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COMPACT ELEMENTS AND SMALLEST FAITHFUL REPRESENTATION OF C^*-ALGEBRAS

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ABSTRACT. Let A be a C^*-algebra which either allows a faithful separable representation or is postliminal. We prove that A then admits a smallest faithful representation if and only if the ideal of compact elements is an essential ideal in A.

INTRODUCTION

A faithful representation (H, φ) of a C^*-algebra A will be called a *smallest faithful representation of* A, if (H, φ) is contained in every faithful representation of A. It follows from [7], Theorem 5.1.5, that any two smallest faithful representations of a given C^*-algebra A are unitarily equivalent. The main result of this paper says that for C^*-algebras which allow a faithful separable representation and for postliminal C^*-algebras, there is a smallest faithful representation if and only if the ideal I of compact elements is an essential ideal, i.e. iff I has the annihilator $\{0\}$ in the algebra. The "if" part of this result stays true for arbitrary C^*-algebras.

The proof of the "if" part can be outlined as follows: The ideal I of compact elements admits a smallest faithful representation (H, φ_I). If I is an essential ideal, then the unique extension of (H, φ_I) to A over H is a smallest faithful representation of A.

The "only if" part of the proof is sketched below: Let (K, ψ) be an irreducible representation of A. Consider the ideal consisting of the elements of A which lie in the kernel of every irreducible representation not unitarily equivalent to (K, ψ). This ideal is non-zero iff (K, ψ) is contained in the smallest faithful representation (H, φ) of A. To every irreducible representation (K, ψ) contained in (H, φ) we can thus associate a simple ideal. Let F be the family of ideals defined in this way. The elements of F are mutually orthogonal. Since A admits a smallest faithful representation, the elements of F do so as well, and the intersection of the annihilators of the elements of F is the zero ideal. Thus, the ideal generated by the union of the elements of F is an essential ideal of A. The elements of F consist entirely of compact elements of A. In the separable case this is an immediate consequence of Rosenberg's

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Theorem. It follows that the ideal generated by the union of the elements of F is contained in I and that I is essential too.

I. Compact elements

I.1 **Definitions and notational conventions.** Let I be a subset of the C^*-algebra A, and suppose (H, φ) is a representation of A. Let K be a linear subspace of H. The linear span of the set $\{\varphi(i)k | i \in I, k \in K\}$ is denoted by $\varphi(I)K$. We denote the closure of $\varphi(I)K$ by $\varphi(I)K^-$. $\varphi(I)H^-$ is called the *essential subspace of I under the representation* (H, φ). Let B be a C^*-subalgebra of the C^*-algebra A, and suppose K is a closed linear subspace of H, invariant for B. Then the restriction of (H, φ) to K and B is denoted by $(H, \varphi)_{B,K}$. We denote $(H, \varphi)_{B,\varphi(B)H^-}$ simply by $(H, \varphi)_B$.

We write $(H, \varphi) \cong (K, \psi)$ to express the fact that (H, φ) and (K, ψ) are unitarily equivalent representations. If (K, ψ) is a subrepresentation of (H, φ) we write $(K, \psi) \leq (H, \varphi)$. $(H_1, \varphi_1) \leq (H_2, \varphi_2)$ and $(H_2, \varphi_2) \leq (H_1, \varphi_1)$ imply $(H_1, \varphi_1) \cong (H_2, \varphi_2)$ (see e.g. [7], Theorem 5.1.5).

An element a of a C^*-algebra A is said to be *compact in* A, if there is a faithful representation (H, φ) of A such that $\varphi(a)$ is a compact operator. We define *the dimension of a in A* to be the minimum rank the image of a takes on under faithful representations of A. We denote this number by $\dim_A a$ and say that *a is finite dimensional in A* if $\dim_A a < \infty$. If p and q are Murray von Neumann equivalent projections of A we write $p \sim q$. If we speak about an ideal of A, it is always meant to be a closed two-sided ideal.

We give a brief account of what needs to be known about compact elements of C^*-algebras in the sequel (see also [1], [5], [11], [12], [13]).

I.2 **Proposition.** *Let A be a C^*-algebra.*

(a) *Let I be an ideal of A. If (H, φ_I) is a non-degenerate representation of I, then there is a unique extension (H, φ_A) of (H, φ_I) defined on all of A.*

If (H, φ_I) is faithful, then $\ker(\varphi_A) = \operatorname{Ann}(I)$. *If, moreover, I is essential, then (H, φ_A) is faithful too.*

(b) *Suppose that B is a hereditary subalgebra of A and that $b \in B$. Then $\dim_B b = \dim_A b$ and b is compact as an element of B iff b is compact as an element of A. In particular this applies in the special case where B is an ideal.*

(c) *The set of compact elements of A is an ideal. It is generated by the set of one-dimensional projections. The function $\dim_A$, defined on A, is invariant under Murray von Neumann Equivalence. The ideal generated by a one-dimensional projection of A is simple. Equivalent one-dimensional projections generate the same ideal. The ideal of compact elements of A is the restricted sum of the simple ideals obtained in this way.*

(d) *Let I be the ideal of compact elements of A. To each simple subideal I_λ of I corresponds one and only one equivalence class of irreducible representations of I, whose restriction to I_λ is non-trivial. This correspondence is bijective. But every representation of I is a direct product of irreducible representations. Together with* (c) *this implies that I admits a smallest faithful representation.*

(e) *Every faithful representation (H, φ) of a C^*-algebra contains a quasi-equivalent, faithful subrepresentation, which maps every n-dimensional element to an n-dimensional operator and every compact element to a compact operator, simultaneously.*

Proof. We must give a few hints on how to prove (c), (d) and (e).

A projection $p \in A$ is one-dimensional iff pAp is a one-dimensional C^*-algebra.

To show the "if" part, let $\tau_p: A \to \mathbf{C}$ be defined by the equation $pap = \tau_p(a)p$ $(a \in A)$. We call τ_p the *associated state of* p *on* A. Now, pAp is a hereditary subalgebra of A. A state on a hereditary subalgebra of A allows exactly one extension to a state on A, and if the former is pure, the latter will be equally so (see e.g. [7], p. 91, Theorem 3.3.9; p. 148, Theorem 5.1.13). Thus τ_p is a pure state on A, and the representation $(H_{\tau_p}, \varphi_{\tau_p})$ of A associated with the pure state τ_p is irreducible. We call $(H_{\tau_p}, \varphi_{\tau_p})$ *the representation of* A *associated with* p. By use of the transitivity theorem of R. Kadison, it can be shown that $\operatorname{rank} \varphi_{\tau_p}(p) = 1$. Moreover, if q is a one-dimensional projection of A, then

$$\varphi_{\tau_p}(q) \neq 0 \Leftrightarrow p \sim q .$$

Now suppose (H, φ) is a representation of A such that $\varphi(p) \neq 0$. Then $(H_{\tau_p}, \varphi_{\tau_p}) \leq (H, \varphi)$. To see this, note that $x \in \operatorname{im}(\varphi(p))$ and $\|x\| = 1$ imply $(\varphi(a)x|x) = \tau_p(a)$ $(a \in A)$. Therefore, if (H, φ) is faithful, then it contains every representation associated with one-dimensional projections of A. Take a quasiequivalent subrepresentation (K, ψ) of (H, φ), such that the multiplicity of every subrepresentation associated with a one-dimensional projection is one. (K, ψ) is faithful, and $\operatorname{rank}(\psi(p)) = 1$ for every projection p with $\dim_{\mathbf{C}} pAp = 1$.

(K, ψ) maps every element of the ideal of compact elements I to a compact operator, since I is generated by the one-dimensional projections of A (see below). Moreover, any representation of I is a direct product of representations associated with one-dimensional projections. Thus I admits a smallest faithful representation.

I is generated by the one-dimensional projections: Reduce the problem to the Hermitian case. Let thus $a \in I$ hermitian and suppose that (H, φ) is a faithful representation of A, such that $\varphi(a)$ is a compact operator. Then there is a strictly decreasing sequence $(\lambda_n)_{\mathbf{N}}$ of positive real numbers with lower bound zero and a sequence $(p_n)_{\mathbf{N}}$ of finite-dimensional projections in $B(H)$ such that the series $\sum_{\mathbf{N}} \lambda_n \cdot p_n$ converges to $\varphi(a)$. Functional calculus shows that $p_n \in \varphi(A)$ for every $n \in \mathbf{N}$. The problem is thus reduced to the case where a is a finite-dimensional projection of A. Then $\dim_{\mathbf{C}} aAa < \infty$, i.e. aAa is of the form $M_{n_1}(\mathbf{C}) \oplus \cdots \oplus M_{n_k}(\mathbf{C})$ with $n_1, \ldots, n_k$ natural numbers. It follows that $a = 1_{aAa}$ is a sum of $n_1 + \cdots + n_k$ one-dimensional projections.

The ideal generated by a one-dimensional projection p is the simple ideal associated with the irreducible representation $(H_{\tau_p}, \varphi_{\tau_p})$ (see Introduction and Lemma II.3). □

II. Smallest faithful representations

II.1 **Lemma.** *Let A be a C^*-algebra allowing of a smallest faithful representation (H, φ). Then (H, φ) is unitarily equivalent to a certain direct sum of irreducible representations of A, every summand appearing with multiplicity one.*

Proof. If $\operatorname{PS}(A)$ denotes the set of pure states on A, then $\bigoplus_{\operatorname{PS}(A)}(H_\tau, \varphi_\tau)$ is a direct sum of irreducible representations which is faithful and therefore contains (H, φ). However, subrepresentations of a direct sum of irreducible representations are unitarily equivalent to direct sums of irreducible representations themselves. In particular, (H, φ) is unitarily equivalent to a direct sum of irreducible representations. The statement about multiplicity one is evident. □

II.2 **Proposition.** *Let $(I_\lambda)_\Lambda$ be a family of ideals of a C^*-algebra A, such that the two following conditions hold*:

(i) $\bigcap_\Lambda \operatorname{Ann}(I_\lambda) = \{0\}$.

(ii) *For any two distinct elements $\mu, \nu \in \Lambda$, $I_\mu \cap I_\nu = \{0\}$.*

Then A admits a smallest faithful representation iff I_λ admits a smallest faithful representation for every $\lambda \in \Lambda$.

Proof. We start with the assumption that for each $\lambda \in \Lambda$, there is a smallest faithful representation $(H_\lambda, \varphi'_\lambda)$ of I_λ. Let $(H_\lambda, \varphi_\lambda)$ denote its unique extension to A. Then $\ker(\varphi_\lambda)$ equals $\operatorname{Ann}(I_\lambda)$. Therefore $\bigcap_\Lambda \ker \varphi_\lambda = \bigcap_\Lambda \operatorname{Ann}(I_\lambda) = \{0\}$. But this means that the direct sum $\bigoplus_\Lambda (H_\lambda, \varphi_\lambda)$ is faithful. Now, for a $\lambda \in \Lambda$ and a faithful representation (H, φ) of A, the restriction of (H, φ) to I_λ is faithful, and therefore contains the smallest faithful representation of I_λ. It follows that (H, φ) contains $(H_\lambda, \varphi_\lambda)$. From the fact that for distinct $\mu, \nu \in \Lambda$ the essential spaces of I_μ, I_ν in H are mutually orthogonal, we conclude that (H, φ) contains $\bigoplus_\Lambda (H_\lambda, \varphi_\lambda)$. Hence, $\bigoplus_\Lambda (H_\lambda, \varphi_\lambda)$ is a smallest faithful representation of A.

We now assume that A admits a smallest faithful representation (H, φ). Let $\mu \in \Lambda$, and suppose (H_μ, φ_μ) is a faithful non-degenerate representation of I_μ. Let (H_μ, ψ_μ) be the unique extension of (H_μ, φ_μ) to A. Then the faithful representation $(H_\mu, \psi_\mu) \oplus (\bigoplus_{\Lambda \setminus \{\mu\}} (H, \varphi)_{(A, \varphi(I_\lambda)H^-)})$ of A contains $(H, \varphi) \cong \bigoplus_\Lambda (H, \varphi)_{(A, \varphi(I_\lambda)H^-)}$. But $(H, \varphi)_{(A, \varphi(I_\mu)H^-)}$ and $(H, \varphi)_{(A, \varphi(I_\lambda)H^-)}$ are disjoint $(\lambda \neq \mu)$. This implies that $(H, \varphi)_{(A, \varphi(I_\mu)H^-)} \leq (H_\mu, \psi_\mu)$ and $(H, \varphi)_{(I_\mu, \varphi(I_\mu)H^-)} \leq (H_\mu, \varphi_\mu)$. $(H, \varphi)_{(I_\mu, \varphi(I_\mu)H^-)}$ is thus a smallest faithful representation of I_μ. □

II.3 **Lemma.** *Let (H, φ) be an irreducible representation of a C^*-algebra A, and suppose I is the intersection of the kernels of the irreducible representations of A, disjoint from (H, φ). Then I is a simple ideal of A. Moreover, if I is non-zero, then* $\operatorname{Ann}(I)$ *and* $\ker(\varphi)$ *coincide.*

Proof. The proof of this lemma is easily found by the reader. □

II.4 **Definition.** We say that a C^*-algebra A is *completely decomposable into its simple ideals*, iff $\bigcap_\Lambda \operatorname{Ann}(I_\lambda) = \{0\}$ for a family $(I_\lambda)_\Lambda$ of simple ideals of A. Note that $(I_\lambda)_\Lambda$ then contains every non-zero simple ideal and satisfies conditions (i) and (ii) of Proposition II.2.

II.5 **Theorem.** *A C^*-algebra A has a smallest faithful representation iff the following conditions are satisfied*:

(i) *A is completely decomposable into its simple ideals.*

(ii) *Every simple ideal of A admits a smallest faithful representation.*

Proof. Suppose A satisfies conditions (i) and (ii). Then Proposition II.2 applies to the family of simple ideals of A, and it follows that A has a smallest faithful representation.

Conversely, suppose A admits a smallest faithful representation (H, φ). Let $(H_\gamma, \varphi_\gamma)_\Gamma$ be a family containing exactly one representative of each equivalence class of irreducible representations of A. $\exists\Omega \subset \Gamma$ s.t. $(H, \varphi) \cong \bigoplus_{\Omega}(H_\omega, \varphi_\omega)$ (see Lemma II.1). Let $I_\mu := \bigcap_{\Gamma\setminus\{\mu\}} \ker(\varphi_\gamma)$ $(\mu \in \Gamma)$. Then I_μ is a simple ideal $(\mu \in \Gamma)$ (see Lemma II.3). Consider $\Lambda := \{\lambda \in \Gamma | I_\lambda \neq \{0\}\}$. Then $\mu \in \Omega \Leftrightarrow \bigoplus_{\Gamma\setminus\{\mu\}}(H_\gamma, \varphi_\gamma)$ is not faithful $\Leftrightarrow \bigcap_{\Gamma\setminus\{\mu\}} \ker(\varphi_\gamma) \neq \{0\} \Leftrightarrow \mu \in \Lambda$, i.e. $\Lambda = \Omega$.

Now, $\bigcap_\Lambda \mathrm{Ann}(I_\lambda) = \bigcap_\Omega \ker(\varphi_\lambda) = \{0\}$, since $\ker \varphi_\lambda = \mathrm{Ann}(I_\lambda)$ $(\lambda \in \Lambda)$ (see Lemma II.3). This implies (i).

(ii) follows from Proposition II.2 and from (i). □

The next lemma is a corollary of Rosenberg's Theorem (see e.g. [6], p. 505).

II.6 **Lemma.** *Let A be a simple C^*-algebra, faithfully representable over a separable Hilbert space and allowing of a smallest faithful representation. Then there is a separable Hilbert space H such that A is $*$-isomorphic to $K(H)$, the algebra of compact operators over H.*

Proof. By simplicity of A, every non-zero representation of A is faithful and contains the smallest faithful representation of A. Therefore, every non-zero irreducible representation of A is unitarily equivalent to the smallest faithful representation of A, hence is unique to within unitary equivalence. Since A is assumed to be separable too, Rosenberg's Theorem applies to A. Consequently, A is $*$-isomorphic to $K(H)$ for some separable Hilbert space H. □

II.7 **Theorem.** *Let A be a C^*-algebra for which there is a faithful separable representation. Then A admits a smallest faithful representation if and only if the ideal of compact elements of A is an essential ideal of A.*

Proof. Suppose the ideal I of compact elements of A is essential. The family $\{I\}$ of ideals satisfies conditions (i) and (ii) of Proposition II.2, and by Proposition I.2(d) I has a smallest faithful representation. It follows from Proposition II.2 that A allows a smallest faithful representation.

On the other hand, if A admits a smallest faithful representation, then, by Theorem II.5, A is decomposable into its simple ideals $(I_\lambda)_\Lambda$, all of which admit a smallest faithful representation. Since A can be represented faithfully over a separable Hilbert space, the smallest faithful representations of the I_λ act on separable Hilbert spaces as well. It follows from Lemma II.6 that I_λ consists entirely of elements which are compact in I_λ $(\lambda \in \Lambda)$. But the ideals I_λ are hereditary subalgebras of A, hence their elements are also compact in A, i.e. $I_\lambda \subset I$ (see Proposition 1.2(b)). Then $\mathrm{Ann}(I_\lambda) \supset \mathrm{Ann}(I)$ $(\lambda \in \Lambda)$. It follows that $\{0\} = (\bigcap_\Lambda \mathrm{Ann}(I_\lambda)) \supset \mathrm{Ann}(I)$. I is therefore an essential ideal of A. □

II.8 **Theorem.** *A postliminal C^*-algebra A admits a smallest faithful representation if and only if the ideal of compact elements is an essential ideal of A.*

Proof. The "if" part of the proof of Theorem II.7 remains valid for general C^*-algebras. A fortiori it applies to postliminal C^*-algebras.

Conversely, if A admits a smallest faithful representation, then, in complete analogy to the "only if" part of the proof of Theorem II.7, it suffices to show that the simple ideals I_λ of A consist entirely of elements which are compact in I_λ. By simplicity of I_λ, the smallest faithful representation $(H_\lambda, \psi_\lambda)$ of I_λ is irreducible $(\lambda \in \Lambda)$. Let $(H_\lambda, \varphi_\lambda)$ be the extension of $(H_\lambda, \psi_\lambda)$ to A.

Then, since $(H_\lambda, \varphi_\lambda)$ is non-zero on I_λ and since A is postliminal, we have: $\{0\} \neq \varphi_\lambda(I_\lambda) \cdot K(H_\lambda) \subset K(H_\lambda) \subset \varphi_\lambda(A)$. Therefore, $\varphi_\lambda^{-1}(\varphi_\lambda(I_\lambda) \cdot K(H_\lambda))$ is a non-zero ideal in I_λ. Then $\varphi_\lambda^{-1}(\varphi_\lambda(I_\lambda) \cdot K(H_\lambda)) = I_\lambda$, since I_λ is simple, and $\psi_\lambda(I_\lambda) \subset K(H_\lambda)$. ψ_λ being faithful implies that every element of I_λ is compact in I_λ. $\square$

II.9 **Historical remark and conclusion.** In his paper [8] Naimark rose the question of whether a C^*-algebra which admits a unique irreducible representation is $*$-isomorphic to the algebra of compact operators over some Hilbert space. Using a partial result provided by Naimark in [8], A. Rosenberg gave a proof of the above statement in the separable case in [10], p. 529. However, as mentioned by Pedersen in [9], p. 255, and by Fell and Doran in [6], p. 506, the general case remains an open problem till today. If Rosenberg's Theorem can be generalized to a class of C^*-algebras closed under the taking of subideals, then Theorem II.7 immediately generalizes to this class of C^*-algebras.

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Bibliography

1. B. A. Barnes, G. J. Murphy, M. R. Smyth, and T. T. West, *Riesz and Fredholm theory in Banach algebras*, Research Notes in Math., vol. 67, Pitman, London, 1982.
2. C. Constantinescu, *Funktionalanalysis*, Autographie zur Vorlesung an der ETH Zürich, 1992.
3. J. Dixmier, *Von Neumann algebras*, North-Holland, Amsterdam, 1981.
4. ——, *C^*-algebras*, North-Holland, Amsterdam, 1982.
5. J. A. Erdos, *On certain elements of C^*-algebras*, Illinois J. Math. **15** (1971), 682–693.
6. J. M. G. Fell and R. S. Doran, *Representations of $*$-algebras, locally compact groups, and Banach $*$-algebraic bundles*, Vol. 1, Academic Press, New York, 1988.
7. G. J. Murphy, *C^*-algebras and operator theory*, Academic Press, San Diego, CA, 1990.
8. M. A. Naimark, *On a problem of the theory of rings with involution*, Uspekhi Mat. Nauk (N.S.) **6** (1951), 160–164. (Russian)
9. G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Math. Soc. Monograph, no. 14, Academic Press, New York, 1979.
10. A. Rosenberg, *The number of irreducible representations of simple rings with no minimal ideals*, Amer. J. Math. **75** (1953), 523–530.
11. K. Vala, *Sur les elements compacts d'une algèbre normée*, Ann. Acad. Sci. Fenn. **407** (1967).
12. K. Ylinen, *Compact and finite-dimensional elements of normed algebras*, Ann. Acad. Sci. Fenn. **428** (1968).
13. ——, *A note on the compact elements of C^*-algebras*, Proc. Amer. Math. Soc. **35** (1972), 305–306.

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