is the Bessel function of the first kind with j_0 its first positive zero.

The proof of Theorem 1.6 relies on analysing the recursion of Lemma 2.2 together with Theorem 2.4.

Finally we will often apply the following trivial upper bound for $V_{g,n}(\ell_1,\dots,\ell_n)$.

Lemma 2.5. *For any $g,n\geqslant 0$ with $2g+n>2$ and $\ell_1,\dots,\ell_n\geqslant 0$*

$$\frac{V_{g,n}(\ell_1,\dots,\ell_n)}{V_{g,n}}\leqslant\prod_{i=1}^n\frac{\sinh\left(\frac{\ell_i}{2}\right)}{\left(\frac{\ell_i}{2}\right)}.$$

This follows from Theorem 2.1 and Lemma 2.3 by the Taylor expansion of $\prod_{i=1}^n\frac{\sinh\left(\frac{\ell_i}{2}\right)}{\left(\frac{\ell_i}{2}\right)}$, e.g. [38, Proposition 3.1].

2.3. Mirzakhani's integration formula. Finally, we give a brief account of Mirzakhani's integration formula which will used in our applications.

We define a k-multicurve to be an ordered k-tuple $\Gamma = (\gamma_1, \dots, \gamma_k)$ of disjoint non-homotopic non-peripheral simple closed curves on $\Sigma_{g,n}$, and we write $[\Gamma] = [\gamma_1, \dots, \gamma_k]$ to denote its homotopy class. The mapping class group $\mathrm{MCG}_{g,n}$ acts on homotopy classes of multicurves and we denote the orbit containing $[\Gamma]$ by

$$\mathcal{O}_\Gamma = \left\{(\sigma \cdot \gamma_1, \dots, \sigma \cdot \gamma_k) \mid \sigma \in \mathrm{MCG}_{g,n}\right\}.$$

Given a simple, non-peripheral closed curve γ on $\Sigma_{g,n}$, for $(X, \varphi) \in \mathcal{T}_{g,n}$ we define $\ell_\gamma(X)$ to be the length of the unique geodesic in the free homotopy class of $\varphi(\gamma)$. Then given a function $f : \mathbb{R}^k_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, for $X \in \mathcal{M}_{g,n}$ we define

$$f^\Gamma(X) \stackrel{\text{def}}{=} \sum_{(\alpha_1,\dots,\alpha_k)\in\mathcal{O}_\Gamma} f\left(\ell_{\alpha_1}(X), \dots, \ell_{\alpha_k}(X)\right).$$

Let $\Sigma_{g,n}(\Gamma)$ denote the result of cutting the surface $\Sigma_{g,n}$ along $(\gamma_1, \dots, \gamma_k)$, then $\Sigma_{g,n}(\Gamma) = \sqcup_{i=1}^s \Sigma_{g_i,c_i,d_i}$ for some $\{(c_i, d_i)\}_{i=1}^s$. Each γ_i gives rise to two boundary components γ_i^1 and γ_i^2 of $\Sigma_{g,n}(\Gamma)$. Given $\boldsymbol{x} = (x_1, \dots, x_k)$, let $\boldsymbol{x}^{(i)}$ denote the tuple of coordinates x_j of $\boldsymbol{x}$ such that γ_j is a boundary component of Σ_{g,c_i,d_i}. We define

$$V_{g,n}(\Gamma, \mathbf{x}) \stackrel{\text{def}}{=} \prod_{i=1}^{s} V_{g_i,c_i+d_i}\left(\boldsymbol{x}^{(i)}\right).$$

We can now state Mirzakhani's integration formula.

Theorem 2.6 (Mirzakhani's Integration Formula [34, Theorem 7.1]). *Given a k-multicurve $\Gamma = (\gamma_1, \dots, \gamma_k)$,*

$$\int_{\mathcal{M}_{g,n}} f^\Gamma(X)\, dX = C(\Gamma) \int_{\mathbb{R}^k_{\geqslant 0}} f(x_1, \dots, x_k)\, V_{g,n}(\Gamma, \mathbf{x})\, x_1 \cdots x_k \mathrm{d}x_1 \cdots \mathrm{d}x_k,$$

where $C(\Gamma) \leqslant 1$ is an explicit constant.

One can see [50, Footnote (2)] for a detailed description of the constant $C(\Gamma)$. We note that if each component of the multicurve Γ is separating then $C(\Gamma) = 1$.

3. Preliminary Volume Computations

In this section, we isolate some technical computations that we will frequently apply throughout the remainder of the paper. First, we begin with noting a computation due to Manin and Zograf as well as estimating the order of the tail of a related series.

Lemma 3.1. *The series*

$$\sum_{i=0}^{\infty} \frac{V_{0,i+3} x_0^{i+1}}{(2\pi^2)^i (i+1)!} = \frac{j_0^2}{4}, \tag{3.1}$$

where j_0 is the first positive zero of the Bessel function J_0. Moreover, for any fixed $g \geq 0$, as $n \to \infty$

$$\sum_{i=\lfloor\sqrt{n}\rfloor}^{\infty} \frac{V_{g,i+3} x_0^{i+1}}{(i+1)!(2\pi^2)^i (i+4)^{\frac{5g}{2}}} \ll_g \frac{1}{n^{\frac{1}{4}}},$$

and

$$\sum_{i=0}^{\infty} \frac{V_{g,i+3} x_0^{i+1}}{(i+1)!(2\pi^2)^i (i+4)^{\frac{5g}{2}}} \ll_g 1.$$

Proof. Defining

$$\Phi_0(x) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} \frac{V_{0,i+3} x^{i+3}}{(2\pi^2)^i (i+3)!},$$

as discussed in Sect. 1.5, it was shown in [23, eq. (0.8)] that $y(x) \stackrel{\text{def}}{=} \Phi_0''(x)$ can be obtained by inverting the Bessel function $x(y) \stackrel{\text{def}}{=} -\sqrt{y} J_0'(2\sqrt{y})$. The first part then follows from evaluating $y(x_0) = \frac{j_0^2}{4}$. For the second part, by Theorem 2.4, as $i \to \infty$,

$$\frac{V_{g,i+3} x_0^{i+1}}{(2\pi^2)^i (i+1)!(i+4)^{\frac{5g}{2}}} = (2\pi^2)^{3g} x_0^{-2} \frac{(i+3)(i+2)}{(i+4)^{\frac{7}{2}}} \left(B_g + O_g\left(\frac{1}{i}\right)\right) \ll_g \frac{1}{(i+4)^{\frac{3}{2}}},$$

and so, as $n \to \infty$ and $i \geq \lfloor \sqrt{n} \rfloor$, we have

$$\sum_{i=\lfloor \sqrt{n} \rfloor}^{\infty} \frac{V_{g,i+3} x_0^{i+1}}{(2\pi^2)^i (i+1)!(i+4)^{\frac{5g}{2}}} \ll_g \sum_{i=\lfloor \sqrt{n} \rfloor}^{\infty} \frac{1}{(i+4)^{\frac{3}{2}}} \ll_g \int_{\sqrt{n}-1}^{\infty} \frac{1}{(x+4)^{\frac{3}{2}}} \mathrm{d}x \ll_g \frac{1}{n^{\frac{1}{4}}}.$$

The final part follows from the limit comparison test by comparing the summand $\frac{V_{g,i+3} x_0^{i+1}}{(i+1)!(2\pi^2)^i (i+4)^{\frac{5g}{2}}}$ with the sequence $\frac{1}{(i+4)^{\frac{3}{2}}}$.

Lemma 3.2. *For any* $0 \leqslant g_1 \leqslant g$ *and* $\varepsilon > 0$, *as* $n \to \infty$,

$$\sum_{i=0}^{\lfloor \frac{n-4}{2} \rfloor} \binom{n-3}{i} \frac{V_{g_1,i+3} V_{g-g_1,n-i-1}}{V_{g,n}} \ll_g \frac{1}{n^{\frac{1}{2}-\varepsilon}}.$$

Proof. We split the summation and consider first when $0 \leqslant i \leqslant \lfloor \sqrt{n} \rfloor$. By Theorem 2.4,

$$\begin{aligned}
&\binom{n-3}{i} \frac{V_{g-g_1,n-i-1}}{V_{g,n}} \\
&= \frac{(n-3)!}{i!(n-3-i)!} \frac{(n-i-1)!}{n!} \left(\frac{n-i}{n+1}\right)^{\frac{5(g-g_1)-7}{2}} \\
&\quad x_0^{i+1} \frac{1}{(2\pi^2)^{3g_1+i+1}(n+1)^{\frac{5g_1}{2}}} \left(1 + O_g\left(\frac{1}{n}\right)\right) \\
&= \frac{i+1}{n} \frac{n-1-i}{n-1} \frac{n-2-i}{n-2} \frac{x_0^{i+1}}{(i+1)!} \frac{1}{(2\pi^2)^{3g_1+i+1}(n+1)^{\frac{5g_1}{2}}} \left(1 + O_g\left(\frac{1}{\sqrt{n}}\right)\right) \\
&\ll_g \frac{1}{\sqrt{n}} \frac{x_0^{i+1}}{(i+1)!} \frac{1}{(2\pi^2)^{3g_1+i+1}} \frac{1}{(n+1)^{\frac{5g_1}{4}}} \frac{1}{(i+4)^{\frac{5g_1}{2}}}.
\end{aligned}$$

And so, using Lemma 3.1

$$\sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\binom{n-3}{i}\frac{V_{g_1,i+3}V_{g-g_1,n-i-1}}{V_{g,n}} \ll_g \frac{1}{n^{\frac{1}{2}+\frac{5g_1}{4}}}\sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\frac{V_{g_1,i+3}x_0^{i+1}}{(2\pi^2)^{3g_1+i+1}(i+1)!(i+4)^{\frac{5g_1}{2}}}$$
$$\ll_g \frac{1}{n^{\frac{1}{2}+\frac{5g_1}{4}}}.$$

For $\lfloor\sqrt{n}\rfloor \leqslant i \leqslant \lfloor\frac{n-4}{2}\rfloor$, we again use Theorem 2.4 to obtain

$$\binom{n-3}{i}\frac{V_{g-g_1,n-i-1}}{V_{g,n}}$$
$$= \frac{(n-3)!}{i!(n-3-i)!}\frac{(n-i-1)!}{n!}\left(\frac{n-i}{n+1}\right)^{\frac{5(g-g_1)-7}{2}}$$
$$x_0^{i+1}\frac{1}{(2\pi^2)^{3g_1+i+1}(n+1)^{\frac{5g_1}{2}}}\left(1+O_g\left(\frac{1}{n}\right)\right)$$
$$\leqslant 2^{\frac{7}{2}}\frac{1}{n}\frac{1}{i!}\frac{x_0^{i+1}}{(2\pi^2)^{3g_1+i+1}(n+1)^{\frac{5g_1}{2}}}\left(1+O_g\left(\frac{1}{n}\right)\right),$$

where we use

$$\left(\frac{n-i}{n+1}\right)^{\frac{5(g-g_1)-7}{2}} \leqslant 2^{\frac{7}{2}},$$

since $i+1 \leqslant \frac{n}{2}-1$ and $n-i \geqslant \frac{n}{2}+2$. Moreover, for any $\varepsilon > 0$,

$$\frac{1}{n} \ll \frac{1}{n^{\frac{1}{2}-\varepsilon}}\frac{1}{(i+4)^{\frac{1}{2}+\varepsilon}}$$

and so

$$\sum_{i=\lfloor\sqrt{n}\rfloor+1}^{\lfloor\frac{n-4}{2}\rfloor}\binom{n-3}{i}\frac{V_{g_1,i+3}V_{g-g_1,n-i-1}}{V_{g,n}}$$
$$\ll_g \frac{1}{n^{\frac{1}{2}-\varepsilon}}\sum_{i=\lfloor\sqrt{n}\rfloor+1}^{\lfloor\frac{n-4}{2}\rfloor}\frac{V_{g_1,i+3}x_0^{i+1}}{(2\pi^2)^{3g_1+i+1}(i+4)^{\frac{1}{2}+\varepsilon}i!(n+1)^{\frac{5g_1}{2}}}$$
$$\ll_g \frac{1}{n^{\frac{1}{2}(1-\varepsilon)}}.$$

The last bound follows from the fact that for $i \geqslant \sqrt{n}$, Theorem 2.4 provides

$$\frac{V_{g_1,i+3}x_0^{i+1}}{(2\pi^2)^{3g_1+i+1}(i+4)^{\frac{1}{2}+\varepsilon}i!(n+1)^{\frac{5g_1}{2}}} \ll_g \frac{1}{(i+4)^{1+\varepsilon}}.$$

Lemma 3.3. *For any $g \geqslant 0$ and $k \leqslant \frac{1}{8}\log(n)$, the following estimates hold as $n \to \infty$,*

$$\sum_{i=0}^{\lfloor \frac{n-4}{2} \rfloor} \binom{n-3-k}{i+1} \frac{V_{g_1,i+3} V_{g-g_1,n-i-1}}{V_{g,n}} = \begin{cases} \frac{j_0^2}{8\pi^2} + O_g\left(\frac{1}{n^{\frac{1}{4}}}\right) & \text{if } g_1 = 0, \\ O_g\left(\frac{1}{n^{\frac{1}{4}}}\right) & \text{if } 1 \leq g_1 \leq g, \end{cases}$$

$$\sum_{i=0}^{\lfloor \sqrt{n} \rfloor} \binom{n-3-k}{i+1} \frac{V_{g_1,i+3} V_{g-g_1,n-i-1}}{V_{g,n}} = \begin{cases} \frac{j_0^2}{8\pi^2} + O_g\left(\frac{1}{n^{\frac{1}{4}}}\right) & \text{if } g_1 = 0, \\ O_g\left(\frac{1}{n^{\frac{5}{4}}}\right) & \text{if } 1 \leq g_1 \leq g, \end{cases}$$

$$\sum_{i=\lfloor \sqrt{n} \rfloor + 1}^{\lfloor \frac{n-4}{2} \rfloor} \binom{n-3-k}{i+1} \frac{V_{g_1,i+3} V_{g-g_1,n-i-1}}{V_{g,n}} \ll_g \frac{1}{n^{\frac{1}{4}}}.$$

Proof. We start with $g_1 = 0$. By Lemma 3.1, we have

$$\sum_{i=0}^{\infty} \frac{V_{0,i+3}}{(i+1)!} \left(\frac{x_0}{2\pi^2}\right)^{i+1} = \frac{j_0^2}{8\pi^2},$$

and so we write

$$\begin{aligned} &\left| \sum_{i=0}^{\lfloor \frac{n-4}{2} \rfloor} \binom{n-3-k}{i+1} \frac{V_{0,i+3} V_{g,n-i-1}}{V_{g,n}} - \sum_{i=0}^{\infty} \frac{V_{0,i+3}}{(i+1)!} \left(\frac{x_0}{2\pi^2}\right)^{i+1} \right| \\ &\leqslant \underbrace{\left| \sum_{i=0}^{\lfloor \sqrt{n} \rfloor} \binom{n-3-k}{i+1} \frac{V_{0,i+3} V_{g,n-i-1}}{V_{g,n}} - \sum_{i=0}^{\lfloor \sqrt{n} \rfloor} \frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!} x_0^{i+1} \right|}_{(1)} \\ &+ \underbrace{\left| \sum_{i=\lfloor \sqrt{n} \rfloor + 1}^{\lfloor \frac{n-4}{2} \rfloor} \binom{n-3-k}{i+1} \frac{V_{0,i+3} V_{g,n-i-1}}{V_{g,n}} \right|}_{(2)} + \underbrace{\left| \sum_{i=\lfloor \sqrt{n} \rfloor + 1}^{\infty} \frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!} x_0^{i+1} \right|}_{(3)}. \end{aligned}$$

For (1), we use Theorem 2.4 to see that because $i \leq \sqrt{n}$ and $k \leq \frac{1}{8}\log(n)$,

$$\begin{aligned} &\binom{n-3-k}{i+1} \frac{V_{g,n-i-1}}{V_{g,n}} \\ &= \frac{(n-3-k)!}{(i+1)!(n-k-i-4)!} \frac{(n-i-1)!}{n!} \left(\frac{n-i}{n+1}\right)^{\frac{5g-7}{2}} \left(\frac{x_0}{2\pi^2}\right)^{i+1} \left(1 + O_g\left(\frac{1}{n}\right)\right) \\ &= \left(\prod_{p=0}^{k+2} \left(1 - \frac{i+1}{n-p}\right)\right) \frac{1}{(i+1)!} \left(\frac{x_0}{2\pi^2}\right)^{i+1} \left(1 + O_g\left(\frac{1}{\sqrt{n}}\right)\right) \\ &= \frac{1}{(i+1)!} \left(\frac{x_0}{2\pi^2}\right)^{i+1} \left(1 + O_g\left(\frac{1}{\sqrt{n}}\right)\right), \end{aligned}$$

and for (2), when $\sqrt{n} < i \le \left\lfloor \frac{n-4}{2} \right\rfloor$,

$$\binom{n-3-k}{i+1}\frac{V_{g,n-i-1}}{V_{g,n}} \leqslant 2^{\frac{7}{2}}\frac{1}{(i+1)!}\left(\frac{x_0}{2\pi^2}\right)^{i+1}\left(1+O_g\left(\frac{1}{n}\right)\right).$$

It follows by Lemma 3.1 that

$$\begin{aligned}
&\left|\sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\binom{n-3-k}{i+1}\frac{V_{0,i+3}V_{g,n-i-1}}{V_{g,n}} - \sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1}\right| \\
&\quad \ll_g \frac{1}{\sqrt{n}}\sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1} \\
&\quad \ll_g \frac{1}{\sqrt{n}}, \\
&\left|\sum_{i=\lfloor\sqrt{n}\rfloor+1}^{\lfloor\frac{n-4}{2}\rfloor}\binom{n-3-k}{i+1}\frac{V_{0,i+3}V_{g,n-i-1}}{V_{g,n}}\right| \ll_g \frac{1}{\sqrt{n}}, \\
&\left|\sum_{i=\lfloor\sqrt{n}\rfloor+1}^{\infty}\frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1}\right| \ll_g \frac{1}{\sqrt{n}}.
\end{aligned}$$

For $1 \le g_1 \le g$, we have

$$\begin{aligned}
&\binom{n-3-k}{i+1}\frac{V_{g-g_1,n-i-1}}{V_{g,n}} \\
&= \frac{(n-3-k)!}{(i+1)!(n-k-i-4)!}\frac{(n-i-1)!}{n!}\left(\frac{n-i}{n+1}\right)^{\frac{5(g-g_1)-7}{2}}\frac{1}{(n+1)^{\frac{5g_1}{2}}} \\
&\left(\frac{x_0}{2\pi^2}\right)^{i+1}\left(\frac{B_{g-g_1}}{B_g}+O_g\left(\frac{1}{n}\right)\right) \\
&\ll_g \frac{1}{(i+1)!(n+1)^{\frac{5g_1}{2}}}\left(\frac{x_0}{2\pi^2}\right)^{i+1}.
\end{aligned}$$

Thus, by Lemma 3.1 and the fact that $i \le \sqrt{n}$

$$\begin{aligned}
\sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\binom{n-3-k}{i+1}\frac{V_{g_1,i+3}V_{g-g_1,n-i-1}}{V_{g,n}} &\ll_g \sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\frac{(i+4)^{\frac{5g_1}{2}}}{(n+1)^{\frac{5g_1}{2}}}\frac{V_{g_1,i+3}}{(i+1)!(i+4)^{\frac{5g_1}{2}}}\left(\frac{x_0}{2\pi^2}\right)^{i+1} \\
&\ll_g \frac{1}{(n+1)^{\frac{5g_1}{4}}}\sum_{i=0}^{\infty}\frac{V_{g_1,i+3}}{(i+1)!(i+4)^{\frac{5g_1}{2}}}\left(\frac{x_0}{2\pi^2}\right)^{i+1} \ll_g \frac{1}{n^{\frac{5}{4}}},
\end{aligned}$$

and

$$\sum_{i=\lfloor\sqrt{n}\rfloor+1}^{\lfloor\frac{n-4}{2}\rfloor}\binom{n-3-k}{i+1}\frac{V_{g_1,i+3}V_{g-g_1,n-i-1}}{V_{g,n}} \ll_g \frac{1}{(n+1)^{\frac{5g_1}{2}}}\sum_{i=\lfloor\sqrt{n}\rfloor+1}^{\lfloor\frac{n-4}{2}\rfloor}\frac{V_{g_1,i+3}}{(i+1)!}\left(\frac{x_0}{2\pi^2}\right)^{i+1}$$

$$\ll_g \sum_{i=\lfloor\sqrt{n}\rfloor+1}^{\infty} \frac{V_{g_1,i+3}}{(i+1)!(i+4)^{\frac{5g_1}{2}}}\left(\frac{x_0}{2\pi^2}\right)^{i+1} \ll_g \frac{1}{n^{\frac{1}{4}}}.$$

The latter claims follow from using the respective estimates already calculated.

Remark 3.4. We note from the proof of Lemma 3.3 that we implicitly also prove the estimate

$$\sum_{i=0}^{n} \frac{1}{(n+1)^{\frac{5g}{2}}} \frac{V_{g,i+3}}{(i+1)!}\left(\frac{x_0}{2\pi^2}\right)^{i+1} \ll_g \frac{1}{n^{\frac{1}{4}}},$$

whenever $g \geq 1$.

4. Asymptotics of Intersection Numbers

In this section we prove Theorems 1.6 and 1.5.

4.1. *A special case.*

We begin with Theorem 1.6, starting with with the special case of $g = 0$, $k = 1$ and $d_1 = 1$.

Proposition 4.1. *As $n \to \infty$,*

$$\frac{[\tau_0^{n-1}\tau_1]_{0,n}}{V_{0,n}} = 12\sum_{i=0}^{\infty} \frac{V_{0,i+3}x_0^{i+1}}{(2\pi^2)^{i+1}(i+1)!} + O\left(\frac{1}{n^{\frac{1}{4}}}\right) = \frac{3j_0^2}{2\pi^2} + O\left(\frac{1}{n^{\frac{1}{4}}}\right).$$

Proof. By Lemma 2.2 we have

$$\begin{aligned}\left[\tau_0^{n-1}\tau_1\right]_{0,n} &= 12 \sum_{I\sqcup J=\{1,\dots,n-3\}} [\tau_0^{|I|+3}]_{0,|I|+3}[\tau_0^{|J|+2}]_{0,|J|+2} \\ &= 12\sum_{i=0}^{n-4}\binom{n-3}{i} V_{0,i+3}V_{0,n-i-1} \\ &= 12\sum_{i=0}^{\lfloor\frac{n-4}{2}\rfloor}\binom{n-3}{i} V_{0,i+3}V_{0,n-i-1} + 12\sum_{i=0}^{\lfloor\frac{n-4}{2}\rfloor-\delta_{n\in 2\mathbb{Z}}}\binom{n-3}{i+1} V_{0,i+3}V_{0,n-i-1},\end{aligned}$$

and the result follows from Lemmas 3.2 and 3.3 with $g = 0$. □

We next continue with the case of $g = 0$, $k = 1$ and induct.

Lemma 4.2. *For $\ell \geq 0$, there exist constants $C_\ell > 0$ such that*

$$\frac{[\tau_0^{n-1}\tau_\ell]_{0,n}}{V_{0,n}} = C_\ell + O\left(\frac{\ell}{n^{\frac{1}{4}}}\right),$$

where the implied constant is independent of ℓ. We have that $C_0 = 1$ and for $\ell \geq 1$,

$$C_{\ell+1} = 8C_\ell\sum_{i=0}^{\infty} \frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1} + 4\sum_{i=0}^{\infty} \frac{[\tau_0^{i+2}\tau_\ell]_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1}.$$

Proof. We proceed by induction on ℓ. We clearly have $C_0 = 1$ and Proposition 4.1 gives $C_1 = 12\sum_{i=0}^{\infty} \frac{V_{i+3}}{(i+1)!} x_0^{i+1}$. Assuming the result for some $\ell \geq 1$, we use Lemma 2.2 to compute

$$\begin{aligned}
&\frac{[\tau_0^{n-1}\tau_{\ell+1}]_{0,n}}{V_{0,n}}\\
&= 8\sum_{i=1}^{n-3}\binom{n-3}{i}\frac{V_{0,i+2}[\tau_0^{n-i-1}\tau_\ell]_{0,n-i}}{V_{0,n}} + 4\sum_{i=1}^{n-3}\binom{n-3}{i}\frac{[\tau_0^{i+1}\tau_\ell]_{0,i+2}V_{0,n-i}}{V_n}.\\
&= 8\left(\sum_{i=0}^{\lfloor\frac{n-4}{2}\rfloor}\binom{n-3}{i}\frac{V_{0,n-i-1}[\tau_0^{i+2}\tau_\ell]_{0,i+3}}{V_{0,n}} + \sum_{i=0}^{\lfloor\frac{n-4}{2}\rfloor-\delta_{n\in 2\mathbb{Z}}}\binom{n-3}{i+1}\frac{V_{0,i+3}[\tau_0^{n-i-2}\tau_\ell]_{0,n-i-1}}{V_{0,n}}\right)\\
&\quad + 4\left(\sum_{i=0}^{\lfloor\frac{n-4}{2}\rfloor}\binom{n-3}{i}\frac{V_{0,i+3}[\tau_0^{n-i-2}\tau_\ell]_{0,n-i-1}}{V_{0,n}} + \sum_{i=0}^{\lfloor\frac{n-4}{2}\rfloor-\delta_{n\in 2\mathbb{Z}}}\binom{n-3}{i+1}\frac{V_{0,n-i-1}[\tau_0^{i+2}\tau_\ell]_{0,i+3}}{V_{0,n}}\right).
\end{aligned}$$

Now, $[\tau_0^{i+2}\tau_\ell]_{0,i+3} \leq [\tau_0^{i+3}]_{0,i+3} = V_{0,i+3}$ and so by Lemma 3.2,

$$\sum_{i=0}^{\lfloor\frac{n-4}{2}\rfloor}\binom{n-3}{i}\frac{V_{0,n-i-1}[\tau_0^{i+2}\tau_\ell]_{0,i+3}}{V_{0,n}} = O\left(\frac{1}{n^{\frac{1}{4}}}\right).$$

Moreover,

$$\begin{aligned}
&\left|\sum_{i=0}^{\lfloor\frac{n-4}{2}\rfloor-\delta_{n\in 2\mathbb{Z}}}\binom{n-3}{i+1}\frac{V_{0,i+3}[\tau_0^{n-i-2}\tau_\ell]_{0,n-i-1}}{V_{0,n}} - C_\ell\sum_{i=0}^{\infty}\frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1}\right|\\
&\quad = O\left(\frac{\ell + C_\ell + 1}{n^{\frac{1}{4}}}\right),
\end{aligned}$$

with the implied constant independent of ℓ. To see this, we write

$$\begin{aligned}
&\left|\sum_{i=0}^{\lfloor\frac{n-4}{2}\rfloor-\delta_{n\in 2\mathbb{Z}}}\binom{n-3}{i+1}\frac{V_{0,i+3}[\tau_0^{n-i-2}\tau_\ell]_{0,n-i-1}}{V_{0,n}} - C_\ell\sum_{i=0}^{\infty}\frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1}\right|\\
&\leqslant \left|\sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\binom{n-3}{i+1}\frac{V_{0,i+3}[\tau_0^{n-i-2}\tau_\ell]_{0,n-i-1}}{V_{0,n}} - C_\ell\sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1}\right|\\
&\quad + C_\ell\sum_{i=\lfloor\sqrt{n}\rfloor+1}^{\infty}\frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1} + \sum_{i=\lfloor\sqrt{n}\rfloor+1}^{\lfloor\frac{n-4}{2}\rfloor-\delta_{n\in 2\mathbb{Z}}}\binom{n-3}{i+1}\frac{V_{0,i+3}[\tau_0^{n-i-2}\tau_\ell]_{0,n-i-1}}{V_{0,n}}.
\end{aligned}$$

Then, the latter two terms are $O\left(\frac{C_\ell}{n^{\frac{1}{4}}}\right)$ and $O\left(\frac{1}{n^{\frac{1}{4}}}\right)$ respectively by Lemmas 3.1 and 3.3 after bounding $[\tau_0^{n-i-2}\tau_\ell]_{0,n-i-1} \leq V_{0,n-i-1}$, where both implied constants are independent of ℓ. For the first term, we use the inductive hypothesis to write

$[\tau_0^{n-i-2}\tau_\ell]_{0,n-i-1} = V_{0,n-i-1}\left(C_\ell + O\left(\frac{\ell}{n^{\frac{1}{4}}}\right)\right)$ for $i \leq \lfloor\sqrt{n}\rfloor$ which, combined with the proof of Lemma 3.3, results in

$$\left|\sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\binom{n-3}{i+1}\frac{V_{0,i+3}[\tau_0^{n-i-2}\tau_\ell]_{0,n-i-1}}{V_{0,n}} - C_\ell\sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1}\right|$$
$$\leqslant C_\ell\left|\sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\binom{n-3}{i+1}\frac{V_{0,i+3}V_{0,n-i-1}}{V_{0,n}} - \sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1}\right|$$
$$+ O\left(\frac{\ell}{n^{\frac{1}{4}}}\sum_{i=0}^{\lfloor\sqrt{n}\rfloor}\binom{n-3}{i+1}\frac{V_{0,i+3}V_{n-i-1}}{V_{0,n}}\right) = O\left(\frac{C_\ell+\ell}{n^{\frac{1}{4}}}\right),$$

where the implied constant is independent of ℓ. We remark that due to the bound $\left[\tau_{d_1},\ldots,\tau_{d_n}\right]_{g,n} \leqslant V_{g,n}$, we automatically get $C_\ell \leqslant 1$ within the inductive hypothesis. By again using $[\tau_0^{n-i-2}\tau_\ell]_{0,n-i-1} \leq [\tau_0^{n-i-1}]_{0,n-i-1} = V_{0,n-i-1}$ and Lemma 3.2 we obtain

$$\sum_{i=0}^{\lfloor\frac{n-4}{2}\rfloor}\binom{n-3}{i}\frac{V_{0,i+3}[\tau_0^{n-i-2}\tau_\ell]_{0,n-i-1}}{V_{0,n}} = O\left(\frac{1}{n^{\frac{1}{4}}}\right),$$

where the implied constant is independent of ℓ, and using arguments identical to the proof of Lemma 3.3 (replacing the role of $V_{0,i+3}$ with $[\tau_0^{i+2}\tau_\ell]_{0,i+3}$) we obtain

$$\sum_{i=0}^{\lfloor\frac{n-4}{2}\rfloor-\delta_{n\in 2\mathbb{Z}}}\binom{n-3}{i+1}\frac{V_{0,n-i-1}[\tau_0^{i+2}\tau_\ell]_{0,i+3}}{V_n} = \sum_{i=0}^{\infty}\frac{[\tau_0^{i+2}\tau_\ell]_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1} + O\left(\frac{1}{n^{\frac{1}{4}}}\right),$$

where again the implied constant does not depend on ℓ. It follows that

$$\frac{[\tau_0^{n-1}\tau_{\ell+1}]_n}{V_{0,n}} = 8C_\ell\sum_{i=0}^{\infty}\frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1} + 4\sum_{i=0}^{\infty}\frac{[\tau_0^{i+2}\tau_\ell]_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x_0^{i+1} + O\left(\frac{\ell+1}{n^{\frac{1}{4}}}\right), \tag{4.1}$$

where, since the number of $O\left(\frac{1}{n^{\frac{1}{4}}}\right)$ and $O\left(\frac{\ell+1}{n^{\frac{1}{4}}}\right)$ terms are independent of ℓ, the implied constant in (4.1) is independent of ℓ, as required.

4.2. Computing the constants C_ℓ. We now seek to compute the constants C_ℓ using the recursive relation of Lemma 4.2. Consider the generating function

$$\Phi_\ell(x) = \sum_{i=3}^{\infty}\frac{[\tau_0^{i-1}\tau_\ell]_{0,i}}{(2\pi^2)^{i-2}i!}x^i,$$

so that

$$\Phi_\ell''(x) = \sum_{i=0}^{\infty}\frac{[\tau_0^{i+2}\tau_\ell]_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x^{i+1}, \qquad \Phi_0''(x) = \sum_{i=0}^{\infty}\frac{V_{0,i+3}}{(2\pi^2)^{i+1}(i+1)!}x^{i+1}.$$

Lemma 4.2 motivates us to consider the recurrence relation

$$\begin{cases} C_\ell(x) = 8C_{\ell-1}(x)\Phi_0''(x) + 4\Phi_{\ell-1}''(x) & \ell \geq 2, \\ C_0(x) = 1, C_1(x) = 12\Phi_0''(x). \end{cases}$$

Introduce the function

$$F(x, y) = \sum_{m=0}^{\infty} C_m(x)y^m,$$

so that

$$F(x, y) - 8\Phi_0''(x)yF(x, y) = 1 + 4\sum_{m=1}^{\infty} \Phi_{m-1}''(x)y^m,$$

and

$$F(x, y) = \frac{1}{1 - 8\Phi_0''(x)y}\left(1 + 4\sum_{m=1}^{\infty} \Phi_{m-1}''(x)y^m\right).$$

Differentiating ℓ times with respect to y and evaluating at $y = 0$ gives the left hand side equal to $\ell! C_\ell$ and the right hand side can be computed. Indeed, the quotient rule gives

$$\frac{\partial^\ell}{\partial y^\ell}\left(\frac{f(x, y)}{g(x, y)}\right) = \sum_{j=0}^{\ell}(-1)^j \binom{\ell}{j} \frac{\partial^{\ell-j} f}{\partial y^{\ell-j}}(x, y)\frac{E_j(x, y)}{(g(x, y))^{j+1}},$$

where

$$E_j(x, y) = j\frac{\partial g}{\partial y}(x, y)E_{j-1}(x, y) - g(x, y)\frac{\partial E_{j-1}}{\partial y}(x, y).$$

Applying this in our case, we compute

$$\frac{\partial^{\ell-j}}{\partial y^{\ell-j}}\left(1 + 4\sum_{m=1}^{\infty} \Phi_{m-1}''(x)y^m\right)(x, 0) = \begin{cases} 1 & j = \ell, \\ 4(\ell - j)!\Phi_{\ell-j-1}''(x) & j = 0, \ldots, \ell - 1, \end{cases}$$
$$(g(x, 0))^{j+1} = 1,$$
$$E_j(x, 0) = (-8\Phi_0''(x))^j j!.$$

So that formally,

$$C_\ell(x) = 4\sum_{j=0}^{\ell-1}(8\Phi_0''(x))^j \Phi_{\ell-j-1}''(x) + (8\Phi_0''(x))^\ell. \tag{4.2}$$

Thus we can understand the constants C_ℓ if we can compute the second derivative of the generating functions $\Phi_\ell(x)$ at x_0. We observe in the following lemma, that the recursive formula Lemma 4.2 implies that the generating functions Φ_ℓ satisfy a family of ODE's.

Lemma 4.3. *For $\ell \geqslant 1$, $x \leqslant x_0$,*

$$\Phi_\ell'''(x) = 8\Phi_0''(x)\Phi_{\ell-1}'''(x) + 4\Phi_0'''(x)\Phi_{\ell-1}''(x).$$

Proof. Recalling that

$$\Phi_\ell(x) = \sum_{i=3}^{\infty} \frac{[\tau_0^{i-1}\tau_\ell]_{0,i}}{(2\pi^2)^{i-2}\, i!} x^i,$$

Lemma 4.2 implies that

$$\begin{aligned}
\Phi_\ell(x) =& \sum_{i=3}^{\infty} \frac{x^i}{(2\pi^2)^{i-2}\, i!}\Bigg(8\sum_{j=1}^{i-3}\binom{i-3}{j}[\tau_0^{j+2}\tau_{\ell-1}]_{0,j+3}[\tau_0^{i-j-1}]_{0,i-j-1} \\
&+ 4\sum_{j=1}^{i-3}\binom{i-3}{j}[\tau_0^{j+1}\tau_{\ell-1}]_{0,j+2}[\tau_0^{i-j}]_{0,i-j}\Bigg) \\
=& \sum_{i=3}^{\infty} \frac{x^i}{(2\pi^2)^{i-2}\, i(i-1)(i-2)}\Bigg(8\sum_{j=1}^{i-3}\frac{[\tau_0^{j+2}\tau_{\ell-1}]_{0,j+3}}{j!}\frac{[\tau_0^{i-j-1}]_{0,i-j-1}}{(i-3-j)!} \\
&+ 4\sum_{j=1}^{i-3}\frac{[\tau_0^{j+1}\tau_{\ell-1}]_{0,j+2}}{j!}\frac{[\tau_0^{i-j}]_{0,i-j}}{(i-3-j)!}\Bigg).
\end{aligned}$$

Then differentiating three times, we obtain

$$\begin{aligned}
\Phi_\ell'''(x) =& \sum_{i=3}^{\infty} \frac{x^{i-3}}{(2\pi^2)^{i-2}}\Bigg(8\sum_{j=1}^{i-3}\frac{[\tau_0^{j+2}\tau_{\ell-1}]_{0,j+3}}{j!}\frac{[\tau_0^{i-j-1}]_{0,i-j-1}}{(i-3-j)!} \\
&+ 4\sum_{j=1}^{i-3}\frac{[\tau_0^{j+1}\tau_{\ell-1}]_{0,j+2}}{j!}\frac{[\tau_0^{i-j}]_{0,i-j}}{(i-3-j)!}\Bigg) \\
=& 8\left(\sum_{n=0}^{\infty}\frac{[\tau_0^{n+2}\tau_{\ell-1}]_{0,n+3}}{(2\pi^2)^{n+1}\, n!}x^n\right)\left(\sum_{n=0}^{\infty}\frac{[\tau_0^{n+2}]_{0,n+2}}{(2\pi^2)^{n}\, n!}x^n\right) \\
&+ 4\left(\sum_{n=0}^{\infty}\frac{[\tau_0^{n+1}\tau_{\ell-1}]_{0,n+2}}{(2\pi^2)^{n}\, n!}x^n\right)\left(\sum_{n=0}^{\infty}\frac{[\tau_0^{n+3}]_{0,n+3}}{(2\pi^2)^{n+1}\, n!}x^n\right).
\end{aligned}$$

It follows that

$$\Phi_\ell'''(x) = 8\Phi_0''(x)\Phi_{\ell-1}'''(x) + 4\Phi_0'''(x)\Phi_{\ell-1}''(x),$$

as claimed.

Lemma 4.4. *For $\ell \geqslant 1$ and $x \leqslant x_0$,*

$$\Phi_\ell''(x) = \frac{2^{3\ell+1}}{\sqrt{\pi}}\frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(\ell+2)}\Phi_0''(x)^{\ell+1}.$$

Proof. Using the recurrence relation

$$\Phi_\ell'''(x) = 8\Phi_0''(x)\Phi_{\ell-1}'''(x) + 4\Phi_0'''(x)\Phi_{\ell-1}''(x)$$

from Lemma 4.3 repeatedly we obtain

$$\begin{aligned}\Phi_\ell'''(x) &= 4\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi_0''(x)\Phi_{\ell-1}''(x)\right) + 4\Phi_0''(x)\Phi_{\ell-1}'''(x)\\ &= 4\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi_0''(x)\Phi_{\ell-1}''(x)\right) + 4\left[8\Phi_0''(x)^2\Phi_{\ell-2}'''(x) + 4\Phi_0'''(x)\Phi_0''(x)\Phi_{\ell-2}''(x)\right]\\ &= 4\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi_0''(x)\Phi_{\ell-1}''(x)\right) + 4\cdot\frac{8}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi_0''(x)^2\Phi_{\ell-2}''(x)\right)\\ &+ 4\left(8-\frac{4}{2}\right)\left[8\Phi_0''(x)^3\Phi_{\ell-2}'''(x) + 4\Phi_0''(x)^2\Phi_0'''(x)\Phi_{\ell-2}''(x)\right]\\ &= \dots\\ &= \sum_{j=1}^{\ell}\left(\prod_{i=1}^{j-1}\left(8-\frac{4}{i}\right)\right)\frac{4}{j}\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi_0''(x)^j\Phi_{\ell-j}''(x)\right).\end{aligned}$$

Then integrating from 0 to x and using that $\Phi_\ell''(0) = 0$ for every ℓ gives that

$$\Phi_\ell''(x) = \sum_{j=1}^{\ell}\left(\prod_{i=1}^{j-1}\left(8-\frac{4}{i}\right)\right)\frac{4}{j}\Phi_0''(x)^j\Phi_{\ell-j}''(x). \tag{4.3}$$

It follows from the recursive formula (4.3) that for every $\ell \geqslant 0$ there is an A_ℓ such that

$$\Phi_\ell''(x) = A_\ell\Phi_0''(x)^{\ell+1}.$$

Inserting this back into the recursive relation we obtain

$$A_\ell(\ell+1)\Phi_0''(x)^\ell\Phi_0'''(x) = 8\ell A_{\ell-1}\Phi_0''(x)^\ell\Phi_0'''(x) + 4A_{\ell-1}\Phi_0'''(x)\Phi_0''(x)^\ell.$$

After cancelling, we then see that

$$A_\ell = \frac{4(2\ell+1)}{\ell+1}A_{\ell-1},$$

and we know that $A_0 = 1$, so that

$$A_\ell = \prod_{m=1}^{\ell}\frac{4(2m+1)}{m+1} = \frac{2^{3\ell+1}}{\sqrt{\pi}}\frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(\ell+2)}.$$

Lemma 4.5. *For each $\ell \geqslant 1$,*

$$C_\ell = \left(\frac{j_0}{\pi}\right)^{2\ell}\frac{(2\ell+1)}{\sqrt{\pi}}\frac{\Gamma(\ell+\frac{1}{2})}{\Gamma(\ell+1)}.$$

Proof. By (4.2) we have

$$C_\ell = 4\sum_{j=0}^{\ell-1}(8\Phi_0''(x_0))^j\Phi_{\ell-j-1}''(x_0) + (8\Phi_0''(x_0))^\ell,$$

and so from Lemma 4.4

$$C_\ell = 8^\ell\Phi_0''(x_0)^\ell\left(\frac{1}{\sqrt{\pi}}\sum_{j=0}^{\ell-1}\frac{\Gamma(\ell-j+\frac{1}{2})}{\Gamma(\ell-j+1)} + 1\right) = 8^\ell\Phi_0''(x_0)^\ell\frac{(2\ell+1)}{\sqrt{\pi}}\frac{\Gamma(\ell+\frac{1}{2})}{\Gamma(\ell+1)}.$$

where the summation is evaluated by

$$\sum_{j=0}^{\ell-1}\frac{\Gamma(\ell-j+\frac{1}{2})}{\Gamma(\ell-j+1)}=\sum_{j=1}^{\ell}\frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+1)}=(2\ell+1)\frac{\Gamma(\ell+\frac{1}{2})}{\Gamma(\ell+1)}-\sqrt{\pi}.$$

with the last equality following by induction in ℓ and using the identity $\Gamma(z+1)=z\Gamma(z)$. The result then follows from the fact that $\Phi_0''(x_0)=\frac{j_0^2}{8\pi^2}$ from Lemma 3.1.

4.3. Proof of Theorems 1.5 and 1.6. To prove the remainder of the intersection number asymptotics, we will again work inductively using the relations of Lemma 4.2 and the computation of the constants C_ℓ.

Proof of Theorem 1.6. We proceed by induction on k. When $k=0$, the result is obvious so let us assume that the result is true for some $1\le k<\frac{1}{8}\log(n)$. We wish to show that for any $\ell\le 3g+n-3-\sum_{i=1}^{k}d_i$ that

$$\frac{[\tau_0^2\tau_0^{n-(k+1)-2}\tau_\ell\tau_{d_1}\cdots\tau_{d_k}]_{g,n}}{V_{g,n}}=C_\ell\prod_{i=1}^{k}C_{d_i}+O_g\left(\frac{\sum_{i=1}^{k}d_i+\ell}{n^{\frac{1}{4}}}\right),$$

where the implied constant is independent of $d_1,\ldots,d_k,\ell$. In the following, implied constants in O's will be independent of $n,d_1,\ldots,d_k$ and ℓ unless specified otherwise. We shall induct on ℓ and note that the base case of $\ell=0$ is simply the inductive hypothesis for the induction in k as $C_0=1$. Let $d'=(\underbrace{0,\ldots,0}_{n-(k+1)-2},d_1,\ldots,d_k)$ so that by Lemma 2.2,

$$\begin{aligned}&\frac{[\tau_0^2\tau_0^{n-(k+1)-2}\tau_\ell\tau_{d_1}\cdots\tau_{d_k}]_{g,n}}{V_{g,n}}\\&=\underbrace{\frac{[\tau_0^4\tau_\ell\prod_{i=1}^{k}\tau_{d_i}]_{g-1,n+2}}{V_{g,n}}}_{(1)}\\&\quad+\frac{8}{V_{g,n}}\underbrace{\sum_{\substack{I\sqcup J=\{1,\ldots,n-3\}\\ g_1+g_2=g}}[\tau_0^2\tau_{\ell-1}\prod_{i\in I}\tau_{d_i'}]_{g_1,|I|+3}[\tau_0^2\prod_{i\in J}\tau_{d_i'}]_{g_2,|J|+2}}_{(2)}\\&\quad+\frac{4}{V_{g,n}}\underbrace{\sum_{\substack{I\sqcup J=\{1,\ldots,n-3\}\\ g_1+g_2=g}}[\tau_0\tau_{\ell-1}\prod_{i\in I}\tau_{d_i'}]_{g_1,|I|+2}[\tau_0^3\prod_{i\in J}\tau_{d_i'}]_{g_2,|J|+3}}_{(3)}.\end{aligned}$$

Term (1) can be controlled easily using Theorem 2.4 since

$$\frac{[\tau_0^4\tau_\ell\prod_{i=1}^{k}\tau_{d_i}]_{g-1,n+2}}{V_{g,n}}\le\frac{V_{g-1,n+2}}{V_{g,n}}$$

$$= \frac{(n+2)!}{n!} \frac{(n+3)^{\frac{5(g-1)-7}{2}}}{(n+1)^{\frac{5g-7}{2}}} x_0^{-2}(2\pi^2)^{-1}\left(1+O_g\left(\frac{1}{n}\right)\right) \ll_g \frac{1}{n^{\frac{1}{2}}}. \tag{4.4}$$

Next we bound (2). Consider the summands where $|I| \leq \left\lfloor \frac{n-4}{2} \right\rfloor$ for which we bound both of the intersection numbers by the respective moduli space volumes

$$[\tau_0^2\tau_{\ell-1}\prod_{i\in I}\tau_{d_i'}]_{g_1,|I|+3}[\tau_0^2\prod_{i\in J}\tau_{d_i'}]_{g_2,|J|+2} \leq V_{g_1,|I|+3}V_{g_2,|J|+2}.$$

The contribution of these summands is thus bounded by

$$\sum_{g_1=0}^{g}\sum_{i=0}^{\left\lfloor \frac{n-4}{2} \right\rfloor}\binom{n-3}{i}\frac{V_{g_1,i+3}V_{g-g_1,n-i-1}}{V_{g,n}} = O_g\left(\frac{1}{n^{\frac{1}{4}}}\right),$$

using Lemma 3.2. For the summands where $|I| > \left\lfloor \frac{n-4}{2} \right\rfloor$, we split into two cases:

1. $n-(k+1)-1,\ldots,n-3 \in I$.
2. At least one of the indices $n-(k+1)-1,\ldots,n-3$ does not lie in I.

For summands where the first case occurs, we have by the inductive hypothesis in ℓ that

$$\begin{aligned}
[\tau_0^2\tau_{\ell-1}\prod_{i\in I}\tau_{d_i'}]_{g_1,|I|+3} &= [\tau_0^{2+|I|-k}\tau_{\ell-1}\prod_{i=1}^{k}\tau_{d_i}]_{g_1,|I|+3} \\
&= V_{g_1,|I|+3}\left(C_{\ell-1}\prod_{i=1}^{k}C_{d_i} + O_g\left(\frac{\sum_{i=1}^k d_i+\ell-1}{n^{\frac{1}{4}}}\right)\right),
\end{aligned}$$

as $|I| > \left\lfloor \frac{n-4}{2} \right\rfloor$, where the implied constant is independent of $d_1,\ldots,d_k,\ell$, and

$$[\tau_0^2\prod_{i\in J}\tau_{d_i'}]_{g_2,|J|+2} = [\tau_0^{|J|+2}]_{g_2,|J|+2} = V_{g_2,|J|+2}.$$

Thus, the contribution of these summands is given by

$$8\sum_{g_1=0}^{g}\sum_{i=\left\lfloor \frac{n-4}{2} \right\rfloor+1}^{n-4}\binom{n-3-k}{i-k}\frac{V_{g_1,i+3}V_{g-g_1,n-i-1}}{V_{g,n}}\left(C_{\ell-1}\prod_{i=1}^{k}C_{d_i}+O_g\left(\frac{\sum_{i=1}^k d_i+\ell-1}{n^{\frac{1}{4}}}\right)\right),$$

since after assigning the indices $n-(k+1)-1,\ldots,n-3$ to I there are $i-k$ elements of I left to choose and $n-3-k$ elements to choose them from. Re-indexing this summation we obtain

$$8\sum_{g_1=0}^{g}\sum_{i=0}^{\left\lfloor \frac{n-4}{2} \right\rfloor-\delta_{n\in 2\mathbb{Z}}}\binom{n-3-k}{i+1}\frac{V_{g-g_1,i+3}V_{g_1,n-i-1}}{V_{g,n}}\left(C_{\ell-1}\prod_{i=1}^{k}C_{d_i}+O_g\left(\frac{\sum_{i=1}^k d_i+\ell-1}{n^{\frac{1}{4}}}\right)\right).$$

which is equal to $8\Phi_0''(x_0)C_{\ell-1}\prod_{i=1}^k C_{d_i} + O_g\left(\frac{\sum_{i=1}^k d_i+\ell}{n^{\frac{1}{4}}}\right)$. To see this, we note that by Lemma 3.3, we have

$$\left|\sum_{i=0}^{\left\lfloor\frac{n-4}{2}\right\rfloor-\delta_{n\in 2\mathbb{Z}}}\binom{n-3-k}{i+1}\frac{V_{0,i+3}V_{g,n-i-1}}{V_{g,n}}C_{\ell-1}\prod_{i=1}^k C_{d_i} - \Phi_0''(x_0)C_{\ell-1}\prod_{i=1}^k C_{d_i}\right|$$
$$\leqslant \delta_{n\in 2\mathbb{Z}}\binom{n-3-k}{\frac{n}{2}-1}\frac{V_{0,\frac{n}{2}+1}V_{g,\frac{n}{2}+1}}{V_{g,n}}C_{\ell-1}\prod_{i=1}^k C_{d_i} + O_g\left(\frac{\sum_{i=1}^k d_i+\ell-1}{n^{\frac{1}{4}}}\right).$$

By Theorem 2.4 we have

$$\binom{n-3-k}{\frac{n}{2}-1}\frac{V_{0,\frac{n}{2}+1}V_{g,\frac{n}{2}+1}}{V_{g,n}}$$
$$= \frac{(n-3-k)!}{\left(\frac{n}{2}-1\right)!\left(\frac{n}{2}-2-k\right)!}\frac{\left(\frac{n}{2}+1\right)!^2}{n!}\left(\frac{n}{2}+2\right)^{\frac{5g-14}{2}}(n+1)^{-\frac{5g-7}{2}}x_0^{-2}(2\pi^2)^{-1}$$
$$\left(B_0 + O_g\left(\frac{1}{n}\right)\right) = O_g\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

Using that each $C_{d_i} \leqslant 1$ (which we get for free by the fact that $\left[\prod \tau_{d_i}\right]_{g,n} \leqslant V_{g,n}$), the $g_1 = g$ term is equal to $8\Phi_0''(x_0)C_{\ell-1}\prod_{i=1}^k C_{d_i} + O_g\left(\frac{\sum_{i=1}^k d_i+\ell}{n^{\frac{1}{4}}}\right)$. For $0 \leq g_1 < g$, we use Lemma 3.3 to obtain

$$\sum_{g_1=0}^{g-1}\sum_{i=0}^{\left\lfloor\frac{n-4}{2}\right\rfloor-\delta_{n\in 2\mathbb{Z}}}\binom{n-3-k}{i+1}\frac{V_{g-g_1,i+3}V_{g_1,n-i-1}}{V_{g,n}}\left(C_{\ell-1}\prod_{i=1}^k C_{d_i} + O_g\left(\frac{\sum_{i=1}^k d_i+\ell-1}{n^{\frac{1}{4}}}\right)\right)$$
$$\ll_g \sum_{g_1=0}^{g-1}\frac{1}{n^{\frac{1}{4}}}\left(C_{\ell-1}\prod_{i=1}^k C_{d_i} + O_g\left(\frac{\sum_{i=1}^k d_i+\ell-1}{n^{\frac{1}{4}}}\right)\right) \ll_g \frac{\sum_{i=1}^k d_i+\ell}{n^{\frac{1}{4}}},$$

and so the claim follows.

Next, we consider the contribution to (2) from the summands where at least one of the indices $n-(k+1)-1,\dots,n-3$ does not lie in I and $|I| > \left\lfloor\frac{n-4}{2}\right\rfloor$. This is equal to

$$8\sum_{a=0}^{k-1}\sum_{g_1=0}^{g}\sum_{\substack{I\sqcup J=\{1,\dots,n-3\}\\ |I|>\left\lfloor\frac{n-4}{2}\right\rfloor\\ I\text{ has exactly } a \text{ indices from}\\ \{n-(k+1)-1,\dots,n-3\}}}\frac{[\tau_0^2\tau_{\ell-1}\prod_{i\in I}\tau_{d_i'}]_{g_1,|I|+3}[\tau_0^2\prod_{i\in J}\tau_{d_i'}]_{g-g_1,|J|+2}}{V_{g,n}}.$$

We use the trivial bound $[\tau_0^2\tau_{\ell-1}\prod_{i\in I}\tau_{d_i'}]_{g_1,|I|+3}[\tau_0^2\prod_{i\in J}\tau_{d_i'}]_{g-g_1,|J|+2} \leq V_{g_1,|I|+3}V_{g-g_1,|J|+2}$ so that the sum has an upper bound of the form

$$8\sum_{a=0}^{k-1}\sum_{g_1=0}^{g}\sum_{i=\left\lfloor\frac{n-4}{2}\right\rfloor+1}^{n-3-(k-a)}\binom{k}{a}\binom{n-3-k}{i-a}\frac{V_{g_1,i+3}V_{g-g_1,n-i-1}}{V_{g,n}}$$

$$= 8\sum_{a=0}^{k-1}\sum_{g_1=0}^{g}\sum_{i=k-a-1}^{\left\lfloor \frac{n-4}{2}\right\rfloor-\delta_{n\in 2\mathbb{Z}}}\binom{k}{a}\binom{n-3-k}{i+1+a-k}\frac{V_{g-g_1,i+3}V_{g_1,n-i-1}}{V_{g,n}}.$$

Using Theorem 2.4 we see that since $k \le \frac{1}{8}\log(n)$ and $i \le \left\lfloor \frac{n-4}{2}\right\rfloor$,

$$\begin{aligned}
&\binom{n-3-k}{i+1+a-k}\frac{V_{g_1,n-i-1}}{V_{g,n}}\\
&\leqslant \frac{1}{(n+1)^{\frac{5(g-g_1)}{2}}}\frac{2^{\frac{7}{2}}}{n-a-3}\frac{1}{i!}\frac{x_0^{i+1}}{(2\pi^2)^{3(g-g_1)+i+1}}\\
&\quad\cdot\left(\prod_{p=0}^{a+2}\frac{n-i-(p+1)}{n-p}\right)\left(\prod_{q=1}^{k-a-1}\frac{i-q}{n-(q+a+3)}\right)\left(1+O_g\left(\frac{1}{n}\right)\right)\\
&\leqslant \frac{C}{(n+1)^{\frac{5(g-g_1)}{2}}}\frac{1}{n^{\frac{3}{8}}}\frac{1}{(i+4)^{\frac{5}{8}}}\frac{1}{i!}\frac{x_0^{i+1}}{(2\pi^2)^{3(g-g_1)+i+1}}\left(1+O_g\left(\frac{1}{n}\right)\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
&8\sum_{a=0}^{k-1}\sum_{g_1=0}^{g}\sum_{i=k-a-1}^{\left\lfloor \frac{n-4}{2}\right\rfloor-\delta_{n\in 2\mathbb{Z}}}\binom{k}{a}\binom{n-3-k}{i+1+a-k}\frac{V_{g-g_1,i+3}V_{g_1,n-i-1}}{V_{g,n}}\\
&\ll_g \frac{1}{n^{\frac{3}{8}}}\sum_{g_1=0}^{g}\frac{1}{(n+1)^{\frac{5(g-g_1)}{2}}}\sum_{a=0}^{k-1}\binom{k}{a}\sum_{i=k-a-1}^{\left\lfloor \frac{n-4}{2}\right\rfloor}(i+4)^{-\frac{5}{8}}\frac{V_{g-g_1,i+3}}{i!}\frac{x_0^{i+1}}{(2\pi^2)^{3(g-g_1)+i+1}}\\
&\ll_g \frac{g2^k}{n^{\frac{3}{8}}} \ll_g \frac{1}{n^{\frac{1}{4}}},
\end{aligned}$$

where we use the fact that

$$\frac{1}{(n+1)^{\frac{5(g-g_1)}{2}}}\frac{V_{g-g_1,i+3}}{i!}\frac{x_0^{i+1}}{(2\pi^2)^{3(g-g_1)+i+1}} \ll_g \frac{(i+3)!}{i!}\frac{(i+4)^{\frac{5(g-g_1)-7}{2}}}{(n+1)^{\frac{5(g-g_1)}{2}}} \ll_g (i+4)^{-\frac{1}{2}}.$$

In conclusion, we find that

$$(2) = 8\Phi_0''(x_0)C_{\ell-1}\prod_{i=1}^{k}C_{d_i} + O\left(\frac{\sum_{i=1}^{k}d_i+\ell}{n^{\frac{1}{4}}}\right).$$

We evaluate (3) in a similar manner. If $|I| > \left\lfloor \frac{n-4}{2}\right\rfloor$, then we use trivial volume bounds on the intersection numbers and obtain an upper bound from this contribution of

$$\frac{4}{V_{g,n}}\sum_{g_1=0}^{g}\sum_{i=\left\lfloor \frac{n-4}{2}\right\rfloor+1}^{n-3}\binom{n-3}{i}V_{g_1,i+2}V_{g-g_1,n-i}$$

$$= \frac{4}{V_{g,n}} \sum_{g_1=0}^{g} \sum_{i=0}^{\lfloor \frac{n-4}{2} \rfloor} \binom{n-3}{i} V_{g-g_1,i+3} V_{g_1,n-i-1} \ll_g \frac{1}{n^{\frac{1}{4}}},$$

by Lemma 3.2. When $|I| \leq \lfloor \frac{n-4}{2} \rfloor$, we split between the case where $n-(k+1)-1,\ldots,n-3 \in J$ or not. When they are in J the contribution comes from the term when $g_1 = 0$ and we have by the inductive hypothesis in k that

$$[\tau_0^3 \prod_{i \in J} \tau_{d_i'}]_{g,|J|+3} = V_{g,|J|+3} \left(\prod_{i=1}^{k} C_{d_i} + O_g \left(\frac{\sum_{i=1}^{k} d_i}{n^{\frac{1}{4}}} \right) \right),$$

and so the contribution from these terms is

$$\frac{4}{V_{g,n}} \sum_{i=1}^{\lfloor \frac{n-4}{2} \rfloor} \binom{n-3-k}{i} [\tau_0^{i+1} \tau_{\ell-1}]_{0,i+2} V_{g,n-i} \left(\prod_{i=1}^{k} C_{d_i} + O \left(\frac{\sum_{i=1}^{k} d_i}{n^{\frac{1}{4}}} \right) \right)$$
$$= \frac{4}{V_{g,n}} \sum_{i=0}^{\lfloor \frac{n-4}{2} \rfloor - 1} \binom{n-3-k}{i+1} [\tau_0^{i+2} \tau_{\ell-1}]_{0,i+3} V_{g,n-i-1} \left(\prod_{i=1}^{k} C_{d_i} + O_g \left(\frac{\sum_{i=1}^{k} d_i}{n^{\frac{1}{4}}} \right) \right).$$

Analogous application of Lemma 3.3 (but replacing $V_{0,i+3}$ by $[\tau_0^{i+2}\tau_{\ell-1}]_{0,i+3}$) results in this being equal to

$$4\Phi_{\ell-1}''(x_0) \prod_{i=1}^{k} C_{d_i} + O \left(\frac{\sum_{i=1}^{k} d_i + \ell}{n^{\frac{1}{4}}} \right).$$

The terms for $g_1 > 0$ can easily be shown to be $O_g \left(\frac{\sum_{i=1}^{k} d_i + \ell}{n^{\frac{1}{4}}} \right)$ identically as for (2) using Lemma 3.3. When at least one of the indices $n-(k+1)-1,\ldots,n-3$ is not in J we bound the intersection numbers by their corresponding moduli space volumes to obtain an upper bound on the contribution by

$$4 \sum_{a=1}^{k} \sum_{g_1=0}^{g} \binom{k}{a} \sum_{i=a}^{\lfloor \frac{n-4}{2} \rfloor} \binom{n-3-k}{i-a} \frac{V_{g_1,i+2} V_{g-g_1,n-i}}{V_{g,n}}$$
$$= 4 \sum_{a=0}^{k-1} \sum_{g_1=0}^{g} \binom{k}{a} \sum_{i=k-a-1}^{\lfloor \frac{n-4}{2} \rfloor - 1} \binom{n-3-k}{i+1+k-a} \frac{V_{g-g_1,i+3} V_{g_1 n-i-1}}{V_{g,n}} \ll_g \frac{1}{n^{\frac{1}{4}}},$$

as before. This means that

$$(3) = 4\Phi_{\ell-1}''(x_0) \prod_{i=1}^{k} C_{d_i} + O_g \left(\frac{1}{n^{\frac{1}{4}}} \right),$$

so that

$$\begin{aligned}
&\frac{[\tau_0^2\tau_0^{n-(k+1)-2}\tau_\ell\tau_{d_1}\cdots\tau_{d_k}]_{g,n}}{V_{g,n}} \\
&= 8\Phi_0''(x_0)C_{\ell-1}\prod_{i=1}^{k}C_{d_i} + 4\Phi_{\ell-1}''(x_0)\prod_{i=1}^{k}C_{d_i} + O_g\left(\frac{\sum_{i=1}^k d_i+\ell}{n^{\frac{1}{4}}}\right) \\
&= C_\ell\prod_{i=1}^{k}C_{d_i} + O_g\left(\frac{\sum_{i=1}^k d_i+\ell}{n^{\frac{1}{4}}}\right),
\end{aligned} \tag{4.5}$$

as required, using the recursive formula from Lemma 4.2 for the final equality. Since the implied constants in the $O_g\left(\frac{1}{n^{\frac{1}{4}}}\right)$ and $O_g\left(\frac{\sum_{i=1}^k d_i+\ell-1}{n^{\frac{1}{4}}}\right)$ terms are independent of $d_1,\ldots,d_k,\ell$, and the number of such terms we combine together is independent of $d_1,\ldots,d_k,\ell$ the implied constant in (4.5) is also independent of $d_1,\ldots,d_k,\ell$.

Proof of Theorem 1.5. Theorem 1.5 follows from Theorem 1.6 and Theorem 2.1 by recognising the Taylor expansion of $\prod_{i=1}^k I_0(x_i)$. We have

$$\begin{aligned}
\frac{V_{g,n}(\ell_1,\ldots,\ell_k,0_{n-k})}{V_{g,n}} &= \sum_{|d|\le 3g+n-3}\prod_{i=1}^{k}\left(\frac{j_0\ell_i}{2\pi}\right)^{2d_i}\frac{1}{\sqrt{\pi}}\frac{\Gamma(d_i+\frac{1}{2})}{\Gamma(d_i+1)}\frac{1}{(2d_i)!} \\
&\quad + O\left(\frac{1}{n^{\frac{1}{4}}}\sum_{|d|\le 3g+n-3}|d|\prod_{i=1}^{k}\frac{\left(\frac{\ell_i}{2}\right)^{2d_i}}{(2d_i+1)!}\right) \\
&= \sum_{d_1=0}^{3g+n-3}\cdots\sum_{d_k=0}^{3g+n-3}\prod_{i=1}^{k}\left(\frac{j_0\ell_i}{2\pi}\right)^{2d_i}\frac{1}{\sqrt{\pi}}\frac{\Gamma(d_i+\frac{1}{2})}{\Gamma(d_i+1)}\frac{1}{(2d_i)!} \\
&\quad - \sum_{3g+n-3<|d|\le k(3g+n-3)}\prod_{i=1}^{k}\left(\frac{j_0\ell_i}{2\pi}\right)^{2d_i}\frac{1}{\sqrt{\pi}}\frac{\Gamma(d_i+\frac{1}{2})}{\Gamma(d_i+1)}\frac{1}{(2d_i)!} \\
&\quad + O\left(\frac{1}{n^{\frac{1}{4}}}\sum_{|d|\le 3g+n-3}|d|\prod_{i=1}^{k}\frac{\left(\frac{\ell_i}{2}\right)^{2d_i}}{(2d_i+1)!}\right).
\end{aligned}$$

But,

$$\left(\frac{j_0\ell_i}{2\pi}\right)^{2d_i}\frac{1}{\sqrt{\pi}}\frac{\Gamma(d_i+\frac{1}{2})}{\Gamma(d_i+1)}\frac{1}{(2d_i)!} = \left(\frac{j_0\ell_i}{4\pi}\right)^{2d_i}\frac{1}{(d_i!)^2},$$

and we recall that

$$I_0(x) = \sum_{d=0}^{\infty}\left(\frac{x}{2}\right)^{2d}\frac{1}{(d!)^2},$$

so that

$$\sum_{d_1=0}^{3g+n-3}\cdots\sum_{d_k=0}^{3g+n-3}\prod_{i=1}^{k}\left(\frac{j_0\ell_i}{2\pi}\right)^{2d_i}\frac{1}{\sqrt{\pi}}\frac{\Gamma(d_i+\frac{1}{2})}{\Gamma(d_i+1)}\frac{1}{(2d_i)!}$$

$$= \prod_{i=1}^{k} \left(I_0 \left(\frac{j_0 \ell_i}{2\pi} \right) - \sum_{d=3g+n-2}^{\infty} \left(\frac{j_0 \ell_i}{4\pi} \right)^{2d} \frac{1}{(d!)^2} \right).$$

We then notice that

$$\sum_{d=3g+n-2}^{\infty} \left(\frac{j_0 \ell_i}{4\pi} \right)^{2d} \frac{1}{(d!)^2} \leq \frac{\exp\left(\left(\frac{j_0 \ell_i}{4\pi} \right)^2 \right)}{(3g+n-2)!}.$$

Using also that $I_0(x) \leq e^x$, we find

$$\begin{aligned}
&\prod_{i=1}^{k} \left(I_0 \left(\frac{j_0 \ell_i}{2\pi} \right) - \sum_{d=3g+n-2}^{\infty} \left(\frac{j_0 \ell_i}{4\pi} \right)^{2d} \frac{1}{(d!)^2} \right) \\
&= \prod_{i=1}^{k} I_0 \left(\frac{j_0 \ell_i}{2\pi} \right) + O\left(\sum_{\substack{I \sqcup J = \{1,\dots,k\} \\ J \neq \emptyset}} \prod_{i \in I} \exp\left(\frac{j_0 \ell_i}{4\pi} \right) \prod_{j \in J} \frac{\exp\left(\left(\frac{j_0 \ell_i}{4\pi} \right)^2 \right)}{(3g+n-2)!} \right) \\
&= \prod_{i=1}^{k} I_0 \left(\frac{j_0 \ell_i}{2\pi} \right) + O\left(\sum_{j=1}^{k} \binom{k}{j} \frac{\exp\left(\frac{j_0 \sum_{i=1}^{k} \max(\ell_i, \ell_i^2)}{4\pi} \right)}{(3g+n-2)!^j} \right) \\
&= \prod_{i=1}^{k} I_0 \left(\frac{j_0 \ell_i}{2\pi} \right) + O_k\left(\frac{\exp\left(\frac{j_0 \sum_{i=1}^{k} \max(\ell_i, \ell_i^2)}{4\pi} \right)}{(3g+n-2)!} \right).
\end{aligned}$$

Using the same estimates, we see that

$$\sum_{3g+n-3 < |d| \leq k(3g+n-3)} \prod_{i=1}^{k} \left(\frac{j_0 \ell_i}{2\pi} \right)^{2d_i} \frac{1}{\sqrt{\pi}} \frac{\Gamma(d_i + \frac{1}{2})}{\Gamma(d_i + 1)} \frac{1}{(2d_i)!} = O_k\left(\frac{\exp\left(\sum_{i=1}^{k} \left(\frac{j_0 \ell_i}{4\pi} \right)^2 \right)}{(3g+n-2)!^k} \right).$$

For the remaining error, we notice that

$$\frac{1}{n^{\frac{1}{4}}} \sum_{|d| \leq 3g+n-3} |d| \prod_{i=1}^{k} \frac{\left(\frac{\ell_i}{2} \right)^{2d_i}}{(2d_i+1)!} \leq \frac{1}{n^{\frac{1}{4}}} \sum_{i=1}^{k} l_i \frac{\partial}{\partial l_i} \left(\prod_{i=1}^{k} \frac{\sinh\left(\frac{\ell_i}{2} \right)}{\frac{\ell_i}{2}} \right) \leqslant \frac{\prod_{i=1}^{k} \cosh\left(\frac{\ell_i}{2} \right)}{n^{\frac{1}{4}}},$$

to obtain

$$\frac{V_{g,n}(\ell_1, \dots, \ell_k, 0_{n-k})}{V_{g,n}} = \prod_{i=1}^{k} I_0 \left(\frac{j_0 \ell_i}{2\pi} \right) + O\left(\frac{\prod_{i=1}^{k} \cosh\left(\frac{\ell_i}{2} \right)}{n^{\frac{1}{4}}} \right).$$

□

5. Proof of Theorem 1.1

The purpose of this section is the prove Theorem 1.1.

Proof of Theorem 1.1. Let $N_2(X, L)$ denote the number of unoriented, primitive closed geodesics on X with length $\leqslant L$ that separate off exactly two cusps and no genus. We first show that for any $L < 2\operatorname{arcsinh}1$, there is a constant $C(L)$ such that

$$\mathbb{P}_{g,n}\left[\frac{N_2(X, L)}{n} < C(L)\right] \to 0 \tag{5.1}$$

as $n \to \infty$.

By our assumption, any geodesic of length $< L$ is simple and we can write

$$N_2(X, L) = \sum_{[\gamma]} \sum_{\alpha \in \mathrm{MCG}\cdot[\gamma]} \mathbb{1}_{\leqslant L}(\ell_\alpha(X)),$$

where the first summation is over all mapping class group orbits of (homotopy classes of) simple closed curves which separate $\Sigma_{g,n}$ into $\Sigma_{0,3}$ and $\Sigma_{g,n-1}$. Since by our definition, the mapping class group respects the labelling of punctures, there are $\binom{n}{2}$ orbits. By Mirzakhani's integration formula, and the fact that $V_{0,3}(x, y, z) = 1$,

$$\begin{aligned}
\mathbb{E}\left[\frac{N_2(X, L)}{n}\right] &= \frac{1}{nV_{g,n}}\binom{n}{2}\int_0^L xV_{g,n-1}(0_{n-n_1}, x)\mathrm{d}x \\
&= \frac{1}{n}\binom{n}{2}\frac{V_{g,n-1}}{V_{g,n}}\int_0^L x\left(I_0\left(\frac{j_0 x}{2\pi}\right) + O_g\left(\frac{1}{n^{\frac{1}{4}}}\cosh\left(\frac{x}{2}\right)\right)\right)\mathrm{d}x \\
&= \frac{1}{n}\binom{n}{2}\frac{V_{g,n-1}}{V_{g,n}}\left(\left(\frac{2\pi}{j_0}\right)^2\int_0^{\frac{j_0 L}{2\pi}} xI_0(x)\mathrm{d}x + O_{g,L}\left(\frac{1}{n^{\frac{1}{4}}}\right)\right).
\end{aligned}$$

To evaluate the integral, we recall that the modified Bessel functions satisfy

$$\begin{aligned}
\frac{2n}{x}I_n(x) &= I_{n-1}(x) - I_{n+1}(x), \\
2I_n'(x) &= I_{n-1}(x) + I_{n+1}(x),
\end{aligned}$$

which are easily derived from similar identities for the Bessel functions $J_n(x)$ (see [2, (9.1.27)]). Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}(xI_1(x)) = I_1(x) + \frac{x}{2}(I_0(x) + I_2(x)) = xI_0(x),$$

so that

$$\left(\frac{2\pi}{j_0}\right)^2\int_0^{\frac{j_0 L}{2\pi}} xI_0(x)\mathrm{d}x = \frac{2\pi L}{j_0}I_1\left(\frac{j_0 L}{2\pi}\right).$$

By Theorem 2.4,

$$\binom{n}{2}\frac{V_{g,n-1}}{nV_{g,n}} = \frac{n-1}{n}\frac{1}{2}\left(\frac{n}{n+1}\right)^{\frac{5g-7}{2}}\left(\frac{x_0}{2\pi^2}\right)\left(1+O_g\left(\frac{1}{n}\right)\right)$$
$$= \frac{x_0}{4\pi^2}\left(1+O_g\left(\frac{1}{n^{\frac{1}{2}}}\right)\right).$$

Then since $x_0 = \frac{1}{2}j_0J_1(j_0)$,

$$\mathbb{E}\left[\frac{N_2(X,L)}{n}\right] = \frac{LJ_1(j_0)I_1\left(\frac{j_0L}{2\pi}\right)}{4\pi}\left(1+O\left(\frac{1}{n^{\frac{1}{4}}}\right)\right).$$

Now we compute the variance.

$$\mathbb{E}\left[\left(\frac{N_2(X,L)}{n}\right)^2\right] = \frac{1}{n^2}\mathbb{E}\left[\sum_{\substack{\gamma\in\mathcal{P}(X)\\ \gamma \text{ separates off exactly 2 cusps}}} \mathbb{1}_{\leqslant L}\left(\ell_\gamma(X)\right)\right]$$
$$+\frac{1}{n^2}\mathbb{E}\left[\sum_{\substack{(\gamma_1,\gamma_2)\in\mathcal{P}(X)\times\mathcal{P}(X)\\ \gamma_1\neq\gamma_2\\ \gamma_1,\gamma_2 \text{ separate off exactly 2 cusps}}} \mathbb{1}_{\leqslant L}\left(\ell_{\gamma_1}(X),\ell_{\gamma_2}(X)\right)\right].$$

The first term on the right hand side is just equal to

$$\frac{\mathbb{E}[N_2(X,L)]}{n^2} = O_{g,L}\left(\frac{1}{n}\right).$$

Since $L < 2\operatorname{arcsinh}1$, any pair of curves $\gamma_1\neq\gamma_2$ with length $\leqslant L$ are disjoint by the Collar Lemma (c.f. [10, Theorem 4.4.6]). We calculate the second term as

$$\frac{1}{n^2}\mathbb{E}\left[\sum_{\substack{(\gamma_1,\gamma_2)\in\mathcal{P}(X)\times\mathcal{P}(X)\\ \gamma_1\neq\gamma_2\\ \gamma_1,\gamma_2 \text{ separate off exactly 2 cusps}}} \mathbb{1}_{\frac{1}{L}}\left(\ell_{\gamma_1}(X),\ell_{\gamma_2}(X)\right)\right]$$
$$= \binom{n}{2,2}\frac{1}{n^2V_{g,n}}\int_0^L\int_0^L x_1x_2V_{g,n-2}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2$$
$$= \binom{n}{2,2}\frac{V_{g,n-2}}{n^2V_{g,n}}\left(\left(\frac{2\pi L}{j_0}I_1\left(\frac{j_0L}{2\pi}\right)\right)^2+O_{g,L}\left(\frac{1}{n^{\frac{1}{4}}}\right)\right),$$

using the same computation for the integral as before. Then by Theorem 2.4,

$$\binom{n}{2,2}\frac{V_{g,n-2}}{n^2V_{g,n}}$$
$$= \frac{1}{4}\frac{(n-2)(n-3)}{n^2}\left(\frac{n-1}{n+1}\right)^{\frac{5g-7}{2}}\left(\frac{x_0}{2\pi^2}\right)^2\left(1+O_g\left(\frac{1}{n}\right)\right)$$

$$= \frac{1}{4}\left(\frac{x_0}{2\pi^2}\right)^2 \left(1 + O_g\left(\frac{1}{n^{\frac{1}{2}}}\right)\right). \tag{5.2}$$

Then

$$\mathrm{Var}\left(\frac{N_2\left(X, L\right)}{n}\right) = \mathbb{E}\left[\left(\frac{N_2\left(X, L\right)}{n}\right)^2\right] - \left(\mathbb{E}\left[\frac{N_2\left(X, L\right)}{n}\right]\right)^2 \ll_{g,L} \frac{1}{n^{\frac{1}{4}}}.$$

Taking $C(L)$ to be any constant $< \frac{L J_1(j_0) I_1\left(\frac{j_0 L}{2\pi}\right)}{4\pi}$, the claim follows by applying Chebyshev's inequality.

By applying a min-max argument, it is shown in [21, Section 5], if $N_2\left(X, \frac{\varepsilon}{6}\right) > k$ then $\lambda_k \leqslant \varepsilon$. Taking $L = \frac{\varepsilon}{6}$, it follows from (5.1) that $\lambda_{C(\frac{\varepsilon}{6})n} < \varepsilon$ with probability tending to 1 as $n \to \infty$.

6. Relative Frequencies of Closed Curves

The purpose of this section is to prove the following.

Theorem 6.1. *Let $L = L(n) > 0$ be any function with $L \to \infty$ as $n \to \infty$ and $L = O\left(\log n\right)$. Then there exists a function $\varepsilon(n)$ with $\varepsilon(n) \to 0$ as $n \to \infty$ such that a Weil-Petersson random surface $X \in \mathcal{M}_{g,n}$ satisfies*

$$N^{\mathrm{s}}(X, L) \leqslant \varepsilon(n) N^{\mathrm{ns}}(X, L)$$

with probability tending to 1 *as $n \to \infty$.*

Outline of the proof. Let $L \to \infty$ as $n \to \infty$ with $L = O\left(\log n\right)$. First we show, using Theorem 1.5, that there is a constant $c_1 < 1$ such that with high probability, the number of simple closed geodesics of length less than L is at most $ne^{c_1 L}$. The fact that $c_1 < 1$ is crucial here. We then want a lower bound for the number of closed geodesics of length up to L which holds with high probability and grows faster than $ne^{c_1 L}$ as $n \to \infty$.

To achieve this, we note that by the estimates of Sect. 5, for any $\varepsilon > 0$, with probability tending to 1 as $n \to \infty$, a random surface in $\mathcal{M}_{g,n}$ has at least $c\left(\varepsilon\right) n$ closed geodesics of length at most ε which separate off a pair of pants with two cusps. This tells us that with high probability on a random surface there are at least $c\left(\varepsilon\right) n$ disjoint subsurfaces which are pairs of pants with two cusps and geodesic boundary with length less than ε. We show in Sect. 6.2 that one can pick ε so that there are at least $e^{c_2 L}$ closed geodesics of length less than L in each subsurface where c_2 satisfies $c_1 < c_2 < 1$. It follows that

$$\frac{N^s\left(X, L\right)}{N^{\mathrm{ns}}\left(X, L\right)} \leqslant \mathrm{const} \cdot e^{(c_1 - c_2)L}$$

with probability tending to 1 as $n \to \infty$.

6.1. Growth of the number of simple closed geodesics.

Lemma 6.2. *Let $L = L(n)$ be any function with $L \to \infty$ as $n \to \infty$ and $L = O(\log n)$. Then there exists a positive constant c_1 with $c_1 < 1$ such that*

$$\mathbb{P}_{g,n}\left[N^s(X, L) \geqslant ne^{c_1 L}\right] \to 0,$$

as $n \to \infty$.

Proof. Let $\Sigma_{g,n}$ denote a genus-g topological surface with n labelled punctures. We have

$$N(X, L) = \sum_{[\gamma]} \sum_{\alpha\in[\gamma]} \mathbb{1}_{\leqslant L}(\ell_\alpha(X)),$$

where the exterior summation is over all mapping class group orbits of homotopy classes of simple closed curves on $\Sigma_{g,n}$. On $\Sigma_{g,n}$ there are the following types of mapping class group orbits:

1. When $g \geq 1$, there is a single mapping class group orbit of non-separating curves that when cut along, reduces the genus by one and adds two boundaries.
2. For each configuration $\left\{\left(g_1, n_1, \left\{c_1^1, \ldots, c_{n_1}^1\right\}\right), \left(g_2, n_2, \left\{c_1^2, \ldots, c_{n_2}^2\right\}\right)\right\}$ where $g_1 + g_2 = g$, $n_1 + n_2 = n$, $2g_i + n_i - 1 > 0$ and $\left\{c_1^1, \ldots, c_{n_1}^1, c_1^2, \ldots, c_{n_2}^2\right\} = \{1, \ldots, n\}$, there is a single mapping class group orbit for the curves that separate the surface into a component with genus g_1, n_1 cusps labelled with $\{c_1^1, \ldots, c_{n_1}^1\}$ and one boundary and a component with genus g_2, n_2 cusps labelled with $\{c_1^2, \ldots, c_{n_2}^2\}$ and one boundary.

We will write $0_m := \underbrace{(0, \ldots, 0)}_{m}$. Using Theorem 2.6, we see that

$$\mathbb{E}_{g,n}\left(\frac{N(X, L)}{n}\right) = \underbrace{\frac{1}{nV_{g,n}} \int_0^L xV_{g-1,n+2}(0_n, x, x)\mathrm{d}x}_{(1)}$$
$$+ \underbrace{\frac{1}{nV_{g,n}} \sum_{g_1=1}^{g-1} \sum_{n_1=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{n_1} \int_0^L xV_{g_1,n_1+1}(0_{n_1}, x)V_{g-g_1,n-n_1+1}(0_{n-n_1}, x)\mathrm{d}x}_{(2)}$$
$$+ \underbrace{\frac{1}{nV_{g,n}} \sum_{n_1=2}^{n} \binom{n}{n_1} \int_0^L xV_{0,n_1+1}(0_{n_1}, x)V_{g,n-n_1+1}(0_{n-n_1}, x)\mathrm{d}x}_{(3)}.$$

For term (1), we use the trivial volume bounds Lemma 2.5 and (4.4) to obtain

$$\frac{1}{nV_{g,n}} \int_0^L xV_{g-1,n+2}(0_{n-2}, x, x)\mathrm{d}x = \frac{e^L}{n^{\frac{1}{4}}} \frac{V_{g-1,n+2}}{nV_{g,n}} = \frac{e^L}{n^{\frac{3}{2}}}.$$

For term (2), we use the trivial bound

$$V_{g_1,n_1+1}(0_{n_1}, x)V_{g-g_1,n-n_1+1}(0_{n-n_1}, x) \leq \frac{\sinh^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2} V_{g_1,n_1+1}V_{g-g_1,n-n_1+1},$$

and employ Theorem 2.4 to write

$$\binom{n}{n_1}\frac{V_{g-g_1,n-n_1+1}}{nV_{g,n}} \ll_g \frac{1}{n_1!}\frac{1}{(n+1)^{\frac{5g_1}{2}}}\frac{x_0^{n_1-1}}{(2\pi^2)^{3g_1+n_1-1}}.$$

Then,

$$\begin{aligned}&\frac{1}{nV_{g,n}}\sum_{g_1=1}^{g-1}\sum_{n_1=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{n_1}\int_0^L xV_{g_1,n_1+1}(0_{n_1},x)V_{g-g_1,n-n_1+1}(0_{n-n_1},x)\mathrm{d}x\\ &\ll_g e^L\sum_{g_1=1}^{g-1}\sum_{n_1=0}^{\lfloor\frac{n}{2}\rfloor}\frac{V_{g_1,n_1+1}}{n_1!}\frac{1}{(n+1)^{\frac{5g_1}{2}}}\frac{x_0^{n_1-1}}{(2\pi^2)^{3g_1+n_1-1}} \ll_g \frac{e^L}{n^{\frac{1}{4}}},\end{aligned}$$

by Remark 3.4. The leading order contribution comes from term (3). To see this, write

$$\begin{aligned}&\Big|\sum_{n_1=2}^{n}\binom{n}{n_1}\int_0^L x\frac{V_{0,n_1+1}(0_{n_1},x)V_{g,n-n_1+1}(0_{n-n_1},x)}{nV_{g,n}}\mathrm{d}x\\ &\qquad-\sum_{n_1=2}^{\infty}\frac{1}{n_1!}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1}\int_0^L xV_{0,n_1+1}(0_{n_1},x)I_0\left(\frac{j_0x}{2\pi}\right)\mathrm{d}x\Big|\\ &\leqslant \underbrace{\Big|\sum_{n_1=2}^{\lfloor\sqrt{n}\rfloor}\binom{n}{n_1}\int_0^L x\frac{V_{0,n_1+1}(0_{n_1},x)V_{g,n-n_1+1}(0_{n-n_1},x)}{nV_{g,n}}\mathrm{d}x - \sum_{n_1=2}^{\lfloor\sqrt{n}\rfloor}\frac{1}{n_1!}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1}\int_0^L xV_{0,n_1+1}(0_{n_1},x)I_0\left(\frac{j_0x}{2\pi}\right)\mathrm{d}x\Big|}_{(a)}\\ &\quad+\underbrace{\sum_{n_1=\lfloor\sqrt{n}\rfloor}^{n}\binom{n}{n_1}\int_0^L x\frac{V_{0,n_1+1}(0_{n_1},x)V_{g,n-n_1+1}(0_{n-n_1},x)}{nV_{g,n}}\mathrm{d}x}_{(b)}\\ &\quad+\underbrace{\sum_{n_1=\lfloor\sqrt{n}\rfloor}^{\infty}\frac{1}{n_1!}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1}\int_0^L xV_{0,n_1+1}(0_{n_1},x)I_0\left(\frac{j_0x}{2\pi}\right)\mathrm{d}x}_{(c)}.\end{aligned}$$

We start with bounding (a) by using that fact that since $n_1 \le \sqrt{n}$, we have $n-n_1+1 \ge \frac{n}{2}$ so that by Theorem 1.5,

$$V_{g,n-n_1+1}(0_{n-n_1},x) = V_{g,n-n_1+1}\left(I_0\left(\frac{j_0x}{2\pi}\right)+O_g\left(\frac{1}{n^{\frac{1}{4}}}\cosh\left(\frac{x}{2}\right)\right)\right).$$

Then by Theorem 2.4,

$$\binom{n}{n_1}\frac{V_{g,n-n_1+1}}{nV_{g,n}} = \frac{n-n_1+1}{n}\frac{1}{n_1!}\left(\frac{n-n_1+2}{n+1}\right)^{\frac{5g-7}{2}}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1}\left(1+O_g\left(\frac{1}{n}\right)\right)$$
$$= \frac{1}{n_1!}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1}\left(1+O_g\left(\frac{1}{n^{\frac{1}{2}}}\right)\right).$$

Thus using Lemma 2.5,

$$(a) \ll_g \frac{1}{n^{\frac{1}{2}}}\sum_{n_1=2}^{\lfloor\sqrt{n}\rfloor}\frac{1}{n_1!}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1}\int_0^L xV_{0,n_1+1}(0_{n_1},x)I_0\left(\frac{j_0x}{2\pi}\right)\mathrm{d}x$$
$$+\frac{1}{n^{\frac{1}{4}}}\sum_{n_1=2}^{\lfloor\sqrt{n}\rfloor}\frac{1}{n_1!}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1}\int_0^L xV_{0,n_1+1}(0_{n_1},x)\cosh\left(\frac{x}{2}\right)\mathrm{d}x$$
$$\ll_g e^L\frac{1}{n^{\frac{1}{4}}}\sum_{n_1=2}^{\lfloor\sqrt{n}\rfloor}\frac{V_{0,n_1+1}}{n_1!}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1} \ll_g \frac{e^L}{n^{\frac{1}{4}}},$$

with the last line following from Lemma 3.1. For (b), we bound

$$V_{0,n_1+1}(0_{n_1},x)V_{g,n-n_1+1}(0_{n-n_1},x) \le V_{0,n_1+1}V_{g,n-n_1+1}\frac{\sinh^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2},$$

and then split the summation for $\lfloor\sqrt{n}\rfloor \le n_1 \le \lfloor\frac{n}{2}\rfloor$ and $\lfloor\frac{n}{2}\rfloor \le n_1 \le n$. In the first case, we use Theorem 2.4 to bound

$$\binom{n}{n_1}\frac{V_{g,n-n_1+1}}{nV_{g,n}} \ll_g \frac{1}{n_1!}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1},$$

and in the second case,

$$\binom{n}{n_1}\frac{V_{0,n_1+1}}{nV_{g,n}} \ll_g \frac{1}{(n-n_1)!}\frac{x_0^{n-n_1-1}}{\left(2\pi^2\right)^{3g+n-n_1-1}}\frac{1}{(n+1)^{\frac{5g}{2}}}.$$

Thus by Remark 3.4 and using Lemma 2.5 for the other volumes,

$$(b) \ll_g e^L\sum_{n_1=\lfloor\sqrt{n}\rfloor}^{\lfloor\frac{n}{2}\rfloor}\frac{V_{0,n_1+1}}{n_1!}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1}+e^L\sum_{n_1=\lfloor\frac{n}{2}\rfloor}^{n}\frac{V_{g,n-n_1+1}}{(n-n_1)!}\left(\frac{x_0}{2\pi^2}\right)^{n-n_1-1}\frac{1}{(n+1)^{\frac{5g}{2}}}$$
$$\ll_g e^L\sum_{n_1=\lfloor\sqrt{n}\rfloor}^{\lfloor\frac{n}{2}\rfloor}\frac{V_{0,n_1+1}}{n_1!}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1}+e^L\sum_{n_1=0}^{n-\lfloor\frac{n}{2}\rfloor}\frac{V_{g,n_1+1}}{n_1!}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1}\frac{1}{(n+1)^{\frac{5g}{2}}} \ll_g \frac{e^L}{n^{\frac{1}{4}}}.$$

For (c), we again use

$$V_{0,n_1+1}(0_{n_1},x) \le V_{0,n_1+1}\frac{\sinh\left(\frac{x}{2}\right)}{\frac{x}{2}},$$

and then apply Lemma 3.1 to obtain

$$(c) \ll_{g,L} \sum_{n_1=\lfloor\sqrt{n}\rfloor}^{\infty}\frac{V_{0,n_1+1}}{n_1!}\left(\frac{x_0}{2\pi^2}\right)^{n_1-1} \ll_g \frac{e^L}{n^{\frac{1}{4}}}.$$

Combining, we see that (3) is asymptotically bounded above

$$(3) \ll_g \sum_{n_1=2}^{\infty} \frac{1}{n_1!} \left(\frac{x_0}{2\pi^2}\right)^{n_1-1} \int_0^L x V_{0,n_1+1}(0_{n_1}, x) I_0\left(\frac{j_0 x}{2\pi}\right) \mathrm{d}x + \frac{e^L}{n^{\frac{1}{4}}}.$$

Applying once more

$$V_{0,n_1+1}(0_{n_1}, x) \leq V_{0,n_1+1} \frac{\sinh\left(\frac{x}{2}\right)}{\frac{x}{2}},$$

and noting by Lemma 3.1 that

$$\sum_{n_1=2}^{\infty} \frac{V_{0,n_1+1}}{n_1!} \left(\frac{x_0}{2\pi^2}\right)^{n_1-1} \ll 1,$$

we see that

$$(3) \ll_g \int_0^L I_0\left(\frac{j_0 x}{2\pi}\right) \sinh\left(\frac{x}{2}\right) dx + \frac{e^L}{n^{\frac{1}{4}}}.$$

Combining the respective bounds on (1), (2) and (3), we obtain,

$$\mathbb{E}_{g,n}\left(\frac{N^{\mathrm{s}}(X, L)}{n}\right) \ll_g \int_0^L I_0\left(\frac{j_0 x}{2\pi}\right) \sinh\left(\frac{x}{2}\right) dx + \frac{e^L}{n^{\frac{1}{4}}}.$$

Since $\frac{j_0}{2\pi} < \frac{1}{2}$, there exists $c_0 < 1$ so that

$$\mathbb{E}_{g,n}\left(\frac{N^{\mathrm{s}}(X, L)}{n}\right) \ll_g e^{c_0 L} + \frac{e^L}{n^{\frac{1}{4}}}.$$

Finally, since $L = O(\log n)$ there exists a $C > 0$ such that $L \leqslant C \log n$ for n sufficiently large. Then taking $c_1 > c_0$ with $c_1 < 1$ and $1 - c_1 < \frac{1}{5C}$, by Markov's inequality,

$$\mathbb{P}_{g,n}\left[N^{\mathrm{s}}(X, L) \geqslant n e^{c_1 L}\right] \ll_g e^{(c_0 - c_1)L} + \frac{e^{(1-c_1)L}}{n^{\frac{1}{4}}} \ll_g e^{(c_0-c_1)L} + n^{C(1-c_1)-\frac{1}{4}} \to 0$$

as $n \to \infty$.

6.2. Counting geodesics in pants. In this subsection we prove the following.

Proposition 6.3. *Let Y_ℓ be a hyperbolic pair of pants with two cusps and one geodesic boundary component of length ℓ. For any $\kappa > 0$ there are constants ℓ_0 and T_0 such that for every Y_ℓ with $\ell \in [\ell_0/2, \ell_0]$ and every $T \geqslant T_0$,*

$$N(Y_\ell, T) \geqslant \frac{1}{10} e^{(1-\kappa)T}.$$

Proposition 6.3 relies on a prime geodesic theorem with precise error terms, due to Naud [40]. For a hyperbolic surface S and $0 \leqslant a < b$, let $N(S, [a, b])$ denote the number of closed geodesics on S with lengths in the interval $[a, b]$.

Theorem 6.4. ([40]) *Let S_ℓ be an infinite-volume hyperbolic pair of pants with two cusps and a funnel of width $\ell < 1$. Then for any $0 < a \leq b < 1$,*

$$N(S_\ell, T) = \mathrm{li}\left(e^{\delta(S_\ell)T}\right) + O\left(e^{\left(\frac{\delta(S_\ell)}{2}+\frac{1}{4}\right)T}\right),$$

where the implied constant is uniform over $\ell \in [a, b]$, and $\delta(S_\ell)$ is the Hausdorff dimension of the limit set of a Fuchsian group Γ for which $S_\ell = \Gamma\backslash\mathbb{H}$.

Proof. This is essentially [40, Theorem 1.2] with some small adaptations to make the error term uniform in ℓ over the compact window. To this end, note firstly that, for example by [32, Theorem 3.1], the Hausdorff dimension, and hence the first resonance $\delta_0(S_\ell) = \delta(S_\ell)(1 - \delta(S_\ell))$ is continuous in ℓ. Moreover, we fix a topological surface Σ and can equip each S_ℓ with a marking $\varphi : \Sigma \to S_\ell$ such that for any free homotopy class of closed curve $[\alpha]$ on Σ, the length $\ell_{[\alpha]}(S_\ell)$ of the geodesic representative of $[\varphi(\alpha)]$ on S_ℓ is continuous in ℓ.

We now follow precisely the notation as in the proof of [40, Theorem 1.2]. Note that for fixed X and Y as in the proof, the number of closed geodesics on S_ℓ can be bounded uniformly as ℓ ranges over $[a, b]$ since the systole of these surface is uniformly bounded below by a. It follows that in [ibid. Equation (2.19)] the geometric (right-hand) side of the formula is locally continuous and hence continuous on $[a, b]$. Coupled with the continuity of the first resonance, we see that the error term

$$\sum_{s\in\mathcal{R}_{S_\ell}\backslash\delta_0(S_\ell)} \hat{g}(s)$$

is also continuous (note that for us, $\mathcal{R}_{S_\ell} = \mathcal{R}^+_{S_\ell} \cup \delta_0(S_\ell)$ since by [8], $\delta_0\,(S_\ell)$ is the only L^2-eigenvalue). The proof now continues identically up to equations (2.27) and (2.28) where the bound on the error term is dependent on ℓ in a pointwise manner of the form:

$$\left|\sum_{s\in\mathcal{R}_{S_\ell}\backslash\delta_0(S_\ell)} \hat{g}(s)\right| \leq C_1(\ell) + C_2(\ell)X^aY^b + C_3(\ell)X^cY^d.$$

But, using the continuity of the left-hand side, the $C_i(\ell)$ on the right-hand side can be made uniform over all $\ell \in [a, b]$. The proof then continues identically as in [40, Theorem 1.2] propagating these uniform bounds throughout the remainder.

We now prove Proposition 6.3.

Proof of Proposition 6.3. By gluing a funnel of width ℓ to Y_ℓ, we obtain a complete surface S_ℓ of infinite area. Since Y_ℓ is the convex core of S_ℓ, any closed geodesic on S_ℓ lies in Y_ℓ, and so it remains to prove the statement for S_ℓ.

Let $\kappa > 0$ be given. By e.g. [32, Theorem 3.1], the Hausdorff dimension $\delta\,(S_\ell)$ is a continuous function in ℓ with $\lim_{\ell\to 0}\delta\,(S_\ell) = 1$. Then we can find an $\ell_0 > 0$ such that

$$\delta\,(S_\ell) > 1 - \kappa$$

for all $\ell < \ell_0$. By Theorem 6.4 we have that

$$N(S_\ell, T) \geqslant \mathrm{li}\left(e^{\delta(S_\ell)T}\right) - O\left(e^{\frac{3}{4}T}\right),$$

where the implied constant is uniform over $\ell \in [\ell_0/2, \ell_0]$. Assuming $\kappa < \frac{1}{4}$, the leading term dominates and there exists a T_0 such that for every $\ell < \ell_0$,

$$N\left(S_\ell, T\right) \geqslant \frac{1}{10} e^{(1-\kappa)T}$$

for all $T > T_0$.

6.3. Proof of Theorem 6.1. We can now proceed with the proof of Theorem 6.1.

Proof. Let $L \to \infty$ with $L = O(\log n)$ be given and c_1 be the constant given by Lemma 6.2. Choose $\kappa < 1 - c_1$ and let $\ell_0(\kappa)$ be given as in Proposition 6.3. By an identical argument to that in the proof of Theorem 1.1, namely the probabilistic bounds on $N_2(X, L)$, there exists a $C = C(\ell_0) > 0$ such that a.a.s.

$$N_2\left(X, [\ell_0/2, \ell_0]\right) \geqslant Cn,$$

where $N_2\left(X, [\ell_0/2, \ell_0]\right)$ counts the number of unoriented primitive closed geodesics on X with lengths in $[\ell_0/2, \ell_0]$ that separate off a pair of pants with two cusps from X when cut along. Then a.a.s. X contains at least Cn disjoint subsurfaces $\{Y_i\}$ with two cusps and a geodesic boundary with length in $[\ell_0/2, \ell_0]$. Therefore by Proposition 6.3,

$$\#\left\{\gamma \in \mathcal{P}(X) \mid \ell_\gamma(X) \leqslant L\right\} \geqslant \sum_i \#\left\{\gamma \in \mathcal{P}(Y_i) \mid \ell_\gamma(Y_i) \leqslant L\right\} \geqslant Cne^{(1-\kappa)L},$$

a.a.s. where $\mathcal{P}(X)$ is the collection of primitive closed geodesic on X. By Lemma 6.2, we have

$$\frac{N^s(X, L)}{N(X, L)} \leqslant \frac{ne^{c_1 L}}{Cne^{(1-\kappa)L}},$$

a.a.s.. Then since $N(X, L) = N^{\mathrm{s}}(X, L) + N^{\mathrm{ns}}(X, L)$ we see

$$\frac{N^s(X, L)}{N^{\mathrm{ns}}(X, L)} = \left(\frac{1}{1 - \frac{N^s(X,L)}{N(X,L)}}\right) \frac{N^s(X, L)}{N(X, L)} \to 0$$

as $n \to \infty$.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors certify that they have no affiliations or involvement with any organization or entity with any financial interest, or non-financial interest in the subject matter or materials discussed in this manuscript.

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