ON PSEUDOSPECTRA AND POWER GROWTH

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ABSTRACT. The celebrated Kreiss matrix theorem is one of several results relating the norms of the powers of a matrix to its pseudospectra (i.e. the level curves of the norm of the resolvent). But to what extent do the pseudospectra actually *determine* the norms of the powers? Specifically, let A, B be square matrices such that, with respect to the usual operator norm $\|\cdot\|$,

$$(*) \quad \|(zI - A)^{-1}\| = \|(zI - B)^{-1}\| \quad (z \in \mathbb{C}).$$

Then it is known that $1/2 \leq \|A\|/\|B\| \leq 2$. Are there similar bounds for $\|A^n\|/\|B^n\|$ for $n \geq 2$? Does the answer change if A, B are diagonalizable? What if $(*)$ holds, not just for the norm $\|\cdot\|$, but also for higher-order singular values? What if we use norms other than the usual operator norm? The answers to all these questions turn out to be negative, and in a rather strong sense.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $N \geq 1$, let $\mathbb{C}^N$ be complex Euclidean N -space, and let $\mathbb{C}^{N \times N}$ be the algebra of complex $N \times N$ matrices. We write $|\cdot|$ for the Euclidean norm on $\mathbb{C}^N$, defined by $|x| := (\sum_1^N |x_j|^2)^{1/2}$, and $\|\cdot\|$ for the associated operator norm on $\mathbb{C}^{N \times N}$, defined by $\|A\| := \sup\{|Ax| : |x| = 1\}$.

It is well known that, given $A \in \mathbb{C}^{N \times N}$, the long-term growth of the norms of powers of A is governed by the spectral radius $\rho(A)$. Indeed, by the spectral radius formula, we have

$$\|A^n\| \geq \rho(A)^n \quad (n \geq 1) \quad \text{and} \quad \lim_{n \rightarrow \infty} \|A^n\|^{1/n} = \rho(A).$$

However, in the shorter term, $\|A^n\|$ may well be significantly larger than $\rho(A)^n$. The recent book of Trefethen and Embree [4] contains an account of these transient effects, illustrated by examples drawn from many different fields. One of the main themes of the book is that, to predict accurately the growth of $\|A^n\|$, it is important to study not only the spectrum of A , but also its pseudospectra, which we now define.

Given $A \in \mathbb{C}^{N \times N}$ and $\epsilon > 0$, the ϵ -pseudospectrum of A is defined to be the set

$$\sigma_\epsilon(A) := \{z \in \mathbb{C} : \|(zI - A)^{-1}\| > \epsilon^{-1}\}.$$

Here and in what follows we adopt the useful convention that $\|(zI - A)^{-1}\| = \infty$ if $z \in \sigma(A)$, the spectrum of A . Thus $\sigma_\epsilon(A)$ shrinks to $\sigma(A)$ as $\epsilon \downarrow 0$. From a knowledge of the pseudospectra of A , it is possible to deduce both upper and lower bounds on the growth of $\|A^n\|$. A well-known result of this type is the Kreiss matrix theorem. We refer to [4] for this and several other such results. In addition, there are efficient methods for numerical computation of pseudospectra (see [4, Chapter IX]), so this approach is highly practical.

The purpose of this note is to show that, in predicting power growth, not even pseudospectra tell the whole story. The issue was already addressed by Greenbaum and Trefethen in [3] (see also

2000 *Mathematics Subject Classification*. Primary 47A10; Secondary 15A18, 15A60, 65F15.

Key words and phrases. matrix, norm, spectral radius, eigenvalue, singular value, pseudospectra.

Research supported by grants from NSERC and the Canada Research Chairs program.

[4, §47]). Suppose that two $N \times N$ matrices A, B have identical pseudospectra, i.e. that

$$(1) \quad \|(zI - A)^{-1}\| = \|(zI - B)^{-1}\| \quad (z \in \mathbb{C}).$$

Does it then follow that $\|p(A)\| = \|p(B)\|$ for every polynomial p ? In particular, can we conclude that $\|A^n\| = \|B^n\|$ for all $n \geq 1$? The answer is no. An example was given in [3] (and again in [4]) of two matrices A, B with identical pseudospectra such that $\|A\| = 1$ and $\|B\| = \sqrt{2}$. But this example leaves several basic questions unresolved:

1.1. What about higher powers? By adapting the Greenbaum–Trefethen example, one can construct, for each $\epsilon > 0$, matrices A, B with identical pseudospectra such that $\|A\|/\|B\| > 2 - \epsilon$. On the other hand, it is known that if A, B satisfy (1), then we must have

$$(2) \quad 1/2 \leq \|A\|/\|B\| \leq 2.$$

(For a proof, see [4, pp.168–169]; see also the remark after Theorem 5.1 below.) Are there similar bounds for $\|A^n\|/\|B^n\|$ for $n \geq 2$? If this were the case, then we could justifiably say that pseudospectra determine power norms, at least up to a constant factor. However, our first result answers this question negatively, and in a fairly strong sense.

Recall that (finite or infinite) sequence (α_k) is called *submultiplicative* if $\alpha_{k+l} \leq \alpha_k \alpha_l$ for all k, l for which the inequality makes sense. For example, the sequence $(\|A^k\|)_{k \geq 1}$ is submultiplicative for every matrix A .

Theorem 1.1. *Let $n \geq 2$, and let $\alpha_2, \dots, \alpha_n$ and $\beta_2, \dots, \beta_n$ be positive submultiplicative sequences. Then there exist $N \geq 1$ and matrices $A, B \in \mathbb{C}^{N \times N}$ such that*

$$(3) \quad \|(zI - A)^{-1}\| = \|(zI - B)^{-1}\| \quad (z \in \mathbb{C})$$

and

$$(4) \quad \|A^k\| = \alpha_k \quad \text{and} \quad \|B^k\| = \beta_k \quad (k = 2, \dots, n).$$

We may take $N = 2n + 3$.

This shows that matrices can have identical pseudospectra and yet their second and higher powers have norms that are completely unrelated to each other.

1.2. What about diagonalizable matrices? The matrices A, B in the Greenbaum–Trefethen example are nilpotent, as are those constructed in the proof of Theorem 1.1 above. Obviously, these are rather special. What happens if, instead, we consider more generic matrices, for example diagonalizable matrices? (By ‘diagonalizable’, we mean similar to a diagonal matrix.) Could it be that, for such matrices at least, the pseudospectra completely determine the power growth? Until now, no counterexample was known. We obtain one by combining the construction in Theorem 1.1 with a perturbation argument.

Theorem 1.2. *Let $n \geq 2$, let $\alpha_2, \dots, \alpha_n$ and $\beta_2, \dots, \beta_n$ be positive submultiplicative sequences, and let $\epsilon > 0$. Then there exist $N \geq 1$ and **diagonalizable** matrices $A, B \in \mathbb{C}^{N \times N}$ such that*

$$(5) \quad \|(zI - A)^{-1}\| = \|(zI - B)^{-1}\| \quad (z \in \mathbb{C})$$

and

$$(6) \quad \alpha_k - \epsilon < \|A^k\| < \alpha_k + \epsilon \quad \text{and} \quad \beta_k - \epsilon < \|B^k\| < \beta_k + \epsilon \quad (k = 2, \dots, n).$$

1.3. What if we use ‘higher-order’ pseudospectra? Given a matrix $A \in \mathbb{C}^{N \times N}$, its *singular values* $s_1(A), \dots, s_N(A)$ are the square roots of the eigenvalues of A^*A , listed in decreasing order. In particular, $s_1(A) = \|A\|$. One of the principal methods for computing the pseudospectra of A is to calculate the singular values of $zI - A$ for $z \in \mathbb{C}$ (once done for one value of z , it is relatively inexpensive to do it for many z), and then use the fact that

$$\|(zI - A)^{-1}\| = s_1\left((zI - A)^{-1}\right) = \frac{1}{s_N(zI - A)}.$$

It is thus reasonable to ask whether, by retaining the other singular values of $(zI - A)^{-1}$, it is possible to determine $\|A^n\|$ for values of $n \geq 2$. The following result gives a partial positive answer.

Theorem 1.3. *Let $N \geq 1$ and let $A, B \in \mathbb{C}^{N \times N}$ be matrices satisfying*

$$(7) \quad s_j\left((zI - A)^{-1}\right) = s_j\left((zI - B)^{-1}\right) \quad (z \in \mathbb{C}, j = 1, \dots, N).$$

Then, for every polynomial p ,

$$(8) \quad \frac{1}{\sqrt{N}} \leq \frac{\|p(A)\|}{\|p(B)\|} \leq \sqrt{N}.$$

It would be interesting to know if these bounds can be improved so as to be independent of N . However, even if this were the case, the theorem would be a bit unrealistic, because it would require us to keep track of all N singular values, which is probably too expensive in practice. Is there an analogous result where, by keeping track of just a few singular values, we can obtain inequalities like (8) at least for polynomials of low degree? The following generalization of Theorem 1.1 gives a negative answer.

Theorem 1.4. *Let $n \geq 2$, let $\alpha_2, \dots, \alpha_n$ and $\beta_2, \dots, \beta_n$ be positive submultiplicative sequences, and let $m \geq 1$. Then there exist $N \geq 1$ and matrices $A, B \in \mathbb{C}^{N \times N}$ such that*

$$(9) \quad s_j\left((zI - A)^{-1}\right) = s_j\left((zI - B)^{-1}\right) \quad (z \in \mathbb{C}, j = 1, \dots, m)$$

and

$$(10) \quad \|A^k\| = \alpha_k \quad \text{and} \quad \|B^k\| = \beta_k \quad (k = 2, \dots, n).$$

We may take $N = (m + 1)(n + 2) - 1$.

1.4. What about other norms? Though the Euclidean-norm case is undoubtedly the most important one, there are instances where it is more appropriate to consider pseudospectra defined with respect to other norms. In [4, §§56,57], several examples are given based on the 1-norm on $\mathbb{C}^N$, defined by $|x|_1 := \sum_{j=1}^N |x_j|$ and the associated operator norm $\|\cdot\|_1$, given by $\|A\|_1 := \sup\{|Ax|_1 : |x|_1 = 1\}$. There is no analogue of the Greenbaum–Trefethen example for this norm, because of the following theorem.

Theorem 1.5. *Let $N \geq 1$ and let $A, B \in \mathbb{C}^{N \times N}$ be matrices satisfying*

$$(11) \quad \|(zI - A)^{-1}\|_1 = \|(zI - B)^{-1}\|_1 \quad (z \in \mathbb{C}).$$

Then $\|A\|_1 = \|B\|_1$.

Can we also deduce that $\|A^n\|_1 = \|B^n\|_1$ for $n \geq 2$? Once again, the answer turns out to be no, and not just for $\|\cdot\|_1$, but for a whole variety of possible norms. To make this precise, it is convenient to introduce some notation and terminology.

Given square matrices A, B , perhaps of different sizes, we shall write $A \oplus B$ for the block matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

A norm $||| \cdot |||$ on $\mathbb{C}^{N \times N}$ will be called *admissible* if it satisfies the following three conditions:

- $||| \cdot |||$ is an algebra norm, i.e. $|||AB||| \leq |||A||| \cdot |||B|||$ for all $A, B \in \mathbb{C}^{N \times N}$ and $|||I||| = 1$;
- every permutation matrix $Q \in \mathbb{C}^{N \times N}$ satisfies $|||Q||| = 1$;
- every block matrix $A \oplus B \in \mathbb{C}^{N \times N}$ satisfies $|||A \oplus B||| = \max(|||A \oplus 0|||, |||0 \oplus B|||)$.

For example, if $|\cdot|_p$ is the usual p -norm on $\mathbb{C}^N$, given by $|x|_p := (\sum_{j=1}^N |x_j|^p)^{1/p}$, then the associated operator norm on $\mathbb{C}^{N \times N}$ is admissible.

The following result is a generalization of Theorem 1.1 to this context.

Theorem 1.6. *Let $n \geq 2$, and let $\alpha_2, \dots, \alpha_n$ and $\beta_2, \dots, \beta_n$ be positive submultiplicative sequences. Then there exist $N \geq 1$ and $A, B \in \mathbb{C}^{N \times N}$ such that, for every admissible norm $||| \cdot |||$ on $\mathbb{C}^{N \times N}$,*

$$(12) \quad |||(zI - A)^{-1}||| = |||(zI - B)^{-1}||| \quad (z \in \mathbb{C})$$

and

$$(13) \quad |||A^k||| = \alpha_k \quad \text{and} \quad |||B^k||| = \beta_k \quad (k = 2, \dots, n).$$

We may take $N = 2n + 3$.

We conclude by remarking that there is at least one well-known norm on $\mathbb{C}^{N \times N}$ for which matrices A, B with identical pseudospectra *do* have identical power growth. This is the Hilbert–Schmidt (or Frobenius) norm, as was shown by Greenbaum and Trefethen in [3]. We shall need their result in §4, where more details will be given. Of course, the Hilbert–Schmidt norm is not an admissible norm in our sense; in fact it fails all three parts of the definition.

The rest of the paper is devoted to the proofs of the six theorems above.

2. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is based on a construction using weighted shifts, which will also serve as a model in several other proofs to follow. It is therefore written in such a way as to be easy to adapt to other situations.

Given $\omega_1, \dots, \omega_n > 0$, we write

$$(14) \quad S(\omega_1, \dots, \omega_n) := \begin{pmatrix} 0 & \omega_1 & 0 & \dots & 0 \\ 0 & 0 & \omega_2/\omega_1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \omega_n/\omega_{n-1} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Lemma 2.1. *Let $\omega_1, \dots, \omega_n$ be a positive submultiplicative sequence, and let $S = S(\omega_1, \dots, \omega_n)$. Then*

$$(15) \quad \|S^k\| = \omega_k \quad (k = 1, \dots, n)$$

and

$$(16) \quad 1 + \max_{1 \leq k \leq n} \frac{\omega_k |z|^k}{2} \leq \|(I - zS)^{-1}\| \leq 1 + \sum_{k=1}^n \omega_k |z|^k \quad (z \in \mathbb{C}).$$

Proof. Let $k \in \{1, \dots, n\}$. Taking Q to be an appropriate cyclic permutation matrix, we have $S^k Q = \text{diag}(\omega_k, \omega_{k+1}/\omega_1, \dots, \omega_n/\omega_{n-k}, 0, \dots, 0)$. Using the submultiplicativity of the sequence $\omega_1, \dots, \omega_n$, we obtain

$$\|S^k Q\| = \max(\omega_k, \omega_{k+1}/\omega_1, \dots, \omega_n/\omega_{n-k}, 0, \dots, 0) = \omega_k.$$

Since $\|Q\| = 1 = \|Q^{-1}\|$ and $\|\cdot\|$ is an algebra norm, it follows that $\|S^k\| = \omega_k$. This proves (15).

For the upper bound in (16), note that $(I - zS)^{-1} = \sum_{k=0}^n z^k S^k$, whence

$$\|(1 - zS)^{-1}\| = \left\| \sum_{k=0}^n z^k S^k \right\| \leq \sum_{k=0}^n |z|^k \|S^k\| = 1 + \sum_{k=1}^n |z|^k \omega_k \quad (z \in \mathbb{C}).$$

For the lower bound, first fix $k \in \{1, \dots, n\}$ and $z \in \mathbb{C}$. Let Q be the permutation matrix that exchanges rows 2 and $k+1$, and let $P := \text{diag}(1, e^{i\theta}, 0, \dots, 0)$, where $\theta = -\arg(z^k)$. Then

$$\overline{P}Q(I - zS)^{-1}QP = \begin{pmatrix} 1 & |z|^k \omega_k \\ 0 & 1 \end{pmatrix} \oplus 0,$$

and hence

$$\|(I - zS)^{-1}\| \geq \left\| \begin{pmatrix} 1 & |z|^k \omega_k \\ 0 & 1 \end{pmatrix} \oplus 0 \right\|.$$

Conjugating by the permutation matrix that swaps the first two rows, we have

$$\left\| \begin{pmatrix} 1 & |z|^k \omega_k \\ 0 & 1 \end{pmatrix} \oplus 0 \right\| = \left\| \begin{pmatrix} 1 & 0 \\ |z|^k \omega_k & 1 \end{pmatrix} \oplus 0 \right\|.$$

Taking averages, it follows that

$$\|(I - zS)^{-1}\| \geq \left\| \begin{pmatrix} 1 & |z|^k \omega_k/2 \\ |z|^k \omega_k/2 & 1 \end{pmatrix} \oplus 0 \right\|.$$

Since $\|\cdot\|$ is always at least as large as the spectral radius, we deduce that

$$\|(I - zS)^{-1}\| \geq \rho\left(\begin{pmatrix} 1 & |z|^k \omega_k/2 \\ |z|^k \omega_k/2 & 1 \end{pmatrix} \oplus 0\right) = 1 + \frac{|z|^k \omega_k}{2}.$$

As this holds for each $k \in \{1, \dots, n\}$ and each $z \in \mathbb{C}$, we obtain the lower bound in (16). $\square$

Proof of Theorem 1.1. First, choose α_1, β_1 large enough so that the sequences $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ are submultiplicative, and set

$$A_0 := S(\alpha_1, \dots, \alpha_n) \quad \text{and} \quad B_0 := S(\beta_1, \dots, \beta_n).$$

Then set $A := A_0 \oplus C_0$ and $B := B_0 \oplus C_0$, where C_0 is the $(n+2) \times (n+2)$ matrix defined by

$$C_0 := S(\Gamma, \gamma^2, \gamma^3, \dots, \gamma^{n+1}).$$

Here Γ, γ are positive numbers to be chosen later. Note that the sequence $\Gamma, \gamma^2, \dots, \gamma^{n+1}$ is submultiplicative provided that $\Gamma \geq \gamma$. Defined in this way, A, B are $(2n+3) \times (2n+3)$ matrices, and we shall show that they satisfy (3) and (4) if γ, Γ are chosen suitably.

We first choose $\gamma > 0$ small enough so that $\gamma^k < \min(\alpha_k, \beta_k)$ ($k = 2, \dots, n$). With this choice,

$$\|A^k\| = \max(\|A_0^k\|, \|C_0^k\|) = \max(\alpha_k, \gamma^k) = \alpha_k \quad (k = 2, \dots, n),$$

and likewise $\|B^k\| = \beta_k$ ($k = 2, \dots, n$).

It remains to choose Γ to ensure that A, B have identical pseudospectra. This will be the case provided that

$$(17) \quad \|(I - zC_0)^{-1}\| \geq \|(I - zA_0)^{-1}\| \quad \text{and} \quad \|(I - zC_0)^{-1}\| \geq \|(I - zB_0)^{-1}\| \quad (z \in \mathbb{C}).$$

By Lemma 2.1, this will be true if

$$\max\left(\frac{\Gamma t}{2}, \frac{\gamma^{n+1}t^{n+1}}{2}\right) \geq \sum_{k=1}^n \alpha_k t^k \quad \text{and} \quad \max\left(\frac{\Gamma t}{2}, \frac{\gamma^{n+1}t^{n+1}}{2}\right) \geq \sum_{k=1}^n \beta_k t^k \quad (t \geq 0).$$

Now there exists t_0 , depending on $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma$, but not on Γ , such that

$$\frac{\gamma^{n+1}t^{n+1}}{2} \geq \sum_{k=1}^n \alpha_k t^k \quad \text{and} \quad \frac{\gamma^{n+1}t^{n+1}}{2} \geq \sum_{k=1}^n \beta_k t^k \quad (t \geq t_0).$$

Hence it will suffice that

$$\frac{\Gamma t}{2} \geq \sum_{k=1}^n \alpha_k t^k \quad \text{and} \quad \frac{\Gamma t}{2} \geq \sum_{k=1}^n \beta_k t^k \quad (0 \leq t \leq t_0).$$

This will certainly be true provided we choose Γ large enough. With this choice, the construction is complete. $\square$

3. PROOF OF THEOREM 1.2

The basic idea is to perturb the construction in the proof of Theorem 1.1. However, keeping track of the norms of the resolvents requires a certain amount of care. We shall need two lemmas.

Lemma 3.1. *Let $V, W \in \mathbb{C}^{N \times N}$. If V is invertible and $\|V - W\| < 1/(2\|V^{-1}\|)$, then W is invertible and*

$$\|V^{-1} - W^{-1}\| \leq 2\|V^{-1}\|^2\|V - W\|.$$

Proof. We have

$$\|I - V^{-1}W\| = \|V^{-1}(V - W)\| \leq \|V^{-1}\| \|V - W\| \leq 1/2.$$

Therefore $V^{-1}W$ is invertible, and hence so too is W .

We also have

$$(18) \quad \|V^{-1} - W^{-1}\| = \|V^{-1}(W - V)W^{-1}\| \leq \|V^{-1}\| \|W - V\| \|W^{-1}\|.$$

Since $\|W - V\| \leq 1/(2\|V^{-1}\|)$, it follows that $\|V^{-1} - W^{-1}\| \leq \|W^{-1}\|/2$, whence $\|W^{-1}\| \leq 2\|V^{-1}\|$. Substituting this back into (18) gives the result. $\square$

In the next lemma, adj denotes adjugate and ρ denotes spectral radius.

Lemma 3.2. *Let $V \in \mathbb{C}^{N \times N}$. Then*

$$\|\text{adj}(V) - (-V)^{N-1}\| \leq 2^N \rho(V) \|V\|^{N-2}.$$

Proof. It suffices to prove this when V is invertible, since invertible matrices are dense in $\mathbb{C}^{N \times N}$.

Let $p(z)$ be the characteristic polynomial of V . Since $p(z) = \prod_{j=1}^N (z - \lambda_j)$, where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of V , we have $p(z) = \sum_{j=0}^N a_j z^j$, where

$$(19) \quad a_N = 1, \quad a_0 = (-1)^N \det(V) \quad \text{and} \quad |a_j| \leq \binom{N}{j} \rho(V)^{N-j} \quad (j = 0, \dots, N).$$

Now by the Cayley–Hamilton theorem $p(V) = 0$. Multiplying by V^{-1} and rearranging gives

$$a_0 V^{-1} + a_N V^{N-1} = - \sum_{j=1}^{N-1} a_j V^{j-1}.$$

Using (19), it follows that

$$\|(-1)^N \det(V) V^{-1} + V^{N-1}\| \leq \sum_{j=1}^{N-1} \binom{N}{j} \rho(V)^{N-j} \|V\|^{j-1}.$$

Since $\det(V) V^{-1} = \text{adj}(V)$ and $\rho(V) \leq \|V\|$, we deduce that

$$\|(-1)^N \text{adj}(V) + V^{N-1}\| \leq \sum_{j=1}^{N-1} \binom{N}{j} \rho(V) \|V\|^{N-j-1} \|V\|^{j-1} \leq 2^N \rho(V) \|V\|^{N-2},$$

whence the result. $\square$

Proof of Theorem 1.2. Define A_0, B_0, C_0 as in the proof of Theorem 1.1. The choice of γ is the same as before, so that $\|(A_0 \oplus C_0)^k\| = \alpha_k$ and $\|(B_0 \oplus C_0)^k\| = \beta_k$ for $k = 2, \dots, n$. This time, however, we choose Γ a little differently, stipulating that $\Gamma \geq \gamma$ and

$$(20) \quad \frac{\Gamma t}{2} \geq \sum_{k=1}^n \alpha_k t^k + 2t \quad \text{and} \quad \frac{\Gamma t}{2} \geq \sum_{k=1}^n \beta_k t^k + 2t \quad (0 \leq t \leq 1/\gamma).$$

The next step is to perturb A_0, B_0, C_0 so as to obtain diagonalizable matrices. To this end, we fix distinct complex numbers $\zeta_1, \dots, \zeta_{n+1}$ of modulus $1/2$, and set

$$D := \text{diag}(\zeta_1, \dots, \zeta_{n+1}) \quad \text{and} \quad D' := \text{diag}(\zeta_1, \dots, \zeta_{n+1}, 0),$$

Then, for each $\delta > 0$, we define

$$A_\delta := A_0 + \delta D, \quad B_\delta := B_0 + \delta D \quad \text{and} \quad C_\delta := C_0 + \delta D'.$$

Each of $A_\delta, B_\delta, C_\delta$ has distinct eigenvalues, so $A_\delta \oplus C_\delta$ and $B_\delta \oplus C_\delta$ are diagonalizable.

By continuity, if $\delta > 0$ is chosen small enough, then

$$\left| \|(A_\delta \oplus C_\delta)^k\| - \alpha_k \right| < \epsilon \quad \text{and} \quad \left| \|(B_\delta \oplus C_\delta)^k\| - \beta_k \right| < \epsilon \quad (k = 2, \dots, n).$$

We next show that, reducing δ if necessary, we have

$$(21) \quad \begin{cases} \|(C_\delta - zI)^{-1}\| \geq \|(A_\delta - zI)^{-1}\| \\ \|(C_\delta - zI)^{-1}\| \geq \|(B_\delta - zI)^{-1}\| \end{cases} \quad (|z| \geq \gamma).$$

By Lemma 2.1, we have

$$\|(I - wC_0)^{-1}\| - \|(I - wA_0)^{-1}\| \geq (1 + \Gamma|w|/2) - (1 + \sum_{k=1}^n |\alpha_k| |w|^k) \quad (w \in \mathbb{C}).$$

From our choice of Γ in (20), it follows that

$$\|(I - wC_0)^{-1}\| - \|(I - wA_0)^{-1}\| \geq 2|w| \quad (|w| \leq 1/\gamma).$$

We now apply Lemma 3.1 with $V = I - wA_0$ and $W = I - wA_\delta$. We find that, if $\delta|w|\|D\| \leq 1/(2\|(I - wA_0)^{-1}\|)$, then

$$\|(I - wA_\delta)^{-1} - (I - wA_0)^{-1}\| \leq 2\|(I - wA_0)^{-1}\|^2 \delta|w|\|D\|.$$

It follows that, if δ is chosen small enough, then

$$\|(I - wA_\delta)^{-1} - (I - wA_0)^{-1}\| \leq |w| \quad (|w| \leq 1/\gamma).$$

Likewise, if δ is small enough, then

$$\|(I - wC_\delta)^{-1} - (I - wC_0)^{-1}\| \leq |w| \quad (|w| \leq 1/\gamma).$$

Putting all of this together, we find that, if δ is sufficiently small, then

$$\|(I - wC_\delta)^{-1}\| \geq \|(I - wA_\delta)^{-1}\| \quad (|w| \leq 1/\gamma),$$

from which (21) follows for A . Evidently, a similar argument applies to B .

The next step is to show that, by reducing δ yet further, we may ensure that

$$(22) \quad \begin{cases} \|(C_\delta - zI)^{-1}\| \geq \|(A_\delta - zI)^{-1}\| \\ \|(C_\delta - zI)^{-1}\| \geq \|(B_\delta - zI)^{-1}\| \end{cases} \quad (|z| \leq \delta).$$

For this we use Lemma 3.2. Applying this lemma with $V = A_\delta - zI$, and recalling that this is an $(n+1) \times (n+1)$ matrix, we obtain

$$\|\text{adj}(A_\delta - zI) - (zI - A_\delta)^n\| \leq 2^{n+1} \rho(A_\delta - zI) \|A_\delta - zI\|^{n-1}.$$

Now $\sigma(A_\delta - zI) = \sigma(\delta D - zI)$, so

$$\rho(A_\delta - zI) = \rho(\delta D - zI) \leq \|\delta D - zI\| \leq \delta + |z|.$$

It follows that

$$\|\text{adj}(A_\delta - zI) - (zI - A_\delta)^n\| \leq 2^{n+1}(\delta + |z|) \|A_\delta - zI\|^{n-1}.$$

Note also that $\sup_{|z| \leq \delta} \|(zI - A_\delta)^n - (-A_0)^n\| = O(\delta)$ as $\delta \rightarrow 0$. Hence there exists a constant K , independent of z, δ , such that

$$\|\text{adj}(A_\delta - zI) - (-A_0)^n\| \leq K\delta \quad (|z| \leq \delta).$$

Similarly, as C_δ is an $(n+2) \times (n+2)$ matrix, there exists a constant K' such that

$$\|\text{adj}(C_\delta - zI) - (-C_0)^{n+1}\| \leq K'\delta \quad (|z| \leq \delta).$$

Since $A_0^n \neq 0$ and $C_0^{n+1} \neq 0$, it follows that, if δ is small enough, then

$$\begin{cases} \|\text{adj}(C_\delta - zI)\| \geq \|C_0^{n+1}\|/2 \\ \|\text{adj}(A_\delta - zI)\| \leq 2\|A_0^n\| \end{cases} \quad (|z| \leq \delta).$$

Now

$$\text{adj}(C_\delta - zI) = \det(C_\delta - zI)(C_\delta - zI)^{-1}, \quad \text{adj}(A_\delta - zI) = \det(A_\delta - zI)(A_\delta - zI)^{-1}$$

and

$$\frac{\det(C_\delta - zI)}{\det(A_\delta - zI)} = \frac{\det(\delta D' - zI)}{\det(\delta D - zI)} = -z.$$

Combining these facts, we obtain that, for sufficiently small $\delta > 0$,

$$\frac{\|(C_\delta - zI)^{-1}\|}{\|(A_\delta - zI)^{-1}\|} \geq \frac{1}{|z|} \frac{\|C_0^{n+1}\|}{4\|A_0^n\|} \geq \frac{1}{\delta} \frac{\|C_0^{n+1}\|}{4\|A_0^n\|} \quad (|z| \leq \delta).$$

Thus, if δ is chosen small enough, then (22) holds for A . The argument for B is similar.

Fix $\delta > 0$ so that (21) and (22) hold. Summarizing what we have achieved so far, if we define $A := A_\delta \oplus C_\delta$ and $B := B_\delta \oplus C_\delta$, then A, B are diagonalizable matrices satisfying (6) and

$$\|(A - zI)^{-1}\| = \|(B - zI)^{-1}\| \quad (z \in \mathbb{C} \setminus Q),$$

where Q is the annulus $\{z \in \mathbb{C} : \delta < |z| < \gamma\}$. It remains to deal with the case $z \in Q$, which we do as follows. Let L be the maximum of $\sup_{z \in Q} \|(A - zI)^{-1}\|$ and $\sup_{z \in Q} \|(B - zI)^{-1}\|$. Cover Q by a finite number of disks of radius $1/L$, with centres $\mu_1, \dots, \mu_m \in Q$ say. Define $E := \text{diag}(\mu_1, \dots, \mu_m)$. Then we have

$$\begin{cases} \|(E - zI)^{-1}\| \geq \|(A - zI)^{-1}\| \\ \|(E - zI)^{-1}\| \geq \|(B - zI)^{-1}\| \end{cases} \quad (z \in Q).$$

Thus, if we replace A, B by $A \oplus E, B \oplus E$ respectively, then they have identical pseudospectra. Evidently the new A, B are still diagonalizable. Finally, as $\mu_1, \dots, \mu_m \in Q$, we have $\|E^k\| \leq \gamma^k \leq \min(\alpha_k, \beta_k)$ for $k = 2, \dots, n$, and so (6) still holds. The construction is complete. $\square$

Remark. The construction yields matrices A, B having eigenvalues of multiplicity at most two. It would be interesting to obtain an example where the eigenvalues were all of multiplicity one.

4. PROOFS OF THEOREMS 1.3 AND 1.4

Theorem 1.3 is an easy consequence of the following result of Greenbaum and Trefethen. Recall that the Hilbert–Schmidt norm of a square matrix A is defined by

$$\|A\|_{HS} := \sqrt{\text{trace}(A^*A)}.$$

Theorem 4.1 ([3, Theorem 3]). *Let $A, B \in \mathbb{C}^{N \times N}$, and suppose that*

$$(23) \quad \|(zI - A)^{-1}\|_{HS} = \|(zI - B)^{-1}\|_{HS} \quad (z \in \mathbb{C}).$$

Then, for every polynomial p ,

$$(24) \quad \|p(A)\|_{HS} = \|p(B)\|_{HS}.$$

Since [3] was never published, we also include a brief proof for the reader's convenience.

Proof. Setting $\zeta = 1/z$, we see that (23) is equivalent to

$$(25) \quad \text{trace}[(I - \bar{\zeta}A^*)^{-1}(I - \zeta A)^{-1}] = \text{trace}[(I - \bar{\zeta}B^*)^{-1}(I - \zeta B)^{-1}] \quad (\zeta \in \mathbb{C}).$$

Expanding, we deduce that, for some $r > 0$,

$$\sum_{k,l \geq 0} \text{trace}(A^{*k}A^l) \bar{\zeta}^k \zeta^l = \sum_{k,l \geq 0} \text{trace}(B^{*k}B^l) \bar{\zeta}^k \zeta^l \quad (|\zeta| < r).$$

Taking $(\frac{\partial}{\partial \zeta})^k (\frac{\partial}{\partial \bar{\zeta}})^l$ of both sides and then setting $\zeta = 0$, we obtain

$$\text{trace}(A^{*k}A^l) = \text{trace}(B^{*k}B^l) \quad (k, l \geq 0).$$

Now, let p be a polynomial, say $p(z) = \sum_{j=0}^n a_j z^j$. Then

$$\text{trace}(p(A)^*p(A)) = \sum_{k,l=0}^n \bar{a}_k a_l \text{trace}(A^{*k}A^l) = \sum_{k,l=0}^n \bar{a}_k a_l \text{trace}(B^{*k}B^l) = \text{trace}(p(B)^*p(B)),$$

whence $\|p(A)\|_{HS} = \|p(B)\|_{HS}$. This completes the proof. $\square$

Proof of Theorem 1.3. Observe that $\|A\|_{HS}^2 = \sum_{j=1}^N s_j(A)^2$. Thus, hypothesis (7) implies that (23) holds, and consequently also (24). For each polynomial p , we therefore have

$$\sum_{j=1}^N s_j(p(A))^2 = \sum_{j=1}^N s_j(p(B))^2.$$

Recalling that the usual operator norm $\|\cdot\|$ is just the first singular value s_1 , we thus obtain

$$\|p(A)\|^2 = s_1(p(A))^2 \leq \sum_{j=1}^N s_j(p(A))^2 = \sum_{j=1}^N s_j(p(B))^2 \leq N s_1(p(B))^2 = N \|p(B)\|^2.$$

This gives the right-hand side of (8), and the left-hand side is proved similarly. $\square$

Remark. In going from (7) to (23), we are losing some information. In fact (7) is equivalent to the following, more complicated version of (25):

$$(26) \quad \text{trace}\left([(I - \bar{\zeta}A^*)^{-1}(I - \zeta A)^{-1}]^n\right) = \text{trace}\left([(I - \bar{\zeta}B^*)^{-1}(I - \zeta B)^{-1}]^n\right) \quad (\zeta \in \mathbb{C}, n \geq 1).$$

Till now, we have not seen how to exploit this.

Proof of Theorem 1.4. We repeat the construction in the proof of Theorem 1.1, defining A_0, B_0, C_0 exactly as in that proof. This time, however, we define

$$A := A_0 \oplus \overbrace{C_0 \oplus \cdots \oplus C_0}^m \quad \text{and} \quad B := B_0 \oplus \overbrace{C_0 \oplus \cdots \oplus C_0}^m.$$

Then $A, B \in \mathbb{C}^{N \times N}$, where $N = (n+1) + m(n+2) = (m+1)(n+2) - 1$. Just as before,

$$\|A^k\| = \max(\|A_0^k\|, \|C_0^k\|, \dots, \|C_0^k\|) = \alpha_k \quad (k = 2, \dots, n),$$

and similarly for B , so (10) holds. Also, since

$$(zI - A)^{-1} = (zI - A_0)^{-1} \oplus \overbrace{(zI - C_0)^{-1} \oplus \cdots \oplus (zI - C_0)^{-1}}^m,$$

and $\|(zI - C_0)^{-1}\| \geq \|(zI - A_0)^{-1}\|$ for all $z \in \mathbb{C}$ (see (17)), it follows that

$$s_j\left((zI - A)^{-1}\right) = \|(zI - C_0)^{-1}\| \quad (z \in \mathbb{C}, j = 1, \dots, m).$$

Likewise, the same is true with A replaced by B . Thus (9) holds, and the proof is complete. $\square$

5. PROOFS OF THEOREMS 1.5 AND 1.6

In fact we shall prove the following slight generalization of Theorem 1.5.

Theorem 5.1. *Let $A, B \in \mathbb{C}^{N \times N}$, and suppose that*

$$(27) \quad \|(I - \zeta A)^{-1}\|_1 = \|(I - \zeta B)^{-1}\|_1 + o(\zeta) \quad \text{as } \zeta \rightarrow 0, \zeta \in \mathbb{C}.$$

Then $\|A\|_1 = \|B\|_1$.

Proof. The norm $\|\cdot\|_1$ has the particularity that $\|A\|_1 = \max(|Ae_1|_1, \dots, |Ae_N|_1)$, where $e_1, \dots, e_N$ is the standard unit vector basis of $\mathbb{C}^N$. Fix a j so that $\|A\|_1 = |Ae_j|_1$. Multiplying A and B by the same unimodular constant, we may suppose that $a_{jj} \geq 0$, in other words the j -th entry in Ae_j is non-negative. It then follows that, for all $t \geq 0$,

$$|(I + tA)e_j|_1 = 1 + t|Ae_j|_1 = 1 + t\|A\|_1.$$

On the other hand, as $t \rightarrow 0^+$, we have

$$|(I + tA)e_j|_1 \leq \|I + tA\|_1 = \|(I - tA)^{-1}\|_1 + o(t) = \|(I - tB)^{-1}\|_1 + o(t) \leq 1 + t\|B\|_1 + o(t).$$

Combining these facts, we deduce that $\|A\|_1 \leq \|B\|_1$. By symmetry $\|B\|_1 \leq \|A\|_1$ as well. $\square$

Remark. This theorem may be viewed as a result about numerical ranges in Banach algebras. For background on numerical ranges, we refer to [1, 2]. Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach algebra with identity 1, and given $a \in \mathcal{A}$, let $\nu_{\mathcal{A}}(a)$ denote the numerical radius of a . It is well known that

$$\nu_{\mathcal{A}}(a) = \limsup_{\zeta \rightarrow 0} \frac{\|1 + \zeta a\|_{\mathcal{A}} - 1}{|\zeta|} \quad (a \in \mathcal{A}),$$

and also that there exists a constant $n(\mathcal{A}) \in [e^{-1}, 1]$, called the *numerical index* of $\mathcal{A}$, such that

$$n(\mathcal{A})^{-1}\|a\|_{\mathcal{A}} \leq \nu_{\mathcal{A}}(a) \leq \|a\|_{\mathcal{A}} \quad (a \in \mathcal{A}).$$

From these facts it follows easily that, if $a, b \in \mathcal{A}$ satisfy

$$\|(1 - \zeta a)^{-1}\| = \|(1 - \zeta b)^{-1}\| + o(\zeta) \quad \text{as } \zeta \rightarrow 0, \zeta \in \mathbb{C},$$

then $\nu_{\mathcal{A}}(a) = \nu_{\mathcal{A}}(b)$, and hence

$$n(\mathcal{A}) \leq \|a\|_{\mathcal{A}}/\|b\|_{\mathcal{A}} \leq n(\mathcal{A})^{-1}.$$

Moreover, it is known that the numerical indices of $(\mathbb{C}^{N \times N}, \|\cdot\|)$ and $(\mathbb{C}^{N \times N}, \|\cdot\|_1)$ are equal to $1/2$ and 1 respectively. We thus recover as special cases both the result (2) mentioned earlier and Theorem 5.1 above.

We now turn to the proof of Theorem 1.6. Recall that the notion of admissible norm on $\mathbb{C}^{N \times N}$ was defined in the introduction, and that the weighted shift $S(\omega_1, \dots, \omega_n)$ was defined in (14).

Lemma 5.2. *Let $\omega_1, \dots, \omega_n$ be a positive submultiplicative sequence, and let $S = S(\omega_1, \dots, \omega_n)$. Then, for every admissible norm $|||\cdot|||$ on $\mathbb{C}^{(n+1) \times (n+1)}$,*

$$(28) \quad |||S^k||| = \omega_k \quad (k = 1, \dots, n)$$

and

$$(29) \quad 1 + \max_{1 \leq k \leq n} \frac{\omega_k |z|^k}{2} \leq |||(I - zS)^{-1}||| \leq 1 + \sum_{k=1}^n \omega_k |z|^k \quad (z \in \mathbb{C}).$$

Proof. Repeat the proof of Lemma 2.1, observing that it is valid for every admissible norm. $\square$

Proof of Theorem 1.6. Repeat the proof of Theorem 1.1, using Lemma 5.2 in place of Lemma 2.1. Note that the choices of γ and Γ depend only on the α_j and β_j , and not on the particular norm. Thus the same pair of matrices A, B works simultaneously for all admissible norms $|||\cdot|||$. $\square$

Acknowledgements. I am greatly indebted to Nick Trefethen for introducing me to this topic, for making available the article [3], and for numerous invaluable discussions. This work was carried out while I was visiting the Mathematical Institute of the University of Oxford and the Oxford University Computing Laboratory, and I am grateful to both institutions for their hospitality.

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