ON FINITENESS CONJECTURES FOR ENDOMORPHISM ALGEBRAS OF ABELIAN SURFACES

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Abstract. It is conjectured that there exist only finitely many isomorphism classes of endomorphism algebras of abelian varieties of bounded dimension over a number field of bounded degree. We explore this conjecture when restricted to quaternion endomorphism algebras of abelian surfaces of GL_2-type over $\mathbb{Q}$ by giving a moduli interpretation which translates the question into the diophantine arithmetic of Shimura curves embedded in Hilbert surfaces. We address the resulting problems on these curves by local and global methods, including Chabauty techniques on explicit equations of Shimura curves.

1. Introduction

Let A/K be an abelian variety defined over a number field K. For any field extension L/K, let $\mathrm{End}_L(A)$ denote the ring of endomorphisms of A defined over L and $\mathrm{End}_L^0(A) = \mathrm{End}_L(A) \otimes \mathbb{Q}$. A conjecture, which may be attributed to Robert Coleman[1], asserts the following.

Conjecture C(e, g) : *Let e, $g \geq 1$ be positive integers. Then, up to isomorphism, there exist only finitely many rings $\mathcal{O}$ over $\mathbb{Z}$ such that $\mathrm{End}_L(A) \simeq \mathcal{O}$ for some abelian variety A/K of dimension g and a field extension L/K of a number field K of degree $[K : \mathbb{Q}] \leq e$.*

The conjecture holds in dimension 1: by the theory of complex multiplication, if E/K is an elliptic curve and L/K is an extension of number fields, then $\mathrm{End}_L(E)$ is either $\mathbb{Z}$ or an order $\mathcal{O}$ in an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ such that the ring class field $H_{\mathcal{O}}$ attached to $\mathcal{O}$ is contained in $K(\sqrt{-d})$. Thus $[H_{\mathcal{O}} : \mathbb{Q}] \leq 2[K : \mathbb{Q}]$. Since $[H_{\mathcal{O}} : \mathbb{Q}] = 2h(\mathcal{O})$, where we let $h(\mathcal{O})$ denote the class number of $\mathcal{O}$, Conjecture C(h, 1) is now a consequence of the Brauer-Siegel Theorem [29, Chapter XVI, Theorem 3], which implies that for given $e \geq 1$, there exist finitely many imaginary quadratic orders $\mathcal{O}$ such that $h(\mathcal{O}) \leq e$.

Assuming the generalized Riemann hypothesis and using similar ideas, Greenberg announced [22] a generalization of the above statement to abelian varieties of arbitrary dimension $g \geq 1$ with multiplication by orders in complex multiplication fields of degree $2g$.

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[1]In a personal communication to the last author, Robert Coleman pointed out that this conjecture was posed by him during a lecture in a slightly weaker form.

Another instance that motivates Coleman's conjecture stems from the celebrated work of Mazur [31], as we now explain. Let E_1, $E_2/\mathbb{Q}$ be elliptic curves without CM over $\mathbb{Q}$ and $A = E_1 \times E_2$. Then, it is easily checked that $\mathrm{End}_{\mathbb{Q}}(A) \simeq \mathbb{Z}\times\mathbb{Z}$ if E_1 and E_2 are not isogenous over $\mathbb{Q}$, and $\mathrm{End}_{\mathbb{Q}}(A) \simeq M_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}), N \mid c\}$ if there is a cyclic isogeny of degree N between E_1 and E_2. By [31, Theorem 1] (for N prime) and the discussion on [31, p. 131] (for arbitrary N), this holds for only finitely many values of N.

As in [35, p. 191], we say that an abelian variety A defined over $\mathbb{Q}$ is *of* GL_2*-type over* $\mathbb{Q}$ if the endomorphism algebra $\mathrm{End}^0_{\mathbb{Q}}(A)$ is a number field E of degree $[E : \mathbb{Q}] = \dim A$. These abelian varieties had been introduced by Ribet in [36, p. 243] (in a slightly more general way) and this terminology is motivated by the fact that if E is a number field of degree $\dim A$ which is contained in $\mathrm{End}^0_{\mathbb{Q}}(A)$, then the action of $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ on the ℓ-adic Tate module associated with A defines a representation with values in $\mathrm{GL}_2(E \otimes \mathbb{Q}_\ell)$. According to [36, p. 244], E must be either a totally real or a complex multiplication number field.

An abelian variety A is called *modular over* $\mathbb{Q}$ if it is a quotient of the Jacobian variety $J_1(N)$ of the modular curve $X_1(N)$ defined over $\mathbb{Q}$. If moreover A is simple over $\mathbb{Q}$, its modularity over $\mathbb{Q}$ is equivalent to the existence of an eigenform $f \in S_2(\Gamma_1(N))$ such that A is isogenous over $\mathbb{Q}$ to the abelian variety A_f attached by Shimura to f. As is well-known, all simple modular abelian varieties A over $\mathbb{Q}$ are of GL_2-type over $\mathbb{Q}$ and the generalized Shimura-Taniyama-Weil Conjecture predicts that the converse is also true (cf. e. g. [35, p. 189]). As was shown by Ribet in [36, Theorem 4.4], this conjecture holds if Serre's Conjecture [44, Conjecture 3.2.4?] is assumed.

As we mentioned, Conjecture $\mathbf{C}(e, g)$ is settled for $e \geq 1$, $g = 1$. For the particular case $e = 1$, we have that if $E/\mathbb{Q}$ is an elliptic curve over $\mathbb{Q}$ and $L/\mathbb{Q}$ is a field extension, then $\mathrm{End}^0_L(E) = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7, 11, 19, 43, 67$ or 163. On the other hand, the case $g \geq 2$ is completely open. The aim of this article is to address the question for quaternion endomorphism algebras of abelian surfaces of GL_2-type over $\mathbb{Q}$.

In general, it is known that if A is an absolutely simple abelian surface of GL_2-type over $\mathbb{Q}$ then, for any number field L, the endomorphism algebra $\mathrm{End}^0_L(A)$ is either a real quadratic field or an indefinite division quaternion algebra over $\mathbb{Q}$ (cf. [35, Proposition 1.1, Theorem 1.2 and Proposition 1.3]).

We recall some basic facts on the arithmetic of quaternion algebras (cf. [49, Ch. I §1 and Ch. III, Theorem 3.1] for these and other details). A quaternion algebra B over $\mathbb{Q}$ is a central simple algebra B of rank 4 over $\mathbb{Q}$. For any $a, b \in \mathbb{Q}^*$, one can define the quaternion algebra $(\frac{a,b}{\mathbb{Q}}) = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij$, where $i^2 = a$, $j^2 = b$ and $ij = -ji$. Any quaternion algebra is isomorphic to $(\frac{a,b}{\mathbb{Q}})$ for some $a, b \in \mathbb{Q}^*$. The reduced discriminant of B is the square-free integer $D = \mathrm{disc}(B) := \prod p$, where p runs through the (finitely many) prime numbers such that $B \otimes \mathbb{Q}_p \not\simeq \mathrm{M}_2(\mathbb{Q}_p)$. Since for any square-free positive integer D there exists a (single up to isomorphism) quaternion algebra B with $\mathrm{disc}(B) = D$, we shall denote it by B_D. We have $D = 1$ for $B = \mathrm{M}_2(\mathbb{Q})$, and this is the only non division quaternion algebra over $\mathbb{Q}$. A quaternion algebra B over $\mathbb{Q}$ is called indefinite if $B \otimes \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ or, equivalently, if D is the product of an *even* number of prime numbers.

For $\alpha \in B$, let $\bar{\alpha}$ denote its conjugate and write $\mathrm{n} : B \to \mathbb{Q}$ and $\mathrm{tr} : B \to \mathbb{Q}$ for the reduced norm and the reduced trace on B, respectively. An order $\mathcal{O}$ in B is a subring of B of rank 4 over $\mathbb{Z}$ such that $\mathrm{n}(\alpha), \mathrm{tr}(\alpha) \in \mathbb{Z}$ for all $\alpha \in \mathcal{O}$. The order is maximal if it is not properly contained in any other order. If B_D is indefinite, there exists a single maximal order in B_D up to conjugation by elements in B_D^*, which we will denote by $\mathcal{O}_D$.

Definition 1.1. *Let $m > 1$ and $D = p_1 \cdots p_{2r}$ for some $r \geq 1$ be square-free integers and let B_D be a quaternion algebra over $\mathbb{Q}$ of discriminant D. We say the pair (D, m) is* modular *over $\mathbb{Q}$ if there exists a modular abelian surface $A/\mathbb{Q}$ such that*

$$\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) \simeq B_D \text{ and } \mathrm{End}^0_{\mathbb{Q}}(A) \simeq \mathbb{Q}(\sqrt{m}).$$

We say the pair (D, m) is premodular *over $\mathbb{Q}$ if there exists an abelian surface A of GL_2-type over $\mathbb{Q}$ such that*

$$\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) \simeq B_D \text{ and } \mathrm{End}^0_{\mathbb{Q}}(A) \simeq \mathbb{Q}(\sqrt{m}).$$

We state a particular consequence of Coleman's Conjecture separately.

Conjecture 1.2. *The set of premodular pairs (D, m) over $\mathbb{Q}$ is finite.*

It is worth noting that, given a fixed quaternion algebra B_D, there are infinitely many real quadratic fields $\mathbb{Q}(\sqrt{m})$ that embed in B_D, since any field $\mathbb{Q}(\sqrt{m})$ with m such that $(\frac{m}{p}) \neq 1$ for all $p|D$ does embed in B_D (cf. [49, Ch. III §5 C]). Thus, the finiteness of premodular pairs (D, m) over $\mathbb{Q}$ for a fixed D is also not obvious a priori.

A further motivation for Conjecture 1.2 is computational. Define the *minimal level* of a modular pair (D, m) as the minimal N such that there exists a newform $f \in S_2(\Gamma_0(N))$ with $(B_D, \mathbb{Q}(\sqrt{m})) \simeq (\mathrm{End}^0_{\overline{\mathbb{Q}}}(A_f), \mathrm{End}^0_{\mathbb{Q}}(A_f))$. The computations below are due to Koike and Hasegawa [23] for $N \leq 3000$. By means of Steins's program *Hecke* implemented in [30], we extended these computations for $N \leq 7000$.

Proposition 1.3. *The only modular pairs (D, m) of minimal level $N \leq 7000$ are:*

(D, m)	$(6, 2)$	$(6, 3)$	$(6, 6)$	$(10, 10)$	$(14, 7)$	$(15, 15)$	$(22, 11)$
N	675	1568	243	2700	1568	3969	5408

In Theorem 1.4 (iv) we show that the above are not the only examples of premodular pairs (D, m) over $\mathbb{Q}$. According to the generalized Shimura-Taniyama-Weil Conjecture in dimension two, these pairs should actually be *modular* pairs.

On the other hand, it is remarkable that not a single example of a pair (D, m) has ever been excluded from being modular or premodular over $\mathbb{Q}$. In this work we present the first examples, either obtained by local methods or by methods using global information, summarised in the following result, which we shall prove by the end of Section 6.

Theorem 1.4.

(i) *If (D, m) is a premodular pair over $\mathbb{Q}$, then $m = D$ or $m = \frac{D}{p}$ for some prime number $p \mid D$ which does not split in $\mathbb{Q}(\sqrt{D/p})$.*

(ii) *Let p, q be odd prime numbers. If $(\frac{q}{p}) = 1$ or $p \equiv 1 \bmod 12$ or $p \equiv q \equiv 1 \bmod 4$, then $(p \cdot q, q)$ is not premodular over $\mathbb{Q}$.*

(iii) *The pairs*

$$(D,m) \in \{(10,2),(15,3),(15,5),(21,3),(26,2),(26,13),(33,11),(38,2),(38,19),(46,23),(51,3),(58,2),(91,91),(106,53),(115,23),(118,59),(123,123),(142,2),(155,5),(155,31),(155,155),(158,158),(159,3),(202,101),(215,43),(326,326),(446,446),(591,3),(1247,43)\}$$

are not premodular over $\mathbb{Q}$.

(iv) *The pairs*

$$(D,m) \in \{(6,2),(6,3),(6,6),(10,5),(10,10),(14,7),(14,14),(15,15),(21,21),(22,2),(22,11),(22,22),(33,33),(34,34),(46,46),(26,26),(38,38),(58,29),(58,58)\}$$

are premodular over $\mathbb{Q}$.

(v) *Let* $D > 546$. *Then there exist only finitely many* $\overline{\mathbb{Q}}$*-isomorphism classes of abelian surfaces* A *of* GL_2*-type over* $\mathbb{Q}$ *such that* $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_D$.

(vi) *For the pairs*

$$(D,m) \in \{(6,2),(6,3),(6,6),(10,5),(10,10),(14,14),(15,15),(21,21),(22,2),(22,11),(22,22),(33,33),(34,34),(46,46)\},$$

there exist infinitely many $\overline{\mathbb{Q}}$*-nonisomorphic abelian surfaces* A *defined over* $\mathbb{Q}$ *such that* $\mathrm{End}^0_{\mathbb{Q}}(A) \simeq \mathbb{Q}(\sqrt{m})$ *and* $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_D$.

(vii) *Up to isomorphism over* $\overline{\mathbb{Q}}$, *there exist exactly two abelian surfaces* $A/\mathbb{Q}$ *with* $\mathrm{End}^0_{\mathbb{Q}}(A) \simeq \mathbb{Q}(\sqrt{7})$ *and* $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_{14}$.

All pairs (D,m) for $D \leq 34$ are covered by Theorem 1.4. As a particularly interesting example, we obtain that there exists no abelian surface A of GL_2-type over $\mathbb{Q}$ such that $\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) \simeq B_{155}$: indeed, by Theorem 1.4 (i) and (iii) none of the pairs $(155,m)$ are premodular over $\mathbb{Q}$.

Note also that it follows from Theorem 1.4 (ii) and the Čebotarev Density Theorem that there actually exist infinitely many pairs $(p\cdot q, q)$ which are not premodular over $\mathbb{Q}$.

The strategy followed in this paper is to prove that the condition for a pair (D,m) to be premodular over $\mathbb{Q}$ is equivalent to the existence of a point in a suitable subset of the set of rational points on an Atkin-Lehner quotient of the Shimura curve canonically attached to $\mathcal{O}_D$.

The article and the proof of Theorem 1.4 are organized as follows: In Section 2 we introduce Shimura curves, Hilbert surfaces and forgetful maps between them. In Section 3 we use the diophantine local properties of Shimura curves to prove parts (i) and (ii) of Theorem 1.4 as a combination of Theorem 3.2 and Proposition 3.4. In Section 4 we prove a descent result on the field of definition of abelian surfaces with quaternionic multiplication. In Corollary 4.10, we show how part (v) follows from our results combined with the work in [38]. Finally, in Sections 5 and 6 we prove the remaining parts of Theorem 1.4 by means of explicit computations and Chabauty techniques on explicit equations of Shimura curves.

2. Towers of Shimura curves and Hilbert surfaces

We recall some basic facts on Shimura varieties and particularly on Shimura curves and Hilbert surfaces. Our main references are [32, Sections 1 and 2], [1, Ch. III] and [14, Sections 1, 2 and 3]. Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ be the algebraic group

over $\mathbb{R}$ obtained by restriction of scalars of the multiplicative group. A Shimura datum is a pair (G, X), where G is a connected reductive affine algebraic group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class in the set of morphisms of algebraic groups $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$, as in [32, Definition 1.4].

Let $\mathbb{A}_f$ denote the ring of finite adeles of $\mathbb{Q}$. As in [32, Section 1.5], for any compact open subgroup U of $G(\mathbb{A}_f)$, let

$$\mathrm{Sh}_U(G, X)(\mathbb{C}) = G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f))/U,$$

which has a natural structure of quasi-projective complex algebraic variety, that we may denote by $\mathrm{Sh}_U(G, X)_{\mathbb{C}}$.

Let (G, X) and (G', X') be two Shimura data and let U, U' be compact open subgroups of $G(\mathbb{A}_f)$ and $G'(\mathbb{A}_f)$, respectively. A morphism $f : G \to G'$ of algebraic groups which maps X into X' and U into U' induces a morphism

$$\mathrm{Sh}_f : \mathrm{Sh}_U(G, X)_{\mathbb{C}} \to \mathrm{Sh}_{U'}(G', X')_{\mathbb{C}}$$

of algebraic varieties (cf. [32, Section 1.6.3]).

In this section, we consider two particular instances of Shimura varieties: Shimura curves attached to an indefinite quaternion algebra and Hilbert surfaces attached to a real quadratic number field.

2.1. **Shimura curves.** Let B_D be an indefinite quaternion algebra over $\mathbb{Q}$ of reduced discriminant D and fix an isomorphism $\Phi : B_D \otimes \mathbb{R} \xrightarrow{\simeq} \mathrm{M}_2(\mathbb{R})$. Let $\mathcal{O}_D \subset B_D$ be a maximal order and let $G/\mathbb{Z}$ be the group scheme $\mathcal{O}_D^*$. We have that $G(\mathbb{Q}) = B_D^*$ and $G(\mathbb{A}_f) = \prod_p \mathcal{O}_{D,p}^*$, where for any prime number p, we let $\mathcal{O}_{D,p} = \mathcal{O}_D \otimes \mathbb{Z}_p$. Let $X = \mathbb{H}^{\pm}$ be the $\mathrm{GL}_2(\mathbb{R})$-conjugacy class of the map $a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. As complex analytical spaces, $\mathbb{H}^{\pm}$ is the union of two copies of Poincaré's upper half plane $\mathbb{H}$.

For any compact open subgroup U of $G(\mathbb{A}_f)$, let $X_{U,\mathbb{C}} = \mathrm{Sh}_U(G, X)_{\mathbb{C}}$ be the Shimura curve attached to the Shimura datum (G, X) and U. It is the union of finitely many connected components of the form $\Gamma_i \setminus \mathbb{H}$, where Γ_i are discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$.

Fix a choice of an element $\mu \in \mathcal{O}_D$ such that $\mu^2 + \delta = 0$ for some $\delta \in \mathbb{Q}^*, \delta > 0$ and let $\varrho_\mu : B_D \to B_D$, $\beta \mapsto \mu^{-1}\bar{\beta}\mu$. For any scheme S over $\mathbb{C}$, let $\mathcal{F}_{U,\mu}(S)$ be the set of isomorphism classes of $(A, \iota, \nu, \mathcal{L})$, where A is an abelian scheme over S, $\iota : \mathcal{O}_D \hookrightarrow \mathrm{End}_S(A)$ is a ring monomorphism, ν is an U-level structure on A and $\mathcal{L}$ is a polarization on A such that the Rosati involution $* : \mathrm{End}_S^0(A) \to \mathrm{End}_S^0(A)$ is ϱ_μ on B_D (cf. [1, p. 128]). As is well known, $X_{U,\mathbb{C}}$ coarsely represents the moduli functor $\mathcal{F}_{U,\mu}$.

A point $[A, \iota, \nu, \mathcal{L}] \in X_{U,\mathbb{C}}(\mathbb{C})$ is called a *Heegner point* or a *CM point* if ι is not surjective or, equivalently, if A is isogenous to the square of an elliptic curve with complex multiplication (cf. [25, pp. 16-17], [40, Definition 4.3]).

The modular interpretation implies that the reflex field of the Shimura datum (G, X) is $\mathbb{Q}$ and that $X_{U,\mathbb{C}}$ admits a canonical model $X_{U,\mathbb{Q}}$ over $\mathbb{Q}$, which is the coarse moduli space for any of the above moduli functors $\mathcal{F}_{U,\mu}$ extended to arbitrary bases over $\mathbb{Q}$ (cf. [1, Ch. III, 1.1-1.4], [32, Section 2]). The isomorphism class of the algebraic curve $X_{U,\mathbb{Q}}$ does not depend on the choice of $\mu \in \mathcal{O}_D$, although its moduli interpretation does depend on μ. This is due to the following remarkable property: given a triplet (A, ι, ν) as above, each choice of an element $\mu \in \mathcal{O}_D$, $\mu^2 + \delta = 0$,

$\delta > 0$, determines a single reduced polarization $\mathcal{L}_\mu$ compatible with (A, ι, ν). Given $\mu_1, \mu_2 \in \mathcal{O}_D$, $\mu_i^2 + \delta_i = 0$, $\delta_i > 0$ for $i = 1, 2$, the natural isomorphism between $\mathcal{F}_{U,\mu_1}$ and $\mathcal{F}_{U,\mu_2}$ is provided by the map $[A, \iota, \nu, \mathcal{L}_{\mu_1}] \mapsto [A, \iota, \nu, \mathcal{L}_{\mu_2}]$.

As a particular case, let $\mathcal{O} \subset \mathcal{O}_D$ be an integral order contained in $\mathcal{O}_D$, and let $\hat{\mathcal{O}}^* = \prod_p \mathcal{O}_p^*$. Let us simply denote by $X_{\mathcal{O},\mathbb{Q}}$ the Shimura curve $X_{U,\mathbb{Q}}$ for $U = \hat{\mathcal{O}}^*$. Again, for any fixed $\mu \in B_D^*$, $\mu^2 + \delta = 0$, it admits the following alternative modular interpretation: $X_{\mathcal{O},\mathbb{Q}}$ coarsely represents the functor $\hat{\mathcal{F}}_{\mathcal{O},\mu} : \mathrm{Sch}/\mathbb{Q} \to \mathrm{Sets}$, sending a scheme S over $\mathbb{Q}$ to the set of isomorphism classes of triplets $(A, \iota, \mathcal{L})$, where $(A, \mathcal{L})$ is a polarized abelian scheme over S as above and $\iota : B_D \hookrightarrow \mathrm{End}_S^0(A)$ is a ring monomorphism such that $\iota(B_D) \cap \mathrm{End}_S(A) = \iota(\mathcal{O})$.

The Atkin-Lehner group of $X_{\mathcal{O},\mathbb{Q}}$ is the normalizer $W_\mathcal{O} = \mathrm{Norm}_{B_D^*}(\mathcal{O})/\mathbb{Q}^* \cdot \mathcal{O}^*$. There is a natural action of $W_\mathcal{O}$ on the functor $\hat{\mathcal{F}}_{\mathcal{O},\mu}$: for any $[\omega] \in W_\mathcal{O}$, we have $\omega : \hat{\mathcal{F}}_{\mathcal{O},\mu}(S) \to \hat{\mathcal{F}}_{\mathcal{O},\mu}(S)$, $(A, \iota, \mathcal{L}) \mapsto (A, \omega^{-1}\iota\omega, \mathcal{L}_\omega)$, where $\mathcal{L}_\omega$ denotes the single reduced polarization compatible with $(A, \omega^{-1}\iota\omega)$. This action induces a natural immersion $W_\mathcal{O} \subseteq \mathrm{Aut}_\mathbb{Q}(X_{\mathcal{O},\mathbb{Q}})$ (cf. [25, Proposition 1.2.6]).

When $\mathcal{O}_D$ is a maximal order in B_D, $W_{\mathcal{O}_D} \simeq (\mathbb{Z}/2\mathbb{Z})^{2r}$, where $2r = \#\{p \text{ prime} : p|D\}$ is the number of ramified primes of D. A full set of representatives of $W_{\mathcal{O}_D}$ is $\{\omega_m : m|D, m > 0\}$, where $\omega_m \in \mathcal{O}_D$, $\mathrm{n}(\omega_m) = m$. As elements of $W_{\mathcal{O}_D}$, these satisfy $\omega_m^2 = 1$ and $\omega_m \cdot \omega_n = \omega_{mn}$ for any two coprime divisors m, $n|D$ (cf. [25, Proposition 1.2.4], [38, Section 1]).

2.2. **Hilbert surfaces.** Let F be a real quadratic extension of $\mathbb{Q}$, let R_F be its ring of integers and let G be the $\mathbb{Z}$-group scheme $\mathrm{Res}_{R_F/\mathbb{Z}}(\mathrm{GL}_2(R_F))$. Since $F \otimes \mathbb{R} \simeq \mathbb{R}^2$, we have $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$. Let $X = \mathbb{H}^\pm \times \mathbb{H}^\pm$. For any compact open subgroup U of $G(\mathbb{A}_f)$, let $H_{U,\mathbb{C}} = \mathrm{Sh}_U(G, X)_\mathbb{C}$ be the Hilbert surface attached to the Shimura datum (G, X) and U as in the first paragraph of [14, Section 2].

The Hilbert surface $H_{U,\mathbb{C}}$ admits, in the same way as $X_{U,\mathbb{C}}$, a canonical model $H_{U,\mathbb{Q}}$ over $\mathbb{Q}$ which is the coarse moduli space of abelian surfaces $(A, j, \nu, \mathcal{L})$ together with a ring homomorphism $j : R_F \to \mathrm{End}(A)$, a U-level structure and a polarization $\mathcal{L}$ on A such that $*_{|j(R_F)}$ is the identity map. When U is the restriction of scalars of $\mathrm{GL}_2(\hat{R})$ for a given quadratic order $R \subseteq R_F$, we write $H_{R,\mathbb{Q}} := \mathrm{Sh}_U(G, X)_\mathbb{Q}$. As in the Shimura curve case, $H_{R,\mathbb{Q}}$ can also be regarded as the coarse moduli space of polarized abelian surfaces with real multiplication by R and no level structure (cf. [14, 3.1]).

A point $P \in H_{U,\mathbb{C}}(\mathbb{C})$ is called a *Heegner point* or a *CM point* if the underlying abelian surface A has complex multiplication in the sense of Shimura-Taniyama (cf. [14, Definition 1.2 and Lemma 6.1]): the endomorphism algebra $\mathrm{End}_\mathbb{C}^0(A)$ contains a quartic CM-field.

2.3. **Forgetful maps.** We consider various forgetful maps between Shimura curves and Hilbert surfaces with level structure.

For any integral quaternion order $\mathcal{O}$ of B_D, let $\hat{\mathcal{O}}^* \subseteq \hat{\mathcal{O}}_D^*$ be the natural inclusion of compact groups. The identity map on the Shimura data $(\mathcal{O}_D^*, \mathbb{H}^\pm)$ induces a morphism

$$X_{\mathcal{O},\mathbb{Q}} \longrightarrow X_{\mathcal{O}_D,\mathbb{Q}}$$

which can be interpreted in terms of moduli as forgetting the level structure: $[A, \iota, \nu, \mathcal{L}] \mapsto [A, \iota, \mathcal{L}]$.

Similarly, for any quadratic order R of F, there is a natural morphism

$$H_{R,\mathbb{Q}} \longrightarrow H_{R_F,\mathbb{Q}}.$$

Finally, let $R \subset \mathcal{O}$ be a real quadratic order *optimally embedded* in $\mathcal{O}$, which means that $R = F \cap \mathcal{O}$, and fix an element $\mu \in B_D^*$, $\mu^2 + \delta = 0$, $\delta \in \mathbb{Q}^*$, $\delta > 0$ symmetric with respect to R (that is, $\varrho_\mu | R = 1_R$). Regard $X_{\mathcal{O},\mathbb{Q}}$ as representing the moduli functor $\hat{\mathcal{F}}_{\mathcal{O},\mu}$. Attached to the pair (R,μ) there is a distinguished forgetful morphism

$$\begin{array}{rccc} \pi_{(R,\mu)}: & X_{\mathcal{O},\mathbb{Q}} & \longrightarrow & H_{R,\mathbb{Q}} \\ & [A, \iota : \mathcal{O} \to \mathrm{End}(A), \mathcal{L}] & \mapsto & [A, \iota_{|R} : R \to \mathrm{End}(A), \mathcal{L}] \end{array}$$

of Shimura varieties which consists on forgetting the ring endomorphism structure in the moduli interpretation of these varieties.

Let R' be a quadratic order of F optimally embedded in $\mathcal{O}_D$. Writing $R = R' \cap \mathcal{O}$, we obtain the following commutative diagram.

$$\begin{array}{ccc} X_{\mathcal{O},\mathbb{Q}} & \longrightarrow & X_{\mathcal{O}_D,\mathbb{Q}} \\ \pi_{(R,\mu)} \downarrow & & \downarrow \pi_{(R',\mu)} \\ H_{R,\mathbb{Q}} & \longrightarrow & H_{R',\mathbb{Q}} \end{array}$$

The main consequence we wish to derive from the above is simply a translation into terms of moduli of the problem posed in Section 1.

Proposition 2.1. *Let B_D be an indefinite division quaternion algebra over $\mathbb{Q}$, let $\mathcal{O}_D$ be a maximal order and let $F = \mathbb{Q}(\sqrt{m})$ for some square-free integer $m > 1$.*

Assume that, for any order R of F, optimally embedded in $\mathcal{O}_D$, and $\mu \in B_D^*$ symmetric with respect to R, the set of rational points of $\pi_{(R,\mu)}(X_{\mathcal{O}_D,\mathbb{Q}})$ in the Hilbert surface $H_{R,\mathbb{Q}}$ consists entirely of Heegner points. Then, the pair (D,m) is not premodular over $\mathbb{Q}$.

Proof. Let $A/\mathbb{Q}$ be an abelian surface such that $\mathrm{End}^0_{\mathbb{Q}}(A) = F = \mathbb{Q}(\sqrt{m})$ and $\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) = B_D$. Let $R = \mathrm{End}_{\mathbb{Q}}(A)$ and $\mathcal{O} = \mathrm{End}_{\overline{\mathbb{Q}}}(A)$, which we will regard as an order in F and an order in B_D respectively. By construction, the order R is optimally embedded in $\mathcal{O}$. Since A is projective over $\mathbb{Q}$, it admits a (possibly non-principal) polarization $\mathcal{L}$ defined over $\mathbb{Q}$. Let $*$ denote the Rosati involution on B_D induced by $\mathcal{L}$. By [41, Theorem 1.2 (4)], we have $* = \varrho_\mu$ for some $\mu \in B_D^*$ with $\mu^2 + \delta = 0$ for some $\delta \in \mathbb{Q}^*$, $\delta > 0$. By choosing an explicit isomorphism $\iota : \mathcal{O} \xrightarrow{\sim} \mathrm{End}_{\overline{\mathbb{Q}}}(A)$, the triplet $(A, \iota, \mathcal{L})$ produces a point P in $X_{\mathcal{O},\mathbb{Q}}(\overline{\mathbb{Q}})$, when we regard the Shimura curve as coarsely representing the functor $\hat{\mathcal{F}}_{\mathcal{O},\mu}$.

Moreover, we have $\iota_{|R} : R \simeq \mathrm{End}_{\mathbb{Q}}(A)$. From the fact that $\mathcal{L}$ is defined over $\mathbb{Q}$, it follows that $*_{|R}$ is an anti-involution on R. Since R is totally real, it follows that $*_{|R}$ is the identity. Hence, the point P is mapped to a point $P_R \in H_{R,\mathbb{Q}}(\mathbb{Q})$ by the forgetful map $\pi_{(R,\mu)} : X_{\mathcal{O},\mathbb{Q}} \to H_{R,\mathbb{Q}}$.

Let $\mathcal{O}_D$ be a maximal order in B_D containing $\mathcal{O}$ and let $R' = F \cap \mathcal{O}_D$, where we regard $F = R \otimes \mathbb{Q}$ as naturally embedded in B_D. By the above commutative diagram of Shimura varieties, we obtain a point $P_{R'} \in H_{R',\mathbb{Q}}(\mathbb{Q})$ which lies in the image of the forgetful map $X_{\mathcal{O}_D,\mathbb{Q}} \to H_{R',\mathbb{Q}}$.

Since $\mathrm{End}^0_{\mathbb{Q}}(A) \simeq B_D$ is a quaternion algebra, it contains no quartic CM-fields and thus $P_{R'}$ is not a Heegner point. This proves the proposition. □

The relevance of Proposition 2.1 to our problem is the following. Firstly, it translates the condition for a pair (D, m) to be premodular over $\mathbb{Q}$ into the existence of a suitable rational point on a projection of a Shimura curve. Secondly, note that (D, m) is a premodular pair over $\mathbb{Q}$ if there exists an abelian surface $A/\mathbb{Q}$ such that $\mathrm{End}_{\mathbb{Q}}(A)$ is an order in $\mathbb{Q}(\sqrt{m})$ and $\mathrm{End}_{\overline{\mathbb{Q}}}(A)$ is an order in the quaternion algebra B_D. Proposition 2.1 reduces our problem to study the set of rational points on the Shimura curve $X_{\mathcal{O}_D,\mathbb{Q}}$ attached to a *maximal* order in B_D. These curves have been extensively studied, rather than the more general curves $X_{\mathcal{O}}$ attached to an arbitrary quaternion order.

3. Atkin-Lehner quotients of Shimura curves

Fix a maximal order $\mathcal{O}_D$ in an indefinite division quaternion algebra B_D of discriminant D and let us simply denote $X_D = X_{\mathcal{O}_D,\mathbb{Q}}$. As explained in Section 2.1, X_D is equipped with the Atkin-Lehner group of involutions $W_D = \{\omega_m : m \mid D\} \subseteq \mathrm{Aut}_{\mathbb{Q}}(X_D)$. For $m \mid D$, let $X_D^{(m)}$ be the quotient curve $X_D/\langle\omega_m\rangle$ and $\pi_m : X_D \to X_D^{(m)}$ the natural projection map.

For any extension field $K/\mathbb{Q}$, let $X_D(K)_h$ denote the subset of Heegner points of $X_D(K)$ and let $X_D(K)_{nh} = X_D(K)\backslash X_D(K)_h$ the set of non-Heegner points over K. Similarly, set $X_D^{(m)}(K)_h = \pi_m(X_D(\overline{\mathbb{Q}})_h) \cap X_D^{(m)}(K)$ and $X_D^{(m)}(K)_{nh} = X_D^{(m)}(K)\backslash X_D^{(m)}(K)_h$.

Proposition 3.1. *Let X_D be the Shimura curve of discriminant D attached to the maximal order $\mathcal{O}_D$ as above. Then,*

(i) $X_D(\mathbb{R}) = \emptyset$.
(ii) *There exists no abelian surface $A/\mathbb{R}$ such that* $\mathrm{End}_{\mathbb{R}}(A) \supseteq \mathcal{O}_D$.

Proof. (i) is [47, Theorem 0] when particularized to Shimura curves. (ii) follows from (i) and the moduli interpretation of X_D described in Section 2.1. □

Theorem 3.2. *Let $m > 1$ be a square-free integer. Assume that the pair (D, m) is premodular over $\mathbb{Q}$. Then,*

(i) *$m|D$ and all prime divisors $p|\frac{D}{m}$ do not split in $\mathbb{Q}(\sqrt{m})$.*
(ii) $X_D^{(m)}(\mathbb{Q})_{nh} \neq \emptyset$.

Proof. Assume that (D, m) is premodular over $\mathbb{Q}$. By Proposition 2.1, there exists an order R of $F = \mathbb{Q}(\sqrt{m})$ optimally embedded in $\mathcal{O}_D$ and $\mu \in \mathcal{O}_D$, $\mu^2 + \delta = 0$, $\delta > 0$, symmetric with respect to R such that the set of rational points of $\pi_{(R,\mu)}(X_{\mathcal{O}_D,\mathbb{Q}})$ in the Hilbert surface $H_{R,\mathbb{Q}}$ contains a non-Heegner point.

Assume first that $m \nmid D$. It was shown in [40, Theorem 4.4], (cf. also [41, Section 6] when $\delta = D$) that there is then a birational equivalence

$$\pi_{(R,\mu)}(X_{\mathcal{O}_D,\mathbb{Q}}) \overset{\sim}{\dashrightarrow} X_D.$$

This birational morphism is defined over $\mathbb{Q}$ and becomes a regular isomorphism when restricted to the set of non-Heegner points. By Proposition 3.1, we obtain a contradiction.

Assume now that $m|D$. Since the anti-involution ϱ_μ restricts to the identity map on R, we have that $B_D \simeq (\frac{-\delta,m}{\mathbb{Q}})$. Again, it follows from [40, Theorem 4.4], that

there is a birational equivalence

$$\pi_{(R,\mu)}(X_{\mathcal{O}_D,\mathbb{Q}}) \overset{\sim}{\dashrightarrow} X_D^{(m)}$$

which is defined over $\mathbb{Q}$ and that becomes a regular isomorphism when restricted to the set of non-Heegner points. We conclude that $X_D^{(m)}(\mathbb{Q})$ must contain a non-Heegner point. Moreover, since F must embed in B, [49, Ch. III §5 C] applies to ensure that all primes $p|D$ do not split in F. □

As an immediate consequence of (i), we obtain the following corollary.

Corollary 3.3. *Given a discriminant D of a division quaternion algebra over $\mathbb{Q}$, the set of modular pairs (D, m) is finite.*

In view of Theorem 3.2, the diophantine properties of these curves are crucial for the understanding of Conjecture 1.2. We first study under what circumstances the curve $X_D^{(m)}$ has no points over some completion of $\mathbb{Q}$.

Proposition below is [43, Theorem 2.7]. Part (i) has also been shown in [8] by using supersingular abelian surfaces.

Proposition 3.4. *Let $X_D^{(m)}$ be as above. Then $X_D^{(m)}(\mathbb{Q}_v) \neq \emptyset$ for all places v of $\mathbb{Q}$ if and only if one of the following conditions holds:*

- *(i)* $m = D$
- *(ii)* $m = D/\ell$ *for a prime* $\ell \neq 2$ *such that*
  - $(\frac{m}{\ell}) = -1$*;*
  - (a) $(\frac{-m}{\ell}) = 1$, $(\frac{-\ell}{p}) \neq 1$ *for all primes* $p \mid m$*, or* (b) $\ell \equiv 1 \bmod 4$, $p \not\equiv 1 \bmod 4$ *for all primes* $p \mid m$*;*
  - *if* $r \geq 2$*, then we have* $(\frac{-m/p}{\ell}) = -1$ *for all odd primes* $p \mid m$*, and if* $2|D$ *we also have either* $(\frac{-m/2}{\ell}) = -1$*, or* $q \equiv 3 \bmod 4$ *for all primes* $q \mid D/2$*;*
  - *for every prime* $p \nmid D$*,* $p < 4g^2$*, there exists some imaginary quadratic field that splits B_D and contains an integral element of norm p or pm.*
- *(iii)* $m = D/2$ *such that*
  - $m \not\equiv 1 \bmod 8$*;*
  - $p \equiv 3 \bmod 4$ *for all* $p \mid m$*, or* $p \equiv 5$ *or* $7 \bmod 8$ *for all* $p \mid m$*;*
  - *if* $r \geq 2$*, then for every prime* $p \mid m$ *we have* $m/p \not\equiv -1 \bmod 8$*;*
  - *for every prime* $p \nmid D$*,* $p < 4g^2$*, there exists some imaginary quadratic field that splits B_D and contains an integral element of norm p or pm.*

As a direct consequence of the combination of Theorem 3.2 and Proposition 3.4, we obtain Theorem 1.4 (i). When $D = p \cdot q$ is the product of two primes, the congruence conditions of parts (ii) and (iii) above simplify notably, as we state in Theorem 1.4 (ii).

As a consequence of part (i), we conclude that global methods alone may enable us to prove that any pair (D, D) is not premodular. We also note that the last item of parts (ii) and (iii) of Proposition 3.4 allows us to produce isolated examples of pairs like $(159, 3), (215, 43), (591, 3)$ and $(1247, 43)$ which are not premodular over $\mathbb{Q}$ and are not covered by Theorem 1.4 (ii).

Finally we remark that [43] studies the failure of the Hasse principle for curves $X_{pq}^{(q)}$ over $\mathbb{Q}$ for suitable collections of pairs of primes p, q. As an example, it is shown in [43, Section 3] that $X_{23\cdot 107}^{(107)}(\mathbb{Q}) = \emptyset$ although it does have rational points

everywhere locally. Hence, we obtain that $(23 \cdot 107, 107)$ is not premodular over $\mathbb{Q}$ but this can not be derived from Theorem 1.4 (ii).

4. A descent theorem on rational models of abelian surfaces with quaternionic multiplication

Let A/K be an abelian variety defined over a number field K and let $\mathcal{L}$ be a polarization on it. Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$ containing K. Let K_0 be the field of moduli of $(A, \mathcal{L})$, that is, the minimal subfield K_0 of K such that for each $\sigma \in \mathrm{Gal}\,(\overline{\mathbb{Q}}/K_0)$ there exists an isomorphism $\mu_\sigma : (A^\sigma, \mathcal{L}^\sigma) \to (A, \mathcal{L})$ of polarized abelian varieties over $\overline{\mathbb{Q}}$. A similar definition works for a triplet $(A, \iota, \mathcal{L})$ where $\iota : R \hookrightarrow \mathrm{End}(A)$ is a monomorphism of rings for a given ring R, by asking the isomorphisms μ_σ to be compatible with the action of R on A (cf. [42, Section 1], for more details).

The following result is due to Weil [50, Theorem 3]. See also the first paragraph of [36, Section 8] for the specific statement for abelian varieties and a generalization in the category of abelian varieties up to isogeny.

Proposition 4.1. *A polarized abelian variety $(A, \mathcal{L})/K$ admits a model over its field of moduli K_0 if and only if for each $\sigma \in \mathrm{Gal}\,(\overline{\mathbb{Q}}/K_0)$ there exists an isomorphism $\mu_\sigma : (A^\sigma, \mathcal{L}^\sigma) \to (A, \mathcal{L})$ such that $\mu_\sigma \mu_\tau^\sigma = \mu_{\sigma\tau}$ for any $\sigma, \tau \in \mathrm{Gal}\,(\overline{\mathbb{Q}}/K_0)$.*

Let B_D be an indefinite division quaternion algebra of reduced discriminant $D = p_1 \cdots p_{2r}$, $r \geq 1$, and let $\mathcal{O}_D$ be a maximal order in B_D. Let $m \geq 1$, $m \mid D$.

Lemma 4.2. *Let $(A, \mathcal{L})/K$ be a polarized abelian surface such that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_D$. Then $\mathrm{Aut}_{\overline{\mathbb{Q}}}(A, \mathcal{L}) = \{\pm 1\}$.*

Proof. Since $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_D$, it follows that $\mathrm{Aut}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_D^* = \{\alpha \in \mathcal{O}_D : n(\alpha) = \pm 1\}$, which is an infinite group (cf. [49, Ch. IV, Theorem 1.1]). Theorem 2.2 in [39] attaches an element $\mu(\mathcal{L}) \in B_D$ to the polarization $\mathcal{L}$ and shows that $\mathrm{Aut}_{\overline{\mathbb{Q}}}(A, \mathcal{L}) \simeq \{\alpha \in \mathcal{O}_D^* : \bar{\alpha}\mu(\mathcal{L})\alpha = \mu(\mathcal{L})\}$ (cf. also [41, Theorem 1.2 (1)]). By [41, Theorem 1.2 (2-3)], $\mu(\mathcal{L})^2 + d = 0$ for some $d \in \mathbb{Z}$, $d \geq D$.

Let $\alpha \in \mathrm{Aut}_{\overline{\mathbb{Q}}}(A, \mathcal{L})$. We first observe that $n(\alpha) = 1$: Indeed, if $n(\alpha) = -1$ then $\bar{\alpha} = -\alpha^{-1}$ and thus $\mu(\mathcal{L})\alpha = -\alpha\mu(\mathcal{L})$. This implies that $B_D \simeq (\frac{-1,-d}{\mathbb{Q}})$, which contradicts the indefiniteness of B_D.

Hence, $\mathrm{Aut}_{\overline{\mathbb{Q}}}(A, \mathcal{L}) \simeq \{\alpha \in \mathcal{O}_D^* : n(\alpha) = 1, \mu(\mathcal{L})\alpha = \alpha\mu(\mathcal{L})\} \simeq S^*$, where $S = \mathcal{O}_D \cap \mathbb{Q}(\mu(\mathcal{L}))$. Since S is an imaginary quadratic order in $\mathbb{Q}(\mu(\mathcal{L})) \simeq \mathbb{Q}(\sqrt{-d})$ for $d \geq D \geq 6$, we obtain that $\mathrm{Aut}_{\overline{\mathbb{Q}}}(A, \mathcal{L}) = \{\pm 1\}$. □

The next Lemma is Theorem 3.4 (C. (1)) of [12].

Lemma 4.3. *Let $A/\mathbb{Q}$ be an abelian surface such that $\mathrm{End}_K(A) \simeq \mathcal{O}_D$ over an imaginary quadratic field K and $\mathrm{End}^0_{\mathbb{Q}}(A) \simeq \mathbb{Q}(\sqrt{m})$. Then A admits a polarization $\mathcal{L} \in H^0(\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{NS}(A_{\overline{\mathbb{Q}}}))$ of degree $d > 0$ if and only if $B_D \simeq (\frac{-Dd,m}{\mathbb{Q}})$.*

The next Lemma is essentially due to Ribet.

Lemma 4.4. *Let $A/\mathbb{Q}$ be an abelian surface such that $\mathrm{End}^0_{\mathbb{Q}}(A) \simeq \mathbb{Q}(\sqrt{m})$ and $\mathrm{End}^0_K(A) \simeq B_D$, where $K = \mathbb{Q}(\sqrt{-\delta})$, $\delta > 0$. Then $B_D \simeq (\frac{-\delta,m}{\mathbb{Q}})$.*

Proof. This is stated verbatim in [10, Theorem 1]. However, note that the statement of [10, Theorem 1] is restricted to *modular* abelian surfaces. By applying

and making explicit [36, Theorem 5.6], it is shown in [18, p. 133], that the same formula still holds for arbitrary abelian surfaces of GL_2-type over $\mathbb{Q}$. □

In [33, Theorem 4.2], Murabayashi proved a descent result for principally polarized simple abelian surfaces with quaternionic multiplication under certain hypotheses. We give an alternative proof of his result that allows us to generalize it to arbitrarily polarized abelian surfaces and which is unconditionally valid.

Theorem 4.5. *There exists an abelian surface $A/\mathbb{Q}$ such that $\mathrm{End}^0_{\mathbb{Q}}(A) \simeq \mathbb{Q}(\sqrt{m})$ and $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_D$ if and only if there exists $Q \in X_D^{(m)}(\mathbb{Q})_{nh}$ such that $\pi_m^{-1}(Q) \subset X_D(K)$ for an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-\delta})$ with*

$$B_D \simeq (\frac{-\delta, m}{\mathbb{Q}}).$$

Proof. **I.** Assume first that there exists an abelian surface $A/\mathbb{Q}$ such that $\mathrm{End}^0_{\mathbb{Q}}(A) \simeq \mathbb{Q}(\sqrt{m})$ and $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_D$.

Let us show how can one attach to A a point $Q \in X_D^{(m)}(\mathbb{Q})_{nh}$. Let $d \geq 1$ be the minimal integer such that $B_D \simeq (\frac{-Dd,m}{\mathbb{Q}})$. We know from Lemma 4.3 that there exists a polarization $\mathcal{L}$ on A of degree d defined over $\mathbb{Q}$. Fix an isomorphism $\iota : \mathcal{O}_D \xrightarrow{\sim} \mathrm{End}_{\overline{\mathbb{Q}}}(A)$ and let $R = \iota^{-1}(\mathrm{End}_{\mathbb{Q}}(A))$, which is isomorphic to a quadratic order of $\mathbb{Q}(\sqrt{m})$. Since $\mathrm{End}^0_{\mathbb{Q}}(A)$ is a real quadratic field, [12, Theorem 3.4 **C**] shows that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathrm{End}_K(A) = \mathcal{O}_D$ for some imaginary quadratic field $K = \mathbb{Q}(\sqrt{-\delta})$.

Let $* : B_D \to B_D$ be the Rosati involution induced by $\mathcal{L}$. By [41, Theorem 1.2 (2-3-4)], $\beta^* = \varrho_\mu(\beta) = \mu^{-1}\bar{\beta}\mu$ for some $\mu \in \mathcal{O}_D$, $\mu^2 + Dd = 0$. If we regard the Shimura curve X_D as coarsely representing $\hat{\mathcal{F}}_{\mathcal{O}_D,\mu}$, the triplet $[A, \iota, \mathcal{L}]$ produces a point P in $X_D(K)$. Note however that the triplet $(A, \iota_{|R}, \mathcal{L})$ is defined over $\mathbb{Q}$. Hence $\pi_{(R,\mu)}(P) \in H_{R,\mathbb{Q}}(\mathbb{Q})$. Since ϱ_μ is symmetric with respect to R, as in the proof of Theorem 3.2, the forgetful morphism $\pi_{(R,\mu)} : X_D \to H_{R,\mathbb{Q}}$ is birationally equivalent to the composition of the projection $\pi_m : X_D \to X_D^{(m)}$ and an immersion of $X_D^{(m)}$ into $H_{R,\mathbb{Q}}$. To be more precise, as stated in [41, Theorem 3.5], there exists a possibly singular curve $\tilde{X} \overset{j}{\hookrightarrow} H_{R,\mathbb{Q}}$ embedded in $H_{R,\mathbb{Q}}$ such that $\pi_{(R,\mu)} = j \cdot b \cdot \pi_m$ where $b : X_D^{(m)} \to \tilde{X}$ is a birational map that becomes an isomorphism away from the set of Heegner points. Hence $Q = \pi_m(P) \in X_D^{(m)}(\mathbb{Q})_{nh}$. Finally, by Lemma 4.4, $B_D \simeq (\frac{-\delta,m}{\mathbb{Q}})$.

II. Conversely, let $K = \mathbb{Q}(\sqrt{-\delta})$ be an imaginary quadratic field. Assume that $B_D \simeq (\frac{-\delta,m}{\mathbb{Q}})$ and let $P \in X_D(K)$ such that $Q = \pi_m(P) \in X_D^{(m)}(\mathbb{Q})_{nh}$ (and thus $\pi_m^{-1}(Q) = \{P, \omega_m(P)\} \subset X_D(K)$). Choose $\mu, \omega \in \mathcal{O}_D$, $\mu^2 = -\delta$, $\omega^2 = m$, $\mu\omega = -\omega\mu$ and let $R = \mathbb{Q}(\omega) \cap \mathcal{O}_D$. Regard the Shimura curve X_D as coarsely representing $\hat{\mathcal{F}}_{\mathcal{O}_D,\mu}$. The element μ determines an embedding of K into B_D. As is stated in [25, Theorem 2.1.3], the point $P \in X_D(K)$ can be represented by the $\overline{\mathbb{Q}}$-isomorphism class of a polarized simple abelian surface with quaternionic multiplication $(A_0, \iota_0, \mathcal{L}_0)$ completely defined over K and such that the Rosati involution that $\mathcal{L}_0$ induces on B_D is ϱ_μ.

As in Part I, the condition $\pi_m(P) \in X_D^{(m)}(\mathbb{Q})_{nh}$ implies that $\pi_{(R,\mu)}(P) \in H_{R,\mathbb{Q}}(\mathbb{Q})$, and this amounts to saying that the field of moduli of $(A_0, \iota_{0|_R}, \mathcal{L}_0)$ is $\mathbb{Q}$.

For any number field $F \subset \overline{\mathbb{Q}}$, let $G_F = \mathrm{Gal}\,(\overline{\mathbb{Q}}/F)$. Let $\sigma \in G_\mathbb{Q} \backslash G_K$. Then there exists an isomorphism $\nu : A_0 \to A_0^\sigma$ such that $\nu^*(\mathcal{L}_0^\sigma) = \mathcal{L}_0$ and $\nu \cdot \omega^{-1} \cdot \alpha \cdot \omega = \alpha^\sigma \cdot \nu$

for all endomorphisms $\alpha \in B_D = \mathrm{End}_K^0(A)$. In particular,

$$\nu \cdot \omega = \omega^\sigma \cdot \nu\,, \quad \nu \cdot \mu = -\mu^\sigma \cdot \nu\,.$$

We split the proof into two parts.

Step 1: We show that ν may be assumed to be defined over K.

To prove this claim, we first note that since $\mathrm{End}_{\overline{\mathbb{Q}}}(A_0) = \mathcal{O}_D$, $\mathrm{Aut}_K(A_0, \mathcal{L}_0) = \{\pm 1\}$ by Lemma 4.2. Let $\rho_\nu : G_K \to \mathrm{Aut}_K(A_0, \mathcal{L}_0) = \{\pm 1\}$ be the group homomorphism defined by $\rho_\nu(\tau) = \nu^{-1} \cdot \nu^\tau$.

Suppose that ν was not defined over K, that is $\rho_\nu(G_K) = \{\pm 1\}$. Let L/K be the quadratic extension such that $G_L = \ker \rho_\nu$. Since L is the minimal field of definition of all homomorphisms in $\mathrm{Hom}(A_0, A_0^\sigma)$ and $\mathrm{Hom}(A_0^\sigma, A_0)$, we deduce that $L/\mathbb{Q}$ is a Galois extension. Since K is imaginary, $L/\mathbb{Q}$ can not be cyclic and there exists a square-free integer $d > 1$ such that $L = K(\sqrt{d})$.

Let $V_K = H^0(A_0, \Omega^1_{A_0/K})$ denote the vector space of regular differentials on A over K. Since $B_D \simeq \mathrm{End}_K^0(A)$, the action of the endomorphisms on V_K induces an embedding $* : B_D \hookrightarrow \mathrm{End}_K(V_K) \simeq \mathrm{M}_2(K)$ and an isomorphism $B_D \otimes K \simeq K + K\mu + K\nu + K\mu\nu \simeq \mathrm{M}_2(K)$. We may choose basis of V_K such that the matrix expressions of ω^* and μ^* acting on V_K are

$$M_m = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}, \quad M_\delta = \begin{pmatrix} \sqrt{-\delta} & 0 \\ 0 & -\sqrt{-\delta} \end{pmatrix},$$

respectively. Indeed, this is possible because, as stated by the Skolem-Noether Theorem (cf. [49, Ch. I, Theorem 2.1]), all automorphisms of $\mathrm{M}_2(K)$ are inner and we have $B_D \otimes K \simeq \mathrm{M}_2(K) = K + KM_\delta + KM_m + KM_\delta M_m$.

Let $N \in \mathrm{GL}_2(K(\sqrt{d}))$ be the matrix expression of $\nu \in \mathrm{Hom}(A_0, A_0^\sigma)$ with respect to this basis of V_K and its Galois conjugate of V_K^σ. Then N satisfies

$$N^\tau = -N\,, \quad M_m \cdot N = N \cdot M_m^\sigma = N \cdot M_m\,, \quad M_\delta \cdot N = -N \cdot M_\delta^\sigma = N \cdot M_\delta\,,$$

for $\tau \in G_K \backslash G_L$. Hence, $N = \sqrt{d} \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}$, $\beta \in K$. Fix σ in $G_{\mathbb{Q}(\sqrt{d})}$, $\sigma \notin G_K$. We have $\nu^\sigma \cdot \nu \in \mathrm{Aut}(A_0, \mathcal{L}_0) = \{\pm 1\}$, thus $N \cdot N^\sigma = \pm\,\mathrm{id}$ and $\beta \cdot \beta^\sigma = 1/d$. Hence, the normal closure F of $K(\sqrt{\beta})/\mathbb{Q}$ is dihedral containing $K(\sqrt{d})$ and $F/\mathbb{Q}(\sqrt{-d \cdot \delta})$ is cyclic. Let $\rho_\beta : G_K \to \{\pm 1\}$ be the surjective morphism such that $\ker \rho_\beta = G_{K(\sqrt{\beta})}$. Attached to the cocycle $\rho_\beta \in H^1(G_K, \{\pm 1\})$ there is a polarized abelian surface $(A_1, \mathcal{L}_1)$ defined over K together with an isomorphism $\lambda : (A_0, \mathcal{L}_0) \to (A_1, \mathcal{L}_1)$ such that $\lambda^\tau = \lambda \cdot \rho_\beta(\tau)$. We claim that $\phi = \lambda^\sigma \cdot \nu \cdot \lambda^{-1} : A_1 \to A_1^\sigma$ is defined over K. Indeed, for any $\tau \in G_K$,

$$\phi^\tau = (\lambda^{\sigma\cdot\tau\cdot\sigma^{-1}})^\sigma \cdot \nu^\tau \cdot (\lambda^{-1})^\tau = \rho_\beta(\sigma \cdot \tau \cdot \sigma^{-1} \cdot \tau^{-1}) \cdot \rho_\nu(\tau) \cdot \phi\,.$$

Since $\sigma \cdot \tau \cdot \sigma^{-1} \cdot \tau^{-1} \in G_{K(\sqrt{\beta})}$ if and only if $\tau \in G_{K(\sqrt{d})}$, we obtain that $\phi^\tau = \phi$.

Moreover, all endomorphisms of A_1 are of the form $\lambda \cdot \varphi \cdot \lambda^{-1}$ with φ in $\mathrm{End}_K(A_0)$. These are all defined over K because $(\lambda \cdot \varphi \cdot \lambda^{-1})^\tau = \delta_\beta(\tau \cdot \tau^{-1})\lambda \cdot \varphi \cdot \lambda^{-1} = \lambda \cdot \varphi \cdot \lambda^{-1}$.

We therefore assume that ν is defined over K.

Step 2: We show that $(A_0, \mathcal{L}_0)$ admits a model over $\mathbb{Q}$ with all its endomorphisms defined over K.

We do so by applying Proposition 4.1. Since $\nu^\sigma \cdot \nu \in \mathrm{Aut}(A_0, \mathcal{L}_0)$, we have $\nu^\sigma \cdot \nu = \epsilon\,\mathrm{id}$ with $\epsilon \in \{\pm 1\}$. Using the same basis of $H^0(A_0, \Omega^1_{A_0/K})$ and $H^0(A_0^\sigma, \Omega^1_{A_0^\sigma/K})$ as

above, the matrix expression $N \in \mathrm{GL}_2(K)$ of ν is such that $M_m \cdot N = N \cdot M_m^\sigma = N \cdot M_m\,, \quad M_\delta \cdot N = -N \cdot M_\delta^\sigma = N \cdot M_\delta\,.$ It follows that

$$N = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}\,, \quad \beta \in K\,.$$

Hence, $\beta \cdot \beta^\sigma = \epsilon$. Since K is imaginary, $\epsilon = 1$. Proposition 4.1 applies to ensure the existence of a polarized abelian surface $(A, \mathcal{L})$ defined over $\mathbb{Q}$ and isomorphic over K to $(A_0, \mathcal{L}_0)$. Since $A \simeq A_0$ over K, we obtain that there is an isomorphism $\iota : \mathcal{O}_D \xrightarrow{\sim} \mathrm{End}_K(A)$.

Finally, we show that $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathbb{Q}(\sqrt{m})$. The triplets $(A, \iota, \mathcal{L})$ and $(A_0, \iota_0, \mathcal{L}_0)$ are isomorphic and the assertion $[A, \iota_{|R}, \mathcal{L}] \in X_D^{(m)}(\mathbb{Q})$ implies that for every $\sigma \in \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\alpha \in R$, $\iota(\alpha)^\sigma = \iota(\alpha) \in \mathrm{End}_{\overline{\mathbb{Q}}}(A)$. Hence $\iota(R) \subset \mathrm{End}_{\mathbb{Q}}(A)$. Thus $\mathbb{Q}(\sqrt{m}) = R \otimes \mathbb{Q} \subseteq \mathrm{End}^0_{\mathbb{Q}}(A) \subseteq \mathrm{End}^0_{\overline{\mathbb{Q}}}(A) = B_D$. Since the only two subalgebras of B_D that contain $\mathbb{Q}(\sqrt{m})$ are $\mathbb{Q}(\sqrt{m})$ and B_D themselves, and the latter can not occur by Proposition 3.1, we obtain that $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathbb{Q}(\sqrt{m})$, as we claimed. □

Remark 4.6. Let $(A, \mathcal{L})$ be a polarized abelian variety over a number field K and let $K_0 \subseteq K$ be its field of moduli. Assume that $\mathrm{Aut}(A, \mathcal{L}) \simeq \{\pm 1\}$. Then, Proposition 4.1 can be rephrased in the language of cohomology of groups as follows. Let $\mathrm{Br}_{K_0} = H^2(\mathrm{Gal}\,(\overline{\mathbb{Q}}/K_0), \overline{\mathbb{Q}}^*)$ denote the Brauer group of K_0. For any choice of isomorphisms $\{\mu_\sigma, \sigma \in \mathrm{Gal}\,(\overline{\mathbb{Q}}/K_0)\}$ as above, the map

$$\begin{array}{rccc} c: & \mathrm{Gal}\,(\overline{\mathbb{Q}}/K_0) \times \mathrm{Gal}\,(\overline{\mathbb{Q}}/K_0) & \longrightarrow & \overline{\mathbb{Q}}^* \\ & (\sigma, \tau) & \mapsto & \mu_\sigma \cdot \mu_\tau^\sigma \cdot \mu_{\sigma\tau}^{-1} \end{array}$$

produces a well-defined continuous cocycle $\xi(A, \mathcal{L}) \in \mathrm{Br}_{K_0}[2]$ in the 2-torsion subgroup of Br_{K_0}. It is easily checked that different choices of μ_σ for $\sigma \in \mathrm{Gal}\,(\overline{\mathbb{Q}}/K_0)$ lead to cocycles that differ from a coboundary. Proposition 4.1 says that $(A, \mathcal{L})$ admits a model over K_0 if and only if $\xi(A, \mathcal{L}) = 1 \in \mathrm{Br}_{K_0}[2]$. By class field theory (cf. [45, Ch. X, §4, §5, §6]), there is a natural identification between $\mathrm{Br}_{K_0}[2]$ and the group $(A_{K_0}, \otimes)$, where we let A_{K_0} denote the set of isomorphism classes of quaternion algebras over K_0. Upon this identification, the proof of Theorem 4.5 shows that if $(A, \iota : \mathcal{O}_D \xrightarrow{\sim} \mathrm{End}_{\overline{\mathbb{Q}}}(A), \mathcal{L})$ is a polarized abelian surface over $\overline{\mathbb{Q}}$ with quaternionic multiplication such that the field of moduli of $(A, \iota_{|R} : R \hookrightarrow \mathrm{End}_{\overline{\mathbb{Q}}}(A), \mathcal{L})$ is $\mathbb{Q}$ for some order $R \subset \mathcal{O}_D$, $R \otimes \mathbb{Q} \simeq \mathbb{Q}(\sqrt{m})$, then the field of moduli of $(A, \iota, \mathcal{L})$ is an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-\delta})$ and $\xi(A, \mathcal{L}) = B_D \otimes (\frac{-\delta, m}{\mathbb{Q}})$ in $\mathrm{Br}_{\mathbb{Q}}$.

Motivated by the above result, we make the following definition.

Definition 4.7. *The subset of descent points of* $X_D^{(m)}(\mathbb{Q})$ *is* $X_D^{(m)}(\mathbb{Q})_d :=$

$$\{Q \in X_D^{(m)}(\mathbb{Q})_{nh} : \pi_m^{-1}(Q) \subset X_D(\mathbb{Q}(\sqrt{-\delta})), B_D \simeq (\frac{-\delta, m}{\mathbb{Q}}) \textit{ for some } \delta > 0\}.$$

$$\text{Set } r_m := \#X_D^{(m)}(\mathbb{Q})\,, \quad rh_m := \#X_D^{(m)}(\mathbb{Q})_h\,, \quad rd_m := \#X_D^{(m)}(\mathbb{Q})_d\,.$$

Note that r_m and rd_m may be $+\infty$. The proof of Theorem 4.5 actually yields more information. Let (D, m) be a premodular pair over $\mathbb{Q}$. One may wonder how many abelian surfaces $A/\mathbb{Q}$ exist up to isomorphism such that $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathbb{Q}(\sqrt{m})$ and $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathcal{O}_D$. We make this precise in what follows.

Definition 4.8. *Let $\mathcal{Q}_{(D,m)}(\mathbb{Q})$ denote the set of $\overline{\mathbb{Q}}$-isomorphism classes of abelian surfaces A defined over $\mathbb{Q}$ such that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_D$ and $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathbb{Q}(\sqrt{m})$.*

As in Section 3, let W_D denote the Atkin-Lehner group acting on X_D. For a premodular pair (D,m) over $\mathbb{Q}$, let $W_{D,m} = W_D/\langle\omega_m\rangle$. As stated in [25, Proposition 3.2.2] or [20, Proposition 5.5], the fixed points of an Atkin-Lehner involution $\omega \in W_D$ acting on X_D are Heegner points. Hence, the group $W_{D,m}$ is naturally a subgroup of $\mathrm{Aut}_{\mathbb{Q}}(X_D^{(m)})$ which freely acts on the set $X_D^{(m)}(\mathbb{Q})_d$.

Theorem 4.9. *Let (D,m) be a premodular pair over $\mathbb{Q}$. There is a canonical one-to-one correspondence*

$$\mathcal{Q}_{(D,m)}(\mathbb{Q}) \longleftrightarrow W_{D,m}\backslash X_D^{(m)}(\mathbb{Q})_d$$

and hence, if $D = p_1 \cdot \ldots \cdot p_{2r}$, then

$$|\mathcal{Q}_{(D,m)}(\mathbb{Q})| = \frac{rd_m}{2^{2r-1}}\,.$$

Proof. Let $[A] \in \mathcal{Q}_{(D,m)}(\mathbb{Q})$ represented by an abelian surface A defined over $\mathbb{Q}$. In Part I of the proof of Theorem 4.5, we show how can one attach a point $Q \in X_D^{(m)}(\mathbb{Q})_d$. Since, as explained in the last two paragraphs of Section 2.1, the group $W_{D,m}$ acts on $Q = [A, \iota_{|R}, \mathcal{L}]$ by fixing the isomorphism class of A and switching ι and $\mathcal{L}$, we deduce that A produces a well-defined point in $W_{D,m}\backslash X_D^{(m)}(\mathbb{Q})_d$. The inverse map from $W_{D,m}\backslash X_D^{(m)}(\mathbb{Q})_d$ onto $\mathcal{Q}_{(D,m)}(\mathbb{Q})$ is constructed in Part II of the proof of Theorem 4.5 and for the same reason as above it does not depend on the choice of $Q \in X_D^{(m)}(\mathbb{Q})_d$ in its orbit under the action by $W_{D,m}$. □

Corollary 4.10. *Let $D > 546$. Then there exist only finitely many $\overline{\mathbb{Q}}$-isomorphism classes of abelian surfaces A of GL_2-type over $\mathbb{Q}$ such that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_D$.*

Proof. For a given discriminant D, assume that there exist infinitely many abelian surfaces A of GL_2-type over $\mathbb{Q}$ up to isomorphism over $\overline{\mathbb{Q}}$ such that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_D$. According to Theorem 1.4 (i) and Theorem 4.9, there exists some $m \mid D$ such that $X_D^{(m)}$ has infinitely many rational points over $\mathbb{Q}$. Since the degree of the map $\pi_m : X_D \to X_D^{(m)}$ is 2, this implies that X_D has infinitely many quadratic points. As is shown in [38, Theorem 9], the largest such discriminant is $D = 546$. □

Remark 4.11. One can check [38, Table 3] to find out which is precisely the list of discriminants D for which we can claim that Corollary 4.10 holds true.

As a refinement of the above considerations, we wonder for which pairs (D,m) there exists a curve $C/\mathbb{Q}$ of genus 2 such that the Jacobian $J(C)$ has multiplication by $\mathbb{Q}(\sqrt{m})$ over $\mathbb{Q}$ and quaternionic multiplication by $\mathcal{O}_D$ over $\overline{\mathbb{Q}}$.

Corollary 4.12. *Let $K = \mathbb{Q}(\sqrt{-\delta})$ be an imaginary quadratic field. Then, there exists a curve $C/\mathbb{Q}$ defined over $\mathbb{Q}$ such that $\mathrm{End}^0_{\mathbb{Q}}(J(C)) \simeq \mathbb{Q}(\sqrt{m})$ and $\mathrm{End}_K(J(C)) \simeq \mathcal{O}_D$ if and only if $\pi_m(X_D(K)) \cap X_D^{(m)}(\mathbb{Q})_{nh} \neq \emptyset$ and*

$$B_D \simeq \left(\frac{-\delta, m}{\mathbb{Q}}\right) \simeq \left(\frac{-D, m}{\mathbb{Q}}\right).$$

Proof. By Theorem 4.5, there exists an abelian surface $A/\mathbb{Q}$ such that $\mathrm{End}^0_{\mathbb{Q}}(A) \simeq \mathbb{Q}(\sqrt{m})$ and $\mathrm{End}_K(A) \simeq \mathcal{O}_D$ if and only if $\pi_m(X_D(K)) \cap X_D^{(m)}(\mathbb{Q})_{nh} \neq \emptyset$ and $B_D \simeq \left(\frac{-\delta,m}{\mathbb{Q}}\right)$. By Lemma 4.3, A admits a principal polarization over $\mathbb{Q}$ if moreover $B_D \simeq \left(\frac{-D,m}{\mathbb{Q}}\right)$. The result now follows because as is well-known (cf. e. g. [21, Theorem 3.1]), an absolutely irreducible abelian surface $A/\mathbb{Q}$ is the Jacobian of a smooth curve $C/\mathbb{Q}$ of genus 2 if and only if A is principally polarizable over $\mathbb{Q}$. □

Next, we illustrate the above results with several examples.

Example 4.13. In [24, Lemma 4.5], Hashimoto and Tsunogai provided a family of curves of genus 2 whose Jacobians have quaternionic multiplication by $\mathcal{O}_6$. These families specialize to infinitely many curves defined over $\mathbb{Q}$. However, one can not expect that to be always possible for a discriminant D even when there is an Atkin-Lehner quotient $X_D^{(m)} \simeq \mathbb{P}^1_{\mathbb{Q}}$. As we pointed out in Section 1, computations due to Hasegawa [23] exhibit B_{14} as a modular quaternion algebra. This is indeed possible because $X_{14}^{(14)} \simeq \mathbb{P}^1_{\mathbb{Q}}$ but there does not exist a curve $C/\mathbb{Q}$ of genus 2 whose Jacobian $J(C)$ is of GL_2-type over $\mathbb{Q}$ and has quaternionic multiplication by $\mathcal{O}_{14}$ over $\overline{\mathbb{Q}}$, because $B_{14} \not\simeq (\frac{-14,2}{\mathbb{Q}})$, $(\frac{-14,7}{\mathbb{Q}})$ nor $(\frac{-14,14}{\mathbb{Q}})$.

Example 4.14. An affine equation of the Shimura curve X_6 is $x^2 + y^2 + 3 = 0$ (cf. Table 1) and the action of w_6 on this model is $(x, y) \mapsto (-x, y)$. We have $X_6^{(6)} \simeq \mathbb{P}^1_{\mathbb{Q}}$ and there exist infinitely many points on $X_6(K)$, $K = \mathbb{Q}(\sqrt{-21})$, mapping to a rational point on $X_6^{(6)}$. Regard X_6 as the coarse moduli space attached to $\hat{\mathcal{F}}_{\mathcal{O}_6,\mu}$ for some $\mu \in \mathcal{O}_6$ such that $\mu^2 + 6 = 0$. As stated in [25, Theorem 1.4.1], the points $P \in X_6(K)$ are represented by *principally* polarized abelian surfaces $P = [A, \iota, \mathcal{L}]$ with quaternionic multiplication. Since K splits B_6, we can assume by [25, Theorem 2.1.3] that $(A, \iota, \mathcal{L})$ are defined over K. Moreover, since the class number $h(K) > 1$, we know by [25, Theorem 3.1.5], [20, Theorem 5.11] that there are no Heegner points on $X_6(K)$. As explained in Section 2.1, this means that the abelian surfaces A are absolutely irreducible. Finally, we have $\omega_6(P) = \omega_6[A, \iota, \mathcal{L}] = [A, \mu^{-1}\iota\mu, \mathcal{L}]$ by [41, Theorem 3.5] (cf. also [41, Section 7, p. 273]). Since $\pi_6(P) \in X_6^{(6)}(\mathbb{Q})$, this implies that $(A, \mathcal{L})$ are isomorphic to their Galois conjugate $(A^\sigma, \mathcal{L}^\sigma)$, for $\mathrm{Gal}\,(K/\mathbb{Q}) = \langle\sigma\rangle$. However, since $(\frac{-21,6}{\mathbb{Q}}) \not\simeq B_6$, we conclude by Theorem 4.5 that there does not exist a model of $(A, \mathcal{L})$ defined over $\mathbb{Q}$.

Example 4.15. Let f be the newform of $S_2(\Gamma_0(243))$ with q-expansion

$$f = q + \sqrt{6}\,q^2 + 4\,q^4 + \dots .$$

The modular abelian surface A_f obtained as an optimal quotient of the Jacobian of $X_0(43)$ satisfies that $\mathrm{End}_K(A_f)$ is a maximal order of the quaternion algebra B_6, where we let $K = \mathbb{Q}(\sqrt{-3})$. By [39, Theorem 7.1] we know that there is a single class of principal polarizations $\mathcal{L}_0$ on A_f/K up to $\overline{\mathbb{Q}}$-isomorphism. Hence $A_f \otimes K$ is the Jacobian of a curve C/K. Since $\mathcal{L}_0$ is isomorphic to its Galois conjugate $\mathcal{L}_0^\sigma$, it follows that C_0 is isomorphic to C_0^σ but, although its Jacobian $\mathrm{Jac}(C_0) = A_f \otimes K$ admits a projective model over $\mathbb{Q}$, the curve C_0 can not be defined over $\mathbb{Q}$ because $B_6 \not\simeq (\frac{-6,6}{\mathbb{Q}})$. In fact, by using similar methods to [21], we obtain the following equation for C_0:

$$y^2 = (2 + 2\sqrt{-3})x^6 + 12(-3 + \sqrt{-3})x^5 - 12(3 + 7\sqrt{-3})x^4 + 4(69 + 7\sqrt{-3})x^3 + 8(-11 + 7\sqrt{-3})x^2 - 18(1 + 5\sqrt{-3}) + 12(2 + \sqrt{-3}).$$

It can be checked that its Igusa invariants are rational and there is a morphism $\nu : C_0 \to C_0^\sigma$ defined over K such that $\nu^\sigma \cdot \nu$ is the hyperelliptic involution.

5. Rational points on quotient Shimura curves $X_D^{(m)}$ of genus ≤ 1

The set of rational Heegner points $X_D^{(m)}(\mathbb{Q})_h$ on an Atkin-Lehner quotient of a Shimura curve is finite and its cardinality rh_m can be computed by using the following formula, which stems from the work of Jordan on complex multiplication[2].

Proposition 5.1. *Let $D = p_1 \cdot \ldots \cdot p_{2r}$ and let $m|D$. For $i = 1$ or 2, let us denote by $\mathcal{R}_i$ the set of orders of imaginary quadratic fields whose class number is i. For any $R \in \mathcal{R}_1$, set $p_R = 2$ when $\operatorname{disc}(R) = -2^k$ and p_R to be the single odd prime dividing $\operatorname{disc}(R)$, otherwise. The number rh_m is given by the following formula:*

$$rh_m = \begin{cases} 2^{2r-1}\#\{R \in \mathcal{R}_1 : \left(\frac{R}{p_i}\right) = -1 \text{ for all } p_i|D\} + \\ 2^{2r-2}\#\{R \in \mathcal{R}_1 : p_R|D\,, \left(\frac{R}{p_i}\right) = -1 \text{ for all } p_i|\frac{D}{p_R}\} + \\ \#\{R \in \mathcal{R}_2 : 2D = -\operatorname{disc}(R)\} & \text{, if } m = D, \\ 2^{r-2}\#\{R \in \mathcal{R}_1 : p_R = \frac{D}{m}\} + \#\{R \in \mathcal{R}_2 : -\frac{D}{\operatorname{disc}(R)} \in \mathbb{Q}^{*2}\} + \\ \frac{(-1)^p+1}{2}\#\{R \in \mathcal{R}_2 : 2D = -\operatorname{disc}(R)\} & \text{, if } m = \frac{D}{p}, \\ 0 & \text{otherwise.} \end{cases}$$

where $\left(\frac{R}{p}\right) = \begin{cases} \left(\frac{K}{p}\right) & \text{if } p \nmid \mathrm{cond}(R) \\ 1 & \text{if } p \mid \mathrm{cond}(R) \end{cases}$ *is the Eichler symbol.*

Proof. This is a direct application of Proposition 5.5 and Corollary 5.13 of [20]. □

In any case, $0 \leq rd_m \leq r_m - rh_m$. The condition $r_m = rh_m$ implies $rd_m = 0$ and allows us to claim the non existence of an abelian surface $A/\mathbb{Q}$ with $\operatorname{End}_{\mathbb{Q}}^0 A = \mathbb{Q}(\sqrt{m})$ and $\operatorname{End}_{\overline{\mathbb{Q}}}^0 A = B_D$. Proving the existence of such an abelian surface, i. e. $rd_m > 0$, requires the knowledge of an equation for X_D and the action of ω_m on it.

There are exactly twelve Shimura curves X_D of genus $g \leq 2$. For all of them, $D = p \cdot q$ with p, q primes and affine equations for these curves are known (cf. [19], [20], [27], [28]). For these equations, the Atkin-Lehner involutions $\omega_p, \omega_q, \omega_{p.q}$ act on the curve, sending (x, y) to $(-x, y)$, $(x, -y)$ and $(-x, y)$ in some suitable order. The next table shows equations, genera and the actions of ω_p and ω_q for these curves.

[2]To be precise, the work in [25, Ch. 3] restricts to complex multiplication by the full ring of integers in imaginary quadratic fields. Since there exist nonmaximal orders of class number 1 or 2, one actually needs to apply the statements of [20, Section 5].

$D = p \cdot q$	g	X_D	$\omega_p(x,y)$	$\omega_q(x,y)$
$2 \cdot 3$	0	$x^2 + y^2 + 3 = 0$	$(-x, -y)$	$(\ \ x, -y)$
$2 \cdot 5$	0	$x^2 + y^2 + 2 = 0$	$(\ \ x, -y)$	$(-x, -y)$
$2 \cdot 11$	0	$x^2 + y^2 + 11 = 0$	$(-x, -y)$	$(\ \ x, -y)$
$2 \cdot 7$	1	$(x^2 - 13)^2 + 7^3 + 2y^2 = 0$	$(-x, \ \ y)$	$(-x, -y)$
$3 \cdot 5$	1	$(x^2 + 3^5)(x^2 + 3) + 3y^2 = 0$	$(-x, \ \ y)$	$(-x, -y)$
$3 \cdot 7$	1	$x^4 - 658x^2 + 7^6 + 7y^2 = 0$	$(-x, -y)$	$(-x, \ \ y)$
$3 \cdot 11$	1	$x^4 + 30x^2 + 3^8 + 3y^2 = 0$	$(-x, \ \ y)$	$(-x, -y)$
$2 \cdot 17$	1	$3x^4 - 26x^3 + 53x^2 + 26x + 3 + y^2 = 0$	$(-\frac{1}{x}, \frac{y}{x^2})$	$(-\frac{1}{x}, \frac{-y}{x^2})$
$2 \cdot 23$	1	$(x^2 - 45)^2 + 23 + 2y^2 = 0$	$(-x, \ \ y)$	$(-x, -y)$
$2 \cdot 13$	2	$y^2 = -2x^6 + 19x^4 - 24x^2 - 169$	$(-x, -y)$	$(-x, \ \ y)$
$2 \cdot 19$	2	$y^2 = -16x^6 - 59x^4 - 82x^2 - 19$	$(-x, -y)$	$(-x, \ \ y)$
$2 \cdot 29$	2	$2y^2 = -x^6 - 39x^4 - 431x^2 - 841$	$(-x, -y)$	$(\ \ x, -y)$

Table 1. Equations and Atkin-Lehner involutions on Shimura curves

Unfortunately, the construction of the above equations does not allow us to distinguish the rational Heegner points among the rational points on the curves $X_D^{(m)}$, unless they are fixed by some Atkin-Lehner involution (however, for $D = 6$ and 10, cf. [11]). This forces the proof of next theorem to be more elaborate.

Theorem 5.2. *For the twelve values of D as above, the triplets (r_m, rh_m, rd_m) take the following values:*

$D = p \cdot q$	(r_p, rh_p, rd_p)	(r_q, rh_q, rd_q)	(r_D, rh_D, rd_D)
$2 \cdot 3$	$(\infty, 1, \infty)$	$(\infty, 1, \infty)$	$(\infty, 8, \infty)$
$2 \cdot 5$	$(\infty, 2, 0)$	$(\infty, 2, \infty)$	$(\infty, 11, \infty)$
$2 \cdot 7$	$(0, 0, 0)$	$(6, 2, 4)$	$(\infty, 8, \infty)$
$2 \cdot 11$	$(\infty, 2, \infty)$	$(\infty, 2, \infty)$	$(\infty, 8, \infty)$
$2 \cdot 13$	$(1, 1, 0)$	$(3, 1, 0)$	$(\infty, 10, > 0)$
$2 \cdot 17$	$(0, 0, 0)$	$(0, 0, 0)$	$(\infty, 8, \infty)$
$2 \cdot 19$	$(1, 1, 0)$	$(3, 1, 0)$	$(\infty, 8, > 0)$
$2 \cdot 23$	$(0, 0, 0)$	$(2, 2, 0)$	$(\infty, 8, \infty)$
$2 \cdot 29$	$(1, 1, 0)$	$(\infty, 2, > 0)$	$(\infty, 13, > 0)$
$3 \cdot 5$	$(\infty, 2, 0)$	$(4, 4, 0)$	$(\infty, 10, \infty)$
$3 \cdot 7$	$(2, 2, 0)$	$(0, 0, 0)$	$(\infty, 10, \infty)$
$3 \cdot 11$	$(0, 0, 0)$	$(2, 2, 0)$	$(\infty, 10, \infty)$

Table 2. Rational, Heegner and descent points

Proof. We split the proof in three parts according to the genus g of X_D.

Case $g = 0$. In all these cases the equation is $x^2 + y^2 = -d$, for some prime $d|D$. For each pair (D, m), the points on $X_D(\sqrt{-\delta})$ of the form $(a, b\sqrt{-\delta})$, $(b\sqrt{-\delta}, a)$ or $(a\sqrt{-\delta}, b\sqrt{-\delta})$, with $a, b \in \mathbb{Q}$ and a square-free integer $\delta \geq 1$, are the only affine points on $X_D(\mathbb{Q}(\sqrt{-\delta}))$ which may provide rational points on $X_D^{(m)}(\mathbb{Q})$, depending on whether ω_m maps (x, y) to either $(x, -y)$, $(-x, y)$ or $(-x, -y)$. Let $(\cdot, \cdot)$ denote the global Hilbert symbol over $\mathbb{Q}$. When $\omega_m(x, y) = (x, -y)$ or $(-x, y)$, such rational points exist if and only if $(\delta, -d) = 1$. Similarly, when $\omega_m(x, y) = (-x, -y)$,

there exist points on $X_D(\mathbb{Q}(\sqrt{-\delta}))$ which project onto a rational point on $X_D^{(m)}(\mathbb{Q})$ if and only if $(-1, d\cdot\delta) = 1$. It is easy to check that, for all pairs $(D,m) \neq (10,2)$, these conditions and the descent condition of Theorem 4.5 have infinitely many solutions for δ. If we let l be a prime number, we may take δ as follows.

(D,m)	δ	Conditions on l
$(6,2)$	$3\,l$	$l \equiv 1 \pmod 8$
$(6,3),(6,6)$	l	$l \equiv 1 \pmod{24}$
$(10,5)$	$2\,l$	$l \equiv 1 \pmod{20}$
$(10,10)$	$2\,l$	$l \equiv 1 \pmod{40}$
$(22,2)$	$11\,l$	$l \equiv 1 \pmod 8$
$(22,11),(22,22)$	l	$l \equiv 1 \pmod{88}$

For the pair $(10,2)$, we have $d = 2$ and the condition $(\delta,-2) = 1$ implies $5 \not| \delta$ and thus $(\frac{-\delta,2}{\mathbb{Q}}) \not\simeq B_{10}$, since $(-\delta,2)_5 = 1$. Moreover, the two points $(1 : \pm\sqrt{-1} : 0)$ at infinity on the curve X_{10} produce a rational point on $X_{10}^{(2)}$ which is Heegner because its preimages are fixed points by the Atkin-Lehner involution ω_5. We conclude that for $D = 10$, $rd_2 = 0$ despite $r_2 = \infty$.

Case $g = 1$. Let us first consider the cases for which $m \neq D$. Note that ω_m does not act as $(x,y) \mapsto (-y,x)$.

Assume first that $D \neq 34$. Then, the genus of $X_D^{(m)}$ is zero except for the cases in which ω_m maps (x,y) to $(-x,-y)$. The latter holds for the pairs $(D,m) = (14,7),(15,5),(21,3)$, $(33,11)$ and $(46,23)$, and in these cases $g(X_D^{(m)}) = 1$. To be more precise, the curves $X_D^{(m)}$ are elliptic curves over $\mathbb{Q}$ and there is a single isogeny class of conductor D in each case. Their Mordell-Weil rank over $\mathbb{Q}$ is 0 and the orders of the group of rational torsion points on them are 6, 4, 2, 2 and 2, respectively. Only for the pair $(D,m) = (14,7)$ do we have $r_m > rh_m$. But in this case, the two rational Heegner points can be recognized from the affine equation of the curve $X_{14}^{(7)}$ because they are fixed points by some Atkin-Lehner involution. It then turns out that the four rational non-Heegner points on $X_{14}^{(7)}$ are the projections of the points $(\pm 8\sqrt{-1}, \pm 56\sqrt{-1}), (\pm 2\sqrt{-2}, \pm 14\sqrt{-2}) \in X_{14}(\overline{\mathbb{Q}})$. Since $\delta = 1, 2$ satisfy the descent condition, we have $rd_7 = 4$.

When ω_m acts as $(x,y) \mapsto (-x,y)$, we have $r_m = 0$ except for $(D,m) = (15,3)$. The affine equation $(X + 3^5)(X + 3) + 3y^2 = 0$ for $X_{15}^{(3)}$ shows that there are no rational points at infinity on this model. Moreover, it turns out that for all $(X,y) \in X_{15}^{(3)}(\mathbb{Q})$, the 5-adic valuation of the X-coordinate is $v_5(X) = 0$. Since $\delta \equiv -X \pmod{\mathbb{Q}^2}$, we have $5 \not| \delta$. Thus, $(\frac{-\delta,3}{\mathbb{Q}}) \not\simeq B_{15}$ because $(-\delta,3)_5 = 1$. It follows that $rd_3 = 0$.

Finally, let us consider the particular case $D = 34$. The equations of the curves $X_{34}^{(2)}$ and $X_{34}^{(17)}$ are

$$v^2 + 3u^4 - 26u^3 + 71u^2 - 104u + 236 = 0\,, \quad z^2 + 3u^2 - 26u + 59 = 0\,,$$

respectively, where $u = x - 1/x$, $v = y(1 + 1/x^2)$ and $z = y/x$. Hence, $X_{34}^{(2)}(\mathbb{R}) = \emptyset$ and $X_{34}^{(17)}(\mathbb{R}) = \emptyset$ and we obtain that $r_m = 0$ in both cases. Note that this also follows from Proposition 3.4.

We now consider the case $m = D$. We have $\omega_m : (x,y) \mapsto (x,-y)$ and the curve $X_D^{(D)}$ admits an affine model of the form $f(x) + dY = 0$, where $f(x) \in \mathbb{Q}[x]$

is monic of degree 4 and $(\frac{-d,D}{\mathbb{Q}}) \simeq B_D$. A point $(x_0, Y_0) \in X_D^{(D)}(\mathbb{Q})$ satisfies the descent condition if and only if $(f(x_0), D) = 1$, that is, $f(x_0) = u_0 - Dv_0^2$ for some $u_0, v_0 \in \mathbb{Q}$. Hence, the descent condition for (x_0, Y_0) turns out to be equivalent to the existence of a rational point on the algebraic surface $S_D : f(x) = u^2 - Dv^2$ with $x = x_0$. For $D = 14, 15, 21, 33, 34$ and 46 we have the following rational points on S_D: $(x_0, u_0, v_0) = (4, 24, 4), (1, 44, 8), (1, 944, 192), (8, 145, 16), (9, 113, 18)$ and $(4, 408, 60)$, respectively. The elliptic curves $E_D : f(x) = u^2 - Dv_0^2$ have at the least three rational points: two rational points at infinity and the affine point (x_0, u_0). It can be easily checked that, for the above values of D, the rational torsion subgroup $E_{D,\mathrm{tors}}(\mathbb{Q})$ of E_D is of order 2, except for $D = 34$, when $E_{D,\mathrm{tors}}(\mathbb{Q})$ has order 1. We conclude that the Mordell-Weil rank of $E_D(\mathbb{Q})$ is greater than 0 and thus $rd_D = \infty$.

Case $g = 2$. The three curves X_D are bielliptic. In Cremona notation [9, Table 1], the three elliptic quotients $X_D/\langle\omega_2\rangle$ are $26B2$, $38B2$, $58B2$, while $X_{26}/\langle\omega_{13}\rangle$, $X_{38}/\langle\omega_{19}\rangle$, $X_{58}/\langle\omega_{58}\rangle$ are $26A1$, $38A1$ and $58A1$, respectively.

For the three curves $X_D^{(2)}$, we have $rh_2 = r_2$ and hence the pairs $(D, 2)$ are not premodular over $\mathbb{Q}$. For $(26, 13)$ and $(38, 19)$, the single rational Heegner point corresponds to the projection of the two points at infinity because they are fixed points by ω_2. The preimages of the other two rational points are $(\pm\sqrt{-5}, \pm 26)$ and $(\pm\sqrt{-5}/2, \pm 19/4)$ for $D = 26$ and 38 respectively. In both cases $(-5, q)_5 = -1$ and hence $rd_q = 0$. For the remaining cases, we have $r_m = \infty$ and we can easily find $rh_m + 1$ rational points on $X_D^{(m)}$ satisfying the descent condition. □

When D is large enough so that the genus of X_D is at least 3, we know no explicit equations describing X_D and Theorem 4.5 can not be directly applied. For those pairs (D, m) such that $X_D^{(m)}(\mathbb{Q}) \neq \emptyset$, one can still show that (D, m) is not premodular over $\mathbb{Q}$ provided $X_D^{(m)}(\mathbb{Q})$ is a finite set and one can prove that $X_D^{(m)}(\mathbb{Q}) = X_D^{(m)}(\mathbb{Q})_h$. The cardinality of $X_D^{(m)}(\mathbb{Q})_h$ can be computed by Proposition 5.1, whereas the computation of $|X_D^{(m)}(\mathbb{Q})|$ is a much more difficult problem.

There exist several Atkin-Lehner quotients $X_D^{(m)}$ which are elliptic curves over $\mathbb{Q}$. It readily follows from comparing this table with [9, Table 1] that six of them are elliptic curves of rank zero (cf. [37, Table 6.4] for the complete list together with Weierstrass models for them)[3]. Namely, these are $X_{35}^{(7)}$, $X_{51}^{(3)}$, $X_{106}^{(53)}$, $X_{115}^{(23)}$, $X_{118}^{(59)}$ and $X_{202}^{(101)}$, which correspond to the elliptic curves $35A1$, $51A2$, $106D1$, $115A1$, $118D1$ and $202A_1$ respectively. In all cases but $(35, 7)$ the number of rational points is $r_m = 1$. Thus $rh_m = 1$ since $r_m - rh_m$ is even whenever $r_m < \infty$. In particular, we get $rd_m = 0$ and thus $(51, 3)$, $(106, 53)$, $(115, 23)$, $(118, 59)$ and $(202, 101)$ are not premodular over $\mathbb{Q}$. We can claim nothing about $(35, 7)$, since $r_m = 3$, $rh_m = 1$ and there is not an explicit equation for X_{35} at our disposal.

Combining the above with Theorems 4.9 and 5.2, together with those pairs exhibited in the last paragraph of Section 3, we obtain Theorem 1.4 (iii-iv-vi-vii) for all pairs but a few ones which deserve more attention: namely, those (D, m) such that $X_D^{(m)}$ is a curve of genus 2 and $X_D^{(m)}(\mathbb{Q}) \neq \emptyset$. Computing the full list of rational points on these curves is a harder task that we address in the next section.

[3]Table 6.4 in [37] contains a mistake: The list of elliptic curves $X_D^{(m)}$ over $\mathbb{Q}$ tabled there is not complete, as $X_{35}^{(7)}$, $X_{51}^{(3)}$ and $X_{115}^{(23)}$ are missing. Indeed, these elliptic curves are $35A1$, $51A2$ and $115A1$, respectively. This error was also reproduced in [38, Table 2], where it was wrongly claimed that these curves failed to have rational points over $\mathbb{Q}_5$, $\mathbb{Q}_{17}$ and $\mathbb{Q}_5$, respectively.

D	m	$X_D^{(m)}$	$X_D^{(m)}(\mathbb{Q})$
91	D	$Y^2 = -X^6 + 19X^4 - 3X^2 + 1$	$(0, \pm 1), (\pm 1, \pm 4), (\pm 3, \pm 28)$
123	D	$Y^2 = -9X^6 + 19X^4 + 5X^2 + 1$	$(0, \pm 1), (\pm 1, \pm 4), (\pm 1/3, \pm 4/3)$
141	D	$Y^2 = 27X^6 - 5X^4 - 7X^2 + 1$	$(\pm 1, \pm 4), (\pm 1/3, \pm 4/9),$ $(0, \pm 1), (\pm 11/13, \pm 4012/2197)$
142	2	$Y^2 = -16X^6 - 87X^4 - 146X^2 - 71$	$\emptyset$
142	D	$Y^2 = 16X^6 + 9X^4 - 10X^2 + 1$	$\pm\infty, (0, \pm 1), (\pm 1, \pm 4), (\pm 1/3, \pm 4/27)$
155	D	$Y^2 = 25X^6 - 19X^4 + 11X^2 - 1$	$\pm\infty, (\pm 1, \pm 4), (\pm 1/3, \pm 4/27)$
158	D	$Y^2 = -8X^6 + 9X^4 + 14X^2 + 1$	$(\pm 1, \pm 4), (0, \pm 1), (\pm 1/3, \pm 44/27)$
254	D	$Y^2 = 8X^6 + 25X^4 - 18X^2 + 1$	$(0, \pm 1), (\pm 1, \pm 2), (\pm 2, \pm 29)$
326	D	$Y^2 = X^6 + 10X^4 - 63X^2 + 4$	$\pm\infty, (0, \pm 2)$
446	D	$Y^2 = -16X^6 - 7X^4 + 38X^2 + 1$	$(0, \pm 1), (\pm 1, \pm 4)$

Table 3. Rational points on the bielliptic $X_D^{(m)}$ of genus 2

6. Covering techniques on bielliptic Shimura curves $X_D^{(m)}$ of genus 2

It was shown in [19, Proposition 4.2] that there exist exactly ten Shimura curve quotients $X_D^{(m)}$ which are bielliptic of genus 2. Applying Proposition 5.1, the triplets (D, m, rh_m) are $(91, 91, 10)$, $(123, 123, 10)$, $(141, 141, 10)$, $(142, 2, 0)$, $(142, 142, 10)$, $(155, 155, 10)$, $(158, 158, 10)$, $(254, 254, 8)$, $(326, 326, 4)$ and $(446, 446, 6)$.

In this section we study the set of rational points on these curves. We first list the $\mathbb{Q}$-rational points on each $X_D^{(m)}$ which are easily found by a short search. Table 3 lists some small rational points on the bielliptic curves X_D^m) of genus 2.

In this section, we show that Table 3 lists all rational points for each $X_D^{(m)}$. The case $D = 142$, $m = 2$ is straightforward: there are no points in $X_{142}^{(2)}(\mathbb{R})$ from which it follows that there are none in $X_{142}^{(2)}(\mathbb{Q})$. For the other values of D, m, each $X_D^{(m)}$ has points everywhere locally and so cannot be resolved in this way. We first recall the techniques from [2],[5],[15],[16],[17], which we summarize here in a simplified form adapted to the curves $X_D^{(m)}$. The fact that each $X_D^{(m)}$ is bielliptic allows a specialized version (cf. [15]) of the same ideas of [6]; similar methods are available for arbitrary hyperelliptic curves, as described in [4] and [5]. Each of the curves $X_D^{(m)}$ is of genus 2 and of the form

$$X_D^{(m)} : \ Y^2 = f_3X^6 + f_2X^4 + f_1X^2 + f_0, \text{ with } f_i \in \mathbb{Z}.$$

Any such curve $X_D^{(m)}$ has a map $(X, Y) \mapsto (X^2, Y)$ from $X_D^{(m)}$ to the elliptic curve $Y^2 = f_3w^3 + f_2w^2 + f_1w + f_0$, and map $(X, Y) \mapsto (1/X^2, Y/X^3)$ from $X_D^{(m)}$ to the elliptic curve $Z^2 = f_0x^3 + f_1x^2 + f_2x + f_3$. The Jacobian of $X_D^{(m)}$ is $\mathbb{Q}$-isogenous to the product of these elliptic curves over $\mathbb{Q}$ which, in all of these examples, each have rank 1 (and no nontrivial torsion). It follows that $\mathrm{Jac}(X_D^{(m)})(\mathbb{Q})$ has rank 2, and so Chabauty techniques [7] cannot be used, since they only apply when the rank of the Mordell-Weil group of the Jacobian is strictly less than the genus of the curve. It is therefore necessary to imitate the technique in [15], which we briefly summarize here in a simplified form suited to these examples. We first fix one of the above two elliptic curves – it does not matter which one; we shall use the

latter elliptic curve, since the resulting models will typically be slightly simpler. Define $E_D^{(m)}, (x_0, Z_0), \phi, t$ as follows.

$$E_D^{(m)}:\ Z^2 = f_0x^3 + f_1x^2 + f_2x + f_3, \tag{1}$$

$$(x_0, Z_0) \text{ generates } E_D^{(m)}(\mathbb{Q}), \tag{2}$$

$$\phi:\ X_D^{(m)} \longrightarrow E_D^{(m)} : (X, Y) \mapsto (1/X^2, Y/X^3), \tag{3}$$

$$t := \text{ root of } f_0x^3 + f_1x^2 + f_2x + f_3, \tag{4}$$

so that $E_D^{(m)}(\mathbb{Q})/2E_D^{(m)}(\mathbb{Q}) = \{\infty, (x_0, Z_0)\}$. Suppose that $(X, Y) \in X_D^{(m)}(\mathbb{Q})$. Then, applying ϕ, we let $x = 1/X^2$ and $Z = Y/X^3$ so that $(x, Z) \in E_D^{(m)}(\mathbb{Q})$. We recall the injective homomorphism $\mu : E_D^{(m)}(\mathbb{Q})/2E_D^{(m)}(\mathbb{Q}) \to \mathbb{Q}(t)^*/(\mathbb{Q}(t)^*)^2$ defined by $\mu(\infty) = 1$ and $\mu((x, Z)) = f_0(x - t)$ from [48, Chapter X, Theorem 1.1]. It follows that $\mu((x, Z))$ equals either 1 or $f_0(x_0 - t)$ in $\mathbb{Q}(t)^*/(\mathbb{Q}(t)^*)^2$. Hence, either $f_0Z^2/(x-t)$ or $(x_0-t)Z^2/(x-t)$ is a square. We can eliminate Z^2 using (1) and simplification yields:

$$\begin{array}{rl} \text{either} & f_0(f_0x^2 + (f_0t + f_1)x + (t^2f_0 + tf_1 + f_2)) \in (\mathbb{Q}(t)^*)^2 \\ \text{or} & (x_0 - t)(f_0x^2 + (f_0t + f_1)x + (t^2f_0 + tf_1 + f_2)) \in (\mathbb{Q}(t)^*)^2. \end{array}$$

Note that we do not really need (x_0, Z_0); we only need the square class of $x_0 - t$. This can already be determined from the 2-Selmer group of $E_D^{(m)}$, without computing an explicit generator of the Mordell-Weil group. In our examples, however, the curve $E_D^{(m)}$ has small coefficients and finding an actual generator is little more work.

Since $x = 1/X^2$ is a square itself, we can multiply either quantity with x without changing its square class. Hence, we have shown that if $(X, Y) \in X_D^{(m)}(\mathbb{Q})$ then there exists $y \in \mathbb{Q}(t)$ such that $(x, y) = (1/X^2, y)$ is a $\mathbb{Q}(t)$-rational point on one of the curves

$$F_D^{(m)} : y^2 = f_0x\big(f_0x^2 + (f_0t + f_1)x + (t^2f_0 + tf_1 + f_2)\big), \tag{5}$$

$$G_D^{(m)} : y^2 = (x_0 - t)x\big(f_0x^2 + (f_0t + f_1)x + (t^2f_0 + tf_1 + f_2)\big). \tag{6}$$

This gives a strategy for trying to prove that we have found all of $X_D^{(m)}(\mathbb{Q})$; it is sufficient find all $(x, y) \in F_D^{(m)}(\mathbb{Q}(t))$ such that $x \in \mathbb{Q}$ and similarly for $G_D^{(m)}$. This can be attempted using the techniques in [2],[5],[15],[16],[17], which apply local techniques to bound the number of such points. Since these articles already contain several worked examples of this type, we merely provide a brief sketch of one case, to give an idea of the general strategy, and to allow the reader to interpret the tabular summary given later. This will be followed by a description of any unusual features of difficult special cases. The full details of the computations are available at:

`http://www.cecm.sfu.ca/~nbruin/shimura/`

Consider $X_{142}^{(142)} : Y^2 = 16X^6 + 9X^4 - 10X^2 + 1$, where we wish to show that the only $\mathbb{Q}$-rational points have X-coordinate $\infty, 0, 1, \pm 1/3$ (we use ∞, depending on context, as the notation both for the point at infinity and its X-coordinate). Then $E_{142}^{(142)}$ of (1) is the elliptic curve $V^2 = x^3 - 10x^2 + 9x + 16$ over $\mathbb{Q}$, which has rank 1, with generator $(x_0, V_0) = (1, 4)$. Under $(X, Y) \mapsto (1/X^2, Y/X^3)$, the known

points in $X_{142}^{(142)}(\mathbb{Q})$ map to $(0,\pm 4)$, ∞, $(1,\pm 4)$ and $(9,\pm 4) = 2(1,\pm 4)$ in $E_{142}^{(142)}(\mathbb{Q})$. Letting t be the cubic number satisfying $t^3-10t^2+9t+16$, the curves (5),(6) become

$$\begin{aligned} F_{142}^{(142)} : y^2 &= x^3 + (t-10)x^2 + (t^2-10t+9)x, \\ G_{142}^{(142)} : y^2 &= (1-t)\big(x^3 + (t-10)x^2 + (t^2-10t+9)x\big). \end{aligned}$$

Our known points in $E_{142}^{(142)}(\mathbb{Q})$ induce $\infty, (0,0), (9,\pm(3t^2-15t)/4) \in F_{142}^{(142)}(\mathbb{Q}(t))$ and $\infty, (0,0), (1,\pm 4) \in G_{142}^{(142)}(\mathbb{Q}(t))$. It is sufficient to show that there are no other points (x,y) in $F_{142}^{(142)}(\mathbb{Q}(t))$ or $G_{142}^{(142)}(\mathbb{Q}(t))$ for which $x \in \mathbb{Q}$. Note that, in each case $\infty, (0,0)$ give the entire torsion group, and so we have a point of infinite order. Furthermore, a standard complete 2-descent or 2-isogeny descent, as recently implemented by N. Bruin in Magma [30] (or for an older version, see [3]), gives a Selmer bound of 1 on the rank, and so both $F_{142}^{(142)}(\mathbb{Q}(t))$ or $G_{142}^{(142)}(\mathbb{Q}(t))$ have rank 1.

Consider, for example, the second curve G_{142}. One can easily check with finite field arguments, that $R = (0,0) + 4(1,4) \in G_{142}(\mathbb{Q}(t))$ is in the kernel of reduction modulo 3, which is inert in $\mathbb{Q}(t)/\mathbb{Q}$ (so that $\langle R\rangle$ is of finite index in $G_{142}(\mathbb{Q}(t))$), and can check that $\langle R\rangle$, $(0,0)+\langle R\rangle$, $(1,4)+\langle R\rangle$ and $(1,-4)+\langle R\rangle$ and the only 4 cosets containing possible $(x,y) \in G_{142}^{(142)}(\mathbb{Q}(t))$ for which $x \in \mathbb{Q}$ (note that, although $(0,0),(1,4)$ do not generate $G_{142}(\mathbb{Q}(t))$, we do have that $(0,0), (1-t, t^2-t-4)$ generate $G_{142}(\mathbb{Q}(t))$ and that $(1,4) = (0,0) + 2(1-t, t^2-t-4)$, so that $(0,0),(1,4)$ can be treated as if they are generators for the purposes of our 3-adic argument). This means that we only need to consider $(x,y) = nR, (0,0)+nR, (1,4)+nR, (1,-4)+nR$, for $n \in \mathbb{Z}_3$, and it is sufficient in each case to show that $n=0$ is the only case where $x \in \mathbb{Q}$. Using the formal exp and log functions in the 3-adic formal group [15], we can express nR as $\exp(n \cdot \log(R))$ and deduce that

$$1/(x\text{-coordinate of } nR) \equiv 8\cdot 3^2 n^2 + (2\cdot 3^2 n^2 + 3^3 n^4)t + (3^2 n^2 + 3^3 n^4)t^2 \pmod{3^4},$$

where each coefficient of a power of t is a power series in n defined over $\mathbb{Z}_3$ whose coefficients converge to 0. If $x \in \mathbb{Q}$ then the coefficients of t and t^2 must be 0. Taking the coefficient of t, we have a power series n for which $n=0$ is a known double root, and for which the coefficient of n^2 has 3-adic absolute value strictly greater than all subsequent coefficients of powers of n. It follows that $n=0$ is the only solution. A 3-adic analysis of $(0,0)+nR, (1,4)+nR, (1,-4)+nR$ also shows that these can only have $\mathbb{Q}$-rational x-coordinate when $n=0$. We know that the only $(x,y) \in G_{142}^{(142)}(\mathbb{Q}(t))$ with $x \in \mathbb{Q}$ are the points with $x = \infty, 0, 1$. Similarly, a 5-adic argument shows that the only $(x,y) \in F_{142}^{(142)}(\mathbb{Q}(t))$ with $x \in \mathbb{Q}$ are the points with $x = \infty, 0, 9$, as required.

These methods can be applied when the ranks of $F_D^{(m)}(\mathbb{Q}(t))$ and $G_D^{(m)}(\mathbb{Q}(t))$ are less than the degree of $\mathbb{Q}(t)$, that is, less than 3. Fortune is in our favour, since the ranks for these examples indeed all turn out to be $0, 1$ or 2. Note that, in the above rank 1 example, there was information to spare, since either the coefficient of t or that of t^2 could be used to bound the number of solutions. For the rank 2 cases, one can still obtain a bound, but the information from both power series must be used.

The following table summarizes the computations.

$D=m$	$F_D^{(m)}$ and $G_D^{(m)}$	r	$x\in\mathbb{Q}$	p
91	$F_D^{(m)}: y^2 = x^3+(t-3)x^2+(t^2-3t+19)x$	1	$\infty, 0, \frac{1}{9}$	5
91	$G_D^{(m)}: y^2 = (1-t)\big(x^3+(t-3)x^2+(t^2-3t+19)x\big)$	1	$\infty, 0, 1, 4$	5
123	$F_D^{(m)}: y^2 = x^3+(t+5)x^2+(t^2+5t+19)x$	1	$\infty, 0, 9$	5
123	$G_D^{(m)}: y^2 = (1-t)\big(x^3+(t+5)x^2+(t^2+5t+19)x\big)$	1	$\infty, 0, 1$	7
141	$F_D^{(m)}: y^2 = x^3+(t-7)x^2+(t^2-7t-5)x$	1	$\infty, 0, 9$	7
141	$G_D^{(m)}: y^2 = (1-t)\big(x^3+(t-7)x^2+(t^2-7t-5)x\big)$	2	$\infty, 0, 1, \frac{169}{121}$	7
142	$F_D^{(m)}: y^2 = x^3+(t-10)x^2+(t^2-10t+9)x$	1	$\infty, 0, 9$	5
142	$G_D^{(m)}: y^2 = (1-t)\big(x^3+(t-10)x^2+(t^2-10t+9)x\big)$	1	$\infty, 0, 1$	3
155	$F_D^{(m)}: y^2 = x^3+(t-11)x^2+(t^2-11t+19)x$	1	$\infty, 0, 9$	7
155	$G_D^{(m)}: y^2 = (t-1)\big(x^3+(t-11)x^2+(t^2-11t+19)x\big)$	2	$\infty, 0, 1$	3
158	$F_D^{(m)}: y^2 = x^3+(t+14)x^2+(t^2+14t+9)x$	2	$\infty, 0, 1, 9$	5
158	$G_D^{(m)}: y^2 = (-3-t)\big(x^3+(t+14)x^2+(t^2+14t+9)x\big)$	0	$\infty, 0$	–
254	$F_D^{(m)}: y^2 = x^3+(t-18)x^2+(t^2-18t+25)x$	0	$\infty, 0$	–
254	$G_D^{(m)}: y^2 = (1-t)\big(x^3+(t-18)x^2+(t^2-18t+25)x\big)$	2	$\infty, 0, 1, \frac{1}{4}$	5
326	$F_D^{(m)}: y^2 = 4\big(4x^3+(4t-63)x^2+(4t^2-63t+10)x\big)$	1	$\infty, 0$	5
326	$G_D^{(m)}: y^2 = -t\big(4x^3+(4t-63)x^2+(4t^2-63t+10)x\big)$	0	$\infty, 0$	–
446	$F_D^{(m)}: y^2 = x^3+(t+38)x^2+(t^2+38t-7)x$	1	$\infty, 0, 1$	3
446	$G_D^{(m)}: y^2 = (5-t)\big(x^3+(t+38)x^2+(t^2+38t-7)x\big)$	0	$\infty, 0$	–

Table 4. Summary of computations

The second column gives the models for the curves $F_D^{(m)}$ and $G_D^{(m)}$, and the third column gives the rank over $\mathbb{Q}(t)$, where the cubic number t is as defined in (4). In all cases, the torsion over $\mathbb{Q}(t)$ consists only of ∞ and $(0,0)$. The fourth column gives the list of all $x\in\mathbb{Q}$ which are x-coordinates of a point (x,y) on the curve and defined over $\mathbb{Q}(t)$; the final column gives a prime p such that a p-adic argument proves that no other such $x\in\mathbb{Q}$ are possible. Of course, the rank 0 cases are trivial and so no such prime is required.

The computations referenced above verify the following theorem.

Theorem 6.1. *The curves $X_D^{(m)}$ have no $\mathbb{Q}$-rational points apart from those given in Table 3.*

We conclude with mention of a few special features of the computations. Recall that in the sketched worked example for the case $D=m=142$, it turned out that the Selmer bound from a complete 2 descent was the same as the rank. However, for the six cases $F_{91}^{(91)}, G_{91}^{(91)}, G_{123}^{(123)}, F_{155}^{(155)}, F_{254}^{(254)}, G_{326}^{(326)}$, this bound is two greater than the actual rank. In order to find a sharp bound, one can perform a complete 2-descent on the 2-isogenous curve. It follows that in each of these cases there are elements of order 2 in the Shafarevich-Tate group over $\mathbb{Q}(t)$.

In the other direction, there are two curves $G_{155}^{(155)}$ and $F_{326}^{(326)}$, where the group generated by images of the known points in $X_D^{(m)}(\mathbb{Q})$ is less than the actual rank, so one must search for the missing independent points of infinite order. For example,

the 2-Selmer bound on the rank of $G_{155}^{(155)}(\mathbb{Q}(t))$ is 2, and the images of the known points in $X_{155}^{(155)}(\mathbb{Q})$ give only $\infty, (0,0), (1,4)$, of which only $(1,4)$ is of infinite order, so that we are missing an independent point of infinite order. In this case, a naive short search discovers the required point $((t^2+10t+25)/4, 13t^2-11t+10)$. The 2-Selmer bound on the rank of $F_{326}^{(326)}(\mathbb{Q}(t))$ is 1, and the images of the known points in $X_{326}^{(326)}(\mathbb{Q})$ do not give any points of infinite order, and in fact the required point is

$$(x,y) = \big(\tfrac{63540t^2-1005167t+228495}{2888}, \tfrac{90341332t^2-1429154471t+325168047}{109744}\big).$$

This could not be found by a naive search, and we needed to use the improved search techniques described in the appendix of [5], and recently implemented by N. Bruin into Magma [30].

As we explain in the last paragraph of Section 6, Theorem 6.1 completes the proof of Theorem 1.4 (iii-iv-vi-vii). As we indicate at the end of Section 3, parts (i-ii) of Theorem 1.4 follow from Theorem 3.2 and Proposition 3.4. Part (v) is proved in Corollary 4.10 and this gives the full proof of Theorem 1.4.

From Theorem 6.1 and an analogue of Theorem 4.9 we can also derive the following result.

Corollary 6.2. *For each of the pairs*

$$(D,m) = (141,141), (142,142), (254,254),$$

the number of $\overline{\mathbb{Q}}$-isomorphism classes of abelian surfaces $A/\overline{\mathbb{Q}}$ that admit an embedding $\iota : \mathbb{Q}(\sqrt{m}) \hookrightarrow \mathrm{End}_{\overline{\mathbb{Q}}}(A))$, whose field of moduli is $\mathbb{Q}$, and such that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_D$, are 2, 1 and 1, respectively.

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