Examining the energy loss in the Inverted Pendulum Model for rocking bodies

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Abstract

Understanding of the mechanisms that may lead to failure of rocking bodies is of significant importance to quantifying and minimizing the associated risk. The most common model to describe the rocking problem is the Inverted Pendulum model (*IPM*) proposed by Housner in 1963, which has been followed by multiple researchers. It is often claimed that the *IPM* is an acceptable model if behaviors such sliding, uplifting, deformability and the three-dimensional nature of the response can be restrained mechanically. However, even in a suitably chosen case where the previous may indeed have a minimal effect in the response, there are still uncertainties with regard to the assumption introduced in the *IPM* of how energy is lost during impacts. This paper deals with this assumption and investigates the effects in the stability of rocking bodies. The effect of this assumption is discussed showing that the associated uncertainty propagates to the conclusion of whether a rocking body would survive or fail when subjected to a ground excitation. A method that quantifies this uncertainty by making use of a dynamic property of the rocking system is introduced.

Introduction

The response of a free-standing body subjected to a base excitation has been of interest to engineers in recent decades. The first dynamic model for rocking was suggested by Housner (Housner, 1963) in 1963 after the Chilean Earthquakes, where a number of slender structures survived, while apparently more stable structures were severely damaged. Housner modeled the response of rocking bodies as inverted pendulums of equivalent properties. For this modelling approach, several assumptions must be introduced: the body and the base are considered to be rigid, sliding and uplift are neglected, the response is planar and the loss of energy which only occurs during impacts is exclusively described by the dimensions of the body.

The effect of some of the previous assumptions on the stability of rocking bodies has been studied in a series of papers. Sliding and uplifting was investigated in (Ishiyama, 1982; Shenton and Jones, 1991), the deformability of the ground was considered in (Psycharis and Jennings, 1983; Koh et al., 1986;

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Palmeri and Makris, 2008). Two models, depending on whether the base of the rocking body is flat or has concentrate supports the Winkler Model (*WM*) and Concentrate Spring Model (*CSM*) respectively, that take into account all the previous phenomena were proposed in (Chatzis and Smyth, 2012*b*). A three dimensional model ignoring sliding, uplift and deformability was proposed in (Zulli et al., 2012) and the extensions of the *WM* and *CSM* into three dimensions were presented in (Chatzis and Smyth, 2012*a*). Those papers have highlighted that each of these assumptions is indeed of significant importance and should not be omitted. Additionally the stability investigations in (Chatzis and Smyth, 2012*b,a*) showed that neglecting sliding and the three-dimensional nature of the response are not necessarily conservative assumptions.

It is often however claimed that the effect of these omitted phenomena can be minimized mechanically. For example, sliding may be minimized using a high friction material between the body and the base, while the effects of deformability are substantially smaller for relatively stiff materials for the body and the support medium, the chance of the body being in a free flight mode is reduced as it becomes more slender and assuming that all the excitations are horizontal, while the out of plane response of the body can be minimized if the perpendicular dimension of the base of the body, the length, is substantially greater to the dimension of the in plane dimension of the base, the width. Nevertheless, even if the effect of the previous phenomena is minimized, the *IPM* as introduced by (Housner, 1963), makes an explicit assumption about the energy lost during impacts: the location of the vertical impulsive force generated during impact is assumed to be at the future rocking corner. Therefore, the angular momentum is conserved about this point.

Existing works in the literature have used alternative assumptions about the loss of energy during impacts (Zhang et al., 2012; Yim et al., 1980; Dimitrakopoulos and DeJong, 2012; Voyagaki et al., 2014; ElGawady et al., 2011). These are usually expressed through the use of alternative values for the reduction of energy factor. While some of those works have attributed the use of different reduction factors to the effect of the ignored phenomena, i.e., sliding, uplift, deformability and the three-dimensional nature of the response, it should be noted that their inclusion would result in a substantially different dynamic behavior including dynamic patterns that could not be predicted by the *IPM* for any assumed value of the reduction of energy factor. This work will illustrate that conserving momentum with respect to the future rocking corner is not the only admissible solution to the problem even if, the examined body is such so that the ignored phenomena have indeed a minimal effect on its response.

It will be shown in this work that the vertical impulse during an impact may be applied in any position between the future rocking corner and the midpoint of the base, without violating the rest of the assumptions. This position can, hence, be considered as a random variable with a related uncertainty. This paper illustrates of how this uncertainty propagates to the stability of a rocking body subjected to a ground excitation.

Stability diagrams are developed for a rigid body subjected to ground excitations of the form of single, double or triple sinusoidal, cosinoidal or square pulses.

81 Conclusions are drawn on the effect of the position of the vertical impulse on
 82 the stability of the body. To quantify the propagation of the uncertainty on the
 83 stability of the body, a probabilistic approach is developed using the dynamic
 84 properties of the stability of a rocking body. The approach is initially formu-
 85 lated for the stability of rocking bodies subjected to single cycloidal pulses as
 86 in (Zhang and Makris, 2001) and then extended to the case of multiple pulses
 87 (Plaut et al., 1996). Finally, the effect of the occurring uncertainty is demon-
 88 strated by generating fragility curves for earthquake inputs generated with a
 89 stochastic method (Shinozuka and Deodatis, 1991).

90 Model and Assumptions

91 Dynamic Model

92 This section provides a brief description of the *IPM* (Housner, 1963). The
 93 rocking body and the properties used in the following equations are defined in
 94 Figure 1. In this paper the convention which is used for the axes is shown in
 95 Figure 1, with x and y being the horizontal and vertical axis respectively, and
 96 θ the rotation. The block can be fully defined by its mass m , its moment of
 97 inertia I_0 about one of the two corners 0 and 0', while b and h are the half-width
 98 and half-height of the body, respectively. Furthermore, R is the distance from
 99 a corner to the center of mass, $R = \sqrt{h^2 + b^2}$. The slenderness of the block
 100 is defined by the angle $\alpha = \arctan \frac{b}{h}$. Depending on the sign of θ the body
 101 rocks with respect to the corresponding corner, and hence can be simulated by
 102 a pendulum whose pin is located at that corner as shown in Figure 1. This of
 103 course implies, that the body cannot slide, or experience a free-flight mode, that
 104 both the body and the ground are rigid, and that the response of the body is
 105 strictly planar, which are assumptions that will be adopted in this paper.

106 [Figure 1 about here.]

107 The motion of the body is defined through three patterns. Patterns A and
 108 B are the pendulum patterns corresponding to $\theta < 0$ and $\theta > 0$ respectively,
 109 while pattern C is the static pattern $\theta = 0$ and $\dot{\theta} = 0$, i.e., when the body is at
 110 rest. The block is subjected to a positive horizontal, $\ddot{x}_g$, and vertical, $\ddot{y}_g$, ground
 111 accelerations. F_x and F_y are D'Alembert fictitious forces defined by $F_x = -m\ddot{x}_g$
 112 and $F_y = -m\ddot{y}_g$. The ground is experiencing only translational accelerations
 113 and not rotational as for example examined in Virgin et al. (1996). If moments
 114 are taken about the center of rotation 0 or 0', the equations of motion, using
 115 Newton's second law, are respectively:

$$\begin{aligned} \ddot{\theta} &= -p^2 \sin(\alpha - \theta) + \frac{m\ddot{x}_g}{I_0} R \cos(\alpha - \theta) - \frac{m\ddot{y}_g}{I_0} R \sin(\alpha - \theta) \text{ for } \theta > 0 \\ \ddot{\theta} &= p^2 \sin(\alpha + \theta) + \frac{m\ddot{x}_g}{I_0} R \cos(\alpha + \theta) + \frac{m\ddot{y}_g}{I_0} R \sin(\alpha + \theta) \text{ for } \theta < 0 \end{aligned} \quad (1)$$

116 where $p^2 = m g R / I_0$.

117 In the static pattern the ground applies to the body horizontal and vertical
118 reactions, f_x and f_y respectively, so that the static equilibrium is retained. The
119 vertical reaction f_y is applied at a distance d measured, by convention, from
120 the right corner 0. Since it is assumed that the ground cannot provide any
121 tensile forces, the position of the vertical reaction force is bounded between the
122 two corners, hence: $0 \leq d \leq 2b$. If in order to retain static equilibrium the
123 obtained value of d is outside those limits, then the static equilibrium cannot be
124 retained as it is assumed that the ground interface is unable to provide tensile
125 forces. Hence, as $d = 0$ or $d = 2b$ the body transitions from the static pattern
126 to patterns B or A , respectively.

127 In order to switch between patterns A and B , an impact occurs as is shown in
128 Figure 2. This transition takes place when $\theta = 0$. During this impact, the body
129 will experience impulsive forces. The corresponding resultants of the horizontal
130 and vertical impulses are defined as J_x and J_y respectively.

131 [Figure 2 about here.]

132 The assumption of conserving momentum with respect to the future rocking
133 corner during impacts, is equivalent to assuming that the vertical impulse acts
134 on that corner. To obtain J_y a distribution of vertical impulses along the base
135 of the body has to be assumed. The restrictions for that distribution are that
136 the distributed impulses are applied at the contact interface, i.e. between the
137 two corners, and that they correspond to compressive impulsive forces. For
138 the purposes of illustration of the following Figure 3 and without any loss of
139 generality, a flat base is assumed for the body and hence the distributed impulses
140 can be applied at any point between the two corners.

141 Figure 3 represents one of the infinitely many admissible distributions that
142 could be considered for the distributed vertical impulses. Since the body is
143 rigid the distribution can be completely represented by the resultant impulse
144 J_y which is located at the centroid of the distribution. This centroid, without
145 any loss of generality, may be at a distance λb measured from the post-impact
146 rocking corner towards the pre-impact rocking corner. The exact geometry
147 of the base plays an important role in the response of the body as has been
148 shown in (Chatzis and Smyth, 2012b; Chatterjee and Ruina, 1998). A non-
149 flat multi-contact base would result into a non-continuous impulse distribution
150 along the base. Nonetheless, this does not influence any of the results of the
151 paper as the continuous nature of the distribution of the impulses is not made
152 use of. For Figure 3 the pre- and post-impact rocking corners are assumed to
153 be $0'$ and 0 respectively. The requirement for all forces from the interface to be
154 compressive, including impulsive forces and the corresponding impulses result
155 in the following constraints: $J_y > 0$ and $0 \leq \lambda \leq 2$. The latter constraint
156 translates to the centroid being between the two corner points.

157 [Figure 3 about here.]

158 Formulating the momentum equations linking the time instances after and
 159 before the impact as in Figure 2, the following three momentum equations are
 160 obtained:

$$\begin{aligned} m\dot{x}^+ &= m\dot{x}^- + J_x \\ m\dot{y}^+ &= m\dot{y}^- + J_y \\ I_c\dot{\theta}^+ &= I_c\dot{\theta}^- - (b - \lambda b)J_y \operatorname{sign}(\dot{\theta}^-) + h J_x \end{aligned} \quad (2)$$

161 where $\dot{x}^+$, $\dot{y}^+$, $\dot{\theta}^+$ and $\dot{x}^-$, $\dot{y}^-$, $\dot{\theta}^-$ are: the horizontal, vertical and angular
 162 velocity after and before impact and I_c is the moment of inertia about the
 163 center of the block. The *sign* in equation (2) accounts for whether the future
 164 rocking corner is 0 or 0', with $\operatorname{sign}(\dot{\theta}^-)$ being + or - respectively. Due to the
 165 pendulum kinematic constraints:

$$\begin{aligned} \dot{x}^+ &= -h\dot{\theta}^+ & \dot{y}^+ &= b\|\dot{\theta}^+\| \\ \dot{x}^- &= -h\dot{\theta}^- & \dot{y}^- &= -b\|\dot{\theta}^-\| \end{aligned} \quad (3)$$

166 Thus, the unknowns of the system are $\dot{\theta}^+$, J_y , J_x and λ . To find a solution
 167 of the system, it is necessary to assume the value of one of the unknowns, for
 168 example, λ . While this is often assumed to be equal to zero in the literature,
 169 it is important to note that there would not be any violation of the momentum
 170 equations assuming any value for λ . But if one takes into account the previous
 171 discussion, any value of $\lambda \in [0, 2]$ could be used. The angular velocity after
 172 impact is defined through the reduction of energy factor, r , which relates the
 173 kinetic energy before and after impact.

$$r = \frac{(\dot{\theta}^+)^2}{(\dot{\theta}^-)^2} = \left(\frac{1 - \frac{3}{4}(2 - \lambda)\sin^2 \alpha}{1 - \frac{3}{4}\lambda\sin^2 \alpha} \right)^2 \quad (5)$$

174 Leading to the equation: $\dot{\theta}^+ = \sqrt{r}\dot{\theta}^-$. It can be seen from equation (5) that
 175 if $\lambda > 1 \Rightarrow r > 1$, hence energy would be generated during the impact, which
 176 is not physically possible in this problem where the ground experiences only
 177 translational accelerations. Thus, two material assumptions, the lack of tensile
 178 forces from the interface and that there cannot be an increase of the energy
 179 during an impact, result in λ taking admissible values in the interval $[0, 1]$. Any
 180 value of $\lambda \in [0, 1]$ is completely compatible with the assumptions made in the
 181 *IPM*. To this point it should be noted that, researchers have suggested the
 182 use of values of smaller reduction of energy factors than the one suggested by
 183 Housner in (Housner, 1963), or equivalently have suggested negative values for
 184 λ , to account for additional loss of energy during impact, potentially due to the
 185 ignored sliding, bouncing, deformability of the ground, or the three dimensional
 186 nature of the response. However, if those assumptions were to be violated, the

dynamic response of the body would include additional dynamic patterns, whose effect on the response cannot be approximated by an *IPM* with any value of λ . In other words, one should either use a model capable of taking into account those phenomena, or if these phenomena can be neglected then an *IPM* can be used assuming that $\lambda \in [0, 1]$. This work will assume the latter.

Finally, it should be noted that the occurring $\sqrt{r} = \frac{1 - \frac{3}{4}(2-\lambda)\sin^2\alpha}{1 - \frac{3}{4}\lambda\sin^2\alpha}$ should be positive. Negative solutions do not make physical sense and they would result in a free-flight pattern that is incompatible with the *IPM*. This paper will deal with bodies of aspect ratio h/b so that $\sqrt{r}$ is positive even for $\lambda = 0$. As algebraically $\sqrt{r}$ becomes larger for larger values of λ , this guarantees that $\sqrt{r} > 0$ for any of the bodies and values of λ examined in this paper.

Numerical Simulation

[Figure 4 about here.]

To integrate this model over time, the dynamic equations in (1) are brought to state-space form and are integrated over time using a variable time stepping Runge-Kutta 4-5 pair as presented in (Dormand and Prince, 1980). This algorithm uses a 4th order Runge-Kutta to propagate the solution over time and performs a comparison versus the prediction obtained from the 5th order Runge-Kutta estimating an error from the difference of the two methods. The time step is varied in order to retain the estimated relative and absolute error within some bounds.

Unlike typical equations of motion, the ones describing the behavior of the *IPM* are not smooth, i.e., infinitely differentiable as there is a switch of the state-space equations from one pattern to the other. Additionally, the velocity state $\dot{\theta}$ is discontinuous around the time instance of an impact. In order to switch from one pattern to the other an event driven function algorithm is suggested in (Shampine and Gordon, 1975), that allows for stopping the integration around a specific time instance at which an event is located, is defined. In this problem the events are the transitions between patterns or the detection of toppling, i.e., failure, of the body, i.e., $\|\theta\| = \pi/2$. Those are located online by the algorithm as implemented in Matlab's ode45 solver with the Events option Mathworks (n.d.). Each time a transition between patterns is located, the algorithm stops the integration and switches the state-space equations used to account for the new pattern. Furthermore, if the detected event was an impact then the velocity of the body is modified according to equation (5) to account for the loss of energy during the impact.

Equation (5) implies that eventually $\dot{\theta}$ tends to zero but never becomes numerically equal to zero. In other words, in theory the body can never go back to the static pattern from one of the two pendulum patterns. This however, is impractical in terms of the time integration and may lead to numerical instabilities when equation (5) is applied to values of the pre-impact velocity, or equivalently the kinetic energy just before the impact E_k , close to the machine tolerance. To avoid that and ensure stability and robustness of the algorithm, a

condition is imposed after an impact event regarding the minimum value of the kinetic energy of the body: if the energy of a body after impact is smaller than this minimum value, then the static pattern is enforced as shown in Figure 4. The integration continues until a new switch is located, or until the stopping criteria for integration are met. These stopping criteria are either that toppling of the body was detected, or that the upper bound of the requested time window of integration was met, i.e., the body survived this excitation. Figure 4 shows the numerical implementation.

Deterministic effect of λ on the stability of the body

As stated in the previous section (Model and Assumptions), λ has an associated uncertainty. The source of this uncertainty is the lack of modeling the contact physics in the IPM, which allows for an infinite number of admissible distributions for the vertical impulses. This can be thought of as a ‘structural’ uncertainty of the *IPM*. This section will examine how different values of λ would affect the stability of a rocking body when subjected to a ground acceleration $\ddot{u}$. In this first section a single cycloidal pulse stability analysis will be conducted, similar to the one presented in (Zhang and Makris, 2001). A free standing block with $p=2.1401$ and $\alpha = 0.2501$ as in (Zhang and Makris, 2001), as shown in Figure 1, is subjected to single cycle sinusoidal, cosinoidal and square pulses for a range of angular frequencies ω_u and amplitude u_0

$$\begin{aligned} \ddot{u} &= 0, & \text{if } t > T \\ \ddot{u} &= u_0 \mathbf{H}(\omega_u t), & \text{if } 0 \leq t \leq T \end{aligned} \quad (6)$$

where $T = \frac{2\pi}{\omega_u}$, $\mathbf{H}$ is either of the functions sin, cos, square for the sinusoidal, cosinoidal and square pulses respectively, with the square function being defined as the following periodic function with period T :

$$\begin{aligned} \text{square}(\omega_u t) &= +1, & \text{if } 0 < t < T/2 \\ \text{square}(\omega_u t) &= -1, & \text{if } T/2 < t < T \end{aligned} \quad (7)$$

As noted in (Zhang and Makris, 2001), three areas can be observed in a stability diagram of a body subjected to single cycloidal pulses: survival, failure after a single impact and failure without impact. The term ‘impact’ in this paper refers to the impact of the body with the ground. Stability diagrams are generated in Figure 5(a) showing these areas for a range of combinations of u_0 and ω_u for a sinusoidal pulse. Note that there are no failures for $\frac{u_0}{\tan(\alpha)g} < 1$, because this is the critical value to initiate rocking. The non-linearity of the problem is apparent by the fact that there are pulses of constant frequency, for which as the amplitude of the pulse, $\frac{u_0}{\alpha g}$, is increased the block goes from the

stable region to the unstable (failure with impact), re-transitions to the stable region and has a final transition to the unstable region (failure without impact).

[Figure 5 about here.]

Thanks to the similarity property described in (Zhang and Makris, 2001; Dimitrakopoulos and DeJong, 2012) the results for the diagram are applicable to bodies which have the same aspect ratio but potentially different size, as expressed through R , as long as the same value of ω_u/p is examined. It should further be noted that, as indicated in this paper, an additional parameter required for ensuring similarity between two such bodies is the assumption of the same value of λ for both of them. In (Zhang and Makris, 2001), it was assumed that the vertical impulse during an impact is applied at the future rocking corner, i.e., $\lambda = 0$. However, as discussed in section Model and Assumptions, λ can attain values between zero and one. Figure 5 shows the stability diagrams for all three types of pulses for values of $\lambda = 0$ and $\lambda = 1$. A significant increase of the failure with impact can be observed in all three cases, as λ goes from zero to one. This is reasonable as larger values of λ correspond to a smaller loss of energy during impacts and specifically the extreme case of $\lambda = 1$ case corresponds to conservation of energy during impacts. For the cosinoidal pulse, as can be seen in Figures 5(b) and 5(e), the single impact instability region is smaller than in Figure 5(a) and 5(d) and the boundary between non-impact and safety regions is steeper. On the other hand the square pulse, presents similarities to the sinusoidal pulse as can be seen in Figure 5(c) and 5(f).

[Figure 6 about here.]

As expected, the area of failure without impact is unaffected by λ . This behaviour can be observed for any aspect ratio h/b and for the purpose of illustrating this, two different aspect ratios are used in Figure 6 for all three pulses. Figure 6 represents the boundaries between survival and failure for different values of λ . As the assumed value of λ is increased the area of failure with impact is expanding by including more pairs of u_0 and ω_u . This significant behaviour means that, for a given pair of values ω_u and u_0 in the area defined between the boundary lines for the areas of failure with impact for $\lambda = 0$ and $\lambda = 1$, a body which fails for a given value of λ will also fail for higher values of λ . Hence, for the case of single cycle pulses, there is a value of $\lambda = \lambda^*$, such that a point on the stability diagram belonging to the predefined region will fail for any $\lambda > \lambda^*$ and survive for $\lambda < \lambda^*$:

$$\begin{aligned} \text{if } \lambda < \lambda^* &\rightarrow \text{SURVIVAL} \\ \text{if } \lambda \geq \lambda^* &\rightarrow \text{FAILURE} \end{aligned} \tag{8}$$

It should also be observed that points in the diagram that fail for $\lambda = 0$ fail for any of the allowable values of λ . Notice that the survival or failure of the remaining points on the stability diagram, outside the area defined earlier, does not depend on the value of λ .

Following the property defined in equation (8), the bisection method is applied to calculate λ^* , i.e., the value of λ signifying the transition between failure and survival. The bisection method detects this transition with the precision of a tolerance imposed (tolerance= 10^{-3}). Figure 7 shows the corresponding values of λ^* for a single sinusoidal, cosinoidal and square pulse. It can be observed that the mapping of λ^* to an (ω_u, u_0) pair depends on the type of the input.

[Figure 7 about here.]

Figure 8 shows a comparison between the three Pulses for the cases of $\lambda = 0$ and $\lambda = 1$. Notably, the square pulse results in a higher range of u_0 and ω_u values that signify failure. On the other hand, the cosinoidal pulse seems to have a usually safer behaviour than the sinusoidal and square pulses. However, there are areas on the diagram for which any of the three inputs may result in failure and the other two to safety. For example, the cosinoidal pulse fails for the case of $\lambda = 0$, $\omega_u = 4p$ and $u_0 = 3.7\alpha g$, while the square pulse survives. Comparing the cosinoidal and the sinusoidal pulses for the case of $\lambda = 0$ for $\omega_u = 4p$ and $u_0 = 5.9\alpha g$, the former fails but not the latter. In general, for a specific value of ω_u , u_0 and λ , due to the non-linearity of the problem, there will be areas where the survival or failure of the body will depend on the type of the input.

[Figure 8 about here.]

At this point, it can be observed that variations of the value of λ significantly change the failure with impact areas. In the literature, it is claimed that $\lambda = 0$, hence according to equation 5 the smallest allowable value of r is used. However, this paper has already demonstrated that the body has a greater tendency to collapse as the value of λ is increased. Hence, assuming that $\lambda = 0$, is a non-conservative assumption. It should further be noted that in Figure 5(a) when ω_u/p is small, the minimum overturning acceleration for failure is given by the lower limit of the failure with impact area. As the frequency of the input pulse is increased, this area disappears and hence the minimum amplitude for failure is given by the boundary of the failure with no impact area. Hence at some frequency, there is a jump of the minimum acceleration for overturning. This is a favorable property as the required amplitude for overturning the body for an input beyond that transition frequency is substantially larger. Nevertheless, the value of this transition frequency increases as λ is increased, as shown in Figure 6, reducing the range of these favorable, in terms of the required amplitude of overturning, frequencies.

For the remainder of this work, the following dimensions will be adopted $p = 2.1401$ and $\alpha = 0.3218$ which correspond to $h=1.5240$ m, $b=0.5080$ m and $h/b = 3$.

341 Quantifying the Probability of Failure.

342 Probabilistic models for the effect of λ on the stability of 343 the body.

344 Based on the previous discussion in section Model and Assumptions, λ is a
345 random variable between zero and one and hence it is associated with some
346 probability of occurrence. As a result, the outcome of a body being subjected to
347 a single cycloidal pulse of a specific u_0 and ω_u is also linked with a corresponding
348 probability of survival/failure. While, there are points in the stability diagram,
349 i.e., pairs of values (ω_u, u_0) , that fail or survive for any of the possible values of
350 λ , there are also points for which, the deterministic property defined on equation
351 (8) correspond to a probability property: the probability of the body failing or
352 surviving is only the probability of λ being smaller or greater than λ^* :

$$\begin{aligned} P(\lambda > \lambda^*) &= \text{PROBABILITY OF FAILURE} \\ P(\lambda < \lambda^*) &= \text{PROBABILITY OF SURVIVAL} \end{aligned} \quad (9)$$

353 In this work, two types of distributions are used for modelling λ : uniform
354 and truncated Gaussian distributions. The latter is defined in terms of the
355 mean, μ , and the standard deviation, σ . Assuming a value of μ closer to a value
356 implies that this is expected as the most probable value of λ , while σ quantifies
357 the dispersion of a set of values. Thus a σ closer to zero indicates that the
358 expected values of λ are closer to the mean. An increase of the value of sigma
359 indicates that there is smaller certainty about the expected value of λ . The
360 limits of μ and σ , for a truncated Gaussian distribution, are $\mu_u = 0.5$ and $\sigma_u =$
361 0.5 , respectively. These values correspond to the uniform distribution, which
362 corresponds to the assumption that any value of lambda between zero and one is
363 equally possible. Equations 10 and 11 refer to the Probability Density Function
364 (PDF), $PDF_U(x)$, and Cumulative Density Function (CDF), $CDF_U(x)$, of the
365 Uniform Distribution.

$$PDF_U(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1, \\ 0 & \text{for } x < 0 \text{ or } x > 1 \end{cases} \quad (10)$$

$$CDF_U(x) = \begin{cases} 0 & \text{for } x < 0, \\ x & \text{for } x \in [0, 1), \\ 1 & \text{for } x \geq 1 \end{cases} \quad (11)$$

366 It is often convenient to define a truncated Gaussian in terms of an untrun-
367 cated Gaussian of μ_u and σ_u . The truncated Gaussian distribution is defined
368 as follow:

$$PDF_T(x; \mu_u, \sigma_u, 0, 1) = \frac{\phi(\xi)}{\Phi(\beta) - \Phi(\alpha)} \quad (12)$$

$$CDF_T(x; \mu_u, \sigma_u, 0, 1) = \frac{\Phi(\xi) - \Phi(\alpha)}{\Phi(\beta) - \Phi(\alpha)} \quad (13)$$

$$\mu = \mu_u + \frac{\phi(\alpha) - \phi(\beta)}{\Phi(\alpha) - \Phi(\beta)} \sigma_u \quad (14)$$

$$\sigma = \sigma_u^2 \left(1 + \frac{\alpha\phi(\alpha) - \beta\phi(\beta)}{\Phi(\alpha) - \Phi(\beta)} - \left(\frac{\phi(\alpha) - \phi(\beta)}{\Phi(\alpha) - \Phi(\beta)} \right)^2 \right) \quad (15)$$

where μ_u and σ_u are the values of the untruncated Gaussian distribution, $\xi = \frac{x - \mu_u}{\sigma_u}$, $\alpha = \frac{0 - \mu_u}{\sigma_u}$ and $\beta = \frac{1 - \mu_u}{\sigma_u}$, $[0, 1]$ are the limited values $[0, 1]$, ϕ and Φ are the *PDF* the *CDF* of the normal Gaussian distribution ($\mu = 0$ and $\sigma = 1$), respectively.

The probability of failure of the body for a range of values of u_0 and ω_u should then be calculated. Monte Carlo simulations can be used to calculate the probability of failure. A number of generated random samples of λ are used to calculate the probability of failure for each value of u_0 and ω_u for a given distribution. The number of generated samples of λ is increased until convergence is achieved. For the case of the uniform distribution, the random values are modelled using the ‘rand’ tool from Matlab which generates uniformly distributed random numbers. For the truncated Gaussian distribution, the algorithm described in Robert (1995) is used to generate samples. The probability of failure is calculated for a pair of u_0 and ω_u , as the sum of all observed failures divided by the number of samples of λ .

Alternatively, the probability of survival or failure can be defined directly using the property defined in equation 9. In this approach, having defined the values of λ^* , for a body of a given aspect ratio and type of input, one can then apply equation 9 for each pair of points (u_0 and ω_0) to generate the corresponding probability of survival/failure in the area of points for which equations 8 and 9 are applicable.

The advantage of the second approach is that the second step, i.e., the transition from the λ^* mapping to the probability of failure does not require any further dynamic simulations, but only the calculation of the associated probabilities, that solely depend on the assumed distributions and the values of λ^* corresponding to each point. This probability can, for the cases examined, be calculated analytically. This results in an overall substantial reduction of calculation cost in comparison to the Monte Carlo approach. It should be noted that the Monte Carlo simulations mentioned in this paper use 1000 samples.

For any single cycloidal pulse input, it can be observed as for example in Figure 5, that the area of failure without impact does not change for different values of λ . Hence, the probability of failure in these areas is 100%. Because λ cannot be greater than one, the area that corresponds to survival for the case of $\lambda = 1$ has a probability of failure of 0%. Hence, as in previous sections, only the points for which equations (8) and (9) are applicable, are analysed. Figure 9 compares the Monte Carlo Simulations versus the defined probabilistic

approach for a single cosinoidal pulse. It can be seen that there is an excellent match between the two methods. In fact the difference between the two methods is controlled by the number of samples used for the Monte Carlo method and the error tolerance chosen for numerically estimating equation (9). By increasing the former and reducing the latter a desired difference tolerance between the two methods can be achieved. The same can be verified regardless of the input, dimensions of the body or the distribution assumed for λ .

[Figure 9 about here.]

Effect of the Uncertainty of λ in the estimate of the outcome.

The probability of failure of the body for a single cosinoidal pulse input, assuming truncated Gaussian distributions of λ corresponding to different values of μ and σ are compared in Figure 10. Increasing μ and keeping σ constant as in Figures 10(a) and 10(c) or 10(b) and 10(d), the area of points with probability of failure close to 100% increases. This is in agreement with what was presented in section Deterministic effect of λ on the stability of the body, as higher values of μ correspond to a higher expected value of λ . On the other hand, increasing σ and keeping μ constant, as in Figures 10(a) and 10(b) or 10(c) and 10(d), results in reducing the size of areas with an almost constant probability of failure.

In Figure 10 there are points whose probability of failure is very close to 50%. A probability of failure of 50% means that there is as much certainty about the outcome using the mode, as that which would occur by flipping a coin. Thus, there is a very large uncertainty to the model's prediction on whether the body is expect to survive or fail. The same practically holds for a range of probabilities close to 50%, e.g. 40 – 60%. Hence points in the diagram with probabilities of failure in that range should be further examined.

[Figure 10 about here.]

In attempting to understand the effect of the assumed distribution of λ on the region of points with the aforementioned range of probabilities of failure, the *CDFs* for assumed Truncated Gaussian distributions of λ occurring from different values of μ and σ are illustrated in Figure 11(a). The vertical axis represents the the probability of λ being greater than the value indicated in the horizontal axis. Those distributions are used in the stability analysis of a body with $\alpha = 0.2501$ and $p = 0.3218$ subjected to a single cycloidal pulse ground excitation with the results shown in 11(b). A point in the region of the stability diagram that satisfies the property of equation 9 would have a probability of failure of $X\%$, if $1 - CDF(\lambda^*) = X\%$. For the reasons explained earlier it is hence beneficial to define in Figure 11(a), the values of λ^* that would correspond to a probability of failure of 40% and 60%, the corresponding values of λ^* will be denoted as λ_{60}^* and λ_{40}^* , respectively.

In Figure 11(a) the values of λ_{40}^* and λ_{60}^* for each distribution are selected. The values are shown in Table 1. In Figure 11(b), the points which correspond

447 to λ^* values equal to λ_{60}^* and λ_{40}^* are represented with lines. These lines are
 448 also the boundaries of the points of this uncertain region between 40% and 60%
 449 probability of failure for each distribution. The dashed line represents the 60%
 450 boundaries and the continuous line the 40% boundary. In Table 1, for $\mu = 0.2$,
 451 the λ_{60}^* value decreases as σ is increased. However, for λ_{40}^* , the decrease is
 452 smaller. From Figure 7(b), it can be observed that the values of λ^* decrease
 453 when moving towards the bottom left corner, and increase while moving towards
 454 the upper right corner. The same behaviour is observed in the Figure 11(b). The
 455 60% line moves towards the bottom left corner as σ is increased and is almost
 456 constant for the 40% line. Hence, the region of the points with probabilities
 457 between 40% and 60% expands towards the bottom left corner. In Table 1,
 458 for $\mu = 0.8$, λ_{60}^* is almost constant whereas λ_{40}^* increases as σ is increased. In
 459 Figure 11(b), the uncertain region extends towards the upper right corner as σ is
 460 increased. In Table 1, for values of $\mu = 0.5$, the probability of λ^* increases and
 461 decreases symmetrically as the uniform distribution is approached, matching
 462 with what is observed in Figure 11(b) where the uncertain region expands in
 463 both directions as σ is increased.

464 It was shown in the previous paragraph, how one can determine the be-
 465 haviour of this uncertain region of points with probabilities between 40% and
 466 60%, using only the λ^* for a specific body and excitation and the CDF for the
 467 assumed distribution of λ .

468 [Table 1 about here.]

469 [Figure 11 about here.]

470 **Stability Analysis: Multiple-Pulses**

471 This section is an extension of single pulses to multiple pulses. To this end, the
 472 body is subjected to double and triple cycloidal pulses, by modifying equation
 473 (6) to:

$$\begin{aligned}
 \ddot{u} &= 0, & \text{if } t > N_c T \\
 \ddot{u} &= u_0 \mathbf{H}(\omega_u t), & \text{if } 0 \leq t \leq N_c T
 \end{aligned} \tag{16}$$

474 where N_c is the number of cycles of the pulse. Figure 12 illustrates areas of
 475 failure with impact for different values of λ subjected to double sinusoidal and
 476 cosinoidal pulses.

477 [Figure 12 about here.]

478 It should be noticed that not all points agree with the property described
 479 on (8). For example, in Figure 12(a) for a double sinusoidal pulse input some
 480 points fail for $\lambda = 0$ while they do not for greater values of λ . And in general,
 481 for a pair of u_0 and ω_u , there are points for which the body fails for a value
 482 of λ , λ_2 , and survives for another value of λ , λ_1 , even when $\lambda_1 > \lambda_2$. The

483 same phenomenon can be observed in Figure 12(b) for a double cosinoidal pulse
 484 input. For $\omega_u = 3.5p$ and $u_0 = 5\alpha g$, $\lambda = 0$ the body fails, whereas for $\lambda = 0.2$ it
 485 survives. Hence, for each ω_u and u_0 , there might be multiple transitions from
 486 survival to failure as the value of λ increases. This is illustrated in Figure 13.
 487 Those transitions essentially correspond to multiple roots in accordance to what
 488 was defined for single pulses in section .

489 Due to equation (8) not being applicable for this case, two methods are
 490 investigated for determining the probability of failure of the body: *i*) a multiroot
 491 finding method *ii*) Monte Carlo Simulations. The latter is practically the same
 492 as explained in section Probabilistic models for the effect of λ on the stability
 493 of the body, where the probability of failure for a point on the stability diagram
 494 is calculated as the sum of all the occurrences of failure over the total number
 495 of samples of λ used.

496 [Figure 13 about here.]

497 A multiroot finding method is used based on a modification of what was
 498 presented in for single pulses. This method determines for a pair of ω_u and u_0
 499 all the corresponding roots, i.e., values of λ , λ^* , for which there is a transition
 500 from survival (S) to failure (F) and vice-versa, as shown in Figure 13. Once all
 501 roots for a pair of values of ω_u and u_0 are found, the probability of survival or
 502 failure is determined depending on whether the body fails or survives for $\lambda = 0$:
 503 Following Figure 13, if the body fails for $\lambda = 0$, the probability of survivals is
 504 defined as in equation (17). Otherwise, the probability of survival is defined as
 505 in equation (18).

$$\begin{aligned} P_S &= P((0 < \lambda < \lambda_1^*) \cup (\lambda_2^* < \lambda < \lambda_3^*) \cup \dots) \\ &= P(0 < \lambda < \lambda_1^*) + P(\lambda_2^* < \lambda < \lambda_3^*) \dots \end{aligned} \quad (17)$$

$$\begin{aligned} P_S &= P((\lambda_1^* < \lambda < \lambda_2^*) \cup (\lambda_3^* < \lambda < \lambda_4^*) \cup \dots) \\ &= P(\lambda_1^* < \lambda < \lambda_2^*) + P(\lambda_3^* < \lambda < \lambda_4^*) \dots \end{aligned} \quad (18)$$

506 [Figure 14 about here.]

507 Figures 14(a) and 14(b) compares Monte Carlo simulations against the mul-
 508 tiroot finding method, for calculating the probability of failure for a body sub-
 509 jected to 2 cycoidal pulses, under the assumption that the values of λ following
 510 the uniform distribution. It can be seen that there is an excellent match between
 511 the two methods and the same can be observed for any underlying assumed
 512 distribution of λ or assumed multi-cycle cycloidal input. Using the multiroot
 513 method suggested, the probability of the body failing or surviving can be easily
 514 and quickly determined without the need for further dynamic simulations once
 515 the roots have been calculated, even if the assumed distribution of λ changes.
 516 Hence, Monte Carlo simulations and the associated computational cost may be
 517 avoided.

[Figure 15 about here.]

Comparing the single pulses against the multiple pulses, it can be noticed that similar trends in terms of the probabilities of failure can be observed. Despite the fact that equations 8 and 9 are not applicable anymore, the probability of failure still depends on the assumed distribution of λ as can be seen in Figures 15(a) and 15(b) where a uniform and Truncated Gaussian distribution of λ are compared when the body is subjected to three cosinoidal pulses.

Stability of Rocking Bodies Subjected to Simulated Earthquakes

The response of a rocking block subjected to cycloidal pulses was examined in sections Deterministic effect of λ on the stability of the body and Probabilistic models for the effect of λ on the stability of the body. However, ground motions cannot in generally be approximated with idealized pulses as the ones examined so far. Hence, there is a need to study the effect of the propagation of the uncertainty of λ on the stability of rocking bodies subjected to earthquakes. For the purpose of this study, artificially generated earthquakes are used. To this end the Spectral Representation method defined in (Shinozuka and Deodatis, 1991) is used together with the Kanai-Tajimi, Clough-Penzien spectrum (Clough and Penzien, 1975):

$$G(\omega) = S_0 \frac{\left(1 + 4\xi_g^2 \left(\frac{\omega}{\omega_g}\right)^2\right)}{\left(1 - \left(\frac{\omega}{\omega_g}\right)^2\right)^2 + 4\xi_g^2 \left(\frac{\omega}{\omega_g}\right)^2} \frac{\left(\frac{\omega}{\omega_f}\right)^4}{\left(1 - \left(\frac{\omega}{\omega_f}\right)^2\right)^2 + 4\xi_f^2 \left(\frac{\omega}{\omega_f}\right)^2} \quad (19)$$

where S_0 denotes the ground intensity, ω_g and ξ_g are the natural frequency and damping ratio of the ground, and ω_f and ξ_f are positive parameters related to a high-pass filter that ensures finite velocities when $\omega = 0$. To introduce the non-stationarity of the process, a time modulation function $\varphi(t)$ developed by Jennings et al. (1968) Jennings et al. (1968) is applied.

$$\varphi(t) = \begin{cases} t/t_1 & t < t_1 \\ 1 & t_1 \leq t \leq t_2 \\ e^{-\beta(t-t_2)} & t > t_2 \end{cases} \quad (20)$$

To generate the records the values used in equations 19 and 20 equation are defined as $\omega_g = 15\text{rad/sec}$, $\xi_g = 0.6$, $\omega_f = 1.5\text{rad/sec}$, $\xi_f = 0.6$, $t_1 = 1$, $t_2 = 9$ and $\beta = 0.2$, corresponding to a firm soil.

[Figure 16 about here.]

To quantify the risk of failure for the body subjected to the generated time histories, the probability of failure is calculated after varying $\sqrt{S_0}$ as a measure

548 of intensity and generating a suitable number of records. For this end, 10000
549 earthquakes are generated for $S_0 = 1$ and are later uniformly scaled by multi-
550 plying the occurring amplitudes with $\sqrt{S_0}$. The probability is calculated using
551 a Monte Carlo analysis as the sum of records for which failure occurs over the
552 total number of records. Depending on the assumed value of λ a different curve
553 is produced as shown in Figure 16.

554 As expected, the probability of failure, for a given λ , increases as the value
555 of the intensity measure, S_0 , increases. However, it should be noted that the
556 probability of failure also increases for a given S_0 when the assumed value of
557 λ is increased. This indicates once again that the use of zero values for λ may
558 potentially underestimate the probability of failure of the body, a conclusion
559 compatible to the results seen in earlier sections of this paper on cycloidal pulses.

560 Conclusions

561 This paper discusses the assumptions involved in modelling the loss of energy
562 during impact in the *IPM*, even if sliding, uplift, ground deformability and the
563 three dimensional nature of the response can indeed be neglected. It is shown
564 that the location of the vertical impulse during impact could be applied at any
565 point between the post-impact rocking corner of the body and the midpoint of
566 the base, without violating any of the assumptions of the model. The effect of
567 the vertical impulse acting at a distance λ (for λ between zero and one) from the
568 future rocking corner was investigated first in terms of the stability of a rocking
569 body subjected to single cycloidal pulses. It was illustrated that the assumption
570 for the value of λ significantly affects the stability of rocking bodies and that
571 the assumption of $\lambda = 0$, common in the literature, is the least conservative
572 assumption compatible with the assumptions of the *IPM*. A property of the
573 dynamic system was highlighted, where increasing the value of λ for a point
574 in the stability diagram results in expanding the area of failure with impact
575 monotonically.

576 Defining λ as random variable, the probability of failure of the body is
577 quantified. A probabilistic method based on the dynamic behavior of the system,
578 defined in equation 9, was introduced to calculate efficiently the probability
579 of failure of a rocking body. The method was verified against Monte-Carlo
580 Simulations. Using this method, it was shown that there may be points in the
581 stability diagram whose probability of failure is close to 50% and hence the
582 prediction of the model on whether the body will survive or fail is associated
583 with a large uncertainty. The size and location of this uncertain region was
584 shown to be dependent on the assumed distribution of λ .

585 The previous were then extended in terms of stability of a body subjected to
586 multiple pulses. A method was developed to calculate the corresponding prob-
587 ability of failure of the body by extending the probabilistic method presented
588 for the case of multiple transitions between survival and failure. It was again
589 shown that the uncertainty in λ has a significant effect in the stability of a rock-
590 ing body. The final example illustrated the dependence of fragility curves for

rocking bodies subjected to artificially generated earthquakes on the assumed value of λ , where again it was shown that the assumption of $\lambda = 0$ may result in underestimating the probability of failure for the body.

This paper shows the need for taking into account the uncertainty related to the assumption of how energy is lost during impacts, as expressed through the range of allowable values of λ that are compatible with the assumptions of the *IPM*. In order to reduce this uncertainty one could conduct experiments to further characterize a suitable distribution of λ , or mechanically modify the body to make the assumption of λ being zero more probable. This however would still not guarantee that λ would be fixed for that body and that for example would not depend on the value of the pre-impact velocities, an assumption that was retained in this paper. Finally, one would still need to take into account that there are several cases where the omitted phenomena of sliding, uplift, deformability and three dimensional nature of the response would affect the behaviour of the body by allowing for dynamic patterns that cannot be approximated from the *IPM*, through the use of any value of λ .

The latter assumptions can be relaxed with suitable models, as for example the spring models proposed in (Chatzis and Smyth, 2012*b*) and (Chatzis and Smyth, 2012*a*). Although such models do require the suitable identification of the parameters of the springs and dampers used to model the behaviour of the body, it should be noted that those could potentially be identified using system identification methods or through material models suitable for each application. In parallel the models would also be capable of taking into account phenomena omitted by the *IPM*. Hence, use of such models may be a viable alternative to the *IPM* and its inherent uncertainty illustrated in this paper.

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Figure 1 – Inverted Pendulum model defined by Housner Housner (1963) for a free-standing rocking block.

Figure 2 – It is assumed, with any loss of generality, that the impact occurs switching from Pattern A to B. At impact the body experiences an horizontal, J_x and vertical, J_y , impulses at λ .

Figure 3 – a) One of the infinitely many admissible distributions to describe the vertical impulse during impact. J_y is the relative vertical impulse. b) The red distribution corresponds to tensile stresses which is a violation of the non-tensile assumption.

Figure 4 – Schematic explanation of the time integration scheme used.

Figure 5 – Overturning Acceleration Spectrum of Free standing block subjected to a single sinusoidal (a, c), cosinoidal (b, d) and square (c, f) acceleration pulse with frequency ω_u , $p=2.1401$ and $\alpha = 0.2501$.

Figure 6 – Boundary regions which separate the area of failure with impact from the failure without impact and survival. The block is subjected to a single sinusoidal (a, d) cosinoidal (b, e) and (c, f) square acceleration pulse with a constant $p = 2.1401$ for $\alpha = 0.3218$ and for $\alpha = 0.2501$.

Figure 7 – Values of λ^* of a free standing block with a constant frequency parameter $p = 2.1401$ subjected to a single sinusoidal for a) $\alpha = 0.3218$ and d) $\alpha = 0.2501$, to a cosinoidal for b) $\alpha = 0.3218$ and e) $\alpha = 0.2501$ and to a square pulse for c) $\alpha = 0.2501$ and f) $\alpha = 0.1244$.

Figure 8 – Overturning Acceleration Spectrum for a single sinusoidal, cosinoidal and square pulse for values of $\lambda = 0$ and $\lambda = 1$ subjected to a rigid body with $p = 2.1401$ and $\alpha = 0.2501$.

Figure 9 – Probability of Failure for one cosinoidal pulse using (a) Monte Carlo Simulation and (b) the Probabilistic approach for a body with $p = 2.1401$ and $\alpha = 0.3218$.

Figure 10 – Probability of Failure for one single cosinoidal pulse using the truncated Gaussian distribution with a) $\mu = 0.2$ and $\sigma = 0.15$, b) $\mu = 0.2$ and $\sigma = 0.5$, c) $\mu = 0.8$ and $\sigma = 0.15$ and d) $\mu = 0.8$ and $\sigma = 0.5$ for a body with $p = 2.1401$ and $\alpha = 0.3218$.

Figure 11 – *CDF* (a) and λ^* boundaries for probabilities between 40–60% (b) for a rigid body subjected to a single cosinoidal pulse for a body with $p = 2.1401$ and $\alpha = 0.3218$.

Figure 12 – Boundaries region which separate the area of failure with impact from the failure without impact and survival for a body with $p = 2.1401$ and $\alpha = 0.3218$. The block is subjected to a) double sinusoidal pulse and b) double cosinoidal pulse.

Figure 13 – Graph explaining the two possible cases that can happen using the multiroot finding method.

Figure 14 – Probabilistic approach for the Uniform Distribution for two cosinoidal pulses using a) Monte Carlo simulation and b) Multiroots finding method for a body with $p = 2.1401$ and $\alpha = 0.3218$.

Figure 15 – Probabilistic approach for the Uniform Distribution for three cosinoidal pulses using the multiroot finding method using a) the uniform distribution and b) the Truncated Gaussian with $\mu = 0.2$, $\sigma = 0.15$ for a body with $p = 2.1401$ and $\alpha = 0.3218$.

Figure 16 – Probability of failure for different values of $\sqrt{S_0}$ using 10.000 artificial earthquakes for different values of λ for a body with $p = 2.1401$ and $\alpha = 0.3218$.

691 **List of Tables**

<small>692</small>	1	λ^* values for 40 – 60% for truncated distributions of different μ	
<small>693</small>		and σ	37

Table 1 – λ^* values for 40 – 60% for truncated distributions of different μ and σ .

μ	σ	λ_{40}^*	λ_{60}^*
0.2	0.06	0.2254	0.1781
	0.15	0.2116	0.1326
0.5	0.1	0.5260	0.4260
	0.25	0.5743	0.4750
0.8	0.06	0.8051	0.7949
	0.14	0.8604	0.7836