Galois Representations attached to Algebraic Automorphic Representations

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For Sophie

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Abstract

This thesis is concerned with the Langlands program; namely the global Langlands correspondence, Langlands functoriality, and a conjecture of Gross in [21].

In chapter 1, we cover the most important background material needed for this thesis. This includes material on reductive groups and their root data, the definition of automorphic representations and a general overview of the Langlands program, and Gross' conjecture concerning attaching l-adic Galois representations to automorphic representations on certain reductive groups G over $\mathbb{Q}$.

In chapter 2, we show that odd-dimensional definite unitary groups satisfy the hypotheses of Gross' conjecture and verify the conjecture in this case using known constructions of automorphic l-adic Galois representations. We do this by verifying a specific case of a generalisation of Gross' conjecture; one should still get l-adic Galois representations if one removes one of his hypotheses but with the cost that their image lies in ${}^{C}G(\overline{\mathbb{Q}}_l)$ as opposed to ${}^{L}G(\overline{\mathbb{Q}}_l)$. Such Galois representations have been constructed for certain automorphic representations on G, a definite unitary group of arbitrary dimension, and there is a map ${}^{C}G(\overline{\mathbb{Q}}_l) \to {}^{L}G(\overline{\mathbb{Q}}_l)$ precisely when G is odd-dimensional.

In chapter 3, which forms the main part of this thesis, we show that $G = \mathrm{U}_n(B)$ where B is a rational definite quaternion algebra satisfies the hypotheses of Gross' conjecture. We prove that one can transfer a cuspidal automorphic representation π of G to a π' on Sp_{2n} (a Jacquet-Langlands type transfer) provided it is Steinberg at some finite place. We also prove this when B is indefinite. One can then transfer π' to an automorphic representaion of GL_{2n+1} using the work of Arthur in [1]. Finally, one can attach l-adic Galois representations to these automorphic representations on GL_{2n+1}, provided we assume π is regular algebraic if B is indefinite, and show that they have orthogonal image.

Contents

Introduction

Let G be a connected reductive group over F, a local or global field of characteristic 0. The Langlands correspondence is meant to parametrise certain representations associated to G; these are admissible representations $\pi : G(F) \to \mathrm{GL}(V)$ when F is local and automorphic representations $\pi : G(\mathbb{A}_F) \to \mathrm{GL}(V)$ when F is global (in both cases V is a complex representation). The parametrising objects are L-homomorphisms

$$\phi : L_F \to {}^L G$$

called L-parameters (the term L-homomorphism $\phi : L_F \to {}^L G$ simply means that ϕ commutes with the projections to the Galois group $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$). The group L_F is the Langlands group of F. When F is local it has the following quite simple form;

$$L_F = \begin{cases} W_F \times \mathrm{SU}(2) & \text{if } F \text{ is non-archimedean} \\ W_F & \text{if } F \text{ is archimedean} \end{cases}$$

where W_F is the Weil group of F. When F is global, however, the existence of L_F is highly conjectural. The purpose of this thesis is to address the correspondence for $F = \mathbb{Q}$ between

$$\{\pi : G(\mathbb{A}) \to \mathrm{GL}(V)\} \longleftrightarrow \{\phi : L_{\mathbb{Q}} \to {}^L G\}$$

Of course this is not yet possible in general as the existence of $L_{\mathbb{Q}}$ is not known. However, as there should be a map $L_{\mathbb{Q}} \to \Gamma_{\mathbb{Q}}$, one can restrict to those automorphic representations which should have l-adic Galois representations attached to them, i.e. those automorphic representations whose associated L-parameter[1] is the pull-back of a Galois representation by the map $L_{\mathbb{Q}} \to \Gamma_{\mathbb{Q}}$. In [21], Gross defines algebraic modular forms on certain reductive groups G which are essentially just automorphic forms on a group G where all arithmetic subgroups of $G(\mathbb{Q})$ are finite (or equivalently any condition in [21, Proposition 1.4]). He conjectures that attached to such objects one should get l-adic Galois representations satisfying certain compatibility conditions (see [21, Conjecture 17.2]). It is this conjecture which guides our study

[1] After choosing an isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$.

of automorphic representations. We will be able to verify Gross' conjecture when G is an odd-dimensional definite unitary group in chapter 2, essentially using existing results, and prove it when G is a certain type of inner form of Sp_{2n} in chapter 3. It is this work in chapter 3 which is the main contribution of this thesis.

When G is a definite unitary group (of arbitrary dimension n), Kottwitz in [25] and Clozel in [15] attach Galois representations from Γ_E to $\mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ where E is the CM field that splits G; these results are based on a version of stable base change relating automorphic representations of $\mathrm{Res}_{E/\mathbb{Q}}\mathrm{GL}_n$ and certain twisted unitary groups over $\mathbb{Q}$. Their result requires a certain local hypothesis; this has been removed by the work of Chenevier and Harris in [14] and Shin in [37], though for our purposes the work of Kottwitz and Clozel will be sufficient. The work of Clozel, Harris, and Taylor in [17] and Buzzard and Gee in [10] extends this representation to one from $\Gamma_{\mathbb{Q}}$ to ${}^{C}G(\overline{\mathbb{Q}}_l)$ (the C-group of G is defined to be the L-group of a certain $\mathbb{G}_m$-extension of G). Provided n is odd, there is a map ${}^{C}G \to {}^{L}G$ giving us a representation from $\Gamma_{\mathbb{Q}}$ to ${}^{L}G(\overline{\mathbb{Q}}_l)$. The work of Buzzard and Gee in [10] will also allow us to extend Gross' conjecture somewhat, being able to remove one of his hypotheses at the cost of getting a Galois representation valued in ${}^{C}G(\overline{\mathbb{Q}}_l)$. When G is an inner form of Sp_{2n} we will aim to transfer our automorphic representation to Sp_{2n}. These inner forms are given by $\mathrm{U}_n(B)$ where B is a rational quaternion algebra. To do this we will first prove a version of the Jacquet-Langlands correspondence from G to Sp_{2n} using Arthur's version of the stable trace formula found in [1]. We will need to assume that the cuspidal automorphic representation of G is Steinberg at one finite place; however this assumption is included in the hypotheses of Gross' conjecture. The group G satisfies Gross' hypotheses when B is definite, though if B is indefinite we will still be able to prove that the automorphic representation transfers to Sp_{2n}. Having got a representation on Sp_{2n} we will be able to get a Galois representation from $\Gamma_{\mathbb{Q}}$ to $\mathrm{GL}_{2n+1}(\overline{\mathbb{Q}}_l)$ using the work of Arthur, Chenevier, Clozel, Harris, Shin, Taylor, and others. Finally, using the work of Bellaïche and Chenevier in [4], we will be able to conclude that the Galois representation has orthogonal image. Again, this will work in both the definite and indefinite cases, though we will need to make a regularity assumption in the indefinite case.

Chapter 1

Background

1.1 Algebraic Groups

In this section we cover the relevant background material on algebraic groups. The primary references for this section are [5], [6], [12], [31], [38], and [41].

1.1.1 Reductive Groups

Definition. *We say that G is an algebraic group if it is an algebraic variety together with*

1. *An element $e \in G$ (identity)*

2. *A morphism (multiplication) $G \times G \to G$, denoted by $(x, y) \mapsto xy$*

3. *A morphism (inversion) $G \to G$, denoted by $x \mapsto x^{-1}$*

with respect to which G is a group. If k is a field we say that G is a k-group if it is a k-variety and multiplication and inversion are defined over k.

One defines a morphism of algebraic groups as a morphism of algebraic varieties which is also a group homomorphism; a k-morphism of algebraic groups is a morphism which is defined over k. It is a fact that $e \in G(k)$ for any field k. We denote the connected component of e in G by G_0; it is a normal subgroup of finite index in G whose cosets are the connected (and indeed the irreducible) components of G. Moreover, if G is defined over k then so is G_0. We say G is connected if it is connected as a variety in which case $G = G_0$.

We now give some examples of algebraic groups.

Example. *Any finite group is an algebraic group.*

Example. GL_n *is an algebraic group over k for any field k:*

$$\mathrm{GL}_n(k) = \{g \in M_n(k) : \det(g) \neq 0\}$$

The above example clearly shows that the k-variety defined by GL_n is affine. We call such an algebraic group an affine algebraic group. One can show that any affine algebraic group is a subgroup of $\mathrm{GL}_n(k)$ for some k. Crucially we have the following examples.

Example.

- $\mathbb{B}_n = \{(a_{ij}) \in \mathrm{GL}_n : a_{ij} = 0 \text{ if } i > j\}$ *(upper triangular matrices)*
- $\mathbb{D}_n = \{(a_{ij}) \in \mathrm{GL}_n : a_{ij} = 0 \text{ if } i \neq j\}$ *(diagonal matrices)*
- $\mathbb{U}_n = \{(a_{ij}) \in \mathbb{B}_n : a_{ii} = 1 \text{ for } 1 \leq i \leq n\}$ *(strictly upper triangular matrices)*

We now introduce some more definitions needed to classify connected affine algebraic groups. Note an algebraic group is solvable if it has the usual composition series but each group appearing in it is an algebraic group.

Definition. *Let G be a connected affine algebraic group.*

1. *We say G is simple if it is not abelian and has no non-trivial normal algebraic subgroups.*
2. *We say G is almost simple if it has no non-trivial normal connected closed subgroups. Equivalently, G is almost simple if it has finite center Z and G/Z is simple.*
3. *We say G is semisimple if it has no non-trivial normal solvable connected closed subgroups.*

Semisimple groups are built up from almost simple ones. An isogeny of algebraic groups is a surjective homomorphism with finite kernel. Two groups H_1 and H_2 are said to be isogenous if there is another group G and isogenies $H_1 \leftarrow G \to H_2$. If G is semisimple then there exist almost simple groups $G_1, \ldots, G_r$ and an isogeny $G_1 \times \cdots G_r \to G$. The problem of classifying simple (or almost simple) groups therefore becomes one of the main problems in the theory. Over an algebraically closed field all of them are isogenous to one on the following list.

Example.

- *Type A_n - corresponding to the groups SL_{n+1} defined as*

$$\mathrm{SL}_{n+1}(k) = \{g \in \mathrm{GL}_{n+1}(k) : \det(g) = 1\}$$

- *Type B_n - corresponding to the groups of the type* SO_{2n+1} *defined as*

$$\mathrm{SO}_{2n+1}(k) = \{g \in \mathrm{SL}_{2n+1}(k) : g^t J g = J\}$$

where $J := J_{2n+1}$ is the matrix with '1's on the anti-diagonal and '0's elsewhere

- *Type C_n - corresponding to the groups of the type* Sp_{2n} *defined as*

$$\mathrm{Sp}_{2n}(k) = \{g \in \mathrm{GL}_{2n}(k) : g^t \Omega g = \Omega\}$$

where $\Omega = \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix}$

- *Type D_n - corresponding to the groups of the type* SO_{2n} *defined as*

$$\mathrm{SO}_{2n}(k) = \{g \in \mathrm{SL}_{2n}(k) : g^t J g = J\}$$

where $J := J_{2n}$

- *Exceptional groups of types E_6, E_7, E_8, F_4, G_2*

From now on we take k to be perfect and let $\overline{k}$ be the algebraic closure of k. We now want to mimic the definitions of the groups $\mathbb{B}_n$ and $\mathbb{D}_n$ for more general affine algebraic groups. We define the following, analogues of $\mathbb{D}_n$ and $\mathbb{B}_n$ respectively.

Definition. 1. *We say an affine algebraic group over k is a torus if it is isomorphic to $\mathbb{D}_n$ over $\overline{k}$ for some n. We say it is split if it is isomorphic to $\mathbb{D}_n$ over k.*

2. *Let G be an affine algebraic group. A Borel subgroup B of G is one which is maximal among the connected solvable subgroups. We define a parabolic subgroup P of G as a subgroup $B \subseteq P \subseteq G$.*

For example if G is GL_n then $\mathbb{B}_n$ is a Borel subgroup and $\mathbb{D}_n$ is a split torus. More generally, for the groups of types A_n, B_n, C_n and D_n one can take B to be the intersection of $\mathbb{B}_n$ with the relevant group. Similarly, the intersection of $\mathbb{D}_n$ with the relevant group is a maximal torus inside the group. We now introduce the class of groups which will be of central concern to us.

Definition. *A connected affine algebraic group is reductive if it has no non-trivial connected normal unipotent subgroups. Any reductive group G is an extension of some semisimple group H by a torus T, i.e. it fits into a short exact sequence*

$$1 \to T \to G \to H \to 1$$

We say that G is split if it contains a split maximal torus.

For example GL_n is a split reductive group with split maximal torus $\mathbb{D}_n$. From now on all algebraic groups we consider will be reductive.

1.1.2 Root Data

In this section we introduce a definition which will allow us to classify reductive groups as well as allowing us to define the dual group of a reductive group; this will be crucial to us later on.

Definition. *A root datum is a quadruple $(X, \Phi, X^\vee, \Phi^\vee)$ where $X, X^\vee$ are free $\mathbb{Z}$-modules of finite rank in duality by a pairing $\langle\ ,\ \rangle : X \times X^\vee \to \mathbb{Z}$, and $\Phi, \Phi^\vee$ are subsets of $X, X^\vee$ in bijection via a map $\alpha \leftrightarrow \alpha^\vee$ satisfying*

- $\langle \alpha, \alpha^\vee \rangle = 2$ *for all* $\alpha \in \Phi$
- $s_\alpha(\Phi) \subseteq \Phi$ *for all* $\alpha \in \Phi$ *where*

$$s_\alpha : X \to X, \;\; x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$$

- *The group W generated by $\{s_\alpha : \alpha \in \Phi\}$ is finite*

The $\alpha \in \Phi$ are called roots and the $\alpha^\vee \in \Phi^\vee$ are called coroots. The group W is called the Weyl group.

We want to attach a root datum to a pair (G, T) where G is a split reductive group and $T \subseteq G$ is a split maximal torus. First we will define an X and $X^\vee$. Let T be a split torus (not neccessarily maximal). Define

$$X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$$

$$X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$$

Note that $\mathbb{G}_m = \mathrm{GL}_1$. There is a natural pairing $\langle\ ,\ \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ defined for $t \in \mathbb{G}_m(k) = k^\times$, $\chi \in X^*(T), \lambda \in X_*(T)$ by $(\chi \circ \lambda)(t) = t^{\langle \chi, \lambda \rangle}$. For a pair (G, T) where T is a split maximal torus inside G we will take $X = X^*(T)$ and $X^\vee = X_*(T)$; note these are both free abelian groups of rank $\dim(T)$. In order to define the roots we will need to define the Lie algebra associated to a reductive group.

Define $k[\epsilon] = k[X]/(X^2)$ to be the ring of dual numbers. There are natural maps $k \to k[\epsilon] \to k$, the first defined by $a \mapsto a + 0 \cdot \epsilon$ and the second by $a + b\epsilon \mapsto a$. If G is an algebraic group over k then we have maps

$$G(k) \to G(k[\epsilon]) \to G(k)$$

We define the Lie algebra of G, denoted either $\mathrm{Lie}(G)$ or $\mathfrak{g}$, as the kernel of the second map. We now give some examples of Lie algebras associated to algebraic groups.

Example. *Let $G = \mathrm{GL}_n$. Then $\mathfrak{gl}_n = M_n(k)$.*

To see this note that $\mathrm{GL}_n(k[\epsilon]) \subseteq \{A + B\epsilon : A, B \in M_n(k)\}$ and therefore

$$\mathfrak{gl}_n \subseteq \{I + B\epsilon : B \in M_n(k)\}$$

Noting that $(I + B\epsilon)(I - B\epsilon) = I$ shows we have equality. We consider two more examples. First we define

$$\mathrm{PGL}_n = \mathrm{GL}_n / Z(\mathrm{GL}_n) = \mathrm{GL}_n / \{\text{scalar matrices}\}$$

which is a simple algebraic group. We wish to calulate $\mathfrak{sl}_2$ and $\mathfrak{pgl}_2$. Clearly $\mathrm{SL}_2(k[\epsilon]) \subset \mathrm{GL}_2(k[\epsilon])$ so

$$\mathfrak{sl}_2 \subseteq \{I + B\epsilon : B \in M_2(k)\}$$

For $I + B\epsilon \in \mathrm{SL}_2(k[\epsilon])$ one must have that

$$\begin{aligned} 1 = \det(I + B\epsilon) = \det \begin{pmatrix} 1 + b_1\epsilon & b_2\epsilon \\ b_3\epsilon & 1 + b_4\epsilon \end{pmatrix} &= (1 + b_1\epsilon)(1 + b_4\epsilon) \\ &= 1 + (b_1 + b_4) = 1 + \mathrm{tr}(B) \end{aligned}$$

Consequently we find that

$$\mathfrak{sl}_2 = \{A \in M_2(k) : \mathrm{tr}(A) = 0\}$$

To determine $\mathfrak{pgl}_2$, and indeed $\mathfrak{pgl}_n$, note that for $(A + B\epsilon), (C + D\epsilon) \in \mathrm{GL}_n(k[\epsilon])$

$$(A + B\epsilon)(C + D\epsilon) = AC + (AD + BC)\epsilon$$

In order for $(A + B\epsilon)$ to have an inverse we see that A must be invertible, in which case the inverse is $(A^{-1} - A^{-1}BA^{-1}\epsilon)$. Therefore

$$\mathrm{GL}_n(k[\epsilon]) = \{A + B\epsilon : A \in \mathrm{GL}_n(k), B \in M_n(k)\}$$

It is easy to see given the equivalence relation on PGL_n that

$$\mathfrak{pgl}_n = M_n(k)/\{\text{scalar matrices}\}$$

We now introduce the definition of roots attached to a pair (G, T) where T is a split maximal torus of G. The group $G(k[\epsilon])$ acts on itself by conjugation which preserves $\mathfrak{g}$. We therefore get an action of $G(k) \subset G(k[\epsilon])$ on $\mathfrak{g}$, and an action of $G(R)$ on $\mathfrak{g}_R = \mathfrak{g} \otimes R$ for any k-algebra R. We therefore get a representation

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

called the adjoint representation. This gives an action of T on $\mathfrak{g}$, and as T is split it is diagonalisable. We therefore get a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_0$ is the subspace of $\mathfrak{g}$ on which T acts trivially and $\mathfrak{g}_\alpha$ is the subspace of $\mathfrak{g}$ on which T acts through the non-trivial character α. We define $\Phi = \Phi(G,T) \subset X^*(T)$ to be this collection of non-trivial characters (and call them roots). We now give some examples of root data attached to reductive groups.

Example. *Let $G = \mathbb{G}_m^r$ for $r \in \mathbb{N}$, and therefore $T = G$. It has root datum (isomorphic to)*

$$(\mathbb{Z}^r, \emptyset, \mathbb{Z}^r, \emptyset)$$

This is fairly obvious as T acts trivally on $\mathfrak{g}$ by conjugation so Φ, and therefore $\Phi^\vee$, must be empty.

Example. *Let $G = \mathrm{SL}_2$, so $T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} : x \neq 0 \right\}$. (G,T) has root datum (isomorphic to)*

$$(\mathbb{Z}, \{2, -2\}, \mathbb{Z}, \{1, -1\})$$

To see this recall $\mathfrak{sl}_2$ consists of trace zero 2×2 matrices. Then $X^*(T) = \mathbb{Z}\chi$ and $X_*(T) = \mathbb{Z}\lambda$ where

$$\chi : \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mapsto x, \ \ \lambda : x \mapsto \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

T acts on $\mathfrak{g} = \mathfrak{sl}_2$ as follows

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} a & x^2 b \\ x^{-2} c & -a \end{pmatrix}$$

Setting $\alpha = 2\chi$ we see that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

so $\Phi = \{2\chi, -2\chi\}$. As we know that $\langle \alpha, \alpha^\vee \rangle = 2$, we see that $\alpha^\vee = \lambda$ and $\Phi^\vee = \{\lambda, -\lambda\}$ giving the claimed root datum.

Example. *Let $G = \mathrm{PGL}_2$, so $T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x_2 \end{pmatrix} : x_1 x_2 \neq 0 \right\} / \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \right\}$. (G,T) has root datum (isomorphic to)*

$$(\mathbb{Z}, \{1, -1\}, \mathbb{Z}, \{2, -2\})$$

In this case $X^*(T) = \mathbb{Z}\chi$ and $X_*(T) = \mathbb{Z}\lambda$ where

$$\chi : \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \mapsto x_1/x_2, \ \ \lambda : x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

The Lie algebra is $\mathfrak{pgl}_2 = M_2(k)/\{\text{scalar matrices}\}$ so T acts as follows

$$\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1^{-1} & 0 \\ 0 & x_2^{-1} \end{pmatrix} = \begin{pmatrix} a & \frac{x_1}{x_2} b \\ \frac{x_2}{x_1} c & -a \end{pmatrix}$$

Clearly the only roots are χ and $-\chi$ which forces the coroots to be 2λ and -2λ giving the required root datum.

It is worth recording a couple of observations at this point. Firstly, in the previous two cases we were able to calculate the coroots easily as a simple consequence of the axioms of root data. However, it is not clear a priori how to attach coroots to more general reductive groups. The general definition uses these two examples; one can show that each root of G arises from a copy of either SL_2 or PGL_2 in G. We then define $\alpha^\vee$ to be the corresponding coroot from the copy. Secondly, the root datum of PGL_2 can be obtained from SL_2 by swapping X with $X^\vee$ and Φ with $\Phi^\vee$. Of course for an abstract root datum we can always do this to obtain a new root datum; we call it the dual root datum. This idea will be crucial to us later on.

We now give an important well-known example, though we omit the details of the construction.

Example. *Let $G = \mathrm{GL}_n$ and $T = \mathbb{D}_n \subseteq G$. One has that*

$$X^*(T) = \bigoplus_{i=1}^n \mathbb{Z}\chi_i, \ \ X_*(T) = \bigoplus_{j=1}^n \mathbb{Z}\lambda_j$$

$$\chi_i : \mathrm{diag}(x_1, \ldots, x_n) \mapsto x_i, \ \ \lambda_j : x \mapsto \mathrm{diag}(1, \ldots, \underbrace{x}_{j^{\text{th}}\text{place}}, \ldots, 1)$$

and

$$\Phi = \{\chi_i - \chi_j : i \neq j\}, \ \ \Phi^\vee = \{\lambda_i - \lambda_j : i \neq j\}$$

where $(\chi_i - \chi_j)^\vee = (\lambda_i - \lambda_j)$ and $\langle \chi_i, \lambda_j \rangle = \delta_{ij}$.

We have sketched how to attach a root datum to a split reductive group. They are particularly important as they classify reductive groups in the following sense.

Theorem ([41])**.** *1. Let G and H be split reductive groups. Then G and H are isomorphic if and only if their root data (with respect to chosen maximal tori) are isomorphic*[1].

2. Given an abstract root datum there exists a split reductive group which has that root datum.

Given this theorem, we can make the following definition.

Definition. *Let G be a reductive group with root datum $(X, \Phi, X^\vee, \Phi^\vee)$. We define the dual group of G, denoted by $\widehat{G}$, to be the split reductive group with root datum $(X^\vee, \Phi^\vee, X, \Phi)$.*

Example.

- $\widehat{\mathrm{GL}}_n = \mathrm{GL}_n$
- $\widehat{\mathrm{Sp}}_{2n} = \mathrm{SO}_{2n+1}$, $\widehat{\mathrm{SO}}_{2n+1} = \mathrm{Sp}_{2n}$
- $\widehat{\mathrm{SO}}_{2n} = \mathrm{SO}_{2n}$

We end this section with a final definition and some examples, a refinement of the idea of a root datum.

Definition. *Let $(X, \Phi, X^\vee, \Phi^\vee)$ be a root datum. One defines a base for this root datum as a subset $\Delta \subset \Phi$ s.t.*

- *Δ is a basis of $X \otimes_{\mathbb{Z}} \mathbb{Q}$*
- *for any $\beta \in \Phi$ there exist constants c_α, either all ≥ 0 or all ≤ 0, such that $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$*

One defines the set of positive roots Φ^+ as those roots which are a positive combination of elements of Δ. We also get a set $\Delta^\vee \subset \Phi^\vee$. We call the sextuple $(X, \Phi, \Delta, X^\vee, \Phi^\vee, \Delta^\vee)$ a based root datum (we will often omit the $\Phi, \Phi^\vee$).

If the root datum is attached to a reductive group it turns out that a choice of a set of positive roots is equivalent to choosing a Borel subgroup. This is best illustrated by an example.

[1]A morphism of root data $(X, \Phi, X^\vee, \Phi^\vee)$ and $(Y, \Psi, Y^\vee, \Psi^\vee)$ is a pair of maps $f : Y \to X$ and $f^\vee : X^\vee \to Y^\vee$ s.t. f maps Ψ bijectively to Φ and $f^\vee(f(\alpha)^\vee) = \alpha^\vee$.

Example. *Let $G = \mathrm{GL}_n$, $T = \mathbb{D}_n$, and $B = \mathbb{B}_n$. Recall the pair (G, T) has roots and coroots*

$$\Phi = \{\chi_i - \chi_j : i \neq j\}, \ \ \Phi^\vee = \{\lambda_i - \lambda_j : i \neq j\}$$

The positive roots (with respect to this choice of Borel) and coroots are those roots where $i < j$. The bases are

$$\Delta = \{\chi_i - \chi_{i+1} : 1 \leq i \leq n-1\}, \ \ \Delta^\vee = \{\lambda_j - \lambda_{j+1} : 1 \leq j \leq n-1\}$$

We now end this section giving the based root data for the groups SO_{2n+1}, Sp_{2n} and SO_{2n} (with respect to the upper triangular Borel subgroup).

Example. *The based root datum of SO_{2n+1} is as follows; X and $X^\vee$ are n-dimensional spanned by e_i and f_j respectively with the obvious inner product and the simple roots and coroots are*

$$\Delta = \{e_i - e_{i+1} : 1 \leq i \leq n-1\} \cup \{e_n\}, \ \ \Delta^\vee = \{f_j - f_{j+1} : 1 \leq j \leq n-1\} \cup \{2f_n\}$$

Example. *The based root datum of Sp_{2n} is as follows; X and $X^\vee$ are n-dimensional spanned by e_i and f_j respectively with the obvious inner product and the simple roots and coroots are*

$$\Delta = \{e_i - e_{i+1} : 1 \leq i \leq n-1\} \cup \{2e_n\}, \ \ \Delta^\vee = \{f_j - f_{j+1} : 1 \leq j \leq n-1\} \cup \{f_n\}$$

Example. *The based root datum of SO_{2n} is as follows; X and $X^\vee$ are n-dimensional spanned by e_i and f_j respectively with the obvious inner product and the simple roots and coroots are*

$$\Delta = \{e_i - e_{i+1} : 1 \leq i \leq n-1\} \cup \{e_{n-1} + e_n\},$$
$$\Delta^\vee = \{f_j - f_{j+1} : 1 \leq j \leq n-1\} \cup \{f_{n-1} + f_n\}$$

1.1.3 Forms of Reductive Groups

We have (roughly) defined a based root datum attached to a split reductive group. However we would like to extend this definition to non-split reductive groups. The crucial observation is that any reductive group is split over an algebraically closed field. Therefore, if we have a reductive group G over k we can base change to $\overline{k}$ and attach a root datum to $G_{\overline{k}} = G \otimes_k \overline{k}$. This process introduces an action of the Galois group $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$ which is trivial if G is split over k (though not conversely). To do this precisely we introduce the following definition.

Definition. *Let G be a reductive group over k. A reductive group H over k is a form of G (or a k-form) if G and H are isomorphic over $\overline{k}$.*

We have already met examples of these. A torus T is defined to be a form of $\mathbb{D}_n$ for some n. We shall need to define a more refined notion of form and to do this we shall need to use group cohomology.

Recall for G either a finite or profinite group and A a (continuous) G-module we have the cohomology groups $H^i(G, A)$. They are determined uniquely by requiring that $H^0(G, A) = A^G$, for any G-homomorphism $A \to B$ there exists a canonical map $H^i(G, A) \to H^i(G, B)$, and for any short exact sequence of G-modules

$$0 \to A \to B \to C \to 0$$

there is a long exact sequence

$$0 \to H^0(G, A) \to H^0(G, B) \to H^0(G, C) \to H^1(G, A) \to \cdots$$

One can also define these groups explicitly. The problem is that we will want to take cohomology of non-abelian groups with a G-action. We will therefore use cohomology (pointed) sets which we will define explicitly, see [19, Chapter 2] for more on what follows. Let G be a group and A a group with a G-action. First define $H^0(G, A) = A^G$ (the fixed points of A under this action). Define a 1-cocycle as a function $c : G \to A$ satisfying

$$c(st) = c(s) \cdot s(c(t)) \ \forall s, t \in G$$

We say that two cocycles c_1, c_2 are equivalent if $\exists a \in A$ s.t.

$$c_1(t) = a^{-1} \cdot c_2(t) \cdot t(a) \ \forall t \in G$$

This is clearly an equivalence relation and we define $H^1(G, A)$ as the pointed set of 1-cocycles modulo this equivalence relation with the trivial cocycle $c(t) = 1 \ \forall t \in G$ as the distinguished point. Note if A is abelian then the definition of 1-cocycle is the same as the normal definition, and moreover

$$c_1 \sim c_2 \iff c_1(t) = a^{-1} \cdot c_2(t) \cdot t(a) \ \forall t \in G \iff c_1(t) \cdot c_2(t)^{-1} = a^{-1} \cdot t(a) \ \forall t \in G$$

i.e. c_1, c_2 are equivalent if and only if their difference is a 1-coboundary. Therefore the pointed set and group definitions of cohomology match up when A is abelian. We now recall that H^1 extends exact sequences in the normal way.

Proposition. *Let G be a group, and A, B, C be groups with a G-action. Suppose there is an exact sequence of groups (with maps G-homomorphisms)*

$$1 \to A \to B \to C \to 1$$

Then $\exists$ an exact sequence of pointed sets[2]

$$1 \to A^G \to B^G \to C^G \to H^1(G, A) \to H^1(G, B) \to H^1(G, C)$$

The reason we are interested in all of this is as follows. Let K/k be a Galois extension, let G be a reductive group over k, and let H be a form of G such that there exists an isomorphism $\phi : G(K) \cong H(K)$. We call such an H a K/k-form of G. We get a Galois action on $\mathrm{Aut}(G_K)$ defined by $(\gamma \cdot f) : g \mapsto \gamma(f(\gamma^{-1}(g)))$ which preserves $\mathrm{Inn}(G_K)$. We have the following proposition.

Proposition. *There exists a bijection of pointed sets between*

$$\{K/k\text{-forms of } G\} \longleftrightarrow H^1(\mathrm{Gal}(K/k), \mathrm{Aut}(G_K))$$

Explicitly, given a K/k-form H pick[3] *an isomorphism $\phi : G(K) \cong H(K)$ and define a cocycle $c \in H^1(\mathrm{Gal}(K/k), \mathrm{Aut}(G_K))$ by*

$$c : \gamma \mapsto \phi^{-1} \circ \gamma \circ \phi \circ \gamma^{-1}$$

Conversely, given $c \in H^1(\mathrm{Gal}(K/k), \mathrm{Aut}(G_K))$ define for a k-algebra R

$$H(R) := \{g \in G(K \otimes_k R) : (c(\gamma) \cdot \gamma)g = g \;\; \forall \gamma \in \mathrm{Gal}(K/k)\}$$

We can now attach a based root datum to a general reductive group G over k. First we take the based root datum of $G_{\overline{k}}$, call it $\mathbf{\Psi}$. There exists a split exact sequence ([38, pg. 10])

$$1 \to \mathrm{Inn}(G_{\overline{k}}) \to \mathrm{Aut}(G_{\overline{k}}) \to \mathrm{Aut}(\mathbf{\Psi}) \to 1$$

This comes with a Galois action of Γ_k defined as follows (see [5, pg. 29]). Let T/k be a maximal torus, split over $\overline{k}$ but not neccessarily over k, contained in a chosen Borel subgroup $B/\overline{k}$. An element $\gamma \in \Gamma_k$ carries $T(\overline{k})$ to $T(\overline{k})$ but does not necessarily carry $B(\overline{k})$ to itself. However, there exists $g_\gamma \in N_G(T)$ such that $g_\gamma\gamma$ does take $B(\overline{k})$ to itself. This is independent of choice of representative in $W = N_G(T)/T$, and gives

[2] Recall the kernel of a map of pointed sets is the subset of elements mapped to the base point.
[3] One can show the map does not depend on this choice.

an automorphism of $\mathbf{\Psi}$ dependent only on γ. We let $\mu_G : \Gamma_k \to \mathrm{Aut}(\mathbf{\Psi})$ be the homomorphism defined by the above. We also get a Galois action on the dual root datum, which due to the short exact sequence gives a Galois action on the dual group $\widehat{G}$. One defines the L-group as this semidirect product. The fixed field of this action is a finite extension E/k, and one may as well simply consider the action of the finite Galois group $\mathrm{Gal}(E/k)$ on the based root datum. For a field $F \supseteq k$ one can define the L-group as

$$^LG(F) = \widehat{G}(F) \rtimes \mathrm{Gal}(E/k)$$

Note that in this thesis, as is common in the literature, we will often[4] use ^LG to denote $^LG(\mathbb{C})$.

We can now complete our analysis of forms of reductive groups. Let $K = \overline{k}$ (or some Galois extension) which splits G. There exists a split exact sequence ([38, pg. 10])

$$1 \to \mathrm{Inn}(G_K) \to \mathrm{Aut}(G_K) \to \mathrm{Aut}(\mathbf{\Psi}) \to 1$$

We therefore get an exact sequence

$$\begin{aligned} 1 \to \mathrm{Inn}(G_K)^{\Gamma_k} \to \mathrm{Aut}(G_K)^{\Gamma_k} \to \mathrm{Aut}(\mathbf{\Psi})^{\Gamma_k} \\ \to H^1(\Gamma_k, \mathrm{Inn}(G_K)) \to H^1(\Gamma_k, \mathrm{Aut}(G_K)) \to H^1(\Gamma_k, \mathrm{Aut}(\mathbf{\Psi})) \end{aligned}$$

This allows us to make the following definition.

Definition. *Let G be a reductive group over k and let H be a form of G. We say H is an inner form of G if the cocycle associated to H by the earlier proposition lies in the image of the natural map $H^1(\Gamma_k, \mathrm{Inn}(G_K)) \to H^1(\Gamma_k, \mathrm{Aut}(G_K))$.*

Note that if H is a form of G then they have the same Galois action on their root data if and only if they are inner forms of each other. In particular, if G and H are inner forms of each other then $^LH = {}^LG$.

It is illustrative to discuss an example of inner forms of reductive groups which will be central to what follows.

Example. *Let $G = B^\times$ where B/k is a quaternion alegbra (see [19, Chapter 1]) and* $\mathrm{char}(k) \neq 2$*; this is defined as follows*[5]

$$B = \left(\frac{a, b}{k}\right) = \langle 1, i, j, ij : i^2 = a, j^2 = b, ij = -ji \rangle_k$$

[4]There is potential for confusion here, as for all the groups G we consider $\widehat{G}$ will be defined over $\mathbb{Q}$ so one can consider ^LG as an algebraic group over $\mathbb{Q}$.

[5]The notation $\langle\ \rangle_k$ means the algebra defined by the k-span of the elements in the brackets subject to the relations described.

where $a, b \in k^\times$. Then G is an inner form of GL_2 *(in fact the converse also holds).*

Let $E = k(\sqrt{a})$. Given $q = x + iy + jz + ijw \in B$ we can embed $B \hookrightarrow M_2(E)$ by

$$q \mapsto \begin{pmatrix} x + y\sqrt{a} & (z + w\sqrt{a})b \\ z - w\sqrt{a} & x - y\sqrt{a} \end{pmatrix} \in M_2(E)$$

Clearly $G(E) \cong \mathrm{GL}_2(E)$ so G is a form of GL_2. Let ϕ be the isomorphism $\mathrm{GL}_2(E) \cong G(E)$. We want to consider the map $c(t) = \phi^{-1}t\phi t^{-1}$ where $t(x + y\sqrt{a}) = x - y\sqrt{a}$ (i.e. t is the non-trivial element of $\mathrm{Gal}(E/k)$), note $c \in H^1(\mathrm{Gal}(E/k), \mathrm{Aut}(G_E))$. As $t^{-1} = t$ we have that

$$\begin{pmatrix} x + y\sqrt{a} & (z + w\sqrt{a})b \\ z - w\sqrt{a} & x - y\sqrt{a} \end{pmatrix} \overset{t}{\mapsto} \begin{pmatrix} \bar{x} - \bar{y}\sqrt{a} & (\bar{z} - \bar{w}\sqrt{a})b \\ \bar{z} + \bar{w}\sqrt{a} & \bar{x} + \bar{y}\sqrt{a} \end{pmatrix}$$

$$\overset{\phi}{\mapsto} \bar{x} - i\bar{y} + j\bar{z} - ij\bar{w} \overset{t}{\mapsto} x - iy + jz - ijw \xmapsto{\phi^{-1}} \begin{pmatrix} x - y\sqrt{a} & (z - w\sqrt{a})b \\ z + w\sqrt{a} & x + y\sqrt{a} \end{pmatrix}$$

In other words we have that

$$c(t)(\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}) = \begin{pmatrix} q_4 & bq_3 \\ b^{-1}q_2 & q_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ b^{-1} & 0 \end{pmatrix} \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

We set $\gamma = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$ and see that $\gamma^{-1} = \begin{pmatrix} 0 & 1 \\ b^{-1} & 0 \end{pmatrix}$, so $c(t) \in \mathrm{Inn}(G_E)$. Therefore G is an inner form of GL_2 as required.

We now introduce a weakening of the definition of a split reductive group which will be crucial to the theory of forms.

Definition. *Let G be a reductive group over k. We say that G is quasi-split if it contains a Borel subgroup defined over k.*

Note that split groups are quasi-split. It is a fact that any reductive group has an inner form which is quasi-split, this form is unique up to isomorphism. Note this means if G is a reductive group there is a quasi-split form G' of G such that ${}^LG = {}^LG'$. We finish with some examples and non-examples.

Example. *$G = B^\times$ is not quasi-split provided $B \ncong M_2(k)$ (i.e. it is not split), for example if $k = \mathbb{R}$ and*

$$B = \left(\frac{-1, -1}{\mathbb{R}}\right)$$

then G is not quasi-split. We denote this quaternion alegbra by $\mathbb{H}$, it is known as Hamilton's quaternions.

We now give an example of a quasi-split group which is not split, and one which will be crucial in what follows. The orthogonal groups over a field k (of characteristic $\neq 2$) are defined as

$$\mathrm{O}_m = \mathrm{O}(J_m) = \{g \in \mathrm{GL}_m : g^t J_m g = J_m\}$$

We previously defined $\mathrm{SO}_m = \mathrm{O}_m \cap \mathrm{SL}_m$ which is a normal subgroup of O_m. When $m = 2n+1$ is odd, $-I_{2n+1} \notin \mathrm{SO}_{2n+1}$ and we have that

$$\mathrm{O}_{2n+1} \cong \mathrm{SO}_{2n+1} \times \{\pm 1\}$$

When $m = 2n$ is even, however, $-I_{2n} \in \mathrm{SO}_{2n}$ and one has an outer automorphism of SO_{2n} given by conjugation by O_{2n}. Let

$$\eta : \Gamma_k \to \{\pm 1\}$$

be a quadratic character of the Galois group. It determines an extension E/k which is quadratic if η is non-trivial. We thus get a form of SO_{2n}, which we denote by $\mathrm{SO}_{2n}(\eta)$, coming from $H^1(\mathrm{Gal}(E/k), \mathrm{Out}(\mathrm{SO}_{2n}(E)))$. If η is trivial this is just the split group SO_{2n} but if η is non-trivial this is a non-inner form of SO_{2n}, it is quasi-split but not split. It can be defined explicitly as follows for non-trivial η;

$$\mathrm{SO}_{2n}(\eta)(k) = \{g \in \mathrm{SO}_{2n}(E) : X^{-1}\tau(g)X = g\}$$

where $X \in \mathrm{O}_{2n}(k) \backslash \mathrm{SO}_{2n}(k)$ and τ is the non-trivial element of $\mathrm{Gal}(E/k)$. For example one can take

$$X = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

The group $\mathrm{SO}_{2n}(\eta)$ is attached to the cocycle which sends τ to conjugation by X. When $n = 1$ it takes the following form

$$\mathrm{SO}_2(\eta)(k) = \left\{ \begin{pmatrix} x + y\sqrt{d} & 0 \\ 0 & x - y\sqrt{d} \end{pmatrix} : x, y \in k,\ x^2 - dy^2 = 1 \right\}$$

where $E = k(\sqrt{d})$. This is clearly quasi-split but not split; it is a non-split torus (as $\mathrm{SO}_2(\eta)(E) \cong \mathrm{SO}_2(E) \cong E^\times$) but $\mathrm{SO}_2(\eta)$ is solvable (as it's abelian) so is a Borel subgroup defined over k.

1.2 Automorphic Representations and the Langlands Program

In this section, and the rest of this thesis, we let $\mathbb{A}$ and $\widehat{\mathbb{Q}}$ denote the adeles and finite adeles of $\mathbb{Q}$ respectively. For a number field F we let $\mathbb{A}_F$ denote the adeles of F.

1.2.1 Automorphic Representations and Automorphic Forms

The primary reference for this section is [8] along with [29].

Definition. *Let G be a locally compact group (e.g. $\mathrm{GL}_n(\mathbb{A})$). Then there exists a positive regular Borel measure $d_L g$ on G that is left invariant under the action of G*

$$\int_G f(xg)d_L g = \int_G f(g)d_L g \ \ \forall x \in G$$

which is unique up to scalars. It is called a left Haar measure. Defining $d_R g = d_L(g^{-1})$ gives a right invariant measure called a right Haar measure. A locally compact group G is unimodular if its left and right Haar measures are equal up to a scalar. Reductive groups are unimodular.

We now let G be a reductive group over $\mathbb{Q}$. We define the non-archimedian Hecke algebra $\mathcal{H}^\infty = C_c^\infty(G(\widehat{\mathbb{Q}}))$ to be the space of smooth (i.e. locally constant) compactly supported functions on $G(\widehat{\mathbb{Q}})$. It is an algebra with respect to convolution

$$(f * g)(x) = \int_{G(\widehat{\mathbb{Q}})} f(y)g(y^{-1}x)dy$$

Picking K_∞ to be a maximal compact subgroup of $G(\mathbb{R})$ we define the archimedian Hecke algebra $\mathcal{H}_\infty = \mathcal{H}(G(\mathbb{R}), K_\infty)$ as the convolution algebra of distributions of $G(\mathbb{R})$ supported on K_∞.

Definition.

1. *A representation (π, V) of $\mathcal{H}^\infty$ is admissible if it is nondegenerate*[6] *and for all open compact subgroups $K \subseteq G(\widehat{\mathbb{Q}})$ the space V^K (the space of vectors fixed by K) is finite dimensional.*

2. *A continuous representation (π, V) of $G(\mathbb{R})$ on a Hilbert space V is admissible if $\pi|_K$ is unitary and each irreducible unitary representation of K occurs in it with finite multiplicity.*

[6] A G-module M is nondegenerate if every element in M is of the form $g_1 \cdot m_1 + \cdots g_k \cdot m_k$ where each $g_i \in G$ and each $m_j \in M$, see [11, pg. 113].

3. *One defines the global Hecke algebra as $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}^\infty$, and an irreducible representation (π, V) of $\mathcal{H}$ is said to be admissible if both components in its tensor product decomposition*[7] *are admissible.*

Let V be a finite-dimensional irreducible complex representation of K_∞. First we give an approximate definition of an automorphic form. We will not need the exact definition for our purposes since this thesis will be primarily concerned with automorphic representations, see [8] for a complete definition.

Definition. *An automorphic form for G of weight V is a function*

$$f : G(\mathbb{Q})\backslash G(\mathbb{A}) \to V$$

s.t.

- $f(gk) = f(g)\ \forall g \in G(\mathbb{A})$ *and* $\forall k \in K_f$ *where* $K_f \subset G(\widehat{\mathbb{Q}})$ *is an open compact subgroup*
- $f(gk_\infty) = k_\infty^{-1} \circ f(g)\ \forall g \in G(\mathbb{A})$ *and* $\forall k_\infty \in K_\infty$
- *various smoothness and boundedness conditions*

An automorphic form is cuspidal if for all $g \in G(\mathbb{A})$

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ng)dn = 0$$

for all unipotent groups N which appear in the decomposition of a parabolic subgroup $P = MN \subset G$; note any parabolic subgroup has a decomposition $P = MN$ where M is a Levi subgroup (a maximal reductive subgroup) and N a unipotent subgroup.

We can now define automorphic representations, see [8, Section 4.6].

Definition. *An automorphic representation of G (or $G(\mathbb{A})$) is an irreducible admissible representation of $\mathcal{H}$ which is isomorphic to a subquotient of a representation of $\mathcal{H}$ in the space of automorphic forms on $G(\mathbb{A})$. We say the automorphic representation is cuspidal if the subquotient is of a representation of $\mathcal{H}$ in the space of cuspidal automorphic forms on $G(\mathbb{A})$.*

It is a crucial fact about automorphic representations that they have a restricted tensor product decomposition; any automorphic representation $\pi = \widehat{\bigotimes}_v \pi_v$ for some admissible representations π_v of $G(\mathbb{Q}_v)$ (see [18]). We remark that the converse problem, i.e. determining which restricted tensor products of local admissible representations are automorphic, is a difficult one.

[7] See [18].

1.2.2 The Langlands Program

Let G be a connected reductive group over F, a local or global field of characteristic 0. The Langlands correspondence is meant to parametrise certain representations associated to G; these are admissible representations $\pi : G(F) \to \mathrm{GL}(V)$ when F is local and automorphic representations $\pi : G(\mathbb{A}_F) \to \mathrm{GL}(V)$ when F is global (in both cases V is a complex representation). The parameterising objects are, L-homomorphisms

$$\phi : L_F \to {}^L G$$

called L-parameters[8], the term L-homomorphism $\phi : L_F \to {}^L G$ simply means that ϕ is a homomorphism which commutes with the projections to the Galois group Γ_F. The group L_F is the Langlands group of F. When F is local it has the following quite simple form;

$$L_F = \begin{cases} W_F \times \mathrm{SU}(2) & \text{if } F \text{ is non-archimedean} \\ W_F & \text{if } F \text{ is archimedean} \end{cases}$$

where W_F is the Weil group of F. We will care about this group in the cases where $F = \mathbb{Q}_p$ and $\mathbb{R}$ (see [39, pg. 6]). Recall the Weil group $W_{\mathbb{Q}_p}$ is the dense subgroup of $\Gamma_{\mathbb{Q}_p}$ consisting of elements which act on $\overline{\mathbb{F}}_p$ as (rational) integral powers of the Frobenius. For $F = \mathbb{R}$, $W_{\mathbb{R}} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$ where

$$jzj^{-1} = \bar{z} \;\; \forall z \in \mathbb{C}^\times \text{ and } j^2 = -1$$

When F is global the existence of L_F is highly conjectural. It should come with maps forming a commutative diagram

$$\begin{tikzcd} L_{F_v} \arrow[r] \arrow[d, hook, "\iota_v"] & \Gamma_{F_v} \arrow[d, hook] \\ L_F \arrow[r] & \Gamma_F \end{tikzcd}$$

where the horizontal maps have connected kernel and dense image but not much else is known. We will primarily be interested in the correspondence

$$\{L\text{-packets of automorphic forms on } G(\mathbb{A})\} \longleftrightarrow \{\phi : L_{\mathbb{Q}} \to {}^L G\}$$

and how it relates to the local correspondence

$$\{L\text{-packets of admissible representations on } G(\mathbb{Q}_v)\} \longleftrightarrow \{\phi : L_{\mathbb{Q}_v} \to {}^L G\}$$

[8]Or more acurately equivalence classes of L-parameters, with the equivalence relation being $\widehat{G}$-conjugacy.

where v is some local place. The term L-packet is introduced because in general each L-parameter will be associated to more than one admissible/automorphic representation. We define the L-packet of a representation π as being the set of representations associated to the same L-parameter as π, and similarly define the L-packet of a parameter as being the set of representations that are associated to it. The local and global correspondences are meant to be compatible in the following sense; if an automorphic representation π corresponds to ϕ then each of its local components π_v corresponds to $\phi_v = \phi \circ \iota_v$. However, as the existence of $L_{\mathbb{Q}}$ is unknown it will not be possible to establish this. We will instead later define a substitute for the global parameters $\phi : L_{\mathbb{Q}} \to {}^L G$ which shall have a localisation in terms of local parameters $\{\phi_v\}_v$ where v runs over all local places.

While the local Langlands correspondence is not known in general, it is known in a few important important cases. Let $F = \mathbb{Q}$ and p be a finite prime. Suppose that π_p is unramified. Then $G_p = G/\mathbb{Q}_p$ is quasi-split and the action of the Weil group $W_{\mathbb{Q}_p}$ on $\widehat{G}$ factors through the quotient of the Weil group by the inertia group, $W_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}$. This quotient is an infinite cyclic group with canonical generator Frob_p. In this unramified setting we have the local Langlands correspondence (see [1, pg. 20]); π_p corresponds to a parameter $\phi_p : L_{\mathbb{Q}_p} \to \widehat{G} \rtimes \langle \mathrm{Frob}_p \rangle$ and the mapping $\pi_p \mapsto c(\pi_p) = \phi_p(\mathrm{Frob}_p)$ is a bijection from the unramified representations of $G(\mathbb{Q}_p)$ to the set of semisimple $\widehat{G}$-orbits in ${}^L G_p$ that project to Frob_p in $W_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}$. Another crucial local case is at the infinite place; the local Langlands correspondence over $\mathbb{R}$ is known for any connected reductive group G (see [30]). Finally, due to Harris and Taylor in [22], the local Langlands correspondence is known for GL_n over $\mathbb{Q}_p$.

Despite being highly conjectural, the Langlands correspondences predict relations between automorphic representations of certain groups which one can try and prove directly. Let G and H be reductive groups over F (again a local or global field of characteristic 0). Suppose additionally that G is quasi-split and that there exists an L-homomorphism $\iota : {}^L H \to {}^L G$. This gives a map

$$\{\phi_H : L_F \to {}^L H\} \longrightarrow \{\phi : L_F \to {}^L G\}$$

$$\phi_H \mapsto \iota \circ \phi_H$$

which, if the conjectural local and global Langlands correspondences held, would induce a map[9]

$$\begin{aligned}&\{(L\text{-packets of) admissible/automorphic representations of } H\}\\ &\qquad\longrightarrow \{(L\text{-packets of) admissible/automorphic representations of } G\}\end{aligned}$$

This conjectural principle is called Langlands functoriality and it will prove central to what follows. It is used by Arthur in [1] to describe a substitute for the global parameters $\phi : L_F \to {}^LG$ where $G = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$ or $\mathrm{SO}_{2n}(\eta)$. Roughly speaking, this substitute is automorphic representations of GL_{2n+1} in the case when G is symplectic and automorphic representations of GL_{2n} when G is orthogonal (this will be made more precise in chapter 3). The other case of Langlands functoriality which will be used prominently is the conjectural transfer

$$\begin{aligned}&\{(L\text{-packets of) automorphic representations on } G(\mathbb{A})\}\\ &\qquad\longrightarrow \{(L\text{-packets of) automorphic representations on } G'(\mathbb{A})\}\end{aligned}$$

where G is a reductive group over $\mathbb{Q}$ and G' is the unique (up to isomorphism) quasi-split inner form of G. As ${}^LG = {}^LG'$, the fact that there should be a transfer is obvious. The case when $G = B^\times$ where B is a rational quaternion algebra and $G' = \mathrm{GL}_2$ was established by Jacquet and Langlands and is refered to as the Jacquet-Langlands correspondence (for GL_2). We will refer to this type of transfer for general reductive groups as a generalised Jacquet-Langlands transfer or a Jacquet-Langlands type transfer.

We return now to the global question concerning classifying automorphic representations of a reductive group G over $\mathbb{Q}$. As mentioned above this is currently well out of reach as it's unknown whether the group $L_\mathbb{Q}$ exists. As opposed to the approach adopted by Arthur mentioned above, one could instead use the conjectured map $L_\mathbb{Q} \to \Gamma_\mathbb{Q}$ (the map is thought to be continuous with connected kernel and dense image). After choosing an isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$, composing a Galois representation $\rho : \Gamma_\mathbb{Q} \to {}^LG(\overline{\mathbb{Q}}_l)$ with the map $L_\mathbb{Q} \to \Gamma_\mathbb{Q}$ gives an L-parameter and, if the Langlands parameterisation holds, an automorphic representation. We can therefore instead try and parametrise a subset of automorphic representations that should in fact have Galois representations attached to them. Of course it is not clear a priori what subset of automorphic representations we should look at. In the following section we

[9] G being quasi-split is neccessary to ensure the existence of this map.

will describe some examples of automorphic representations which should lie in this subset.

1.3 Gross' Conjecture

Let G be a connected reductive group over $\mathbb{Q}$. In this section we define algebraic modular forms or algebraic automorphic forms, considered by Gross in [21]. We then state Gross' conjecture about attaching Galois representations to algebraic automorphic forms.

1.3.1 Algebraic Modular Forms

Let S be the maximal split torus in the center of G and let S' be the maximal quotient of G which is a split torus. They both have the same dimension (we denote it n) and there is an isogeny of tori $S \to G \to S'$. There is a decomposition

$$\mathbb{A}^{\times} = \mathbb{Q}^{\times} \times \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^{\times}$$

Fixing an isomorphism $S' \cong \mathbb{G}_m^n$ gives a continuous homomorphism

$$G(\mathbb{A}) \to S'(\mathbb{A}) \cong \mathbb{G}_m(\mathbb{A})^n \to (\mathbb{R}_{>0})^n$$

where the last map is induced from the natural projection of $\mathbb{A}^{\times} \to \mathbb{R}_{>0}$. We denote the kernel of this map by $G(\mathbb{A})_1$ which clearly contains $G(\mathbb{Q})$. As stated in [21, Proposition 1.3], but originally shown in [7], the quotient $G(\mathbb{Q})$ is discrete in the locally compact group $G(\mathbb{A})_1$ and the quotient space $G(\mathbb{Q})\backslash G(\mathbb{A})_1$ has finite Haar measure; it is compact if and only if S is a maximal split torus in G over $\mathbb{Q}$.

Example. *If $G = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$, or $\mathrm{SO}_{2n}(\eta)$, then S is trivial and $n = 0$. Therefore $G(\mathbb{A})_1 = G(\mathbb{A})$, $G(\mathbb{Q})$ is discrete in the locally compact group $G(\mathbb{A})$, and the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})$ has finite Haar measure but is not compact unless $G = \mathrm{SO}_{2n}(\eta)$ and $\eta \neq 1$.*

We now state an abreviated form of [21, Proposition 1.4]. Recall any reductive group G can be embedded in GL_N for some N (see [6] or [31]). We define[10]

$$G(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathrm{GL}_N(\mathbb{Z})$$

Two subgroups A, B are said to be commensurable if $A \cap B$ has finite index in both A and B. A subgroup of $G(\mathbb{Q})$ is said to be arithmetic if it is commensurable with $G(\mathbb{Z})$.

[10]Of course this definition depends on the choice of embedding.

Proposition. *The following three conditions are equivalent:*

- *every arithmetic subgroup of* $G(\mathbb{Q})$ *is finite*
- S *is a maximal split torus in* G *over* $\mathbb{R}$
- $G(\mathbb{R})_1 = G(\mathbb{R}) \cap G(\mathbb{A})_1$ *is a maximal compact subgroup of* $G(\mathbb{R})$

A proof can be found in Gross' work and he also gives more equivalent conditions in [21, Proposition 1.4] but we will only need to focus on these three; it is worth noting that most of the proposition was known much earlier (see [34, Chapter 6]).

Definition. *We say that* G *is* $\mathbb{R}$*-compact if it satisfies any of the equivalent conditions of the proposition above, or more generally any of the equivalent conditions in [21, Proposition 1.4].*

It is for groups of this type that we define algebraic modular forms.

Example. $G = B^\times$ *where* B *is a definite quaternion algebra is* $\mathbb{R}$*-compact.*

To see this, suppose that $B = \left(\frac{a,b}{\mathbb{Q}}\right)$ (note $a, b < 0$ as B is definite) and let $E = \mathbb{Q}(\sqrt{a})$. The group $G(\mathbb{Q})$ is isomorphic to the group of matrices

$$\left\{ \begin{pmatrix} x + y\sqrt{a} & (z + w\sqrt{a})b \\ z - w\sqrt{a} & x - y\sqrt{a} \end{pmatrix} : (w, x, y, z) \neq (0, 0, 0, 0) \right\} \subset \mathrm{GL}_2(E)$$

The maximal split torus S in the center of G is the center which is isomorphic to $\mathbb{G}_m$. As E is imaginary quadratic S is the maximal split torus in G over $\mathbb{R}$ (it is not over E or $\mathbb{C}$ as this then contains $\mathbb{D}_2$).

We now define algebraic modular forms, or algebraic automorphic forms. The definition appears in [21, Section 4] but again had been around for much longer.

Definition. *Let* G *be a connected reductive group over* $\mathbb{Q}$ *which is* $\mathbb{R}$*-compact and let* V *be an irreducible representation of* G *over* $\mathbb{Q}$*. An algebraic modular form for* G *of weight* V *is a function*

$$f : G(\mathbb{A}) \to V$$

s.t.

- $f(\gamma g) = \gamma \circ f(g)$ $\forall g \in G(\mathbb{A})$ *and* $\forall \gamma \in G(\mathbb{Q})$
- $f(gk) = f(g)$ $\forall g \in G(\mathbb{A})$ *and* $\forall k \in G(\mathbb{R})_0 \times K_f$ *where* $K_f \subset G(\widehat{\mathbb{Q}})$ *is an open compact subgroup*

These are clearly similar objects to automorphic forms, but are simpler for the following reason. The double coset space

$$G(\mathbb{Q})\backslash G(\mathbb{A})/(G(\mathbb{R})_0 \times K_f)$$

is finite. Fixing representatives for this double coset space $\{g_\alpha\}$ we can define artihmetic subgroups

$$\Gamma_\alpha = G(\mathbb{Q}) \cap g_\alpha(G(\mathbb{R})_0 \times K_f)g_\alpha^{-1}$$

Any algebraic modular form f of level K_f is therefore completely determined by its value on the finitely many elements $\{g_\alpha\}$, note that the Γ_α are finite. In fact, the space of algebraic modular forms of level K_f of weight V is isomorphic to

$$\bigoplus_\alpha V^{\Gamma_\alpha}$$

where V^{Γ_α} is the subspace of Γ_α invariants; for more on all of this see [21, pg. 69].

We will now detail the correspondence between algebraic modular forms and automorphic forms, at least in the case we will be most concerned with. Suppose G now has trivial torus S and $G(\mathbb{R})$ is connected[11]. The relation between algebraic modular forms and automorphic forms is as follows. Let f be an algebraic modular form of level K_f and weight V. We define

$$F_\infty(g) = g_\infty^{-1} \circ f(g) \in V \otimes \mathbb{R}$$

where $g_\infty \in G(\mathbb{R})$ is the infinite component of $g \in G(\mathbb{A})$. The map $f \mapsto F_\infty$ produces an automorphic form of level K_f and weight V. Conversely, if F is an automorphic form of level K_f and weight V (where V is in fact a rational representation) then f defined by

$$f(g) = F(g^\infty)$$

(where $g^\infty \in G(\widehat{\mathbb{Q}})$ is the finite component of $g \in G(\mathbb{A})$) is an algebraic modular form of the same weight and level, and moreover

$$F_\infty(g) = g_\infty^{-1} \circ f(g) = g_\infty^{-1} \circ F(g^\infty) = F(g^\infty g_\infty) = F(g)$$

so there is a bijection between algebraic modular forms and automorphic forms with values in a rational represention. Of course if the Langlands program holds then such objects should have L-parameters associated to them, but in fact Gross conjectures

[11] Neither of these assumptions are neccessary but they will hold for most of the groups that we will be interested in and simplify the correspondence.

something stronger.

As via the dictionary above algebraic modular forms become automorphic forms on $G(\mathbb{A})$ (those from section 1.2.1), they have a Hecke action defined by double coset operators as follows. Given a double coset $K\hat{g}K$ where $\hat{g} \in G(\widehat{\mathbb{Q}})$ we write

$$K\hat{g}K = \bigsqcup_{i=1}^{n} \hat{g}_i K$$

and then define for an algebraic modular form of weight V and level K

$$T(\hat{g})f(g) = \sum_{i=1}^{n} f(g\hat{g}_i K)$$

1.3.2 Twisting Elements

Let G be a connected reductive group over $\mathbb{Q}$. If it is $\mathbb{R}$-compact then we have defined algebraic modular forms on G in section 1.3.1. Before we can (roughly) state Gross' conjecture we need to state a further requirement on our group G. Let $(X, \Delta, X^\vee, \Delta^\vee)$ be the based root datum associated to G.

Definition. *We say that G has a twisting element if there exists an element $\eta \in X$ which is Galois invariant and*

$$\langle \eta, \alpha^\vee \rangle = 1 \ \forall \alpha^\vee \in \Delta^\vee$$

This property is always satisfied by half the sum of positive roots ρ but this is not always in X. Note that G has a twisting element if and only if all its inner forms do, and if ρ is in X then any form of G has a twisting element (as ρ is always Galois invariant).

Example. GL_2 *(and its inner forms) possesses a twisting element, specifically if the root datum is defined as in section 1.1.2 then we can take $\eta = \chi_1$.*

Note that $\rho = \frac{1}{2}(\chi_1 - \chi_2) \notin X$ but G still posesses a twisting element.

Example. *If n is odd then all forms G of* GL_n *possess a twisting element.*

To see this, note that the the set of positive roots for GL_n are

$$\Phi^+ = \{\chi_i - \chi_j : 1 \leq i < j \leq n\}$$

Therefore

$$2\rho = (n-1)\chi_1 + (n-3)\chi_2 + \cdots + (1-n)\chi_n = \sum_{i=1}^{n}(n-(2i-1))\chi_i$$

so $\rho \in X$ if and only if n is odd. As noted above, this does not neccessarily mean that G does not possess a twisting element if n is even. For example if $G = \mathrm{GL}_n$ then one can always take

$$\eta_0 = (n-1)\chi_1 + (n-2)\chi_2 + \cdots + \chi_{n-1} = \sum_{i=1}^{n}(n-i)\chi_i$$

Any twisting element is of the form $\eta_0 + m(\chi_1 + \cdots + \chi_n)$ but this is not neccessarily Galois invariant for an arbitrary form G. One sees that

$$\eta_0 = \rho + \left(\frac{n-1}{2}\right)(\chi_1 + \cdots + \chi_n)$$

We will later see that if n is even then not all forms of G possess a twisting element.

Example. *If G is a form of SO_{2n+1} then it does not possess a twisting element.*

To see this, recall that X and $X^\vee$ are spanned by e_i and f_j respectively with the obvious inner product and $2f_n \in \Delta^\vee$. Therefore for any $x \in X$ $\langle x, 2f_n\rangle \in 2\mathbb{Z}$ so it is impossible for a twisting element to exist.

Example. *If G is a form of Sp_{2n} then $\rho \in X$ so G possesses a twisting element.*

In this case X is spanned by e_i and we have

$$\Delta = \{e_i - e_{i+1} : 1 \leq i \leq n-1\} \cup \{2e_n\}$$

and the set of positive roots is

$$\Phi^+ = \{(e_i - e_j), (e_i + e_j) : 1 \leq i < j \leq n\} \cup \{2e_i : 1 \leq i \leq n\}$$

Therefore

$$2\rho = 2ne_1 + 2(n-1)e_2 + \cdots 2e_n = 2\sum_{i=1}^{n}(n+1-i)e_i \implies \rho \in X$$

Example. *If G is a form of SO_{2n} then $\rho \in X$ so G possesses a twisting element.*

In this case X is spanned by e_i and we have

$$\Delta = \{e_i - e_{i+1} : 1 \leq i \leq n-1\} \cup \{e_{n-1} + e_n\}$$

and the set of positive roots is

$$\Phi^+ = \{(e_i - e_j), (e_i + e_j) : 1 \leq i < j \leq n\}$$

Therefore

$$2\rho = 2(n-1)e_1 + 2(n-2)e_2 + \cdots 2e_{n-1} = 2\sum_{i=1}^{n}(n-i)e_i \implies \rho \in X$$

We will consider the the existence of twisting elements in more detail in appendix A. We finish this section by noting that having a twisting element bears no relation to being $\mathbb{R}$-compact. For example, if G is an inner form of GL_2 then it always has a twisting element but is only $\mathbb{R}$-compact if it is a definite quaternion algebra. Suppose now G is an inner form of GL_n. We claim that G is only $\mathbb{R}$-compact if either $G = \mathbb{G}_m$ or $G = B^\times$ where B is a definite quaternion algebra. As G is an inner form of GL_n, $G = A^\times$ where A is a central simple algebra of dimension n^2. Wedderburn's theorem says that every central simple algebra A over $\mathbb{Q}$ is isomorphic to a matrix algebra over a division algebra, i.e. $A \cong M_k(D)$. This means that $G \cong \mathrm{GL}_k(D)$. By definition of a central simple algebra, A must have center $\mathbb{Q}$ so G must have center $\mathbb{Q}^\times$. Hence $S = \mathbb{G}_m$. It is well known that $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ as there are only 2 non-isomorphic division algebras over $\mathbb{R}$, these are $\mathbb{R}$ and Hamilton's quaternions $\mathbb{H}$. Therefore, as A has index n, we either have $G(\mathbb{R}) \cong \mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_{n/2}(\mathbb{H})$ (the latter is obviously only possible if n is even). However, these contain the split tori $(\mathbb{R}^\times)^n$ and $(\mathbb{R}^\times)^{n/2}$ respectively so S can only be a maximal split torus over $\mathbb{R}$ if $n = 1$ or 2. Therefore either $G \cong \mathbb{G}_m$ or $G(\mathbb{R}) \cong \mathbb{H}^\times$, this latter case is precisely the condition that A is a definite quaternion algebra. Thus for $n > 2$ these inner forms are not $\mathbb{R}$-compact.

1.3.3 The Conjecture

We now give a rough statement of Gross' conjecture in [21]. As opposed to stating it about attaching Galois representations to algebraic modular forms, we will state it about attaching Galois representations to automorphic representations. It follows from [21, Proposition 8.5] that the algebraic modular forms of weight V we wish to attach Galois representations to correspond to automorphic representations π where $\pi_\infty \cong V \otimes \mathbb{C}$ and π_q is Steinberg for all $q \in S$ where S is a finite set of primes of size at least two containing all ramified places (see [10, pg. 37]). Recall the local Langlands correspondence is known at the unramified places. Gross' conjecture becomes the following.

Conjecture (Gross)**.** *Let l be a finite prime and $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ be an isomorphism. Let G be a group which is $\mathbb{R}$-compact and suppose that half the sum of the positive roots lies in the root datum for G. Let π be an automorphic representation of the type described above with finite set S (again as described above). Then there exists a continuous Galois representation*

$$\rho_\pi : \Gamma_{\mathbb{Q}} \to {}^L G(\overline{\mathbb{Q}}_l)$$

s.t. at all finite places $p \notin S \cup \{l\}$ we have that

$$\phi_{\pi_p} = \iota_l(\rho_\pi|_{W_{\mathbb{Q}_p}})$$

where ϕ_{π_p} is the Langlands parameter of π_p, the local component of π at p.

Assuming that half the sum of positive roots lies in the root datum shouldn't be neccessary, one should just have to assume it has a twisting element η. However, for the two main cases we will consider half the sum of the positive roots will lie in the root datum and this simplifies the statement of the conjecture. Gross actually conjectures something stronger. Firstly, he states that if $\tau \in \Gamma_{\mathbb{Q}}$ is a complex conjugation then $\rho_\pi(\tau)$ is $\widehat{G}(\overline{\mathbb{Q}}_l)$-conjugate to $(\eta(-1), \tau) \in {}^L G(\overline{\mathbb{Q}}_l)$; note here we regard the twisting element η as a cocharacter of $\widehat{T}$. Secondly, he states that actually one can get the image of ρ_π lying inside ${}^L G(E_\lambda)$ for any prime $\lambda | l$; here E is an explicitely defined number field depending on π. However, we will not be too concerned with these parts of the conjecture though we will be able to deal with them in some cases. Finally, it is worth noting that we will not have to make use of Gross' Steinberg condition in the cases we consider. We will instead use weaker local hypotheses making an assumption about only one finite place.

Chapter 2

Unitary Groups and C-Groups

2.1 Unitary Groups

2.1.1 The Definition

Let $E/\mathbb{Q}$ be a quadratic extension, and for $x \in E$ let $\bar{x} \in E$ denote the image of x under the non-trivial element of $\mathrm{Gal}(E/\mathbb{Q})$. A Hermitian space for $E/\mathbb{Q}$ is a finite-dimensional E-vector space W together with a pairing $\langle\ ,\ \rangle : W \times W \to E$ which is linear in the first variable and skew-symmetric. We define G by

$$G(\mathbb{Q}) = \{u \in \mathrm{Aut}_E(W) : \langle ux, uy\rangle = \langle x, y\rangle\ \ \forall x, y \in W\}$$

If $n = \dim_E(W)$ then it is clear that $G \subseteq \mathrm{GL}_n$. Moreover, $G(E) \cong \mathrm{GL}_n(E)$ so G is connected, reductive, and a form of GL_n. Now let $E = \mathbb{Q}(\sqrt{-d})$ where d is a positive square-free integer and suppose that the Hermitian form which defines G is definite at infinity, we say that G is definite. The group $G(\mathbb{R})$ is then (isomorphic to) the usual group of unitary matrices which is compact, hence G is $\mathbb{R}$-compact. Finally note that $G(\mathbb{R})$ is connected[1].

Let G be a definite unitary group defined over $\mathbb{Q}$ with $E = \mathbb{Q}(\sqrt{-d})$ the splitting field of G ($d > 0$ and is a square-free integer). We know what happens at infinity. For a finite prime p, we see that the following can happen.

- If p is split in E then $G(\mathbb{Q}_p) \cong \mathrm{GL}_n(\mathbb{Q}_p)$ so is split.
- If p is inert or ramified then $G(\mathbb{Q}_p)$ is a true unitary group. If n is odd this is quasi-split and unique up to isomorphism. If n is even there are two possibilities (up to isomorphism); one quasi-split and one not.

[1] This follows, for example, from [21, Proposition 2.4] as the maximal split torus in the center of G is trivial.

Note if n is odd then G is quasi-split at each finite place, if n is even then things are slightly more complicated though the group will still be quasi-split at almost all finite places (see [3, Section 1] for a more detailed discussion of the local behaviour of unitary groups over $\mathbb{Q}$).

As $G(\overline{\mathbb{Q}}) \cong \mathrm{GL}_n(\overline{\mathbb{Q}})$ the based root datum of G is that of GL_n, i.e.

$$(\mathbb{Z}^n, \{\chi_i - \chi_{i+1} : i = 1, \ldots, n-1\}, \mathbb{Z}^n, \{\lambda_i - \lambda_{i+1} : i = 1, \ldots, n-1\})$$

However there is a non-trivial Galois action on the root datum given by τ, the non-trivial element of $\mathrm{Gal}(E/\mathbb{Q})$. The Galois action sends

$$a_1\chi_1 + \cdots a_n\chi_n \mapsto -a_n\chi_1 - \cdots - a_1\chi_n$$

so it sends $\chi_i - \chi_{i+1} \mapsto \chi_{n-i} - \chi_{n+1-i}$.

2.1.2 The Dual Group and Twisting Element

As GL_n is self-dual we see that

$${}^L G = \mathrm{GL}_n \rtimes \langle \tau \rangle$$

To see what the action of τ is, define $a \in \mathrm{GL}_n$ to be the element with alternating $+1, -1$s on the anti-diagonal and zeroes everywhere else. Let z^t denote the transpose of z, so z^{-t} denotes the transpose of z^{-1} or equivalently the inverse of z^t. Then

$$\tau(g) = ag^{-t}a^{-1}$$

We now look for an element η as described by Gross. Sadly, if n is even this is not possible. For η to be invariant under τ it must be of the following form

$$\eta = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_i(\chi_i - \chi_{n+1-i}) \;\; a_i \in \mathbb{Z}$$

For n even (say $n = 2k$ for $k \in \mathbb{N}$) we'd see that $\langle \eta, \lambda_k - \lambda_{k+1} \rangle = 2a_k \neq 1$ so no such η exists. However, for $n = 2k+1$ where k is a non-negative integer, we can set

$$\eta = \sum_{i=1}^{k} (k+1-i)(\chi_i - \chi_{n+1-i})$$

In this case it is easy to see this is the unique choice of η (provided $n > 1$). In fact η is half the sum of the positive roots of G, we saw this in section 1.3.2, so[2]

$$\eta = \left(\frac{n-1}{2}\right)\chi_1 + \left(\frac{n-3}{2}\right)\chi_2 + \cdots + \left(\frac{3-n}{2}\right)\chi_{n-1} + \left(\frac{1-n}{2}\right)\chi_n$$

[2]For $n \geq 3$.

We will see that if G is an odd-dimensional definite unitary group then one can attach Galois representations with image in ${}^LG$ to automorphic representations of G. We will also be able to get Galois representations in the even-dimensional case, however the image will not be in ${}^LG$.

2.1.3 Galois Representations into GL_n

Provided n is odd, we are trying to find Galois representations of the form

$$\rho : \Gamma_{\mathbb{Q}} \to {}^LG(\overline{\mathbb{Q}}_l) = \mathrm{GL}_n(\overline{\mathbb{Q}}_l) \rtimes \langle \tau \rangle$$

To do this we first want Galois representations from Γ_E to $\mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ for a general n. Let G be as before. Note that for the primes p which split in E one has $G(\mathbb{Q}_p) \cong \mathrm{GL}_n(\mathbb{Q}_p)$. The following theorem can be found as expressed below in [3], but is obtained from [14] and [37].

Theorem. *Let π be an automorphic representation of G. Then for each prime l there exists a unique semi-simple Galois representation*

$$\rho : \Gamma_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

such that for each prime p which splits in E with $\mathfrak{p}|p$, $\rho|_{W_{E_\mathfrak{p}}}$ corresponds under the local Langlands correspondence to π_p, seen as a representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ by the isomorphism determined by $\mathfrak{p}$. Moreover, one has that $\rho^\tau \cong \chi_l^{1-n}\rho^\vee$ where χ_l is the l-adic cyclotomic character.

Note that we will actually only need a weaker result of Kottwitz and Clozel, in [25] and [15] respectively. They assume that the automorphic representation π is discrete at some finite split place.

Now Γ_E is of index 2 in $\Gamma_{\mathbb{Q}}$ and the non-trivial element τ in the quotient exchanges the conjugacy classes of $\mathrm{Frob}_\mathfrak{p}$ and $\mathrm{Frob}_{\mathfrak{p}'}$ if $p = \mathfrak{p}\mathfrak{p}'$. We want to lift ρ after extending the image to ${}^LG$. Specifically, as

$$\Gamma_{\mathbb{Q}} = \Gamma_E \rtimes \langle \tau \rangle$$

we would like to extend the definition to have $\Gamma_{\mathbb{Q}} \to {}^LG(\overline{\mathbb{Q}}_l) = \mathrm{GL}_n(\overline{\mathbb{Q}}_l) \rtimes \langle \tau \rangle$, i.e. we want a representation from $\Gamma_{\mathbb{Q}}$ to ${}^LG(\overline{\mathbb{Q}}_l)$ which restricts to ρ. The question is how can we do this, and indeed why won't this work when n is even? We will use [17] and

[10]. The outline is thus. The work of [17] extends the Galois representation ρ to a representation

$$r : \Gamma_{\mathbb{Q}} \to \mathcal{G}_n(\overline{\mathbb{Q}}_l)$$

where $\mathcal{G}_n$ will be defined later. Then, using a homomorphism $\mathcal{G}_n \to {}^C G$ we will have representations into the C-group of G (again this will be defined later). All this can be done for an arbitrary n. When n is odd we will be able to get a map ${}^C G \to {}^L G$ but this will not be possible in the even-dimensional case.

2.1.4 Extending Galois Representations to $\mathcal{G}_n$

In this section we define the group $\mathcal{G}_n$, as defined in [17], and explain how we can extend our Galois representation to it. The group $\mathcal{G}_n$ is the semi-direct product of $\mathrm{GL}_n \times \mathbb{G}_m$ with $\langle \tau \rangle$, with the action defined by

$$(x, y)^\tau = (yx^{-t}, y)$$

To extend our Galois representation we use lemma 2.1.4 of [17, pgs. 10-11]. Recall we have a

$$\rho : \Gamma_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

s.t. $\rho^\tau \cong \chi_l^{1-n} \rho^\vee$. This means there is some invertible matrix $J \in \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ s.t.

$$J\rho(\tau g \tau^{-1})J^{-1} = \chi_l(g)^{1-n}\rho(g)^{-t} \;\; \forall g \in \mathrm{Gal}(\overline{E}/E)$$

One then defines the perfect pairing from lemma 2.1.4 by $\langle x, y \rangle = x^t J y$. By its definition one sees that

$$\langle \rho(g)x, \rho(\tau g \tau^{-1})y \rangle = \chi_l^{1-n}(g)\langle x, y \rangle$$

Moreover, if ρ is absolutely irreducible one sees that $\langle y, x \rangle = \epsilon \langle x, y \rangle$ where $\epsilon^2 = 1$, this means J is symmetric or anti-symmetric (as $J^t = \epsilon J$). Suppose now that π is cuspidal. While ρ is not always absolutely irreducible, it is if π is square-integrable at some prime p split in E. In particular, if π is discrete at some finite split place p (e.g. Steinberg at p) then ρ is absolutely irreducible. This means we only need the weaker result of Kottwitz and Clozel for finding ρ as discussed in section 2.1.3. As $\langle \; , \; \rangle$ is a perfect pairing it must be non-degenerate. We will need the following fact; let n be odd, $A \in M_n(R)$ for some integral domain R with characteristic $\neq 2$, and suppose $A^t = -A$. Then A is not invertible. To see this note

$$\det(A) = \det(A^t) = \det(-A) = (-1)^n \det(A) = -\det(A)$$

so $\det(A) = 0$. As we know that J is invertible, this shows if n is odd then $\epsilon = 1$ and J is symmetric. If n is even then a priori either possibility can occur. In fact $\epsilon = 1$ if n is even as well. To see why this holds, note that π is square-integrable at some finite split place. As mentioned previously, the Galois representation associated to π is absolutely irreducible. Moreover π is stable, i.e. its base change to GL_n/E is cuspidal (see [3, Section 4.3] for a discussion on this, the Galois representation result is in [40]). Hence $\epsilon = 1$ due to [4, Theorem 1.2]. It is worth noting at this point that if a cuspidal automorphic representation π of G is stable it is expected that its associated Galois representation is absolutely irreducible, but this is not yet known in general.

Now $\chi^{1-n}(\tau) = (-1)^{n-1}$. One defines $\mu : \Gamma_{\mathbb{Q}} \to \mathbb{Z}_l^\times$ by $\mu(g) = \chi^{1-n}(g)\ \forall g \in \Gamma_E$ and $\mu(\tau) = -1$. Then we can apply [17, Lemma 2.1.1] to get an extension

$$r : \Gamma_{\mathbb{Q}} \to \mathcal{G}_n(\overline{\mathbb{Q}}_l)$$

defined by $r(g) = (\rho(g), \chi_l^{1-n}(g))$ for $g \in \Gamma_E$ and $r(\tau) = (J^{-1}, 1)\tau$.

2.2 C-Groups

2.2.1 The Definition

We now briefly describe the important facts about C-groups, the reference for this is [10]. Recall that in Gross' work he assumes the existence of a certain element η which we call a twisting element. Unfortunately, as we have seen, not all reductive groups have a twisting element. Buzzard and Gee show in [10] that for any connected reductive group G there always exists a central $\mathbb{G}_m$-extension which possesses a twisting element. Specifically, there is a group $\widetilde{G}$ possessing a twisting element which fits into an exact sequence

$$1 \to \mathbb{G}_m \to \widetilde{G} \to G \to 1$$

and there is a natural bijection between the set of splittings of this sequence and twisting elements of G. One then defines ${}^C G = {}^L \widetilde{G}$. Of course to calculate this when G is a definite unitary group we must find out what $\widehat{\widetilde{G}}$ is. One finds there is a surjection

$$\widehat{G} \times \mathbb{G}_m \to \widehat{\widetilde{G}}$$

with (central) kernel of order 2 generated by $(2\rho(-1), -1)$, where 2ρ is the sum of the positive roots of G and we regard it as a cocharacter of $\widehat{T} \subseteq \widehat{G}$.

One finds that

$$^CG = (\mathrm{GL}_n \times \mathbb{G}_m)/((-1)^{n-1}, -1) \rtimes \langle\tau\rangle, \text{ where } (x,y)^\tau = (ax^{-t}a^{-1}, y)$$

where $a \in \mathrm{GL}_n$ is the element with alternating $+1, -1$s on the anti-diagonal and zeroes everywhere else. We write elements of ^CG as triples (g, μ, γ) with $g \in \mathrm{GL}_n$, $\mu \in \mathbb{G}_m$ and $\gamma \in \mathrm{Gal}(E/\mathbb{Q})$. Of course if n is odd one has a map

$$^CG \to {}^LG; (g, \mu, 1) \mapsto (g, 1), (1, 1, \tau) \mapsto (1, \tau)$$

so getting C-group representations will give us L-group ones. This is to be expected, as half the sum of positive roots being in the character group makes being L- and C-algebraic equivalent and this holds if n is odd (see [10] for more on this).

2.2.2 Getting C-group representations

We want to get a representation

$$\rho' : \Gamma_\mathbb{Q} \to {}^CG(\overline{\mathbb{Q}}_l)$$

In [10, Section 8.3] a homomorphism is defined as follows

$$j : {}^CG \to \mathcal{G}_n$$

$$(x, y, 1) \mapsto (xy^{1-n}, y^{2(1-n)}), (1, 1, \tau) \mapsto (a, (-1)^{n-1})\tau$$

They then define a map

$$j' : \mathcal{G}_n \times \mathbb{G}_m \to {}^CG$$

defined on the identity component by[3]

$$((x, \mu), \lambda) \mapsto (x\lambda^{(n-1)/2}, \lambda^{1/2}, 1)$$

such that $j' \circ (j \times d) = \mathrm{id}_{^CG}$ ($d : {}^CG \to \mathbb{G}_m$ is defined in their paper) and define a Galois representation $j' \circ (r \times \chi_l)$. We will instead give ρ' explicitly. One has that

$$\rho'(g) = (\chi_l^{(n-1)/2}(g)\rho(g), \chi_l^{1/2}(g), 1) \ \ \forall g \in \Gamma_E$$

$$\rho'(\tau) = ((-1)^{(n-1)/2} J^{-1}a^{-1}, (-1)^{1/2}, \tau)$$

It is easy to see that

$$\rho'(\tau^2) = ((-1)^{(n-1)/2} J^{-1}a^{-1}, (-1)^{1/2}, \tau) \cdot ((-1)^{(n-1)/2} J^{-1}a^{-1}, (-1)^{1/2}, \tau)$$

[3]Note that this is well-defined and independent of the choice of square root due to the definition of ^CG.

$$= ((-1)^{(n-1)/2}J^{-1}a^{-1}, (-1)^{1/2}, 1) \cdot ((-1)^{-(n-1)/2}aJ^ta^ta^{-1}, (-1)^{1/2}, 1)$$

$$= ((-1)^{n-1}, -1, 1) = (1,1,1) \in {}^CG$$

since $J^t = J$ and $a^t = a^{-1} = (-1)^{n-1}a$. To check the action of τ is compatible, note on the one hand that

$$\rho'(\tau g \tau^{-1}) = (\chi_l^{(n-1)/2}(g)\rho(\tau g \tau^{-1}), \chi_l^{1/2}(g), 1)$$

On the other hand

$$\rho'(\tau)\rho'(g)\rho'(\tau^{-1}) = ((-1)^{(n-1)/2}J^{-1}a^{-1}, (-1)^{1/2}, \tau)\cdot$$
$$(\chi_l^{(n-1)/2}(g)\rho(g), \chi_l^{1/2}(g), 1) \cdot ((-1)^{(n-1)/2}J^{-1}a^{-1}, (-1)^{1/2}, \tau) =$$

$$((-1)^{(n-1)/2}J^{-1}a^{-1}, (-1)^{1/2}, \tau) \cdot ((-1)^{(n-1)/2}\chi_l^{(n-1)/2}(g)\rho(g)J^{-1}a^{-1}, (-1)^{1/2}\chi_l^{1/2}(g), \tau)$$

$$= ((-1)^{(n-1)/2}J^{-1}a^{-1}, (-1)^{1/2}, 1)\cdot$$
$$((-1)^{-(n-1)/2}\chi_l^{-(n-1)/2}(g)a\rho(g)^{-t}J^ta^ta^{-1}, (-1)^{1/2}\chi_l^{1/2}(g), 1)$$

$$= (\chi_l^{-(n-1)/2}(g)J^{-1}\rho(g)^{-t}J(-1)^{n-1}, -\chi_l^{1/2}(g), 1)$$

$$= (\chi_l^{(n-1)/2}(g)\rho(\tau g\tau^{-1}), \chi_l^{1/2}(g), 1) \cdot ((-1)^{n-1}, -1, 1)$$

$$= \rho'(\tau g \tau^{-1}) \in {}^CG$$

as required.

2.2.3 The Odd-Dimensional Case

In this subsection let n be odd, say $n = 2k+1$. Using the previously defined map ${}^CG \to {}^LG$ we get an L-group representation $\rho_\pi : \Gamma_{\mathbb{Q}} \to {}^LG(\overline{\mathbb{Q}}_l)$. Specifically it is defined by

$$\rho_\pi(g) = (\rho(g)\chi_l^k(g), 1) \ \forall g \in \Gamma_E$$

$$\rho_\pi(\tau) = ((-1)^kJ^{-1}a, \tau)$$

where ρ is the representation of Γ_E given by the theorem of section 2.1.3. Just to check this makes sense, note that (as $a = a^{-1}$ and $J^t = J$)

$$\rho_\pi(\tau)\rho_\pi(g)\rho_\pi(\tau^{-1}) = ((-1)^kJ^{-1}a, \tau) \cdot (\rho(g)\chi_l^k(g), 1) \cdot ((-1)^kJ^{-1}a, \tau)$$

$$= ((-1)^kJ^{-1}a, \tau) \cdot ((-1)^k\chi_l^k(g)\rho(g)J^{-1}a, \tau)$$

$$= ((-1)^kJ^{-1}a, 1) \cdot ((-1)^{-k}\chi_l^{-k}(g)a\rho(g)^{-t}J^ta^{-t}a^{-1}, 1)$$

$$= (\chi_l^{-k}(g)J^{-1}\rho(g)^{-t}J, 1) = (\chi_l^k(g)\rho(\tau g\tau^{-1}), 1) = \rho_\pi(\tau g\tau^{-1})$$

and

$$\rho_\pi(\tau^2) = ((-1)^k J^{-1}a, \tau) \cdot ((-1)^k J^{-1}a, \tau)$$

$$= ((-1)^k J^{-1}a, 1) \cdot ((-1)^{-k} aJ^t a^{-t} a^{-1}, 1) = (1, 1)$$

as required. This is our Galois representation which was predicted to exist by Gross.

We can also check the form of complex conjugation, recall this is part of Gross' conjecture. We have that $\rho_\pi(\tau) = ((-1)^k J^{-1}a, \tau)$ and we want to show this is $\widehat{G}$-conjugate to $(\eta(-1), \tau)$. What exactly does it mean to be $\widehat{G}$-conjugate in this case? It means that $\exists g \in \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ s.t.

$$(\eta(-1), \tau) = (g^{-1}, 1) \cdot ((-1)^k J^{-1}a, \tau) \cdot (g, 1)$$

$$= ((-1)^k g^{-1}J^{-1}a, \tau) \cdot (g, \tau) \cdot (1, \tau) = ((-1)^k g^{-1}J^{-1}g^{-t}a, \tau)$$

So we need to show that there is a $g \in \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ s.t.

$$g^t J g = (-1)^k a\eta(-1)$$

Recall that

$$\eta(-1) = \begin{pmatrix} (-1)^k & & & & & & & & \\ & (-1)^{k-1} & & & & & & & \\ & & \ddots & & & & & & \\ & & & -1 & & & & & \\ & & & & 1 & & & & \\ & & & & & -1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & (-1)^{k-1} & \\ & & & & & & & & (-1)^k \end{pmatrix}$$

so we have that $a\eta(-1)$ is the matrix with 1's on the anti-diagonal. This matrix is symmetric and invertible (therefore so is $(-1)^k a\eta(-1)$), so by the general theory of symmetric non-degenerate bilinear forms, as J is also symmetric and invertible, $\exists g, h \in \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ (not necessarily orthogonal) s.t.

$$g^t J g = I_n = h^t((-1)^k a\eta(-1))h$$

so we have that

$$(gh^{-1})^t J(gh^{-1}) = (-1)^k a\eta(-1)$$

We have therefore proved the following theorem.

Theorem 2.2.1. *Let l be a finite prime and $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ be an isomorphism. Let π be a cuspidal automorphic representation of G, an odd-dimensional definite unitary group over $\mathbb{Q}$, which is discrete at one finite split place. Then there exists a continuous Galois representation*

$$\rho_\pi : \Gamma_{\mathbb{Q}} \to {}^L G(\overline{\mathbb{Q}}_l)$$

s.t. at almost all finite places $p \neq l$ we have that

$$\phi_{\pi_p} = \iota_l(\rho_\pi|_{W_{\mathbb{Q}_p}})$$

where ϕ_{π_p} is the Langlands parameter of π_p, the local component of π at p. Moreover, $\rho_\pi(\tau)$ is $\widehat{G}$-conjugate to $(\eta(-1), \tau)$ where η is half the sum of the positive roots of G.

2.2.4 Extending Gross' Conjecture

We previously remarked that the work of Buzzard and Gee allows one to always adjoin a twisting element by passing to a central $\mathbb{G}_m$-extension $\widetilde{G}$. In fact we can say more. By the proof of [10, Proposition 5.3.3], we know that there exists a central isogeny

$$\phi : \widetilde{G} \twoheadrightarrow G \times \mathbb{G}_m$$

with kernel of order 2. Suppose that Γ is an arithmetic subgroup of $\widetilde{G}(\mathbb{Q})$. Then $\phi(\Gamma)$ lies inside an arithmetic subgroup of $G \times \mathbb{G}_m$ by [34, Theorem 4.1].
Now $|(G \times \mathbb{G}_m)(\mathbb{Z})| = 2|G(\mathbb{Z})| < \infty$, so we have $\phi(\Gamma)$ is finite. Therefore as ϕ has kernel of order 2 we have

$$|\Gamma| \leq 2|\phi(\Gamma)| < \infty$$

so $\widetilde{G}$ has the property that all arithmetic subgroups of $\widetilde{G}(\mathbb{Q})$ finite (i.e. it's $\mathbb{R}$-compact), and has a twisting element (all we needed here was that ϕ has finite kernel). Note that $\widetilde{G}$ must be reductive, since it is an extension of a reductive group by a reductive group. To see that $\widetilde{G}$ is connected, recall it fits into an exact sequence

$$1 \to \mathbb{G}_m \to \widetilde{G} \to G \to 1$$

where $\mathbb{G}_m$ and G are connected, hence so is $\widetilde{G}$.

Note that as the exact sequence is central, we have that automorphic representations on G with trivial central character correspond exactly to those on $\widetilde{G}$ with trivial central character. The consequence of all of this is as follows. Let G be an $\mathbb{R}$-compact reductive group. Then to an automorphic representation of G which satisfies the hypotheses of Gross' conjecture one should be able to attach an l-adic Galois representation

$$\rho : \Gamma_{\mathbb{Q}} \to {}^{C}G(\overline{\mathbb{Q}}_l)$$

Moreover, this gives an L-group representation precisely when G possesses a twisting element. We have verified this in the case when G is a definite unitary group. For more on this see [10].

Chapter 3

Symplectic Groups and their Inner Forms

3.1 The Groups

3.1.1 Definitions

We now give the main definitions needed in this chapter. Let F be a field of characteristic zero. We define

$$\mathrm{GSp}(M) = \{g \in \mathrm{GL}_{2n} : g^t M g = c(g) M\}$$

where $n \in \mathbb{N}, c(g) \in \mathbb{G}_m$, and $M \in \mathrm{GL}_{2n}$ is an invertible skew-symmetric matrix. We take $\mathrm{GSp}_{2n} := \mathrm{GSp}(\Omega)$ where one takes

$$\Omega = \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix}$$

where J_n is the matrix

$$J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in \mathrm{GL}_n(F)$$

We define

$$\mathrm{Sp}(M) = \{g \in \mathrm{GSp}(M) : c(g) = 1\}$$

and take $\mathrm{Sp}_{2n} := \mathrm{Sp}(\Omega)$.

The endoscopic groups H of a reductive group G/F are quasi-split groups such that

$$\widehat{H} = \mathrm{Cent}(s, \widehat{G})^o$$

where s is a semi-simple element of $\widehat{G}$ and here we are taking the connected component of the centraliser. We consider equivalence classes of these groups with the equivalence relation given by translation by $Z(\widehat{G})$ and conjugation by $\widehat{G}$. We say an endoscopic group H is elliptic if $Z(\widehat{H})^{\Gamma_F} \subset Z(\widehat{G})$. We do not need to give the precise general definition, and instead give the elliptic endoscopic groups in the cases we are interested in. The elliptic endoscopic groups of Sp_{2n} are[1] (up to equivalence)

$$H = \mathrm{Sp}_{2n_1} \times \mathrm{SO}_{2n_2}(\eta)$$

where $n_1 + n_2 = n$, η is a quadratic character, and η is non-trivial if $n_2 = 1$ and trivial if $n_2 = 0$ (see [1, Section 1.2]). The elliptic endoscopic groups of $\mathrm{SO}_{2n}(\eta)$ are

$$H = \mathrm{SO}_{2n_1}(\eta_1) \times \mathrm{SO}_{2n_2}(\eta_2)$$

with $n_1 + n_2 = n$, $\eta_1\eta_2 = \eta$, and η_i is non-trivial if $n_i = 1$ and trivial if $n_i = 0$. We obtain the following L-groups

$${}^L\mathrm{GL}_k = \mathrm{GL}_k,\ {}^L\mathrm{Sp}_{2n} = \mathrm{SO}_{2n+1},\ {}^L\mathrm{SO}_{2m}(\eta) = \mathrm{SO}_{2m} \rtimes \mathrm{Gal}(E/F)$$

In the case where E/F is quadratic, the action of $\mathrm{Gal}(E/F)$ is given by conjugation by an element of $\mathrm{O}_{2m}(\mathbb{C})\backslash\mathrm{SO}_{2m}(\mathbb{C})$. The definition of H means that $\widehat{H} \subseteq \widehat{G}$. If $G = \mathrm{Sp}_{2n}$ then there is an embedding ${}^LH \hookrightarrow \widehat{\mathrm{Sp}_{2n}}$ and it is worth noting that elements conjugate under $\mathrm{O}_{2n_2}(\mathbb{C})$ in $\widehat{H}$ become conjugate in $\widehat{\mathrm{Sp}_{2n}}$ under this embedding.

Let $B/\mathbb{Q}$ be a quaternion algebra and let R be a commutative $\mathbb{Q}$-algebra. If $g = (q_{ij})$ with $q_{ij} \in B$ we define $g^* = (\bar{q}_{ji})$ where $\bar{q}$ is the standard quaternion involution. We define algebraic groups over $\mathbb{Q}$ by

$$\mathrm{GU}_n(B)(R) = \{g \in \mathrm{GL}_n(B) \otimes_{\mathbb{Q}} R : g^*g = \lambda(g)I_n,\ \lambda(g) \in R\}$$

$$\mathrm{U}_n(B)(R) = \{g \in \mathrm{GU}_n(B)(R) : \lambda(g) = 1\}$$

If $g = (q_{ij}) \in \mathrm{GL}_n(B)$ then we see that the jth entry on the diagonal of g^*g is $\sum_{i=1}^n N(q_{ij}) \in \mathbb{Q}$ (note $N(q) = q\bar{q}$ is the usual quaternion norm). Hence $\lambda(g) \in R^\times$ if $g \in \mathrm{GU}_n(B)(R)$ and in fact λ is a homomorphism (over $\mathbb{Q}$) of algebraic groups $\mathrm{GU}_n(B) \to \mathbb{G}_m$ and $\mathrm{U}_n(B)$ is the kernel. We will generally consider quaternions as being elements of $M_2(\overline{\mathbb{Q}})$ and therefore consider the rational points of these groups as being subgroups of $\mathrm{GL}_{2n}(\overline{\mathbb{Q}})$; note that the 'conjugate transpose' g^* is different from the standard one on $2n \times 2n$ matrices. Finally we define the group

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} w & -\bar{z} \\ z & \bar{w} \end{pmatrix} : z, w \in \mathbb{C},\ |z|^2 + |w|^2 = 1 \right\}$$

which appears in the local Langlands correspondence in the definition of L_F for F a non-archimedean local field.

[1] Recall the relevant special orthogonal groups $\mathrm{SO}_{2n}(\eta)$ were defined section 1.1.3.

3.1.2 The Root Datum of Sp_{2n}

We now construct the root datum of Sp_{2n} (and its inner forms), note that as the group is split there is no Galois action. For the maximal split torus T we may take matrices of the form

$$\begin{pmatrix} u_1 & & & & & \\ & \ddots & & & & \\ & & u_n & & & \\ & & & u_n^{-1} & & \\ & & & & \ddots & \\ & & & & & u_1^{-1} \end{pmatrix}$$

and for $t \in T$ and $1 \leq i \leq n$ define $e_i(t) = u_i$. These characters form a basis of $X^*(T)$. One also defines cocharacters of T, which we label $f_1, \ldots, f_n$, in such a way that the natural pairing $\langle\ ,\ \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ evaluates as follows

$$\langle e_i, f_j \rangle = \delta_{ij}$$

One gets the following roots

$$\{\pm(e_i - e_j), \pm(e_i + e_j) : 1 \leq i < j \leq n\} \cup \{\pm 2e_i : 1 \leq i \leq n\}$$

The upper triangular matrices form a Borel subgroup and one obtains the following as the simple roots and associated simple coroots

$$\alpha_i = e_i - e_{i+1} \text{ for } 1 \leq i < n,\ \alpha_n = 2e_n$$

$$\alpha_i^\vee = f_i - f_{i+1} \text{ for } 1 \leq i < n,\ \alpha_n^\vee = f_n$$

The element

$$\eta = \sum_{i=1}^{n} (n+1-i)e_i = ne_1 + (n-1)e_2 + \cdots 2e_{n-1} + e_n$$

satisfies the property that $\langle \eta, \alpha^\vee \rangle = 1$ for all simple coroots $\alpha^\vee$ (it is also the unique η with this property). Note in fact that η is equal to half the sum of the positive roots of Sp_{2n}. As inner forms of Sp_{2n} have the same based root datum as Sp_{2n} all of these groups possess a twisting element. Of course Sp_{2n} isn't $\mathbb{R}$-compact, but if we can find inner forms which are $\mathbb{R}$-compact then one would expect to be able to find automorphic Galois representations with image in the L-group, in this case that means they would have orthogonal image.

3.1.3 Inner Forms

We want to consider what the inner forms of Sp_{2n}, and indeed GSp_{2n} are. We will prove a result concerning inner forms of Sp_{2n} and GSp_{2n} but first we will state the following fact. Suppose F is an arbitrary field with $\mathrm{char}(F) \neq 2$. Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and let $W \in \mathrm{GL}_{2n}(F)$ be the skew-symmetric matrix given by n w's on the diagonal. For any skew-symmetric matrix $A \in \mathrm{GL}_{2n}(F)$ there exists a matrix $Q \in \mathrm{GL}_{2n}(F)$ such that $QAQ^t = W$, but Q is not neccessarily orthogonal. In all cases one sees that the groups $\mathrm{GSp}(W)$ and $\mathrm{GSp}(A)$ are isomorphic, as are $\mathrm{Sp}(W)$ and $\mathrm{Sp}(A)$. In particular $\mathrm{GSp}(W) \cong \mathrm{GSp}_{2n}$ and $\mathrm{Sp}(W) \cong \mathrm{Sp}_{2n}$. In fact in this case it is easy to explicitly give the isomorphism. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_{2n}\}$ be the standard basis of column vectors in F^{2n} and let P be the matrix

$$P = (\mathbf{e}_1, \mathbf{e}_{2n}, \mathbf{e}_2, \mathbf{e}_{2n-1}, \cdots, \mathbf{e}_n, \mathbf{e}_{n+1})$$

One sees that P is orthogonal (since it's a permutation matrix), and a simple calculation shows that $PWP^{-1} = PWP^t = \Omega$. Therefore the map

$$x \mapsto PxP^t$$

gives the isomorphisms $\mathrm{GSp}(W) \xrightarrow{\sim} \mathrm{GSp}_{2n}$ and $\mathrm{Sp}(W) \xrightarrow{\sim} \mathrm{Sp}_{2n}$. We can now state the following proposition.

Proposition 3.1.1. *Let B be a rational quaternion algebra. Then $\mathrm{GU}_n(B)$ is an inner form of GSp_{2n}, and $\mathrm{U}_n(B)$ is an inner form of Sp_{2n}. Moreover, if B is definite then $\mathrm{U}_n(B)$ is $\mathbb{R}$-compact.*

Proof. Let $\widetilde{G} = \mathrm{GU}_n(B)$ and $G = \mathrm{U}_n(B)$ where B is a rational quaternion algebra, say

$$B = \left(\frac{a, b}{\mathbb{Q}}\right) := \langle 1, i, j, ij : i^2 = a, j^2 = b, ij = -ji \rangle_{\mathbb{Q}}$$

We may assume that $a, b \in \mathbb{Z}$ and are square-free. Let $E = \mathbb{Q}(\sqrt{a})$. Given $q = x + iy + jz + ijw \in B$ we get an element

$$q = \begin{pmatrix} x + y\sqrt{a} & (z + w\sqrt{a})b \\ z - w\sqrt{a} & x - y\sqrt{a} \end{pmatrix} \in M_2(E)$$

Note we have

$$g \in \widetilde{G} \iff g^*g = \lambda(g) I_{2n}$$

where g^* is the conjugate transpose, and such a g is in G if and only if $\lambda(g) = 1$. By $\bar{g}$ we simply mean performing the standard quaternion involution $q \mapsto \bar{q}$ to all its entries. Regarding $B \subset M_2(E)$ this looks like

$$q = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \mapsto \bar{q} = \begin{pmatrix} q_4 & -q_2 \\ -q_3 & q_1 \end{pmatrix}$$

If $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then one sees that

$$w^{-1} \begin{pmatrix} q_1 & q_3 \\ q_2 & q_4 \end{pmatrix} w = \bar{q}$$

If we let W be as before, and note that W^{-1} is the $2n \times 2n$ matrix with n w^{-1}'s on the diagonal, we see that

$$W^{-1} g^t W = g^*$$

Note that $W^{-1} = W^t = -W$. Therefore, if $g^* g = \lambda(g) I_{2n}$ then $g^t W g = \lambda(g) W$ so we have an isomorphism[2] $\phi : (\mathrm{GSp}_{2n})_E \cong \widetilde{G}_E$, i.e. $\widetilde{G}$ is a form of GSp_{2n}. Moreover, this isomorphism clearly restricts to one identifying $(\mathrm{Sp}_{2n})_E \cong G_E$ so they are also forms of each other.

In order to show that $\widetilde{G}$ and G are inner forms of GSp_{2n} and Sp_{2n} respectively we will show that their associated cocycles have image in the inner automorphisms of the relevant group. We therefore want to consider for the cocycle c the map $c(t) = \phi^{-1} t \phi t^{-1}$ where $t \in \mathrm{Gal}(E/\mathbb{Q})$ is the non-trivial element of the Galois group defined by $t(x + y\sqrt{a}) = x - y\sqrt{a}$. When $n = 1$ we call this $c_2(t)$. One sees that the action of this is

$$c_2(t)(\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}) = \begin{pmatrix} q_4 & bq_3 \\ b^{-1}q_2 & q_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ b^{-1} & 0 \end{pmatrix} \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

We set $\gamma = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$ and see that $\gamma^{-1} = \begin{pmatrix} 0 & 1 \\ b^{-1} & 0 \end{pmatrix}$. Now let Γ and Γ^{-1} be the $2n \times 2n$ matrices consisting of n γ's and γ^{-1}'s on the diagonal respectively. We find $c(t)(g) = \Gamma^{-1} g \Gamma$. Moreover,

$$\Gamma^t W \Gamma = \begin{pmatrix} \gamma^t w \gamma & & \\ & \ddots & \\ & & \gamma^t w \gamma \end{pmatrix}$$

[2]As GSp_{2n} is isomorphic to $\mathrm{GSp}(W)$ over $\mathbb{Q}$.

and
$$\gamma^t w \gamma = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = (-b) \cdot w$$
so $\Gamma^t W \Gamma = (-b) \cdot W$, i.e. $\Gamma \in \mathrm{GSp}(W)(\mathbb{Q})$. Thus we have shown that $\widetilde{G}$ is an inner form of GSp_{2n}.

In order to conclude that $\mathrm{U}_n(B)$ is an inner form of Sp_{2n} we need to be able to show that we can take Γ to be in $\mathrm{Sp}(W)(\overline{\mathbb{Q}})$. Multiplying the matrix Γ by a scalar matrix won't change the conjugation, so conjugating instead by $\Gamma' = (-b)^{-1/2}\Gamma$ gives what we want.

We now determine which of these inner forms of Sp_{2n} are $\mathbb{R}$-compact, i.e. all of their arithmetic subgroups are finite. Recall arithmetic subgroups are those commensurable with $G(\mathbb{Z})$. It is therefore enough to show that $G(\mathbb{Z})$ is finite. Note that
$$B^\times(\mathbb{Z}) = \{ \begin{pmatrix} x + y\sqrt{a} & (z + w\sqrt{a})b \\ z - w\sqrt{a} & x - y\sqrt{a} \end{pmatrix} : x, y, z, w \in \mathbb{Z}, \;\; x^2 - ay^2 - bz^2 + abz^2 = \pm 1 \}$$
which clearly has size at most 8 if B is definite and is infinite otherwise. Assume B is definite (so $a, b < 0$). Now for $g \in G(\mathbb{Z})$ we know that $g^* g = I_{2n}$. If $g = (q_{ij})$ then we see that the jth entry on the diagonal of $g^* g$ is $\sum_{i=1}^n N(q_{ij})$. As the q_{ij} must have 'integer values' (i.e. be in $B(\mathbb{Z})$) we see that the norms must have integer values. As B is definite $N(q_{ij}) \geq 0$ and is 0 if and only if $q_{ij} = 0$. Therefore exactly one q_{ij} is non-zero for each j and this entry has norm one. Therefore elements of $G(\mathbb{Z})$ consist of matrices with precisely one non-zero quaternion in each row and column which is of the form $x + iy + jz + ijw$ with $x^2 - ay^2 - bz^2 + abz^2 = 1$ and $w, x, y, z \in \mathbb{Z}$. There are at most 8 possible values for these which makes $|G(\mathbb{Z})| \leq 8^n n! < \infty$. Therefore the inner forms $G = \mathrm{U}_n(B)$ where B is definite are $\mathbb{R}$-compact. $\square$

It is worth noting that the groups above are described in [41, pg. 56].

3.2 Symplectic and Orthogonal Automorphic Representations: the work of Arthur

In this section we discuss Arthur's work in [1] on the local and global Langlands correspondence for symplectic and orthogonal groups. In order to keep consistency with the notation of the relevant sections of [1], we redefine the L-group of a reductive group G over a field F as follows;
$${}^L G = \widehat{G} \rtimes \Gamma_F$$

Not only are we now taking the semidirect product with the whole absolute Galois group, but when we write ${}^LG$ and $\widehat{G}$ we really mean the group of complex points. For example, this means we obtain the following L-groups for our groups over $\mathbb{Q}$;

$$ {}^L\mathrm{Sp}_{2n} = \mathrm{SO}_{2n+1}(\mathbb{C}) \times \Gamma_{\mathbb{Q}},\ {}^L\mathrm{SO}_{2m}(\eta) = \mathrm{SO}_{2m}(\mathbb{C}) \rtimes \Gamma_{\mathbb{Q}} $$

3.2.1 Twisted Endoscopy

In order to attach Galois representations to π we will need to consider the theory of twisted endoscopy. We will start off by following [1, Section 1.2] to give the definition.

Consider a continuous semisimple representation

$$ \rho : \Lambda_{\mathbb{Q}} \to \mathrm{GL}_N(\mathbb{C}) $$

where $\Lambda_{\mathbb{Q}}$ is a topological group with a continuous mapping $\Lambda_{\mathbb{Q}} \to \Gamma_{\mathbb{Q}}$ with connected kernel and dense image. We say that ρ is self-dual if ρ is equivalent to

$$ \rho^\vee(\gamma) := \rho(\gamma)^{-t}\ \forall \gamma \in \Lambda_{\mathbb{Q}} $$

where g^t denotes the transpose of g. As ρ is semisimple we can decompose it as $\rho = l_1\rho_1 \oplus \cdots \oplus l_m\rho_m$ where ρ_i are inequivalent irreducible representations and the l_i are their multiplicities; one has that $\sum_{i=1}^m l_i n_i = N$. We say ρ is elliptic if $l_i = 1$ and ρ_i is self-dual for all i. Suppose ρ is elliptic. Then there exists $A_i \in \mathrm{GL}_{n_i}(\mathbb{C})$ such that

$$ \rho_i^\vee(\gamma) = A_i \rho_i(\gamma) A_i^{-1} $$

Applying the involution $\theta : g \mapsto g^{-t}$ we see

$$ \rho_i(\gamma) = A_i^{-t} \rho_i^\vee(\gamma) A_i^t = A_i^{-t} A_i \rho_i(\gamma)(A_i^{-t}A_i)^{-1} $$

As ρ_i is irreducible the matrix $A_i^{-t}A_i$ is a scalar, so $A_i^t = c_i A_i$. Taking transposes of each side we see that $c_i^2 = 1$. Therefore $c_i = \pm 1$ and ρ_i has image contained in either an orthogonal or symplectic group depending on whether $c_i = 1$ or -1. One can therefore decompose $\rho = \rho_O \oplus \rho_S$ where the components are direct sums of the components with $c_i = 1$ and -1 respectively. By conjugating in $\mathrm{GL}_N(\mathbb{C})$ we can assume that they have image lying in the orthogonal and symplectic groups defined in chapter 1 (see [1, pg. 8]). Note that if A is an invertible $n_S \times n_S$ skew-symmetric matrix we have that

$$ \det(A) = \det(A^t) = \det(-A) = (-1)^{n_S} \det(A) $$

so n_S is even, say $n_S = 2n'_S$. After conjugating we get that

$$\mathrm{Im}(\rho) \leq \mathrm{O}_{n_O}(\mathbb{C}) \times \mathrm{Sp}_{2n'_S}(\mathbb{C})$$

The symplectic part ρ_S has image lying in the connected group $\widehat{G}_S = \mathrm{Sp}_{2n'_S}(\mathbb{C})$ which is the dual group of $G_S = \mathrm{SO}_{2n'_S+1}$. The orthogonal part is more complicated by the fact that $\mathrm{O}_{n_O}(\mathbb{C})$ is disconnected. By composing ρ_O with the projection to

$$\mathrm{O}_{n_O}(\mathbb{C})/\mathrm{SO}_{n_O}(\mathbb{C}) \cong \{\pm 1\}$$

we get a character η of $\Lambda_{\mathbb{Q}}$ of order 1 or 2. As the kernel of the map $\Lambda_{\mathbb{Q}} \to \Gamma_{\mathbb{Q}}$ is connected we can identify η as a character of $\Gamma_{\mathbb{Q}}$ of order 1 or 2 which in turn determines an extension $E/\mathbb{Q}$ of degree 1 or 2 respectively.

The orthogonal part of the representation ρ is affected by whether N is odd or even. Let us first assume that $N = 2n + 1$. Since n_S is even this means n_O is odd, say $n_O = 2n'_O + 1$. The matrix $-I_{n_O}$ represents the non-identity component of O_{n_O} and the orthogonal group is a direct product

$$\mathrm{O}_{2n'_O+1}(\mathbb{C}) = \mathrm{SO}_{2n'_O+1}(\mathbb{C}) \times \{\pm 1\}$$

We therefore set $G_O = \mathrm{Sp}_{2n'_O}$, so $\widehat{G}_O = \mathrm{SO}_{2n'_O+1}(\mathbb{C})$. The image of ρ_O is contained in

$${}^L(G_O)_{E/\mathbb{Q}} = \widehat{G}_O \times \mathrm{Gal}(E/\mathbb{Q}) \hookrightarrow \mathrm{O}_{2n'_O+1}(\mathbb{C})$$

where E is the extension determined by η.

Let us now suppose that $N = 2n$. Then n_O is even, say $n_O = 2n'_O$, and we now have a non-trivial outer automorphism of $\mathrm{SO}_{2n'_O}(\mathbb{C})$ given by conjugation by $\mathrm{O}_{2n'_O}(\mathbb{C})$. We therefore set $G_O = \mathrm{SO}_{2n'_O}(\eta)$; this group is always quasi-split but is only split if η is trivial. One has that $\widehat{G}_O = \mathrm{SO}_{2n'_O}(\mathbb{C})$ and the image of ρ_O is contained in

$${}^L(G_O)_{E/\mathbb{Q}} = \widehat{G}_O \rtimes \mathrm{Gal}(E/\mathbb{Q}) \hookrightarrow \mathrm{O}_{2n'_O}(\mathbb{C})$$

where again E is the extension determined by η. When E is degree 2 the Galois action is given by conjugation by the non-trivial element of $\mathrm{O}_{2n'_O}(\mathbb{C})/\mathrm{SO}_{2n'_O}(\mathbb{C})$.

For any N we call[3] $G = G_O \times G_S$ a θ-twisted endoscopy group for GL_N; note that by

[3]Note that the O and S refer to the fact that $\widehat{G}_O$ and $\widehat{G}_S$ are orthogonal and symplectic respectively. G_S is always orthogonal. G_O is orthogonal when N is even and symplectic when N is odd.

taking $N = 2n+1$, $n'_O = n$, and $n'_S = 0$, we see that Sp_{2n} is a θ-twisted endoscopy group of GL_{2n+1}. Similarly, SO_{2n+1} and $\mathrm{SO}_{2n}(\eta)$ are θ-twisted endoscopy groups of GL_{2n}. The representation ρ factors through

$$^{L}G_{E/\mathbb{Q}} = {}^{L}(G_O)_{E/\mathbb{Q}} \times {}^{L}(G_S)_{E/\mathbb{Q}} \hookrightarrow \mathrm{GL}_N(\mathbb{C})$$

Note this construction in fact only depends on the decomposition $n_O + n_S = N$ and the character η. It uniquely determines an embedding

$$\xi = \xi_{O,S,\eta} : {}^{L}G = \widehat{G} \rtimes \Gamma_{\mathbb{Q}} \hookrightarrow {}^{L}\mathrm{GL}_N = \mathrm{GL}_N(\mathbb{C}) \times \Gamma_{\mathbb{Q}}$$

The overlap with normal endoscopy is as follows. One can form the group

$$\widetilde{G}^{+}(N) = \mathrm{GL}_N \rtimes \langle \theta \rangle$$

From the representation ρ we can define a semi-simple element $s = J_{O,S}^{-1} \rtimes \theta$ in the dual group of $\widetilde{G}^{+}(N)$ where

$$J_{O,S} = \begin{pmatrix} & & -J_{n'_S} \\ & J_{n_O} & \\ J_{n'_S} & & \end{pmatrix}$$

The triplet (G, s, ξ) is an endoscopic datum for $\mathrm{GL}_N \rtimes \theta$. Those obtained by the above procedure are elliptic and all elliptic endoscopic data are of this form. We denote the elliptic endoscopic data of $\mathrm{GL}_N \rtimes \theta$ by $\widetilde{\mathcal{E}}_{\mathrm{ell}}(N)$ and all of its endoscopic data by $\widetilde{\mathcal{E}}(N)$ (see [1, Section 1.2]). We set $\widetilde{\mathcal{E}}_{\mathrm{sim}}(N) \subseteq \widetilde{\mathcal{E}}_{\mathrm{ell}}(N)$ to be those G such that N_O or N_S vanishes and call such G simple. Finally, note although Sp_{2n} is a θ-twisted endoscopic group of GL_{2n+1} there is a slight subtlety to this as there are two choices for the embedding ξ depending on the character η.

3.2.2 Parameters

The definition of the twisted endoscopic group associated to a representation

$$\rho : \Lambda_{\mathbb{Q}} \to \mathrm{GL}_N(\mathbb{C})$$

is clear. However it is not clear which topological group $\Lambda_{\mathbb{Q}}$ to apply it to or if the classification depends on this choice. Self-dual automorphic representations of GL_N, or automorphic representations of $\widetilde{G}^{+}(N)$, should be parametrized by L-parameters; that is certain L-homomorphisms $L_{\mathbb{Q}} \to \mathrm{GL}_N(\mathbb{C})$ which are self-dual. If we knew that $L_{\mathbb{Q}}$ existed we could apply the definitions of twisted endoscopy with $\Lambda_{\mathbb{Q}} = L_{\mathbb{Q}}$. The problem is that we don't know $L_{\mathbb{Q}}$ exists so we have to make do with something else in the global case. We briefly describe Arthur's alternative following [1, Sections 1.3 & 1.4].

We start off with defining various types of unitary automorphic representations of GL_N. We let $\mathcal{A}_{\mathrm{cusp}}(N)$ denote the unitary (irreducible) cuspidal automorphic representations of GL_N, $\mathcal{A}_2(N)$ denote the unitary discrete automorphic representations, and let $\mathcal{A}(N)$ denote all representations unitarily induced from unitary irreducible representations in the discrete spectrum of a Levi subgroup. We have the following chain of subsets;

$$\mathcal{A}_{\mathrm{cusp}}(N) \subset \mathcal{A}_2(N) \subset \mathcal{A}(N)$$

We wish to parametrise these subsets using a corresponding chain

$$\Phi_{\mathrm{sim}}(N) \subset \Psi_{\mathrm{sim}}(N) \subset \Psi(N)$$

The definitions of these are as follows. Firstly we simply set $\Phi_{\mathrm{sim}}(N) = \mathcal{A}_{\mathrm{cusp}}(N)$. We then let $\Psi_{\mathrm{sim}}(N)$ denote the set of formal tensor products $\psi = \mu \boxtimes \nu$ where $N = kn$, $\mu \in \mathcal{A}_{\mathrm{cusp}}(k)$, and ν is the unique irreducible representation of SU_2 of degree n. Moeglin and Waldspurger proved that there is a bijection between $\mathcal{A}_2(N)$ and pairs (n, μ) where $N = kn$ and $\mu \in \mathcal{A}_{\mathrm{cusp}}(k)$ (see [32, pg. 606]). This gives a bijection between $\Psi_{\mathrm{sim}}(N) \leftrightarrow \mathcal{A}_2(N)$, we define Π_ψ as being the image of ψ under this bijection. We then define $\Psi(N)$ as the set of formal unordered direct sums

$$\psi = l_1\psi_1 \boxplus \cdots \boxplus l_m\psi_m$$

for $l_i \in \mathbb{N}$ and distinct $\psi_i \in \Psi_{\mathrm{sim}}(N_i)$ where $\sum_{i=1}^m l_i N_i = N$. Given ψ we take Π_ψ to be the induced representation

$$\mathrm{Ind}_P^{\mathrm{GL}_N}(\underbrace{\Pi_{\psi_1} \otimes \cdots \otimes \Pi_{\psi_1}}_{l_1} \otimes \cdots \otimes \underbrace{\Pi_{\psi_m} \otimes \cdots \otimes \Pi_{\psi_m}}_{l_m})$$

where P is the parabolic subgroup with Levi subgroup $(\mathrm{GL}_{N_1})^{l_1} \times \cdots \times (\mathrm{GL}_{N_m})^{l_m}$. We have a bijection between $\Psi(N)$ and $\mathcal{A}(N)$ given by $\psi \mapsto \Pi_\psi$.

We need to define the notion of an isobaric sum of automorphic representations. A general irreducible automorphic representation of GL_N can be characterised as one of the irreducible constituents of an induced representation

$$\mathrm{Ind}_P^{\mathrm{GL}_N}(\pi_1 \otimes \cdots \otimes \pi_m)$$

where P is the parabolic subgroup with Levi subgroup $\mathrm{GL}_{N_1} \times \cdots \times \mathrm{GL}_{N_m}$ where $\sum_{i=1}^m N_i = N$, and the π_i are cuspidal (not necessarily unitary or distinct) representations of GL_{N_i}. We can isolate a canonical constituent $\pi = \widehat{\bigotimes}\pi_v$ where π_v is obtained from the local Langlands parameter

$$\phi_v = \phi_{1,v} \oplus \cdots \oplus \phi_{m,v}$$

and $\phi_{i,v} \leftrightarrow \pi_{i,v}$ under the local Langlands correspondence for GL_{N_i}. We write

$$\pi = \pi_1 \boxplus \cdots \boxplus \pi_m$$

and say π is an isobaric sum of the π_i. Note if they are all unitary then in fact $\pi = \mathrm{Ind}_P^{\mathrm{GL}_N}(\pi_1 \otimes \cdots \otimes \pi_m)$. It is worth noting that, due to Moeglin and Waldspurger in [32], representations in $\mathcal{A}_2(N)$ are isobaric; if one has associated parameter $\psi = \mu \boxtimes \nu$ then it has isobaric sum

$$\mu\left(\frac{n-1}{2}\right) \boxplus \mu\left(\frac{n-3}{2}\right) \boxplus \cdots \boxplus \mu\left(\frac{3-n}{2}\right) \boxplus \mu\left(\frac{1-n}{2}\right)$$

where $\mu(s) : x \mapsto \mu(x)|x|^s$ for $x \in \mathrm{GL}_k(\mathbb{A})$. Note that such an isobaric sum cannot be said to be an induced representation as the factors $\mu(s)$ are not always unitary. The representations of $\mathcal{A}(N)$ are also isobaric, we will write that

$$\Pi_\psi = l_1\Pi_{\psi_1} \boxplus \cdots \boxplus l_m\Pi_{\psi_m}$$

even though the representations Π_{ψ_k} are not necessarily cuspidal.

We want to define conjugacy class data for our automorphic representations and parameters (see [1, pgs 20-21]). Let G be an arbitrary connected reductive group over $\mathbb{Q}$ and $\pi = \widehat{\bigotimes}_v \pi_v$ be an automorphic representation of G. Recall almost all of the π_v are unramified. If π_v is unramified then $v = p$ for some finite prime p, G_p is quasi-split, and the action of the Weil group $W_{\mathbb{Q}_p}$ on $\widehat{G}$ factors through the quotient of the Weil group by the inertia group, $W_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}$. This quotient is an infinite cyclic group with canonical generator Frob_p. We have the local Langlands correspondence in this case; π_p corresponds to a parameter $\phi_p : L_{\mathbb{Q}_p} \to \widehat{G} \rtimes \langle \mathrm{Frob}_p \rangle$ and the mapping $\pi_p \mapsto c(\pi_p) = \phi_p(\mathrm{Frob}_p)$ is a bijection from the unramified representations of $G(\mathbb{Q}_p)$ to the set of semisimple $\widehat{G}$-orbits in ${}^LG_p$ that project to Frob_p in $W_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}$. Let $c_p(\pi)$ be the image of $c(\pi_p)$ under the embedding of ${}^LG_p \hookrightarrow {}^LG$ up to conjugation. Thus to π, if it's unramified outside a finite set S, we can define

$$c^S(\pi) = \{c_p(\pi) : p \notin S\}$$

We define an equivalence relation on these (note S can vary) by saying two are equivalent if they are equal at all but finitely many places; we let $c(\pi)$ denote the equivalence class of a $c^S(\pi)$. When $G = \mathrm{GL}_N$, for a $\psi \in \Psi(N)$ we define $c(\psi) = c(\Pi_\psi)$.

We are going to be interested in parameters which are self-dual with respect to the automorphism θ defined before. This maps

$$\psi \mapsto \psi^\vee = l_1(\mu_1^\vee \boxtimes \nu_1) \boxplus \cdots \boxplus l_m(\mu_m^\vee \boxtimes \nu_m)$$

(the irreducible representations of SU_2 are self-dual.) Note that $\Pi_\psi^\vee = \Pi_{\psi^\vee}$. We denote the set of self-dual parameters by $\widetilde{\Psi}(N)$. These correspond to self-dual automorphic representations of GL_N. Given a $\psi \in \widetilde{\Psi}(N)$ we let K_ψ be the set parametrising its simple components and form a partition $K_\psi = I_\psi \sqcup J_\psi \sqcup J_\psi^\vee$ where I_ψ consists of the self-dual components and J_ψ consists of orbit representatives for the remaining components. We define $\widetilde{\Psi}_{\mathrm{sim}}(N)$ as those parameters where $m = 1$ and $l_1 = 1$, and let $\widetilde{\Phi}_{\mathrm{sim}}(N)$ be the parameters in $\widetilde{\Psi}_{\mathrm{sim}}(N)$ where ν is trivial. Crucially for us there is a bijection between $\widetilde{\Phi}_{\mathrm{sim}}(N)$ and the cuspidal self-dual automorphic representations of GL_N. We have the following theorem due to Arthur.

Theorem ([1, Theorem 1.4.1])**.** *Given $\phi \in \widetilde{\Phi}_{\mathrm{sim}}(N)$ $\exists$ a unique $G_\phi = (G_\phi, s_\phi, \xi_\phi) \in \widetilde{\mathcal{E}}_{\mathrm{ell}}(N)$ s.t. for some discrete automorphic representation π of G_ϕ we have*

$$c(\phi) = \xi_\phi(c(\pi))$$

Moreover, G_ϕ is simple.

In other words, this theorem says that a cuspidal self-dual automorphic representation of GL_N comes from either a symplectic or an orthogonal group.

The previous theorem allows Arthur to define a replacement for $L_\mathbb{Q}$. Given $\psi \in \widetilde{\Psi}(N)$ we can decompose it as

$$\psi = (\boxplus_{i \in I_\psi} l_i \psi_i) \boxplus (\boxplus_{j \in J_\psi} l_j(\psi_j \boxplus \psi_{j^\vee}))$$

For $i \in I_\psi$ we apply the theorem to $\mu_i \in \widetilde{\Phi}_{\mathrm{sim}}(k_i)$ and get a datum H_i. If $j \in J_\psi$ we set $H_j = \mathrm{GL}_{k_j}$. We then form the fibre product

$$\mathcal{L}_\psi = \prod_{k \in I_\psi \cup J_\psi} ({}^L H_k \to \Gamma_\mathbb{Q})$$

For $i \in I_\psi$ we let

$$\tilde{\mu}_i : {}^L H_i \to \mathrm{GL}_{k_i}(\mathbb{C}) \times \Gamma_\mathbb{Q}$$

be the standard embedding, and for $j \in J_\psi$ define the embedding

$$\tilde{\mu}_j : {}^L H_j \to \mathrm{GL}_{2k_j}(\mathbb{C}) \times \Gamma_{\mathbb{Q}}$$

$$h_j \times \sigma \mapsto (h_j \oplus J_{k_j} h_j^{-t} J_{k_j}^{-1}) \times \sigma$$

We then define the map

$$\tilde{\psi} : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C}) \times \Gamma_{\mathbb{Q}}$$

$$\tilde{\psi} = (\bigoplus_{i \in I_\psi} l_i(\tilde{\mu}_i \otimes \nu_i)) \oplus (\bigoplus_{j \in J_\psi} l_j(\tilde{\mu}_j \otimes \nu_j))$$

Let $G \in \widetilde{\mathcal{E}}_{\mathrm{sim}}(N)$. We define the set $\widetilde{\Psi}(G)$ as those $\psi \in \widetilde{\Psi}(N)$ such that $\widetilde{\psi}$ factors through ${}^L G$, i.e. there exists an L-homomorphism

$$\widetilde{\psi}_G : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

such that $\xi \circ \widetilde{\psi}_G = \widetilde{\psi}$ where $\xi : {}^L G \hookrightarrow {}^L \mathrm{GL}_N$ is the embedding of L-groups associated to G. Of course $\widetilde{\Psi}(G)$ depends on the choice of embedding ξ so in the case of $G = \mathrm{Sp}_{2n}$ or $\mathrm{SO}_{2n}(\eta)$ there are really two such sets. Clearly by its definition $\widetilde{\Psi}(G) \subseteq \widetilde{\Psi}(N)$. We define $\widetilde{\Psi}_2(G)$ as those $\psi \in \widetilde{\Psi}(G)$ that are elliptic.

We conclude this section by describing which parameters ψ are in $\widetilde{\Psi}(\mathrm{Sp}_{2n})$. A parameter $\psi \in \widetilde{\Psi}(N)$ comes with a partition $K_\psi = I_\psi \sqcup J_\psi \sqcup J_\psi^\vee$. If $j \in J_\psi$ the component $l_j(\tilde{\psi}_j \oplus \tilde{\psi}_j^\vee)$ of $\tilde{\psi}$ has image contained in the following group;

$${}^L(\underbrace{\mathrm{GL}_{N_j} \times \cdots \times \mathrm{GL}_{N_j}}_{l_j})$$

For $i \in I_\psi$ where $l_i = 2l_i'$ the component $l_i \tilde{\psi}_i$ takes values in the group

$${}^L(\underbrace{\mathrm{GL}_{N_i} \times \cdots \times \mathrm{GL}_{N_i}}_{l_i'})$$

These indices therefore impose no constraint as the above components are both contained in the orthogonal and symplectic groups via the following maps (indeed they are both Levi subgroups of them)

$$\begin{array}{ll} \mathrm{GL}_n \hookrightarrow \mathrm{Sp}_{2n} \text{ and } \mathrm{SO}_{2n} & \qquad \mathrm{GL}_n \hookrightarrow \mathrm{SO}_{2n+1} \\ A \mapsto \begin{pmatrix} A & \\ & J_n A^{-t} J_n^{-1} \end{pmatrix} & \qquad A \mapsto \begin{pmatrix} A & & \\ & 1 & \\ & & J_n A^{-t} J_n^{-1} \end{pmatrix} \end{array}$$

We therefore turn our attention to a component $\psi_i = \mu_i \boxtimes \nu_i$ with $2 \nmid l_i$ and $i \in I_\psi$. The component $\mu_i \in \widetilde{\Phi}_{\mathrm{sim}}(k_i)$ gives rise to a datum H_i and its dual group $\widehat{H}_i$. The n_i-dimensional representation ν_i of $\mathrm{SL}_2(\mathbb{C})$ determines a group $\widehat{K}_i \subset \mathrm{GL}_{n_i}(\mathbb{C})$; it is symplectic if n_i is even and orthogonal if n_i is odd. We have the tensor product embedding

$$\widehat{H}_i \times \widehat{K}_i \to \mathrm{GL}_{N_i}(\mathbb{C})$$

which has orthogonal image if (and only if) $\widehat{H}_i$ and $\widehat{K}_i$ are of the same type. We say our ψ_i is orthogonal if this holds and require our ψ to have all ψ_i with $i \in I_\psi$ and $2 \nmid l_i$ orthogonal. This condition is precisely what we need for $\psi \in \widetilde{\Psi}(\mathrm{Sp}_{2n})$. Of course there is also the distinction over the choice of embedding. For each $\psi_i = \mu_i \boxtimes \nu_i$ with $2 \nmid l_i$ and $i \in I_\psi$ we can find the central character η_i of μ_i of order 1 or 2 which we identify with a character of $\Gamma_{\mathbb{Q}}$. The character which determines the embedding of ${}^L\mathrm{Sp}_{2n} \hookrightarrow {}^L\mathrm{GL}_{2n+1}$ is then defined by

$$\eta = \prod_{i \in I_{\psi,O}} (\eta_i)^{l_i}$$

where $I_{\psi,O}$ are the indices in I_ψ which have orthogonal ψ_i. For a $\psi \in \widetilde{\Psi}(\mathrm{Sp}_{2n})$ this is equal to the product over the characters η_i with $i \in I_{\psi,O}$ such that $2 \nmid l_i$.

Note that one can carry out the same procedure above for any $G \in \widetilde{\mathcal{E}}_{\mathrm{sim}}(N)$. One says that $\psi \in \widetilde{\Psi}(G)$ if the ψ_i with $i \in I_\psi$ and $2 \nmid l_i$ are orthogonal if $\widehat{G}$ is orthogonal[4] and symplectic if $\widehat{G}$ is symplectic. Similarly, for a $G = G_O \times G_S \in \widetilde{\mathcal{E}}_{\mathrm{ell}}(N)$ we say that $\psi \in \widetilde{\Psi}(G)$ if $\psi = \psi_O \boxplus \psi_S$ with $\psi_O \in \widetilde{\Psi}(G_O)$ and $\psi_S \in \widetilde{\Psi}(G_S)$. For more on all of this see [1, pgs. 33-35].

3.2.3 The Main Local and Global Results

We can now state the main local and global theorems of [1]. These will be crucial in stating the restricted versions of the stable trace formula we will need later in this chapter. First we will deal with the main local theorem. Since the local Langlands correspondence over $\mathbb{R}$ has already been proved for an arbitrary reductive group we will just give the results for $\mathbb{Q}_p$. Let $G \in \widetilde{\mathcal{E}}_{\mathrm{sim}}(N)$ (later we will also let it be an elliptic endoscopic group H of Sp_{2n}). Let $E/\mathbb{Q}_p$ be the smallest extension which G is split over, in our cases it is trivial unless $G = \mathrm{SO}_{2n}(\eta)$ and η is non-trivial in which case it is degree 2. We define the Langlands L-group of $\mathbb{Q}_p$ by

$$L_{\mathbb{Q}_p} = W_{\mathbb{Q}_p} \times \mathrm{SU}(2)$$

[4]The embedding is defined in a similar way to the Sp_{2n} case.

where SU(2) is the special unitary group. We have two families

$$\Phi(G) = \{\phi : L_{\mathbb{Q}_p} \to {}^L G\},\ \Pi(G) = \{\pi : G(\mathbb{Q}_p) \to \mathrm{GL}(V)\}$$

The $\phi \in \Phi(G)$ are continuous semisimple L-homomorphisms which means that the following diagram commutes (the maps to $\mathrm{Gal}(E/\mathbb{Q}_p)$ are just the normal projections);

$$\begin{tikzcd} L_{\mathbb{Q}_p} \arrow[r, "\phi"] \arrow[d] & {}^L G \arrow[dl] \\ \mathrm{Gal}(E/\mathbb{Q}_p) & \end{tikzcd}$$

and they are taken up to conjugacy in $\widehat{G}$. The set $\Pi(G)$ consists of irreducible admissible representations of $G(\mathbb{Q}_p)$ taken up to equivalence. We then define quotients

$$\widetilde{\Phi}(G) = \Phi(G)/\sim,\ \widetilde{\Pi}(G) = \Pi(G)/\sim$$

where the equivalence relations are trivial if $G = \mathrm{Sp}_{2n}$ or SO_{2n+1}. If $G = \mathrm{SO}_{2n}(\eta)$ it is defined as follows; $\widetilde{\Phi}(G)$ consists of the $\mathrm{O}_{2n}(\mathbb{C})/\mathrm{SO}_{2n}(\mathbb{C})$-orbits in $\Phi(G)$ defined by conjugation on ${}^L G$ and $\widetilde{\Pi}(G)$ consists of the $\mathrm{O}_{2n}(\mathbb{Q}_p)/\mathrm{SO}_{2n}(\mathbb{Q}_p)$-orbits in $\Pi(G)$ defined by conjugation on $G(\mathbb{Q}_p)$. We define subsets

$$\widetilde{\Phi}_{\mathrm{bdd}}(G) = \{\phi \in \widetilde{\Phi}(G) : \mathrm{Im}(\phi) \text{ is bounded}\} \subset \widetilde{\Phi}(G)$$

$$\widetilde{\Pi}_{\mathrm{temp}}(G) = \{\pi \in \widetilde{\Pi}(G) : \pi \text{ is tempered}\} \subset \widetilde{\Pi}(G)$$

It will be useful to define further intermediary subsets. We define

$$\widetilde{\Pi}_{\mathrm{unit}}(G) = \{\pi \in \widetilde{\Pi}(G) : \pi \text{ is unitary}\}$$

and one has that $\widetilde{\Pi}_{\mathrm{temp}}(G) \subset \widetilde{\Pi}_{\mathrm{unit}}(G) \subset \widetilde{\Pi}(G)$. We want to define a corresponding intermediary subset of $\widetilde{\Phi}(G)$. We define

$$\widetilde{\Psi}(G) = \{\psi : L_{\mathbb{Q}_p} \times \mathrm{SU}(2) \to {}^L G :\ \mathrm{Im}(\psi) \text{ is bounded}\}$$

Clearly we can regard $\widetilde{\Phi}_{\mathrm{bdd}}(G) \subset \widetilde{\Psi}(G)$ by setting ϕ to be trivial on the extra SU(2) factor. Moreover, for any $\psi \in \widetilde{\Psi}(G)$ we can define a parameter $\phi_\psi \in \widetilde{\Phi}(G)$ by

$$\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right)$$

where $|w|$ is the pullback to $L_{\mathbb{Q}_p}$ of the absolute value on $W_{\mathbb{Q}_p}$ and we identify ψ with its analytic extension to $L_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C})$. The boundedness condition on ψ and the fact that a homomorphism from $\mathrm{SL}_2(\mathbb{C})$ to a complex group is determined by its diagonal weight means the mapping $\psi \mapsto \phi_\psi$ is injective. We can therefore regard $\widetilde{\Psi}(G) \subset \widetilde{\Phi}(G)$. We have the following theorem due to Arthur.

Theorem ([1]). *To any* $\psi \in \widetilde{\Psi}(G)$ *one can associate a finite set* $\widetilde{\Pi}_\psi$ *along with a mapping of sets*

$$\widetilde{\Pi}_\psi \to \widetilde{\Pi}_{\text{unit}}(G)$$

and a canonical mapping

$$\pi \mapsto \langle \cdot, \pi \rangle \quad \pi \in \widetilde{\Pi}_\psi$$

from $\widetilde{\Pi}_\psi$ *to the group of characters on the finite abelian group*[5] $\mathcal{S}_\psi$ *such that* $\langle \cdot, \pi \rangle = 1$ *if* G *and* π *are unramified. If* $\psi = \phi \in \widetilde{\Phi}_{\text{bdd}}(G)$ *then the map* $\widetilde{\Pi}_\phi \to \widetilde{\Pi}_{\text{unit}}(G)$ *has trivial fibres and its image lies in* $\widetilde{\Pi}_{\text{temp}}(G)$ *so we can regard* $\widetilde{\Pi}_\phi \subset \widetilde{\Pi}_{\text{temp}}(G)$*. Moreover*

$$\widetilde{\Pi}_{\text{temp}}(G) = \bigsqcup_{\phi \in \widetilde{\Phi}_{\text{bdd}}(G)} \widetilde{\Pi}_\phi$$

We have not listed all of the properties the finite sets $\widetilde{\Pi}_\psi$, called L-packets, should satisfy as the only thing that will matter for our purposes is the existence of the finite sets. For the full description of the L-packets see [1, Theorem 1.5.1 & Theorem 2.2.1].

The theorem gives a correspondence between $\widetilde{\Phi}_{\text{bdd}}(G) \leftrightarrow \widetilde{\Pi}_{\text{temp}}(G)$. This correspondence is the local Langlands correspondence when $G = \mathrm{Sp}_{2n}$ or SO_{2n+1}. If H is an elliptic endoscopic group of Sp_{2n} recall that $H = \mathrm{Sp}_{2n_1} \times \mathrm{SO}_{2n_2}(\eta)$. We can do all of the above with H instead. The equivalence relation is conjugation by O_{2n_2} on the $\mathrm{SO}_{2n_2}(\eta)$ component. We therefore don't quite have the normal correspondence we'd want in this case, instead we have it for orbits of size 2. However, as we have an embedding ${}^LH \hookrightarrow {}^L\mathrm{Sp}_{2n}$, conjugation by O_{2n_2} becomes normal conjugacy in $\widehat{\mathrm{Sp}}_{2n}$ under this embedding. We therefore have a well-defined map

$$\widetilde{\Phi}(H) \to \Phi(\mathrm{Sp}_{2n})$$

and similar maps between the previously defined subsets.

The local theorem is crucial for properly describing the global theorem, it will be enough for our purposes to assume that $F = \mathbb{Q}$. Suppose that $\psi \in \widetilde{\Psi}(G)$ is a global parameter for $G \in \widetilde{\mathcal{E}}_{\text{sim}}(N)$ (again, later we will also let it be an elliptic endoscopic group H of Sp_{2n}). This has a localisation

$$\psi_v : L_{\mathbb{Q}_v} \times \mathrm{SU}(2) \to {}^LG_v$$

[5] See [1, Section 1.4] for the definition of the group.

for all places v (see [1, Section 1.4]). For our global parameter ψ we can define the global packet

$$\widetilde{\Pi}_\psi = \left\{ \bigotimes_v \pi_v : \pi_v \in \widetilde{\Pi}_{\psi_v} \text{ and } \langle \cdot, \pi_v \rangle = 1 \text{ for almost all } v \right\}$$

A representation in the global packet therefore defines a character[6]

$$\langle x, \pi \rangle = \prod_v \langle x_v, \pi_v \rangle \text{ for } x \in \mathcal{S}_\psi$$

We have the following theorem, essentially [1, Theorem 1.5.2].

Theorem ([1]). *There is a $\mathcal{H}(G)$-module isomorphism*

$$L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})) \cong \bigoplus_{\psi \in \widetilde{\Psi}_2(G)} \left(\bigoplus_{\pi \in \widetilde{\Pi}_\psi \ s.t. \ \langle \cdot, \pi \rangle = \epsilon_\psi} m_\psi \pi \right)$$

where

- m_ψ *equals* 1 *or* 2
- $\epsilon_\psi : \mathcal{S}_\psi \to \{\pm 1\}$ *is a linear character explicitely defined in terms of* ψ *(see [1, pgs. 47-48])*

3.3 Langlands Functoriality for Symplectic Inner Forms

In this section we introduce Arthur's stable trace formula, state how to simplify it to suit our purposes, and prove a Jacquet-Langlands type result on transfering representations from $\mathrm{U}_n(B)$ to Sp_{2n}. We will prove a slightly stronger result when B is definite, recall this is precisely when $\mathrm{U}_n(B)(\mathbb{R}) \ncong \mathrm{Sp}_{2n}(\mathbb{R})$.

3.3.1 Arthur's Stable Trace Formula

Let $G/\mathbb{Q}$ be an arbitrary reductive group, let G' be its unique quasi-split inner form, and let H be an elliptic endoscopic group of G (and G'). Let ρ be an irreducible admissible complex representation of $H(\mathbb{Q}_v)$. The fundamental lemma proved by Ngo along with a result of Waldspurger in [42] states that there is a transfer

[6] Again see [1, Section 1.4] for the definition of the group $\mathcal{S}_\psi$.

$$C_c^\infty(G(\mathbb{Q}_v)) \to C_c^\infty(H(\mathbb{Q}_v))$$
$$f \mapsto f^H$$

which satisfies a certain identity involving orbital integrals. We have no need to state this here, but refer the reader to [33] for an exposition on the subject. This also works for G' instead of G. The fundamental lemma can be applied to the case where $H = G'$. The spectral side of the trace formula for G, which is the bit we will be interested in, consists of sums of traces of various operators $\pi(f)$ where π is an automorphic representation and the operator $\pi(f)$ is defined by

$$\pi(f) = \int_{G(\mathbb{A})} f(g)\pi(g)dg$$

We will not sum over all automorphic representations though; we will restrict our attention to the discrete spectrum and also make some further restrictions in the sum. We define for $f \in C_c^\infty(G(\mathbb{A}))$

$$I^G_{\mathrm{disc},t}(f) = \sum a^G_{\mathrm{disc}}(\pi)\mathrm{tr}\pi(f)$$

where $a^G_{\mathrm{disc}}(\pi)$ is a complex number associated to each automorphic representation (for the definition see [1, Section 3.1]) and the sum is over all representations that occur in the discrete part of the trace formula which have infinitesimal character with norm of its imaginary part[7] equal to t; this properly contains the set of discrete automorphic representations with such infinitesimal characters. We have a similar expression for the quasi-split inner form G'. In order to prove a transfer from G to G' we want to compare these two formulas. The distributions are unstable (i.e. is not supported on stable orbital integrals) but we can write them in terms of stable distributions using the stable trace formula. This states that for any connected reductive group G there exist stable linear forms $S^H_{\mathrm{disc},t}$ such that

$$I^G_{\mathrm{disc},t}(f) = \sum_H \iota(G,H)S^H_{\mathrm{disc},t}(f^H) \tag{3.3.1}$$

where the sum runs over all elliptic endoscopic groups and the $\iota(G,H)$ are constants associated to each H. Again we also get a similar formula for G'. Note as endoscopic groups are quasisplit all the groups appearing on the right hand side of (3.3.1) are quasi-split. For $G = G'$ (i.e. quasi-split) one has for $f' \in C_c^\infty(G'(\mathbb{A}))$ that

$$S^{G'}_{\mathrm{disc},t}(f') = I^{G'}_{\mathrm{disc},t}(f') - \sum_{H \neq G'} \iota(G',H)S^H_{\mathrm{disc},t}(f'^H) \tag{3.3.2}$$

[7]See [1, Section 3.1] for more details on this definition.

This equation is used as inductive definition of the $S^H_{\mathrm{disc},t}$. It is worth noting that $\iota(G', H) = \iota(G, H)$ (see [23, pg. 633]), and from now on we will only write $\iota(G', H)$ in our formulas.

From now on let $G = \mathrm{U}_n(B)$ where B is a non-split rational quaternion algebra, let $G' = \mathrm{Sp}_{2n}$, and let $H = \mathrm{Sp}_{2n_1} \times \mathrm{SO}_{2n_2}(\eta)$ be an elliptic endoscopic group of G (and G'). Note that the groups G are anisotropic, i.e. they contain no non-trivial $\mathbb{Q}$-split tori, so all automorphic representations of G are cuspidal. In order to prove the transfer from G to G' we will use a refined version of the stable trace formula used by Arthur in [1, Section 3]. Recall the definition of a conjugacy class c attached to an automorphic representation of G. We define the distribution $I^G_{\mathrm{disc},t,c}$ in the same way as $I^G_{\mathrm{disc},t}$ but with the additional requirement that every automorphic representation appearing in the sum has conjugacy class c. For the definition of $S^H_{\mathrm{disc},t,c}$ see [1, Section 3.3].

Corollary 3.3.2 (b) of [1] gives the following identity;

$$I^{G'}_{\mathrm{disc},t,c}(f') = \sum_H \iota(G', H) S^H_{\mathrm{disc},t,c}(f'^H) \tag{3.3.3}$$

Similarly, applying Lemma 3.3.1 (b) of [1] to G along with equation (3.3.9) of [1] gives

$$I^{G}_{\mathrm{disc},t,c}(f) = \sum_H \iota(G', H) S^H_{\mathrm{disc},t,c}(f^H) \tag{3.3.4}$$

Both of these are vaild for any test functions f and f'. We now take f' to be a matching test function associated to f via the fundamental lemma[8]. Rearranging equations (3.3.3) and (3.3.4) to make $S^{G'}_{\mathrm{disc},t,c}(f')$ the subject and equating the two expressions gives

$$\begin{aligned} &I^{G}_{\mathrm{disc},t,c}(f) - \sum_{H \neq G'} \iota(G', H) S^H_{\mathrm{disc},t,c}(f^H) \\ &\qquad = I^{G'}_{\mathrm{disc},t,c}(f') - \sum_{H \neq G'} \iota(G', H) S^H_{\mathrm{disc},t,c}(f'^H) \end{aligned} \tag{3.3.5}$$

This is the form of the trace formula which we will use to prove our transfer.

[8]If $f = \otimes_v f_v$ then $f' = \otimes_v f'_v$ where f'_v is a test function associated to f_v by the fundamental lemma; when $G(\mathbb{Q}_v) \cong G'(\mathbb{Q}_v)$ then we can take $f_v = f'_v$

3.3.2 The Jacquet-Langlands transfer

Suppose π_0 is an automorphic[9] representation of G with associated pair (t,c). Therefore $I^G_{\mathrm{disc},t,c}$ is not identically zero so at least one of the other terms in equation (3.3.5) must not be identically zero. Proposition 3.4.1 of [1] therefore implies that $(t,c) = (t(\psi), c(\psi))$ for some $\psi \in \widetilde{\Psi}(2n+1)$. Equation (3.3.5) can be rewritten as

$$I^G_{\mathrm{disc},t(\psi),c(\psi)}(f) - \sum_{H \neq G'} \iota(G',H) S^H_{\mathrm{disc},\psi}(f^H)$$
$$= I^{G'}_{\mathrm{disc},\psi}(f') - \sum_{H\neq G'} \iota(G',H) S^H_{\mathrm{disc},\psi}(f'^H) \quad (3.3.6)$$

in the notation of [1, Section 3.3]. Moreover, since both sides of this equation equal $S^{G'}_{\mathrm{disc},\psi}(f')$, at least one of $S^H_{\mathrm{disc},\psi}(f^H)$ or $S^H_{\mathrm{disc},\psi}(f'^H)$ (here H could equal G') is non-zero. Theorem 4.1.2 of [1] tells us that $\psi \in \widetilde{\Psi}(G')$. If we knew that $\psi \in \widetilde{\Psi}_{\mathrm{sim}}(G')$, so $\psi \notin \widetilde{\Psi}(H)$ for all $H \neq G'$, then the proper endoscopic terms in (3.3.6) would vanish by [1, Theorem 4.1.2]. We will adopt a different approach.

We let S be the finite even set of places where B is not split, note $G(\mathbb{Q}_v) \cong G'(\mathbb{Q}_v)$ for all $v \notin S$. We assume that at some finite place $q \notin S$ $\pi_{0,q}$ is the Steinberg representation. We write

$$\pi_0 = \pi_{0,S} \otimes \pi_{0,q} \otimes \tau_0^{S_q} = \pi_{0,S} \otimes \tau_0^S$$

where $S_q = S \cup \{q\}$. Set f_q to be a pseudo-coefficient of the Steinberg representation; as $q \notin S$ note that $f'_q = f_q$. As f_q is stabilizing the proper endoscopic contribution to the trace formula vanishes (see [27]). One obtains

$$I^G_{\mathrm{disc},t(\psi),c(\psi)}(f_S f_q f^{S_q}) = I^{G'}_{\mathrm{disc},\psi}(f'_S f_q f^{S_q}) \quad (3.3.7)$$

For an automorphic form π of G, we define

$$a^G_{\mathrm{disc},\psi}(\pi) = \begin{cases} a^G_{\mathrm{disc}}(\pi) & \text{if } \pi \text{ is associated to the pair } (t(\psi), c(\psi)) \\ 0 & \text{otherwise} \end{cases}$$

and define $a^{G'}_{\mathrm{disc},\psi}(\pi')$ in the same way[10] for π' in the discrete spectrum of G'. Rewriting (3.3.7) we get

$$\sum_{\pi_S,\pi_q,\tau^{S_q}} \left\{ a^G_{\mathrm{disc},\psi}(\pi_S \otimes \pi_q \otimes \tau^{S_q}) \mathrm{tr}\pi_S(f_S) \mathrm{tr}\pi_q(f_q) \right\} \mathrm{tr}\tau^{S_q}(f^{S_q})$$
$$= \sum_{\pi'_S,\pi_q,\tau^{S_q}} \left\{ a^{G'}_{\mathrm{disc},\psi}(\pi'_S \otimes \pi_q \otimes \tau^{S_q}) \mathrm{tr}\pi'_S(f'_S) \mathrm{tr}\pi_q(f_q) \right\} \mathrm{tr}\tau^{S_q}(f^{S_q})$$

[9]Note since G is anisotropic over $\mathbb{Q}$ all its automorphic representations are cuspidal.

[10]We make these definitions to simplify the notation in the equations that follow.

Here the sum on the left is over all π_S, π_q, τ^{S_q} such that $\pi_S \otimes \pi_q \otimes \tau^{S_q}$ is in the discrete automorphic spectrum and the sum on the right is defined similarly. Note in the above we have picked f_q, but f_S and f^{S_q} are arbitrary. Using linear independence of characters for $G(\mathbb{A}^{S_q})$ we get for any fixed τ^{S_q} that

$$\sum_{\pi_S,\pi_q} a^G_{\mathrm{disc},\psi}(\pi_S \otimes \pi_q \otimes \tau^{S_q})\mathrm{tr}\pi_S(f_S)\mathrm{tr}\pi_q(f_q)$$
$$= \sum_{\pi'_S,\pi_q} a^{G'}_{\mathrm{disc},\psi}(\pi'_S \otimes \pi_q \otimes \tau^{S_q})\mathrm{tr}\pi'_S(f'_S)\mathrm{tr}\pi_q(f_q) \quad (3.3.8)$$

Note that we are still free to choose f_S.

We now use an argument appearing in [28, pgs. 5-6], see [24, pg. 637] for the neccessary results. Since f_q is the pseudo-coefficient of the Steinberg representation one has that $\mathrm{tr}\pi_q(f_q) = 0$ for all unitary π_q unless π_q is Steinberg or the trivial representation; note all the representations appearing in Arthur's discrete part of the trace formula are unitary. For any place $v \notin S$, $G(\mathbb{Q}_v) \cong \mathrm{Sp}_{2n}(\mathbb{Q}_v)$ so is non-compact. If $v = p$ is a finite prime then $G(\mathbb{Q})G(\mathbb{Q}_p)$ is dense in $G(\mathbb{A})$ so an automorphic representation that is trivial at p must be trivial, the same thing is true for $G' = \mathrm{Sp}_{2n}$. Recall our (cuspidal) automorphic representation π_0 is not trivial as it is Steinberg at q. Therefore $\pi_{0,p}$ must be non-trivial for all $p \notin S$. Applying equation (3.3.8) to the fixed representation $\tau_0^{S_q}$ (which is non-trivial at all finite places) we see that none of the π_q appearing in the equation can be trivial. Therefore the only non-vanishing traces at q occur when π_q is the Steinberg representation. We therefore have

$$\sum_{\pi_S} a^G_{\mathrm{disc},\psi}(\pi_S \otimes \pi_{0,q} \otimes \tau_0^{S_q})\mathrm{tr}\pi_S(f_S) = \sum_{\pi'_S} a^{G'}_{\mathrm{disc},\psi}(\pi'_S \otimes \pi_{0,q} \otimes \tau_0^{S_q})\mathrm{tr}\pi'_S(f'_S) \quad (3.3.9)$$

As the left hand side of (3.3.9) is not identically zero (since we started with π_0 to begin with) we can find an f_S such that the left hand side doesn't vanish. This will transfer to a function f'_S such that the right hand side doesn't vanish. We therefore have for some $\pi' = \pi'_S \otimes \pi_{0,q} \otimes \tau_0^{S_q}$ that

$$a^{G'}_{\mathrm{disc},\psi}(\pi'_S \otimes \pi_{0,q} \otimes \tau^{S_q})\mathrm{tr}\pi'_S(f'_S) \neq 0$$

so there is an automorphic representation π' in the discrete spectrum of G' such that $\pi_{0,v} \cong \pi'_v$ for all $v \notin S$. By [16, Proposition 4.10], a representation in the discrete spectrum which is tempered at some finite place must be cuspidal (note the result over $\mathbb{R}$ is proved in [43]). As π' is Steinberg at q, and the Steinberg representation is tempered, π' must be cuspidal. To summarise we have proved the following.

Theorem 3.3.1. *Let $G = \mathrm{U}_n(B)$ for B a rational quaternion algebra, let $G' = \mathrm{Sp}_{2n}$, and let S be the set of places where B is not split. Let π_0 be a (cuspidal) automorphic representation of G which is Steinberg at some finite place outside S. Then there exists a cuspidal automorphic representation π' of G' such that $\pi_{0,v} \cong \pi'_v$ for all $v \notin S$.*

The above result holds for any B but we will need a stronger result in the definite case. In order to attach Galois representations we will need to ensure that π' has the same infinitesimal character as π_0. When B is indefinite $\infty \notin S$ so π' has the same infinitesimal character as π_0. When B is definite, however, we cannot conclude this from the above. In order to ensure this we will have to consider the local Langlands correspondence at the infinite place.

3.3.3 The Local Langlands Correspondence over $\mathbb{R}$

In order to attach Galois representations later on, we will need π_0 to be regular algebraic. The definition of this is as follows. In [10], Buzzard and Gee note there are two different definitions of an automorphic representation being algebraic; L- and C-algebraic. These notions coincide precisely when half the sum of the positive roots lies in X, part of the root datum (see section 1.3.2). This is satisfied for inner forms of Sp_{2n} (including Sp_{2n}) so we can label automorphic representations of these groups as algebraic without any ambiguity. In this case an automorphic representation is algebraic if its infinitesimal character has integer coefficients (i.e. it's integral), it is regular algebraic if its infinitesimal character has distinct integer coefficients (the distinct coefficients mean the character is regular).

We will now assume that $G(\mathbb{R}) \ncong \mathrm{Sp}_{2n}(\mathbb{R})$ so $\infty \in S$, i.e. $G = \mathrm{U}_n(B)$ where B is definite. In order to control the infinitesimal character of π' we need to consider the local Langlands correspondence at infinity in more detail. We will therefore consider the Weil group of $\mathbb{R}$. Recall it is defined as $W_{\mathbb{R}} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$ where

$$jzj^{-1} = \bar{z} \ \forall z \in \mathbb{C}^\times \text{ and } j^2 = -1$$

The results of Langlands in [30] show that irreducible admissible representations of $G'(\mathbb{R})$ (for an arbitrary G') are partitioned into finite L-packets Π_ϕ parametrised by admissible homomorphisms

$$\phi : W_{\mathbb{R}} \to {}^L G'$$

Now for our G, $G(\mathbb{R})$ is compact so its L-packets are singletons $\{\pi_\phi\}$ and the finite-dimensional irreducible representations π_ϕ are parametrised by discrete L-parameters

ϕ. If $G' = \mathrm{Sp}_{2n}$ the local Langlands correspondence gives maps between $\{\pi_\phi\} \leftrightarrow \Pi_\phi$ (a correspondence between L-packets of G and G') which preserves infinitesimal characters.

We wish to describe the discrete parameters $\phi : W_{\mathbb{R}} \to \mathrm{SO}_{2n+1}(\mathbb{C})$ which describe all the irreducible admissible representations of $G(\mathbb{R})$ (all of which are discrete). By following [5, Example 10.5], we see that for $z \in \mathbb{C}$ we have (up to conjugacy) that

$$\phi(z) = \begin{pmatrix} (z\bar{z}^{-1})^{a_1} & & & & & & \\ & \ddots & & & & & \\ & & (z\bar{z}^{-1})^{a_n} & & & & \\ & & & 1 & & & \\ & & & & (z\bar{z}^{-1})^{-a_n} & & \\ & & & & & \ddots & \\ & & & & & & (z\bar{z}^{-1})^{-a_1} \end{pmatrix}$$

where $\nu = (a_1, \ldots, a_n)$ is the infinitesimal character of ϕ (see [1, pg. 69]). As ϕ is discrete this satisfies $a_i \in \mathbb{Z}$ and $a_1 > \cdots > a_n > 0$, i.e. ν is integral[11] and regular. In particular, this means that any irreducible admissible representation of $G(\mathbb{R})$ has regular integral infinitesimal character and any (discrete) automorphic representation of G is regular algebraic.

We have proved a Jacquet-Langlands type transfer from $\mathrm{U}_n(B)$ to Sp_{2n}. We would like to prove additionally that if π_0 is the automorphic representation of G then π'_∞ is a local Langlands transfer of $\pi_{0,\infty}$; i.e. that they have the same associated L-parameter. To do this we will recall some facts about certain types of test functions.

For the rest of this subsection only, let G be an arbitrary connected semisimple group over $\mathbb{Q}$ with quasi-split inner form G'. Recall for a discrete representation τ of $G(\mathbb{R})$ there exists a pseudo-coefficient f_τ; this satisfies the property that for all tempered representations σ of $G(\mathbb{R})$

$$\mathrm{tr}\sigma(f_\tau) = \begin{cases} 1 & \text{if } \sigma \cong \tau \\ 0 & \text{o/w} \end{cases}$$

Moreover, as an immediate consequence of [9, Proposition 4.13], for any irreducible admissible representation σ of $G(\mathbb{R})$ one has $\mathrm{tr}\sigma(f_\tau) = 0$ unless σ has the same infinitesimal character as τ (this is also discussed in [26]). The problem is that there may be some non-tempered representations of $G(\mathbb{R})$ with the same infinitesimal

[11] See [5, pg. 46].

character as some discrete representations.

We need to know how pseudo-coefficients transfer under the fundamental lemma to G'. Due to [36, Theorem 5.1], one finds that $f_\tau \mapsto f'$ where

$$f' = \sum_{\tau'} c_\tau f_{\tau'}$$

where the τ' are the discrete representations of $G'(\mathbb{R})$ in the L-packet associated to τ; c_τ is a non-zero constant depending only on τ. Consequently, a pseudo-coefficient of $\mathrm{U}_n(B)$ for B definite transfers to a linear combination of pseudo-coefficients.

3.3.4 The Definite Case

We can now state our enhanced theorem for $G = \mathrm{U}_n(B)$ where B is a definite rational quaternion algebra.

Theorem 3.3.2. *Let $G = \mathrm{U}_n(B)$ for B a definite rational quaternion algebra, let $G' = \mathrm{Sp}_{2n}$, and let S be the set of places where B is not split. Let π_0 be a (cuspidal) automorphic representation of G which is Steinberg at some place outside S. Then there exists a regular algebraic cuspidal automorphic representation π' of G' such that $\pi_{0,v} \cong \pi'_v$ for all $v \notin S$. Moreover, π'_∞ is in the L-packet associated to $\pi_{0,\infty}$ by the local Langlands correspondence over $\mathbb{R}$.*

Proof. Recall we have the following equation from (3.3.9), where we let $S = V \cup \{\infty\}$ (where V is a set of primes of odd cardinality)

$$\sum_{\pi_\infty, \pi_V} a^G_{\mathrm{disc},\psi}(\pi_\infty \otimes \pi_V \otimes \pi_{0,q} \otimes \tau^{S_q}) \mathrm{tr}\pi_\infty(f_\infty) \mathrm{tr}\pi_V(f_V)$$
$$= \sum_{\pi'_\infty, \pi'_V} a^{G'}_{\mathrm{disc},\psi}(\pi'_\infty \otimes \pi'_V \otimes \pi_{0,q} \otimes \tau^{S_q}) \mathrm{tr}\pi'_\infty(f'_\infty) \mathrm{tr}\pi'_V(f'_V) \quad (3.3.10)$$

Note as each representation is Steinberg at q they are all cuspidal. We now choose f_∞ to be a pseudo-coefficient for the representation $\pi_{0,\infty}$. Since all the $G(\mathbb{R})$ representations are discrete and its L-packet are singletons, one has

$$\sum_{\pi_V} a^G_{\mathrm{disc},\psi}(\pi_{0,\infty} \otimes \pi_V \otimes \pi_{0,q} \otimes \tau^{S_q}) \mathrm{tr}\pi_V(f_V)$$
$$= \sum_{\pi'_\infty} \sum_{\pi'_V} a^{G'}_{\mathrm{disc},\psi}(\pi'_\infty \otimes \pi'_V \otimes \pi_{0,q} \otimes \tau^{S_q}) \mathrm{tr}\pi'_\infty(f'_\infty) \mathrm{tr}\pi'_V(f'_V) \quad (3.3.11)$$

Since [36, Theorem 5.1] tells us that f'_∞ is a linear combination of pseudo-coefficients for the discrete representations of $G'(\mathbb{R})$ in the L-packet associated to $\pi_{0,\infty}$ via the local Langlands correspondence, and these all have the same infinitesimal character as $\pi_{0,\infty}$, all π'_∞ appearing in the right hand side of (3.3.11) with $\mathrm{tr}\pi'_\infty(f'_\infty) \neq 0$ have the same regular integral infinitesimal character as $\pi_{0,\infty}$. As the left hand side is non-zero by assumption we can pick an f_V such that it doesn't vanish, we therefore find some cuspidal automorphic representation $\pi' = \pi'_\infty \otimes \pi'_V \otimes \pi_{0,q} \otimes \tau^{S_q}$ of G' such that $\pi_{0,p} \cong \pi'_p$ for all $p \notin S$ and, moreover, π'_∞ has the same infinitesimal character as $\pi_{0,\infty}$.

To finish the proof we will again use the work of Arthur. By [1, Theorem 1.5.2] one can attach to π' a self-dual representation Π in the discrete spectrum of GL_{2n+1}. This global correspondence is compatible with the local correspondences stated in [1, Theorem 1.5.1 & Theorem 2.2.1], i.e. the local L-parameters of π' and Π are the same when identified by the embedding $\mathrm{SO}_{2n+1} \hookrightarrow \mathrm{GL}_{2n+1}$. The representation π' is Steinberg at q, so its L-parameter is tempered. Hence Π is tempered at q (in fact it will be Steinberg), so Π is cuspidal by [16, Proposition 4.10]. But Π has the same infinitesimal character as π' and π_0, so is regular algebraic. In summary, Π is a regular algebraic self-dual cuspidal representation of GL_{2n+1} so is tempered at infinity by [15, Section 3.2]. The local-global compatibility of Arthur's work shows that π'_∞ is tempered. As $\mathrm{tr}\pi'_\infty(f'_\infty) \neq 0$ and f'_∞ is a linear combination of pseudo-coefficients for the discrete representations of $G'(\mathbb{R})$ in the L-packet associated to $\pi_{0,\infty}$ via the local Langlands correspondence, π'_∞ must lie in this L-packet.

□

We remarked in this proof that the representation Π of GL_{2n+1} is actually Steinberg at q. This is not critical to our argument, but we provide two explanations of why this is true. Let G be a connected reductive group over $\mathbb{Q}_q$. The L-parameter of the Steinberg representation is the extension of the 'principal homomorphism[12]' which is trivial on $W_{\mathbb{Q}_q}$ (see [20], this is 'well-known'). This principal parameter satisfies a number of properties, one of which is that its image is contained in a unique Borel subgroup of $\widehat{G}$. In the case of GL_{2n+1} it is simply the $(2n+1)$-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ which is orthogonal, hence the Steinberg representation of Sp_{2n} over $\mathbb{Q}_q$ transfers to the Steinberg representation of GL_{2n+1} over $\mathbb{Q}_q$. The local-global

[12] A homomorphism $\mathrm{SL}_2 \to \widehat{G}$.

compatibility of Arthur's work then tells us that Π is Steinberg at q.

Indeed one can instead see that Steinberg transfers to Steinberg from Arthur's work directly. Let St_N denote the Steinberg representation of GL_N over $\mathbb{Q}_q$ where $N = 2n+1$. The Arthur parameter associated to it is trivial on the Weil group and on the $\mathrm{SU}(2)$ component is the maximal representation into GL_N (see [1, pg. 16] for the construction of Arthur's local parameters from the Weil-Deligne version). This is orthogonal, so defines a parameter ϕ for Sp_{2n}. The group $\mathcal{S}_\phi$ Arthur associates to this is clearly trivial so ϕ defines a unique tempered representation π of Sp_{2n} over $\mathbb{Q}_q$. Now [1, Theorem 2.2.1] gives twisted character relations between Sp_{2n} and GL_N; it says that to associated[13] test functions f and f_N one has that

$$\mathrm{tr}\, \pi(f) = \widetilde{\mathrm{tr}}\, \mathrm{St}_N(f_N)$$

where the tilde denotes the twisted trace (see [1, Chapter 2] for precise definitions of all of these things). Crucially, the pseudo-coefficients of the Steinberg representation for GL_N and Sp_{2n} are associated to each other. This follows as functions f and f_N being associated to each other means stable orbital integrals of f equal twisted stable orbital integrals of f_N, and these orbital integrals are computed in [13, Proposition 3.8 & Corollary 3.10] (the non-twisted case follows from [24]). It follows that $\mathrm{tr}\pi(f) \neq 0$ as the twisted trace of the pseudo-coefficient f_N on GL_N is non-zero (again see [13, Proposition 3.8]), and therefore π is Steinberg since f traces in no other tempered representation.

3.4 Getting Galois Representations

3.4.1 Galois Representations for Sp_{2n}

We now describe the twisted endoscopic transfer from Sp_{2n} to GL_{2n+1}. We have the following theorem; as stated it is essentially theorem 5.1.2 in [35] but the result is due to Arthur in [1].

Theorem ([1])**.** *Let π be regular algebraic cuspidal automorphic representation of Sp_{2n}. Then $\exists$ cuspidal self-dual automorphic representations $\Pi_1, \dots, \Pi_m$ of $\mathrm{GL}_{n_1}, \dots, \mathrm{GL}_{n_m}$ and positive integers k_i s.t. $\sum_{i=1}^m k_i n_i = 2n+1$, π has associated parameter $\psi = \psi_1 \boxplus \cdots \boxplus \psi_m \in \widetilde{\Psi}(2n+1)$ with $J_\psi = \emptyset$, and $\psi_i = \Pi_i \boxtimes \nu_i$ where ν_i has degree k_i. Moreover, $\Pi_i|\cdot|^{(1-k_i)/2}$ is regular L-algebraic (in the sense of Buzzard and Gee in [10]).*

[13] In a way analgous to the fundamental lemma but this time for twisted endoscopy.

We now attach Galois representations to π (see [35, Corollary 5.1.7]). Given the Π_i described above to each regular L-algebraic cuspidal automorphic representation $\Pi_i|\cdot|^{(1-k_i)/2}$ we can attach continuous semisimple Galois representations $\rho_i : \Gamma_{\mathbb{Q}} \to \mathrm{GL}_{n_i}(\overline{\mathbb{Q}}_l)$ using results in [14],[22], and [37], see [2, Theorems 1.1 & 1.2] for the precise result needed. We then define a Galois representation from $\Gamma_{\mathbb{Q}}$ to $\mathrm{GL}_{2n+1}(\overline{\mathbb{Q}}_l)$ by

$$\rho = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} \rho_i \chi_l^{1-j}$$

where χ_l is the l-adic cyclotomic character.

We want to analyse the possible image of this Galois representation. Arthur's theorem says $\psi = \psi_1 \boxplus \cdots \boxplus \psi_m \in \widetilde{\Psi}(\mathrm{Sp}_{2n})$ and it is elliptic. Recall that for such a ψ to be in $\widetilde{\Psi}(\mathrm{Sp}_{2n})$ we must have that each ψ_i has orthogonal image. Consider the Galois representation

$$r_i = \bigoplus_{j=1}^{k_i} \rho_i \chi_l^{1-j}$$

so $\rho = \bigoplus_{i=1}^{m} r_i$. Note that

$$(\rho_i \chi_l^{1-j})^\vee \cong \rho_i \chi_l^{1-k_i} \chi_l^{j-1} = \rho_i \chi_l^{j-k_i}$$

We can then pair up the representations in the form $\rho_i \chi_l^{1-j} \oplus \rho_i \chi_l^{j-k_i}$. Every representation can be paired up like this except for $\rho_i \chi_l^{(1-k_i)/2}$ if k_i is odd. The image of each pair can be embedded in GL_{n_i} which in turn is embedded in a Levi subgroup of both the symplectic and special orthogonal groups. Therefore the only possible obstacle to our ρ being orthogonal is the factor $\rho_i \chi_l^{(1-k_i)/2}$ left over for odd k_i, this being the Galois representation associated to Π_i. Suppose k_i is odd and define $\sigma_i := \rho_i \chi_l^{(1-k_i)/2}$. As $\sigma_i^\vee \cong \sigma_i$ it is orthogonal as an immediate consequence of [4, Corollary 1.3]. As ρ is odd dimensional, we can always project to the special orthogonal group if neccessary. We therefore have the following proposition from the discussion above along with the local-global compatibility of Arthur's parameters, see [1, Section 1.4].

Proposition 3.4.1. *Let l be a finite prime and $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ be an isomorphism. Let π be a regular algebraic cuspidal automorphic representation of Sp_{2n}. Then there exists a Galois representation*

$$\rho : \Gamma_{\mathbb{Q}} \to \mathrm{SO}_{2n+1}(\overline{\mathbb{Q}}_l)$$

s.t. at all finite places $p \neq l$ where π is unramified we have that

$$\phi_{\pi_p} = \iota_l(\rho_\pi|_{W_{\mathbb{Q}_p}})$$

where ϕ_{π_p} is the Langlands parameter of π_p, the local component of π at p.

3.4.2 Combining the results

We would like to attach Galois representations to (cuspidal) automorphic representations π of $G = \mathrm{U}_n(B)$ where B is a rational quaternion algebra. The obvious strategy is to transfer to a representation π' of Sp_{2n} and then apply proposition 3.4.1 to get a Galois representation. If B is split at infinity then $G(\mathbb{R}) \cong \mathrm{Sp}_{2n}(\mathbb{R})$. Therefore, if we assume that the automorphic representation π of G is regular algebraic it will transfer to a suitable representation π' of Sp_{2n} by theorem 3.3.1. However, if B is definite then $G(\mathbb{R}) \ncong \mathrm{Sp}_{2n}(\mathbb{R})$ so we need to control what the representation at infinity transfers to. By applying theorem 3.3.2 to π we get a regular algebraic π' which we can again apply proposition 3.4.1 to. All of this means we have the following theorem.

Theorem 3.4.2. *Let l be a finite prime and $\iota_l : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ be an isomorphism. Let $G = \mathrm{U}_n(B)$ where B is a rational quaternion algebra, and let S be the set of places where $G(\mathbb{Q}_v) \ncong \mathrm{Sp}_{2n}(\mathbb{Q}_v)$. Let π be a (cuspidal) automorphic representation of G which is Steinberg at some finite place outside S. If B is indefinite assume further that π is regular algebraic. Then there exists a Galois representation*

$$\rho_\pi : \Gamma_{\mathbb{Q}} \to \mathrm{SO}_{2n+1}(\overline{\mathbb{Q}}_l)$$

s.t. at all finite places $p \notin S \cup \{l\}$ where π is unramified we have that

$$\phi_{\pi_p} = \iota_l(\rho_\pi|_{W_{\mathbb{Q}_p}})$$

where ϕ_{π_p} is the Langlands parameter of π_p, the local component of π at p.

It is worth ending with some remarks concerning the Galois representations we obtain. The construction first transfers $\pi \mapsto \pi'$, then transfers $\pi' \mapsto \Pi$, and finally attaches a Galois representation to Π. As noted in the proof of theorem 3.3.2, Π is in fact cuspidal since it is tempered at q (in fact it's Steinberg at q). Therefore we do not need the more complicated construction of section 3.4.1, the Galois representation attached to π is simply the Galois representation attached to the regular algebraic self-dual cuspidal representation Π of GL_{2n+1}. Moreover, one does not need the work of Bellaïche and Chenevier to conclude these Galois representations are orthogonal. As remarked in their paper ([4, pg. 4]), the Galois representation is known to be irreducible if Π is square-integrable at some finite place by the works of Harris-Taylor ([22]) and Taylor-Yoshida ([40]). Our Π is Steinberg at q, so the Galois representation is irreducible and odd-dimensional, hence orthogonal.

Appendix A

Twisting Elements and ρ

A.1 Introduction

In Gross' original conjecture, twisting elements play an important role. Recall they are defined as follows.

Definition. *Let G be a reductive group over $\mathbb{Q}$ with based root datum $(X, \Delta, X^\vee, \Delta^\vee)$. We say that G has a twisting element if there exists an element $\eta \in X$ which is Galois invariant and*

$$\langle \eta, \alpha^\vee \rangle = 1 \;\; \forall \alpha^\vee \in \Delta^\vee$$

This property is always satisfied by half the sum of positive roots ρ but this is not always in X. However, for the root data of type B_n, C_n, and D_n, half the sum of positive roots is in X if and only if they have a twisting element. This phenomenon continues for the exceptional root data. It turns out that the following occurs.

Proposition A.1.1. *All the classical based root data have half the sum of positive roots in X except for B_n, A_n for n even, and E_7. Moreover, B_n and E_7 cannot possess a twisting element.*

This means the only possible ambiguity about whether to pick $\eta = \rho$ is for A_n when n is even. We now go through the list of classical root data and calculate whether they can possess a twisting element.

A.2 The Infinite Families of Based Root Data

The calculations for the infinite families of based root data have been done elsewhere in this thesis. We give them again here for completeness.

A.2.1 Type A_n

For the root datum of type A_n, X and $X^\vee$ are n-dimensional and are spanned by elements χ_i and λ_j respectively. One gets the following for a set of simple roots and associated simple coroots

$$\Delta = \{\chi_i - \chi_{i+1} : 1 \leq i < n\}, \ \ \Delta^\vee = \{\lambda_j - \lambda_{j+1} : 1 \leq j < n\}$$

One finds that Φ^+ consists of roots of the form $\chi_i - \chi_j$ where $i < j$. Therefore one sees that

$$2\rho = (n-1)\chi_1 + (n-3)\chi_2 + \cdots (1-n)\chi_n = \sum_{i=1}^{n}(n-(2i-1))\chi_i$$

Consequently $\rho \in X$ iff n is odd. This does not mean however that a twisting element cannot exist. Indeed one can take for a twisting element

$$\eta_0 = (n-1)\chi_1 + (n-2)\chi_2 + \cdots + \chi_{n-1} = \sum_{i=1}^{n}(n-i)\chi_i$$

Any twisting element is of the form $\eta_0 + m(\chi_1 + \cdots + \chi_n)$ but this is not necessarily Galois invariant. One sees that

$$\eta_0 = \rho + \left(\frac{n-1}{2}\right)(\chi_1 + \cdots + \chi_n)$$

We have already seen in section 2.1.2 that if n is even GL_n possesses forms, namely unitary groups of dimension n, which do not possess a twisting element.

A.2.2 Type B_n

For the root datum of type B_n, X and $X^\vee$ are n-dimensional and are spanned by elements e_i and f_j respectively. One gets the following for a set of simple roots and associated simple coroots

$$\Delta = \{e_i - e_{i+1} : 1 \leq i < n\} \cup \{e_n\}, \ \ \Delta^\vee = \{f_j - f_{j+1} : 1 \leq j < n\} \cup \{2f_n\}$$

Because of the element $2f_n \in \Delta^\vee$ we see that $\langle x, 2f_n\rangle \in 2\mathbb{Z}$ for all $x \in X$. It is therefore impossible for any twisting element to exist, irrespective of what the Galois action is.

A.2.3 Type C_n

For the root datum of type C_n, the dual datum to B_n, X and $X^\vee$ are n-dimensional and are spanned by elements e_i and f_j respectively. One gets the following for a set of simple roots and associated simple coroots

$$\Delta = \{e_i - e_{i+1} : 1 \leq i < n\} \cup \{2e_n\}, \ \ \Delta^\vee = \{f_j - f_{j+1} : 1 \leq j < n\} \cup \{f_n\}$$

One finds that

$$\Phi^+ = \{(e_i - e_j), (e_i + e_j) : 1 \leq i < j \leq n\} \cup \{2e_i : 1 \leq i \leq n\}$$

and therefore

$$\rho = ne_1 + (n-1)e_2 + \cdots + 2e_{n-1} + e_n = \sum_{i=1}^{n} (n+1-i)e_i \in X$$

A.2.4 Type D_n

For the root datum of type D_n, X and $X^\vee$ are n-dimensional and are spanned by elements e_i and f_j respectively. One gets the following for a set of simple roots and associated simple coroots

$$\Delta = \{e_i - e_{i+1} : 1 \leq i < n\} \cup \{e_{n-1} + e_n\}, \ \ \Delta^\vee = \{f_j - f_{j+1} : 1 \leq j < n\} \cup \{f_{n-1} + f_n\}$$

One finds that

$$\Phi^+ = \{(e_i - e_j), (e_i + e_j) : 1 \leq i < j \leq n\}$$

and therefore

$$\rho = (n-1)e_1 + (n-2)e_2 + \cdots + e_{n-1} = \sum_{i=1}^{n} (n-i)e_i \in X$$

A.3 The Exceptional Root Data

A.3.1 Type E_8

For E_8, X is the set of vectors spanned by $\{e_i : 1 \leq i \leq 8\}$ where either all the coordinates are in $\mathbb{Z}$ or all the coordinates are in $\mathbb{Z} + 1/2$. One has

$$\Phi = \{\pm e_i \pm e_j : 1 \leq i < j \leq 8\} \cup \left\{ \frac{1}{2} \sum_{i=1}^{8} (-1)^{a_i} e_i : \sum_{i=1}^{8} a_i \equiv 0 \mod 2 \right\}$$

One can take $\Delta = \{\beta_1, \ldots, \beta_8\}$ where

$$\beta_1 = \frac{1}{2}(-e_1 - e_8 + \sum_{i=2}^{7} e_i), \quad \beta_2 = -e_1 - e_2$$

$$\beta_i = e_{i-2} - e_{i-1} \text{ for } 3 \leq i \leq 8$$

For the positive roots one gets

$$\Phi^+ = \{\pm e_i - e_j : 1 \leq i < j \leq 8\} \cup \left\{ \frac{1}{2}(-e_8 + \sum_{i=1}^{7}(-1)^{a_i} e_i) : \sum_{i=1}^{8} a_i \equiv 1 \mod 2 \right\}$$

Consequently one find that

$$2\rho = -2(7e_8 + 6e_7 + \cdots e_2) - \frac{1}{2}(2^5(e_1 + e_8) + 2^5(e_8 - e_1))$$

$$\implies \rho = -(23e_8 + 6e_7 + 5e_6 + \cdots + e_2) \in X$$

A.3.2 Type E_7

To get a base for E_7 one deletes β_8 from the base for E_8 obtaining $\Delta = \{\beta_1, \cdots, \beta_7\}$. As all the roots are of length two with respect to the natural inner product, it is clear that E_7 (and also E_8) is self-dual. Suppose there exists a twisting element

$$\eta = \sum_{i=1}^{8} m_i e_i$$

By taking its inner product with $\beta_i^\vee$ for $3 \leq i \leq 7$ one sees that

$$m_1 - m_2 = m_2 - m_3 = \cdots = m_5 - m_6 = 1$$

Taking the inner product with $\beta_2^\vee$ we get that $-m_1 - m_2 = 1$ which implies that

$$(m_1, m_2, m_3, m_4, m_5, m_6) = (0, -1, -2, -3, -4, -5)$$

This also means that η must have all integer entries (as opposed to all half-integer entries). Finally taking the inner product with $\beta_1^\vee$ gives

$$\frac{1}{2}(-m_1 - m_8 + \sum_{i=2}^{7} m_i) = 1$$

$$\implies m_7 - m_8 = 2 + \sum_{i=2}^{6}(-m_i) = 17$$

However, η must be in X which has basis Δ. Note that the only element of the base which involves either e_7 or e_8 is β_1. For η to have integral entries it must have a multiple of $2\beta_1$ in it. But this means that $17 = m_7 - m_8 \equiv 0 \mod 2$ which is a contradiction. Therefore no twisting element can exist.

It is worth noting that if one does the above calculation for E_8 instead one still has

$$(m_1, m_2, m_3, m_4, m_5, m_6) = (0, -1, -2, -3, -4, -5)$$

and $m_7 - m_8 = 17$. Taking the inner product of η with $\beta_8^\vee$ gives $m_6 - m_7 = 1$ so $m_7 = -6$ and $m_8 = -23$. We thus get η is half the sum of the positive roots of E_8.

A.3.3 Type E_6

To get a base for E_6 one deletes β_7 and β_8 from the base for E_8 obtaining $\Delta = \{\beta_1, \cdots, \beta_6\}$. One finds that

$$\Phi^+ = \{\pm e_i - e_j : 1 \leq i < j \leq 5\} \cup \left\{ \frac{1}{2}(e_6 + e_7 - e_8 + \sum_{i=1}^{5} (-1)^{c_i} e_i : \sum_{i=1}^{5} c_i \equiv 1 \mod 2 \right\}$$

and therefore

$$2\rho = -2(4e_5 + 3e_4 + 2e_3 + e_2) + \frac{1}{2}(2^3(e_6 + e_7 - e_1 - e_8) + 2^3(e_6 + e_7 + e_1 - e_8))$$

$$\implies \rho = -(4e_5 + 3e_4 + 2e_3 + e_2) + 4(e_6 + e_7 - e_8) \in X$$

A.3.4 Type F_4

Similarly to the E_n, X consists of the span of $\{e_i : 1 \leq i \leq 4\}$ where either all the coordinates are in $\mathbb{Z}$ or all the coordinates are in $\mathbb{Z} + 1/2$. One gets the following root system and base

$$\Phi = \{\pm e_i : 1 \leq i \leq 4\} \cup \{\pm e_i \pm e_j : 1 \leq i < j \leq 4\} \cup \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}$$

$$\Delta = \left\{ e_1 - e_2, e_2 - e_3, e_3, \frac{1}{2}(-e_1 - e_2 - e_3 + e_4) \right\}$$

We therefore get positive roots

$$\Phi^+ = \{e_i : 1 \leq i \leq 4\} \cup \{e_i - e_j : 1 \leq i < j \leq 3\} \\ \cup \{e_4 \pm e_i : 1 \leq i \leq 3\} \cup \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 + e_4) \right\}$$

and consequently

$$2\rho = (e_1 + e_2 + e_3 + e_4) + 2(2e_1 + e_2) + 2 \cdot 3e_4 + \frac{1}{2} \cdot 8e_4$$

$$\implies \rho = \frac{1}{2}(e_1 + e_2 + e_3 + e_4) + (2e_1 + e_2 + 5e_4) \in X$$

A.3.5 Type G_2

X consists of vectors v in the $\mathbb{Z}$-span of $\{e_1, e_2, e_3\}$ such that the normal inner product of v with $c := e_1 + e_2 + e_3$ is zero. One gets the following root system

$$\Phi = \{e_i - e_j : i \neq j\} \cup \{\pm(c - 3e_i) : 1 \leq i \leq 3\}$$

and base $\Delta = \{\alpha, \beta\}$ where

$$\alpha = e_1 - e_2, \;\; \beta = c - 3e_1 = -2e_1 + e_2 + e_3$$

Noticing that

$$\alpha + \beta = e_3 - e_1, \;\; 2\alpha + \beta = e_3 - e_2$$

$$3\alpha + \beta = c - 3e_2, \;\; 3\alpha + 2\beta = 3e_3 - c$$

one gets the following set of positive roots

$$\Phi^+ = \{e_1 - e_2, e_3 - e_1, e_3 - e_2, c - 3e_1, c - 3e_2, 3e_3 - c\}$$

Consequently

$$2\rho = \alpha + \beta + (\alpha + \beta) + (2\alpha + \beta) + (3\alpha + \beta) + (3\alpha + 2\beta) = 10\alpha + 6\beta$$

$$\implies \rho = 5\alpha + 3\beta = -e_1 - 2e_2 + 3e_3 \in X$$

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