Prime ideals of Iwasawa algebras over Solvable Groups

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A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2020

Abstract

In this thesis, we will explore the non-commutative Iwasawa algebra $\mathcal{O}G$ of a uniform pro-p group G over the ring of integers $\mathcal{O}$ of a finite extension $K\backslash\mathbb{Q}_p$. Our aim is to explore the prime ideal structure of $\mathcal{O}G$, and our ultimate goal is to prove that all prime ideals have a standard form for which we have a complete classification. We will focus on the case where G is solvable, and we will divide the problem into two cases, prime ideals containing p and prime ideals not containing p.

In the former case, we may quotient out by p and study the mod-p Iwasawa algebra kG, where k is the residue field of K. For G nilpotent, the classification in this case was completed by Ardakov, using the theory of Mahler expansions of continuous automorphisms of G. In the first half of this thesis, we will recap this theory, and show how it can be generalised to complete the classification for prime ideals in kG over solvable groups G of the form $\mathbb{Z}_p^d \rtimes \mathbb{Z}_p$.

In the latter case, we may tensor with K and study the rational Iwasawa algebra KG. The theory of Mahler expansions ultimately fails in this case, so we will instead explore methods involving p-adic representation theory. We focus on the case where $G \cong \mathbb{Z}_p^d \rtimes \mathbb{Z}_p$ is nilpotent, and we will complete the classification for faithful primitive ideals of KG in this case for $p > 2$. Our main technique will be to study the representation theory of the completed enveloping algebra $\widehat{U(\mathcal{L})}_K$ of the associated Lie algebra $\mathcal{L}$ of G, and to describe all primitive ideals of this algebra using a class of canonical representations.

Acknowledgments

I would like to thank my supervisor, Professor Konstantin Ardakov, for all the help and guidance given to me throughout my DPhil, and for the many ideas and suggestions that have helped direct my research. I would also like to thank EPSRC for funding this project.

I give particular thanks to my friends and DPhil colleagues Richard Mathers, Joshua Ciappara, Ioan Stanciu and James Timmins for providing an excellent working environment, for a multitude of helpful conversations and discussions, and for their invaluable contributions to my postgraduate life. Additionally, I give thanks to Andreas Bode and Billy Woods for their assistance with multiple difficulties I encountered during my project, and also to my extended officemates including Jo French, Tom Zielinski, Filip Zivanovic and Nadav Gropper.

I am also truly grateful to the students and staff at Brasenose college, most notably Anirudh Sridhar, Cresente Molina and Nicolas Perrault, for their continued encouragement and for aiding me in every aspect of my life in Oxford.

Finally, and most importantly, I would like to thank my family for all their support since I started my DPhil and before.

Contents

Chapter 1

Introduction

Throughout, we will fix a prime number p, and a finite extension K of $\mathbb{Q}_p$. Let $\mathcal{O} = \mathcal{O}_K$ be the valuation ring of K, let $\pi \in \mathcal{O}$ be a uniformiser, and let k be the residue field of K.

1.1 Completed group algebras and Iwasawa algebras

Let G be a profinite group in the sense of [17, Definition 1.1], and let R be a commutative, pseudocompact topological ring in the sense of [12], i.e. the limit of an inverse system of Artinian rings, so we could of course take $R = \mathcal{O}$ or $R = k$. We define the *completed group algebra* of G with coefficients in R as:

$$RG := \varprojlim_{U \trianglelefteq_o G} R[G/U] \tag{1.1}$$

where $R[H]$ denotes the standard group algebra, and the inverse limit is taken as U runs over all open normal subgroups of G, which form a topological basis of the neighbourhoods of the identity in G. We can think of RG as a certain topological completion of $R[G]$.

We will assume that G is a compact p-adic Lie group, meaning that it is locally homeomorphic to $\mathbb{Z}_p^d$. Using [17, Theorem 8.32] we see that this just means that G is profinite and contains an open subgroup that is a *uniform pro-p group* in the sense of [17, Definition 4.1].

If G is a compact p-adic Lie group, the completed group algebra $\mathcal{O}G$ is sometimes called the *$\mathcal{O}$-Iwasawa algebra* of G. This is a Noetherian, semilocal algebra, and it is of fundamental importance in the representation theory of compact p-adic Lie groups, since studying continuous, $\mathcal{O}$-linear representations of G requires us to study modules over $\mathcal{O}G$ with a pro-discrete topology by [32, Corollary 19.3].

In this thesis, we are interested in classifying the two-sided ideals of $\mathcal{O}G$, which has been an ongoing project in non-commutative ring theory for many years.

Note: The p-adic field K is *not* pseudocompact with respect to its canonical topology, so we do not define KG as $\varprojlim_{U \trianglelefteq_o G} K[G/U]$.

The usual approach to studying $\mathcal{O}G$ is to consider it separately through the lens of characteristic p and characteristic 0. More specifically, we consider two further rings:

- The *mod-p Iwasawa algebra* kG, this can be realised as the quotient $\mathcal{O}G/(\pi)$.
- The *Iwasawa algebra of continuous K-distributions* $KG := \mathcal{O}G \otimes_{\mathcal{O}} K$.

Studying prime ideals in both these algebras is equivalent to studying prime ideals in $\mathcal{O}G$, because prime ideals in kG are in bijection with prime ideals in $\mathcal{O}G$ containing p, and prime ideals in KG are in bijection with prime ideals in $\mathcal{O}G$ not containing p.

Also, to underline the importance of KG as an algebra, recall from [33, Section 2] the definition of the *locally analytic K-distribution algebra* of G, $D(G, K)$, and recall from [34, Theorem 5.1] that it is a *Fréchet-Stein algebra.* The distribution algebra is essential within the study of locally analytic representations of G, since admissible, locally analytic K-representations of G arise as coadmissible modules over $D(G, K)$.

The Iwasawa algebra KG embeds into $D(G, K)$ as a dense subalgebra, and in fact $D(G, K)$ is faithfully-flat over KG by [34, Theorem 4.11]. This suggests that understanding the structure of KG would help towards understanding $D(G, K)$, and since KG is Noetherian and $D(G, K)$ generally is not, this approach would be easier than approaching $D(G, K)$ directly.

Now, since any compact p-adic Lie group G contains a uniform open normal subgroup U, it follows that we can realise RG as a *crossed product* $RG = RU * \frac{G}{U}$, for $R = \mathcal{O}, k, K$, and $\frac{G}{U}$ is a finite group. Therefore, we may restrict our attention to the study of uniform groups, which can make our calculations easier.

The Iwasawa algebra of a non-abelian group was first defined in 1965 by Lazard in [25], where he used it to compute the continuous cohomology of a compact p-adic Lie group, but these objects were not well studied until the early 2000s when it was observed by Coates, Schneider and Sujatha in [13] that classifying prime c-ideals in an Iwasawa algebra would have applications to the representation theory of the Galois group of an elliptic curve without complex multiplication, and other related areas which have applications within the Langlands programme. This work initiated the project of obtaining a complete classification of the prime ideals in $\mathcal{O}G$.

The most obvious examples of prime ideals are *augmentation ideals* of the form $(H-1)\mathcal{O}G$ for some closed, normal subgroup H of G, and *centrally generated ideals*. The ultimate aim of this research is to prove that all prime ideals in $\mathcal{O}G$ can be reduced to this form.

Definition 1.1.1. *Let $R = \mathcal{O}, k$ or K, and let G be a uniform pro-p group. We say that a prime ideal P of RG is* standard *if there exists a closed, normal subgroup H*

of G such that:

- $G_0 := \frac{G}{H}$ *is torsionfree.*
- $H - 1 \subseteq P$.
- *The image of P in* RG_0 *is centrally generated.*

We say that P is virtually standard *if* $P \cap RU$ *is a finite intersection of standard prime ideals of RU for some open normal subgroup U of G.*

Note: If P is a prime ideal in $\mathcal{O}G$ containing p, then P is (virtually) standard in $\mathcal{O}G$ if and only if it is (virtually) standard in kG. Similarly, if $p \notin P$ then P is (virtually) standard in $\mathcal{O}G$ if and only if it is (virtually) standard in KG.

Since the centre of RG is $RZ(G)$ for G uniform by [2, Corollary A], for an ideal P to be centrally generated just means that $P = (P \cap RZ(G))RG$, i.e. that P is *controlled* by $Z(G)$. Using this, we deduce the following result:

Theorem 1.1.2. *Let G be a uniform group, and let P be a standard prime ideal of* $\mathcal{O}G$. *Then P is* completely prime, *i.e.* $\frac{\mathcal{O}G}{P}$ *is a domain.*

Proof. If $p \in P$, this follows from [1, Theorem 8.6], and we will see in Appendix B.2 that the proof of this generalises to the case where $p \notin P$. □

The main conjecture in the study of the ideal structure of non-commutative Iwasawa algebras, which is the essence of [4, Question N, Question O], is that virtually standard prime ideals characterise the entire prime spectrum of $\mathcal{O}G$ when G is solvable and uniform:

Conjecture 1.1.3. *Let G be a solvable, uniform pro-p group. Then every prime ideal of* $\mathcal{O}G$ *is virtually standard, and every prime ideal containing p is standard.*

Note: This conjecture is generally untrue if G is not solvable. For example, if G is semisimple then KG will contain many faithful ideals of finite codimension, which cannot be standard. We do suspect, however, that all prime ideals of infinite codimension are standard in this case.

Using Theorem 1.1.2, it would follow from this conjecture that prime ideals in $\mathcal{O}G$ are typically completely prime. Moreover, given any prime ideal P, after quotienting by a smaller augmentation ideal, it would follow that P is uniquely determined by a prime ideal in $\mathcal{O}Z(G)$, thus reducing the problem of ideal classification to a commutative setting.

Conjecture 1.1.3 is trivially true when G is an abelian group, and the smallest non-abelian example for which it holds is groups of rank 2 in characteristic p [38]. In this thesis, we will take steps towards proving the conjecture in general.

1.2 Results to date in characteristic p

It is often easier to work with the mod-p Iwasawa algebra kG than with the whole of $\mathcal{O}G$, and many more results have been established in this case.

The first steps towards a proof of Conjecture 1.1.3 were made by Ardakov, Zhang and Wei in [8], for G a group of Chevalley type, i.e. a uniform pro-p group whose Lie algebra has a Chevalley basis in the sense of [36, Theorem 1].

Recall that a one-sided ideal I of a ring A is *reflexive* if the canonical map $I \to \mathrm{Hom}_A(Hom_A(I, A), A)$ is an isomorphism, and if I is two-sided then I is *reflexive* if it is reflexive as a left and a right ideal.

Theorem 1.2.1 ([8, Corollary to Theorem A])**.** *Assume that $p \geq 5$ and let G be a uniform pro-p group of Chevalley type, where $p \nmid n+1$ whenever the root system of*

G contains an irreducible component of type A_n. Then 0 is the only reflexive prime ideal of kG.

It follows trivially that all reflexive prime ideals in kG are standard, which is weak evidence for Conjecture 1.1.3 in characteristic p for groups of Chevalley type.

Theorem 1.2.1 is the amongst the strongest results obtained to date for ideal classification in kG for G semisimple. The key technique in the proof was to define a canonical set of kG^p-linear derivations of kG, and to prove that they all fixed the reflexive ideal P. It follows that P is generated by a subset of kG^p, and applying induction gives that $P = 0$. This idea formed the inspiration for the idea of *Mahler expansions* which we will explore in Chapter 3.

The notion of Mahler expansions was fully developed by Ardakov in [1], focusing on the case where the group G is nilpotent, culminating in the proof of the strongest result obtained to date in characteristic p:

Theorem 1.2.2 ([1, Theorem A]). *Let G be a uniform, nilpotent group, and let P be a prime ideal in kG. Then P is standard.*

This verifies Conjecture 1.1.3 for nilpotent groups in characteristic p, and the first aim of this thesis is to extend this to more general classes of solvable groups.

1.3 Results to date in characteristic 0

Studying the Iwasawa algebra KG proves to be more difficult, and many of the techniques used in characteristic p fail to carry across. However, the structure of KG opens new avenues of exploration that are not available to us in characteristic p.

First, recall that if R is a ring and M is a (left) R-module, we define the *annihilator* of M to be the set $\mathrm{Ann}_R M := \{r \in R : rM = 0\}$. This is a two-sided ideal of R.

Definition 1.3.1. *Let R be a K-algebra. We say that a two sided ideal P of R is:*

- Primitive *if $P = \mathrm{Ann}_R(M)$ for some irreducible R-module M.*
- Weakly rational *if the centre $Z(R/P)$ is an algebraic field extension of K.*

It is well known that all primitive ideals are prime, and it is easy to show that all maximal ideals are primitive.

For G uniform, the algebras $\mathcal{O}G$ and kG are scalar local, i.e. they both contain a unique primitive (and maximal) ideal, so studying primitive ideals is trivial. On the other hand, KG has a rich primitive ideal structure, which we can focus on classifying. It follows from [18, Theorem 1.1(1)] that every primitive ideal in KG is weakly rational.

A useful technique we can exploit when studying KG is *Lie theory.* If we let $\mathcal{L}$ be the associated $\mathbb{Z}_p$-Lie algebra of G, then it follows from [6, Theorem 10.4] that the *affinoid enveloping algebra* $\widehat{U(\mathcal{L})}_K := \varprojlim_{n\to\infty} U(\mathcal{L})/p^nU(\mathcal{L}) \otimes_{\mathbb{Z}_p} K$ arises as a topological completion of KG. Using this result, we can use the representation theory of $\mathcal{L}$ to explore the primitive ideal structure of KG.

This notion is particularly useful when we consider G to be split-semisimple, in which case $\mathcal{L}$ has a root space decomposition, and there is a well developed theory of Verma modules and simple highest weight modules (see e.g. [20] for details). This was generalised to an analogous representation theory of $\widehat{U(\mathcal{L})}_K$ by Ardakov and Wadsley in [7]:

Theorem 1.3.2 ([7, Theorem 5.4])**.** *Let $\mathbb{G}$ be a connected, simply connected, split-semisimple affine algebraic group over $\mathbb{Z}_p$, let G be an open uniform subgroup of $\mathbb{G}(\mathbb{Z}_p)$, and let $\mathcal{L}$ be the $\mathcal{O}$-Lie algebra of G, with Cartan subalgebra $\mathcal{T}$. Then for any*

$\mathcal{O}$-linear character $\lambda : \mathcal{T} \to \mathcal{O}$, the affinoid Verma module $\widehat{M(\lambda)}_K$ *is faithful over KG.*

Here, $\widehat{M(\lambda)}$ is a p-adic completion of the classical Verma module. This is a $\widehat{U(\mathcal{L})}_K$-module, and as in the classical case it gives rise to a unique simple quotient $\widehat{L(\lambda)}_K$.

It is believed that $\widehat{M(\lambda)}_K$ can be replaced by $\widehat{L(\lambda)}_K$ in the statement of the previous theorem, whenever $\widehat{L(\lambda)}_K$ is infinite dimensional, and this would help towards proving Conjecture 1.1.3 for semisimple groups in characteristic 0, by describing all primitive ideals in KG in terms of annihilators of $\widehat{L(\lambda)}_K$.

It should follow from the work of Stanciu in [35] that all primitive ideals in $\widehat{U(\mathcal{L})}_K$ arise as annihilators of simple highest weight modules $\widehat{L(\lambda)}_K$, so using the dense embedding of KG into $\widehat{U(\mathcal{L})}_K$ together with faithfulness of $\widehat{L(\lambda)}_K$ over KG, it should follow that 0 is the only primitive ideal in KG of infinite codimension, i.e. all prime ideals of infinite codimension are standard.

Remark: The completion $\widehat{U(\mathcal{L})}_K$ is not faithfully flat over KG, so in general it is difficult to translate information about the ideal structure of $\widehat{U(\mathcal{L})}_K$ to KG. However, in [6, Section 10], an inverse system of Noetherian Banach algebras D_{p^n} is constructed, each of which arise as a completion of KG, can be realised as a finite crossed product of the affinoid enveloping algebra, and the limit of this system is the distribution algebra $D(G, K)$, which is faithfully flat over KG.

However, we are instead concerned with solvable groups, where the theory of highest weight modules does not carry over. The second aim of this thesis is to generalise this theory to the solvable case, and to this end we will follow the approach of Dixmier in [16] and construct a class of irreducible, induced representations of $\widehat{U(\mathcal{L})}_K$ which characterise primitive ideals, and study the action of KG on these representations.

1.4 Main Theorems

Theorem 1.2.2 establishes Conjecture 1.1.3 in characteristic p in the case where G is nilpotent, and our first main result generalises this to another class of solvable groups:

Given a uniform group G, we say that G is *abelian-by-procyclic* if it is isomorphic to a semidirect product $\mathbb{Z}_p^d \rtimes \mathbb{Z}_p$ for some $d \in \mathbb{N}$. Clearly abelian-by-procyclic groups are solvable.

Theorem A. *Let G be a uniform, abelian-by-procyclic group. Then every prime ideal of kG is standard.*

This result has already been established for groups of the form $\mathbb{Z}_p \rtimes \mathbb{Z}_p$ in [38, Theorem 7.1]. Using [31], we see that all solvable uniform groups of rank 3 or less are abelian-by-procyclic, so Theorem A verifies Conjecture 1.1.3 in characteristic p for all solvable groups of rank 3 or less.

Our second main result is the key step in the proof of Theorem A, and this result is a *control theorem*:

Definition 1.4.2. *Let $R = \mathcal{O}, k$ or K, and I be a right ideal of RG:*

1. Define $I^\dagger := \{g \in G : g - 1 \in I\}$, a closed subgroup of G, and if I is two sided it is normal in G. We say I is faithful *if $I^\dagger = 1$, i.e. if the natural map $G \to RG/I, g \mapsto g + I$ is injective.*

2. We say that a closed subgroup $H \leq_c G$ controls *I if $I = (I \cap RH)RG$. Define the* controller subgroup *of I by $I^\chi := \bigcap\{U \leq_o G : U$ controls $I\}$.*

It follows from [3, Theorem A] that a closed subgroup H of G controls an ideal $I \trianglelefteq RG$ if and only if $I^\chi \subseteq H$, so in particular I^χ controls I.

To prove that a prime ideal P in RG is standard, our usual approach is to quotient out by $P^\dagger$ to obtain a faithful prime ideal of RG_0 where $G_0 = \frac{G}{P^\dagger}$, and then to prove that this ideal is centrally generated, i.e. controlled by $Z(G)$. Thus the important step is to prove that faithful, prime ideals in RG are controlled by $Z(G)$.

Note: This is particularly useful in characteristic p, where $P^\dagger$ is always isolated by [1, Lemma 5.3(c)], and thus G_0 remains torsionfree.

The usual approach is to prove that under the right conditions, P is controlled by a proper subgroup of G, and apply an inductive argument. However, for this argument to work, we need to work in greater generality than uniform groups, because these can often have closed subgroups that are *not* uniform. So for our main control theorem, we will assume instead that G is *p-valuable* in the sense of [25, III 2.1.2]:

Theorem B. *Let G be a non-abelian, p-valuable, abelian-by-procyclic group, and let P be a faithful, prime ideal of kG. Then P is controlled by a proper, open subgroup of G.*

This result is actually all that is required to prove Theorem A, and we will give the proof of both these results at the end of Chapter 4. The main theory we need for the proof of Theorem B is the theory of Mahler expansions from [1], which we will recap in Chapter 3, and in Chapter 4 we will explore how to overcome the difficulties with applying this theory in the non-nilpotent setting, using non-commutative valuation theory and Lie theory. Most of this work has already been published in [23].

Unfortunately, the theory of Mahler expansions is not very useful in characteristic 0, even if we assume that the group is nilpotent. Our third main result is the strongest we can obtain for ideals in KG using Mahler expansions alone, and we will prove this in Chapter 3.

Theorem C. *Let G be a uniform group, and let P be a faithful, weakly rational ideal of KG. Then P is controlled by $C_G(Z_2(G))$, the centraliser of $Z_2(G)$ in G.*

Here $Z_2(G)$ is the *second centre* of G, i.e. the preimage of the centre of $\frac{G}{Z(G)}$ in G. This theorem is not useful if $Z_2(G) = Z(G)$, in which case $G = C_G(Z_2(G))$ and the result is trivial. However, there are large classes of solvable groups for which $C_G(Z_2(G))$ is a proper subgroup, in particular all nilpotent groups. In fact, if G has nilpotency class 2, i.e. $(G,(G,G)) = 1$, then $C_G(Z_2(G)) = Z(G)$, so it follows from Theorem C that all faithful primitive ideals in KG are standard.

In Chapter 5, we will take an interlude from Iwasawa algebras, and focus solely on the affinoid enveloping algebra $\widehat{U(\mathcal{L})}_K$ of the associated $\mathbb{Z}_p$-Lie algebra $\mathcal{L}$ of G. Following the approaches in [16], for each $\mathcal{O}$-linear form λ of $\mathcal{L}$, we will define a representation of $\widehat{U(\mathcal{L})}_K$ known as a *Dixmier module*, denoted $\widehat{D(\lambda)}$, which is essentially a generalisation of the affinoid Verma module $\widehat{M(\lambda)}$ for a semisimple Lie algebra. We will prove that this is irreducible, and define the corresponding *Dixmier annihilator* to be the primitive ideal $I(\lambda) := \mathrm{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}$.

Our fourth main result characterises weakly rational ideals of $\widehat{U(\mathcal{L})}_K$ in terms of Dixmier annihilators:

Theorem D. *Let $\mathfrak{g}$ be a finite dimensional, nilpotent K-Lie algebra, and let $\mathcal{L}$ be an $\mathcal{O}$-Lie lattice in $\mathfrak{g}$. Then if P is a weakly rational ideal of $\widehat{U(\mathcal{L})}_K$, there exists $n \in \mathbb{N}$ such that $P \cap \widehat{U(p^n\mathcal{L})}_K$ is a Dixmier annihilator.*

It follows from Theorem D that all primitive ideals in KG can be described in terms of Dixmier annihilators, from which we deduce the following useful corollary:

Corollary 1.4.5. *Let G be a nilpotent, uniform pro-p group such that every faithful Dixmier annihilator in KG is controlled by $Z(G)$. Then any faithful, primitive ideal in KG is maximal and standard.*

We will prove this corollary in Chapter 6. The essential idea of the proof is to show that the annihilator inside KG of a Dixmier module is centrally generated, and therefore standard, and use Theorem D to obtain the general case.

Theorem E. *Let G be a nilpotent, abelian-by-procyclic uniform pro-p group, for $p > 2$. Then all primitive ideals of KG are maximal and virtually standard.*

We will prove this result at the end of Chapter 6, and this will establish a version of Conjecture 1.1.3 in characteristic 0 for nilpotent, abelian-by-procyclic groups, specifically a version concerning primitive ideals as opposed to general prime ideals.

To prove Theorem E, the idea is we show that all Dixmier annihilators in KG are centrally generated, and apply Corollary 1.4.5. To this end, we will study the action of the abelian normal subgroup $C_G(Z_2(G))$ on a Dixmier module, since the kernel of this action completely determines the KG-Dixmier annihilator using Theorem C.

Chapter 2

Preliminaries

Notation: We write $H \leq_o G, H \leq_c G$ to mean that H is respectively an open subgroup and a closed subgroup of G. Also, we write $H \trianglelefteq_c^i G$ to mean H is a closed, isolated normal subgroup of G, i.e. $\frac{G}{H}$ is torsionfree. We write $(\cdot,\cdot)$ to denote the group commutator.

2.1 Uniform and p-valuable groups

A *uniform pro-p group* G is a torsionfree, finitely topologically generated pro-p group such that $(G,G) \subseteq G^p$ (or G^4 if $p=2$). Note that any uniform pro-p group is a compact p-adic Lie group, in fact it follows from [17, Theorem 8.32] that a topological group G is a compact p-adic Lie group if and only if it is profinite and contains an open, uniform subgroup.

However, it is possible for a uniform group G to contain subgroups, even open subgroups, which are not uniform, so we need to work in greater generality.

A *p-valuation* on a group G is a map $\omega : G \to \mathbb{R} \cup \{\infty\}$ such that for all $g,h \in G$:

- $\omega(g^{-1}h) \geq \min\{\omega(g), \omega(h)\}$.
- $\omega((g,h)) \geq \omega(g) + \omega(h)$.
- $\omega(g) = \infty$ if and only if $g = 1$.

- $\omega(g^p) = \omega(g) + 1$.
- $\omega(g) > \frac{1}{p-1}$.

And (G, ω) is p-saturated if $\omega(g) > 1 + \frac{1}{p-1}$ if and only if $g = h^p$ for some $h \in G$. In Appendix B.1, we describe the notion of p-valuations in full, and we will recap it only briefly here.

Throughout this thesis, we will usually take G to be a group carrying a complete p-valuation ω such that (G, ω) has a finite *ordered basis* $\underline{g} = \{g_1, \cdots, g_d\}$ such that for every $g \in G$, $g = \underline{g}^\alpha := g_1^{\alpha_1} \cdots g_d^{\alpha_d}$ for some unique $\alpha_i \in \mathbb{Z}_p$, and we call d the rank of G. In this case, we say that G is *p-valuable*, and note that $G \cong \mathbb{Z}_p^d$ as topological spaces (and as groups if G is abelian).

The important property of the $\mathcal{O}$-Iwasawa algebra of a p-valuable group G, is that we have an isomorphism of $\mathcal{O}$-modules between $\mathcal{O}G$ and the power series ring $\mathcal{O}[[b_1, \cdots, b_d]]$, where each g_i is sent to $b_i + 1$.

Examples: 1. A uniform pro-p group carries a p-valuation ω where for all $g \in G$, $\max\{n \in \mathbb{N} : g \in G^{p^n}\} = \omega(g) - 1$ (or $\omega(g) - 2$ if $p = 2$). Then (G, ω) is p-valuable, and in fact p-saturated.

2. Given a p-valuable group (G, ω), and an automorphism $\varphi \in \text{Aut}(G)$, we say that φ is *ω-bounded* if $\{\omega(\varphi(g)g^{-1}) - \omega(g) : g \in G\}$ is bounded below, and we define the *degree* of φ to be $\deg(\varphi) = \inf\{\omega(\varphi(g)g^{-1}) - \omega(g) : g \in G\} \in \mathbb{R}$.

Following [1, Definition 4.5], we define $\text{Aut}^\omega(G) := \{\varphi \in \text{Aut}^\omega(G) : deg(\varphi) > \frac{1}{p-1}\}$, and it follows from [1, Corollary 4.5] that $\text{Aut}^\omega(G)$ is a subgroup of $\text{Aut}(G)$, and in fact if G is p-saturated then $\text{Aut}^\omega(G)$ is a p-saturated group, with p-valuation deg.

2.2 Abelian-by-procyclic Groups

Given an abstract group H, we say that H is *metabelian* if the commutator subgroup (H, H) is abelian. Clearly if H is metabelian then H is solvable. In this thesis, we will be particularly focused on studying the Iwasawa algebras for a particular class of metabelian p-valuable groups.

Definition 2.2.1. *Let H be a group, $n \in \mathbb{N}$. Define the* n'th centre $Z_n(H)$ *of H inductively by $Z_0(H) := 1$, and for $n > 0$, $Z_n(H) := \{h \in H : (h, H) \subseteq Z_{i-1}(H)\}$. We define the* upper central series *of H to be the ascending chain of subgroups:*

$$1 = Z_0(H) \subseteq Z_1(H) \subseteq Z_2(H) \subseteq \cdots$$

We say that H is nilpotent *if the upper central series terminates at H, i.e. $Z_n(H) = H$ for some $n \in \mathbb{N}$. The* nilpotency class *of H is the smallest integer n such that $Z_n(H) = H$.*

Note that each $Z_n(H)$ is a normal subgroup of H, and $Z_1(H) = Z(H)$. By definition, the second centre $Z_2(H)$ of H is equal to $\{h \in H : (g, H) \subseteq Z(H)\}$, and we define the *centraliser of the second centre* to be:

$$C_H(Z_2(H)) := \{h \in H : (h, Z_2(H)) = 1\}. \tag{2.1}$$

Observe the following properties of $C_G(Z_2(G))$ for G a p-valuable group:

- $C_G(Z_2(G))$ is a closed, isolated normal subgroup of G.
- $C_G(Z_2(G))$ is a proper subgroup of G precisely when $Z_2(G) \neq Z(G)$, e.g. if G is nilpotent.
- If G has nilpotency class 2 then $C_G(Z_2(G)) = Z(G)$.
- If G is nilpotent then $(G, G) \subseteq C_G(Z_2(G))$.

Lemma 2.2.2. *For G any p-valuable group, $U \leq_o G$, $C_U(Z_2(U)) = C_G(Z_2(G)) \cap U$.*

Proof. Firstly, we will show that $Z_2(U) \subseteq Z_2(G)$:

If $g \in Z_2(U)$ then $(g, U) \subseteq Z(U)$ by definition, and $Z(U) \subseteq Z(G)$ by [1, Lemma 8.4(b)], so $(g, U) \subseteq Z(G)$. But we know that $G^{p^n} \subseteq U$ for some $n \in \mathbb{N}$, so for all $h \in G$, $(g, h^{p^n}) \in Z(G)$.

But $Z(G)$ is an isolated normal subgroup of G by [1, Lemma 8.4(a)], so by [25, IV.3.4.2], $\frac{G}{Z(G)}$ carries a p-valuation $\overline{\omega}$. Since $(g, h^{p^n}) \in Z(G)$, we have that $\overline{\omega}((g, h^{p^n})) = \infty$, but $\overline{\omega}((g, h^{p^n})) = \overline{\omega}(g, h) + n$ by [32, Proposition 25.1], and hence $\overline{\omega}((g, h)) = \infty$. Thus $(g, h) \in Z(G)$ for all $h \in G$, and $g \in Z_2(G)$ as required.

Now, given $g \in C_U(Z_2(U))$, $(g, Z_2(U)) = 1$, so if $h \in Z_2(G)$ then $h^{p^n} \in Z_2(G) \cap U = Z_2(U)$, so $(g, h^{p^n}) = 1$. So since G is p-valued, applying [32, Proposition 25.1] again gives that $(g, h) = 1$, thus $g \in C_G(Z_2(G))$ and $C_U(Z_2(U)) \subseteq C_G(Z_2(G))$. Conversely, if $g \in C_G(Z_2(G)) \cap U$ then $(g, Z_2(G)) = 1$, so since $Z_2(U) \subseteq Z_2(G)$, $(g, Z_2(U)) = 1$ and $g \in C_U(Z_2(U))$ as required. $\square$

Now, the class of metabelian groups we will be particularly interested in in characteristic p are *abelian-by-procyclic groups*:

Definition 2.2.3. *A compact p-adic Lie group G is* abelian-by-procyclic *if there exists $H \trianglelefteq_c^i G$ such that H is abelian and $\frac{G}{H} \cong \mathbb{Z}_p$. For G non-abelian, H is unique, and we call it the* principal subgroup *of G. It follows easily that $Z(G) \subseteq H$ and $(G, G) \subseteq H$, i.e. G is metabelian.*

Note that if G is abelian-by-procyclic and nilpotent, with principal subgroup H, then it is straightforward to see that $H = C_G(Z_2(G))$.

If G is non-abelian, p-valuable abelian-by-procyclic with principal subgroup H, then using Lemma B.1.2 we can choose an ordered basis $\{h_1, \cdots, h_d, X\}$ for G such that

$\{h_1, \cdots, h_d\}$ is an ordered basis for H, and it follows that $G \cong H \rtimes \mathbb{Z}_p$. So since H is abelian, $G \cong \mathbb{Z}_p^d \rtimes \mathbb{Z}_p$.

Moreover, as described above, we can choose an element $X \in G\backslash H$ such that for every $g \in G$, $g = hX^\alpha$ for some unique $h \in H$, $\alpha \in \mathbb{Z}_p$. We call X a *procyclic element* in G.

Lemma 2.2.4. *Let G be a non-abelian p-valuable abelian-by-procyclic group with principal subgroup H, and let C be a closed subgroup of G. Then C is abelian-by-procyclic. Furthermore, if N is a closed, isolated normal subgroup of G, then $\frac{G}{N}$ is p-valuable, abelian-by-procyclic.*

Proof. Let $H' := C \cap H$, then clearly H' is a closed, isolated, abelian normal subgroup of C. Now, $\frac{C}{H'} = \frac{C}{H \cap C} \cong \frac{CH}{H} \leq \frac{G}{H} \cong \mathbb{Z}_p$, hence $\frac{C}{H'}$ is isomorphic to a closed subgroup of $\mathbb{Z}_p$, i.e. it is isomorphic to either 0 or $\mathbb{Z}_p$.

If $\frac{C}{H'} = 0$ then $C = H' = H \cap C$ so $C \subseteq H$. Hence C is abelian, and therefore abelian-by-procyclic. If $\frac{C}{H'} \cong \mathbb{Z}_p$ then C is abelian-by-procyclic by definition.

For any $N \trianglelefteq_o^i G$, it follows from [25, IV.3.4.2] that $\frac{G}{N}$ is p-valuable, and clearly $\frac{HN}{N}$ is a closed, abelian normal subgroup of $\frac{G}{N}$.

Consider the natural surjection of groups $\frac{G}{H} \twoheadrightarrow \frac{G}{HN} \cong \frac{G/N}{HN/N}$:

If the kernel of this map is zero, then it is an isomorphism, so $\frac{G/N}{HN/N} \cong \frac{G}{H} \cong \mathbb{Z}_p$, so $\frac{G}{N}$ is abelian-by-procyclic by definition.

If the kernel is non-zero, then $\frac{G}{HN}$ is a non-trivial quotient of $\frac{G}{H} \cong \mathbb{Z}_p$, and hence is finite, giving that HN is open in G. So $\frac{HN}{N}$ is abelian and open in $\frac{G}{N}$, and it follows that $\frac{G}{N}$ is abelian, and hence abelian-by-procyclic. $\square$

2.3 Non-commutative valuations

Let R be a ring, throughout this thesis, we assume that all rings are unital. In Appendix A.2, we define a *ring filtration* on R to be a map $w : R \to \mathbb{R} \cup \{\infty\}$ such that for all $x, y \in R$:

- $w(x+y) \geq \min\{w(x), w(y)\}$,
- $w(xy) \geq w(x) + w(y)$,
- $w(1) = 0$ and $w(0) = \infty$.

Also recall from Appendix A.2 the definition of a *Zariskian filtration* w on R, and a *w-regular element* $x \in R$. Also recall that w is a *valuation* if every element of R is w-regular, i.e. $w(xy) = w(x) + w(y)$ for all $x, y \in R$. We will now see some examples of filtrations which we use throughout:

Examples: 1. If I is a two-sided ideal of R, the *I-adic* filtration on R is given by $F_0R = R$ and $F_nR = I^n$ for all $n > 0$. If $I = \pi R$ for some normal element $\pi \in R$, we call this the *π-adic filtration*.

2. If R carries a filtration w, then w extends naturally to $M_k(R)$ via $w((a_{i,j})) = \min\{w(a_{i,j}) : i, j = 1, \cdots, k\}$ – the *standard matrix filtration*.

3. If R carries a filtration w and $I \trianglelefteq R$, we define the *quotient filtration* $\overline{w} : R/I \to \mathbb{Z} \cup \{\infty\}, r + I \mapsto \sup\{w(r+y) : y \in I\}$. In this case, $\text{gr}_{\overline{w}}\ R/I = \text{gr}\ R/\text{gr}\ I$.

4. Let (G, ω) be a p-valuable group with ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$. We say that ω is an *abelian* p-valuation if $\omega(G) \subseteq \frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{Z}$, and $\omega((g,h)) > \omega(g) + \omega(h)$ for all $g, h \in G$.

If ω is abelian, then using [34, section 4] and the natural $\mathcal{O}$-module isomorphism $\mathcal{O}G \cong \mathcal{O}[[b_1, \cdots, b_d]]$, we see that $\mathcal{O}G$ carries a complete, Zariskian filtration w given by:

$$w(\sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha b_1^{\alpha_1} \cdots b_d^{\alpha_d}) = \inf\{v_p(\lambda_\alpha) + \sum_{i \leq d} n\alpha_i \omega(g_i) : \alpha \in \mathbb{N}^d\}.$$

Note that this is not an integer valued filtration, but it takes values in $\frac{1}{e}\mathbb{Z}$, where e is the ramification index of K. So we may replace w by ew to obtain an equivalent, integer valued filtration, which can be given explicitly by:

$$w(\sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha b_1^{\alpha_1} \cdots b_d^{\alpha_d}) = \inf\{v_\pi(\lambda_\alpha) + \sum_{i \leq d} en\alpha_i \omega(g_i) : \alpha \in \mathbb{N}^d\}.$$

Furthermore, using [34, Lemma 4.3], we see that the associated graded ring is isomorphic to $k[t, t_1, \cdots, k_d]$, where $k := \mathcal{O}/\pi\mathcal{O}$ is the residue field, $t_i = \text{gr}(b_i) = \text{gr}(g_i - 1)$ and $t^e \in k\text{gr}(p)$. We call w the *Lazard filtration* on $\mathcal{O}G$.

5. Again, let (G, ω) be a p-valuable group, and assume that ω is abelian. It follows that the algebras $kG = \mathcal{O}G/(\pi)$ and $KG = \mathcal{O}G \otimes_{\mathcal{O}} K$ inherit filtrations w_k and w_K from $\mathcal{O}G$, where w_k is the quotient filtration and $w_K(r \otimes \pi^{-k}) = w(r) - k$. We also call these *Lazard filtrations*, and their associated graded rings have the form:

$$\text{gr}_{w_k}\, kG \cong k[X_1, \cdots, X_d].$$
$$\text{gr}_{w_K}\, KG \cong k[t, t^{-1}, X_1, \cdots, X_d].$$

Note that kG and $\mathcal{O}G$ are complete with respect to their respective Lazard filtrations, but KG is not. In fact, we will see later that its completion will have the form of a p-adically complete enveloping algebra.

Now, recall that a Noetherian domain D is a *non-commutative discrete valuation ring (DVR)* if for any element $x \in Q(D)$, either $x \in D$ or $x^{-1} \in D$ (here $Q(D)$ is the usual ring of quotients of D). Note that in this case, D is local, and every ideal of D is two-sided of the form $J(D)^n = \mu^n D$ for any $\mu \in J(D) \backslash J(D)^2$.

Lemma 2.3.1. *Let D be a non-commutative DVR, let v' be the $J(D)$-adic filtration on $Q(D)$, and let $R := M_n(D)$ for some $n \in \mathbb{N}$. Let v be the standard matrix filtration on $Q(R) = M_n(Q(D))$ corresponding to the v', then given $r \in R$, r is v-regular if and only if r is normal in R, i.e. $rR = Rr$.*

Proof. Firstly, if r is normal then $rR = Rr$ is a two-sided ideal of R, so $rR = M_n(J(D)^m)$ for some $m \in \mathbb{N}$, and $v(r) = m$. Suppose first that $m = 0$, i.e. $rR = R$, and hence r is a unit in R, and $v(r^{-1}) = 0$. Thus for any $y \in Q(R)$, $v(y) = v(r^{-1}ry) \geq v(r^{-1}) + v(ry) = v(ry) \geq v(y)$, forcing equality. Thus $v(ry) = v(y) = v(r) + v(y)$, and similarly $v(yr) = v(y) + v(r)$ and r is v-regular.

More generally, if $v(r) = m$ then $r = \mu^m s$ for some $s \in R$ with $v(s) = 0$, so since $rR = \mu^m R$ it follows that $sR = R$ and hence s is v-regular by the above, and it follows that r is v-regular as required.

Conversely, if $r \in R$ is v-regular, then it is not a zero divisor in $Q(R)$, and hence it is a unit since $Q(R)$ is Artinian. If $v(r) = m$ then $r = \mu^m s$ for some $s \in R$ v-regular with $v(s) = 0$. Thus $v(s^{-1}) = -v(s) = 0$, so s is a unit in R, i.e. $sR = Rs = R$. Therefore $r = \mu^m s$ is normal in R. □

This lemma motivates the following definition:

Definition 2.3.2. *Let Q be a simple, Artinian ring. A* non-commutative valuation *on Q is a Zariskian filtration $v : Q \to \mathbb{Z} \cup \{\infty\}$ such that if $\widehat{Q}$ is the completion of Q with respect to v, then $\widehat{Q} \cong M_k(Q(D))$ for some complete non-commutative DVR D, and v is induced by the matrix filtration induced by the natural $J(D)$-adic filtration.*

2.4 Crossed Products

Given a ring R and a group G, recall from [29, Definition 1.5.8] that a *crossed product* of R with G, denoted $R * G$, is a ring extension $R \subseteq S$, free as a left R-module with basis $\{\bar{g} : g \in G\} \subseteq S^\times$ in bijection with G such that for each $g, h \in G$:

- $\bar{g}R = R\bar{g}$ and

- $\bar{g}R\bar{h}R = \overline{gh}R$.

Also, recall from [30, Definition 1.1] that for any crossed product $S = R*G$ there is a group homomorphism $\sigma : G \to \mathrm{Aut}(R)$, known as the *action*, and a map $\tau : G\times G \to R^\times$ known as the *twist*, such that for all $g, h \in G$, $r \in R$:

- $\bar{g}r = \sigma(g)(r)\bar{g}$ and

- $\overline{g}\overline{h} = \tau(g, h)\overline{gh}$.

The crossed product is completely determined by its action and twist, in other words for any homomorphism $\sigma : G \to \mathrm{Aut}(R)$ and any map $\tau : G \times G \to R^\times$ satisfying the conditions of [30, Lemma 1.1], there is a crossed product $R*G$ with action σ and twist τ.

We prove some basic, algebraic properties of crossed products in Appendix A.1, but the following result is particularly essential:

Proposition 2.4.1. *Let R be a ring with a complete, positive, Zariskian valuation $w : R \to \mathbb{N} \cup \{\infty\}$, let F be a finite group, and let $S = R * F$ be a crossed product with action σ and twist γ. Suppose that $w(\sigma(g)(r)) = w(r)$ for all $g \in F$, $r \in R$.*

Then w extends to a complete, positive, Zariskian filtration $w' : S \to \mathbb{N}\cup\{\infty\}$ defined by $w'(\sum_{g\in F} r_g\overline{g}) = \min\{w(r_g) : g \in G\}$, and $gr_{w'}\ S \cong (gr_w\ R) * F$.

Proof. From the definition it is clear that $w'(s_1 + s_2) \geq \min\{w'(s_1), w'(s_2)\}$. So to prove that w defines a ring filtration, it remains to check that $w'(s_1s_2) \geq w'(s_1) + w'(s_2)$.

In fact, using the additive property, we only need to prove that $w'(r\overline{g}s\overline{h}) \geq w'(r\overline{g}) + w'(s\overline{h})$ for all $r, s \in R$, $g, h \in F$. We will in fact show that equality holds here:

$w'(r\overline{g}s\overline{h}) = w'(r\sigma(g)(s)\overline{g}\overline{h}) = w'(r\sigma(g)(s)\gamma(g,h)\overline{gh}) = w(r\sigma(g)(s)\gamma(g,h))$

$= w(r) + w(\sigma(g)(s)) + w(\gamma(g,h))$ (since w is a valuation)

$= w(r) + w(s)$.

The last equality follows because $w(\sigma(g)(s)) = w(s)$ by assumption, and since R is positively filtered and $\gamma(g,h)$ is a unit in R, it must have value zero. Clearly $w(r) + w(s) = w'(r\overline{g}) + w'(s\overline{h})$ so we are done.

Hence w' is a well-defined ring filtration, clearly $w'(r) = w(r)$ for all $r \in R$, and $w'(\overline{g}) = 0$ for all $g \in F$. We can define $\theta : \text{gr}_w\ R \to \text{gr}_{w'}\ S, r + F_{n+1}R \mapsto r + F_{n+1}S$, which is a well defined, injective ring homomorphism.

Given $s \in S$, $s = \sum\limits_{g \in F} r_g\overline{g}$, so let $A_s := \{g \in F : w(r_g) = w'(s)\}$. Then:

$$\text{gr}(s) = \sum_{g \in A_s} r_g\overline{g} + F_{w'(s)+1}S = \sum_{g \in A_s}(r_g + F_{w'(s)+1}S)(\overline{g} + F_1 S) = \sum_{g \in A_s}\theta(\text{gr}(r_g))\text{gr}(\overline{g}).$$

Hence $\text{gr}_{w'}\ S$ is finitely generated over $\theta(\text{gr}\ R)$ by $\{\text{gr}(\overline{g}) : g \in F\}$. This set forms a basis, hence $\text{gr}_{w'}\ S$ is free over $\theta(\text{gr}\ R)$, and it is clear that each $\text{gr}(\overline{g})$ is a unit in gr S, and they are in bijection with the elements of F.

Therefore gr S is Noetherian, and clearly $R * F$ is complete with respect to w'. Hence w' is Zariskian by [28, Chapter II, Theorem 2.1.2].

Finally, gr$(r\overline{g})$gr$(s\overline{h})$ =gr$(r\overline{g}s\overline{h})$ since we have shown that $w'(r\overline{g}s\overline{h}) = w'(r\overline{g}) + w'(s\overline{h})$, so it is readily checked that (gr R)gr$(\overline{g})$gr$(\overline{h})$ = ((gr R)(gr$(\overline{g})$)((gr R)(gr$(\overline{h})$), and clearly (gr R)(gr$(\overline{g})$) = (gr$(\overline{g})$)(gr R).

Therefore $\text{gr}_{w'}\ S = (\text{gr}_w\ R) * F$. □

2.5 The $C(G, \mathcal{O})$-action

Fix G a compact p-adic Lie group, and define:

$$C(G, \mathcal{O}) := \{f : G \to \mathcal{O} : f \text{ continuous}\}.$$

Then $C(G, \mathcal{O})$ is an $\mathcal{O}$-algebra with pointwise addition and multiplication. Also, for each $n \in \mathbb{N}$, define:

$$C_n = C(G, \tfrac{\mathcal{O}}{\pi^n\mathcal{O}}) := \{f : G \to \tfrac{\mathcal{O}}{\pi^n\mathcal{O}} : f \text{ locally constant}\}.$$

Then each C_n is an $\mathcal{O}$-algebra, and there exists a surjective map:

$$c_{n+1,n} : C_{n+1} \to C_n, f \mapsto (h : G \to \tfrac{\mathcal{O}}{\pi^n\mathcal{O}}, g \mapsto f(g) + \pi^n\mathcal{O}).$$

Furthermore, there exists a surjective map

$$c_n : C(G, \mathcal{O}) \to C_n, f \mapsto (h : G \to \tfrac{\mathcal{O}}{\pi^n\mathcal{O}}, g \mapsto f(g) + \pi^n\mathcal{O}),$$

and clearly $c_{n+1,n} \circ c_{n+1} = c_n$ for all n.

Lemma 2.5.1. *$C(G, \mathcal{O}) = \varprojlim_{n\in\mathbb{N}} C_n$ in the category of $\mathcal{O}$-algebras.*

Proof. Firstly, note that $C(G, \mathcal{O})$ is π-adically complete, and thus

$$C(G, \mathcal{O}) = \varprojlim_{n\in\mathbb{N}} C(G, \mathcal{O})/\pi^n C(G, \mathcal{O}). \tag{2.2}$$

Consider the morphism of $\mathcal{O}$-algebras:

$$\Theta : C(G, \mathcal{O}) \to C_n, f \mapsto (f' : G \to \tfrac{\mathcal{O}}{\pi^n\mathcal{O}}, g \mapsto f(g) + \pi^n\mathcal{O})$$

It is clear that this map is surjective, and if $\Theta(f) = 0$ then $f(g) \in \pi^n\mathcal{O}$ for all $g \in G$, so $f \in \pi^n C(G,\mathcal{O})$. Thus $\ker(\Theta) = \pi^n C(G,\mathcal{O})$, so $C(G,\mathcal{O})/\pi^n C(G,\mathcal{O}) \cong C_n$, and the result follows from (2.2). $\square$

For convenience, set $A_n := \frac{\mathcal{O}}{\pi^n\mathcal{O}}$ for each $n \in \mathbb{N}$, and note that $\mathcal{O}G/\pi^n\mathcal{O}G \cong A_nG$.

Recall from [3, Proposition 2.5] that there exists an action $\rho_n : C_n \to End_{\mathcal{O}}(A_nG)$ for each n such that if $U \leq_o G$ and $f \in C_n$ is constant on the cosets of U, and $r \in A_nG$ with $r = \sum_{g\in G//U} s_g g$ for some $s_g \in A_nU$, then $\rho_n(f)(r) = \sum_{g\in G//U} f(g)s_g g$. Here $G//U$ denotes a complete set of coset representatives for U in G.

Consider the canonical homomorphisms:

$$\nu_n : \mathrm{End}_{\mathcal{O}}(A_nG) \to \mathrm{Hom}(\mathcal{O}G, A_nG),$$

$$f \mapsto (g : \mathcal{O}G \to \mathcal{O}G/\pi^n\mathcal{O}G, r \mapsto f(r + \pi^n\mathcal{O}G)).$$

$$\mu_{n+1,n} : \mathrm{Hom}(\mathcal{O}G, A_{n+1}G) \to \mathrm{Hom}(\mathcal{O}G, A_nG),$$

$$f \mapsto (g : \mathcal{O}G \to \mathcal{O}G/\pi^n\mathcal{O}G, r \mapsto f(r) + \pi^n\mathcal{O}G).$$

These give rise to the following commutative diagram for each $n \in \mathbb{N}$:

$$\begin{array}{ccccc}
C_n & \xrightarrow{\rho_n} & \mathrm{End}_{\mathcal{O}}(A_nG) & \xrightarrow{\nu_n} & \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}G, A_nG) \\
{\scriptstyle c_{n+1,n}}\uparrow & & & & {\scriptstyle \mu_{n+1,n}}\uparrow \\
C_{n+1} & \xrightarrow{\rho_{n+1}} & \mathrm{End}_{\mathcal{O}}(A_{n+1}G) & \xrightarrow{\nu_{n+1}} & \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}G, A_{n+1}G)
\end{array}$$

Now, using a similar argument to Lemma 2.5.1, we get

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}G, \mathcal{O}G) = \varprojlim_{n\in\mathbb{N}} \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}G, A_nG)$$

so it follows that there is a unique map from $C(G,\mathcal{O}) = \varprojlim_{n\in\mathbb{N}} C_n$ to $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}G,\mathcal{O}G) = \mathrm{End}_{\mathcal{O}}\mathcal{O}G$ making the corresponding diagrams commute.

Definition 2.5.2. *Define* $\rho : C(G, \mathcal{O}) \to \mathrm{End}_{\mathcal{O}}(\mathcal{O}G)$ *to be the unique morphism defined above. We call this the* canonical action *of* $C(G, \mathcal{O})$ *on* $\mathcal{O}G$*.*

Note that for each $n \in \mathbb{N}$, $f \in C_n$, $g \in G$, $\rho_n(f)(g) = f(g)g$, and it follows that for each $f \in C(G, \mathcal{O})$, we still have that $\rho(f)(g) = f(g)g$. Also, note that if $f \in C(G, \mathcal{O})$ is locally constant, then $\rho(f)$ is the same as the endomorphism defined in [3, Proposition 2.5], and we have the following result:

Proposition 2.5.3 ([3, Proposition 2.8])**.** *For each* $U \leq_o G$*, let* $C_U := \{f \in C(G, \mathcal{O}) : f \text{ is constant on the cosets of } U\}$*. Then* C_U *is an* $\mathcal{O}$*-subalgebra of* $C(G, \mathcal{O})$*, and for any right ideal* I *of* $\mathcal{O}G$*,* I *is controlled by* U *if and only if* $\rho(C_U)(I) \subseteq I$*, i.e. if and only if for all* $g \in G$*,* $\rho(\delta_{Ug})(I) \subseteq I$*, where* δ_{Ug} *is the characteristic function of the coset* Ug*.*

Since the canonical action of $C(G, \mathcal{O})$ on $\mathcal{O}G$ preserves the ideal (π), it follows that it induces an action on kG which coincides with the action of $C^\infty(G, k)$ defined in [3], so the previous proposition still applies when $\mathcal{O}$ is replaced by k.

Now let us assume that G is p-valuable, and let $\underline{g} = \{g_1, \cdots, g_d\}$ be an ordered basis for G. For each $\alpha \in \mathbb{Z}_p^d$, define $i_{\underline{g}}^{(\alpha)} : G \to \mathcal{O}, \underline{g}^\beta \mapsto \binom{\beta}{\alpha}$, where $\binom{\beta}{\alpha} := \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_d}{\alpha_d}$.

Clearly $i_{\underline{g}}^{(\alpha)} \in C(G, \mathcal{O})$, so let $\partial_{\underline{g}}^{(\alpha)} := \rho(i_{\underline{g}}^{(\alpha)}) \in End_{\mathcal{O}}(\mathcal{O}G)$. We call this the α-*quantized divided power*. For each $i = 1, \cdots, d$, if e_i is the standard i'th basis vector, we let $\partial_i := \partial_{\underline{g}}^{(e_i)}$.

Proposition 2.5.4. *Suppose that* I *is a right ideal of* kG *and* $\partial_j(I) \subseteq I$ *for some* $j \in \{1, \cdots, d\}$*. Then* I *is controlled by a proper open subgroup of* G*.*

Proof. Recall from [1, Lemma 7.13] that if V is an open normal subgroup of G with ordered basis $\{g_1, \cdots, g_{s-1}, g_s^p, \cdots, g_r^p, g_{r+1}, \cdots, g_d\}$ for some $1 \leq s \leq r \leq d$, then for each $g \in G$, $\rho(\delta_{Vg})$ can be expressed as a polynomial in $\partial_s, \cdots, \partial_r$. So it follows from

Proposition 2.5.3 that if $\partial_i(I) \subseteq I$ for all $i = s, \cdots, r$, then I is controlled by V.

Since we know that $\partial_j(I) \subseteq I$, it remains to show that we can find a proper, open normal subgroup U of G with ordered basis $\{g_1, \cdots, g_{j-1}, g_j^p, g_{j+1}, \cdots, g_d\}$, and it will follow that U controls I.

Notation: Given d variables $x_1, \cdots, x_d$, we will write $\underline{x}$ to denote the set $\{x_1, \cdots, x_d\}$, $\underline{x}_{(j,p)}$ to denote the same set, but with x_j replaced by x_j^p, and $\underline{x}_{(j)}$ to denote the set with x_j removed altogether. We write $\underline{x}^\alpha$ to denote $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$.

Let U be the subgroup of G generated topologically by the set $\underline{g}_{j,p}$. It is clear that this subgroup contains G^p, and hence it is open in G. Let us suppose, for contradiction, that it contains an element $u = \underline{g}^\alpha$ where $\alpha \in \mathbb{Z}_p^d$ and $p \nmid \alpha_j$.

Recall from [17, Definition 1.8] that the *Frattini subgroup* $\phi(G)$ of G is defined as the intersection of all maximal open subgroups of G, and since G is a pro-p group, it follows from [17, Proposition 1.13] that $\phi(G)$ contains G^p and $[G, G]$.

Hence $\phi(G)$ is an open normal subgroup of G, and $\frac{G}{\phi(G)}$ is abelian.

Since $\underline{g}_{(j,p)}$ generates U, it is clear that $\underline{g\phi(G)}_{(p,j)}$ generates $\frac{U\phi(G)}{\phi(G)}$. Therefore $u\phi(G) = \underline{g\phi(G)}_{(j,p)}^\beta$ for some $\beta \in \mathbb{Z}_p^d$.

But we know that $u = \underline{g}^\alpha$, so $u\phi(G) = \underline{g\phi(G)}^\alpha$. So since $\frac{G}{\phi(G)}$ is abelian, it follows that $g_j^{\alpha_j - p\beta_j}\phi(G) = \underline{g\phi(G)}_j^\gamma$ for some $\gamma \in \mathbb{Z}_p^d$.

But since $p \nmid \alpha_j$, $\alpha_j - p\beta_j$ is a p-adic unit, and hence $g_j\phi(G) = \underline{g\phi(G)}_j^\delta$, where $\delta_i = \gamma_i(\alpha_i - p\beta_i)^{-1}$. Therefore $\frac{G}{\phi(G)}$ is generated by $\underline{g\phi(G)}_{(j)} = \frac{\underline{g}_{(j)}\phi(G)}{\phi(G)}$.

It follows from [17, Proposition 1.9] that G is generated by $\underline{g}_j$, which has size $d-1$, and this is a contradiction since the rank d of G is the minimal cardinality of a generating set. Therefore, every $u \in U$ has the form $\underline{g}_{j,p}^{\beta}$ for some $\beta \in \mathbb{Z}_p^d$, i.e $\{g_1, \cdots, g_j^p, \cdots, g_d\}$ is an ordered basis for U. □

Lemma 2.5.5. *For any ordered basis $\{g_1, \cdots, g_d\}$, $i = 1, \cdots, d$, ∂_i is a k-linear derivation of kG.*

Proof. The proof of the previous proposition shows that $\{g_1, \cdots, g_i^p, \cdots, g_d\}$ is an ordered basis for a proper, open normal subgroup U of G. Hence every element $r \in kG$ can be written as $\sum\limits_{0 \leq j < p} r_j g_i^j$ for some unique $r_j \in kU$, and $\partial_i(r) = \sum\limits_{0 \leq j < p} j r_j g_i^j$.

Since ∂_i is k-linear, to prove that it is a derivation it remains to prove that $\partial_i(sg_i^j tg_i^l) = sg_i^j \partial_i(tg_i^l) + \partial_i(sg_i^j)tg_i^l$ for all $s, t \in kU$, $0 \leq j, l < p$.

But $\partial_i(sg_i^j tg_i^l) = \partial_i((sg_i^j tg_i^{-j})g_i^{j+l}) = (j + l)(sg_i^j tg_i^{-j})g_i^{j+l} = jsg_i^j tg_i^l + lsg_i^j tg_i^l = sg_i^j \partial_i(tg_i^l) + \partial_i(sg_i^j)tg_i^l$ as required. □

2.6 Mahler Expansions

The theory of Mahler expansions of automorphisms $\varphi \in \mathrm{Aut}^{\omega}(G)$ was developed in [1, Section 6], purely in a characteristic p setting. We now extend this to the general case.

Fix G a p-valuable group, let A be a $\mathcal{O}$ algebra carrying a complete, separated, $\mathcal{O}$-linear filtration $v : A \to \mathbb{Z} \cup \{\infty\}$, and suppose there is a continuous map of $\mathcal{O}$-algebras $\tau : \mathcal{O}G \to A$. First we will need the following well-known result [25, III.1.2.4], known as *Mahler's Theorem*:

Theorem 2.6.1. *Let M be a complete, topological $\mathbb{Z}_p$-module, let $d \in \mathbb{N}$, and let $f : \mathbb{Z}_p^d \to M$ be any continuous map. Then for each $\alpha \in \mathbb{N}^d$, there exist coefficients $m_\alpha(f) \in M$ such that $m_\alpha(f) \to 0$ in M as $\alpha \to \infty$, and for every $\beta \in \mathbb{Z}_p^d$, $f(\beta) = \sum\limits_{\alpha \in \mathbb{N}^d} m_\alpha(f)\binom{\beta}{\alpha}$.*

Note: When we write $\binom{\beta}{\alpha}$ we mean $\binom{\beta_1}{\alpha_1}\cdots\binom{\beta_d}{\alpha_d}$.

So, given an automorphism $\varphi \in \mathrm{Aut}^{\omega}(G)$, fix an ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$ for (G, ω), and consider the map $f : \mathbb{Z}_p^d \to A, \beta \mapsto \tau(\varphi(\underline{g}^{\beta})\underline{g}^{-\beta})$.

Since A is a complete $\mathbb{Z}_p$-module, we can apply Theorem 2.6.1 to obtain elements $m_\alpha(\varphi, \underline{g}) = m_\alpha(f) \in A$ for every $\alpha \in \mathbb{N}^d$ such that $m_\alpha(\varphi, \underline{g}) \to 0$ as $\alpha \to \infty$ and $f(\beta) = \tau(\varphi(\underline{g}^{\beta})\underline{g}^{-\beta}) = \sum\limits_{\alpha\in\mathbb{N}} m_\alpha(\varphi, \underline{g})\binom{\beta}{\alpha}$ for every $\beta \in \mathbb{Z}_p^d$, or in other words:

$$\tau\varphi(\underline{g}^{\beta}) = \sum_{\alpha\in\mathbb{N}^d} m_\alpha(\varphi, \underline{g})\binom{\beta}{\alpha}\tau(\underline{g}^{\beta}). \tag{2.3}$$

We call $m_\alpha(\varphi, \underline{g})$ the *α-Mahler coefficient* of φ, and there is in fact an explicit formula for these coefficients:

$$m_\alpha(\varphi, \underline{g}) = \sum_{\gamma\leq\alpha}(-1)^{\alpha-\gamma}\binom{\alpha}{\gamma}\tau(\varphi(\underline{g}^{\gamma})\underline{g}^{-\gamma}). \tag{2.4}$$

Recall the definition of the quantized divided power $\partial_{\underline{g}}^{(\alpha)} \in \mathrm{End}_{\mathcal{O}}(\mathcal{O}G)$ for each $\alpha \in \mathbb{N}^d$ from the previous section, and recall that $\partial_{\underline{g}}^{(\alpha)}(\underline{g}^{\beta}) = \binom{\beta}{\alpha}\underline{g}^{\beta}$ for every $\beta \in \mathbb{Z}_p^d$. It follows from (2.3) that for every $g \in G$:

$$\tau\varphi(g) = \sum_{\alpha\in\mathbb{N}^d} m_\alpha(\varphi, \underline{g})\tau\partial_{\underline{g}}^{(\alpha)}(g). \tag{2.5}$$

Since $\partial_{\underline{g}}^{(\alpha)}$ is $\mathcal{O}$-linear, it is clear that this identity also holds for any $g \in \mathcal{O}[G]$ after we extend φ linearly.

In fact, φ extends to a continuous, $\mathcal{O}$-linear automorphism of $\mathcal{O}G$ by the proof of [1, Lemma 6.6], so we may consider the homomorphism $\tau\varphi : \mathcal{O}G \to A$. Since φ and τ are continuous, this is a continuous homomorphism.

Also, since A is complete and $m_\alpha(\varphi, \underline{g}) \to 0$ as $\alpha \to \infty$, it follows that the sum $\sum_{\alpha \in \mathbb{N}^d} m_\alpha(\varphi, \underline{g}) \tau \partial_{\underline{g}}^{(\alpha)}(r)$ converges for every $r \in \mathcal{O}G$. So since $\partial_{\underline{g}}^{(\alpha)}$ is continuous and $\mathcal{O}$-linear, we see using (2.5) that:

$$\tau\varphi = \sum_{\alpha \in \mathbb{N}^d} m_\alpha(\varphi, \underline{g}) \tau \partial_{\underline{g}}^{(\alpha)} \tag{2.6}$$

as continuous $\mathcal{O}$-linear endomorphisms $\mathcal{O}G \to A$. We call this expansion for $\tau\varphi$ the *Mahler expansion of* φ.

2.7 Lie theory

As in the classical theory of Lie groups, to each compact p-adic Lie group G we can associate a $\mathbb{Q}_p$-Lie algebra $\mathfrak{g}$, indeed it is shown in [32] that $\mathfrak{g}$ can be regarded as the tangent space to G at the identity, completely analogously.

However, following [17], we instead use a more algebraic construction. This is a well known theory, which we outline fully in Appendix C.1, where we define the $\mathbb{Z}_p$-Lie algebra $\mathcal{L}_G = \log(G)$ of a p-saturated group G to be the subset of the completion $\widehat{\mathbb{Q}_p G}$ with respect to the Lazard filtration consisting of elements of the form $\log(g)$ for $g \in G$.

Also, recall from Appendix B.1 that every p-valuable group G can be embedded as an open subgroup into the p-saturable group $Sat(G)$, and we define the $\mathbb{Q}_p$-Lie algebra of G to be $\mathfrak{g}_G := \mathcal{L}_{Sat(G)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Given an automorphism $\varphi \in \mathrm{Aut}^\omega(Sat(G))$, consider the image φ_* of φ under the *transport of structure functor*, i.e. for any $u \in \mathcal{L} = \log(Sat(G))$,

$$\varphi_*(u) = \log(\varphi(\exp(u))).$$

This is a $\mathbb{Z}_p$-linear derivation of $\mathcal{L}$, and recall from [1, Proposition 4.6] that $\deg(\varphi_* - 1) > \frac{1}{p-1}$, where $\deg(\sigma)$ denotes the degree of a linear endomorphism of $\mathcal{L}$, as defined in Appendix C.1. This gives rise to the following definition:

Definition 2.7.1. *Let* $\varphi \in \mathrm{Aut}^{\omega}(G)$*, and let* $\tilde{\varphi}$ *be the extension to* $Sat(G)$*. Then define the* logarithm *of* φ *as:*

$$z(\varphi) : Sat(G) \to Sat(G), g \mapsto \exp\left(\left(\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}(\tilde{\varphi}_* - 1)^n\right)(\log(g))\right) \tag{2.7}$$

This definition makes sense because since $\deg(\tilde{\varphi}_* - 1) > \frac{1}{p-1}$, the series $\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}(\tilde{\varphi}_* - 1)^n$ converges as an endomorphism of $\mathcal{L}$. We will see later that given certain conditions on φ, we can raise $z(\varphi)$ to sufficiently high p'th powers to ensure that its image lies in G.

2.8 Affinoid Enveloping Algebras

In Appendix D, we define an affinoid K-algebra as a quotient of the commutative Tate algebra $K\langle t_1, \cdots, t_d\rangle$ of convergent power series in K. In this section, we define a non-commutative equivalent of affinoid algebras. First, recall the following definition [6, Definition 2.7]:

Definition 2.8.1. *Let* V *be a* K*-vector space. An* $\mathcal{O}$-lattice *in* V *is an* $\mathcal{O}$*-submodule* N *of* V *such that* $\bigcap_{n\in\mathbb{N}} \pi^n N = 0$ *and* $N \otimes_{\mathcal{O}} K = V$*.*

Note that every $\mathcal{O}$-lattice N in V induces an exhaustive, separated filtration w_N on V given by

$$w_N(u) := \sup\{n \in \mathbb{Z} : u \in \pi^n N\}.$$

This is the π*-adic filtration* on V associated to N. If V is finite dimensional, then V and N are complete with respect to w_N, and if N_1, N_2 are different lattices then w_{N_1} and w_{N_2} are topologically equivalent. More generally, the completion of N with respect to w_N is the $\mathcal{O}$-module defined by:

$$\widehat{N} := \varprojlim_{n\in\mathbb{N}} N/\pi^n N \tag{2.8}$$

while the completion of V with respect to w_N is the K-vector space $\widehat{V} = \widehat{N}_K := \widehat{N} \otimes_{\mathcal{O}} K$.

Now, let $\mathfrak{g}$ be a finite dimensional K-Lie algebra, and let $\mathcal{L}$ be an $\mathcal{O}$-Lie lattice in $\mathfrak{g}$, i.e. $\mathcal{L}$ is an $\mathcal{O}$-lattice in $\mathfrak{g}$ and it is closed under the Lie bracket. Note that the enveloping algebra $U(\mathcal{L})$ is an $\mathcal{O}$-lattice in $U(\mathfrak{g})$.

Definition 2.8.2. *Define the* affinoid enveloping algebra *of $\mathcal{L}$ with coefficients in $\mathcal{O}$ to be $\widehat{U(\mathcal{L})}$, the completion of $U(\mathcal{L})$ with respect to its π-adic filtration. Also, define the* affinoid enveloping algebra *of $\mathcal{L}$ with coefficients in K to be $\widehat{U(\mathcal{L})}_K := \widehat{U(\mathcal{L})} \otimes_{\mathcal{O}} K$ – the completion of $U(\mathfrak{g})$ with respect to the π-adic filtration associated to $U(\mathcal{L})$.*

The following lemma is a straightforward consequence of the Poincaré-Birkoff-Witt theorem, see e.g. [26, Proposition 2.5.1] for the proof.

Lemma 2.8.3. *If we let $\{x_1, \cdots, x_d\}$ be a K-basis for $\mathfrak{g}$ which forms an $\mathcal{O}$-basis for $\mathcal{L}$, then $\widehat{U(\mathcal{L})}$ is isomorphic as an $\mathcal{O}$-module to the Tate algebra $\mathcal{O}\langle x_1, \cdots, x_d\rangle$, and hence $\widehat{U(\mathcal{L})}_K$ is isomorphic to $K\langle x_1, \cdots, x_d\rangle$ as a K-vector space.*

Now, let M be a $\mathfrak{g}$-representation, i.e. a $U(\mathfrak{g})$-module, and let N be an $\mathcal{O}$-lattice in M such that $\mathcal{L} \cdot N \subseteq N$, and suppose that N is π-adically complete. Then it follows that M has the structure of a $\widehat{U(\mathcal{L})}_K$-module. Unless otherwise stated, we will assume that all modules are left modules.

Proposition 2.8.4. *Let M be a finitely generated $\widehat{U(\mathcal{L})}_K$-module. Then M contains an $\mathcal{O}$-lattice N such that N is π-adically complete and $\mathcal{L} \cdot N \subseteq N$.*

Proof. $M = \widehat{U(\mathcal{L})}_K m_1 + \cdots + \widehat{U(\mathcal{L})}_K m_s$, so let $N := \widehat{U(\mathcal{L})} m_1 + \cdots + \widehat{U(\mathcal{L})} m_s$, clearly N is an $\mathcal{O}$-lattice in M.

Note that since $\widehat{U(\mathcal{L})}$ is π-adically complete and gr $\widehat{U(\mathcal{L})} \cong k[t, u_1, \cdots, u_d]$ is Noetherian, it follows from [28, Ch. 2, Theorem 2.1.2] that the π-adic filtration on $\widehat{U(\mathcal{L})}$ is

Zariskian, and hence any left submodule of $\widehat{U(\mathcal{L})}^s$ is closed in $\widehat{U(\mathcal{L})}^s$.

Since $N \cong \frac{\widehat{U(\mathcal{L})}^s}{J}$ for some left submodule J of $\widehat{U(\mathcal{L})}$, it follows that N is π-adically complete. □

2.9 Completions of KG

Recall from section 2.7 that we define the associated Lie algebra $\mathcal{L}_G$ of a p-saturable group G to be a subset the completion $\widehat{\mathbb{Q}_pG}$ of $\mathbb{Q}_pG$ with respect to the Lazard filtration. In this section, we will explore how we can describe completions of the rational Iwasawa algebra KG using this Lie algebra. Throughout, we will assume that $p > 2$.

The most obvious completion of KG is with respect to the Lazard filtration w_K. The following result allows us to realise this as an affinoid enveloping algebra.

Theorem 2.9.1. *Let G be a p-saturable group. Then the completion $\widehat{KG}$ of KG with respect to the Lazard filtration is isomorphic to an affinoid enveloping algebra $\widehat{U(\mathcal{L})}_K$, where $\mathcal{L}$ is an $\mathcal{O}$-Lie lattice in $\mathfrak{g}_G \otimes_{\mathbb{Q}_p} K$ containing $\mathcal{L}_G \otimes_{\mathbb{Z}_p} \mathcal{O}$. Indeed, if G is uniform, then $\mathcal{L} = \frac{1}{p}\mathcal{L}_G \otimes_{\mathbb{Z}_p} \mathcal{O}$.*

Proof. This follows from [25, Théorème IV 3.2.5] in the case where $K = \mathbb{Q}_p$. By extension of scalars, we get an embedding $KG \hookrightarrow \widehat{U(\mathcal{L})}_K$ of K-algebras, continuous with respect to the Lazard filtration on KG and the π-adic filtration on $\widehat{U(\mathcal{L})}_K$. Since the latter is complete, it follows that this is a dense embedding.

The same argument was repeated in the case where G is uniform in [6, Theorem 10.4], in which case $\mathcal{L}$ was taken to be $\frac{1}{p}\mathcal{L}_G \otimes_{\mathbb{Z}_p} \mathcal{O}$. □

This is a useful result, because from the point of view of calculation, it is easier to work inside the affinoid enveloping algebra than the Iwasawa algebra. This is not always the best completion, however, since it it not faithfully flat over KG.

Perhaps a better choice for a completion of KG would be the *distribution algebra* $D(G,K)$ of G with coefficients in K in the sense of [34]. In this case, the natural dense embedding $KG \to D(G,K)$ is faithfully flat by [34, Theorem 4.11], but unfortunately $D(G,K)$ is not Noetherian, so it would be difficult in practice to extract general ring-theoretic information from $D(G,K)$.

However, for each $n \geq 0$, consider the crossed products $D_{p^n} = D_{p^n}(G) := \widehat{U(p^n\mathcal{L})}_K * \frac{G}{G^{p^n}}$ as defined in [6, Proposition 10.6], which arise as a Banach completions of KG with respect to the extension of the dense embedding $KG^{p^n} \to \widehat{U(p^n\mathcal{L})}_K$ to $KG = KG^{p^n} * \frac{G}{G^{p^n}}$. These algebras give rise to an inverse system:

$$KG \to \cdots \to D_{p^3} \to D_{p^2} \to D_p \to D_0 = \widehat{U(\mathcal{L})}_K$$

and we strongly believe that $D(G,K) = \varprojlim_{n\to\infty} D_{p^n}$. So since $D(G,K)$ is faithfully flat over KG, we want to approximate $D(G,K)$ using the Noetherian Banach algebras D_{p^n}, and thus limit how much information we lose. The following result makes this more precise:

Proposition 2.9.2. *Let P be a primitive ideal of KG, then for all sufficiently high $n \in \mathbb{N}$, there exists a primitive ideal Q_n of $D_{p^n} = \widehat{U(p^n\mathcal{L})}_K * \frac{G}{G^{p^n}}$ such that $Q_n \cap KG = P$.*

Proof. Since P is primitive, $P = \operatorname{Ann}_{KG} M$ for some irreducible KG-module M. Using [6, Proposition 10.6(e), Corollary 10.11], we see that for n sufficiently high, $\widehat{M} := D_{p^n} \otimes_{KG} M \neq 0$.

Since M is irreducible and $\widehat{M} \neq 0$, the natural map $M \to \widehat{M}, m \mapsto 1 \otimes m$ is injective. And since D_{p^n} is a Banach completion of KG with respect to some filtration w, it follows that $\widehat{M}$ is a completion of $M = KGm$ with respect to the filtration $v(rm) = \sup\{w_n(r+y) : y \in KG \text{ and } ym = 0\}$.

Therefore, if $r \in P$, i.e. $rM = 0$, then passing to limits shows that $r\widehat{M} = 0$, so $P \subseteq \operatorname{Ann}_{KG} \widehat{M} = (\operatorname{Ann}_{D_{p^n}} \widehat{M}) \cap KG$.

Now, since D_{p^n} is Noetherian and $\widehat{M}$ is a finitely generated D_{p^n}-module, we can choose a maximal submodule $U \leq \widehat{M}$, and let $M' := \widehat{M}/U$ – an irreducible D_{p^n}-module. Also note that U is finitely generated by Noetherianity of D_{p^n}, so completeness of D_{p^n} implies that U is complete, and hence closed in $\widehat{M}$.

Since M is irreducible, the composition $M \hookrightarrow \widehat{M} \twoheadrightarrow M'$ is either injective or zero. If it is zero then $M \subseteq U$, and since U is closed in $\widehat{M}$, it follows that $\widehat{M} \subseteq U$ and $M' = 0$. This contradiction implies that the composition is injective.

Finally, let $Q_n = \operatorname{Ann}_{D_{p^n}} M'$, then Q_n is a primitive ideal of D_{p^n}, and $P \subseteq \operatorname{Ann}_{KG} \widehat{M} \subseteq \operatorname{Ann}_{KG} M' = Q_n \cap KG$. Also, if $r \in Q_n \cap KG$ then $rM' = 0$, so since $M \subseteq M'$, $rM = 0$ and $r \in P$. Thus $P = Q_n \cap KG$ as required. □

Remark: The proof of [6, Proposition 10.6(e), Corollary 10.11] cited in the above proof was proved in [6] only in the case where $K/\mathbb{Q}_p$ is an unramified extension, but the proof carries across.

Lemma 2.9.3. *Let A be a free abelian pro-p group of rank d, $\mathcal{A} := \frac{1}{p}\log(A)$. Then $\frac{A}{A^p} = C_1 \times \cdots \times C_d$ where each $C_i = \langle c_i \rangle = \langle g_i A^p \rangle$ is a cyclic group of order p, and $D_p = D_p(A)$ is an iterated crossed product:*

$$D_p = \widehat{U(p\mathcal{A})_K} * C_1 * \cdots * C_d$$

where for each $i = 1, \cdots, d$ $\overline{c_i}^r = \overline{c_i^r}$ for $0 \leq r < p$, and $\overline{c_i}^p = g_i^p$.

Proof. Firstly, it is clear that since $A = \mathbb{Z}_p^d$ that $\frac{A}{A^p} = \frac{\mathbb{Z}_p^d}{(p\mathbb{Z}_p)^d} = (\frac{\mathbb{Z}_p}{p\mathbb{Z}_p})^d = C_1 \times \cdots \times C_d$ as required.

For the second statement, it suffices to prove that $KA = KA^p * C_1 * \cdots * C_d$, and that this decomposition satisfies the same properties, since it will be preserved after passing to the completion.

Choose a $\mathbb{Z}_p$-basis $\{g_1, \cdots, g_d\}$ for A, and we may assume that $C_i = \langle c_i \rangle$ where $c_i = g_i A^p$. Then every element $r \in KA$ has the form $\sum\limits_{\alpha \in [p-1]^d} r_\alpha g_1^{\alpha_1} \cdots g_d^{\alpha_d}$ for some $r_\alpha \in KA^p$, and r is sent to $\sum\limits_{\alpha \in [p-1]^d} r_\alpha \overline{c_1^{\alpha_1} \cdots c_d^{\alpha_d}}$ under the isomorphism $KA \to KA^p * \frac{A}{A^p}$.

So, since g_i, g_j and $g_i g_j$ are sent to $\overline{c_i}$, $\overline{c_j}$ and $\overline{c_i c_j}$ respectively, it follows that $\overline{c_i c_j} = \overline{c_i} \cdot \overline{c_j}$ for each i, j. Hence $KA = KA^p * C_1 * \cdots * C_d$.

Finally, for $0 \leq r < p$, g_i^r is sent to $\overline{c_i^r}$, and hence $\overline{c_i}^r = \overline{c_i^r}$, and $g_i^p \in KA^p$ is sent to g_i^p, so $\overline{c_i}^p = g_i^p$ as required. $\square$

2.10 The Adjoint algebraic group

Let $\mathfrak{g}$ be a finite dimensional nilpotent K-Lie algebra, and let $\mathcal{L}$ be an $\mathcal{O}$-Lie lattice in $\mathfrak{g}$. In Appendix C.2, we define the *adjoint algebraic groups* associated to $\mathfrak{g}$ and $p\mathcal{L}$. These are group functors, and we denote them by $\mathbb{G}$ and $\mathbb{G}_0$ respectively.

Then $\mathbb{G}$ is an affine K-scheme, isomorphic to $\mathbb{A}_K^m$ where $m = \dim_K \operatorname{ad}(\mathfrak{g}) = \dim_K \frac{\mathfrak{g}}{Z(\mathfrak{g})}$. Therefore, we show in Appendix D that we can identify the rigid analytification $\mathbb{G}^{an}$ with $\mathbb{G}$. Also, for every finite extension F/K, $\mathbb{G}_0(\mathcal{O}_F)$ is an open affinoid subgroup of $\mathbb{G}(F)$, when considered as a rigid variety, isomorphic to the polydisc $\mathbb{D}_0^m(F)$. Therefore, we can consider $\mathbb{G}_0$ as an open affinoid subdomain of $\mathbb{G}$.

Given $\lambda \in \mathfrak{g}^*$, let X be the coadjoint orbit of λ in $\mathfrak{g}^*$, and let S be the stabiliser of λ in $\mathbb{G}$, i.e. $S(F) := \{g \in \mathbb{G}(F) : g \cdot \lambda = \lambda\}$, a subgroup functor of $\mathbb{G}$.

Lemma 2.10.1. *There exists an isomorphism of $\overline{K}$-varieties $\mathbb{G} \cong S \times X$ such that the map $\mathbb{G} \to X, g \mapsto g \cdot \lambda$ is just the natural projection $S \times X \to X$.*

Proof. Since $\mathbb{G}$ is affine and S is closed in $\mathbb{G}$, we see using [10, Theorem II.6.8] that the quotient variety $\mathbb{G}/S$ exists. Since the orbit map $\mathbb{G} \to X$ is surjective and K has characteristic 0, it follows that this map is *separable* in the sense of [10, AG.8.2], and hence using [10, Theorem AG.17.3] and [10, Proposition II.6.7] it follows that $X = \mathbb{G}/S$ and that the map $\mathbb{G} \to X$ is the quotient map. Since X is closed in $\mathfrak{g}^*$, it follows that $\mathbb{G}/S$ is affine.

Now, using [22, I.5.6(1)] we see that $\mathbb{G} \times S \cong \mathbb{G} \times_{\mathbb{G}/S} \mathbb{G}$ as varieties, and using this isomorphism, it follows that the natural map $\mathbb{G} \to \mathbb{G}/S$ is an S-torsor in the sense of [14, Ch.III Definition 4.1.3].

Therefore, since S is unipotent and $\mathbb{G}/S$ is affine, it follows from [14, Ch.IV Proposition 4.3.7(b)] that the torsor $\mathbb{G} \to \mathbb{G}/S$ is trivial, i.e. $\mathbb{G} \cong S \times \mathbb{G}/S = S \times X$ as varieties and the map $\mathbb{G} \to X$ is just the projection to the second factor. □

For each finite extension F/K, there exists a canonical action of $\mathbb{G}_0(\mathcal{O}_F)$ by continuous automorphisms on the affinoid enveloping algebra $\widehat{U(\mathcal{L})}_F$, simply by applying the exp series to the endomorphisms $\mathrm{ad}(u) : \widehat{U(\mathcal{L})}_F \to \widehat{U(\mathcal{L})}_F$ for $u \in p\mathcal{L}_F$, which is well defined since $\mathrm{ad}(u)^{p^n}$ converges to 0 as $n \to \infty$.

Note: 1. The action of $\mathbb{G}_0$ on $\widehat{U(\mathcal{L})}_K$ preserves two-sided ideals.

2. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ and $\mathcal{A} := \mathfrak{a} \cap \mathcal{L}$, then the action of $\mathbb{G}_0(\mathcal{O})$ on $\widehat{U(\mathcal{L})}_K$ restricts to $\widehat{U(\mathcal{A})}_K$. We say that an ideal I of $\widehat{U(\mathcal{H})}_K$ is $\mathbb{G}_0$-*invariant* if $g(I) = I$ for all $g \in \mathbb{G}_0(\mathcal{O})$, and I is $\mathbb{G}_0$-*prime* if it is $\mathbb{G}_0$-invariant and for any $\mathbb{G}_0$-invariant ideals A,B of $\widehat{U(\mathcal{H})}_K$, $AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$.

Proposition 2.10.2. *Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$, and let $\mathcal{H} := \mathfrak{h} \cap \mathcal{L}$. Then if P is a prime ideal of $\widehat{U(\mathcal{L})}_K$ then $Q := P \cap \widehat{U(\mathcal{H})}_K$ is a $\mathbb{G}_0$-prime ideal of $\widehat{U(\mathcal{H})}_K$, so in particular, it is semiprime.*

Proof. Clearly Q is $\mathbb{G}_0$-invariant, so suppose that A_1, A_2 are $\mathbb{G}_0$-invariant ideals of $\widehat{U(\mathcal{H})}_K$ such that $A_1A_2 \subseteq Q$:

Note that for any $u \in \pi\mathcal{L}$, $\exp(\text{ad}(u)) - 1$ sends $\widehat{U(\mathcal{L})}$ to $\pi\widehat{U(\mathcal{L})}$, so it follows that we can write $\text{ad}(u)$ as $\log(\exp(\text{ad}(u)))$. Since $\exp(\text{ad}(u))(A_i) = A_i$ for both i, it follows that $\text{ad}(u)(A_i) \subseteq A_i$, and hence A_1, A_2 are $\mathfrak{g}$-invariant.

It follows that $\widehat{U(\mathcal{L})}_K A_i = A_i\widehat{U(\mathcal{L})}_K$ are two-sided ideals of $\widehat{U(\mathcal{L})}_K$, and $\widehat{U(\mathcal{L})}_K A_1 \widehat{U(\mathcal{L})}_K A_2 = \widehat{U(\mathcal{L})}_K A_1 A_2 \widehat{U(\mathcal{L})}_K \subseteq \widehat{U(\mathcal{L})}_K Q \subseteq P$. Therefore, since P is prime, we may assume without loss of generality that $\widehat{U(\mathcal{L})}_K A_1 \subseteq P$, and hence $A_1 \subseteq Q$ as required.

Finally, if we let $P_1, \cdots, P_r$ be the minimal primes above Q, then for every $g \in \mathbb{G}_0(\mathcal{O})$, $g(P_i) = P_j$ for some j. Therefore $\sqrt{Q} := P_1 \cap \cdots \cap P_r$ is $\mathbb{G}_0$-invariant, so since Q is $\mathbb{G}_0$-prime, it follows that $\sqrt{Q} = Q$ and Q is semiprime as required. □

2.11 J-ideals

Recall that we are interested in primitive ideals and weakly rational ideals in KG. In this section, we will explore how we generalise these concepts.

Proposition 2.11.1. *Let P be a primitive ideal of KG, then $Z(KG/P)$ is finite dimensional over K, and hence P is weakly rational.*

Proof. By definition, $P = \text{Ann}_{KG} M$ for some irreducible KG-module M. Then it follows from [18, Theorem 1.1(1)] that the algebra $\text{End}_{K[G]}M$ of G-equivariant K-linear endomorphisms of M is a finite dimensional K-algebra.

Now, there exists a well defined, injective homomorphism of K-algebras:

$$Z(KG/P) \to \text{End}_{K[G]}M, z+P \mapsto (\phi_z : M \to M, m \mapsto zm)$$

so it follows immediately that $Z(KG/P)$ is finite dimensional over K. $\square$

It is a useful condition for $Z(KG/P)$ to be a finite field extension of K, however it will often necessary for us to work in slightly more generality:

Definition 2.11.2. *Let $Z := Z(G)$. Then given a right ideal I of KG, we say that I is a J-*ideal *of KG if $KZ(G)/I \cap KZ(G)$ is finite dimensional over K.*

Using Proposition 2.11.1, it is clear that primitive ideals are J-ideals.

Lemma 2.11.3. *Let P be a prime J-ideal of KG. Then $P \cap KZ(G)$ is a maximal ideal of $KZ(G)$.*

Proof. Since P is prime in KG, $Q := P \cap KZ(G)$ is prime in $KZ(G)$. So since $KZ(G)/P \cap KZ(G)$ is a domain, finite dimensional over K, it is in fact a finite field extension of K, hence $P \cap KZ(G)$ is maximal in KG. $\square$

Lemma 2.11.4. *Suppose G is nilpotent, and let $P \subseteq Q$ be prime J-ideals of KG with P faithful. Then Q is also faithful.*

Proof. Let $F_1 = KZ(G)/P \cap KZ(G)$, $F_2 = KZ(G)/Q \cap KZ(G)$, then F_1, F_2 are finite field extensions of K, and clearly the natural surjection $KG/P \twoheadrightarrow KG/Q$ reduces to a field extension $F_1 \hookrightarrow F_2$.

Since $Q^\dagger = \{g \in G : g - 1 \in Q\}$ is a normal subgroup of G, using nilpotence of G we see that if $Q^\dagger \neq 1$, then there exists $z \in Q^\dagger \cap Z(G)$ with $z \neq 1$. Thus $z + P, 1 + P \in F_1 \subseteq F_2$, and $z + Q = 1 + Q$, which implies that $z + P = 1 + P$ and hence $z - 1 \in P$. So since P is faithful, $z = 1$ – contradiction.

Therefore, $Q^\dagger = 1$, and hence Q is faithful. $\square$

Lemma 2.11.5. *Let U be an open normal subgroup of G, P a prime ideal of KU such that PKG is a J-ideal of KG. Then P is a J-ideal of KU.*

Proof. Since $PKG \cap KU = P$ by Lemma B.2.1(ii), we have an injection of KU-modules $KU/P \hookrightarrow KG/PKG$, and this map sends $\frac{KZ(U)}{KZ(U)\cap P}$ to $\frac{KZ(U)+KZ(G)\cap PKG}{KZ(G)\cap PKG}$.

But $Z(U) \subseteq Z(G)$ by [1, Lemma 8.4(b)], so this image is contained in $KZ(G)/KZ(G)\cap PKG$, which is finite dimensional over K by the definition of a J-ideal. Hence $KZ(U)/KZ(U) \cap P$ is finite dimensional over K as required. □

Lemma 2.11.6. *Suppose I is a J-ideal in KG and $I \subseteq I'$ for some right ideal I' of KG. Then I' is a J-ideal of KG.*

Proof. Setting $Z := Z(G)$, since I is a J-ideal, $KZ/I \cap KZ$ is finite dimensional over K. But $I \subseteq I'$, so there is a surjection $KG/I \to KG/I'$ of KG-modules, and $KZ/KZ \cap I'$ is the image of $KZ/KZ \cap I$ under this map.

So since $KZ/KZ \cap I'$ is the image of a finite dimensional K-vector space under a K-linear map, it is also finite dimensional over K, making I' a J-ideal. □

Now, recall from Definition B.2.2 how we define a non-splitting prime ideal of KG, as well as a virtually non-splitting right ideal.

Theorem 2.11.7. *Let A be a closed subgroup of G, and suppose that all faithful, virtually non-splitting right J-ideals of KG are controlled by A. Then all faithful, prime J-ideals of KG are controlled by A.*

Proof. This is similar to the proof of [1, Theorem 5.8].

Let P be a faithful, prime J-ideal of KG, and let $P = I_1 \cap \cdots \cap I_m$ be an essential decomposition for P in the sense of [29, Definition 2.2.1], with each I_j a virtually prime right ideal of KG, as defined in Definition B.2.2, and $I_1, \cdots, I_m$ forming a single G-orbit.

Setting $r = 1$, $I_1 = P$, it is clear that such a decomposition exists, so we will assume that r is maximal such that a decomposition of this form exists. We know that r is finite because KG/P has finite uniform dimension in the sense of [29, Chapter 2].

So, by Theorem B.2.4, each I_j is a virtually non-splitting right ideal of KG, and since $P \subseteq I_j$ it follows from Lemma 2.11.6 that I_j is a J-ideal. Furthermore, since P is faithful, it follows from Lemma B.2.5 that each I_j is faithful. Therefore, by assumption, I_j is controlled by A, so $I_j = (I_j \cap KA)KG$ for each j.

Since $P = I_1 \cap \cdots \cap I_r$, we have that $(P \cap KA)KG = ((I_1 \cap KA) \cap \cdots \cap (I_r \cap KA))KG = (I_1 \cap KA)KG \cap \cdots \cap (I_r \cap KA)KG$ by Lemma B.2.1(i). So since $(I_j \cap KA)KG = I_j$ for each j, $(P \cap KA)KG = I_1 \cap \cdots \cap I_r = P$, thus P is controlled by A as required. □

Chapter 3

Applications of Mahler Expansions

In this chapter, we will examine Mahler expansions in depth, and explore how they can be used to deduce control theorems for faithful prime ideals in the Iwasawa algebra $\mathcal{O}G$.

Throughout the chapter, we will fix a complete, p-valued group (G, ω) of finite rank, and an automorphism $\varphi \in \mathrm{Aut}^{\omega}(G)$.

3.1 Mahler Automorphisms

First, suppose that A is an $\mathcal{O}$-algebra carrying a complete, separated, $\mathcal{O}$-linear filtration $v : A \to \mathbb{Z} \cup \{\infty\}$, and that $\tau : \mathcal{O}G \to A$ is a continuous, $\mathcal{O}$-linear ring homomorphism, and we will suppose further that the restriction of τ to G is injective.

Given an ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$ for G, recall expression (2.5) for the expansion of $\tau\varphi(g)$ in terms of the Mahler coefficients $m_\alpha(\varphi, \underline{g})$.

Even though there is a formula for Mahler coefficients (2.4), in practice they are very difficult to calculate. Therefore, we will explore a particular class of automorphisms for which there is a more workable formula.

Definition 3.1.1. *Suppose $\underline{g} = \{g_1, \cdots, g_d\}$ is an ordered basis for G, and $\varphi \in \mathrm{Aut}^{\omega}(G)$. We say that φ is a* Mahler automorphism with respect to $\underline{g}$ *if its Mahler* *coefficients satisfy:*

$$m_\alpha(\varphi, \underline{g}) = \tau(\varphi(g_1)g_1^{-1} - 1)^{\alpha_1} \cdots \tau(\varphi(g_d)g_d^{-1} - 1)^{\alpha_d} \text{ for all } \alpha \in \mathbb{Z}_p^d.$$

The following proposition simplifies the task of finding Mahler automorphisms. For convenience, we write $\psi(g) := \varphi(g)g^{-1}$ for $g \in G$:

Proposition 3.1.2. *Given an ordered basis $\underline{g}$, the following are equivalent:*

i. φ is a Mahler automorphism with respect to $\underline{g} = \{g_1, \cdots, g_d\}$.

ii. $\psi(\underline{g}^\beta) = \prod_{i=1}^d \psi(g_i)^{\beta_i}$ for all $\beta \in \mathbb{N}^d$.

iii. $\psi(g_i)$ commutes with g_j for all $j \leq i$.

Proof. ($ii \implies i$) Given by the proof of [1, Lemma 6.7]

($i \implies ii$) Recall from (2.4) that $m_\alpha(\varphi, \underline{g}) = \sum_{\beta \leq \alpha} (-1)^{\alpha-\beta}\binom{\alpha}{\beta}\tau\psi(g_1^{\beta_1}g_2^{\beta_2}\cdots g_d^{\beta_d})$, so suppose that for all $\alpha \in \mathbb{N}^d$:

$$m_\alpha(\varphi, \underline{g}) = \tau(\psi(g_1) - 1)^{\alpha_1}\tau(\psi(g_2) - 1)^{\alpha_2} \cdots \tau(\psi(g_d) - 1)^{\alpha_d}.$$

Given $i = 1, \cdots, d$, we will first prove that $\psi(g_i^{\beta_i}) = \psi(g_i)^{\beta_i}$ for any $\beta_i \in \mathbb{N}$.

This is clear if $\beta_i = 0$, so suppose, for induction that $\psi(g_i^k) = \psi(g_i)^k$ for all $0 \leq k < \beta_i$. Then setting $\alpha = (0, 0, \cdots, \beta_i, \cdots, 0)$, we have that $m_\alpha(\varphi, \underline{g}) = \tau(\psi(g_i) - 1)^{\beta_i}$ by assumption, and this is equal to $\sum_{k \leq \beta_i} \binom{\beta_i}{k}(-1)^{\beta_i - k}\psi(g_i)^k$ using binomial expansion. Also, $m_\alpha(\varphi, \underline{g}) = \sum_{k \leq \beta_i} (-1)^{\beta_i - k}\binom{\beta_i}{k}\tau\psi(g_i^k)$. So it follows that:

$$0 = m_\alpha(\varphi, \underline{g}) - \tau(\psi(g_i) - 1)^{\beta_i} = \sum_{k \leq \beta_i} (-1)^{\beta_i - k}\binom{\beta_i}{k}(\tau\psi(g_i^k) - \tau\psi(g_i)^k) = \tau\psi(g_i^{\beta_i}) - \tau\psi(g_i)^{\beta_i}.$$

Therefore, $\tau\psi(g_i^{\beta_i}) = \tau\psi(g_i)^{\beta_i}$, so since τ is injective when restricted to G, it follows that $\psi(g_i^{\beta_i}) = \psi(g_i)^{\beta_i}$.

Now, suppose that for some m with $1 \leq m < d$, we have that $\psi(g_1^{\beta_1} \cdots g_m^{\beta_m}) = \psi(g_1)^{\beta_1} \cdots \psi(g_m)^{\beta_m}$ for all $\beta_i \in \mathbb{N}$. We have proved this for $m = 1$, so we will now show that it holds for $m+1$ and apply induction.

Also, we may suppose that for all $k < \beta_{m+1}$, $\psi(g_1^{\beta_1} \cdots g_m^{\beta_m} g_{m+1}^k) = \psi(g_1^{\beta_1} \cdots g_m^{\beta_m})\psi(g_{m+1})^k$, because it is clear that this holds for $\beta_{m+1} = 1$, so we can apply a second induction.

Let $\alpha = (\beta_1, \beta_2, \cdots, \beta_{m+1}, 0, \cdots, 0)$, then by assumption, we know that $m_\alpha(\varphi, \underline{g}) = \tau(\psi(g_1) - 1)^{\beta_1} \tau(\psi(g_2) - 1)^{\beta_2} \cdots \tau(\psi(g_{m+1}) - 1)^{\beta_{m+1}}$, so expanding this expression out using the binomial theorem, we get:

$$m_\alpha(\varphi, \underline{g}) = \sum_{\gamma \leq \alpha} (-1)^{\alpha-\gamma} \tbinom{\alpha}{\gamma} \tau\psi(g_1)^{\gamma_1} \cdots \tau\psi(g_{m+1})^{\gamma_{m+1}}.$$

Also, we know that $m_\alpha(\varphi, \underline{g}) = \sum_{\gamma \leq \alpha} (-1)^{\alpha-\gamma} \tbinom{\alpha}{\gamma} \tau\psi(g_1^{\gamma_1} g_2^{\gamma_2} \cdots g_{m+1}^{\gamma_{m+1}})$, so taking the difference, we get:

$$0 = m_\alpha(\varphi, \underline{g}) - \prod_{1 \leq i \leq m+1} \tau(\psi(g_i) - 1)^{\beta_i} =$$

$$\sum_{\gamma \leq \alpha} (-1)^{\alpha-\gamma} \binom{\alpha}{\gamma} (\tau\psi(g_1^{\gamma_1} g_2^{\gamma_2} \cdots g_{m+1}^{\gamma_{m+1}}) - \tau\psi(g_1)^{\gamma_1} \cdots \tau\psi(g_{m+1})^{\gamma_{m+1}})$$

So since $\psi(g_1^{\gamma_1} g_2^{\gamma_2} \cdots g_{m+1}^{\gamma_{m+1}}) = \psi(g_1)^{\gamma_1} \cdots \tau\psi(g_{m+1})^{\gamma_{m+1}}$ whenever $\gamma_{m+1} < \beta_{m+1}$, it follows that $\tau\psi(g_1^{\beta_1} g_2^{\beta_2} \cdots g_{m+1}^{\beta_{m+1}}) - \tau\psi(g_1)^{\beta_1} \cdots \tau\psi(g_{m+1})^{\beta_{m+1}} = 0$, and hence $\psi(g_1^{\beta_1} g_2^{\beta_2} \cdots g_{m+1}^{\beta_{m+1}}) = \psi(g_1)^{\beta_1} \cdots \psi(g_{m+1})^{\beta_{m+1}}$ by injectivity of τ restricted to G.

($ii \implies iii$) Given $i \leq j$, we have that $\psi(g_i g_j) = \psi(g_i)\psi(g_j)$, so by the definition of ψ, $\varphi(g_i g_j)(g_i g_j)^{-1} = \varphi(g_i) g_i^{-1} \varphi(g_j) g_j^{-1}$.

Therefore, $\varphi(g_i)\varphi(g_j) g_j^{-1} g_i^{-1} = \varphi(g_i) g_i^{-1} \varphi(g_j) g_j^{-1}$, so $\varphi(g_j) g_j^{-1} g_i^{-1} = g_i^{-1} \varphi(g_j) g_j^{-1}$, i.e. $g_i^{-1}\psi(g_j) = \psi(g_j) g_i^{-1}$.

So $\psi(g_j)$ commutes with g_i^{-1}, and hence it commutes with g_i.

($iii \implies ii$) This is clear from the definition of ψ. □

Remarks: 1. It need not be true that if φ is a Mahler automorphism then φ^n is also a Mahler automorphism. However, we will usually impose this condition.

2. It follows from the proposition that the Mahler automorphism property is independent of the choice of A and τ.

Examples: 1. Suppose that $\varphi \in \mathrm{Aut}^\omega(G)$ is trivial mod centre, i.e. $\varphi(g)g^{-1} \in Z(G)$ for all $g \in G$. Then for any ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$, $\varphi(g_i)g_i^{-1}$ is central, and so commutes with $g_1, \cdots, g_i$.

So by the proposition, φ is a Mahler automorphism with respect to any basis. In fact, the proposition shows that if φ is a Mahler automorphism with respect to any basis, then φ is trivial mod centre. But it is possible for φ to be a Mahler automorphism with respect to *some* basis, and yet not be trivial mod centre.

2. It is not very useful to our purposes for our automorphism to be a Mahler automorphism only with respect to some arbitrary ordered basis, as we may not have that much freedom of choice. In general, we need there to be some canonical form that we can exploit:

Definition 3.1.3. *Let $Z = Z(G)$, and suppose there exists a closed, isolated normal subgroup H of G such that $Z \subseteq H$ and $\psi(g) = \varphi(g)g^{-1} \in Z(H)$ for all $g \in G$. We say that φ is an H-*Mahler automorphism *with respect to an ordered basis $\{Hg_1, \cdots, Hg_t\}$ for $\frac{G}{H}$ if $\psi(g_i) \in Z$ for each i.*

Furthermore, we say that φ is a strong *H-Mahler automorphism with respect to this basis if $\psi(g_i) = 1$ for each i.*

Now, suppose that $\{h_1, \cdots, h_d\}$ is any ordered basis for H, then it follows that $\{h_1, \cdots, h_d, g_1, \cdots, g_t\}$ is an ordered basis for (G, ω). So if φ is an H-Mahler automorphism with respect to $\{Hg_1, \cdots, Hg_t\}$, then since $\psi(h_i) \in Z(H)$ for every i, it follows that $\psi(h_i)$ commutes with $h_1, \cdots, h_i$ for every $i \leq d$. Also, $\psi(g_j) \in Z(G)$ for every $j \leq t$, so $\psi(g_j)$ commutes with $h_1, \cdots, h_d, g_1, \cdots, g_j$.

Therefore, it follows from Proposition 3.1.2 that φ is a Mahler automorphism with respect to $\{h_1, \cdots, h_d, g_1, \cdots, g_t\}$.

We will make use of this example throughout the chapter, since once we have established a canonical subgroup H, we have complete freedom over the choice of ordered basis for H.

3. **(Key example)** Let G be a non-abelian, abelian-by-procyclic group, in the sense of Definition 2.2.3, with principal subgroup H and procyclic element X. So for any basis $\{h_1, \cdots, h_d\}$ for H, $\{h_1, \cdots, h_d, X\}$ is an ordered basis for G.

Let φ be the inner automorphism of G induced from conjugation by X. Then since $\frac{G}{H}$ is abelian, $\varphi(g)g^{-1} \in H$ for all $g \in G$, and clearly $\varphi(X)X^{-1} = 1$, so it follows that φ is a strong H-Mahler automorphism with respect to the basis $\{HX\}$ for $\frac{G}{H}$. Furthermore, if G is not nilpotent, or indeed if it has nilpotency class greater than 2, then φ is not trivial mod centre.

In the next chapter, we will focus solely on this example, and use the theory we develop in this chapter to prove Theorem B.

4. (Non-example) Let $H := \langle X, Y, Z \rangle$ where Z is central and $XYX^{-1} = YZ^p$ be the Heisenberg group, and let $G = H \rtimes \langle U \rangle$ where $UXU^{-1} = X^r$, $UYU^{-1} = Y^r$, $UZU^{-1} = Z^{r^2}$ for some $r \in \mathbb{Z}_p$ with $r \neq 1$ and $r \equiv 1$ $(mod\ p)$.

Then G has no H-Mahler automorphisms, and hence no Mahler automorphisms that respect the canonical structure of the group.

3.2 Mahler Approximations

We will now explore some properties of Mahler automorphisms that will be useful in our analysis. For now, we will assume that (G, ω) is p-saturated, and recall the definition of the logarithm $z(\varphi)$ of φ from Definition 2.7.1. Then we have the following useful result, analogous to [1, Proposition 4.9]:

Lemma 3.2.1. *Let $\varphi \in \mathrm{Aut}^{\omega}(G)$ be an automorphism, and suppose that there exists $g \in G$ such that for every $n \in \mathbb{N}$, g commutes with $\varphi^n(g)$. Then:*

- *$z(\varphi)(g)$ commutes with $\varphi^n(g)$ for all n.*
- *For sufficiently high $m \in \mathbb{N}$, $z(\varphi)(g)^{p^m} \equiv \varphi^{p^m}(g)g^{-1} \pmod{G^{p^{2m}}}$.*
- *$z(\varphi)(g) = \lim_{m\to\infty} (\varphi^{p^m}(g)g^{-1})^{p^{-m}}$.*
- *For all $n \in \mathbb{N}$, $z(\varphi^{p^n})(g) = z(\varphi)(g)^{p^n}$.*

Proof. Since G is p-saturated, $\mathcal{L} = \log(G)$ is a saturated Lie algebra, so recall the definition of the Lie automorphism φ_* of $\mathcal{L}$, and that $\varphi_*(\log(g)) = \log(\varphi(g))$ for all $g \in G$. So using Definition 2.7.1, we see that

$$\log(z(\varphi)(g)) = \left(\sum_{n\geq 1} \tfrac{(-1)^{n+1}}{n}(\varphi_* - 1)^n\right)(\log(g)) = \log(\varphi_*)(\log(g)).$$

Now, since $\varphi^n(g)$ commutes with g for all $n \in \mathbb{N}$, it follows easily that $\varphi^n(g)$ and $\varphi^m(g)$ commute for all $n, m \in \mathbb{N}$. But $(\varphi_* - 1)(\log(g)) = \log(\varphi(g)) - \log(g) = \log(\varphi(g)g^{-1})$ since $\varphi(g)$ and g commute, and an inductive argument shows that $(\varphi_* - 1)^n(\log(g)) = \log(a)$ where a is a product of elements of the form $\varphi^t(g)^s$ for integers t, s.

It follows immediately that $\varphi^n(g)$ commutes with $\log(z(\varphi)(g))$ for all n, and hence it commutes with $z(\varphi)(g) = \exp(\log(z(\varphi)(g)))$ for all n, so we deduce the first statement.

For the second statement, we will show that $\log(z(\varphi)(g)^{p^m}) \equiv \log(\varphi^{p^m}(g)g^{-1}) \ (mod\ p^{2m}\mathcal{L})$, and since $\varphi^{p^m}(g)g^{-1}$ commutes with $z(\varphi)(g)^{p^m}$, the statement will follow.

Firstly, $\log(z(\varphi)(g)^{p^m}) = p^m \log(z(\varphi)(g)) = p^m \log(\varphi_*)(\log(g))$, so set $u := \log(g) \in \mathcal{L}$ so that $g = \exp(u)$.

Then since g commutes with $\varphi^{p^m}(g)$, we see that $\log(\varphi^{p^m}(g)g^{-1}) = \log(\varphi^{p^m}(g)) - \log(g) = \varphi_*^{p^m}(u) - u$. But

$$\varphi_*^{p^m}(u) = \exp(\log(\varphi_*^{p^m}))(u) = \sum_{n\geq 0} \frac{p^{nm}}{n!}(\log(\varphi_*))^n(u) = u + p^m \log(\varphi_*)(u) + p^{2m}\beta_m$$

for some $\beta_m \in \mathcal{L}$. Thus $\log(\varphi^{p^m}(g)g^{-1}) - \log(z(\varphi)(g)^{p^m}) = \varphi_*^{p^m}(u) - u - p^m \log(\varphi_*)(u) = p^{2m}\beta_m \in p^{2m}\mathcal{L}$ as required.

Since $\varphi^{p^m}(g)g^{-1}$ commutes with $z(\varphi)(g)$ for all m and $z(\varphi)(g)^{p^m} = \varphi^{p^m}(g)g^{-1}a_m^{p^{2m}}$ for some $a_m \in G$, it follows that $z(\varphi)(g) = (\varphi^{p^m}(g)g^{-1})^{p^{-m}} a_m^{p^m}$. Since G is p-valuable, $a_m^{p^m} \to 1$ as $m \to \infty$, so taking the limit on both sides as $m \to \infty$ gives that $z(\varphi)(g) = \lim\limits_{m\to\infty} (\varphi^{p^m}(g)g^{-1})^{p^{-m}}$ as required.

For the final statement, we know that $z(\varphi^{p^n})(g) = \lim\limits_{m\to\infty} (\varphi^{p^{m+n}}(g)g^{-1})^{p^{-m}}$, while $z(\varphi)(g)^{p^n} = \left(\lim\limits_{m\to\infty} (\varphi^{p^m}(g)g^{-1})^{p^{-m}}\right)^{p^n} = \lim\limits_{m\to\infty} (\varphi^{p^m}(g)g^{-1})^{p^{n-m}} = \lim\limits_{m\to\infty} (\varphi^{p^{(m-n)+n}}(g)g^{-1})^{p^{n-m}}$, and clearly these limits are equal as required. $\square$

Now, recall that a p-valuable group G embeds as an open subgroup into the p-saturated group $Sat(G)$. Given $\varphi \in \mathrm{Aut}^{\omega}(G)$, assume that φ^n is a Mahler automorphism with respect to some ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$ for G for all $n \in \mathbb{N}$, and let $\tilde{\varphi}$ be the canonical extension of φ to $Sat(G)$.

Then since $\varphi^n(g_i)$ commutes with g_i for each i by Proposition 3.1.2, it follows from Lemma 3.2.1 that $z(\tilde{\varphi}^{p^m})(g_i) = z(\tilde{\varphi})(g_i)^{p^m}$ for each i. Since G is open in $Sat(G)$, we can choose $m \in \mathbb{N}$ such that $Sat(G)^{p^m} \subseteq G$, and hence $z(\tilde{\varphi}^{p^m})(g_i) \in G$ for each i.

Now, as in the previous section, fix an $\mathcal{O}$-algebra A with a separated, $\mathcal{O}$-linear filtration $v : A \to \mathbb{Z} \cup \{\infty\}$, and let $\tau : \mathcal{O}G \to A$ be a continuous $\mathcal{O}$-linear ring homomorphism whose restriction to G is injective. Note that since τ is continuous, the subgroup $V_r = \{g \in G : v(\tau(g-1)) > r\}$ is open in G, so it follows that for each $r \geq 0$ there exists $m \in \mathbb{N}$ such that $v(\tau(g^{p^m} - 1)) > r$ for all $g \in G$.

Definition 3.2.2. *Suppose $\varphi \in \mathrm{Aut}^{\omega}(G)$ and φ^n is a Mahler automorphism with respect to $\underline{g} = \{g_1, \cdots, g_d\}$ for every $n \in \mathbb{N}$. Fix an integer $m_1 \geq 0$, which we call the* initial power*, satisfying:*

- *For each $i = 1, \cdots, d$, $z(\tilde{\varphi}^{p^{m_1}})(g_i) \in G$.*
- *If $\tau(p) = 0$ then $v(\tau(g^{p^{m_1}} - 1)) > 0$ for all $g \in G$.*
- *If $\tau(p) \neq 0$ then $v(\tau(g^{p^{m_1}} - 1)) > v(\tau(p))$ for all $g \in G$.*

Note: *We are free to choose m_1 to be arbitrarily high.*

For each $m \in \mathbb{N}$, define the m'th Mahler approximation function *to be $u_m : G \to Sat(G), g \mapsto z(\varphi^{p^m})(g)$, and if m_1 is the initial power, define $u := u_{m_1}$.*

The following result, completely analogous to [1, Proposition 7.7], will be particularly useful to us, as it allows us to approximate Mahler coefficients very accurately.

Proposition 3.2.3. *Let $\underline{g} = \{g_1, \cdots, g_d\}$ be an ordered basis for G such that φ^n is a Mahler automorphism with respect to $\underline{g}$ for every $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, $i = 1, \cdots d$,* *set* $q_{i,m} := \tau(u(g_i)^{p^m} - 1)$*, and write* $\underline{q}_m^\alpha := q_{1,m}^{\alpha_1} \cdots q_{d,m}^{\alpha_d}$ *for any* $\alpha \in \mathbb{N}^d$*. Then for all* $\alpha \in \mathbb{N}^d$*,* $m \geq m_1$*:*

- $v(m_\alpha(\varphi^{p^{m+m_1}}, \underline{g}) - \underline{q}_m^\alpha) \geq p^{2m+m_1}$ *if* $\tau(p) = 0$*.*
- $v(m_\alpha(\varphi^{p^{m+m_1}}, \underline{g}) - \underline{q}_m^\alpha) \geq (2m + m_1)v(\tau(p))$ *if* $\tau(p) \neq 0$*.*

Proof. By the definition of a Mahler automorphism, for each $m \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$, $m_\alpha(\varphi^{p^m}, \underline{g}) = \tau(\varphi^{p^m}(g_1)g_1^{-1} - 1)^{\alpha_1} \cdots \tau(\varphi^{p^m}(g_d)g_d^{-1} - 1)^{\alpha_d}$. So for each $i = 1, \cdots d$, $m \in \mathbb{N}$, define $e_{i,m} = \tau(\varphi^{p^{m+m_1}}(g_i)g_i^{-1} - 1) - q_{i,m}$, and:

$$m_\alpha(\varphi^{p^{m+m_1}}, \underline{g}) = (q_{1,m} + e_{1,m})^{\alpha_1} \cdots (q_{d,m} + e_{d,m})^{\alpha_d} = \underline{q}_m^\alpha + \delta_{m,\alpha}$$

where $\delta_{m,\alpha}$ is a sum of products of length $|\alpha|$ in $q_{i,m}$ and $e_{i,m}$, each containing at least one $e_{i,m}$, so it remains only to prove that $v(e_{i,m}) \geq p^{2m-m_1}$ if $\tau(p) = 0$, and $v(e_{i,m}) \geq (2m - m_1)v(\tau(p))$ if $\tau(p) \neq 0$.

Firstly, using Lemma 3.2.1, we see that for each i, $\varphi^{p^{m+m_1}}(g_i)g_i^{-1} = z(\tilde{\varphi})(g_i)^{p^{m+m_1}} a_m^{p^{2(m+m_1)}}$ for some $a_m \in Sat(G)$. Thus:

$$e_{i,m} = \tau(\varphi^{p^{m+m_1}}(g_i)g_i^{-1} - 1) - q_{i,m} = \tau(\varphi^{p^{m+m_1}}(g_i)g_i^{-1} - 1) - \tau(u(g_i)^{p^m} - 1)$$

$$= \tau(\varphi^{p^{m+m_1}}(g_i)g_i^{-1} - z(\tilde{\varphi}^{p^{m_1}})(g_i)^{p^m}) = \tau(z(\tilde{\varphi}^{p^{m_1}})(g_i)^{p^m} a_m^{p^{2(m+m_1)}} - z(\tilde{\varphi}^{p^{m_1}})(g_i)^{p^m})$$

$$= \tau(z(\tilde{\varphi}^{p^{m_1}})(g_i)^{p^m})\tau(a_m^{p^{2(m+m_1)}} - 1).$$

Since τ is continuous and g^{p^m} converges to 1 in G for all $g \in G$, it follows that $v(\tau(g)) = 0$ for all g, and hence $v(\tau(z(\tilde{\varphi}^{p^{m_1}})(g_i)^{p^m})) = 0$, so we now only need to consider the value of $\tau(a_m^{p^{2(m+m_1)}} - 1)$.

If $\tau(p) = 0$ then A is an $\mathbb{F}_p$-algebra, so $\tau(a_m^{p^{2(m+m_1)}} - 1) = \tau(a_m^{p^{m_1}} - 1)^{p^{2m+m_1}}$. So since $v(a_m^{p^{m_1}} - 1) > 0$ by Definition 3.2.2, this has value at least p^{2m+m_1} as required.

If $\tau(p) \neq 0$ then $v(\tau(a_m^{p^{m_1}} - 1)) > v(\tau(p))$ by Definition 3.2.2, so it follows from Lemma A.2.3 that $v(\tau(a_m^{p^{2(m+m_1)}} - 1)) \geq (2m+m_1)v(\tau(p)) + v(\tau(a_m^{p^{m_1}} - 1)) \geq (2m+m_1)v(\tau(p))$ as required. $\square$

Now, recall from expression (2.6) that for every $y \in \mathcal{O}G$, $\tau\varphi^{p^m}(y) = \sum_{\alpha\in\mathbb{N}^d} m_\alpha(\varphi^{p^m}, \underline{g})\tau\partial_{\underline{g}}^{(\alpha)}(y)$. So if we assume that φ^{p^m} is a Mahler automorphism with respect to $\underline{g}$, and define $q_{i,m} := \tau(u(g_i)^{p^m} - 1)$, then if we set $\varepsilon_m(y) := \sum_{\alpha\in\mathbb{N}^d} (m_\alpha(\varphi^{p^{m+m_1}}, \underline{g}) - \underline{q}_m^\alpha)\tau\partial_{\underline{g}}^{(\alpha)}(y)$, we get that:

$$\tau\varphi^{p^{m+m_1}}(y) = \sum_{\alpha\in\mathbb{N}^d} \underline{q}_m^\alpha \tau\partial_{\underline{g}}^{(\alpha)}(y) + \varepsilon_m(y). \tag{3.1}$$

and it follows from the proposition that $v(\varepsilon_m(y))$ grows much faster with m than $v(\underline{q}_m)$.

Setting $\partial_i := \partial_{\underline{g}}^{(e_i)}$, where e_i is the standard i'th basis vector, we can rewrite (3.1) as:

$$(\tau\varphi^{p^{m+m_1}} - \tau)(y) = q_{1,m}\tau\partial_1(y) + \cdots + q_{d,m}\tau\partial_d(y) + \sum_{|\alpha|\geq 2} \underline{q}_m^\alpha \tau\partial_{\underline{g}}^{(\alpha)}(y) + \varepsilon_m(y). \tag{3.2}$$

Now, if we choose a faithful prime ideal P of $\mathcal{O}G$, then using [1, Theorem C], we can equip the Goldie ring of quotients $Q(\mathcal{O}G/P)$ with a non-commutative valuation v such that the natural map $\tau : \mathcal{O}G \to Q(\mathcal{O}G/P)$ is continuous, and we can of course take A to be the completion of $Q(\mathcal{O}G/P)$. Moreover, since P is faithful, the restriction of τ to G is injective by definition.

If we fix $\varphi \in \mathrm{Aut}^\omega(G)$ such that $\varphi(P) = P$, then for every $y \in P$, $(\tau\varphi^{p^{m+m_1}} - \tau)(y) = 0$. So if we assume that φ is a Mahler automorphism, our expression (3.2) becomes:

$$0 = q_{1,m}\tau\partial_1(y) + \cdots + q_{d,m}\tau\partial_d(y) + \sum_{|\alpha|\geq 2} \underline{q}_m^\alpha \tau\partial_{\underline{g}}^{(\alpha)}(y) + \varepsilon_m(y). \tag{3.3}$$

3.3 Convergence Arguments

So far, this theory is almost identical to the theory developed in [1, Section 7], which was used to prove a control theorem for prime ideals in kG, invariant under some φ trivial mod centre.

The approach, roughly speaking, is to consider our Mahler expansion (3.3), and to divide out by some expression involving the $q_{i,m}$'s and take the limit as $m \to \infty$. Under the right circumstances, we want to show that this limit of the right hand side is $\tau\partial_i(y)$ for some $i = 1, \cdots, d$, so since $y \in P$ was arbitrary, this means that $\tau\partial_i(P) = 0$, i.e. $\partial_i(P) \subseteq P$, and a control theorem will follow from Proposition 2.5.4.

This approach worked very well in characteristic p for φ trivial mod centre, culminating in the proof of [1, Theorem B], which yielded a complete classification of prime ideals in kG for G nilpotent. But this approach does not generalise easily.

The first drawback is that if φ is not trivial mod centre, then the Mahler approximations $\tau(u(g) - 1)$ need not be central, and hence they may not be v-regular, or even necessarily invertible. This is a problem because it means we may not be able to divide out by the $q_{i,m}$ in our expression, and even if we could there is no guarantee that the division would not affect the convergence of the higher order terms and destroy our convergence argument.

In Chapter 4, we will explore methods of choosing our non-commutative valuation v in such a way that we can ensure that the $q_{i,m}$ are v-regular, if we are working in characteristic p, i.e. if our prime ideal P contains p.

However, a more serious issue arises if we are working in characteristic 0, even if we assume that φ is trivial mod centre. This issue may prove, in general, to be fatal.

The problem lies in the growth rate of the Mahler approximations $q_{i,m}$ with m. Recall that these have the form $\tau(u(g_i)^{p^m} - 1)$. So if P contains p, i.e. P is a prime ideal in kG, then this is equal to $\tau(u(g_i) - 1)^{p^m}$. It follows that the Mahler approximations grow exponentially with m, so if we divide out by lower order terms, this will not affect convergence of the higher order terms.

However, if $p \notin P$ then $q_{i,m} = \tau(u(g_i)^{p^m} - 1)$ grows linearly with m by Lemma A.2.3, which means dividing out by lower order terms becomes a problem, and we cannot always guarantee that this will not affect the higher order terms. In fact, in general, it should.

To see this problem more explicitly, let us rewrite our expression (3.3) in matrix form:

$$\begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} q_{1,m} & q_{2,m} & \cdots & q_{d,m} \\ q_{1,m+1} & q_{2,m+1} & \cdots & q_{d,m+1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ q_{1,m+d-1} & q_{2,m+d-1} & \cdots & q_{d,m+d-1} \end{pmatrix} \begin{pmatrix} \tau\partial_1(y) \\ \tau\partial_2(y) \\ \cdot \\ \cdot \\ \tau\partial_d(y) \end{pmatrix} + \underline{\varepsilon}_m(y) \tag{3.4}$$

where the error term $\underline{\varepsilon}_m(y)$ grows at roughly the rate of $q_{1,m}^2$.

The approach in [1, Section 7] was to multiply (3.4) by the inverse of the $d \times d$ matrix T_m with (i, j)-entry $q_{i,m+j-1}$, as featured in the expression. In characteristic p, T_m has the form of a *Smith matrix*, and it follows from [8, Lemma 1.1] that its inverse grows at a strictly smaller rate than $q_{1,m}^{-2}$, so after multiplying by the inverse and passing to the limit, the error term tends to zero, and we conclude that $\tau\partial_i(y) = 0$

for all i as we require.

In characteristic 0, however, $q_{1,m}^{-2}$ grows linearly, and even the most optimistic estimates on the growth of T_m^{-1} will not be any smaller, even in the case where $d = 2$, so the argument fails.

This problem demonstrates that it does not help to consider multiple expressions simultaneously in characteristic zero, so in the next section, we will explore a case where we can use a single expression to deduce a convergence argument.

3.4 Using Compactness

For now, we will fix our faithful, prime ideal P of $\mathcal{O}G$, and we will assume that $p \notin P$. Thus $Q(\mathcal{O}G/P)$ is a K-algebra, and we assume it carries a non-commutative valuation v, and that $\tau : \mathcal{O}G \to Q(\mathcal{O}G/P)$ is continuous. We will also fix $\varphi \in \mathrm{Aut}^{\omega}(G)$, and a closed, isolated, central subgroup A of G satisfying:

- $\varphi \neq 1$.
- $\varphi(P) = P$
- $\varphi(g)g^{-1} \in A$ for all $g \in G$.
- $\mathcal{O}A/\mathcal{O}A \cap P$ is finitely generated over $\mathcal{O}$.

In this case, φ is trivial mod centre, and thus for all $n \in \mathbb{N}$, φ^n is a Mahler automorphism with respect to any ordered basis by Proposition 3.1.2, so we deduce expression (3.3).

Recall from Definition 3.2.2 the Mahler approximation function $u = z(\varphi^{p^{m_1}}) : G \to G$, where m_1 is the initial power. Then since $p \notin P$, we see that $v(\tau(u(g) - 1)) > v(p)$

for all $g \in G$ by the definition of m_1.

Also, since φ is trivial mod centre, we see that $u = z(\varphi^{p^{m_1}})$ is a group homomorphism by [1, Proposition 4.9(c)], and since $\varphi^{p^{m+m_1}}(g)g^{-1} \in A$ for all $g \in G$ and A is isolated, we get $(\varphi^{p^{m+m_1}}(g)g^{-1})^{p^{-m}} \in A$ for all m. So since A is closed, we see using Lemma 3.2.1 that $u(g) = z(\varphi^{p^{m_1}})(g) = \lim_{m\to\infty}(\varphi^{p^{m+m_1}}(g)g^{-1})^{p^{-m}} \in A$.

For now, fix any ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$ for G, and it follows that for all $m \in \mathbb{N}$, $i = 1, \cdots, d$:

$$q_{i,m} = \tau(u(g_i)^{p^m} - 1) \in \mathcal{O}A/\mathcal{O}A \cap P.$$

Define $\lambda := \inf\{v(\tau(u(g) - 1)) : g \in G\} > v(p)$, then it follows from Lemma A.2.3 that if $v(\tau(u(g) - 1)) = \lambda$ then $v(\tau(u(g)^{p^m} - 1)) = mv(p) + \lambda$ for all $m \in \mathbb{N}$.

Lemma 3.4.1. *$\lambda < \infty$ and $\lambda = \min\{v(\tau(u(g_i) - 1)) : i = 1, \cdots, d\}$*

Proof. Since P is faithful and v is separated, it follows from the proof of [1, Proposition 7.5] that $\lambda < \infty$. Note that it is essential here that $\varphi \neq 1$, otherwise the result is untrue.

Clearly $\lambda \leq \min\{v(\tau(u(g_i) - 1)) : i = 1, \cdots, d\}$, so suppose this is a strict inequality, i.e. $v(\tau(u(g_i) - 1)) > \lambda$ for all i.

Since u is a group homomorphism, for any $g = \underline{g}^{\alpha} \in G$,

$$v(\tau(u(g) - 1)) = v(\tau(u(g_1)^{\alpha_1} \cdots u(g_d)^{\alpha_d} - 1)) \geq v(\alpha_1 q_{1,0} + \cdots + \alpha_d q_{d,0}) > \lambda$$

and this is a contradiction because $\lambda := \inf\{v(\tau(u(g)-1)) : g \in G\}$, and this infimum must be attained by discreteness of v. $\square$

Now, fix $R := \mathcal{O}A/\mathcal{O}A \cap P$, then R is a subring of the K-algebra $Q(\mathcal{O}G/P)$. So let F be the K-span of R, then since R is a domain, finitely generated over $\mathcal{O}$, F is a finite field extension of K, and clearly $F \subseteq Z(Q(\mathcal{O}G/P))$.

Let $\mathcal{V}$ be the valuation ring for F, and let $\mu \in \mathcal{V}$ be a uniformiser. Since F is central in $Q(\mathcal{O}G/P)$, it follows from the definition of a non-commutative valuation that $v|_F$ is a valuation on F, and hence $v|_F$ is a scalar multiple of the standard μ-adic valuation on F. Therefore $\mathcal{V} = \{\beta \in F : v(\beta) > 0\}$.

Now, choose i such that $q_{i,0}$ has value λ, and we may assume without loss of generality that $i = 1$. Then since $\lambda > v(p)$, $v(q_{1,m}) = mv(p)+\lambda$ for all m by Lemma A.2.3.

The key property of $\mathcal{V}$ which we can exploit is compactness, which implies that any sequence in $\mathcal{V}$ has a convergent subsequence. Using this notion, we obtain the following result:

Proposition 3.4.2. *For any ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$, set $q_{i,m} := \tau(u(g_i)^{p^m} - 1)$ and assume without loss of generality that $v(q_{1,0}) = \lambda$. Then for each $i = 1, \cdots, d$, the sequence $q_{1,m}^{-1}q_{i,m}$ has a subsequence converging as $m \to \infty$ to some $\beta_i \in \mathcal{V}$ with $(\beta_1\tau\partial_1 + \cdots + \beta_d\tau\partial_d)(P) = 0$.*

Proof. For each m, $q_{1,m}$ is v-regular of value $\lambda + mv(p)$, so since $v(q_{i,m}) \geq \lambda + mv(p)$, it follows that $v(q_{1,m}^{-1}q_{i,m}) \geq \lambda + mv(p) - \lambda - mv(p) = 0$, so $q_{1,m}^{-1}q_{i,m} \in \mathcal{V}$.

Therefore, using compactness of $\mathcal{V}$, we can choose a subsequence $\underline{a} := (n_1, n_2, \cdots)$ with $n_1 < n_2 < n_3 < \cdots$ such that $q_{1,n_m}^{-1}q_{i,n_m}$ converges in $\mathcal{V}$ as $m \to \infty$.

Let $\beta_i := \lim_{m\to\infty} q_{1,n_m}^{-1}q_{i,n_m} \in \mathcal{V}$.

Now, given $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq 2$; $v(q_{1,m}^{-1}\underline{q}_m^{\alpha}) = v(\underline{q}_m^{\alpha}) - v(q_{1,m}) \geq m|\alpha|v(p) + |\alpha|\lambda - mv(p) - \lambda \geq mv(p) + \lambda \to \infty$ as $m \to \infty$.

Also, using Proposition 3.2.3, $v(q_{1,m}^{-1}\varepsilon_m) = v(\varepsilon_m) - v(q_{1,m}) \geq (2m+m_1)v(p) - mv(p) - \lambda = (m+m_1)v(p) - \lambda \to \infty$ as $m \to \infty$.

Therefore, $q_{1,m}^{-1}\underline{q}_m^{\alpha} \to 0$ as $m \to \infty$, $q_{1,m}^{-1}\varepsilon_m \to 0$ as $m \to \infty$. So, dividing out our expression (3.3) by $q_{1,m}$, we obtain:

$$0 = q_{1,m}^{-1}q_{1,m}\tau\partial_1(y) + q_{1,m}^{-1}q_{2,m}\tau\partial_2(y) + \cdots + q_{1,m}^{-1}q_{d,m}\tau\partial_d(y) + \sum_{|\alpha|\geq 2} q_{1,m}^{-1}\underline{q}_m^{\alpha}\tau\partial_{\underline{g}}^{(\alpha)}(y) + q_{1,m}^{-1}\varepsilon_m(y) \tag{3.5}$$

and considering the subsequence associated with $\underline{a} = (n_1, n_2, \cdots)$, and taking the limit, we get that $\beta_1\tau\partial_1(y) + \beta_2\tau\partial_2(y) + \cdots + \beta_d\tau\partial_d(y) = 0$.

Since our choice of $y \in P$ was arbitrary, it follows that $(\beta_1\tau\partial_1 + \cdots + \tau\partial_d)(P) = 0$ as required. $\square$

Fix $\beta_1, \cdots, \beta_d$ as in the statement of Proposition 3.4.2, note that they depend on the choice of ordered basis $\underline{g}$, and define:

$$h_{\underline{g}} : \mathcal{O}G \to Q(\mathcal{O}G/P), x \mapsto (\beta_1\tau\partial_1 + \beta_2\tau\partial_2 + \cdots + \beta_d\tau\partial_d)(x).$$

Then $h_{\underline{g}} \in \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}G, Q(\mathcal{O}G/P))$, and it follows from Proposition 3.4.2 that $h_{\underline{g}}(P) = 0$, and thus $h_{\underline{g}} \in \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}G/P, Q(\mathcal{O}G/P))$.

Also, since each β_i lies in $F = (\mathcal{O}A/\mathcal{O}A \cap P) \otimes_{\mathcal{O}} K$, it follows that the image of $\mathcal{O}G/P$ under $h_{\underline{g}}$ lies in $(\mathcal{O}G/P) \otimes_{\mathcal{O}} K = KG/(P \otimes_{\mathcal{O}} K)$. So since it is clear that each ∂_i extends to a K-linear endomorphism of KG, we may assume that $h_{\underline{g}}$ lies in $\mathrm{Hom}_{\mathcal{O}}(KG/(P \otimes_{\mathcal{O}} K), KG/(P \otimes_{\mathcal{O}} K)) = \mathrm{End}_{\mathcal{O}}(KG/(P \otimes_{\mathcal{O}} K))$, and hence it makes sense to raise $h_{\underline{g}}$ to integer powers.

We now need some technical results:

Lemma 3.4.3. *Fix $R = \frac{\mathcal{O}A}{\mathcal{O}A \cap P}$, then there exists $s \in \mathbb{N}$ such that $\pi^s \mathcal{V} \subseteq R$. Also, there exists $t \in \mathbb{N}$ such that if $x \in \mathcal{V}$ and $v(x) > 0$ then $x^t \in \pi R$.*

Proof. Since R is a lattice in F, which is a finite dimensional $\mathbb{Q}_p$-vector space, it follows that R is a free $\mathbb{Z}_p$-module of rank $\dim_{\mathbb{Q}_p} F$. This is also the rank of $\mathcal{V}$, and it follows that R has finite index in $\mathcal{V}$, and hence $p^l \mathcal{V} \subseteq R$ for some $l \in \mathbb{N}$, and the first statement follows.

Now, if $v(x) > 0$, then $x = \pi^{-k} r$ for some $r \in R$, $k \in \mathbb{N}$, and $v(r) - kv(\pi) \geq 1$. We know that there exists $s \in \mathbb{N}$ such that $\pi^s \mathcal{V} \subseteq R$, so choose $t \in \mathbb{N}$ such that $t \geq (s+1)v(\pi)$.

Then $x^t = \pi^{-kt} r^t = \pi^{s+1}(\pi^{-(kt+s+1)} r^t)$, and note that:

$$v(\pi^{-(kt+s+1)} r^t) = t(v(r) - kv(\pi)) - (s+1)v(\pi) \geq t - (s+1)v(\pi) \geq 0.$$

Thus $\pi^{-(kt+s+1)} r^t \in \mathcal{V}$, so since $\pi^{s+1} \mathcal{V} \subseteq \pi R$ and $x^t = \pi^{s+1}(\pi^{-(kt+s+1)} r^t)$, it follows that $x^t \in \pi R$ as required. $\square$

Now, recall from the definition of $\partial_1, \cdots, \partial_d$, that if $g = \underline{g}^\alpha \in G$ for some $\alpha \in \mathbb{Z}_p^d$, then $\partial_i(g) = \alpha_i g$. So if we let $k_{g,\underline{g}} := \beta_1 \alpha_1 + \beta_2 \alpha_2 + \cdots + \beta_d \alpha_d \in \mathcal{V}$, then $h_{\underline{g}}(g) = k_{g,\underline{g}} g$ for all $g \in G$.

Lemma 3.4.4. *Using Lemma B.1.2, choose an ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$ such that for some $r \leq d$, $\{g_1, \cdots, g_r\}$ is a basis for A. Then for each $i = 1, \cdots, d$, we have that $\partial_i(cr) = c\partial_i(r) + \partial_i(c)r$ for every $c \in \mathcal{O}A$, $r \in \mathcal{O}G$. In particular, ∂_i restricts to a derivation of $\mathcal{O}A$.*

Proof. Since each ∂_i is a continuous K-linear map of $\mathcal{O}G$, it suffices to prove this identity for $c \in \mathcal{O}[A]$, $r \in \mathcal{O}[G]$. In fact, we may assume further that $c \in A$ and $r \in G$, since the property will clearly be preserved after passing to K-linear combinations.

So, by the definition of our ordered basis, $c = g_1^{\alpha_1} \cdots g_r^{\alpha_r}$ and $r = g_1^{\gamma_1} \cdots g_d^{\gamma_d}$ for some $\alpha_i, \gamma_i \in \mathbb{Z}_p$, and since $g_1, \cdots, g_r \in A$ and A is central in G, $cr = g_1^{\alpha_1+\gamma_1} \cdots g_r^{\alpha_r+\gamma_r} g_{r+1}^{\gamma_{r+1}} \cdots g_d^{\gamma_d}$.

For convenience, we will define $\alpha_i = 0$ for every $i > r$. In which case $\partial_i(c) = \alpha_i c$ and $\partial_i(r) = \gamma_i r$ for every i, and $\partial_i(cr) = (\alpha_i + \gamma_i)cr$.

Therefore, $\partial_i(cr) = (\alpha_i + \gamma_i)cr = \alpha_i cr + \gamma_i cr = \partial_i(c)r + c\partial_i(r)$ as required. □

Lemma 3.4.5. *For every ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$ of G such that $\{g_1, \cdots, g_r\}$ is a basis for A for some $r \leq d$, $h_{\underline{g}}$ is F-linear.*

Proof. Since $h_{\underline{g}}(a) = k_{a,\underline{g}}a$ for every $a \in A$, and $k_{a,\underline{g}} \in F$, it follows that $h_{\underline{g}}$ sends $\mathcal{O}A$ to $F = \mathcal{O}A/(\mathcal{O}A \cap P) \otimes_{\mathcal{O}} K$, and hence it sends F to F, i.e. $h_{\underline{g}}$ restricts to a K-linear endomorphism of F.

Now, using Lemma 3.4.4, we see that for every $c \in \mathcal{O}A$, $r \in \mathcal{O}G$, $i = 1, \cdots, d$, $\partial_i(cr) = c\partial_i(r) + \partial_i(c)r$. So since $\beta_1, \cdots, \beta_d$ are central, it is clear that this identity is also satisfied by $h_{\underline{g}} = \beta_1\tau\partial_1 + \cdots + \beta_d\tau\partial_d$.

Using this identity, and the fact that $F = \mathcal{O}A/(\mathcal{O}A \cap P) \otimes_{\mathcal{O}} K$, we see that to prove $h_{\underline{g}}$ is F-linear, it remains only to prove that $h_{\underline{g}}(F) = 0$.

Since $h_{\underline{g}}$ restricts to a K-linear derivation of $\mathcal{O}A$, we will show, in fact, that all K-linear derivations of F are zero, and the result will follow.

Suppose that $\alpha \in F$, then since F is a finite extension of K, α is the root of some polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ with coefficients in K, and we will assume that n is minimal. Let δ be a K-linear derivation of F, then:

$$0 = \delta(0) = \delta(a_0 + a_1\alpha + \cdots + a_n\alpha^n) = a_1\delta(\alpha) + \cdots + a_n\delta(\alpha^n)$$

$$= a_1\delta(\alpha) + \cdots + na_n\alpha^{n-1}\delta(\alpha) = \delta(\alpha)(a_1 + 2a_2\alpha + \cdots + na_n\alpha^{n-1}).$$

So if $\delta(\alpha) \neq 0$ then $a_1 + 2a_2\alpha + \cdots + na_n\alpha^{n-1} = 0$, contradicting minimality of n. Hence $\delta(\alpha) = 0$, meaning that $\delta = 0$. $\square$

From now on, we will assume that there is an $r \leq d$ such that $\{g_1, \cdots, g_r\}$ is an ordered basis for A, and hence we may apply the previous results.

Now, let f be the degree of the residue field of F, then for all $\beta \in \mathcal{V}$:

$$\beta^{p^f-1} \equiv \begin{cases} 1 & v(\beta) = 0 \\ 0 & v(\beta) > 0 \end{cases} \pmod{\mu\mathcal{V}} \tag{3.6}$$

It follows easily that for all $n \in \mathbb{N}$:

$$\beta^{p^n(p^f-1)} \equiv \begin{cases} 1 & v(\beta) = 0 \\ 0 & v(\beta) > 0 \end{cases} \pmod{\mu^{n+1}\mathcal{V}} \tag{3.7}$$

Using Lemma 3.4.3, choose an integer $t_0 \geq 0$ such that $\mu^{t_0+1}\mathcal{V} \subseteq \pi R$ where $R = \mathcal{O}A/\mathcal{O}A \cap P$ and $x^{p^{t_0}} \in \pi R$ for all $x \in \mathcal{V}$ with $v(x) > 0$. Then examining (3.7) shows that $\beta^{p^{t_0}(p^f-1)} \in R \subseteq \mathcal{O}G/P$ for all $\beta \in \mathcal{V}$.

Also, using Lemma 3.4.5, we see that $h_{\underline{g}}$ is F-linear. So since $k_{g,\underline{g}} \in F$, it follows that $h_{\underline{g}}^n(g) = k_{g,\underline{g}}^n g$ for all $g \in G$, $n \in \mathbb{N}$. So if we define $w_{\underline{g}} := h_{\underline{g}}^{p^{t_0}(p^f-1)}$, then since $k_{g,\underline{g}}^{p^{n+t_0}(p^f-1)} \in \mathcal{O}G/P$ for all $g \in G$, $n \in \mathbb{N}$, we have that $w_{\underline{g}}^{p^n}(g) = k_{g,\underline{g}}^{p^{n+t_0}(p^f-1)} g \in$

$\mathcal{O}G/P$, and thus $w_{\underline{g}}$ sends $\frac{\mathcal{O}G}{P}$ to $\frac{\mathcal{O}G}{P}$.

Remark: π is not a unit in $\mathcal{O}G/P$, because if $1+\pi x \in P$ for some $x \in \mathcal{O}G$ then $1 = (1+\pi x)(1-\pi x+\pi^2x^2-\pi^3x^3+\cdots) \in P$, which is impossible. Hence $\pi\frac{\mathcal{O}G}{P} \neq \frac{\mathcal{O}G}{P}$.

Fixing $g \in G$, $n \in \mathbb{N}$, using (3.7) we have:

$$w_{\underline{g}}^{p^n}(g) \equiv \begin{cases} 1 & v(k_{g,\underline{g}}) = 0 \\ 0 & v(k_{g,\underline{g}}) > 0 \end{cases} \ (mod\ \pi^{n+1}\tfrac{\mathcal{O}G}{P}).$$

So, setting $S := \frac{\mathcal{O}G}{P}$ for convenience, consider the composition:

$$\mathcal{O}G \xrightarrow{\tau} S \xrightarrow{w_{\underline{g}}^{p^n}} S$$

Let ι_n be this composition, then $\iota_n(\pi^m\mathcal{O}G) \subseteq \pi^m S$ for all m, and $\iota_n(g) \equiv \begin{cases} g & v(k_{g,\underline{g}}) = 0 \\ 0 & v(k_{g,\underline{g}}) > 0 \end{cases}$ $(mod\ \pi^{n+1}S)$. Hence $\iota_n(r) \equiv \iota_{n+1}(r)\ (mod\ \pi^{n+1}\mathrm{S})$ for all $n \in \mathbb{N}$, $r \in \mathcal{O}G$.

Therefore, since S is π-adically complete, there exists a continuous, $\mathcal{O}$-linear morphism $\iota : \mathcal{O}G \to S$ such that $\iota(P) = 0$, $\iota(x) \equiv \iota_n(x)\ (mod\ \pi^{n+1}S)$ for all $n \in \mathbb{N}$, hence for all $g \in G$:

$$\iota(g) = \begin{cases} g & v(k_{g,\underline{g}}) = 0 \\ 0 & v(k_{g,\underline{g}}) > 0 \end{cases} \tag{3.8}$$

3.5 A Weak Control Theorem in characteristic 0

Now that we have seen how to employ Mahler expansions in characteristic zero to deduce a convergence argument, we will show how to use these results to prove a control theorem.

Recall the assumptions we made on the automorphism $\varphi \in \mathrm{Aut}^{\omega}(G)$ and the central subgroup $A \trianglelefteq_c^i G$ at the beginning of the previous section. Also recall the definitions of τ, v and u, and recall that we define $\lambda := \inf\{v(\tau(u(g)-1)) : g \in G\}$, with

$1 \le \lambda < \infty$ by Lemma 3.4.1. We now assume further that φ acts trivially on A, i.e. $\varphi(a) = a$ for all $a \in A$.

Let $U := \{g \in G : v(\tau(u(g) - 1)) > \lambda\}$, then using the proof of [1, Lemma 7.6] we see that U is a proper, open subgroup of G containing G^p. Since $u(g) = \lim_{n\to\infty} (\varphi^{p^{m_1+m}}(g)g^{-1})^{p^{-m}}$ by Lemma 3.2.1, and $\varphi(a) = a$ for all $a \in A$, it follows that $u(a) = 1$ for all a, and hence $A \subseteq U$.

Therefore, after choosing an ordered basis for $\frac{G}{A}$ and applying Lemma B.1.2, we may fix an ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$ for G such that $\{g_1, \cdots, g_s\}$ is an ordered basis for A and $\{g_1, \cdots, g_r, g_{r+1}^p, \cdots, g_d^p\}$ is an ordered basis for U for some $s \le r \le d$. We want to prove that P is controlled by U.

Recall from the previous section the definition of $k_{g,\underline{g}} \in \mathcal{V}$ for each $g \in G$:

Lemma 3.5.1. $U = \{g \in G : v(k_{g,\underline{g}}) > 0\}$.

Proof. Using Proposition 3.4.2, we see that for each $i = 1, \cdots, d$, $\beta_i = \lim_{m\to\infty} q_{1,n_m}^{-1} q_{i,n_m}$, as n_m runs over some subsequence $\underline{a} = (n_1, n_2, \cdots\cdots)$, where $q_{i,m} := \tau(u(g_i)^{p^m} - 1)$.

By Lemma A.2.3, $v(q_{i,m}) = \lambda + mv(p)$ for $i \le r$, and $v(q_{i,m}) > \lambda + mv(p)$ for all $i > r$. Hence $v(\beta_i) = 0$ for all $i > r$, $v(\beta_i) > 0$ for all $i \le r$.

Given $g \in U$, $g = g_1^{\alpha_1} \cdots g_r^{\alpha_r} g_{r+1}^{p\alpha_{r+1}} \cdots g_d^{p\alpha_d}$ for $\alpha_i \in \mathbb{Z}_p$, so:

$$v(k_{g,\underline{g}}) = v(\alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_r\beta_r + p\alpha_{r+1}\beta_{r+1} + \cdots + p\alpha_d\beta_d) > 0.$$

Conversely, if $v(k_{g,\underline{g}}) > 0$, suppose $g = \underline{g}^{\alpha}$, so $k_{g,\underline{g}} = \beta_1\alpha_1 + \beta_2\alpha_2 + \cdots + \beta_d\alpha_d$.

By the definition of β_i, this means that $v(q_{1,n_m}^{-1}(\alpha_1 q_{1,n_m} + \cdots + \alpha_d q_{d,n_m})) > 0$ for sufficiently high m, and hence $v(\alpha_{r+1}q_{r+1,n_m} + \cdots + \alpha_d q_{d,n_m}) > \lambda + n_m v(p)$.

But since u is a group homomorphism, it is easily seen that $\tau(u(g)^{p^{n_m}} - 1) \equiv \alpha_1 q_{1,n_m} + \cdots + \alpha_d q_{d,n_m} \pmod{\lambda + n_m v(p) + 1}$, so $v(\tau(u(g)^{p^{n_m}} - 1)) > \lambda + n_m v(p)$ for sufficiently high m.

But $v(\tau(u(g)^{p^{n_m}} - 1)) = v(\tau(u(g) - 1)) + n_m v(p)$ by Lemma A.2.3, and hence $v(\tau(u(g) - 1)) > \lambda$ and $g \in U$ as required. $\square$

Now, recall from the previous section the definition of the continuous $\mathcal{O}$-linear morphism $\iota : \mathcal{O}G \to \frac{\mathcal{O}G}{P}$. Then using the lemma and expression (3.8), we deduce that $\iota(g) = \begin{cases} g & g \notin U \\ 0 & g \in U \end{cases}$.

Define $f : G \to \mathcal{O}, g \mapsto \begin{cases} 1 & g \notin U \\ 0 & g \in U \end{cases}$, then clearly $f \in C(G, \mathcal{O})$ is locally constant, so the endomorphism $\rho(f) \in End_{\mathcal{O}}(\mathcal{O}G)$ is well defined, and

$$\rho(f)(g) = f(g)g = \begin{cases} g & g \notin U \\ 0 & g \in U \end{cases}.$$

Therefore $\tau\rho(f) = \iota$ when restricted to $\mathcal{O}[G]$, so since ι is continuous, τ is continuous, and $\rho(f)$ is continuous, it follows that $\tau\rho(f) = \iota$. Hence $\tau\rho(f)(P) = 0$ and $\rho(f)(P) \subseteq P$.

Now we can prove a control theorem:

Theorem 3.5.2. *Let P be a faithful, prime ideal of $\mathcal{O}G$, and suppose there exists a closed, isolated, central subgroup A of G and an automorphism $\varphi \in \mathrm{Aut}^{\omega}(G)$ such that:*

- $\varphi \neq 1$.

- *$\varphi(P) = P$.*

- *$\varphi(g)g^{-1} \in A$ for all $g \in G$.*

- *$\mathcal{O}A/P \cap \mathcal{O}A$ is a finitely generated $\mathcal{O}$-module.*

- *$\varphi(a) = a$ for all $a \in A$.*

Then P is controlled by a proper, open subgroup of G.

Proof. Since $U = \{g \in G : v(\tau(u(g) - 1)) > \lambda\}$ is a proper, open subgroup of G, it suffices to prove that P is controlled by U.

Firstly, suppose that $C = \{x_1, \cdots, x_t\}$ is a complete set of coset representatives for U in G, then for all $r \in \mathcal{O}G$, $r = \sum\limits_{i \leq t} r_i x_i$ for some unique $r_i \in \mathcal{O}U$.

Suppose we can choose C such that if $r \in P$ then $r_1 \in P \cap \mathcal{O}U$. So let $r \in P$, then since $rx_i^{-1}x_1 \in P$ for all $i = 1 \cdots, t$ and $rx_i^{-1}x_1$ has x_1 component r_i, it follows that $r_i \in P \cap \mathcal{O}U$ for each i, and hence P is controlled by U.

So it remains to prove that we can choose such a set C of coset representatives such that if $\sum\limits_{i \leq t} r_i x_i \in P$, then at least one of the r_i lies in $P \cap \mathcal{O}U$.

Now, since U has ordered basis $\{g_1, \cdots, g_r, g_{r+1}^p, \cdots, g_d^p\}$ it follows that

$$C = \{g_{r+1}^{b_{r+1}} \cdots g_d^{b_d} : 0 \leq b_i < p\}$$

is a complete set of coset representatives for U in G, so for each $\underline{b} \in [p-1]^{d-r}$, let $g_{\underline{b}} = g_{r+1}^{b_{r+1}} \cdots g_d^{b_d}$ (here $[p-1] = \{0, 1, \cdots, p-1\}$).

Then if $r = \sum\limits_{\underline{b} \in [p-1]^{d-r}} r_{\underline{b}} g_{\underline{b}} \in P$, then $\rho(f)(r) = \sum\limits_{\underline{b} \in [p-1]^{d-r}} f(g_{\underline{b}}) r_{\underline{b}} g_{\underline{b}}$ by the proof of [1, Proposition 2.5], and this also lies in P since $\rho(f)(P) \subseteq P$.

But $f(g_{\underline{b}}) = 1$ if $\underline{b} \neq 0$, and $f(g_{\underline{0}}) = 0$ hence $\rho(f)(r) = \sum_{\underline{b}\in[p-1]^{d-r}\setminus\{\underline{0}\}} r_{\underline{b}} g_{\underline{b}} \in P$.

Therefore, $r_{\underline{0}} g_{\underline{0}} = r - \rho(f)(r) \in P$, and thus $r_{\underline{0}} \in P \cap \mathcal{O}U$ as required. $\square$

Now that we have established a control theorem for ideals in $\mathcal{O}G$ satisfying the appropriate finiteness property, we can now turn our attention to KG. Recall from Definition 2.11.2 the definition of a J-ideal of KG:

Theorem 3.5.3. *Let G be a p-valuable group, and let P be a faithful, prime J-ideal of KG. Then P is controlled by $C_G(Z_2(G))$.*

Proof. We will assume first that P is non-splitting in the sense of Definition B.2.2(i). Let $H := P^\chi$ be the controller subgroup of P in G, then $Q := P \cap KH$ is a faithful, prime ideal of KH by Proposition B.2.3, and since H is the smallest subgroup of G controlling P, Q is not controlled by any proper subgroup of H.

Also, note that H is a normal subgroup of G by the proof of [1, Lemma 5.2], so for any $g \in G$, $(g, H) \subseteq H$, and clearly $A := Z(G) \cap H$ is a closed, isolated, central subgroup of H.

Since P is a J-ideal of KG, $KZ(G)/KZ(G) \cap P$ is finite dimensional over K by definition, and $KA/KA \cap Q \subseteq KZ(G)/KZ(G) \cap P$. Therefore, $KA/KA \cap Q$ is finite dimensional over K, and hence $\mathcal{O}A/\mathcal{O}A \cap Q$ is finitely generated as an $\mathcal{O}$-module.

So, given $g \in Z_2(G)$, $(g, H) \subseteq Z(G) \cap H = A$, so let φ be the automorphism of H induced by conjugation by g, then $\varphi(Q) = Q$, $\varphi(h)h^{-1} \in A$ for all $h \in H$, and $\varphi(a) = a$ for all $a \in A$. So applying Theorem 3.5.2 gives that if $\varphi \neq 1$, then Q is controlled by a proper subgroup of H – contradiction. Therefore $\varphi = 1$, i.e. g centralises H.

Since our choice of g was arbitrary, we have now proved that $Z_2(G)$ centralises H, and hence H is contained in the centraliser $C_G(Z_2(G))$ of $Z_2(G)$ in G as required. Thus P is controlled by $C_G(Z_2(G))$.

Now suppose that $I \trianglelefteq_r KG$ is a faithful and virtually non-splitting right J-ideal of KG. Then $I = PKG$ for some open subgroup U of G, and some faithful, non-splitting prime P of KU, and P is a J-ideal of KU by Lemma 2.11.5.

We have proved that P is controlled by $C_U(Z_2(U))$, and $C_U(Z_2(U)) = C_G(Z_2(G)) \cap U$ by Lemma 2.2.2, and hence I is controlled by $C_G(Z_2(G))$. So, using Theorem 2.11.7, it follows that every faithful, prime J-ideal of KG is controlled by $C_G(Z_2(G))$ as required. □

Now we can complete the proof of our main control theorem in characteristic 0:

Proof of Theorem C. Clearly all weakly rational ideals of KG are J-ideals, so it follows from Theorem 3.5.3 that all faithful, weakly rational ideals of KG are controlled by $C_G(Z_2(G))$ as required. □

3.6 Growth Rates

For the rest of the chapter, we will assume that our faithful prime ideal P of $\mathcal{O}G$ contains p, or in other words P is a faithful, prime ideal of kG. Again, fix a non-commutative valuation v on $Q(kG/P)$ such that the natural map $\tau : kG \to Q(kG/P)$ is continuous.

Since we are interested in convergence, we want to consider the growth of the values of the Mahler approximations:

Definition 3.6.1. *Let Q be a ring with a filtration $v : Q \to \mathbb{Z} \cup \{\infty\}$. Define $\rho : Q \to \mathbb{R} \cup \{\infty\}, x \to \lim_{n\to\infty} \frac{v(x^n)}{n}$. This is the* growth rate function *of Q with respect to v. The proof of [**?**, Lemma 1] shows that this is well defined.*

Lemma 3.6.2. *Let Q be a ring with a filtration $v : Q \to \mathbb{Z} \cup \{\infty\}$, and let ρ be the corresponding growth rate function. Then for all $x, y \in Q$:*

i. $\rho(x^n) = n\rho(x)$ for all $n \in \mathbb{N}$.

ii. If x and y commute then $\rho(x + y) \geq \min\{\rho(x), \rho(y)\}$ and $\rho(xy) \geq \rho(x) + \rho(y)$.

iii. $\rho(x) \geq v(x)$ and ρ is invariant under conjugation.

iv. If Q is simple and Artinian and v is separated, then $\rho(x) = \infty$ if and only if x is nilpotent.

v. If x is v-regular and commutes with y, then $\rho(x) = v(x)$ and $\rho(xy) = v(x) + \rho(y)$.

Proof. i and ii are given by the proof of [**?**, Lemma 1].

iii. For each $n \in \mathbb{N}$, $\frac{v(x^n)}{n} \geq \frac{nv(x)}{n} = v(x)$, and so $\rho(x) \geq v(x)$.

Given $u \in Q^\times$, $\rho(uxu^{-1}) = \lim_{n\to\infty} \frac{v((uxu^{-1})^n)}{n} = \lim_{n\to\infty} \frac{v(u(x^n)u^{-1})}{n} \geq \lim_{n\to\infty} \frac{v(u)+v(x^n)+v(u^{-1})}{n} = \lim_{n\to\infty} \frac{v(x^n)}{n} + \lim_{n\to\infty} \frac{v(u)}{n} + \lim_{n\to\infty} \frac{v(u^{-1})}{n} = \rho(x)$.

Hence $\rho(x) = \rho(u^{-1}uxu^{-1}u) \geq \rho(uxu^{-1}) \geq \rho(x)$ – forcing equality. Therefore ρ is invariant under conjugation.

iv. Clearly if x is nilpotent then $\rho(x) = \infty$.

First suppose that x is a unit, then for any $y \in Q$, $v(y) = v(x^{-1}xy) \geq v(x^{-1}) + v(xy)$, and so $v(xy) \leq v(y) - v(x^{-1})$. It follows using induction that for all $n \in \mathbb{N}$, $v(x^n y) \leq v(y) - nv(x^{-1})$, and hence $\frac{v(x^n y)}{n} \leq \frac{v(y)}{n} - v(x^{-1})$.

Taking $y = 1$, it follows easily that $\rho(x) \leq -v(x^{-1})$, and since v is separated, this is less than ∞.

Now, since Q is simple and Artinian, we have that $Q \cong M_l(D)$ for some division ring D, $l \in \mathbb{N}$. So applying Fitting's Lemma [21, section 3.4], we can find a unit $u \in Q^\times$ such that uxu^{-1} has standard Fitting block form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where A and B are square matrices over D, possibly empty, A is invertible and B is nilpotent.

If x is not nilpotent then uxu^{-1} is not nilpotent, and hence A is non-empty. Therefore, $\rho(uxu^{-1}) = \rho(A)$, and since A is invertible, $\rho(A) < \infty$. So by part *iii*, $\rho(x) = \rho(uxu^{-1}) < \infty$.

v. Since x is v-regular, $v(x^n) = nv(x)$ for all n, so clearly $\rho(x) = v(x)$. Also, $\rho(xy) = \lim_{n\to\infty} \frac{v((xy)^n)}{n} = \lim_{n\to\infty} \frac{v(x^n y^n)}{n} = \lim_{n\to\infty} \frac{v(x^n)+v(y^n)}{n} = \lim_{n\to\infty} \frac{v((x^n)}{n} + \lim_{n\to\infty} \frac{v(y^n)}{n} = \rho(x) + \rho(y) = v(x) + v(y)$. $\square$

So far, the theory of Mahler automorphisms we have developed is very general, but now we will impose some further conditions on our automorphism φ:

Lemma 3.6.3. *Suppose that G contains a closed, isolated normal subgroup H containing $Z(G)$ such that $\varphi(g)g^{-1} \in H$ for all $g \in G$. Suppose further that for all $n \in \mathbb{N}$, φ^n is an H-Mahler automorphism with respect to some ordered basis $\{Hg_{r+1}, \cdots, Hg_d\}$ for $\frac{G}{H}$ in the sense of Definition 3.1.3.*

Then if $u = u_{m_1} : G \to Sat(G)$ is the Mahler approximation function, then for any ordered basis $\{g_1, \cdots, g_r\}$ for H, $u(g_i) \in Z(H)$ for all $i = 1, \cdots, d$, so if $q_i := \tau(u(g_i) - 1)$ then $q_1, \cdots, q_d$ commute. Moreover, if φ is a strong H-Mahler automorphism then $q_{r+1} = \cdots = q_d = 0$.

Proof. We know that φ is a Mahler automorphism with respect to $\underline{g} = \{g_1, \cdots, g_d\}$, so using Lemma 3.2.1 we see that $u(g_i) = \lim_{n\to\infty}(\varphi^{p^{m_1+n}}(g_i)g_i^{-1})^{p^{-n}}$ for each $i = 1, \cdots, d$.

So since H is a closed, isolated subgroup of G containing $Z(G) \subseteq H$, and $\varphi(g_i)g_i^{-1} \in Z(H)$ for each i, it follows that $u(g_i) \in Z(H)$, and hence $q_1, \cdots, q_d$ commute.

Furthermore, if φ^{p^m} is a strong H-Mahler automorphism, then since $\varphi^{p^m}(g_i)g_i^{-1} = 1$ for all $i > r$, it follows again from Lemma 3.2.1 that $u(g_i) = 1$, and hence $q_{r+1} = \cdots = q_d = 0$. $\square$

Also, if φ is an H-Mahler automorphism, then $\varphi(g)g^{-1} \in Z(H)$ for all $g \in G$, so it follows that the restriction of φ to H is trivial mod centre, and hence the restriction of $u = z(\varphi^{p^{m_1}})$ to H is a group homomorphism by [1, Proposition 4.9(c)].

Given $y \in P$, recall our Mahler expansion (3.3) for a Mahler automorphism φ with respect to some basis $\underline{g} = \{g_1, \cdots, g_d\}$:

$$0 = q_{1,m}\tau\partial_1(y) + \cdots + q_{d,m}\tau\partial_d(y) + \sum_{|\alpha|\geq 2} \underline{q}_m^\alpha \tau\partial_{\underline{g}}^{(\alpha)}(y) + \varepsilon_m(y)$$

where $q_{i,m} = \tau(u(g_i)^{p^m} - 1)$ for each i, and $\varepsilon_m(y) = \sum_{\alpha\in\mathbb{N}^d}(m_\alpha(\varphi^{p^m}, \underline{g}) - \underline{g}^{\alpha p^m})\tau\partial_{\underline{g}}^{(\alpha)}$.

So setting $q_i := q_{i,0} = \tau(u(g_i) - 1)$, since we are working in characteristic p it follows that $q_{i,m} = q_i^{p^m}$ for all i, m, and using Proposition 3.2.3 we have that $v(\varepsilon_m(y)) \geq p^{2m+m_1}$. So we can rewrite our expansion:

$$0 = q_1^{p^m}\tau\partial_1(y) + \cdots + q_d^{p^m}\tau\partial_d(y) + \sum_{|\alpha|\geq 2} \underline{q}^{\alpha p^m}\tau\partial_{\underline{g}}^{(\alpha)}(y) + \varepsilon_m(y). \tag{3.9}$$

So if $\{g_1, \cdots, g_r\}$ is an ordered basis for H and φ is an H-Mahler automorphism with respect to $\{Hg_{r+1}, \cdots, Hg_d\}$, then $q_1, \cdots, q_d$ commute by Lemma 3.6.3. We will assume further that φ is a strong H-Mahler automorphism, and hence $q_i = 0$ for all $i > r$.

Let ρ be the growth rate function corresponding to v in the sense of Definition 3.6.1, and define $\lambda := \inf\{\rho(\tau(u(h)-1)) : h \in H\}$. We see using Definition 3.2.2 and Lemma 3.6.2(iii) that $\lambda \geq 1$.

Lemma 3.6.4. *If $\varphi \neq 1$ then $\lambda < \infty$ and for any ordered basis $\{g_1, \cdots, g_r\}$ for H, $\lambda = \min\{\rho(\tau(u(g_i)-1)) : i = 1, \cdots, r\}$.*

Proof. Suppose that $\lambda = \infty$, then $\rho(\tau(u(h)-1)) = \infty$ for all $h \in H$, so since $Q(kG/P)$ is simple and Artinian and v is separated, it follows from Lemma 3.6.2(iv) that $\tau(u(h)-1)$ is nilpotent for all $h \in H$, i.e. $0 = \tau(u(h)-1)^{p^m} = \tau(u(h)^{p^m}-1)$ for some $m \in \mathbb{N}$.

So since P is faithful, this means that $u(h)^{p^m} = 1$, which means that $u(h) = 1$ for all $h \in H$ since G is torsionfree. Since $\varphi^n(h)h^{-1} \in Z(H)$, we can apply Lemma 3.2.1 to get that $u(h) = z(\varphi^{p^{m_1}})(h) = z(\varphi)(h)^{p^{m_1}} = 1$, and hence $z(\varphi)(h) = 1$, i.e. $\exp\left(\left(\sum\limits_{n\geq 1}\frac{(-1)^{n+1}}{n}(\tilde{\varphi}_* - 1)^n\right)(\log(h))\right) = 1$ in $Sat(G)$ by Definition 2.7.1.

This means that $\log(\varphi_*) = \sum\limits_{n\geq 1}\frac{(-1)^{n+1}}{n}(\tilde{\varphi}_* - 1)^n$ sends $\log(H)$ to 0, and hence $\varphi_* = \exp(\log(\varphi_*))$ acts trivially on $\log(H)$, and it follows that φ is trivial when restricted to H.

But φ is a strong H-Mahler automorphism with respect to some ordered basis $\{Hg_{r+1}, \cdots, Hg_d\}$ for $\frac{G}{H}$, so $\varphi(g_i) = g_i$ for all $i > r$. So since $\varphi|_H = 1$, it follows that $\varphi = 1$ – contradiction.

Therefore $\lambda < \infty$, and the proof that $\lambda = \min\{\rho(\tau(u(g_i)-1)) : i = 1, \cdots, r\}$ is identical to the proof of Lemma 3.4.1, using Lemma 3.6.2(ii) and the fact that $\tau(u(h_1)-1), \cdots, \tau(u(h_r)-1)$ commute. □

It follows from this lemma that for some $i = 1, \cdots, r$, $\rho(q_i) = \lambda$, and clearly for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq 2$, $\rho(\underline{q}^{\alpha}) \geq 2\lambda$. So, for clarity, we write our Mahler expansion (3.9) using Big O notation:

$$0 = q_1^{p^m}\tau\partial_1(y) + ... + q_r^{p^m}\tau\partial_r(y) + O(q^{p^m}). \tag{3.10}$$

Where $q \in Q = Q(\widehat{kG/P})$ with $\rho(q) > \rho(q_i) \geq \lambda$ for all i, i.e. the error term in (3.9) grows at a faster rate than q^{p^m}.

During the remainder of this chapter, we will see how analysing expressions of this form can help us to deduce a control theorem in characteristic p.

3.7 Growth Preserving Polynomials

For the rest of this chapter, we will fix a closed, isolated normal subgroup H of G, containing $Z(G)$. Contrary to previous notation, we will let d denote the rank of H. Fix $\varphi \in \text{Aut}^{\omega}(G)$ with $\varphi \neq 1$, $\varphi(P) = P$, $\varphi(g)g^{-1} \in Z(H)$ for all $g \in G$, and we will suppose that φ is a strong H-Mahler automorphism with respect to some ordered basis $\{Hg_1, \cdots, Hg_e\}$ for $\frac{G}{H}$.

Consider a polynomial of the form $f(x) = a_0x + a_1x^p + a_2x^{p^2} + \cdots + a_rx^{p^r}$, where $a_i \in \tau(kZ(H))$ for each i, $r \geq 0$, we call r the *p-degree* of f. Note that $f : \tau(kZ(H)) \to \tau(kZ(H))$ is $\mathbb{F}_p$-linear.

Then for each $m \in \mathbb{N}$, $i = 0, \cdots, r$, consider expression (3.9) above, with m replaced by $m + i$, and multiply by $a_i^{p^m}$ to obtain:

$$0 = (a_iq_1^{p^i})^{p^m}\tau\partial_1(y) + ... + (a_iq_d^{p^i})^{p^m}\tau\partial_d(y) + \sum_{|\alpha|\geq 2}(a_i\underline{q}^{\alpha p^i})^{p^m}\tau\partial_{\underline{g}}^{(\alpha)}(y) + a_i^{p^m}\varepsilon_{m+i}(y).$$

Sum all these expressions as i ranges from 0 to r we deduce:

$$0 = f(q_1)^{p^m}\tau\partial_1(y) + ... + f(q_d)^{p^m}\tau\partial_d(y) + \sum_{|\alpha|\geq 2} f(\underline{q}^{\alpha})^{p^m}\tau\partial_{\underline{g}}^{(\alpha)}(y) + \delta_m(y) \qquad (3.11)$$

where $\delta_m(y) = a_0^{p^m}\varepsilon_m(y) + a_1^{p^m}\varepsilon_{m+1}(y) + \cdots + a_r^{p^m}\varepsilon_{m+r}(y)$, so it follows from Proposition 3.2.3 that $v(\delta_m(y)) \geq p^{2m+m_1}$.

Recall that we define $\lambda := \inf\{\rho(\tau(u(h)-1)) : h \in H\}$, and we know that $\lambda < \infty$ by Lemma 3.6.4. Using this, we make the following definition:

Definition 3.7.1. *We say that* $f(t) = a_0t + a_1t^p + \cdots + a_rt^{p^r}$ *is a* growth preserving polynomial*, or* GPP*, if:*

i. $\rho(f(q)) \geq p^r\lambda$ *for all* $q \in \tau(kZ(H))$ *with* $\rho(q) \geq \lambda$.

ii. $\rho(f(q)) > p^r\lambda$ *for all* $q \in \tau(kZ(H))$ *with* $\rho(q) > \lambda$.

We say that a GPP f *is* trivial *if for all* $q = \tau(u(h)-1)$, $\rho(f(q)) > p^r\lambda$.

Furthermore, f *is a* special *GPP if* f *is not trivial, and for any* $q = \tau(u(h)-1)$ *with* $\rho(f(q)) = p^r\lambda$, *we have that* $f(q)^{p^k}$ *is* v*-regular for sufficiently high* k.

Example: $f(t) = t$ is clearly a GPP. In general it need not be special, and it is not trivial, because if $\rho(q) = \lambda$ then $\rho(f(q)) = \rho(q) = \lambda$.

Note that if f is a growth preserving polynomial, then for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq 2$, $\rho(\underline{q}^{\alpha}) > \lambda$ so $\rho(f(\underline{q}^{\alpha})) > p^r\lambda$. Moreover, since $v(\delta_m(y)) > p^{2m+m_1}$, it follows that $\delta_m(y)$ also grows at a faster rate than $p^r\lambda$. Thus we may rewrite (3.11) using Big O notation:

$$0 = f(q_1)^{p^m}\tau\partial_1(y) + ... + f(q_d)^{p^m}\tau\partial_d(y) + O(f(q)^{p^m}) \qquad (3.12)$$

where $q \in \tau(kZ(H))$ and $\rho(q) > \lambda$.

For any GPP $f(t)$, define $U_f := \{h \in H : \rho(f(\tau(u(h) - 1))) > p^r\lambda\}$.

Lemma 3.7.2. *If $f(t)$ is a GPP, then U_f is an open subgroup of H containing H^p. Moreover, $U_f = H$ if and only if f is trivial.*

Proof. It is clear from the definition that $U_f = H$ if and only if $\rho(f(\tau(u(h)-1))) > p^r\lambda$ for all $h \in H$, i.e. if and only if f is trivial.

Given $h, h' \in H$, let $q = \tau(u(h) - 1)$, $q' = \tau(u(h') - 1)$. Then

$$\tau(u(hh') - 1) = \tau((u(h) - 1)(u(h') - 1) + (u(h) - 1) + (u(h') - 1))) = qq' + q + q'.$$

Therefore $f(\tau(u(hh') - 1)) = f(qq') + f(q) + f(q)$ using $\mathbb{F}_p$-linearity of f. But $\rho(q), \rho(q') \geq \lambda$ by the definition of λ, so $\rho(qq') \geq 2\lambda > \lambda$, so by the definition of a GPP, $\rho(f(qq')) > p^r\lambda$.

We know that $\rho(f(q)), \rho(f(q')) > p^r\lambda$, thus $\rho(f(qq')+f(q)+f(q')) > p^r\lambda$. Therefore, $\rho(f(\tau(u(hh') - 1))) > p^r\lambda$ and $hh' \in U_f$ as required.

Also, $\tau(u(h^{-1})-1) = -\tau(u(h^{-1}))\tau(u(h)-1) = -\tau(u(h^{-1})-1)\tau(u(h)-1)-\tau(u(h)-1)$, so by the same argument it follows that $h^{-1} \in U_f$, and U_f is a subgroup of H.

Finally, for any $h \in H$ $\rho(\tau(u(h^p) - 1)) = \rho(\tau(u(h) - 1)^p) \geq p\lambda > \lambda$, hence $\rho(f(\tau(u(h^p) - 1))) > p^r\lambda$ and $h^p \in U_f$. Therefore U_f contains H^p and U_f is an open subgroup of H. □

In particular, for $f(t) = t$, let $U := U_f = \{h \in H : \rho(\tau(u(h) - 1)) > \lambda\}$. Then U is an open subgroup of H containing H^p by Lemma 3.7.2, and since f is non-trivial, it is a proper subgroup.

For the rest of this section, fix a non-trivial GPP f of p degree r. Then U_f is a proper open subgroup of H containing H^p by Lemma 3.7.2, so fix a basis $\{h_1, \cdots, h_d\}$ for H such that $\{h_1^p, \cdots, h_t^p, h_{t+1}, \cdots, h_d\}$ is an ordered basis for U_f.

Set $q_i := \tau(u(h_i)-1) \in \tau(kZ(H))$ for each i, so that for $i \leq t$, $\rho(f(q_i)) = p^r\lambda$, and for $i > t$, $\rho(f(q_i)) > p^r\lambda$ i.e. $f(q_i)^{p^m} = O(f(q)^{p^m})$ for some $q \in \tau(kZ(H))$ with $\rho(q) > \lambda$. Define:

$$S_m := \begin{pmatrix} f(q_1)^{p^m} & f(q_2)^{p^m} & \cdot & \cdot & f(q_t)^{p^m} \\ f(q_1)^{p^{m+1}} & f(q_2)^{p^{m+1}} & \cdot & \cdot & f(q_t)^{p^{m+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(q_1)^{p^{m+t-1}} & f(q_2)^{p^{m+t-1}} & \cdot & \cdot & f(q_t)^{p^{m+t-1}} \end{pmatrix}, \underline{\partial} := \begin{pmatrix} \tau\partial_1(y) \\ \tau\partial_2(y) \\ \cdot \\ \cdot \\ \cdot \\ \tau\partial_t(y) \end{pmatrix}$$

Then we can rewrite (3.12) as:

$$0 = S_m \cdot \underline{\partial} + \begin{pmatrix} O(f(q)^{p^m}) \\ O(f(q)^{p^{m+1}}) \\ \cdot \\ \cdot \\ \cdot \\ O(f(q)^{p^{m+t-1}}) \end{pmatrix} \tag{3.13}$$

and multiplying by the adjoint matrix $adj(S_m)$ and applying standard linear algebra gives:

$$0 = \det(S_m)\underline{\partial} + adj(S_m) \begin{pmatrix} O(f(q)^{p^m}) \\ O(f(q)^{p^{m+1}}) \\ \cdot \\ \cdot \\ \cdot \\ O(f(q)^{p^{m+t-1}}) \end{pmatrix}. \tag{3.14}$$

Lemma 3.7.3. *Suppose that f is special. Then for each $i, j \leq t$, the (i, j)-entry of $adj(S_m)$ has value at least $\frac{p^t-1}{p-1}p^{m+r}\lambda - p^{m+r+j-1}\lambda$ for sufficiently high m.*

Proof. By definition, the (i, j)-entry of $adj(S_m)$ is (up to sign) the determinant of the matrix $(S_m)_{i,j}$ obtained by removing the j'th row and i'th column of S_m. This determinant is a sum of elements of the form:

$$f(q_{k_1})^{p^m} f(q_{k_2})^{p^{m+1}} \cdots \widehat{f(q_{k_j})^{p^{m+j-1}}} \cdots f(q_{k_t})^{p^{m+t-1}}$$

for $k_i \leq t$, where the hat indicates that the j'th term in this product is omitted.

Since f is special, $f(q_{k_s})^{p^m}$ is v-regular for $m >> 0$. So since $\rho(f(q_{k_s})) = p^r\lambda$, it follows that $f(q_{k_s})^{p^m}$ is v-regular of value $p^{m+r}\lambda$.

Therefore, this (i,j)-entry has value at least $(1 + p + \cdots + \widehat{p^{j-1}} + \cdots + p^{t-1})p^{m+r}\lambda$ $= \frac{p^t-1}{p-1}p^{m+r}\lambda - p^{m+r+j-1}\lambda$. □

Lemma 3.7.4. *Let* $\Delta := \prod_{\alpha \in \mathbb{P}^{t-1}\mathbb{F}_p} (\alpha_1 f(q_1) + \cdots + \alpha_t f(q_t))$, *where* $\mathbb{P}^{t-1}\mathbb{F}_p$ *is a complete set of representatives for the space* $\mathbb{F}_p^t$ *under the equivalence relation* $\alpha \equiv \beta$ *if* $\alpha = \mu\beta$ *for some* $\mu \in \mathbb{F}_p$.

Then there exists $\delta \in \tau(kH)$, *which is a product of length* $\frac{p^t-1}{p-1}$ *in elements of the form* $f(\tau(u(h) - 1))$, $h \in H \backslash U_f$, *such that* $\rho(\Delta - \delta) > \frac{p^t-1}{p-1}p^r\lambda$.

Proof. For each $\alpha \in \mathbb{F}_p^t \backslash \{0\}$, we have that $\alpha_1 f(q_1) + \cdots + \alpha_t f(q_t) = f(\alpha_1 q_1 + \cdots + \alpha_t q_t)$ using linearity of f.

Using expansions inside kH, we see that $\alpha_1 q_1 + \cdots + \alpha_t q_t = \tau(u(h_1^{\alpha_1} \cdots h_t^{\alpha_t}) - 1) + \epsilon_\alpha$ for some $\epsilon_\alpha \in \tau(kH)$ such that $\rho(\epsilon_\alpha) \geq 2\lambda$. Hence:

$$f(\alpha_1 q_1 + \cdots + \alpha_t q_t) = f(\tau(u(h_1^{\alpha_1} \cdots h_t^{\alpha_t}) - 1)) + f(\epsilon_\alpha).$$

So setting $h_\alpha := u(h_1^{\alpha_1} \cdots h_t^{\alpha_t})$, since $\alpha_i \neq 0$ for some i, it follows from $\mathbb{F}_p$-linear independence of $h_1, \cdots, h_t$ modulo U_f, that $\rho(f(\tau(u(h_\alpha) - 1))) = p^r\lambda$, and hence $h_\alpha \in H \backslash U_f$.

Set $\delta := \prod_{\alpha\in\mathbb{P}^{t-1}\mathbb{F}_p} f(\tau(u(h_\alpha)-1))$. Then $\Delta = \prod_{\alpha\in\mathbb{P}^{t-1}\mathbb{F}_p} (\alpha_1 f(q_1) + \cdots + \alpha_t f(q_t))$

$$= \prod_{\alpha\in\mathbb{P}^{t-1}\mathbb{F}_p} f(\tau(u(h_\alpha)-1)) + \epsilon_\alpha = \delta + \epsilon$$

Where ϵ is a sum of products over all α in $f(\tau(u(h_\alpha)-1))$ and ϵ_α, with each product containing at least one ϵ_α.

Since the length of each of these products is $\frac{p^t-1}{p-1}$, and each term has growth rate at least $p^r\lambda$, with one or more having growth rate at least $2p^r\lambda$, it follows that $\rho(\epsilon) > \frac{p^t-1}{p-1}p^r\lambda$ as required. □

The following result, which will be fundamental in the next chapter, underlines the importance of special GPPs.

Theorem 3.7.5. *Let G be a p-valuable group, and let P be a faithful prime ideal of kG. Suppose further that we have:*

- *A non-commutative valuation v on $Q(kG/P)$.*
- *A closed, isolated normal subgroup H of G.*
- *A strong H-Mahler automorphism $\varphi \in \mathrm{Aut}^\omega(G)$ such that $\varphi(P) = P$.*
- *A special growth preserving polynomial f.*

Then P is controlled by a proper open subgroup of G.

Proof. Since f is special, we have that for some $k > 0$, $f(q_i)^{p^k}$ is v-regular for each $i \leq t$, and thus $v(f(q_i)^{p^k}) = \rho(f(q_i)^{p^k}) = p^{r+k}\lambda$ and for all $m \geq k$, $v(f(q_i)^{p^m}) = p^{r+m}\lambda$.

Also, since $\rho(f(q)) > p^r\lambda$, we can choose $c > 0$ such that $\rho(f(q)) > p^r\lambda + c$, and hence $v(f(q)^{p^m}) > p^{m+r}\lambda + p^m c$ for sufficiently high m.

Consider our Mahler expansion (3.14):

$$0 = \det(S_m)\underline{\partial} + adj(S_m)\begin{pmatrix} O(f(q)^{p^m}) \\ O(f(q)^{p^{m+1}}) \\ \cdot \\ \cdot \\ \cdot \\ O(f(q)^{p^{m+t-1}}) \end{pmatrix}.$$

We will analyse this expression to prove that $\tau\partial_i(P) = 0$ for some $i \leq t$, and it will follow from Proposition 2.5.4 that P is controlled by a proper open subgroup of G as required.

Consider the i'th entry of the vector

$$adj(S_m)\begin{pmatrix} O(f(q)^{p^m}) \\ O(f(q)^{p^{m+1}}) \\ \cdot \\ \cdot \\ \cdot \\ O(f(q)^{p^{m+t-1}}) \end{pmatrix}.$$

This has the form $adj(S_m)_{i,1}O(f(q)^{p^m}) + adj(S_m)_{i,2}O(f(q)^{p^{m+1}}) + \cdots + adj(S_m)_{i,t}O(f(q)^{p^{m+t-1}})$.

By Lemma 3.7.3, we know that $v(adj(S_m)_{i,j}) \geq \frac{p^t-1}{p-1}p^{m+r}\lambda - p^{m+r+j-1}\lambda$ for $m >> 0$, and hence:

$v(adj(S_m)_{i,j}O(f(q)^{p^{m+j-1}})) \geq v(adj(S_m)_{i,j}) + v(O(f(q)^{p^{m+j-1}}))$

$\geq \frac{p^t-1}{p-1}p^{m+r}\lambda - p^{m+r+j-1}\lambda + p^{m+r+j-1}\lambda + p^{m+j-1}c = \frac{p^t-1}{p-1}p^{m+r}\lambda + p^{m+j-1}c$ for each j.

Hence this i'th entry has value at least $\frac{p^t-1}{p-1}p^{m+r}\lambda + p^m c$.

Therefore, the i'th entry of our expression (3.14) has the form $0 = \det(S_m)\tau\partial_i(y) + \epsilon_{i,m}$, where $v(\epsilon_{i,m}) \geq \frac{p^t-1}{p-1}p^{m+r}\lambda + p^m c$.

Now, take $\Delta := \det(S_0)$, and it follows that $\det(S_m) = \Delta^{p^m}$ for each m. Also, using [8, Lemma 1.1(ii)] we see that:

$$\Delta = \beta \cdot \prod_{\alpha \in \mathbb{P}^{t-1}\mathbb{F}_p} (\alpha_1 f(q_1) + \cdots + \alpha_t f(q_t)) \text{ for some } \beta \in \mathbb{F}_p.$$

Therefore, by Lemma 3.7.4, we can find an element $\delta \in \tau(kH)$, which up to scalar multiple is a product of length $\frac{p^t-1}{p-1}$ in elements of the form $f(\tau(u(h)-1))$, with $\rho(f(\tau(u(h)-1))) = p^r\lambda$ for each h, such that $\rho(\Delta - \delta) > \frac{p^t-1}{p-1}p^r\lambda$.

Hence we can find $c' > 0$ such that for all $m >> 0$, $v((\Delta - \delta)^{p^m}) \geq \frac{p^t-1}{p-1}p^{m+r}\lambda + p^m c'$.

Therefore, $0 = \det(S_m)\tau\partial_i(y) + \epsilon_{i,m} = \Delta^{p^m}\tau\partial_i(y) + \epsilon_{i,m} = \delta^{p^m}\tau\partial_i(y) + (\Delta - \delta)^{p^m}\tau\partial_i(y) + \epsilon_{i,m}$.

Again, since f is a special GPP, if $h \in H$ and $\rho(f(\tau(u(h)-1))) = p^r\lambda$, it follows that $f(\tau(u(h)-1))^{p^k}$ is v-regular for $k >> 0$.

So since δ is a product of $\frac{p^t-1}{p-1}$ elements of the form $f(\tau(u(h)-1))$ of growth rate $p^r\lambda$, it follows that for some $k \in \mathbb{N}$, δ^{p^k} is v-regular of value $\frac{p^t-1}{p-1}p^{r+k}\lambda$.

Therefore for all $m \geq k$, δ^{p^m} is v-regular of value $\frac{p^t-1}{p-1}p^{r+m}\lambda$, and dividing out by δ^{p^m} gives that for each $i = 1, \cdots, t$:

$$0 = \tau\partial_i(y) + \delta^{-p^m}(\Delta - \delta)^{p^m}\tau\partial_i(y) + \delta^{-p^m}\epsilon_{i,m}.$$

But for $m >> 0$, $v(\delta^{-p^m}(\Delta - \delta)^{p^m}) \geq \frac{p^t-1}{p-1}p^{m+r}\lambda + p^m c' - \frac{p^t-1}{p-1}p^{m+r}\lambda = p^m c'$, and $v(\delta^{-p^m}\epsilon_{i,m}) \geq \frac{p^t-1}{p-1}p^{m+r}\lambda + p^m c - \frac{p^t-1}{p-1}p^{m+r}\lambda = p^m c$, hence the right hand side of this expression converges to $\tau\partial_i(y)$.

Therefore, $\tau\partial_i(y) = 0$, and since this holds for all $y \in P$, we have that $\tau\partial_i(P) = 0$ for each $i = 1, \cdots, t$. □

So to prove a control theorem, it remains only to prove the existence of a special growth preserving polynomial.

3.8 Central Simple Algebras

In the final two sections of this chapter, we will deal with a special case. Since $Q(kG/P)$ is simple, its centre is a field, so we will now assume that it is finite dimensional over its centre, i.e. $Q(kG/P)$ is a *central simple algebra* (CSA).

Fix a non-commutative valuation v on Q, which we know exists by [1, Theorem C]. Then by definition, the completion Q of $Q(kG/P)$ with respect to v is isomorphic to $M_n(Q(D))$ for some complete, non-commutative DVR D. But since $Q(kG/P)$ is finite dimensional over its centre, the same property holds for Q, and hence $Q(D)$ is finite dimensional over its centre.

Let $F := Z(Q(D))$, $s := \dim_F(Q(D))$, $R := F \cap D$. Then F is a field, R is a commutative DVR, and $Q(D) \cong F^s$ as F-vector spaces. Let $\mu \in R$ be a uniformiser, and let $t := v(\mu) > 0$.

Lemma 3.8.1. *Let $\{y_1, \cdots, y_s\}$ be an F-basis for $Q(D)$ with $0 \leq v(y_i) \leq t$ for all i. Then there exists $l \in \mathbb{N}$ such that if $v(r_1y_1 + \cdots + r_sy_s) \geq l$ then $v(r_i) > 0$ for some i with $r_i \neq 0$.*

Proof. First, note that the field F is complete with respect to the non-archimedean valuation v, and $Q(D) \cong F^s$ carries two filtrations as a F-vector space, which both restrict to v on F. One is the natural valuation v on $Q(D)$, the other is given by $v_0(r_1y_1 + ... + r_sy_s) = \min\{v(r_i) : i = 1, ..., s\}$.

But it follows from [15, Proposition 2.27] that any two norms on F^s are topologically equivalent. Hence v and v_0 induce the same topology on $Q(D)$. It is easy to see that any subspace of F^s is closed with respect to v_0, and hence also with respect to v.

So, suppose for contradiction that for each $m \in \mathbb{N}$, there exist $r_{i,m} \in K$, not all zero, with $v(r_{i,m}) \leq 0$ if $r_{i,m} \neq 0$, and $v(r_{1,m}y_1 + ... + r_{s,m}y_s) \geq m$.

Then there exists i such that $r_{i,m} \neq 0$ for infinitely many m, and we can assume without loss of generality that $i = 1$. So from now on, we assume that $r_{1,m} \neq 0$ for all m.

Dividing out by $r_{1,m}$ gives that $v(y_1 + t_{2,m}y_2 + ... + t_{s,m}y_s) \geq m - v(r_{1,m}) \geq m$, where $t_{i,m} = r_{1,m}^{-1}r_{i,m} \in K$.

Hence $\lim_{m\to\infty}(y_1 + t_{2,m}y_2 + ... + t_{s,m}y_s) = 0$, and thus $\lim_{n\to\infty}(t_{2,m}y_2 + ... + t_{s,m}y_s)$ exists and equals $-y_1$. But since $\mathrm{Span}_F\{y_2, ..., y_s\}$ is closed in F^s, this means that $y_1 \in \mathrm{Span}_F\{y_2, ..., y_s\}$ – contradiction. □

Proposition 3.8.2. *There exists a basis $\{x_1, ..., x_s\} \subseteq D$ for $Q(D)$ over F such that $D = Rx_1 \oplus .... \oplus Rx_s$.*

Proof. It is clear that we can find an F-basis $\{y_1, ..., y_s\} \subseteq D$ for $Q(D)$ such that $v(y_i) < t$ for all i, just by rescaling elements of some arbitrary basis. Therefore, by Lemma 3.8.1, there exists $l \in \mathbb{N}$ such that if $v(r_1y_1 + \cdots + r_st_s) \geq l$ then $v(r_i) > 0$ for some i with $r_i \neq 0$.

Choose $m \in \mathbb{N}$ such that $tm > l$. Then given $x \in D\backslash\{0\}$, $v(x) \geq 0$ so $v(\mu^m x) \geq tm > l$. So if $\mu^m x = r_1y_1 + ... + r_sy_s$ then $v(r_i) \geq 0$ for some $r_i \neq 0$.

It follows from an easy inductive argument that $\mu^{ms}x \in Ry_1 \oplus ..... \oplus Ry_s$, and hence $\mu^{ms}D \subseteq Ry_1 \oplus ..... \oplus Ry_s$.

But $Ry_1 \oplus ..... \oplus Ry_s$ is a free R-module, and R is a commutative PID, hence any R-submodule is also free. Hence $\mu^{ms}D = R(\mu^{ms}x_1) \oplus .... \oplus R(\mu^{ms}x_e)$ for some $x_i \in D$.

It follows easily that $e = s$, $\{x_1, ..., x_s\}$ is an F-basis for $Q(D)$, and $D = Rx_1 \oplus .... \oplus Rx_s$. □

Now, we restrict our non-commutative valuation v on $\widehat{Q} \cong M_n(Q(D))$ to $Q(D)$, and by definition this is the natural $J(D)$-adic valuation. Using Proposition 3.8.2, we fix a basis $\{x_1, \cdots, x_s\} \subseteq D$ for $Q(D)$ over F, with $D = Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_s$.

Proposition 3.8.3. *Let F' be any finite extension of F; then v extends to F'. Let v' be the standard matrix filtration of $M_s(F')$ with respect to v.*

Then there is a continuous embedding of F-algebras $\phi : Q(D) \hookrightarrow M_s(F')$ such that

$$v'(\phi(x)) \leq v(x) < v'(\phi(x)) + 2t \text{ for all } x \in Q(D).$$

Hence applying the functor M_n to ϕ gives us a continuous embedding $M_n(\phi) : \widehat{Q} \hookrightarrow M_{ns}(F')$ such that $v'(M_n(\phi)(x)) \leq v(x) < v'(M_n(\phi)(x)) + 2t$ for all $x \in \widehat{Q}$.

Proof. It is clear that the embedding $F \to F'$ is an isometry, so it suffices to prove the result for $F' = F$.

Again, define $v_0 : Q(D) \to \mathbb{Z} \cup \{\infty\}, \sum_{i=1}^{s} r_i x_i \mapsto \min\{v(r_i) : i = 1, ..., s\}$, it is readily checked that this is a separated filtration of F-vector spaces, and clearly $v(x) \geq v_0(x)$ for all $x \in Q(D)$.

Then if $x = r_1x_1 + ... + r_sx_s \in Q(D)$ with $0 \leq v(x) < t$, then $v(r_i) \geq 0$ for all i because D is an R-lattice by Lemma 3.8.2. Since $v(x_i) \geq 0$ for all i, $v(r_j) < t$ for some j, so since $r_j \in R$, this means that $v(r_j) = 0$, and hence $v_0(x) = 0$.

So if $x \in Q(D)$ with $v(x) = l$, then $at \leq l < (a+1)t$ where $a = \lfloor \frac{l}{t} \rfloor$, and hence $0 \leq v(\mu^{-a}x) < l$, so $v_0(\mu^{-a}x) = 0$.

Thus $\mu^{-a}x = r_1x_1 + ... + r_sx_s$ with $v(r_i) \geq 0$ for all i, $v(r_j) = 0$ for some i, and hence $v_0(x) = ta$, so $v(x) < v_0(x) + t$.

So it follows that $v_0(x) \leq v(x) < v_0(x)+t$ for all $x \in Q(D)$, in particular $0 \leq v(x_i) < t$ for all i. Hence the identity map $(Q(D), v) \to (Q(D), v_0)$ is bounded.

Now, consider the map $\phi : Q(D) \to \mathrm{End}_F(Q(D)), x \mapsto (Q(D) \to Q(D), d \mapsto x \cdot d)$, this is an injective F-algebra homomorphism.

Also, $\mathrm{End}_F(Q(D))$ carries a natural filtration of F-algebras given by

$$v'(\psi) := \min\{v_0(\psi(x_i)) : i = 1, ..., s\} \text{ for each } \psi \in \mathrm{End}_F(Q(D)).$$

Using the usual isomorphism $\mathrm{End}_F(Q(D)) \cong M_s(F)$, this is just the standard matrix filtration, and it is readily seen that

$$v'(\psi) = \inf\{v_0(\psi(d)) : d \in Q(D), 0 \leq v(d) < t\} \text{ for all } \psi \in \mathrm{End}_F(Q(D)).$$

So if $v_0(x) = r$ then since $v_0(1) = 0$ and $v_0(\phi(x)(1)) = v_0(x \cdot 1) = r$, it follows that $v'(\phi(x)) \leq r$. But for all $i = 1, ..., s$:

$$v_0(x \cdot x_i) > v(x \cdot x_i) - t \geq r + v(x_i) - t \geq r - t, \text{ hence } v'(\phi(x)) \geq r - t.$$

Therefore $v'(\phi(x)) \leq v_0(x) \leq v'(\phi(x)) + t$ for all x, so ϕ is bounded, and hence continuous.

Finally, since for all $x \in Q(D)$, $v_0(x) \leq v(x) \leq v_0(x) + t$ and $v'(\phi(x)) \leq v_0(x) \leq v'(\phi(x)) + t$, it follows that $v'(\phi(x)) \leq v(x) \leq v'(\phi(x)) + 2t$. $\square$

Recall from Definition 3.6.1 the growth rate functions ρ and ρ' of $\widehat{Q}$ and $M_{ns}(F')$ with respect to v and v' respectively. Then using Proposition 3.8.3, we see that for all $x \in \widehat{Q}$:

$\rho(x) = \lim_{n\to\infty} \frac{v(x^n)}{n} \leq \lim_{n\to\infty} \frac{v'(x^n)+2t}{n} = \rho'(x) \leq \rho(x)$ – forcing equality.

Therefore $\rho' = \rho$ when restricted to $\widehat{Q}$.

Now, fix a closed, isolated normal subgroup H of G, and an H-Mahler automorphism φ of G. Then given any basis $\{h_1, \cdots, h_d\}$ for H, setting $q_i := \tau(u(h_i) - 1)$, recall our Mahler expansion (3.10):

$$0 = q_1^{p^m} \tau\partial_1(y) + ... + q_d^{p^m} \tau\partial_d(y) + O(q^{p^m}).$$

Where $\rho(q) > 2\rho(q_i)$ for all i.

We may embed $Q(D)$ continuously into $M_s(F')$ for any finite extension F' of $F = Z(Q(D))$ by Proposition 3.8.3, and since each q_i is a square matrix over $Q(D)$, by choosing F' appropriately, we may ensure that they can be reduced to Jordan normal form inside $M_{ns}(F')$.

But since F' has characteristic p, after raising to sufficiently high p'th powers, a Jordan block becomes diagonal. So we may choose $m_0 \in \mathbb{N}$ such that $q_i^{p^{m_0}}$ is diagonalisable for each i.

But $q_1, \cdots, q_d$ commute, and it is well known that commuting matrices can be simultaneously diagonalised. Hence there exists $a \in M_{ns}(F')$ invertible such that $aq_i^{p^{m_0}}a^{-1}$ is diagonal for each i.

So, let $t_i := aq_ia^{-1}$, then after multiplying (3.10) on the left by a, we get:

$$0 = t_1^{p^m} a\tau\partial_1(y) + ... + t_d^{p^m} a\tau\partial_d(y) + O(aq^{p^m}).$$

Note that since $t_i^{p^{m_0}}$ is diagonal for each i, $\rho'(t_i^{p^{m_0}}) = v'(t_i^{p^{m_0}})$, and $\rho'(t_i) = \rho'(q_i)$ since growth rates are invariant under conjugation by Lemma 3.6.2(iii). Since $\rho' = \rho$ on $\widehat{Q}$, it follows that $\rho(q_i^{p^{m_0}}) = v'(t_i^{p^{m_0}})$.

Moreover, $v'(t_i^{p^m}) = \rho(q_i^{p^m}) = p^m\rho(q_i)$ for each $m \geq m_0$.

Also, recall the definition of the initial power m_1 from Definition 3.2.2, and that $q_i = \tau(u(h_i) - 1) = \tau(u_0(h_i) - 1)^{p^{m_1}}$ for each i. So after replacing m_1 by $m_1 + m_0$ we may ensure that each t_i is diagonal, and hence $v'(t_i) = \rho(q_i)$.

Recall that $\lambda = \inf\{v(\rho(\tau(u(h) - 1))) : h \in H\}$, and let

$$U := \{h \in H : \rho(\tau(u(h) - 1)) > \lambda\},$$

then U is a proper open subgroup of H containing H^p by Lemma 3.7.2. Fix an ordered basis $\{h_1, \cdots, h_d\}$ for H such that $\{h_1^p, \cdots, h_r^p, h_{r+1}, \cdots, h_d\}$ is a basis for U, $q_i = \tau(u(h_i) - 1)$, $t_i = aq_ia^{-1}$ as above.

Then it follows that for all $i \leq r$, $v'(t_i) = \rho(q_i) = \lambda$, and for $i > r$, $v'(t_i) > \lambda$, so we have:

$$0 = t_1^{p^m} a\tau\partial_1(y) + ... + t_r^{p^m} a\tau\partial_r(y) + O(aq^{p^m}) \qquad (3.15)$$

where $\rho(q) > \lambda$. In the next section, we will show how to analyse this expression.

3.9 Using Linear Dependence

Definition 3.9.1. *Given $\nu \in \mathbb{Z}$, $c_1, \cdots, c_m \in M_{ns}(F')$ with $v'(c_i) = \nu$ for some i, we say that $c_1, \cdots, c_m$ are $\mathbb{F}_p$-*linearly independent modulo ν^+ *if for any $\alpha_1, \cdots, \alpha_m \in \mathbb{F}_p$, not all zero, $v'(\alpha_1 c_1 + \cdots + \alpha_m c_m) \leq \nu$.*

Lemma 3.9.2. *$t_1, \cdots, t_r$ are $\mathbb{F}_p$-linearly dependent modulo λ^+.*

Proof. Suppose, for contradiction, that $v'(\alpha_1 t_1 + \cdots + \alpha_r t_r) > \lambda$ for some $\alpha_i \in \mathbb{F}_p$, not all zero, then using Lemma 3.6.2(iii) we see that

$\rho(\alpha_1 q_1 + \cdots + \alpha_r q_r) = \rho(a(\alpha_1 q_1 + \cdots + \alpha_r q_r)a^{-1}) = \rho(\alpha_1 t_1 + \cdots + \alpha_r t_r) = v'(\alpha_1 t_1 + \cdots + \alpha_r t_r) > \lambda$.

But since $q_i = \tau(u(h_i) - 1)$ for each i, we can see using expansions in kG that $\alpha_1 q_1 + \cdots + \alpha_r q_r = \tau(u(h_1^{\alpha_1} \cdots h_r^{\alpha_r}) - 1) + O(q_i q_j)$, and clearly $\rho(O(q_i q_j)) > \lambda$, and hence $\rho(\tau(u(h_1^{\alpha_1} \cdots h_r^{\alpha_r}) - 1)) > \lambda$.

But $U = \{h \in H : \rho(\tau(u(h) - 1)) > \lambda\} = \langle h_1^p, \cdots, h_r^p, h_{r+1}, \cdots, h_d \rangle$, so since p does not divide every α_i, it follows that $h_1^{\alpha_1} \cdots h_r^{\alpha_r} \notin U$, and hence $\rho(\tau(u(h_1^{\alpha_1} \cdots h_r^{\alpha_r}) - 1)) = \lambda$ – contradiction. □

Notation: For each $i = 1, \cdots, ns$, denote by e_i the diagonal matrix with 1 in the i'th diagonal position, 0 elsewhere.

Proposition 3.9.3. *Suppose $d_1, \cdots, d_r \in M_{ns}(F')$ are diagonal, $v'(d_i) = \lambda$ for each i, and suppose that for all $m \in \mathbb{N}$ we have:*

$$0 = d_1^{p^m} a_1 + \cdots + d_r^{p^m} a_r + O(aq^{p^m})$$

where $a_i, a, q \in M_{ns}(F')$, $\rho(q) > \lambda$.

Suppose further that for some $j \in \{1, \cdots, ns\}$, *the* j*'th entries of* $d_1, \cdots, d_r$ *are* $\mathbb{F}_p$*-linearly independent modulo* λ^+. *Then* $e_j a_i = 0$ *for all* $i = 1, \cdots, r$.

Proof. Firstly, since $d_{1,j}, \cdots, d_{r,j}$ are $\mathbb{F}_p$-linearly independent modulo λ^+, it follows immediately that $e_j d_1, \cdots, e_j d_r$ are $\mathbb{F}_p$-linearly independent modulo λ^+. And:

$$0 = (e_j d_1)^{p^m} a_1 + \cdots + (e_j d_r)^{p^m} a_r + O(e_j a q^{p^m}).$$

For convenience, set $d_i' := e_j d_i$, and in a similar vein to the proof of Theorem 3.7.5, define the following matrices:

$$D_m := \begin{pmatrix} d_1'^{p^m} & d_2'^{p^m} & . & . & d_r'^{p^m} \\ d_1'^{p^{m+1}} & d_2'^{p^{m+1}} & . & . & d_r'^{p^{m+1}} \\ . & . & . & . & . \\ . & . & . & . & . \\ d_1'^{p^{m+t-1}} & d_2'^{p^{m+t-1}} & . & . & d_r'^{p^{m+t-1}} \end{pmatrix}, \underline{a} := \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_r \end{pmatrix}$$

Then we can rewrite our expression as:

$$0 = D_m \cdot \underline{a} + \begin{pmatrix} O(e_j a q^{p^m}) \\ O(e_j a q^{p^{m+1}}) \\ . \\ . \\ . \\ O(e_j a q^{p^{m+t-1}}) \end{pmatrix}$$

and multiplying by the adjoint matrix $adj(D_m)$ gives:

$$0 = \det(D_m)\underline{a} + adj(D_m) \begin{pmatrix} O(e_j q^{p^m}) \\ O(e_j a q^{p^{m+1}}) \\ . \\ . \\ . \\ O(e_j a q^{p^{m+t-1}}) \end{pmatrix} \tag{3.16}$$

and using an identical argument to the proof of Lemma 3.7.3, we see that the (i, j)-entry of $adj(D_m)$ has value at least $\frac{p^r-1}{p-1} p^m \lambda - p^{m+j-1}\lambda$.

Since $\rho(q) > \lambda$, fix $c > 0$ such that $\rho(q) > \lambda + c$, and hence $v'(e_j a q^{p^m}) \geq p^m\lambda + p^m c + v(a)$ for all sufficiently high m. Then we see that the i'th entry of the vector

$$adj(D_m)\begin{pmatrix} O(e_jaq^{p^m}) \\ O(e_jaq^{p^{m+1}}) \\ . \\ . \\ . \\ O(e_jaq^{p^{m+t-1}}) \end{pmatrix}$$

has value at least $\frac{p^r-1}{p-1}p^m\lambda + p^mc + v(a)$ for $m >> 0$.

So examining the i'th entry of our expression (3.16) gives that $0 = \det(D_m)a_i + \epsilon_{i,m}$, where $v'(\epsilon_{i,m}) \geq \frac{p^t-1}{p-1}p^{m+r}\lambda + p^mc + v(a)$.

Let $\Delta := \det(D_0)$, then $det(D_m) = \Delta^{p^m}$ for all $m \in \mathbb{N}$, and using [8, Lemma 1.1(ii)] we see that

$$\Delta = \beta \cdot \prod_{\alpha \in \mathbb{P}^{r-1}\mathbb{F}_p} (\alpha_1 d_1' + \cdots + \alpha_r d_r') \text{ for some } \beta \in \mathbb{F}_p.$$

Since $d_1', \cdots, d_r'$ are $\mathbb{F}_p$-linearly independent modulo λ^+, each term in this product has value λ, and moreover is a diagonal matrix, with only the j'th diagonal entry non-zero.

Let δ be the j'th diagonal entry of Δ. Then $\delta \in F'$, $\delta^{-1}\Delta = e_j$, and $v(\delta) = \sum_{\alpha \in \mathbb{P}^{r-1}\mathbb{F}_p} \lambda = \frac{p^r-1}{p-1}\lambda$. So:

$$0 = \delta^{-p^m}\Delta^{p^m}a_i + \delta^{-p^m}\epsilon_{i,m} = e_ja_i + \delta^{-p^m}\epsilon_{i,m}$$

and $v'(\delta^{-p^m}\epsilon_{i,m}) = v'(\epsilon_{i,m}) - p^m\frac{p^r-1}{p-1}\lambda \geq \frac{p^r-1}{p-1}p^m\lambda + p^mc + v(a) - \frac{p^m-1}{p-1}p^m\lambda = v(a) + p^mc \to \infty$ as $m \to \infty$.

Hence $\delta^{-p^m}\epsilon_{i,m} \to 0$ and $e_ja_i = 0$ as required. □

Now, consider again the maps $\partial_1, \cdots, \partial_r : kG \to kG$. These are k-linear derivations of kG by Lemma 2.5.5, and we want to prove that $\partial_i(P) = 0$ for some i.

Lemma 3.9.4. *Let $\delta : kG \to kG$ be any k-linear derivation of kG. Then if $c\tau\delta(P) = 0$ for some $0 \neq c \in M_{ns}(F')$ then $\tau\delta(P) = 0$*

Proof. Let $I = \{a \in M_{ns}(F') : a\tau\delta(P) = 0\}$, then it is clear that I is a left ideal of $M_{ns}(F')$, and $I \neq 0$ since $0 \neq c \in I$. We want to prove that $1 \in I$, and hence $\tau\delta(P) = 0$.

We will first prove that I is right $\widehat{Q}$-invariant:

Given $r \in kG$, $y \in P$, $\delta(ry) = r\delta(y) + \delta(r)y$ since δ is a derivation. So $\tau\delta(ry) = \tau(r)\tau\delta(y) + \tau\delta(r)\tau(y) = \tau(r)\tau\delta(y)$. Therefore, for any $a \in I$, $a\tau(r)\tau\delta(y) = a\tau\delta(ry) = 0$ since $ry \in P$. Thus $a\tau(r) \in I$, and it follows that I is right kG/P-invariant.

Given $s \in kG$, regular mod P (i.e. $\tau(s)$ is a unit in $Q(kG/P)$), we have that $I\tau(s) \subseteq I$. Hence we have a descending chain $I \supseteq I\tau(s) \supseteq I\tau(s)^2 \supseteq \cdots$ of right ideals of $M_{ns}(F')$. So since $M_{ns}(F')$ is Artinian, it follows that $I\tau(s)^n = I\tau(s)^{n+1}$ for some $n \in \mathbb{N}$, so dividing out by $\tau(s)^{n+1}$ gives that $I\tau(s)^{-1} = I$.

Therefore, I is right $Q(kG/P)$-invariant, and passing to the completion gives that it is right $\widehat{Q}$-invariant as required.

This means that $I \cap \widehat{Q}$ is a two sided ideal of the simple ring $\widehat{Q} \cong M_n(Q(D))$. We will prove that $I \cap \widehat{Q} \neq 0$, and it will follow that $I \cap \widehat{Q} = \widehat{Q}$ and thus $1 \in I$.

We know that $\widehat{Q} \cong M_n(Q(D))$ and $Q(D) \hookrightarrow M_s(F')$. Since $Q(D)$ is a division ring, we must have that $M_s(F')$ is free as a right $Q(D)$-module, so let $\{x_1, \cdots, x_t\}$ be a basis for $M_s(F')$ over $Q(D)$. It follows easily that $\{x_1 I_{ns}, \cdots, x_t I_{ns}\}$ is a basis for $M_{ns}(F')$ over $M_n(Q(D)) = \widehat{Q}$.

Now, $c \in I$ and $c \neq 0$, so $c = x_1c_1 + \cdots + x_tc_t$ for some $c_i \in \widehat{Q}$, not all zero, and $c\tau\delta(y) = 0$ for all $y \in P$.

Therefore $0 = c\tau\delta(y) = x_1(c_1\tau\delta(y)) + x_2(c_2\tau\delta_2(y)) + \cdots + x_t(c_t\tau\delta(y))$, so it follows from $\widehat{Q}$-linear independence of $x_1 I_{ns}, \cdots, x_t I_{ns}$ that $c_i\tau\delta(y) = 0$ for all i, and hence $c_i \in I \cap \widehat{Q}$.

So choose i such that $c_i \neq 0$, and since $c_i \in I \cap \widehat{Q}$, we have that $I \neq 0$ as required. □

Theorem 3.9.5. *Let $\delta_1, \cdots, \delta_r : kG \to kG$ be k-linear derivations of kG, and suppose that there exist matrices $a, q, d_1, \cdots, d_r \in M_{ns}(F')$ such that a is invertible, the d_i are diagonal of value λ, $\rho(q) > \lambda$ and for all $y \in P$:*

$$0 = d_1^{p^m} a\tau\delta_1(y) + d_2^{p^m} a\tau\delta_2(y) + \cdots + d_r^{p^m} a\tau\delta_r(y) + O(aq^{p^m}).$$

Suppose further that $d_1, \cdots, d_r$ are $\mathbb{F}_p$-linearly independent modulo λ^+, then $\tau\delta_i(P) = 0$ for all i.

Proof. We will use induction on r. First suppose that $r = 1$.

Then since $0 = d_1^{p^m} a\tau\delta_1(y) + O(aq^{p^m})$, it follows immediately from Proposition 3.9.3 that $e_j a\tau\delta_1(y) = 0$ for any $j = 1, \cdots, ns$ such that $v(d_{1,j}) = \lambda$, and this holds for all $y \in P$.

Since a is a unit, $e_j a \neq 0$, so using Lemma 3.9.4, we see that $\tau\delta_1(P) = 0$ as required.

Now suppose, for induction, that the result holds for all $t < r$:

Assume first that there exists $j = 1, \cdots, ns$ such that $d_{1,j}, \cdots, d_{r,j}$ are $\mathbb{F}_p$-linearly independent modulo λ^+. Then using Proposition 3.9.3 and Lemma 3.9.4 again, we see that $e_j a\tau\delta_i(y) = 0$ for all $i = 1, \cdots, r$, $y \in P$, and hence $\tau\delta_i(P) = 0$ for all i as required.

Hence we may assume that all the corresponding entries of $d_1, \cdots, d_r$ are $\mathbb{F}_p$-linearly dependent modulo λ^+, i.e. given $j = 1, \cdots, ns$, we can find $\beta_1, \cdots, \beta_r \in \mathbb{F}_p$ such that $v(\beta_1 d_{1,j} + \cdots + \beta_r d_{r,j}) > \lambda$. We can of course choose j such that $v(d_{i,j}) = \lambda$ for some i.

Without loss of generality, we may assume that for some $1 \leq t < r$, $d_{1,j}, \cdots, d_{t,j}$ are $\mathbb{F}_p$-linearly independent mod λ^+, and that $d_{t+1,j}, \cdots, d_{r,j}$ can be expressed as $\mathbb{F}_p$-linear combinations of $d_{1,j}, \cdots, d_{t,j}$ modulo $\lambda + 1$.

So, for each $i = 1, \cdots, r$, $e_j d_i = \sum_{k=1}^{t} \beta_{i,k} e_j d_k + \epsilon_i$ for some $\beta_{i,j} \in \mathbb{F}_p$, $\epsilon_i \in M_{ns}(F')$ diagonal with $v'(\epsilon_i) > \lambda$.

Multiplying our expression by e_j gives:

$0 = e_j d_1^{p^m} a\tau\delta_1(y) + \cdots + e_j d_r^{p^m} a\tau\delta_r(y) + O(e_j a q^{p^m})$

$= \sum_{i=1}^{r} e_j d_i^{p^m} a\tau\delta_i(y) + O(e_j a q^{p^m})$

$= \sum_{i=1}^{r} e_j \sum_{k=1}^{t} \beta_{i,k} d_k^{p^m} a\tau\delta_i(y) + \sum_{i=1}^{r} \epsilon_i^{p^m} a\tau\delta_i(y) + O(aq^{p^m})$

$= e_j d_1^{p^m} a \sum_{i=1}^{r} \beta_{i,1}\tau\delta_i(y) + \cdots + e_j d_t^{p^m} a \sum_{i=1}^{r} \beta_{i,t}\tau\delta_i(y) + \sum_{i=1}^{r} \epsilon_i^{p^m} a\tau\delta_i(y) + O(aq^{p^m})$.

Now, set $\delta_k' := \sum_{i=1}^{r} \beta_{i,k}\tau\delta_i(y)$, $d_i' := e_j d_i$ for each $i = 1, \cdots, t$. Then the δ_i' are k-linear derivations of kG, and since ϵ is diagonal and $v'(\epsilon) > \lambda$, it follows that $\rho(\epsilon) > \lambda$, and so $\sum_{i=1}^{r} \epsilon_i^{p^m} a\tau\delta_i(y) + O(aq^{p^m}) = O(aq'^{p^m})$ for some q' with $\rho(q') > \lambda$. Hence:

$$0 = d_1'^{p^m} a\tau\delta_1'(y) + d_2'^{p^m} a\tau\delta_2'(y) + \cdots + d_t'^{p^m} a\tau\delta_t'(y) + O(aq'^{p^m}).$$

So since $e_j d_1, \cdots, e_j d_t$ are $\mathbb{F}_p$-linearly independent module λ^+, it follows from induction that $\tau\delta_k'(P) = 0$ for all i, i.e. for all $y \in P$, $\sum_{i=1}^{r} \beta_{i,k}\tau\delta_i(y) = 0$.

Therefore, we may assume without loss of generality that for all $y \in P$, $\tau\delta_r(y) = \beta_1\tau\delta_1(y) + \cdots + \beta_{r-1}\tau\delta_{r-1}(y)$ for some $\beta_i \in \mathbb{F}_p$, not all zero. Hence:

$$0 = d_1^{p^m} a\tau\delta_1(y) + \cdots + d_r^{p^m} a\tau\delta_r(y) + O(aq^{p^m})$$

$$= (d_1 + \beta_1 d_r)^{p^m} a\tau\delta_1(y) + \cdots + (d_{r-1} + \beta_{r-1}d_r)^{p^m} a\tau\delta_{r-1}(y) + O(aq^{p^m})$$

But $d_1 + \beta_1 d_r, \cdots, d_{r-1} + \beta_{r-1}d_r$ are $\mathbb{F}_p$-linearly independent modulo λ^+, so it follows from induction that $\tau\delta_i(P) = 0$ for all $i = 1, \cdots, r-1$.

Therefore, we have $0 = d_r^{p^m} a\tau\delta_r(y) + O(aq^{p^m})$, so applying induction again gives that $\tau\delta_r(P) = 0$ as required. □

Corollary 3.9.6. *Let G be a p-valuable group, and let P be a faithful prime ideal of kG such that $Q(kG/P)$ is a CSA. If there exists a closed, isolated normal subgroup H of G and a strong H-Mahler automorphism $\varphi \in \text{Aut}^\omega(G)$ such that $\varphi(P) = P$, then P is controlled by a proper open subgroup of G.*

Proof. We know that $t_1, \cdots, t_r$ are $\mathbb{F}_p$-linearly independent modulo λ^+ by Lemma 3.9.2, so applying Theorem 3.9.5 with $\delta_i = \partial_i$ and $d_i = t_i$, it follows that $\tau\partial_i(P) = 0$ for all $i = 1, \cdots, r$.

Hence P is controlled by a proper open subgroup of G by Proposition 2.5.4. □

Chapter 4

The Abelian-by-procyclic case in characteristic p

Now that we have explored the usefulness of Mahler expansions in characteristic p, we will see how to apply this in the abelian-by-procyclic case, where we can exploit the canonical Mahler automorphisms.

4.1 Construction of a valuation

We have seen why non-commutative valuations are useful, but to ultimately prove Theorem B, we will need to define a particular non-commutative valuation on $Q(kG/P)$ that allows for the construction of a special growth preserving polynomial. In this section, we will show how to construct this valuation, following a similar argument to the proof of [1, Theorem C].

Let R be a prime, Noetherian ring, and let $w : R \to \mathbb{N} \cup \{\infty\}$ be a positive Zariskian filtration such that $\mathrm{gr}_w R$ is finitely generated as a module over a central, graded, Noetherian subring A, and we will assume that the positive part $A_{>0}$ of A is not nilpotent, and hence we may fix a minimal prime ideal $\mathfrak{q}$ of A with $\mathfrak{q} \not\supseteq A_{>0}$. Define:

$$T = \{X \in A \backslash \mathfrak{q} : X \text{ is homogeneous } \}.$$

Then T is central, and hence localisable in gr R, and the left and right localisations agree.

Lemma 4.1.1. *Let $\mathfrak{q}' := T^{-1}\mathfrak{q}$, then $\mathfrak{q}'$ is a nilpotent ideal of $T^{-1}A$ and:*

i. There exists $Z \in T$, homogeneous of positive degree, such that $\frac{T^{-1}A}{\mathfrak{q}'} \cong (\frac{T^{-1}A}{\mathfrak{q}'})_0[\overline{Z}, \overline{Z}^{-1}]$, where $\overline{Z} := Z + \mathfrak{q}'$.

ii. The quotient $\frac{(T^{-1}A)_{\geq 0}}{Z(T^{-1}A)_{\geq 0}}$ is Artinian, and $T^{-1}A$ is gr-Artinian, i.e. every descending chain of graded ideals terminates.

Proof. Since A is a graded, commutative, Noetherian ring, this is identical to the proof of [1, Proposition 3.2]. □

Since gr R is finitely generated over A, it follows that $T^{-1}\text{gr } R$ is finitely generated over $T^{-1}A$. So using this lemma, we see that $T^{-1}\text{gr } R$ is gr-Artinian.

Let $S := \{r \in R : \text{gr}(r) \in T\}$, then since w is Zariskian, S is localisable by [27, Corollary 2.2], and there exists a Zariskian filtration w' on $S^{-1}R$ such that $\text{gr}_{w'}\ S^{-1}R \cong T^{-1}\text{gr } R$, and if $r \in R$ then $w'(r) \geq w(r)$ with equality if $r \in S$.

Furthermore, w' satisfies $w'(s^{-1}r) = w'(r) - w(s)$ for all $r \in R$, $s \in S$.

Now, since R is prime, the proof of [1, Lemma 3.3] shows that $S^{-1}R = Q(R)$, so let Q' be the completion of $Q(R)$ with respect to w'.

Let $U := F_0Q'$, which is Noetherian by [28, Ch.II Lemma 2.1.4], and it follows that $\text{gr}_{w'}\ U \cong (T^{-1}\text{gr } R)_{\geq 0}$, and since gr $Q' = T^{-1}\text{gr } R$ is gr-Artinian, Q' is Artinian.

Lemma 4.1.2. *There exists a regular, normal element $z \in J(U) \cap Q'^{\times}$ such that $\frac{U}{zU}$ is Artinian, and for all $n \in \mathbb{Z}$, $F_{nw'(z)}Q' = z^nU$, hence the z-adic filtration on Q' is topologically equivalent to w'. Moreover, U has Krull dimension 1 on both sides.*

Proof. Recall the element $Z \in T^{-1}A$ from Lemma 4.1.1(i), then we can choose an element $z \in U$ such that $\text{gr}_{w'}(z) = Z$. Since w' is Zariskian and Z has positive degree,

$z \in F_1 Q' \subseteq J(U)$.

Furthermore, since $\bar{Z} = Z + \mathfrak{q}'$ is a unit in $\frac{T^{-1}A}{\mathfrak{q}'}$ and $\mathfrak{q}'$ is nilpotent, it follows that Z is a unit in $T^{-1}A$, and hence in $T^{-1}\text{gr } R = \text{gr } Q'$.

This means that z is not a zero divisor in Q', and hence it is a unit since Q' is Artinian. Also, for all $u \in Q'$, $w'(zuz^{-1}) = w'(u)$ since Z is central in gr Q', and hence $zUz^{-1} = U$. Therefore z is normal in U.

Since $(T^{-1}\text{gr } R)_{\geq 0}$ is finitely generated over $(T^{-1}A)_{\geq 0}$, it follows that $\frac{\text{gr } U}{Z\text{gr } U}$ is finitely generated over the image of $\frac{(T^{-1}A)_{\geq 0}}{Z(T^{-1}A)_{\geq 0}} \to \frac{\text{gr } U}{Z\text{gr } U}$.

This image is gr-Artinian by Lemma 4.1.1(ii) and hence $\frac{\text{gr } U}{Z\text{gr } U}$ is also gr-Artinian.

Therefore, since gr $\frac{U}{zU} = \frac{\text{gr } U}{Z\text{gr } U}$ under the quotient filtration, it follows that $\frac{U}{zU}$ is Artinian, and the proof of [1, Proposition 3.4] gives us that U has Krull dimension at most 1 on both sides, and that $F_{nw'(z)}Q' = z^n U$ for all $n \in \mathbb{Z}$. □

So, after passing to a simple quotient $\widehat{Q}$ of Q', since $Q(R)$ is simple it follows that the map $Q(R) \to \widehat{Q}$ is injective, and the image is dense with respect to the quotient filtration, so we can think of $\widehat{Q}$ as a topological completion of $Q(R)$.

Now, setting $V := \widehat{Q}_{\geq 0}$ as the image of U in $\widehat{Q}$, we can choose a maximal order $\mathcal{B}$ in $\widehat{Q}$, which is equivalent to V in the sense of [29, Definition 1.9]. Such an order exists by [1, Theorem 3.11], and it is Noetherian.

Furthermore, let $z \in J(U)$ be the regular, normal element from Lemma 4.1.2, and let $\overline{z} \in J(V)$ be the image of z in V, then $\mathcal{B} \subseteq \overline{z}^{-r}V$ for some $r \in \mathbb{N}$ by [1, Proposition 3.7].

It follows from [1, Theorem 3.6] that $\mathcal{B} \cong M_n(D)$ for some complete non-commutative DVR D, and hence $\widehat{Q} \cong M_n(Q(D))$. So let v be the $J(\mathcal{B})$-adic filtration, i.e. the filtration induced from the valuation on D. Then v is topologically equivalent to the $\overline{z}$-adic filtration on $\widehat{Q}$.

It is clear from the definition that the restriction of v to $Q(R)$ is a non-commutative valuation, and the proof of [1, Theorem C] shows that $(R, w) \to (Q(R), v)$ is continuous.

Note that our construction depends on a choice of minimal prime ideal $\mathfrak{q}$ of A. So altogether, we have proved the following:

Theorem 4.1.3. *Let R be a prime, Noetherian ring with a Zariskian filtration $w : R \to \mathbb{N} \cup \{\infty\}$ such that gr_w R is finitely generated as a module over a central, graded, Noetherian subring A, and the positive part $A_{>0}$ of A is not nilpotent.*

Then for every minimal prime ideal $\mathfrak{q}$ of A with $\mathfrak{q} \not\supseteq A_{>0}$, there exists a corresponding non-commutative valuation $v_{\mathfrak{q}}$ on $Q(R)$ such that the inclusion $(R, w) \to (Q(R), v_{\mathfrak{q}})$ is continuous.

In particular, if P is a prime ideal of kG, then $R = kG/P$ carries a natural Zariskian filtration, given by the quotient of the Lazard filtration on kG, and gr $R \cong \frac{\text{gr } kG}{\text{gr } P}$ is commutative, and if $P \neq J(kG)$ then $(\text{gr } R)_{\geq 0}$ is not nilpotent by [1, Lemma 7.2].

Hence we may apply Theorem 4.1.3 to obtain a non-commutative valuation v on $Q(kG/P)$ such that the natural map $\tau : (kG, w) \to (Q(kG/P), v)$ is continuous.

4.2 Normal elements

Now that we have defined our non-commutative valuation, we need to explore its properties. Again, gr R is finitely generated over a central, graded, Noetherian subring A, and $\mathfrak{q}$ is a minimal prime ideal of A, not containing $A_{>0}$. Recall first the data that we used in the construction of $v_{\mathfrak{q}}$:

- w' – a Zariskian filtration on $Q(R)$ such that $w'(r) \geq w(r)$ for all $r \in R$, with equality if $\mathrm{gr}_w(r) \in A \backslash \mathfrak{q}$. Moreover, if $\mathrm{gr}_w(r) \in A \backslash \mathfrak{q}$ then r is w'-regular.

- Q' – the completion of $Q(R)$ with respect to w'.

- U – the positive part of Q', a Noetherian ring.

- z – a regular, normal element of $J(U)$ such that $z^n U = F_{nw'(z)} Q'$ for all $n \in \mathbb{Z}$.

- $v_{z,U}$ – the z-adic filtration on Q', topologically equivalent to w'.

- $\widehat{Q}$ – a simple quotient of Q'.

- V – the positive part of $\widehat{Q}$, which is the image of U in $\widehat{Q}$.

- $\overline{z}$ – the image of z in V.

- $v_{\overline{z},V}$ – the $\overline{z}$-adic filtration on $\widehat{Q}$, topologically equivalent to the quotient filtration.

- $\mathcal{B}$ – a maximal order in $\widehat{Q}$, equivalent to V, satisfying $\mathcal{B} \subseteq \overline{z}^{-r} V$ for some $r \geq 0$, isomorphic to $M_n(D)$ for some non-commutative DVR D.

- $v_{\overline{z},\mathcal{B}}$ – the $\overline{z}$-adic filtration on $\mathcal{B}$.

- $v_{\mathfrak{q}}$ – the $J(\mathcal{B})$-adic filtration on $\widehat{Q}$, topologically equivalent to $v_{\overline{z},\mathcal{B}}$.

From now on, we will assume further that R is an $\mathbb{F}_p$-algebra.

Recall from Lemma 2.3.1 that an element $r \in \mathcal{B}$ is $v_{\mathfrak{q}}$-regular if and only if r is normal in $\mathcal{B}$. Since it is important in the definition of a special GPP for us to establish v-regularity, we will now explore how to construct normal elements.

Lemma 4.2.1. *Given $r \in R$ such that $\mathrm{gr}(r) \in A\backslash\mathfrak{q}$, we have:*

i. r is normal in U, a unit in Q' and for any $u \in U$, $w'(rur^{-1} - u) > w'(u)$.

ii. $v_{\overline{z},V}(r) = v_{z,U}(r)$.

Proof. i. Since $r \in S = \{s \in R : \mathrm{gr}(s) \in A\backslash\mathfrak{q}\}$ and $Q(R) = S^{-1}R$, r is a unit in Q', and we know that $w'(r) = w(r)$. Given $u \in U$, we want to prove that $rur^{-1} \in U$, thus showing that r is normal in U.

We know that $U = F_0Q'$ is the completion of the positive part $F_0Q(R)$ of $Q(R)$ by definition, and we may assume that u lies in $Q(R)$, i.e. $u = s^{-1}t$ for some $s \in S$, $t \in R$, and $w'(u) = w'(t) - w(s) \geq 0$.

But $\mathrm{gr}(r), \mathrm{gr}(s) \notin \mathfrak{q}$, and hence $\mathrm{gr}(r)\mathrm{gr}(s) \neq 0$, which means that $w(rs) = w(r)+w(s)$. Therefore $w'(r^{-1}ur) = w'((sr)^{-1}tr) = w'(tr) - w(rs) \geq w'(t)+w'(r)-w(r)-w(s) = w'(t) - w'(s) \geq 0$, and so $r^{-1}ur \in U$ as required.

Furthermore, since $\mathrm{gr}(r) \in A$ is central in $\mathrm{gr}\, R$, $w'(ru-ur) > w'(u)+w'(r)$, and thus $w'(rur^{-1}-u) = w'((ru-ur)r^{-1}) \geq w'(ru-ur)-w'(r) > w'(u)+w'(r)-w'(r) = w'(u)$.

ii. Let $t := v_{z,U}(r)$.

So $r \in z^tU\backslash z^{t+1}U = F_{tw'(z)}Q'\backslash F_{(t+1)w'(z)}Q'$, and hence $w'(r) = tw'(z) + j$ for some $0 \leq j < w'(z)$.

Since $\mathrm{gr}_{w'}(r) \in A\backslash\mathfrak{q}$, we have that $w'(r^{-1}) = -w'(r) = -tw'(z) - j$.

Let $\overline{r}$ be the image of r in $\widehat{Q}$. Then since $r \in z^tU$, it is clear that $\overline{r} \in \overline{z}^tV$, hence $v_{\overline{z},V}(r) \geq t$, so it remains to prove that $v_{\overline{z},V}(r) \leq t$.

Suppose that $\overline{r} \in \overline{z}^{t+1}V$, i.e. $r - z^{t+1}u$ maps to zero in $\widehat{Q}$ for some $u \in U$, and hence $z^{-t}r - zu = z^{-t}(r - z^{t+1}u)$ also maps to zero.

Let $a = z^{-t}r$, $b = -zu$. Then $w'(b) \geq w'(u) + w'(z) \geq w'(z)$, $w'(a^{-1}) \geq w'(r^{-1}) + tw'(z) = -tw'(z) - j + tw'(z) = -j$, so $w'(a^{-1}b) \geq w'(z) - j > w'(z) - w'(z) = 0$, and therefore $(a^{-1}b)^n \to 0$ as $n \to \infty$.

So by completeness of Q', the series $\sum_{n\geq 0}(-1)^n(a^{-1}b)^n a^{-1}$ converges in Q', and the limit is the inverse of $a+b$, hence $a + b = z^{-t}r - zu$ is a unit in Q'.

Therefore a unit in Q' maps to zero in $\widehat{Q}$ – contradiction.

Hence $\overline{r} \notin \overline{z}^{t+1}V$, so $v_{\overline{z},V}(r) \leq t$ as required. □

Proposition 4.2.2. *Let $u \in U$ be regular and normal, then u is a unit in Q'. Furthermore, if $w'(uau^{-1} - a) > w'(a)$ for all $a \in Q'$, then setting $\overline{u}$ as the image of u in V, we have that $\overline{u}^{p^m}$ is $v_{\mathfrak{q}}$-regular for sufficiently high $m \in \mathbb{N}$.*

Proof. Since u is regular in U, it is not a zero divisor, so it follows that u is not a zero divisor in Q', and hence a unit since Q' is Artinian.

Since u is normal in U, i.e. $uU = Uu$, it follows that $\overline{u}V = V\overline{u}$, so $\overline{u}$ is normal in V. We want to prove that for m sufficiently high, $\overline{u}^{p^m}$ is normal in $\mathcal{B} = F_0\widehat{Q}$, and it will follow from Lemma 2.3.1 that it is $v_{\mathfrak{q}}$-regular.

We know that $w'(uau^{-1} - a) > w'(a)$ for all $a \in Q'$, so let $\theta : Q' \to Q'$ be the conjugation action of u, then $(\theta - id)(F_nQ') \subseteq F_{n+1}Q'$ for all $n \in \mathbb{Z}$.

Therefore, for all $k \in \mathbb{N}$, $(\theta - id)^k(F_nQ') \subseteq F_{n+k}Q'$.

Since Q' is an $\mathbb{F}_p$-algebra, it follows that $(\theta^{p^m} - id)(F_nQ') = (\theta - id)^{p^m}(F_nQ') \subseteq F_{n+p^m}Q'$, and clearly θ^{p^m} is conjugation by $\overline{u}^{p^m}$. So fix $k \in \mathbb{N}$ such that $p^k \geq w'(z)$. Then we know that $z^nU = F_{nw'(z)}U$, so $(\theta^{p^k} - id)(z^nU) \subseteq F_{nw'(z)+p^k}U \subseteq F_{(n+1)w'(z)}U = z^{n+1}U$.

Hence we have that for all $a \in Q'$, $v_{z,U}(u^{p^k}au^{-p^k} - a) > v_{z,U}(a)$, and it follows immediately that $v_{\overline{z},V}(\overline{u}^{p^k}a\overline{u}^{-p^k} - a) > v_{\overline{z},V}(a)$ for all $a \in \widehat{Q}$.

For convenience, let $v := \overline{u}^{p^k} \in V$. We know that $v_{\overline{z},V}(vav^{-1} - a) > v_{\overline{z},V}(a)$ for all $a \in \widehat{Q}$, and we want to prove that v^{p^m} is normal in $\mathcal{B}$ for m sufficiently high.

Let $I = \{x \in V : qx \in V$ for all $q \in \mathcal{B}\}$, then I is a two-sided ideal of V, and since $\mathcal{B} \subseteq \overline{z}^{-r}V$, we have that $\overline{z}^rV \subseteq I$. Also note that $\mathcal{B}I \subseteq I$, because if $q, s \in \mathcal{B}$, $x \in I$ then $sx \in V$ and $q(sx) = (qs)x \in V$, thus $sx \in I$.

Let $\psi : \widehat{Q} \to \widehat{Q}$ be conjugation by v, then we know that $(\psi - id)(\overline{z}^nV) \subseteq \overline{z}^{n+1}V$, and hence $(\psi - id)^s(V) \subseteq \overline{z}^sV$ for all s. Choose $m \in \mathbb{N}$ such that $p^m \geq r$, then $(\psi^{p^m} - id)(V) = (\psi - id)^{p^m}(V) \subseteq \overline{z}^{p^m}V \subseteq \overline{z}^rV \subseteq I$.

Therefore, for all $a \in V$, $v^{p^m}av^{-p^m} - a \in I$, and in particular, for all $a \in I$, $v^{p^m}av^{-p^m} \in I$, so $v^{p^m}Iv^{-p^m} \subseteq I$. So set $b := v^{p^m}$, then it follows from Noetherianity of V that $bIb^{-1} = I$.

Finally, consider the subring $\mathcal{B}' := b^{-1}\mathcal{B}b$ of $\widehat{Q}$ containing V, then since $\mathcal{B}$ is a maximal order equivalent to V, it follows immediately that $\mathcal{B}'$ is equivalent to V, and that $\mathcal{B}'$ is also maximal.

Given $c \in \mathcal{B}'$, $c = b^{-1}qb$ for some $q \in \mathcal{B}$. So given $x \in I$, $cx = b^{-1}qbx = b^{-1}qbxb^{-1}b \in$

$b^{-1}Ib = I$, so $c \in O_l(I) := \{q \in \widehat{Q} : qI \subseteq I\}$, and hence $\mathcal{B}' \subseteq O_l(I)$.

But $O_l(I)$ is an order in $\widehat{Q}$, equivalent to V, and since $\mathcal{B}I \subseteq I$ this order contains $\mathcal{B}$. Since $\mathcal{B}$ and $\mathcal{B}'$ are maximal orders and are both contained in $O_l(I)$, it follows that $O_l(I) = \mathcal{B} = \mathcal{B}' = b\mathcal{B}b^{-1}$.

Therefore $b = v^{p^m} = \overline{u}^{p^{m+k}}$ is normal in $\mathcal{B}$ as required. □

In particular, it is clear that $z \in U$ satisfies the property that $w'(zaz^{-1} - a) > w'(a)$ for all $a \in Q'$, thus $\overline{z}^{p^m}$ is normal in $\mathcal{B}$ for large m.

The next result will be very useful to us later when we want to compare values of elements in $Q(kG/P)$ based on their values in kG.

Theorem 4.2.3. *Given $r \in R$ such that $gr_w(r) \in A\backslash\mathfrak{q}$, there exists $m \in \mathbb{N}$ such that r^{p^m} is $v_{\mathfrak{q}}$-regular inside $\widehat{Q}$. Also, if $s \in R$ with $w(s) > w(r)$ then for sufficiently high m, $v_{\mathfrak{q}}(s^{p^m}) > v_{\mathfrak{q}}(r^{p^m})$.*

Moreover, if $w(s) = w(r)$ and $gr_w(s) \in \mathfrak{q}$ then we also have that $v_{\mathfrak{q}}(s^{p^m}) > v_{\mathfrak{q}}(r^{p^m})$ for sufficiently high m.

Proof. Since $\mathrm{gr}_w(r) \in A\backslash\mathfrak{q}$, it follows from Lemma 4.2.1(i) that r is normal and regular in U, and $w'(rur^{-1} - u) > w'(u)$ for all $u \in U$. So for $m \in \mathbb{N}$ sufficiently high, r^{p^m} is $v_{\mathfrak{q}}$-regular by Proposition 4.2.2.

Note that since $\mathrm{gr}_w(r) \in A\backslash\mathfrak{q}$, we have that $w'(r) = w(r)$. In fact, since $\mathrm{gr}_w(r)$ is not nilpotent, we actually have that $w'(r^n) = w(r^n) = nw(r)$ for all $n \in \mathbb{N}$. So if $w(s) > w(r)$, then for any n, $w'(s^n) \geq nw(s) > nw(r) = w'(r^n)$.

Moreover, if $w(s) = w(r)$ and $\mathrm{gr}_w(s) \in \mathfrak{q}$, then since $\mathfrak{q}' = T^{-1}\mathfrak{q}$ is nilpotent by Lemma 4.1.1, it follows that for n sufficiently high, $w'(s^n) > nw'(s)$, and hence $w'(s^n) > nw(s) = nw(r) = w(r^n) = w'(r^n)$.

So, in either case, after replacing r and s by high p'th powers of r and s if necessary, we may assume that $w'(s) > w'(r)$, i.e. $w'(s) \geq w'(r) + 1$.

It follows that for every $M > 0$, we can find $m \in \mathbb{N}$ such that $w'(s^{p^m}) \geq w'(r^{p^m}) + M$. First we will prove the same result for $v_{z,U}$:

Given $M > 0$, let $N = w'(z)(M + 1)$, so that $M = \frac{1}{w'(z)}N - 1$, then choose m such that $w'(s^{p^m}) \geq w'(r^{p^m}) + N$, and let $l := v_{z,U}(s^{p^m})$, $t := v_{z,U}(r^{p^m})$.

So $s^{p^m} \in z^l U \backslash z^{l+1}U = F_{lw'(z)}Q' \backslash F_{(l+1)w'(z)}Q'$, and $r^{p^m} \in z^t U \backslash z^{t+1}U = F_{lw'(z)}Q' \backslash F_{(l+1)w'(z)}Q'$.

Hence $(l + 1)w'(z) \geq w'(s^{p^m}) \geq lw'(z)$ and $(t + 1)w'(z) \geq w'(r^{p^m}) \geq tw'(z)$.

Therefore, $v_{z,U}(s^{p^m}) = l = \frac{1}{w'(z)}((l + 1)w'(z)) - 1 \geq \frac{1}{w'(z)}w'(s^{p^m}) - 1$

$\geq \frac{1}{w'(z)}(w'(r^{p^m}) + N) - 1 \geq \frac{1}{w'(z)}(tw'(z) + N) - 1$

$= t + \frac{1}{w'(z)}N - 1 = t + M = v_{z,U}(r^{p^m}) + M$ as required.

Now, since $\mathrm{gr}_w(r) \in A \backslash \mathfrak{q}$, we have that $v_{\overline{z},V}(r^{p^m}) = v_{z,U}(r^{p^m})$ for all m by Lemma 4.2.1(ii).

Therefore, since $v_{\overline{z},V}(s^{p^m}) \geq v_{z,U}(s^{p^m})$ for all m, it follows that for every $M > 0$, there exists $m \in \mathbb{N}$ such that $v_{\overline{z},V}(s^{p^m}) \geq v_{\overline{z},V}(r^{p^m}) + M$.

Now we will consider $v_{\overline{z},\mathcal{B}}$, the $\overline{z}$-adic filtration on $\widehat{Q}$.

Recall that $V \subseteq \mathcal{B} \subseteq \overline{z}^{-r}V$, and thus $z^n V \subseteq z^n \mathcal{B} \subseteq z^{n-r}V$ for all n. Hence

$v_{\overline{z},V}(v) - r \leq v_{\overline{z},\mathcal{B}}(v) \leq v_{\overline{z},V}(v)$ for all $v \in V$.

For any $M > 0$, choose m such that $v_{\overline{z},V}(s^{p^m}) \geq v_{\overline{z},V}(r^{p^m}) + M + r$. Then:

$$v_{\overline{z},\mathcal{B}}(s^{p^m}) \geq v_{\overline{z},V}(s^{p^m}) - r \geq v_{\overline{z},V}(r^{p^m}) + M + r - r \geq v_{\overline{z},\mathcal{B}}(r^{p^m}) + M.$$

Now, using Proposition 4.2.2, we know that we can find $k \in \mathbb{N}$ such that $x := \overline{z}^{p^k}$ is normal in $\mathcal{B}$, i.e. $x\mathcal{B} = \mathcal{B}x$ is a two-sided ideal of $\mathcal{B}$. Then since $\mathcal{B} \cong M_n(D)$ for some non-commutative DVR D, it follows that $x\mathcal{B} = J(\mathcal{B})^a$ for some $a \in \mathbb{N}$, and $x^m\mathcal{B} = J(\mathcal{B})^{am}$ for all m.

So, choose $m \in \mathbb{N}$ such that r^{p^m} is $v_{\mathfrak{q}}$-regular, $v_{\mathfrak{q}}(r^{p^m}) \geq a$ and $v_{\overline{z},\mathcal{B}}(s^{p^m}) \geq v_{\overline{z},\mathcal{B}}(r^{p^m}) + p^k$. Then suppose that $v_{\mathfrak{q}}(r^{p^m}) = n$, i.e. $r^{p^m} \in J(\mathcal{B})^n \backslash J(\mathcal{B})^{n+1}$ and $n \geq a$.

We have that $n = qa + t$ for some $q, t \in \mathbb{N}$, $0 \leq t < a$, so $q \geq 1$ and $qa \leq n < n+1 \leq (q+1)a$. Therefore:

$$r^{p^m} \in J(\mathcal{B})^n \subseteq J(\mathcal{B})^{qa} = x^q\mathcal{B} = \overline{z}^{p^k q}\mathcal{B}, \text{ and so } v_{\overline{z},\mathcal{B}}(r^{p^m}) \geq p^k q.$$

Hence $v_{\overline{z},\mathcal{B}}(s^{p^m}) \geq v_{\overline{z},\mathcal{B}}(r^{p^m}) + p^k \geq p^k q + p^k = p^k(q+1)$, so $s^{p^m} \in \overline{z}^{p^k(q+1)}\mathcal{B} = x^{q+1}\mathcal{B} = J(\mathcal{B})^{a(q+1)} \subseteq J(\mathcal{B})^{n+1}$.

Therefore $v_{\mathfrak{q}}(s^{p^m}) \geq n + 1 > n = v_{\mathfrak{q}}(r^{p^m})$.

Furthermore, for all $l \in \mathbb{N}$, $v_{\mathfrak{q}}(s^{p^{m+l}}) \geq p^l v_{\mathfrak{q}}(s^{p^m}) > p^l v_{\mathfrak{q}}(r^{p^m}) = v_{\mathfrak{q}}(r^{p^{m+l}})$ – the last equality holds since r^{p^m} is $v_{\mathfrak{q}}$-regular.

Hence $v_{\mathfrak{q}}(s^{p^n}) > v_{\mathfrak{q}}(r^{p^n})$ for all sufficiently high n as required. □

4.3 The Extended commutator subgroup

From now on, we will fix G a p-valuable, non-abelian, abelian-by-procyclic group with principal subgroup H, procyclic element X. We will also assume for now that G has *split-centre*, i.e. $1 \to Z(G) \to G \to \frac{G}{Z(G)} \to 1$ is a split exact sequence of groups.

Firstly, recall from Appendix C.1 how we define a valuation $w : \mathcal{L} \to \mathbb{R} \cup \{\infty\}$ on a $\mathbb{Z}_p$-Lie algebra $\mathcal{L}$. Also recall that we define $\epsilon := \begin{cases} 1 & p > 2 \\ 2 & p = 2 \end{cases}$, and we say a free $\mathbb{Z}_p$-Lie algebra $\mathcal{L}$ of finite rank is *powerful* if $[\mathcal{L}, \mathcal{L}] \subseteq p^\epsilon \mathcal{L}$. If $\mathcal{L}$ is powerful and w is a valuation on $\mathcal{L}$, then w corresponds to a p-valuation ω on the uniform group $\exp(\mathcal{L})$ defined by $\omega(g) := w(\log(g))$. We use this construction to prove the following result:

Proposition 4.3.1. *Let G be a non-abelian, uniform, abelian-by-procyclic group with split-centre, let $\mathcal{L} := \log(G)$, and let $V := \exp([\mathcal{L}, \mathcal{L}]) \subseteq H^p$. Then there exists a basis $\{h_1, \cdots, h_d\}$ for H, $r \leq d$ such that $\{h_{r+1}, \cdots, h_d\}$ is a basis for $Z(G)$ and $\{h_1^{p^{t_1}}, \cdots, h_r^{p^{t_r}}\}$ is a basis for V for some $t_i \geq \epsilon$.*

Moreover, there exists an abelian p-valuation ω on G such that (i) $\{h_1, \cdots, h_d, X\}$ is an ordered basis for (G, ω), and (ii) $\omega(h_1^{p^{t_1}}) = \omega(h_2^{p^{t_2}}) = \cdots = \omega(h_r^{p^{t_r}}) > \omega(X)$.

Proof. First, note that since G has split centre, we have that $G \cong Z(G) \times \frac{G}{Z(G)}$. In fact, since $Z(G) \subseteq H$, we have that $H \cong Z(G) \times H'$ for some $H' \leq H$, normal and isolated in G.

Set $\mathcal{L} := \log(G)$, $\mathcal{H} := \log(H)$, $x := \log(X)$. Then $\mathcal{L}$ is a powerful $\mathbb{Z}_p$-Lie algebra, free of finite rank, and $\mathcal{L} = \mathcal{H} \rtimes \mathbb{Z}_p x$. Also, it follows from the preceding paragraph that $\mathcal{H} = Z(\mathcal{L}) \oplus \mathcal{H}'$, where $\mathcal{H}' := \log(H')$, and clearly $[\mathcal{L}, \mathcal{L}] = [x, \mathcal{H}] = [x, \mathcal{H}']$.

Note that $\mathrm{ad}(x) : \mathcal{H}' \to [\mathcal{L}, \mathcal{L}]$ is an injective, $\mathbb{Z}_p$-linear map, since its kernel is $\mathcal{H}' \cap Z(\mathcal{L}) = 1$. Since $[\mathcal{L}, \mathcal{L}] \subseteq \mathcal{H}'$, it follows that $[\mathcal{L}, \mathcal{L}]$ has the same rank as $\mathcal{H}'$, so by the Elementary Divisors Theorem, there exists a basis $\{v_1, \cdots, v_r\}$ for $\mathcal{H}'$ such that $\{p^{t_1}v_1, ..., p^{t_r}v_r\}$ is a basis for $[\mathcal{L}, \mathcal{L}]$ for some $t_i \geq 0$. And since $\mathcal{L}$ is powerful, we have in fact that $t_i \geq \epsilon$ for each i.

Let $\{v_{r+1}, \cdots, v_d\}$ be any basis for $Z(\mathcal{L})$, and set $h_i := \exp(v_i)$ for each $i = 1, \cdots d$. It follows that $\{h_1^{p^{t_1}}, ..., h_r^{p^{t_r}}\}$ is a basis for V, and that $\{h_{r+1}, \cdots, h_d\}$ is a basis for $Z(G)$ as required.

Now, the proof of [32, Lemma 26.13] shows that if ω is any p-valuation on G and we choose $c > 0$ with $\omega(g) > c + \frac{1}{p-1}$ for all $g \in G$, then $\omega_c(g) := \omega(g) - c$ defines a new p-valuation on G satisfying $\omega_c((g, h)) > \omega_c(g) + \omega_c(h)$, which preserves ordered bases.

So if ω is an *integer valued* p-valuation satisfying i and ii, then take $c := \frac{1}{e}$ for any integer $e \geq 2$ and ω_c will also satisfy i and ii. Also $\omega_c(G) \subseteq \frac{1}{e}\mathbb{Z}$ and $\omega_c((g, h)) > \omega_c(g) + \omega_c(h)$ for all $g, h \in G$, i.e. ω_c is abelian.

Therefore, it remains to show that we can define an integer valued p-valuation on G satisfying i and ii.

Assume without loss of generality that $t_1 \geq t_i$ for all $i = 1, \cdots, r$. Choose $a \in \mathbb{Z}$ with $a > \epsilon$, and set $a_i := a + t_1 - t_i$ for each i, so that $a_i + t_i = a_j + t_j$ for all $i, j = 1, \cdots, r$.

For convenience, set $v_{d+1} := x$, and for each $i > r$, set $a_i = \epsilon$. Then define:

$$w : \mathcal{L} \to \mathbb{Z} \cup \{\infty\}, \sum_{i=1,\cdots,d} \alpha_i v_i \mapsto \min\{v_p(\alpha_i) + a_i : i = 1, \cdots, d\}.$$

We will prove that w is a valuation on $\mathcal{L}$, and that $w(p^{t_i}v_i) = w(p^{t_j}v_j) > w(x)$ for all $i, j \leq r$. Then by defining ω on G by $\omega(g) = w(\log(g))$, the result will follow.

Firstly, the property that $w(p^{t_i}v_i) = w(p^{t_j}v_j) > w(x)$ is clear, since $w(p^{t_i}v_i) = t_i + a_i = t_j + a_j = w(p^{t_j}v_j)$ for all $i, j < d$, and $a_i + t_i = a + t_1 \geq a > \epsilon = w(x)$.

It is also clear from the definition of w that $w(u+v) \geq \min\{w(u), w(v)\}$, $w(\alpha u) = v_p(\alpha) + w(u)$, $w(u) = \infty$ if and only if $u = 0$, and $w(u) > \frac{1}{p-1}$ for all $u, v \in \mathcal{L}$, $\alpha \in \mathbb{Z}_p$.

Therefore it remains to prove that $w([u,v]) \geq w(u) + w(v)$, and it is straightforward to show that it suffices to prove this for basis elements.

So since $v_{r+1}, \cdots, v_d$ are central, we need only to show that $w([x, v_i]) \geq w(x) + w(v_i)$ for all $i \leq r$.

We have that $[x, v_i] = \alpha_{i,1}p^{t_1}v_1 + ... + \alpha_{i,r}p^{t_r}v_r$ for some $\alpha_{i,j} \in \mathbb{Z}_p$, so:

$w([x, v_i]) = \min_{j=1,\cdots,r}\{v_p(\alpha_{i,j}) + t_j + a_j\} = \min_{j=1,\cdots,r}\{v_p(\alpha_{i,j})\} + t_i + a_i \geq a_i + t_i \geq a_i + \epsilon = w(v_i) + w(x)$. □

Remark: This result strongly depends on uniformity of G, it does not hold in general. For example, if $p > 2$ and $\mathcal{L} = \text{Span}_{\mathbb{Z}_p}\{x, y, z\}$ with $[y, z] = 0$, $[x, y] = py$, $[x, z] = y + pz$, then $\mathcal{L}$ is not powerful, and there is no valuation w on $\mathcal{L}$ that equates the values of basis elements for $[\mathcal{L}, \mathcal{L}]$.

Now, we know that $G = H \rtimes \langle X \rangle$ is p-valuable, so for each $m \in \mathbb{N}$, define $G_m := H \rtimes \langle X^{p^m} \rangle$. It is immediate that G_m is an open, normal subgroup of G, and that it is non-abelian, p-valuable, abelian-by-procyclic with principal subgroup H, procyclic element X^{p^m} and split centre.

Lemma 4.3.2. *There exists $m \in \mathbb{N}$ such that G_m is a uniform group.*

Proof. Recall that G is an open subgroup of the p-saturated group $Sat(G)$, i.e there exists $t \in \mathbb{N}$ with $Sat(G)^{p^t} \subseteq G$. Choose any such t and let $m := t + \epsilon$.

Given $h \in H$, since h and $X^{p^m}hX^{-p^m}$ commute, it follows from Lemma C.1.1 that $(X^{p^m}, h) = \exp(\sum_{n\geq 1} \frac{1}{n!}(\mathrm{ad}(p^m x)^n(v))$ where $x = \log(X)$ and $v = \log(h)$ lie in $\log(Sat(G))$.

We want to prove that $(X^{p^m}, h) \in H^{p^\epsilon} = G^{p^\epsilon} \cap H$, so since $Sat(G)^{p^m} = Sat(G)^{p^{t+\epsilon}} \subseteq G^{p^\epsilon}$, it suffices to prove that $\frac{1}{n!}ad(p^m x)^n(v) \in p^m \log(Sat(G))$ for all $n \geq 1$.

Clearly, for each n, $\mathrm{ad}(p^m x)^n(v) = [p^m x, \mathrm{ad}(p^m x)^{n-1}(u)]$, so we only need to prove that $\mathrm{ad}(p^m x)^{n-1}(u) \in p^{v_p(n!)} \log(Sat(G))$, in which case:

$$\frac{1}{n!}\,\mathrm{ad}(p^m x)^n(v) = \frac{p^m}{n!}[x, \mathrm{ad}(p^m x)^{n-1}(u)] \in \frac{p^{v_p(n!)+m}}{n!} \log(Sat(G)) \subseteq p^m \log(Sat(G)).$$

Let w be a saturated valuation on $\log(Sat(G))$, i.e. if $w(x) > \frac{1}{p-1} + 1$ then $x = py$ for some $y \in \log(Sat(G))$.

Then since $w(\mathrm{ad}(p^m x)^{n-1}(u)) \geq (n-1)w(p^m x) + w(u) > \frac{n-1}{p-1} + \frac{1}{p-1}$, it follows that $\mathrm{ad}(p^m x)^{n-1}(u) = p^k v$ for some $v \in \log(Sat(G))$, $k \geq \frac{n-1}{p-1}$.

We will show that $k \geq v_p(n!)$, and it will follow that $\mathrm{ad}(p^m x)^{n-1}(u) = p^k v \in p^{v_p(n!)} \log(Sat(G))$.

If $n = a_0 + a_1 p + \cdots + a_r p^r$ for some $0 \leq a_i < p$, then let $s(n) = a_0 + a_1 + \cdots + a_r$. We know from [25, I 2.2.3] that $v_p(n!) = \frac{n-s(n)}{p-1}$.

Suppose that $v_p(n!) > \frac{n-1}{p-1}$, i.e. $\frac{n-s(n)}{p-1} > \frac{n-1}{p-1}$, and hence $s(n) < 1$. This means that $s(n) = 0$ and hence $n = 0$ – contradiction.

Therefore $k \geq \frac{n-1}{p-1} \geq v_p(n!)$ as required. $\square$

So from now on, fix $c \in \mathbb{N}$ minimal such that G_c is uniform, and let $\mathcal{L} := \log(G_c)$ – a powerful $\mathbb{Z}_p$-subalgebra of $\log(Sat(G))$.

Lemma 4.3.3. $Z(G) \cap \exp([\mathcal{L}, \mathcal{L}]) = 1$

Proof. Since G has split centre, G_c must also have split centre, so $1 \to Z(G_c) \to G_c \to \frac{G_c}{Z(G_c)} \to 1$ is split-exact by definition. So since G_c is p-saturable and exp and log are isomorphisms of categories by [25, IV.3.2.6], it follows that $0 \to \log(Z(G_c)) \to \mathcal{L} \to \frac{\mathcal{L}}{\log(Z(G_c))} \to 0$ is split exact.

But since $Z(G) \subseteq H \subseteq G_c$ and $Z(G_c) = Z(G) \cap G_c$ by [1, Lemma 8.4(a)], we have that $Z(G_c) = Z(G)$. It is also easily seen that $\log(Z(G_c)) = Z(\mathcal{L})$. Therefore $\mathcal{L}$ has split centre, and hence $Z(\mathcal{L}) \cap [\mathcal{L}, \mathcal{L}] = 0$.

So if $g \in Z(G) \cap \exp([\mathcal{L}, \mathcal{L}])$ then $\log(g) \in Z(\mathcal{L}) \cap [\mathcal{L}, \mathcal{L}] = 0$, so $g = 1$. $\square$

Using this lemma, we make the following definition:

Definition 4.3.4. *Define the* extended commutator subgroup *of G to be*

$$c(G) := (Z(G) \times \exp([\mathcal{L}, \mathcal{L}])) \rtimes \langle X^{p^c} \rangle \subseteq G_c.$$

Recall that if we define $\varphi : G \to G$ to be the automorphism induced by conjugation by X, then φ is a strong H-Mahler automorphism in the sense of Definition 3.1.3. Also, it is clear that $\varphi \neq 1$ if G is non-abelian, and since φ is inner, it is clear that $\varphi(P) = P$ for all prime ideals P of kG, so we may apply the results of the previous chapter.

In particular, recall the definition of the Mahler approximation function $u_m = z(\varphi^{p^m}) : G \to Sat(G)$, and recall from Lemma 3.2.1 that for $h \in H$, $u_m(h) = \lim\limits_{n\to\infty} (\varphi^{p^{n+m}}(h)h^{-1})^{p^{-n}} = \lim\limits_{n\to\infty} (X^{p^{m+n}} h X^{-p^{m+n}} h^{-1})^{p^{-n}} = \lim\limits_{n\to\infty} (X^{p^{m+n}} h^{p^n} X^{-p^{m+n}} h^{-p^n})^{p^{-2n}} = \exp([p^n \log(x), \log(h)])$ by [25, IV 3.2.3], so we have a Lie theoretic description of our Mahler approximations.

Proposition 4.3.5. *If G is any p-valuable, abelian-by-procyclic group with split centre, then:*

i. $c(G)$ is an open normal subgroup of G.

ii. There exists a basis $\{k_1, k_2, \cdots, k_d\}$ for H such that $\{k_{r+1}, \cdots, k_d\}$ is a basis for $Z(G)$ and $\{u_c(k_1), u_c(k_2), \cdots, u_c(k_r), k_{r+1}, \cdots, k_d, X^{p^c}\}$ is a basis for $c(G)$.

iii. We may choose this basis $\{k_1, \cdots, k_d\}$ such that for each $i \leq r$, there exist $\alpha_{i,j} \in \mathbb{Z}_p$ with $p \mid \alpha_{i,j}$ for $j < i$ and $\alpha_{i,i} = 1$, such that $Xu_c(k_i)X^{-1} = u_c(k_1)^{\alpha_{i,1}} \cdots u_c(k_r)^{\alpha_{i,r}}$.

iv. There is an abelian p-valuation ω on $c(G)$ such that $\{u_c(k_1), \cdots, u_c(k_r), k_{r+1}, \cdots, k_d, X^{p^c}\}$ is an ordered basis for $(c(G), \omega)$ and $\omega(u_c(k_i)) = \omega(u_c(k_j)) > \omega(X^{p^c})$ for all $i, j \leq r$.

Proof. Let $V = \exp([\mathcal{L}, \mathcal{L}])$, and let $x = \log(X) \in \log(Sat(G))$.

If $h \in V$ then $h = \exp([p^c x, u])$ for some $u \in \log(H)$, i.e. $u = \log(k)$, and so:

$$h = \exp([\log(X^{p^c}), \log(k)]) = \lim_{n\to\infty}(X^{p^{n+c}} k^{p^n} X^{-p^{n+c}} k^{-p^n})^{p^{-2n}} \text{ by [25, IV. 3.2.3].}$$

Thus $XhX^{-1} = \lim_{n\to\infty}(X^{p^{n+c}}(XkX^{-1})^{p^n} X^{-p^{n+c}} (XkX^{-1})^{-p^n})^{p^{-2n}} = \exp([\log(X^{p^c}), \log(XkX^{-1})])$.

Clearly this lies in V, and hence V is normal in G.

Using Lemma C.1.1, it is straightforward to show that for all $h \in H$, $(X^{p^c}, h) \in V$, therefore:

$$hX^{p^c}h^{-1} = (X^{p^c}, h)^{-1} X^{p^c} \in V \rtimes \langle X^{p^c} \rangle.$$

It follows that $c(G) = (Z(G) \times V) \rtimes \langle X^{p^c} \rangle$ is normal in G.

Using Proposition 4.3.1, we may choose a basis $\{h_1, \cdots, h_d\}$ for H such that $\{h_{r+1}, \cdots, h_d\}$ is a basis for $Z(G)$ and $\{h_1^{p^{t_1}}, \cdots, h_r^{p^{t_r}}\}$ is a basis for V. Therefore $c(G)$ has basis $\{h_1^{p^{t_1}}, \cdots, h_r^{p^{t_r}}, h_{r+1}, \cdots, h_d, X^{p^{c+1}}\}$, and hence it is open in G as required.

Now, for each $i = 1, \cdots, d$, let $u_i = \log(h_i)$, then $\{u_1, \cdots, u_d\}$ is a $\mathbb{Z}_p$-basis for $\log(H)$, and $\{p^{t_1}u_1, \cdots, p^{t_r}u_r\}$ is a basis for $[\mathcal{L}, \mathcal{L}]$.

Therefore, for each i, $p^{t_i}u_i = [p^c x, v_i]$ for some $v_i \in \log(H)$, in fact we may assume that $v_i \in \mathrm{Span}_{\mathbb{Z}_p}\{u_1, \cdots, u_r\}$, and it follows that $\{v_1, \cdots, v_r\}$ forms a basis for $\mathrm{Span}_{\mathbb{Z}_p}\{u_1, \cdots, u_r\}$.

Let $k_i := \exp(v_i)$ for each $i = 1, \cdots r$, and for $i > r$ set $k_i := h_i$. Then we know that

$$u_c(k_i) = \exp([p^c \log(X), \log(k_i)]) = \exp([p^c x, v_i]) = \exp(p^{t_i}u_i) = h_i^{p^{t_i}} \text{ for each } i \leq r.$$

It follows that $\{u_c(k_1), \cdots, u_c(k_r), k_{r+1}, \cdots, k_d, X^{p^c}\}$ is a basis for $c(G)$, thus giving part *ii*.

Now, V is normal in G, and clearly V^p is also normal, so consider the action ψ of X on the r-dimensional $\mathbb{F}_p$-vector space $\frac{V}{V^p}$, i.e. $\psi(hV^p) = (XhX^{-1})V^p$. It is clear that this action ψ is $\mathbb{F}_p$-linear.

Furthermore, since G_c is uniform and $X^{p^c} \in G_c$, we have that $\psi^{p^c} = id$, i.e. $(\psi - id)^{p^c} = 0$. Therefore ψ has a 1-eigenvector in $\frac{V}{V^p}$, and it follows that we may choose a basis for $\frac{V}{V^p}$ such that ψ is represented by an upper-triangular matrix, with 1's on the diagonal.

This basis is obtained by transforming $\{u_c(k_1),\cdots,u_c(k_r)\} = \{h_1^{p^{t_1}},\cdots,h_r^{p^{t_r}}\}$ by an invertible matrix over $\mathbb{Z}_p$. The new basis will also have the same form $\{u_c(k_1'),\cdots,u_c(k_r')\} = \{h_1'^{p^{t_1}},\cdots,h_r'^{p^{t_r}}\}$ as described by *ii*, and it will satisfy *iii* as required.

Finally, using Proposition 4.3.1, we see that there is an abelian p-valuation ω on the uniform group G_c such that $\omega(u_c(k_i)) = \omega(u_c(k_j)) > \omega(X^{p^c})$ for all $i,j \leq r$, and of course ω restricts to $c(G)$, which gives us part *iv*. □

We call a basis $\{k_1,\cdots,k_d\}$ for H satisfying conditions *ii* and *iii* in this Proposition a *J-basis* for H.

4.4 The $c(G)$-filtration

From now on, fix a J-basis $\{k_1,\cdots,k_d\}$ for the principal subgroup H, then Proposition 4.3.5 gives us an abelian p-valuation ω on the extended commutator subgroup $c(G)$ that equates the values of $u(k_i)$ and $u(k_j)$ for each i,j. Unfortunately, using the standard Lazard filtration w on kG, we do not get that $w(u(k_i)-1) = w(u(k_j)-1)$, so we need to define a new filtration.

Theorem 4.4.1. *Let G be a non-abelian, p-valuable, abelian-by-procyclic group with split centre. Let $c(G)$ be the extended commutator subgroup, and let $\{k_1,\cdots,k_d\}$ be a J-basis for H.*

Then there exists a complete, Zariskian filtration $w : kG \to \mathbb{N}\cup\{\infty\}$, which we call the $c(G)$-filtration, *such that:*

i. For all $i,j = 1,\cdots,r$, $w(u_c(k_i)-1) = w(u_c(k_j)-1) = \theta$ for some integer $\theta > 0$.

ii. The associated graded $\mathrm{gr}\ kG \cong k[T_1,\cdots,T_{d+1}] * \frac{G}{c(G)}$, where $T_i = \mathrm{gr}(u_c(k_i)-1)$ for $i \leq r$, $T_i = \mathrm{gr}(k_i - 1)$ for $r+1 \leq i \leq d$ and $T_{d+1} = \mathrm{gr}(X^{p^c}-1)$. Each T_i has positive degree, and $deg(T_i) = \theta$ for $i = 1,\cdots,r$.

iii. Set $\bar{X} := \mathrm{gr}(X)$*. Then* T_r *is central, and for each* $i < r$*,* $\bar{X}T_i\bar{X}^{-1} = T_i + D_i$ *for some* $D_i \in \mathrm{Span}_{\mathbb{F}_p}\{T_{i+1}, \cdots, T_r\}$*.*

Also, let $A := (k[T_1, \cdots, T_{d+1}])^{\frac{G}{c(G)}}$ *be the ring of invariants, then* A *is Noetherian, central in* $\mathrm{gr}\ kG$*, and* $k[T_1, \cdots, T_{d+1}]$ *is finitely generated over* A*.*

Proof. Set $U = c(G) = \langle u_c(k_1), \cdots, u_c(k_r), k_{r+1}, \cdots, k_d, X^{p^c}\rangle$. Then U is an open, normal subgroup of G by Proposition 4.3.5(i), and hence $kG \cong kU * \frac{G}{U}$.

Using Proposition 4.3.5(iv), we choose an abelian p-valuation ω on U such that $\frac{1}{e}\theta := \omega(u_c(k_i)) = \omega(u_c(k_j)) > \omega(X^{p^c})$ for all $i, j \leq r$, where $\theta > 0$ is an integer. Then we can define the Lazard valuation w on kU with respect to ω.

Since $\{u_c(k_1), \cdots, u_c(k_r), k_{r+1}, \cdots, k_d, X^{p^c}\}$ is an ordered basis for (U, ω), it follows from the definition of w that:

$$w(u_c(k_j) - 1) = e\omega(u_c(k_j)) = e\omega(u_c(k_i)) = w(u_c(k_i) - 1) = \theta \text{ for all } i, j \leq r.$$

Furthermore, we have that if $V := \exp([\mathcal{L}, \mathcal{L}]) \subseteq U$ and $r \in kV$ with $w(r) > 0$, then $w(r) \geq \theta$.

We want to apply Proposition 2.4.1 and extend w to $kG \cong kU * \frac{G}{U}$. So we only need to verify that for all $g \in G$, $r \in kU$, $w(grg^{-1}) = w(r)$, and it suffices to verify this property for $r = u_c(k_1) - 1, \cdots, u_c(k_r) - 1, k_{r+1} - 1, \cdots, k_d - 1, X^{p^c} - 1$, since they form a topological generating set for kU.

Since $k_{r+1}, \cdots, k_d \in Z(G)$, it is obvious that $w(g(k_l - 1)g^{-1}) = w(k_l - 1)$ for each $r + 1 \leq l \leq d$, $g \in G$.

For each $j \leq r$, $gu_c(k_j)g^{-1} \in V$, thus:

$w(gu_c(k_j)g^{-1}-1) \geq \theta = w(u_c(k_j)-1)$ and it follows easily that equality holds.

Finally, $g = hX^{\beta}$ for some $h \in H$, $\beta \in \mathbb{Z}_p$, so

$gX^{p^c}g^{-1}-1 = hX^{p^c}h^{-1}-1 = ((h,X^{p^c})-1)(X^{p^c}-1)+((h,X^{p^c})-1)+(X^{p^c}-1).$

Hence $w(g(X^{p^c}-1)g^{-1}) \geq \min\{w((h,X^{p^c})-1), w(X^{p^c}-1)\}$, with equality if $w((h,X^{p^c})-1) \neq w(X^{p^c}-1)$. But since $(h,X^{p^c}) \in V$, we have that

$$w((h,X^{p^c})-1) \geq \theta = e\omega(u_c(k_i)) > e\omega(X^{p^c}) = w(X^{p^c}-1)$$

and hence $w(gX^{p^c}g^{-1}-1) = w(X^{p^c}-1)$ as required. Note that it is here that we need the fact that $\omega(u_c(k_i)) > \omega(X^{p^c})$.

Therefore we can apply Proposition 2.4.1, and extend w to kG so that $\text{gr}_w\ kG \cong (\text{gr}_w\ kU) * \frac{G}{U}$, and we have that $\text{gr}_w\ kU \cong k[T_1, ..., T_d]$ as usual, where $T_i = \text{gr}(u_c(k_i)-1)$ has degree θ for $i \leq r$, $T_i = \text{gr}(k_i - 1)$ for $r+1 \leq i \leq d$ and $T_{d+1} = \text{gr}(X^{p^c}-1)$ as required.

Using Proposition 4.3.5(iii), we see that $Xu_c(k_i)X^{-1} = u_c(k_1)^{\alpha_{1,i}} \cdots u_c(k_r)^{\alpha_{1,i}}$ where $p \mid \alpha_j$ for $j < i$ and $\alpha_{i,i} = 1$.

Hence $\bar{X}T_i\bar{X}^{-1} = \overline{\alpha}_{i,1}T_1 + \cdots + \overline{\alpha}_{i,r}T_r = T_i + \overline{\alpha}_{i+1,1}T_{i+1} + \cdots + \overline{\alpha}_{i,r}T_r$ for each $i \leq r$, thus giving part iii.

Now, every element $u \in \text{gr}\ kU$ is a root of the polynomial $\prod_{g \in \frac{G}{c(G)}} (s - gug^{-1}) \in A[s]$, hence $\text{gr}\ kU$ is integral over A.

So since $k \subseteq A$ and $\text{gr}\ kU \cong k[T_1, ..., T_{d+1}]$ is a finitely generated k-algebra, we have that $\text{gr}\ kU$ is a finitely generated A-algebra, and hence finitely generated as an A-module by the integral property.

So it follows that $\mathrm{gr}_w\ kG$ is finitely generated as a right A-module. Furthermore, since $\mathrm{gr}\ kU$ is Noetherian and commutative, it follows from [19, Theorem 2] that A is Noetherian.

Furthermore, it is easy to show that the twist $\frac{G}{U} \times \frac{G}{U} \to (\mathrm{gr}\ kU)^{\times}$ of the crossed product is trivial, so it follows that if $r \in \mathrm{gr}\ kU$ is invariant under the action of $\frac{G}{U}$ then it is central. Hence A is central in $\mathrm{gr}\ kG$. $\square$

Note: For any $h \in H$, we have that $(u_c(h)-1)+F_{\theta+1}kG \in \mathrm{Span}_{\mathbb{F}_p}\{T_1, \cdots, T_r\}$, and it is equal either to 0 or $\mathrm{gr}(u_c(h)-1)$.

Now, let P be a faithful prime ideal of kG, and let w be the $c(G)$-filtration on kG as defined in the previous theorem. Then w induces the quotient filtration $\overline{w}$ on kG/P.

By [28, Ch.II Corollary 2.1.5], P is closed in kG, and hence kG/P is complete, and $\mathrm{gr}_{\overline{w}}\ kG/P \cong \frac{\mathrm{gr}\ kG}{\mathrm{gr}\ P}$ is Noetherian. Therefore $\overline{w}$ is Zariskian by [28, Ch.II Theorem 2.1.2], and clearly $\tau : kG \to kG/P$ is continuous.

For convenience, set $\overline{T} := T+\ \mathrm{gr}\ P \in \mathrm{gr}\ kG/P$ for all $T \in \mathrm{gr}\ kc(G) = k[T_1, \cdots, T_{d+1}]$.

Let $\overline{A} := \frac{A+\mathrm{gr}\ P}{\mathrm{gr}\ P}$ be the image of A in $\mathrm{gr}\ kG/P$, and let $A' := (k[\overline{T}_1, \cdots, \overline{T}_{d+1}])^{\frac{G}{c(G)}}$ be the ring of $\frac{G}{c(G)}$-invariants in $k[\overline{T}_1, \cdots, \overline{T}_{d+1}] = \frac{k[T_1,\cdots,T_{d+1}]+\mathrm{gr}\ P}{\mathrm{gr}\ P}$.

Since $\frac{G}{c(G)}$-invariant elements in $\mathrm{gr}\ kG$ are $\frac{G}{c(G)}$-invariant modulo $\mathrm{gr}\ P$, it is clear that $\overline{A} \subseteq A' \subseteq \frac{k[T_1,\cdots,T_{d+1}]+\mathrm{gr}\ P}{\mathrm{gr}\ P}$.

Then since $k[T_1, \cdots, T_{d+1}]$ is finitely generated over A by Theorem 4.4.1, it follows that A' is finitely generated over the Noetherian ring $\overline{A}$, hence A' is Noetherian.

Therefore, kG/P is a prime ring with a Zariskian filtration $\overline{w}$ such that gr kG/P is finitely generated over a central, Noetherian subring A'. Hence we may apply Theorem 4.1.3 to produce a non-commutative valuation on $Q(kG/P)$.

4.5 A Special Case

Later in this chapter, we will prove Theorem B in full generality, and our approach will be to construct a special growth preserving polynomial and apply Theorem 3.7.5 but first we need to deal with a special case:

Fix a J-basis $\{k_1, \cdots, k_d\}$ for H, and a Zariskian filtration w on kG satisfying the conditions of Theorem 4.4.1. Using the notation of this theorem, we have that $T_r \in A$ and $\bar{X}T_i\bar{X}^{-1} = T_i + D_i$ for some $D_i \in \text{Span}_{\mathbb{F}_p}\{T_{i+1}, \cdots, T_r\}$ for all $i < r$.

Assumption: We suppose that for each $i < r$, D_i is nilpotent modulo gr P.

Thus for sufficiently high m, $\bar{X}T_i^{p^m}\bar{X}^{-1} \equiv (T_i + D_i)^{p^m} = T_i^{p^m} + D_i^{p^m} \equiv T_i^{p^m}$ (mod gr P), i.e. $\overline{T}_i^{p^m} \in A'$

Fix an integer m_0 such that $\overline{T}_i^{p^{m_0}} \in A'$ for all $i \leq r$.

Proposition 4.5.1. *Suppose that for each $i = 1, \cdots, r$, T_i is nilpotent modulo gr P, i.e. $\overline{T}_i$ is nilpotent. Then $Q(kG/P)$ is a central simple algebra.*

Proof. Using Theorem 4.4.1(*ii*), every element of gr kG has the form

$$\sum_{g \in \frac{G}{c(G)}} \Big(\sum_{\alpha \in \mathbb{N}^{d+1}} \lambda_\alpha T_1^{\alpha_1} \cdots T_{d+1}^{\alpha_{d+1}} \Big) g$$

where $\lambda_\alpha = 0$ for all but finitely many α.

Therefore, it follows immediately from nilpotence of $\overline{T}_1, \cdots, \overline{T}_r$ that $\frac{\text{gr } kG}{\text{gr } P}$ is finitely generated over $\frac{k[T_{r+1},\cdots,T_{d+1}]+\text{gr } P}{\text{gr } P}$.

But since $Z(G) = \langle k_{r+1}, \cdots, k_d\rangle$ by Proposition 4.3.5(ii), it follows that under the quotient filtration, gr $\frac{k(Z(G)\times\langle X^{p^c}\rangle)+P}{P} \cong \frac{k[T_{r+1},\cdots,T_{d+1}]+\text{gr } P}{\text{gr } P}$.

So since gr kG/P is finitely generated over gr $\frac{k(Z(G)\times\langle X^{p^c}\rangle)+P}{P}$, and $\frac{k(Z(G)\times\langle X^{p^c}\rangle)+P}{P}$ is closed in kG/P, it follows from [28, Ch.I Theorem 5.7] that kG/P is finitely generated over $\frac{k(Z(G)\times\langle X^{p^c}\rangle)+P}{P}$.

But $k(Z(G)\times\langle X^{p^c}\rangle)$ is commutative, so kG/P is finitely generated as a right module over a commutative subring. Therefore, by [29, Corollary 13.1.14(iii)], kG/P satisfies a polynomial identity.

So since kG/P is prime, it follows from Posner's theorem [29, Theorem 13.6.5], that $Q(kG/P)$ is a central simple algebra. □

Note: This proof relies on the split centre property, without which we would not be able to argue that $\frac{k[T_{r+1},\cdots,T_{d+1}]+\text{gr } P}{\text{gr } P}$ arises as the associated graded of some commutative subring of kG/P.

Since we can prove Theorem B in the case where $Q(kG/P)$ is a CSA using Corollary 3.9.6, we may assume that $Q(kG/P)$ is not a CSA. So by the proposition, we know that there exists $s \le r$ such that $\overline{T}_s$ is not nilpotent.

Since we know that $\overline{T}_s^{p^{m_0}} \in A'$, it follows there exists a minimal prime ideal $\mathfrak{q}$ of A' such that $\overline{T}_s^{p^{m_0}} \notin \mathfrak{q}$. Using Theorem 4.1.3, we let $v = v_{\mathfrak{q}}$ be the non-commutative valuation on $Q(kG/P)$ corresponding to $\mathfrak{q}$, and let ρ be the growth rate function of v.

So, $\overline{T}_s^{p^{m_0}} = \mathrm{gr}\tau(u_c(k_s)-1)^{p^{m_0}} \in A'\backslash\mathfrak{q}$. So setting $\tau : kG \to Q(kG/P)$ as the natural map, we see using Theorem 4.2.3 that $\tau(u_c(k_s)-1)^{p^k}$ is v-regular for some $k \geq m_0$, and hence $\rho(\tau(u_c(k_s)-1)^{p^k}) = v(\tau(u_c(k_s)-1)^{p^k})$.

Recall from Definition 3.2.2 our definition of the initial power m_1, and we may assume that $m_1 \geq m_0$. Also recall the Mahler approximation function $u := u_{m_1} = z(\varphi^{p^{m_1}})$, and we see using Lemma 3.6.4 that $\lambda := \inf\{\rho(\tau(u(g)-1)) : g \in G\} < \infty$.

Lemma 4.5.2. *Let $h \in H$ such that $\rho(\tau(u(h)-1)) = \lambda$. Then $\tau(u(h)-1)^{p^m}$ is v-regular for sufficiently high m.*

Proof. It is clear that $w(u_c(h)-1) \geq \theta = w(u_c(k_s)-1)$, so let $T(h) := u_c(h)-1+F_{\theta+1}kG \in \mathrm{Span}_{\mathbb{F}_p}\{T_1,\cdots,T_r\}$. Then $T(h) = \mathrm{gr}(u_c(h)-1)$ if and only if $w(u_c(h)-1) = \theta$, otherwise $T(h) = 0$.

We know that $\mathrm{gr}(u_c(k_s)-1) = T_s \notin \mathrm{gr}\ P$, and hence $\overline{w}(\tau(u_c(k_s)-1)) = w(u_c(k_s)-1)$, giving that $\overline{w}(\tau(u_c(h)-1)) \geq w(u_c(h)-1) \geq w(u_c(k_s)-1) = \overline{w}(\tau(u_c(k_s)-1))$.

Also, $T(h)^{p^{m_0}} + \mathrm{gr}\ P \in A'$, and if $T(h)^{p^{m_0}} + \mathrm{gr}\ P \notin \mathfrak{q}$ then $T(h)^{p^{m_0}} + \mathrm{gr}\ P = \mathrm{gr}_{\overline{w}}(\tau(u_c(h)-1)^{p^{m_0}}) \in A'\backslash\mathfrak{q}$, so it follows from Theorem 4.2.3 that $\tau(u(h)-1)^{p^m}$ is v-regular for $m >> 0$.

So, suppose for contradiction that $T(h)^{p^{m_0}} + \mathrm{gr}\ P \in \mathfrak{q}$:

If $T(h)^{p^{m_0}} + \mathrm{gr}\ P = 0$ then $\overline{w}(\tau(u_c(h)-1)^{p^{m_0}}) > p^{m_0}\theta = \overline{w}(\tau(u_c(k_s)-1)^{p^{m_0}})$, and if $T(h)^{p^{m_0}} + \mathrm{gr}\ P \neq 0$ then $T(h)^{p^{m_0}} + \mathrm{gr}\ P = \mathrm{gr}_{\overline{w}}(\tau(u_c(h)-1)^{p^{m_0}}) \in \mathfrak{q}$. In either case, using Theorem 4.2.3, it follows that for m sufficiently high:

$$v(\tau(u_c(h)-1)^{p^m}) > v(\tau(u_c(k_s)-1)^{p^m}).$$

Therefore, $v(\tau(u(h)-1)) > v(\tau(u(k_s)-1))$, so since $\tau(u(k_s)-1)^{p^m}$ is v-regular:

$\rho(\tau(u(h)-1)) \geq \frac{1}{p^m}v(\tau(u(h)-1)^{p^m}) > \frac{1}{p^m}v(\tau(u(k_s)-1)^{p^m}) = \rho(\tau(u(k_s)-1)) \geq \lambda$

– contradiction. □

Recall the definition of a growth preserving polynomial (GPP) from Section 2.4, and recall that the identity map is a non-trivial GPP.

Proposition 4.5.3. *Suppose that $Q(kG/P)$ is not a CSA, and that D_i is nilpotent mod gr P for all $i < r$. Then $id : \tau(kH) \to \tau(kH)$ is a special GPP with respect to some non-commutative valuation on $Q(kG/P)$.*

Proof. This is immediate from Definition 3.7.1 and Lemma 4.5.2. □

Therefore, we can assume from now on that for some i, D_i is not nilpotent mod gr P.

Remark: We have now proved Theorem B in the case where G is uniform, because $D_i = 0$ for all i in this case. For general p-valuable G, however, we cannot assume this. For example, if $p > 2$ and $G = \langle X, Y, Z\rangle$ where Y and Z commute, $XYX^{-1} = Y^r$, $XZX^{-1} = (YZ)^r$, $r = e^p \in \mathbb{Z}_p$, then G is non-uniform, $c = 1$, and $\{Z, Y^{p-1}Z^p\}$ is a J-basis for $H = \langle Y, Z\rangle$. In this case, $\bar{X}T_2\bar{X}^{-1} = T_2$, $\bar{X}T_1\bar{X}^{-1} = T_1 + T_2$, i.e. $D_2 = 0$, $D_1 = T_2$.

4.6 The Reduction Coefficients

Again, fix a J-basis $\{k_1, \cdots, k_d\}$ for H, let w be the $c(G)$-filtration on kG and let $T_i := \text{gr}_w(u_c(k_i) - 1)$ for each $i \leq r$, $T_i := \text{gr}_w(k_i - 1)$ for $i > r$. We know that $T_r, \cdots, T_d$ are central, and $D_i := \bar{X}T_i\bar{X}^{-1} - T_i \in \text{Span}_{\mathbb{F}_p}\{T_{i+1}, \cdots, T_r\}$ for each $i < r$ by Theorem 4.4.1(iii).

We now assume that not all the D_i are nilpotent modulo gr P, so let $s < r$ be maximal such that D_s is not nilpotent, i.e. for all $i > s$, $\overline{T}_i^{p^m} \in A'$ for sufficiently high m.

We fix m_0 such that $\overline{T}_i^{p^{m_0}} \in A'$ for all $i > s$, and we may assume that $m_1 \geq m_0$, where m_1 is the initial power as defined in Definition 3.2.2.

By definition, we know that $D_s \in \mathrm{Span}_{\mathbb{F}_p}\{T_{s+1}, \cdots, T_r\}$, and hence $\overline{D}_s^{p^{m_0}} \in A'$. So since D_s is not nilpotent mod gr P, we can fix a minimal prime ideal $\mathfrak{q}$ of A' such that $\overline{D}_s^{p^{m_0}} \in A' \backslash \mathfrak{q}$.

From now on, we will fix $v = v_{\mathfrak{q}}$ the corresponding non-commutative valuation given by Theorem 4.1.3, and let ρ be the growth rate function corresponding to v.

Define a function L of commuting variables x and y by:

$$L(x, y) := x^p - xy^{p-1}. \tag{4.1}$$

Moreover, for commuting variables $x, y_1, y_2, \cdots, y_n$, define the iterated function

$$L^{(n)}(x, y_1, y_2, \cdots, y_n) := L(L(L(\cdots(L(x, y_1), y_2), \cdots), y_n) \tag{4.2}$$

and for $n = 0$, we define $L^{(n)}(x, y_1, \cdots, y_n) := x$.

We can readily see that for any commutative $\mathbb{F}_p$-algebra S, $y_1, \cdots, y_n \in S$, $L^{(n)}(-, y_1, \cdots, y_n)$ is $\mathbb{F}_p$-linear.

Lemma 4.6.1. *Let S be an $\mathbb{F}_p$-algebra, and let $y_1, \cdots, y_n \in S$ commute. Then there exist $a_0, a_1, \cdots, a_{n-1} \in S$ such that $L^{(n)}(x, y_1, \cdots, y_n) = a_0x + a_1x^p + \cdots + a_{n-1}x^{p^{n-1}} + x^{p^n}$ for all x commuting with $y_1, \cdots, y_n$.*

Proof. Both statements are trivially true for $n = 0$, so assume that they hold for some $n \geq 0$ and proceed by induction on n:

So $L^{(n)}(x, y_1, \cdots, y_n) = a_0 x + a_1 x^p + \cdots + a_{n-1} x^{p^{n-1}} + x^{p^n}$, and:

$$L^{(n+1)}(x, y_1, \cdots, y_{n+1}) = L^{(n)}(x, y_1, \cdots, y_n)^p - L^{(n)}(x, y_1, \cdots, y_n) y_{n+1}^{p-1}$$

$$= (a_0 x + \cdots + a_{n-1} x^{p^{n-1}} + x^{p^n})^p - (a_0 x + \cdots + a_{n-1} x^{p^{n-1}} + x^{p^n}) y_{n+1}^{p-1}$$

$$= (-a_0 y_{n+1}^{p-1})x + (a_0^p - a_1 y_{n+1}^{p-1})x^p + \cdots + (a_{n-2}^p - a_{n-1} y_{n+1}^{p-1})x^{p^{n-1}} + (a_{n-1}^p - y_{n-1}^{p-1})x^{p^n} + x^{p^{n+1}}.$$

So setting $b_0 = (-a_0 y_{n+1}^{p-1})$, $b_i = (a_{i-1}^p - a_i y_{n+1}^{p-1})$ for $1 \leq i \leq n$ (taking $a_n := 1$), we have that $L^{(n+1)}(x, y_1, \cdots, y_{n+1}) = b_0 x + b_1 x^p + \cdots + b_n x^{p^n} + x^{p^{n+1}}$ as required. $\square$

Now, let $B_s := D_s$, and for each $1 \leq i < s$, let $B_i := L^{(s-i)}(D_i, B_s, \cdots, B_{i+1})$.

Lemma 4.6.2. *For each $i \leq s$, $L^{(s-i+1)}(\overline{T}_i, \overline{B}_s, \cdots, \overline{B}_i)^{p^{m_0}} \in A'$, so in particular $\overline{B}_i^{p^{m_0}} \in A'$.*

Proof. Note that $C \in k[T_1, \cdots, T_d]$ is central if and only if it is invariant under the action of $\bar{X}$.

Also, $L(C, C) = C^p - CC^{p-1} = 0$, and if D is $\bar{X}$-invariant, then $\bar{X} L(C, D) \bar{X}^{-1} = L(\bar{X} C \bar{X}^{-1}, D)$

Notation: Let $Y' := \overline{Y}^{p^{m_0}}$ for any $Y \in \text{gr } kG$.

We will proceed by downwards induction on i, starting with $i = s$. Clearly $B_s' = D_s' \in \text{Span}_{\mathbb{F}_p}\{T_{s+1}', \cdots, T_r'\}$ is invariant under the action of $\bar{X}$, so:

$\bar{X} L(T_s', B_s') \bar{X}^{-1} = L(\bar{X} T_s' \bar{X}^{-1}, B_s') = L(T_s' + B_s', B_s') = L(T_s', B_s') + L(B_s, B_s) = L(T_s', B_s')$.

Therefore $L(T'_s, B'_s)$ is $\bar{X}$-invariant and the result holds.

Suppose we have the result for all $s \geq j > i$.

Then $B'_i = L^{(s-i)}(D'_i, B'_s, \cdots, B'_{i+1})$, and $D'_i \in \mathrm{Span}_{\mathbb{F}_p}\{T'_{i+1}, \cdots, T'_r\}$.

Using linearity of $L(-, y)$ we have that

$$B'_i \in \mathrm{Span}_{\mathbb{F}_p}\{L^{(s-i)}(T'_j, B'_s, \cdots, B'_{i+1}) : j = i+1, \cdots, r\}$$

therefore B'_i is $\bar{X}$-invariant by the inductive hypothesis.

Also, since $B'_s, \cdots, B'_{i+1}$ are $\bar{X}$-invariant, we have that:

$$\bar{X}L^{(s-i)}(T'_i, B'_s, \cdots, B'_{i+1})\bar{X}^{-1} = L^{(s-i)}(\bar{X}T'_i\bar{X}^{-1}, B'_s, \cdots, B'_{i+1})$$

$$= L^{(s-i)}(T'_i + \bar{X}T'_i\bar{X}^{-1} - T'_i, B'_s, \cdots, B'_{i+1}) = L^{(s-i)}(T'_i, B'_s, \cdots, B'_{i+1}) + B_i$$

The final equality follows from linearity of $L^{(s-i)}(-, B'_s, \cdots, B'_{i+1})$ and the fact that $D'_i = \bar{X}T'_i\bar{X}^{-1} - T'_i$.

Set $C := L^{(s-i)}(T'_i, B'_s, \cdots, B'_{i+1})$, so that

$$L^{(s-i+1)}(T'_i, B'_s, \cdots, B'_i) = L(C, B'_i) = C^p - CB'^{p-1}_i, \text{ and } \bar{X}C\bar{X}^{-1} = C + B'_i.$$

Then:

$$\bar{X}L(C, B'_i)\bar{X}^{-1} = L(\bar{X}C\bar{X}^{-1}, B'_i) = L(C + B'_i, B'_i) = L(C, B'_i) + L(B'_i, B'_i) = L(C, B'_i)$$

Hence $L(C, B'_i) = L^{(s-i+1)}(T'_i, B'_s, \cdots, B'_i)$ is $\bar{X}$-invariant as required. □

It follows immediately from this Lemma that $L^{(s)}(\overline{T}, \overline{B}_s, \cdots, \overline{B}_1)^{p^{m_0}} \in A'$ for all $T \in \mathrm{Span}_{\mathbb{F}_p}\{T_1, \cdots, T_r\}$ (i.e. for all $T = (u_c(h)-1) + F_{\theta+1}kG$).

Now, for each $i \leq s$, $D_i \in \mathrm{Span}_{\mathbb{F}_p}\{T_{i+1}, \cdots, T_r\}$, so either $D_i = 0$ or $D_i = \mathrm{gr}_w(u_c(f_i) - 1)$ for some $f_i \in H$ with $w(u_c(f_i) - 1) = \theta$.

Definition 4.6.3. *Define $y_s := u_c(f_s) - 1$, and for each $1 \leq i < s$, define $y_i \in kH$ inductively by:*

$$y_i := \begin{cases} L^{(s-i)}(u_c(f_i) - 1, y_s, \cdots, y_{i+1}) & D_i \neq 0 \\ 0 & D_i = 0 \end{cases}$$

And define $b_i := \tau(y_i)^{p^{m_1-c}} \in Q(kG/P)$, we call these b_i the reduction coefficients*.*

For convenience, we will replace m_1 by $m_1 + c$, so that $\tau(u(g) - 1) = \tau(u_c(g) - 1)^{p^{m_1}}$ for all $g \in G$, and $b_i = \tau(y_i)^{p^{m_1}}$. Since $B_s \notin \mathrm{gr}\ P$, it is clear that $\mathrm{gr}_{\overline{w}}(b_s) = \overline{B}_s^{p^{m_1}}$.

Since $\mathrm{gr}_{\overline{w}}(b_s) = \overline{B}_s^{p^{m_1}} \in A' \backslash \mathfrak{q}$, it follows from Theorem 4.2.3 that $b_s^{p^k}$ is v-regular for some $k \in \mathbb{N}$. After replacing m_1 by $m_1 + k$, we may also assume that b_s is v-regular.

Lemma 4.6.4. *Given $h \in H$, let $t := u_c(h) - 1$, $T := t + F_{\theta+1}kG \in gr\ kG$. Then for each $i \leq s$:*

- $w(L^{(s-i)}(t, y_s, \cdots, y_{i+1})) \geq p^{s-i}\theta$,
- $w(L^{(s-i)}(t, y_s, \cdots, y_{i+1})) = p^{s-i}\theta$ *if and only if* $\mathrm{gr}(L^{(s-i)}(t, y_s, \cdots, y_{i+1})) = L^{(s-i)}(T, B_s, \cdots B_{i+1})$,
- $w(L^{(s-i)}(t, y_s, \cdots, y_{i+1})) > p^{s-i}\theta$ *if and only if* $L^{(s-i)}(T, B_s, \cdots B_{i+1}) = 0$.

In particular, for $y_i \neq 0$, $w(y_i) \geq p^{s-i}\theta$, with equality if and only if $B_i = \mathrm{gr}(y_i)$, otherwise $B_i = 0$.

Proof. We will use downwards induction on i, with $i = s$ as the base case:

Since $i = s$, $L^{(s-i)}(t, y_s, \cdots, y_{i+1}) = t$, and clearly $w(t) \geq \theta = p^{s-s}\theta$, and equality holds if and only if $\mathrm{gr}(t) = T$, and otherwise $T = 0$ as required.

Now suppose the result holds for some $i \leq s$, so let $c := L^{(s-i)}(t, y_s, \cdots, y_{i+1})$, $C := L^{(s-i)}(T, B_s, \cdots, B_{i+1})$. Then by induction, $w(c) \geq p^{s-i}\theta$, with equality if and only if $C = \mathrm{gr}(c)$, and $w(y_i) \geq p^{s-i}\theta$.

So $w(L^{(s-i+1)}(t, y_s, \cdots, y_i)) = w(c^p - cy_i^{p-1}) \geq \min\{w(c^p), w(cy_i^{p-1})\}$
$\geq \min\{pw(c), w(c)+(p-1)w(y_i)\} \geq p^{s-i+1}\theta$ as required. In particular, this argument shows that if $w(c) > p^{s-i}\theta$ then $w(c^p - cy_i^{p-1}) > p^{s-i+1}\theta$.

Therefore, if $w(L^{(s-i+1)}(t, y_s, \cdots, y_i)) = p^{s-i+1}\theta$ then $w(c) = p^{s-i}\theta$, and so $C = \mathrm{gr}(c) = c + F_{p^{s-i}\theta}kG$ by induction. In this case,

$$\mathrm{gr}(L^{(s-i+1)}(t, y_s, \cdots, y_i)) = (c^p - cy_i^{p-1}) + F_{p^{s-i+1}\theta}kG.$$

Also, since $c, y_i \in kc(G)$ and w is a valuation on $kc(G)$, we have that $w(c^p) = pw(c) = p^{s-i+1}\theta$ and $w(cy_i^{p-1}) = w(c) + (p-1)w(y_i) \geq p^{s-i+1}\theta$.

If $w(y_i) > p^{s-i}\theta$ then $c^p - cy_i^{p-1} + F_{\theta+1}kG = c^p + F_{\theta+1}kG = C^p$. But since $B_i = 0$ by induction, this means that $c^p - cy_i^{p-1} + F_{p^{s-i+1}\theta+1} = C^p - CB_i^{p-1}$.

Whereas, if $w(y_i) = p^{s-i}\theta$ then $B_i = \mathrm{gr}(y_i) = y_i + F_{p^{s-i}\theta+1}kG$ by assumption, so $c^p - cy_i^{p-1} + F_{p^{s-i+1}\theta+1}kG = C^p - CB_i^{p-1}$ as required.

Finally, if $w(L^{(s-i+1)}(t, y_s, \cdots, y_i)) > p^{s-i+1}\theta$ then $w(c^p - cy_i^{p-1}) > p^{s-i+1}\theta$. Clearly if $C = 0$ then $L^{(s-i+1)}(T, B_s, \cdots, B_i) = C^p - CB_i^{p-1} = 0$, so we may assume that $C \neq 0$, and hence $C = \mathrm{gr}(c) = c + F_{p^{s-i}\theta+1}kG$.

So since $w(c^p - cy_i^{p-1}) > p^{s-i+1}\theta$, it follows that $C^p - CB_i^{p-1} = 0$ as required. $\square$

Using this Lemma, we see that $\overline{w}(b_i) \geq p^{s-i}\theta$, with equality if and only if $\text{gr}_{\overline{w}}(b_i) = \overline{B}_i^{p^{m_1}}$.

Notation: For each $0 \leq i \leq s$, $q \in Q(kG/P)$ commuting with $b_1, \cdots, b_s$, define $L_i(q) := L^{(s-i)}(q, b_s, \cdots, b_{i+1})$.

e.g. $L_s(q) = q$, $L_{s-1}(q) = q^p - qb_s^{p-1}$, $L_{s-2}(q) = q^{p^2} - q^p(b_s^{p^2-p} - b_{s-1}^{p-1}) + qb_s^{p-1}b_{s-1}^{p-1}$.

Again, recall that $\lambda = \inf\{\rho(\tau(u(g) - 1)) : g \in G\} < \infty$ by Lemma 3.6.4.

Lemma 4.6.5. *For each $i \leq s$, $\rho(b_i) \geq p^{s-i}\lambda$, and it follows that $\rho(L_i(\tau(u(h)-1)) \geq p^{s-i}\lambda$ for all $h \in H$.*

Moreover, if $\rho(b_i) = p^{s-i}\lambda$ then $b_i^{p^m}$ is v-regular for m sufficiently high.

Proof. For $i = s$ the first statement is clear, because $b_s = \tau(u(f_s) - 1)$, so $\rho(b_s) \geq \lambda = p^{s-s}\lambda$ by definition. So we will proceed again by downwards induction on i.

The inductive hypothesis states that $\rho(b_{i+1}) \geq p^{s-i-1}\lambda$, and $\rho(L_{i+1}(\tau(u(h) - 1))) \geq p^{s-i-1}\lambda$ for all $h \in H$.

Thus $\rho(L_i(\tau(u(h) - 1))) = \rho(L_{i+1}(\tau(u(h) - 1))^p - L_{i+1}(\tau(u(h) - 1))b_{i+1}^{p-1})$

$\geq \min\{p \cdot p^{s-i-1}\lambda, p^{s-i-1}\lambda + (p-1)p^{s-i-1}\lambda\} = p^{s-i}\lambda$ for all h.

By definition, $b_i = L^{(s-i)}(\tau(u(f_i) - 1), b_s, \cdots, b_{i+1}) = L_i(\tau(u(f_i) - 1))$, so

$$\rho(b_i) = \rho(L_i(\tau(u(f_i) - 1))) \geq p^{s-i}\lambda, \text{ as required.}$$

For the second statement, suppose that $\rho(b_i) = p^{s-i}\lambda$:

Then if $\overline{w}(b_i) > p^{s-i+m_1}\theta = \overline{w}(b_s^{p^{s-i}})$, then since $\text{gr}_{\overline{w}}(b_s) = \overline{B}_s^{p^{m_1}} \in A'\backslash\mathfrak{q}$, it follows that $v(b_i^{p^m}) > v(b_s^{p^{s-i+m}})$ for $m >> 0$ by Theorem 4.2.3.

So using v-regularity of b_s, we see that $\rho(b_i) > \rho(b_s^{p^{s-i}}) \geq p^{s-i}\lambda$ – contradiction.

Therefore, by Lemma 4.6.4, we see that $\overline{w}(b_i) = p^{s-i+m_1}\theta$ and $\text{gr}_{\overline{w}}(b_i) = \overline{B}_i^{p^{m_1}}$.

We know that $\overline{B}_i^{p^{m_1}} \in A'$, so suppose that $\overline{B}_i^{p^{m_1}} \in \mathfrak{q}$. Then since $\overline{w}(b_i) = p^{s-i}\theta = \overline{w}(b_s^{p^{s-i}})$, it follows again from Theorem 4.2.3 that $v(b_i^{p^m}) > v(b_s^{p^{s-i+m}})$ for $m >> 0$, and hence $\rho(b_i) > p^{s-i}\rho(b_s) \geq p^{s-i}\lambda$ – contradiction.

Hence $\overline{B}_i^{p^{m_0}} = \text{gr}_{\overline{w}}(b_i) \in A'\backslash\mathfrak{q}$, so $b_i^{p^k}$ is v-regular for some $k \in \mathbb{N}$ by Theorem 4.2.3. $\square$

Now, using Lemma 4.6.1, we know that $L_i(x) = L^{(s-i)}(x, b_s, \cdots, b_{i+1}) = a_0x + a_1x^p + \cdots + a_{s-i-1}x^{p^{s-i-1}} + x^{p^{s-i}}$ for some $a_j \in \tau(kH)$.

Proposition 4.6.6. *For each $i \leq s$, L_i is a growth preserving polynomial of p-degree $s-i$, and L_s is not trivial.*

Proof. Firstly, it is clear that $L_s = id$, and so L_s is a non-trivial GPP.

We first want to prove that for all $q \in \tau(kH)$, if $\rho(q) \geq \lambda$ then $\rho(L_i(q)) \geq p^{s-i}\lambda$, with strict inequality if $\rho(q) > \lambda$. We know that this holds for $i = s$, so as in the proof of Lemma 4.6.5, we will use downwards induction on i.

So suppose that $\rho(L_{i+1}(q)) \geq p^{s-i-1}\lambda$, with strict inequality if $\rho(q) > \lambda$. Then:

$L_i(q) = L_{i+1}(q)^p - L_{i+1}(q)b_{i+1}^{p-1}$, so $\rho(L_i(q)) \geq \min\{\rho(L_{i+1}(q)^p), \rho(L_{i+1}(q)b_{i+1}^{p-1})\}$.

But $\rho(L_{i+1}(q)^p) \geq p \cdot p^{s-i-1}\lambda = p^{s-i}\lambda$, and since $\rho(b_{i+1}) \geq p^{s-i}\lambda$ by Lemma 4.6.5, $\rho(L_{i+1}(q)b_{i+1}^{p-1}) \geq \rho(L_{i+1}(q)) + (p-1)\rho(b_{i+1}) \geq p^{s-j-1}\lambda + (p-1)p^{s-i-1}\lambda = p^{s-i}\lambda$.

By the inductive hypothesis, both these inequalities are strict if $\rho(q) > \lambda$, and thus L_i is a GPP as required. □

So all that remains is to prove that one of the L_i is special.

4.7 Mahler Expansions again

In order for L_i to be special it must be non-trivial, but there is no reason why this should be true in general. Therefore, we must now revisit Mahler expansions one final time to deal with the special case that for some i, L_i is a trivial GPP, i.e. $\rho(L_i(\tau(u(h)-1))) > p^{s-i}\lambda$ for all $h \in H$.

We know that L_s is not trivial, so we can fix $j \leq s$ such that L_j is non-trivial and L_{j-1} is trivial. We will first need the following technical results:

Lemma 4.7.1. *Let A be a k-algebra, with filtration w such that A is complete with respect to w. Suppose $a \in A$ and $w(a^p - a) > 0$, then $a^{p^m} \to b \in A$ with $b^p = b$ as $m \to \infty$.*

Proof. Let $\varepsilon := a^p - a$, then $w(\varepsilon) > 0$, a commutes with ε, and $a^p = a + \varepsilon$.

Therefore, since $char(k) = p$, $a^{p^2} = a^p + \varepsilon^p = a + \varepsilon + \varepsilon^p$, and it follows from induction that for all $m \in \mathbb{N}$, $a^{p^{m+1}} = a + \varepsilon + \varepsilon^p + \cdots + \varepsilon^{p^m}$.

But $\varepsilon^{p^m} \to 0$ as $m \to \infty$ since $w(\varepsilon) > 0$, so since A is complete, the sum $\sum\limits_{m\geq 0} \varepsilon^{p^m}$ converges in A, and hence $a^{p^m} \to a + \sum\limits_{m\geq 0} \varepsilon^{p^m} \in A$.

So let $b := a + \sum\limits_{m\geq 0} \varepsilon^{p^m}$, then $b^p = a^p + (\sum\limits_{m\geq 0} \varepsilon^{p^m})^p = a + \varepsilon + \sum\limits_{m\geq 1} \varepsilon^{p^m} = a + \sum\limits_{m\geq 0} \varepsilon^{p^m} = b$ as required. □

Proposition 4.7.2. *Let $Q = Q(\widehat{kG/P})$, and let $\delta_1, \cdots, \delta_r : kG \to kG$ be derivations such that $\tau\delta_i(P) \neq 0$ for all i. Set $N := \{(a_1, \cdots, a_r) \in Q^r : (a_1\tau\delta_1 + \cdots + a_r\tau\delta_r)(P) = 0\}$.*

Then N is a Q-bisubmodule of Q^r, and either $N = 0$ or there exist $\alpha_1, \cdots, \alpha_r \in Z(Q)$, not all zero, such that for all $(a_1, \cdots, a_r) \in N$, $\alpha_1 a_1 + \cdots + \alpha_r a_r = 0$.

Proof. Since v is a non-commutative valuation, we have that Q is simple and Artinian, and the proof that N is a Q-bisubmodule of Q^r is similar to the proof of Lemma 3.9.4. For the second statement, we will proceed using induction on r.

First suppose that $r = 1$, then N is a two sided ideal of the simple ring Q, so it is either 0 or Q. But if $N = Q$ then $1 \in Q$ so $\tau\delta_1(P) = 0$ – contradiction. Hence $N = 0$.

Now suppose that the result holds for $r - 1$ for some $r > 1$. If $N \neq 0$ then there exists $(a_1, \cdots, a_r) \in N$ with $a_i \neq 0$ for some i, and we may assume without loss of generality that $i = 1$.

So, let $A := \{a \in Q : (a, a_2, \cdots, a_r) \in N \text{ for some } a_i \in Q\}$, then clearly A is a two-sided ideal of Q, so $A = 0$ or $A = Q$. But $A \neq 0$ since $a_1 \in A$ and $a_1 \neq 0$.

Therefore $A = Q$, and hence we have that for all $b \in Q$, $(b, b_2, \cdots, b_r) \in N$ for some $b_i \in Q$.

Let $N' = \{(a_2, \cdots, a_r) \in Q^{r-1} : (a_2\tau\delta_2 + \cdots + a_r\tau\delta_r)(P) = 0\}$. Suppose first that $N' = 0$.

Then if for some $q \in Q$, $(q, x_2, \cdots, x_r), (q, x_2', \cdots, x_r') \in N$ for $x_i, x_i' \in Q$, we have that $(x_2 - x_2', \cdots, x_r - x_r') \in N' = 0$, and hence $x_i = x_i'$ for all i.

Hence there is a unique $(1, \beta_2, \cdots, \beta_r) \in N$.

Given $x \in Q$, $(x, x\beta_2, \cdots, x\beta_r)$, $(x, \beta_2 x, \cdots, \beta_r x) \in N$, and so $([x, \beta_2], \cdots, [x, \beta_r]) \in N'$. Hence $[x, \beta_i] = 0$ for all i, so $\beta_i \in Z(Q)$.

Moreover, if $(a_1, \cdots, a_r) \in N$, then since $(a_1, a_1\beta_2, \cdots, a_1\beta_r) \in N$, it follows that $a_i = \beta_i a_1$ for all $i > 1$, and since $\tau\delta_1(P) \neq 0$, it is clear that $\beta_i \neq 0$ for some i, thus giving the result.

So from now on, we may assume that $N' \neq 0$, so by the inductive hypothesis, this means that there exist $\alpha_2, \cdots, \alpha_r \in Z(Q)$, not all zero, such that for all $(a_2, \cdots, a_r) \in N'$, $\alpha_2 a_2 + \cdots + \alpha_r a_r = 0$.

Again, suppose we have that $(a, x_2, \cdots, x_r), (a, x'_2, \cdots, x'_r) \in N$ for some $a, x_i, x'_i \in Q$. Then clearly $(x_2 - x'_2, \cdots, x_r - x'_r) \in N'$, and hence $\alpha_2(x_2 - x'_2) + \cdots + \alpha_r(x_r - x'_r) = 0$, i.e. $\alpha_2 x_2 + \cdots + \alpha_r x_r = \alpha_2 x'_2 + \cdots + \alpha_r x'_r$.

So, given $q \in Q$, $(1, x_2, \cdots, x_r) \in N$, we have that $(q, qx_2, \cdots, qx_r), (q, x_2 q, \cdots, x_r q) \in N$, and hence

$$\alpha_2 q x_2 + \cdots + \alpha_r q x_r = \alpha_2 x_2 q + \cdots + \alpha_r x_r q$$

i.e $[q, \alpha_2 x_2 + \cdots + \alpha_r x_r] = 0$.

Since this holds for all $q \in Q$, it follows that $\alpha_2 x_2 + \cdots + \alpha_r x_r \in Z(Q)$, so let $-\alpha_1$ be this value.

In fact, for any such $(1, x'_2, \cdots, x'_r) \in N$, $\alpha_2 x'_2 + \cdots + \alpha_r x'_r = \alpha_2 x_2 + \cdots + \alpha_r x_r = -\alpha_1$, so $-\alpha_1 \in Z(Q)$ is unchanged, regardless of our choice of x_i.

Finally, suppose that $(a_1, \cdots, a_r), (1, x_2, \cdots, x_r) \in N$, then $(a_1, a_1x_2, \cdots, a_1x_r) \in N$, and hence $(a_2 - a_1x_2, \cdots, a_r - a_1x_r) \in N'$. Thus $\alpha_2(a_2 - a_1x_2) + \cdots + \alpha_r(a_r - a_1x_r) = 0$, i.e.

$$\alpha_2 a_2 + \cdots + \alpha_r a_r = a_1(\alpha_2 x_2 + \cdots + \alpha_r x_r) = -\alpha_1 a_1.$$

Therefore $\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_r a_r = 0$, and $\alpha_i \in Z(Q)$ as required. □

Let $U_f := \{h \in H : \rho(f(\tau(u(h) - 1))) > p^r\lambda\}$ for any GPP f of p-degree r. Recall from Lemma 3.7.2 that U_f is an open subgroup of H containing H^p, and that it is proper in H if and only if f is non-trivial. For each $i \leq s$, define $U_i := U_{L_i}$.

Then since L_{j-1} is trivial and L_j is not, we know that $U_{j-1} = H$ and U_j is a proper subgroup of H.

Lemma 4.7.3. *There exists $k \in \mathbb{N}$ such that $b_j^{p^k}$ is v-regular of value $p^{k+s-j}\lambda$, and for any $h \in H \backslash U_j$, $(L_j(\tau(u(h) - 1)b_j^{-1})^{p^m} \to c \in Q(\widehat{kG/P})$ with $c \neq 0$ and $c^p = c$.*

Proof. Since L_{j-1} is trivial, we know that for each $h \in H$, $\rho(L_{j-1}(\tau(u(h) - 1))) > p^{s-j+1}\lambda$.

Choose $h \in H \backslash U_j$, i.e. $\rho(L_j(\tau(u(h) - 1)) = p^{s-j}\lambda$. Setting $q := \tau(u(h) - 1)$ for convenience, we have:

$$\rho(L_{j-1}(q)) = \rho(L_j(q)^p - L_j(q)b_j^{p-1}) > p^{s-j+1}\lambda.$$

But $\rho(L_j(q)^p - L_j(q)b_j^{p-1}) \geq \min\{\rho(L_j(q)^p), \rho(L_j(q)b_j^{p-1})\}$, with equality if $\rho(L_j(q)^p) \neq \rho(L_j(q)b_j^{p-1})$.

So if $\rho(b_j) > p^{s-j}\lambda$, then we have that:

$$\rho(L_j(q)b_j^{p-1}) > \rho(L_j(q)) + (p - 1)p^{s-j}\lambda = p^{s-j+1}\lambda.$$

But $\rho(L_j(q)^p) = p\rho(L_j(q)) = p^{s-j+1}\lambda$, and hence $\rho(L_{j-1}(\tau(u(h)-1))) = \min\{\rho(L_j(q)^p), \rho(L_j(q)b_j^{p-1})\}$ $= p^{s-j+1}\lambda$ – contradiction.

Therefore, $\rho(b_j) \leq p^{s-j}\lambda$, so using Lemma 4.6.5, we see that $\rho(b_j) = p^{s-j}\lambda$, and $b_j^{p^k}$ is v-regular for some k, and thus $v(b_j^{p^k}) = \rho(b_j^{p^k}) = p^{k+s-j}\lambda$.

Now, $\rho((L_j(q)^{p^k} b_j^{-p^k})^p - (L_j(q)^{p^k} b_j^{-p^k})) = \rho(b_j^{-p^{k+1}}(L_j(q)^p - L_j(q)b_j^{p-1})^{p^k})$

$= \rho(L_{j-1}(q)^{p^k}) - pv(b_j^{p^k}) > p^{s-j+k+1}\lambda - p^{s-j+k+1}\lambda = 0$

This means that $v(((L_j(q)b_j^{-1})^p - (L_j(q)b_j^{-1}))^{p^m}) > 0$ for $m >> 0$, so it follows from Lemma 4.7.1 that $L_j(\tau(u(h)-1))b_j^{-1})^{p^m}$ converges to $c \in Q(\widehat{kG/P})$ with $c^p = c$.

Finally, since $\rho(L_j(\tau(u(h)-1))^{p^k} b_j^{-p^k}) = 0$, it follows that $c \neq 0$. $\square$

Theorem 4.7.4. *If L_{j-1} is trivial and L_j is not trivial, then P is controlled by a proper open subgroup of G.*

Proof. Since U_j is a proper subgroup of H containing H^p, we can choose an ordered basis $\{h_1, \cdots, h_d\}$ for H such that $\{h_1^p, \cdots, h_t^p, h_{t+1}, \cdots, h_d\}$ is an ordered basis for U_j.

Consider our Mahler expression (3.12) with $f = L_j$ and $q_i = \tau(u(h_i) - 1)$:

$$0 = L_j(q_1)^{p^m}\tau\partial_1(y) + \cdots + L_j(q_d)^{p^m}\tau\partial_d(y) + O(L_j(q)^{p^m}) \tag{4.3}$$

where $y \in P$ is arbitrary and $\rho(q) > \lambda$, hence $\rho(L_j(q)) > p^{s-j}\lambda$ since L_j is a GPP. Note that we also have:

$$\rho(L_j(q_i)) = p^{s-j}\lambda \text{ for all } i \leq t, \text{ and } \rho(L_j(q_i)) > p^{s-j}\lambda \text{ for all } i > t.$$

Using Lemma 4.7.3, we see that $b_j^{p^k}$ is v-regular of value p^{k+s-j} for some k, and for each $i \leq t$, $(L_j(q_i)^{p^k} b_j^{-p^k})^{p^m} \to c_i \neq 0$ as $m \to \infty$, with $c_i^p = c_i$. Clearly $c_1, \cdots, c_r$ commute.

So, divide out our expression (4.3) by $b_j^{p^m}$, which is v-regular of value $p^{m+s-j}\lambda$ to obtain:

$$0 = (b_j^{-1} L_j(q_1))^{p^m} \tau\partial_1(y) + \cdots + (b_j^{-1} L_j(q_{d-1}))^{p^m} \tau\partial_{d-1}(y) + O((b_j^{-1} L_j(q))^{p^m}). \quad (4.4)$$

Take the limit as $m \to \infty$ and the higher order terms will converge to zero. Hence the expression converges to $c_1\tau\partial_1(y) + \cdots + c_t\tau\partial_t(y)$. Therefore, since $y \in P$ was arbitrary, $(c_1\tau\partial_1 + \cdots + c_t\tau\partial_t)(P) = 0$.

Now, using Proposition 2.5.4, we know that if $\tau\partial_i(P) = 0$ for some $i \leq t$ then P is controlled by a proper open subgroup of G. So we will suppose, for contradiction, that $\tau\partial_i(P) \neq 0$ for all $i \leq t$.

Let $N := \{(a_1, \cdots, a_t) \in Q(\widehat{kG/P}) : (a_1\tau\partial_1 + \cdots + a_t\tau\partial_t)(P) = 0\}$, then $0 \neq (c_1, \cdots, c_t) \in N$, so $N \neq 0$. Therefore, using Proposition 4.7.2, we see that $c_1, \cdots, c_t$ are $Z(Q)$-linearly dependent.

So, we can find some $1 < r \leq t$ such that $c_1, \cdots, c_r$ are $Z(Q)$-linearly dependent, but no proper subset of $\{c_1, \cdots, c_r\}$ is $Z(Q)$-linearly dependent. It follows that we can find $\alpha_2, \cdots, \alpha_r \in Z(Q)\backslash\{0\}$ such that $c_1 + \alpha_2 c_2 + \cdots + \alpha_r c_r = 0$.

Therefore, since $c_i^p = c_i$ for all i, we also have that:

$$c_1 + \alpha_2^p c_2 + \cdots + \alpha_r^p c_r = (c_1 + \alpha_2 c_2 + \cdots + \alpha_r c_r)^p = 0.$$

Hence $(\alpha_2^p - \alpha_2)c_2 + \cdots + (\alpha_r^p - \alpha_r)c_r = 0$.

So using minimality of $\{c_1, \cdots, c_r\}$, this means that $\alpha_i^p = \alpha_i$ for each i, and it follows that $\alpha_i \in \mathbb{F}_p$ for each i, i.e. $c_1, \cdots, c_t$ are $\mathbb{F}_p$-linearly dependent.

So, we can find $\beta_1, \cdots, \beta_t \in \mathbb{F}_p$, not all zero, such that $\beta_1 c_1 + \cdots + \beta_t c_t = 0$, or in other words:

$$\lim_{m\to\infty}(L_j(\beta_1 q_1 + \cdots + \beta_t q_t)^{p^k} b_j^{-p^k})^{p^m} = 0.$$

Therefore, $\rho(L_j(\beta_1 q_1 + \cdots + \beta_t q_t)^{p^k} b_j^{-p^k}) > 0$, and hence $\rho(L_j(\beta_1 q_1 + \cdots + \beta_t q_t)) > v(b_j) = p^{s-j}\lambda$.

But $\beta_1 q_1 + \cdots + \beta_t q_t = \tau(u(h_1^{\beta_1} \cdots h_t^{\beta_t}) - 1) + \varepsilon$, where $\rho(\varepsilon) > \lambda$, and we know that $\rho(L_j(\tau(u(h_1^{\beta_1} \cdots h_t^{\beta_t}) - 1))) = p^{s-j}\lambda$ by the definition of U_j, and $\rho(L_j(\varepsilon)) > p^{s-j}\lambda$ since L_j is a GPP.

Hence $\rho(L_j(\beta_1 q_1 + \cdots + \beta_t q_t)) = \rho(L_j(\tau(u(h_1^{\beta_1} \cdots h_t^{\beta_t}) - 1))) = p^{s-j}\lambda$ – contradiction.

Therefore P is controlled by a proper open subgroup of G. $\square$

4.8 Control Theorem

We may now suppose that for all $i \leq s$, L_i is not trivial. In particular, L_0 is a non-trivial GPP of p-degree s.

Proposition 4.8.1. *L_0 is a special growth preserving polynomial.*

Proof. Since L_0 is non-trivial, we may choose $h \in H$ such that $\rho(L_0(\tau(u(h) - 1))) = p^s\lambda$. We want to prove that $L_0(\tau(u(h) - 1))^{p^k}$ is v-regular for some k. Let $T = u_c(h) - 1 + F_{\theta+1}kG \in \text{Span}_{\mathbb{F}_p}\{T_1, \cdots, T_r\}$.

We know that $L^{(s)}(\overline{T}, \overline{B}_s, \cdots, \overline{B}_1)^{p^{m_1}}$ lies in A' by Lemma 4.6.2. If $L^{(s)}(\overline{T}, \overline{B}_s, \cdots, \overline{B}_1)^{p^{m_0}} \in \mathfrak{q}$ then we may assume that $L^{(s)}(\overline{T}, \overline{B}_s, \cdots, \overline{B}_1)^{p^{m_1}} = 0$, so using Lemma 4.6.4 we see that $\overline{w}(L^{(s)}(\tau(u(h)-1), b_s, \cdots, b_1)) > p^{s+m_1}\theta = \overline{w}(b_s^{p^s})$.

So again, since $\mathrm{gr}_{\overline{w}}(b_s) = \overline{B}_s \in A' \backslash \mathfrak{q}$, it follows from Theorem 4.2.3 that

$$v(L^{(s)}(\tau(u(h)-1), b_s, \cdots, b_1)^{p^m}) > v(b_s^{p^{m+s}}) \text{ for } m >> 0.$$

Hence $\rho(L_0(\tau(u(h)-1))) = \rho(L^{(s)}(\tau(u(h)-1), b_s, \cdots, b_1)) > \rho(b_s^{p^s}) \geq p^s\lambda$ – contradiction.

Therefore, we have that $L^{(s)}(\overline{T}, \overline{B}_s, \cdots, \overline{B}_1)^{p^{m_1}} \in A' \backslash q$, and hence it is equal to $\mathrm{gr}_{\overline{w}}(L^{(s)}(\tau(u(h)-1), b_s, \cdots, b_1))$ by Lemma 4.6.4.

It follows from Theorem 4.2.3 that for $m >> 0$, $L_0(\tau(u(h)-1))^{p^m} = L^{(s)}(\tau(u(h)-1), b_s, \cdots, b_1)^{p^m}$ is v-regular, and hence L_0 is a special GPP by Definition 4.6.6. $\square$

Now we can finally prove our main control theorem in all cases. But we first need the following technical result.

Lemma 4.8.2. *Let $G = H \rtimes \langle X \rangle$ be an abelian-by-procyclic group. Then G has split centre if and only if $(G, G) \cap Z(G) = 1$.*

Proof. It is clear that if G has split centre then $(G, G) \cap Z(G) = 1$. Conversely, suppose that $(G, G) \cap Z(G) = 1$, and consider the $\mathbb{Z}_p$-module homomorphism:

$$H \to H, h \mapsto (X, h).$$

The kernel of this map is precisely $Z(G)$, therefore $(X, H) \cong \frac{H}{Z(G)}$. So since $Z(G) \cap (X, H) = 1$, it follows that $Z(G) \times (X, H)$ has the same rank as H, hence it is open in H.

Recall from [39, Definition 1.6] the definition of the *isolator* $i_G(N)$ of a closed, normal subgroup N of G, and recall from [39, Proposition 1.7, Lemma 1.8] that it is a closed, isolated normal subgroup of G, and that N is open in $i_G(N)$.

Let $C = i_G((X, H)) \leq H$, then it is clear that $Z(G) \cap C = 1$ and that $Z(G) \times C$ is isolated.

Therefore, since $Z(G) \times (X, H)$ is open in H, it follows that $Z(G) \times C = H$, and hence $G = Z(G) \times C \rtimes \langle X \rangle$, and G has split centre. □

Proof of Theorem B. Let $Z_2(G) := \{g \in G : (g, G) \subseteq Z(G)\}$, this is a closed subgroup of G containing $Z(G)$. Suppose first that $Z_2(G) \neq Z(G)$.

Then choose $g \in Z_2(G)\backslash Z(G)$, then $(g, G) \subseteq Z(G)$, so if we take $\psi \in Inn(G)$ to be conjugation by g, then ψ is trivial mod centre, and clearly $\psi(P) = P$. So it follows that P is controlled by a proper, open subgroup of G by [1, Theorem B].

So from now on, we may assume that $Z_2(G) = Z(G) \subseteq H$.

Suppose that $(X, h) \in Z(G)$ for some $h \in H$, then clearly $(h, G) \subseteq Z(G)$, so $h \in Z_2(G) = Z(G)$, giving that $(X, h) = 1$. It follows that $Z(G) \cap (G, G) = 1$, and hence G has split centre by Lemma 4.8.2.

Therefore, using Theorem 4.4.1, we can choose a J-basis $\{k_1, \cdots, k_d\}$ for H and let w be the corresponding $c(G)$-filtration on kG such that $\text{gr}_w\, kG \cong k[T_1, \cdots, T_{d+1}] * \frac{G}{c(G)}$, where $T_i = \text{gr}(u_c(k_i) - 1)$ for $i \leq r$, $T_i = \text{gr}(k_i - 1)$ for $i > r$, $T_r, \cdots, T_d$ are central and $\bar{X}T_i\bar{X}^{-1} = T_i + D_i$ for some $D_i \in \text{Span}_{\mathbb{F}_p}\{T_{i+1}, \cdots, T_r\}$ for all $i < r$.

If $Q(kG/P)$ is a CSA, then since conjugation by X is an inner strong H-Mahler automorphism, the result follows from Corollary 3.9.6, so we may assume that $Q(kG/P)$ is not a CSA.

Hence if each D_i is nilpotent mod gr P, then $id : \tau(kH) \to \tau(kH)$ is a special GPP with respect to some non-commutative valuation by Proposition 4.5.3. Therefore, by Theorem 3.7.5, P is controlled by a proper open subgroup of G as required.

If D_s is not nilpotent mod gr P for some $s < r$, then we can construct GPP's $L_s, \cdots, L_0$ with respect to some non-commutative valuation using Proposition 4.6.6, and L_s is non-trivial.

If L_{j-1} is trivial and L_j is non-trivial for some $0 < j \leq s$, then the result follows from Theorem 4.7.4. Whereas if all the L_i are non-trivial, then L_0 is a special GPP by Proposition 4.8.1, and the result follows again from Theorem 3.7.5. □

4.9 Proof of Theorem A

Now we can conclude with our main classification result for prime ideals in kG for G abelian-by-procyclic.

Proof of Theorem A. Let P be a prime ideal of kG. Let $P^\dagger = \{g \in G : g - 1 \in P\}$, and recall from [1, Lemma 5.3(c)] that $P^\dagger$ is a closed, isolated normal subgroup of G, and hence $G_0 = \frac{G}{P^\dagger}$ is uniform, abelian-by-procyclic by Lemma 2.2.4, and the image $P_0 := \frac{P}{(P^\dagger - 1)kG}$ of P in kG_0 is a faithful prime ideal of kG_0.

So to prove that P is standard, it remains only to prove that P_0 is centrally generated, i.e. controlled by $Z(G_0)$. We will show, more generally, that for G a p-valuable, abelian-by-procyclic group, all faithful prime ideals in kG are controlled by $Z(G)$.

Recall that a prime ideal P is *non-splitting* if for all $U \leq_o G$ controlling P, $P \cap kU$ is prime in kU. We will suppose first that our faithful prime ideal P is non-splitting.

Consider the controller subgroup P^χ of P. Using [1, Proposition 5.5] and the non-splitting property, we see that $Q := P \cap kP^\chi$ is a faithful prime ideal of kP^χ. Also, since P^χ is the smallest subgroup of G controlling P by [3, Theorem A], it follows that Q is not controlled by any proper subgroup of P^χ.

We know that P^χ is a closed, normal subgroup of G, so it follows from Lemma 2.2.4 that P^χ is abelian-by-procyclic. If P^χ is non-abelian, then applying Theorem B gives that Q is controlled by a proper open subgroup U of P^χ – contradiction.

Therefore, P^χ is abelian. So for any $g \in G$, if $\phi \in Aut^\omega(P^\chi)$ is defined by $\phi(h) = ghg^{-1}$, then $\phi(h)h^{-1} \in P^\chi = Z(P^\chi)$ for all $h \in P^\chi$. Therefore, since $\phi(Q) = Q$ and Q is not controlled by any proper subgroup of P^χ, it follows from [1, Theorem B] that $\phi = 1$. Since this is true for all $g \in G$, it follows that P^χ is central in G, i.e. $P^\chi \subseteq Z(G)$ and P is controlled by $Z(G)$ as required.

So, we conclude that any faithful, non-splitting prime ideal of kG is controlled by $Z(G)$. Now suppose that I is a faithful, *virtually non-splitting* right ideal of kG, i.e. $I = PkU$ for some open subgroup U of G, P a faithful, non-splitting prime ideal of kU. Using Lemma 2.2.4, we see that U is p-valuable, abelian-by-procyclic, so by the above discussion, P is controlled by $Z(U)$, and in fact, $Z(U) = Z(G) \cap U$ by [1, Lemma 8.4]. Therefore, since $I \cap kU = P$ by [1, Lemma 5.1(ii)]:

$$I = PkG = (P \cap kZ(U))kUkG = (I \cap kU \cap kZ(G))kG = (I \cap kZ(G))kG.$$

So since G is p-valuable and every faithful, virtually non-splitting right ideal of kG is controlled by $Z(G)$, it follows from [1, Theorem 5.8, Proposition 5.9] that every faithful prime ideal of kG is controlled by $Z(G)$ as required. $\square$

Chapter 5

Affinoid Dixmier Modules

We will now explore some Lie theory, which will be fundamental to the proof of our final main results. Fix $\mathfrak{g}$ a finite dimensional K-Lie algebra, and $\mathcal{L}$ an $\mathcal{O}$-Lie lattice in $\mathfrak{g}$. In this chapter, we will explore further the properties of the affinoid enveloping algebra $\widehat{U(\mathcal{L})}_K$, ultimately leading to the proof of Theorem D.

5.1 Polarisations

For now, we will examine the classical picture. So let F be a field of characteristic 0, and let $\mathfrak{h}$ be a finite dimensional Lie algebra over F. First recall the following definition [16, 1.12.8]:

Definition 5.1.1. *Given a linear form $\lambda \in \mathfrak{h}^*$, define a* polarisation *of $\mathfrak{h}$ at λ to be a solvable subalgebra $\mathfrak{b}$ of $\mathfrak{h}$ such that if V is a subspace of $\mathfrak{h}$ and $\mathfrak{b} \subseteq V$, then $\lambda([V, V]) = 0$ if and only if $V = \mathfrak{b}$.*

In particular, if $\mathfrak{b}$ is a polarisation of $\mathfrak{h}$ at λ, then $\lambda([\mathfrak{b}, \mathfrak{b}]) = 0$, i.e. λ restricts to a character of $\mathfrak{b}$. Note that polarisations need not always exist.

Given $\lambda \in \mathfrak{h}^*$, define $\mathfrak{h}^\lambda := \{x \in \mathfrak{h} : \lambda([x, \mathfrak{h}]) = 0\}$.

Lemma 5.1.2. *Given $\lambda \in \mathfrak{h}^*$, if $\mathfrak{b}$ is a polarisation of $\mathfrak{h}$ at λ then:*

- $\dim_F \mathfrak{b} = \frac{1}{2}(\dim_F \mathfrak{h} + \dim_F \mathfrak{h}^\lambda)$.

- *If* $\mathfrak{b}'$ *is a subalgebra of* $\mathfrak{h}$ *such that* $\lambda([\mathfrak{b}', \mathfrak{b}']) = 0$ *and* $\dim_F \mathfrak{b}' = \frac{1}{2}(\dim_F \mathfrak{h} + \dim_F \mathfrak{h}^\lambda)$*, then* $\mathfrak{b}'$ *is a polarisation of* $\mathfrak{h}$ *at* λ*.*

- $\mathfrak{b}$ *contains every ideal* $\mathfrak{a}$ *of* $\mathfrak{h}$ *such that* $\lambda([\mathfrak{h}, \mathfrak{a}]) = 0$*. In particular,* $\mathfrak{b}$ *contains* $Z(\mathfrak{h})$ *and every ideal* $\mathfrak{a}$ *such that* $\lambda(\mathfrak{a}) = 0$*.*

Proof. The first two statements follow from [16, 1.12.1]. For the final statement, if $\mathfrak{a}$ is an ideal of $\mathfrak{h}$ such that $\lambda([\mathfrak{h}, \mathfrak{a}]) = 0$, then $\lambda([\mathfrak{b} + \mathfrak{a}, \mathfrak{b} + \mathfrak{a}]) \subseteq \lambda([\mathfrak{b}, \mathfrak{b}]) + \lambda([\mathfrak{b}, \mathfrak{a}]) + \lambda([\mathfrak{a}, \mathfrak{a}]) = 0$. Hence $\mathfrak{b}+\mathfrak{a} = \mathfrak{b}$ by the definition of a polarisation, and hence $\mathfrak{a} \subseteq \mathfrak{b}$. □

Examples: 1. If λ is a character of $\mathfrak{h}$, i.e. $\lambda([\mathfrak{h}, \mathfrak{h}]) = 0$, then $\mathfrak{h}$ is a polarisation of $\mathfrak{h}$ at λ. In particular, if $\mathfrak{h}$ is abelian, then for any $\lambda \in \mathfrak{h}^*$, $\mathfrak{h}$ is a polarisation of $\mathfrak{h}$ at λ, in fact it is the only polarisation.

2. If $\mathfrak{h} = \mathfrak{a} \rtimes Fx$ for some abelian subalgebra $\mathfrak{a}$ of $\mathfrak{h}$, $x \in \mathfrak{h}$, then for any $\lambda \in \mathfrak{h}^*$, if $\lambda([\mathfrak{h}, \mathfrak{h}]) \neq 0$ then $\mathfrak{a}$ is a polarisation of $\mathfrak{h}$ at λ.

3. If F is algebraically closed, and $\mathfrak{h}$ is semisimple, then $\lambda \in \mathfrak{h}^*$ has a polarisation $\mathfrak{b}$ if and only if λ is *regular* in the sense of [16, 1.11.6]. In this case $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{h}$ by [16, Proposition 1.12.18], and hence λ is a character of a Cartan subalgebra.

In our case, we will be interested in the case where $\mathfrak{h}$ is solvable. The following result ensures that polarisations always exist in this case:

Proposition 5.1.3. *Suppose that* $\mathfrak{h}$ *contains a chain of ideals* $0 = \mathfrak{h}_0 \subseteq \mathfrak{h}_1 \subseteq \cdots \subseteq \mathfrak{h}_n = \mathfrak{h}$ *such that* $\dim_F \mathfrak{h}_i = i$ *for each* i*, i.e.* $\mathfrak{h}$ *is* completely solvable*. Given* $\lambda \in \mathfrak{h}^*$*, if we set* $\lambda_i := \lambda|_{\mathfrak{h}_i}$*, then the subalgebra* $\mathfrak{b}_\lambda := \mathfrak{h}_1^{\lambda_1} + \cdots + \mathfrak{h}_d^{\lambda_d}$ *is a polarisation of* $\mathfrak{h}$ *at* λ*.*

Proof. This is the proof of [16, Proposition 1.12.18]. □

We call this polarisation $\mathfrak{b}_\lambda$ a *standard polarisation* of $\mathfrak{h}$ at λ, note that it depends on the choice of chain $0 = \mathfrak{h}_0 \subseteq \mathfrak{h}_1 \subseteq \cdots \subseteq \mathfrak{h}_n = \mathfrak{h}$.

Note: 1. If $\mathfrak{h}$ is nilpotent, a chain of this form always exists, and for general solvable $\mathfrak{h}$, we can pass to a finite extension F' of F to find such a chain in $\mathfrak{h}_{F'} := \mathfrak{h} \otimes_F F'$.

2. Given any abelian ideal $\mathfrak{a}$ of $\mathfrak{h}$, we can find a chain such that $\mathfrak{b}_\lambda$ will contain $\mathfrak{a}$.

Now, for any $\lambda \in \mathfrak{h}^*$ and any polarisation $\mathfrak{b}$ of $\mathfrak{h}$ at λ, since λ restricts to a character of $\mathfrak{b}$, it follows that $F_\lambda := Fv$ is a $U(\mathfrak{b})$-module via the $\mathfrak{b}$-action $x \cdot v := \lambda(x)v$. This gives rise to the following definition:

Definition 5.1.4. *Let $\lambda \in \mathfrak{h}^*$, and let $\mathfrak{b}$ be a polarisation of $\mathfrak{h}$ at λ. We define the* $\mathfrak{b}$-Dixmier module *of $\mathfrak{h}$ at λ to be the $U(\mathfrak{h})$-module:*

$$D(\lambda) = D(\lambda)_\mathfrak{b} := U(\mathfrak{h}) \otimes_{U(\mathfrak{b})} F_\lambda. \tag{5.1}$$

Note that $D(\lambda)$ is a cyclic $U(\mathfrak{h})$ module, generated by a vector v_λ on which $U(\mathfrak{b})$ acts by scalars.

This definition is useful, because in the case where F is algebraically closed and $\mathfrak{h}$ is semisimple, these Dixmier modules are precisely the well-known Verma modules, which are fundamental within the representation theory of semisimple Lie algebras. So we may think of Dixmier modules as a generalisation of Verma modules.

Now we will return to the p-adic case. Recall from Definition 2.8.2 the definition of the affinoid enveloping algebra $\widehat{U(\mathcal{L})}_K$, we now want to generalise the notion of a Dixmier module over $U(\mathfrak{g})$ to a completed version over $\widehat{U(\mathcal{L})}_K$:

Given a linear form $\lambda \in \mathfrak{g}^*$, we now make the additional assumption that $\lambda(\mathcal{L}) \subseteq \mathcal{O}$, i.e. $\lambda \in \mathcal{L}^* \subseteq \mathfrak{g}^*$. First note that if $\mathfrak{b}$ is a polarisation of $\mathfrak{g}$ at λ, then $\mathcal{B} := \mathfrak{b} \cap \mathcal{L}$ is a Lie lattice in $\mathfrak{b}$, and $K_\lambda := Kv_\lambda$ is a one dimensional module over $\widehat{U(\mathcal{B})}_K$ via the natural extension of the action of $U(\mathfrak{b})$ to the completion, which is well defined since $\pi^n U(\mathcal{L}) \cdot \mathcal{O}v_\lambda \subseteq \pi^n \mathcal{O}v_\lambda$ for all $n \in \mathbb{N}$.

Definition 5.1.5. *Let $\lambda \in \mathcal{L}^*$ and let $\mathfrak{b}$ be a polarisation of $\mathfrak{g}$ at λ. Define the* $\mathfrak{b}$-affinoid Dixmier module *of $\mathcal{L}$ at λ to be the $\widehat{U(\mathcal{L})}_K$-module defined by:*

$$\widehat{D(\lambda)} = \widehat{D(\lambda)}_{\mathfrak{b}} := \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B})}_K} K_\lambda. \tag{5.2}$$

Notation: If it is unclear what the ground field K is, we may sometimes write $\widehat{D(\lambda)}_K$ for $\widehat{D(\lambda)}$. Also, if it is unclear which lattice $\mathcal{L}$ we are considering, we may sometimes write $\widehat{D(\lambda)}_{\mathcal{B}}$ instead of $\widehat{D(\lambda)}_{\mathfrak{b}}$.

Note that as in the classical case, $\widehat{D(\lambda)}$ is a cyclic $\widehat{U(\mathcal{L})}_K$-module, so $\widehat{D(\lambda)}_{\mathfrak{b}} = \widehat{U(\mathcal{L})}_K v_\lambda$, and $\widehat{U(\mathcal{B})}_K$ acts by scalars on v_λ. In particular, using Proposition 2.8.4, we see that $\widehat{D(\lambda)}$ is π-adically complete with respect to some lattice. In fact it is a π-adic completion of the classical Dixmier module $D(\lambda)$.

Examples: 1. If $\mathfrak{g}$ is split semisimple with Borel subalgebra $\mathfrak{b}$, the *affinoid Verma module* $\widehat{V(\lambda)}$ arises as a Dixmier module. This still has a unique simple quotient $\widehat{L(\lambda)}$.

2. If $\mathfrak{g}$ is abelian, or more generally if λ is a character of $\mathfrak{g}$, then $\widehat{D(\lambda)} = K$ always.

3. If $\mathfrak{g} = \mathfrak{a} \rtimes Kx$, for $\mathfrak{a}$ abelian, then if λ is not a character of $\mathfrak{g}$, $\widehat{D(\lambda)} \cong K\langle t\rangle$, where x acts by t, and each $u \in \mathfrak{a}$ acts by a polynomial in $K[\frac{d}{dt}]$.

5.2 Reducing Quadruples

We will now recall some more classical theory. Again, let F be a field of characteristic 0. The following definition ([16, 4.7.7])) will be very useful to us throughout:

Definition 5.2.1. *Let* $\mathfrak{h}$ *be an* F*-Lie algebra. A* reducing quadruple *of* $\mathfrak{h}$ *is a 4-tuple* $(x, y, z, \mathfrak{h}')$ *where:*

- $0 \neq x, y, z \in \mathfrak{h}$ *and* $\mathfrak{h}'$ *is an ideal of* $\mathfrak{h}$ *of codimension 1,*
- $y, z \in \mathfrak{h}'$ *and* $x \notin \mathfrak{h}'$*,*
- z *is central in* $\mathfrak{h}$ *and* y *is central in* $\mathfrak{h}'$*,*
- $[x, y] = \alpha z$ *for some* $0 \neq \alpha \in F$*.*

In this section, we will prove some technical results concerning the relationship between reducing quadruples and polarisations that we will need in our main argument. The first two can be found in the proof of [16, Theorem 6.1.1]:

Proposition 5.2.2. *Suppose* $\mathfrak{h}$ *is a nilpotent* F*-Lie algebra, and that* $0 = \mathfrak{h}_0 \subseteq \mathfrak{h}_1 \subseteq \cdots \subseteq \mathfrak{h}_n = \mathfrak{h}$ *is a chain of ideals in* $\mathfrak{h}$ *with* $\dim_F \mathfrak{h}_i = i$ *for each* i*. Also let* $\lambda \in \mathfrak{h}^*$ *and let* $\mathfrak{b}_\lambda$ *be the standard polarisation of* $\mathfrak{h}$ *at* λ *with respect to this chain. If we suppose that* $\lambda(\mathfrak{a}) \neq 0$ *for all non-zero ideals* $\mathfrak{a}$ *of* $\mathfrak{h}$ *then:*

(i) There exist $x, y, z \in \mathfrak{h}$*,* $\mathfrak{h}' \trianglelefteq \mathfrak{h}$ *such that* $(x, y, z, \mathfrak{h}')$ *is a reducing quadruple for* $\mathfrak{h}$*.*
(ii) $\mathfrak{b}_\lambda \subseteq \mathfrak{h}'$*.*
(iii) $\mathfrak{b}_\lambda$ *is a standard polarisation of* $\mathfrak{h}'$ *at* $\lambda|_{\mathfrak{g}'}$*.*

Proof. Fix $j \geq 0$ minimal such that $\mathfrak{h}_j$ is not central, and thus $\mathfrak{h}_{j-1}$ is central.

(i) Firstly, for any $0 \neq z$ in the centre of $\mathfrak{h}$, Kz is an ideal of $\mathfrak{h}$, and hence $\lambda(z) \neq 0$. Therefore, $\lambda : Z(\mathfrak{h}) \to F$ is injective, meaning that $Z(\mathfrak{h})$ must have dimension 0 or 1.

But since $\mathfrak{h}$ is nilpotent, the centre cannot be 0, hence $Z(\mathfrak{h}) = Kz$ for some $z \in Z(\mathfrak{h})$.

Since $\mathfrak{h}_j = \mathfrak{h}_{j-1} \oplus Fy$ for some $y \in \mathfrak{h}_j$, it follows that $\mathfrak{h}_j$ is abelian. Also, by our definition of $\mathfrak{h}_j$, y is not central.

Define $\mathfrak{h}' := \{u \in \mathfrak{h} : \lambda([u, \mathfrak{h}_j]) = 0\}$. It is clear that this is an ideal of $\mathfrak{h}$, and if $\mathfrak{h}' = \mathfrak{h}$ then $\lambda([\mathfrak{h}, \mathfrak{h}_j]) = 0$, which means that $[\mathfrak{h}, \mathfrak{h}_j] = 0$ since it is an ideal, and so $\mathfrak{h}_j$ is central in $\mathfrak{h}$ – contradiction. Hence $\mathfrak{h}'$ is a proper ideal of $\mathfrak{h}$, and note that $\mathfrak{h}'$ contains the centre of $\mathfrak{h}$.

Since $\mathfrak{h}_{j-1} \subseteq Z(\mathfrak{h}) = Fz$, it follows that $Fy \oplus Fz$ is an ideal of $\mathfrak{h}$, so for all $x \in \mathfrak{h}$, $[x, y] = \alpha y + \beta z$ for some $\alpha, \beta \in F$. Hence $(\text{ad}(x))^n(y) = \alpha^n y + \alpha^{n-1}\beta z$, so since $\mathfrak{h}$ is nilpotent, it follows that $\alpha = 0$, i.e. $[y, \mathfrak{h}] \subseteq Fz$.

Since $\lambda(z) \neq 0$, it follows that $\lambda([u, y]) = 0$ if and only if $[u, y] = 0$, and hence $\mathfrak{h}' = \ker(\text{ad}(y))$. So since the image of $\text{ad}(y)$ has dimension at most 1, this means that $\mathfrak{h}'$ has codimension 1 in $\mathfrak{h}$, so $\mathfrak{h} = \mathfrak{h}' \oplus Fx$ for some $x \in \mathfrak{h}$.

Finally, it is clear that y is central in $\mathfrak{h}' = \ker(\text{ad}(y))$, so since y is not central in $\mathfrak{h}$, $[x, y] = \alpha z$ for some $0 \neq \alpha \in F$. Thus $(x, y, z, \mathfrak{g}')$ is a reducing quadruple.

(ii) Let $\mathfrak{h}'_i := \mathfrak{h}_i \cap \mathfrak{h}'$, then $0 = \mathfrak{h}'_0 \subseteq \cdots \subseteq \mathfrak{h}'_n = \mathfrak{h}'$. So choose a subsequence $0 = j_0 < j_1 < \cdots < j_m = n$ such that $\dim_F \mathfrak{h}_{j_k} = k$ for all k. For convenience, set $\mathfrak{t}_k := \mathfrak{h}'_{j_k}$, and set $\mu := \lambda|_{\mathfrak{h}'}$.

Note that if $u \in \mathfrak{h}_i^{\lambda_i}$ for some i, then $u \in \mathfrak{h}_i$ and $\lambda([u, \mathfrak{h}_i]) = 0$. If $i < j$ then $\mathfrak{h}_i$ is central, thus u is central so $u \in \mathfrak{h}'$. Whereas if $i \geq j$ then $[u, \mathfrak{h}_j] \subseteq [u, \mathfrak{h}_i]$, and hence $\lambda([u, \mathfrak{h}_j]) = 0$ and $u \in \mathfrak{h}'$. Therefore, $\mathfrak{h}_i^{\lambda_i} \subseteq \mathfrak{h}'$ for each i, and hence $\mathfrak{b}_\lambda \subseteq \mathfrak{h}'$.

(iii) If $u \in \mathfrak{h}_i^{\lambda_i}$ then since $u \in \mathfrak{h}_i \cap \mathfrak{h}' = \mathfrak{h}'_i$ we have that $\lambda([u, \mathfrak{h}'_i]) = 0$, and hence $u \in \mathfrak{h}_i'^{\lambda_i}$. Therefore, $\mathfrak{b}_\lambda = \mathfrak{h}_0^{\lambda_0} + \cdots + \mathfrak{h}_n^{\lambda_n} \subseteq \mathfrak{h}_0'^{\lambda_0} + \cdots + \mathfrak{h}_n'^{\lambda_n} = \mathfrak{t}_0^{\mu_0} + \cdots + \mathfrak{t}_m^{\mu_m} = \mathfrak{b}_\mu$. But since $\mathfrak{b}_\lambda$ is a polarisation of $\mathfrak{h}$ at λ and $\lambda([\mathfrak{b}_\mu, \mathfrak{b}_\mu]) = 0$, this means that $\mathfrak{b}_\lambda = \mathfrak{b}_\mu$, i.e. $\mathfrak{b}_\lambda$ is a standard polarisation of $\mathfrak{h}'$ at μ. □

The next result ensures that after quotienting out by a suitable ideal, we can always ensure that our linear form λ satisfies the hypotheses of Proposition 5.2.2.

Lemma 5.2.3. *Let $\lambda \in \mathfrak{h}^*$, let $0 = \mathfrak{h}_0 \subseteq \mathfrak{h}_1 \subseteq \cdots \subseteq \mathfrak{h}_n = \mathfrak{h}$ be a chain of ideals in $\mathfrak{h}$ with $\dim_F \mathfrak{h}_i = i$ for each i, and let $\mathfrak{b}_\lambda = \mathfrak{h}^{\lambda_0} + \mathfrak{h}_1^{\lambda_1} + \cdots + \mathfrak{h}_n^{\lambda_n}$ be the standard polarisation at λ corresponding to this chain.*

Given an ideal $\mathfrak{a}$ of $\mathfrak{h}$ such that $\lambda(\mathfrak{a}) = 0$, let μ be the linear form of $\mathfrak{h}^q := \mathfrak{h}/\mathfrak{a}$ induced by λ, and let $\mathfrak{b}^q := (\frac{\mathfrak{h}_0+\mathfrak{a}}{\mathfrak{a}})^{\mu_0} + \cdots + (\frac{\mathfrak{h}_n+\mathfrak{a}}{\mathfrak{a}})^{\mu_n}$, where μ_i is the restriction of μ to $\frac{\mathfrak{h}_i+\mathfrak{a}}{\mathfrak{a}}$. Then:

(i) $\mathfrak{a} \subseteq \mathfrak{b}_\lambda$.

(ii) $\mathfrak{b}^q$ is a standard polarisation of $\mathfrak{h}^q$ at μ.

(iii) $\mathfrak{b}^q = \mathfrak{b}_\lambda/\mathfrak{a}$.

Proof. Let $\mathfrak{h}_i^q := \frac{\mathfrak{h}_i+\mathfrak{a}}{\mathfrak{a}}$ for each $i = 1, \cdots, n$. Then $0 = \mathfrak{h}_0^q \subseteq \mathfrak{h}_1^q \subseteq \cdots \subseteq \mathfrak{h}_n^q = \mathfrak{h}^q$ is a chain of ideals in $\mathfrak{h}^q$. So we choose a subsequence $0 = j_0 < j_2 < \cdots < j_m = n$ of $0, 1, \cdots, n$ such that for each k, $\dim_F \mathfrak{h}_{j_k}^q = k$.

For convenience, set $\mathfrak{t}_k := \mathfrak{h}_{j_k}^q$, then it is clear that each $\mathfrak{h}_i^q$ appears in $\mathfrak{t}_1, \cdots, \mathfrak{t}_m$.

(i) Since $\mathfrak{a}$ is an ideal of $\mathfrak{h}$, $[\mathfrak{h}, \mathfrak{a}] \subseteq \mathfrak{a}$, so $\lambda([\mathfrak{h}, \mathfrak{a}]) = 0$. So since $\mathfrak{h}_n = \mathfrak{h}$, this means that $\mathfrak{a} \subseteq \{x \in \mathfrak{h} : \lambda([x, \mathfrak{h}_n]) = 0\} = \mathfrak{h}_n^{\lambda_n} \subseteq \mathfrak{b}_\lambda$ as required.

(ii) Consider the standard polarisation $\mathfrak{b}_\mu = \mathfrak{t}_0^{\mu_0} + \cdots + \mathfrak{t}_m^{\mu_m}$ of $\mathfrak{h}^q$ at μ, we will prove that $\mathfrak{b}_\mu = \mathfrak{b}^q$.

If $x \in \mathfrak{t}_k^{\mu_k} = \{y \in \mathfrak{t}_k : \mu([y, \mathfrak{t}_k]) = 0\}$, then $x = z + \mathfrak{a}$ for some $z \in \mathfrak{h}_{j_k}$, and clearly $\lambda([z, \mathfrak{h}_{j_k}]) = 0$, so $z \in \mathfrak{h}_{j_k}^{\lambda_{j_k}}$. Therefore $\mathfrak{t}_k^{\mu_k} \subseteq \frac{\mathfrak{h}_{j_k}^{\lambda_{j_k}} + \mathfrak{a}}{\mathfrak{a}}$ for each k, and hence $\mathfrak{b}_\mu \subseteq \mathfrak{b}'$.

Conversely, if $z \in \mathfrak{h}_{j_k}^{\lambda_{j_k}}$, then it is clear that since $\lambda(\mathfrak{a}) = 0$, we have that $\mu([z+\mathfrak{a}, h_k]) = 0$, and so $z + \mathfrak{a} \in \mathfrak{t}_k^{\mu_k}$. Thus $\mathfrak{t}_k^{\mu_k} = \frac{\mathfrak{h}_{j_k}^{\lambda_{j_k}} + \mathfrak{a}}{\mathfrak{a}}$ and $\mathfrak{b}_\mu = \mathfrak{b}^q$.

(*iii*) Since $\mathfrak{b}_\lambda = \mathfrak{h}_0^{\lambda_0} + \cdots + \mathfrak{h}_n^{\lambda_n}$ and $\mathfrak{a} \subseteq \mathfrak{b}_\lambda$, it follows that

$$\mathfrak{b}_\lambda/\mathfrak{a} = \frac{\mathfrak{b}_\lambda + \mathfrak{a}}{\mathfrak{a}} = \frac{\mathfrak{h}_0^{\lambda_0} + \cdots + \mathfrak{h}_n^{\lambda_n} + \mathfrak{a}}{\mathfrak{a}} = (\frac{\mathfrak{h}_0 + \mathfrak{a}}{\mathfrak{a}})^{\mu_0} + \cdots + (\frac{\mathfrak{h}_n + \mathfrak{a}}{\mathfrak{a}})^{\mu_n} = \mathfrak{b}^q. \qquad \square$$

While the previous results mainly concern restrictions of polarisations to reducing quadruples, the next result is more concerned with extending them.

Lemma 5.2.4. *Suppose that $\mathfrak{h}$ has reducing quadruple $(x, y, z, \mathfrak{h}')$, and that μ is a linear form of $\mathfrak{h}'$ with polarisation $\mathfrak{b} \subseteq \mathfrak{h}'$ such that $\mu(z) \neq 0$. Then for any extension λ of μ to $\mathfrak{h}$, $\mathfrak{b}$ is a polarisation of $\mathfrak{h}$ at λ.*

Proof. Let V be a vector subspace of $\mathfrak{h}$ such that $\mathfrak{b} \subseteq V$ and $\lambda([V, V]) = 0$. We will show that $V = \mathfrak{b}$, and it will follow that $\mathfrak{b}$ is a polarisation by Definition 5.1.1.

It will suffice to show that $V \subseteq \mathfrak{h}'$ and simply apply the fact that $\mathfrak{b}$ is a polarisation of $\mathfrak{h}'$. So given an element $\beta x + a \in V$, with $\beta \in F$, $a \in \mathfrak{h}'$, we will prove that $\beta = 0$.

Since $\mathfrak{b}$ is a polarisation of $\mathfrak{h}'$ and y is central in $\mathfrak{h}'$, we must have that $y \in \mathfrak{b} \subseteq V$, and therefore $\mu([\beta x + a, y]) = 0$. But $[a, y] = 0$ and $[x, y] = \alpha z$ where $\alpha \neq 0$, thus $0 = \beta\mu([x, y]) = \alpha\beta\mu(z)$. So since $\alpha, \mu(z) \neq 0$, it follows that $\beta = 0$ as required. $\square$

The final classical result we need can be found in the proof of [16, Theorem 6.1.4]:

Proposition 5.2.5. *Suppose that $\mathfrak{h}$ is nilpotent with $n := \dim_F \mathfrak{h} > 3$, and suppose that $\mathfrak{h}$ has a reducing quadruple $(x, y, z, \mathfrak{h}')$. We assume further that $\lambda \in \mathfrak{h}^*$ and $\lambda(\mathfrak{a}) \neq 0$ for all non-zero ideals $\mathfrak{a}$ of $\mathfrak{h}$. Then for any polarisation $\mathfrak{b}$ of $\mathfrak{h}$ at λ, there* *exists a polarisation* $\mathfrak{b}'$ *at* λ*, contained in* $\mathfrak{h}'$*, and a proper subalgebra* $\mathfrak{t} \subsetneq \mathfrak{h}$ *such that* $\mathfrak{b}, \mathfrak{b}' \subseteq \mathfrak{t}$.

Proof. Firstly, note that $\mathfrak{h}^\lambda = \{u \in \mathfrak{h} : \lambda([u, \mathfrak{h}]) = 0\}$ is contained in $\mathfrak{b}$, otherwise $\mathfrak{b} \subsetneq \mathfrak{b} + \mathfrak{h}^\lambda$ and $\lambda([\mathfrak{b} + \mathfrak{h}^\lambda, \mathfrak{b} + \mathfrak{h}^\lambda]) = 0$, contradicting the definition of a polarisation.

If $\mathfrak{b} \subseteq \mathfrak{h}'$, then taking $\mathfrak{b}' = \mathfrak{b}$, $\mathfrak{t} = \mathfrak{h}'$, the statement is trivially true, so we may assume that $\mathfrak{b} \not\subseteq \mathfrak{h}'$.

Since y is central in $\mathfrak{h}'$ but not in $\mathfrak{h}$, it is clear that $\mathfrak{h}' = \ker(\mathrm{ad}(y))$ and $Fz = \mathrm{im}(\mathrm{ad}(y))$. So since there exists $u \in \mathfrak{b} \backslash \mathfrak{h}'$, we have that $[u, y] \neq 0$, i.e. $[u, y] = \beta z$ for some $0 \neq \beta \in F$. But since Fz is a non-zero ideal of $\mathfrak{h}$, $\lambda(z) \neq 0$, and thus $\lambda([u, y]) \neq 0$. So since $\lambda([\mathfrak{b}, \mathfrak{b}]) = 0$ and $u \in \mathfrak{b}$, this means that $y \notin \mathfrak{b}$.

Let $\mathfrak{b}' := (\mathfrak{b} \cap \mathfrak{h}') \oplus Fy$. This is a subalgebra of $\mathfrak{h}$, and clearly it is contained in $\mathfrak{h}'$. Also, $\mathfrak{b} \cap \mathfrak{h}'$ has codimension 1 in $\mathfrak{b}$, therefore $\dim_F \mathfrak{b}' = \dim_F \mathfrak{b} \cap \mathfrak{h}' + 1 = \dim_F \mathfrak{b}$. But it is clear that $\lambda([\mathfrak{b}', \mathfrak{b}']) = 0$, so by Lemma 5.1.2, this means that $\mathfrak{b}'$ is a polarisation of $\mathfrak{h}$ at λ.

Now, let $\mathfrak{t} := \mathfrak{b} \oplus Fy$. Since $Fz = [y, \mathfrak{h}] \subseteq \mathfrak{b}$, this is a subalgebra of $\mathfrak{h}$, and clearly it contains $\mathfrak{b}$ and $\mathfrak{b}'$, so we only need to prove that $\mathfrak{t} \neq \mathfrak{h}$, so assume for contradiction that $\mathfrak{t} = \mathfrak{h}$. This means that $\mathfrak{b}$ has codimension 1 in $\mathfrak{h}$, so $\dim_F \mathfrak{b} = n - 1$. But $\dim_F \mathfrak{b} = \frac{1}{2}(n + \dim_F \mathfrak{h}^\lambda)$ by Lemma 5.1.2, and thus $\dim_F \mathfrak{h}^\lambda = n - 2$.

Furthermore, if $\beta x + \gamma y \in \mathfrak{h}^\lambda$, then $\beta\lambda([x, y]) = \gamma\lambda([y, x]) = 0$, which is only possible if $\beta = \gamma = 0$ since $\lambda(z) = \lambda(\alpha^{-1}[x, y]) \neq 0$. So $\mathrm{Span}_F\{x, y\} \cap \mathfrak{h}^\lambda = 0$, and therefore, $\mathfrak{h} = Fx \oplus Fy \oplus \mathfrak{h}^\lambda$.

Let $\mathfrak{a} := \ker(\lambda) \cap \mathfrak{h}^\lambda$. Then since $z \in \mathfrak{h}^\lambda$ and $\lambda(z) \neq 0$, it follows that $\mathfrak{a}$ has codimension 1 in $\mathfrak{h}^\lambda$, which means that $\dim_F \mathfrak{a} = n - 3$.

It is clear that $\lambda(\mathfrak{a}) = 0$, so we will finish by proving that $\mathfrak{a}$ is an ideal of $\mathfrak{h}$, and this will imply that $\mathfrak{a} = 0$, and hence $n - 3 = 0$ and $n = \dim_F \mathfrak{h} = 3$ – contradicting our assumption.

By the definition of $\mathfrak{h}^\lambda$, it is clear that $\lambda([\mathfrak{h}^\lambda, \mathfrak{h}]) = 0$ and so $[\mathfrak{h}^\lambda, \mathfrak{a}] \subseteq \mathfrak{h}^\lambda \cap \ker(\lambda) = \mathfrak{a}$. So since $\mathfrak{h} = Fx \oplus Fy \oplus \mathfrak{h}^\lambda$, it remains to prove that $[y, \mathfrak{a}] \subseteq \mathfrak{a}$ and $[x, \mathfrak{a}] \subseteq \mathfrak{a}$.

Since $\mathfrak{a} \subseteq \mathfrak{h}'$, we have that $[y, \mathfrak{a}] = 0 \subseteq \mathfrak{a}$, and if we choose $u \in \mathfrak{b}$ such that $u \notin \mathfrak{h}'$, then $\mathfrak{b} = \mathfrak{h}^\lambda \oplus Fu$, so since $\mathfrak{h}$ is nilpotent and $\mathfrak{b}$ is a subalgebra, it follows that $[u, \mathfrak{h}^\lambda] \subseteq \mathfrak{h}^\lambda$, and hence $[u, \mathfrak{a}] \subseteq \mathfrak{h}^\lambda$. Also, since $\lambda([\mathfrak{b}, \mathfrak{b}]) = 0$, it follows that $[u, \mathfrak{a}] \subseteq \ker(\lambda)$, hence $[u, \mathfrak{a}] \subseteq \mathfrak{a}$.

But since $u \notin \mathfrak{h}'$, we have that $u = \beta x + \gamma y$, where $\beta \neq 0$, so it follows immediately that $[x, \mathfrak{a}] \subseteq \mathfrak{a}$ as required. □

5.3 Irreducibility of $\widehat{D(\lambda)}$

We will now examine the affinoid Dixmier module $\widehat{D(\lambda)}_{\mathfrak{b}}$ for a suitable polarisation $\mathfrak{b}$, and in this section, we will prove that we can choose $\mathfrak{b}$ such that $\widehat{D(\lambda)}$ is irreducible.

Firstly, we generalise some results from the classical theory to an affinoid setting:

Lemma 5.3.1. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$ such that $\mathfrak{a} \subseteq \mathfrak{h}$. Let $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{a}$, $\mathfrak{h}_1 := \mathfrak{h}/\mathfrak{a}$. Also, set $\mathcal{H} := \mathcal{L} \cap \mathfrak{h}$, $\mathcal{A} := \mathcal{L} \cap \mathfrak{a}$, $\mathcal{L}_1 := \mathcal{L}/\mathcal{A}$, $\mathcal{H}_1 := \mathcal{H}/\mathcal{A}$, which are Lie lattices in $\mathfrak{h}$, $\mathfrak{a}$, $\mathfrak{g}_1$ and $\mathfrak{h}_1$ respectively. Then:*

(i) There is a continuous surjection of K-algebras $\widehat{U(\mathcal{L})}_K \twoheadrightarrow \widehat{U(\mathcal{L}_1)}_K$ induced by the surjection $\mathcal{L} \twoheadrightarrow \mathcal{L}_1$. The kernel of this surjection is $\mathfrak{a}\widehat{U(\mathcal{L})}_K$

(ii) If M is a finitely generated $\widehat{U(\mathcal{H}_1)}_K$-module, then M has the structure of a $\widehat{U(\mathcal{H})}_K$-module via the surjection in (i), and $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M \cong \widehat{U(\mathcal{L}_1)}_K \otimes_{\widehat{U(\mathcal{H}_1)}_K} M$ as $\widehat{U(\mathcal{L})}_K$-modules.

Proof. (i) It is clear that the surjection $\mathcal{L} \twoheadrightarrow \mathcal{L}_1$ induces a surjection $U(\mathcal{L}) \twoheadrightarrow U(\mathcal{L}_1)$ sending $\pi^n U(\mathcal{L})$ to $\pi^n U(\mathcal{L}_1)$, so this yields a continuous map $\widehat{U(\mathcal{L})} \to \widehat{U(\mathcal{L}_1)}$.

If we fix a basis $\{x_1, \cdots, x_d\}$ for $\mathcal{L}$ such that $\{x_{r+1}, \cdots, x_d\}$ is a basis for $\mathcal{A}$, then using Lemma 2.8.3, we see that every element of $\widehat{U(\mathcal{L}_1)}_K$ has the form $\sum\limits_{\alpha \in \mathbb{N}^r} \lambda_\alpha (x_1 + \mathcal{A})^{\alpha_1} \cdots (x_r + \mathcal{A})^{\alpha_r}$, where $\lambda_\alpha \in \mathcal{O} \to 0$ as $\alpha \to \infty$. Clearly under the map $\widehat{U(\mathcal{L})} \to \widehat{U(\mathcal{L}_1)}$, x_i maps to $x_i + \mathcal{A}$ for each i, and hence the map is surjective.

Moreover, we can write any element of $\widehat{U(\mathcal{L})}$ as $\sum\limits_{\alpha \in \mathbb{N}^r} x_1^{\alpha_1} \cdots x_r^{\alpha_r} c_\alpha$ for some $c_\alpha \in \widehat{U(\mathcal{A})}$ converging to zero, and this maps to 0 if and only if c_α maps to 0 for each α. But each c_α has the form $\sum\limits_{\beta \in \mathbb{N}^{d-r}} \mu_\beta x_{r+1}^{\beta_{r+1}} \cdots x_d^{\beta_d}$, and this maps to zero if and only if $\mu_0 = 0$, i.e. $c_\alpha \in \widehat{U(\mathcal{A})}\mathcal{A}$. Hence the kernel of the surjection is $\widehat{U(\mathcal{L})}\mathcal{A}$ and part (i) follows.

(ii) Let $\phi : \widehat{U(\mathcal{L})}_K \to \widehat{U(\mathcal{L}_1)}_K$ be the surjection from part (i), and define a map:

$$\Theta : \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M \to \widehat{U(\mathcal{L}_1)}_K \otimes_{\widehat{U(\mathcal{H}_1)}_K} M, r \otimes m \mapsto \phi(r) \otimes m.$$

It is clear that this is a well defined map of $\widehat{U(\mathcal{L})}_K$-modules, we want to prove that it is an isomorphism.

Every element $s \in \widehat{U(\mathcal{L}_1)}_K$ can be written uniquely in the form

$$s = \sum_{\alpha \in \mathbb{N}^r} \lambda_\alpha (x_1 + \mathcal{A})^{\alpha_1} \cdots (x_r + \mathcal{A})^{\alpha_r} = \phi(\sum_{\alpha \in \mathbb{N}^r} \lambda_\alpha x_1^{\alpha_1} \cdots x_r^{\alpha_r}),$$

so there is a unique element in $K\langle x_1, \cdots, x_r \rangle$ that maps onto s under ϕ. We call this element $\phi^{-1}(s)$, and it is clear that this defines a K-linear map $\phi^{-1} : \widehat{U(\mathcal{L}_1)} \to$

$K\langle x_1, \cdots, x_r\rangle$.

Therefore, we can define a K-linear map $\Psi : \widehat{U(\mathcal{L}_1)}_K \otimes_{\widehat{U(\mathcal{H}_1)}_K} M \to \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M$ sending $s \otimes m$ to $\phi^{-1}(s) \otimes m$. We can show that this is well defined by choosing an appropriate basis for $\mathcal{H}_1$ that extends to a basis for $\mathcal{L}_1$, and clearly it is a right inverse to Θ.

Using the fact that $\widehat{U(\mathcal{L})}_K$ is isomorphic as a K-vector space to $K\langle x_1, \cdots, x_r\rangle\langle x_{r+1}, \cdots, x_d\rangle$, and $x_{r+1}, \cdots, x_d \in \mathcal{A} \subseteq \mathcal{H}$, we see that every simple tensor $s \otimes m \in \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M$ can be written as an infinite sum of simple tensors $s_n \otimes m_n$ converging to zero as $n \to \infty$, with $s_n \in K\langle x_1, \cdots, x_r\rangle$. We know this sum converges by Proposition 2.8.4.

Therefore, for any $s \in \widehat{U(\mathcal{L})}_K$, $m \in M$, $\Psi\Theta(s \otimes m) = \sum_{n\in\mathbb{N}} \Psi\Theta(s_n \otimes m_n) = \sum_{n\in\mathbb{N}} \Psi(\phi(s_n) \otimes m_n)$, and since $s_n \in K\langle x_1, \cdots, x_s\rangle$ for each n, $\phi^{-1}(\phi(s_n)) = s_n$, and hence $\Psi\Theta(s \otimes m) = s \otimes m$. Thus Ψ and Θ are mutually inverse bijections. $\square$

Now, recall from [5] that if A is a Banach K-algebra, and M is a left A-module, π-adically complete with respect to some lattice $N \subseteq M$, then we may define the *Tate module*:

$$M\langle t_1, \cdots, t_d\rangle := \{\sum_{\alpha\in\mathbb{N}^d} t_1^{\alpha_1} \cdots t_d^{\alpha_d} s_\alpha : s_\alpha \in M, s_\alpha \to 0 \text{ as } |\alpha| \to \infty\}.$$

Note that we don't necessarily give $M\langle t_1, \cdots, t_d\rangle$ the structure of an A-module, a priori it is just a K-vector space.

Proposition 5.3.2. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, and let $\mathcal{H} := \mathfrak{h} \cap \mathcal{L}$, so $\mathcal{H}$ is a Lie lattice in $\mathfrak{h}$. Suppose that M is a finitely generated $\widehat{U(\mathcal{H})}_K$-module. Then if $r = \dim_K \mathfrak{g}/\mathfrak{h}$, there is an isomorphism of K-vector spaces $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M \cong M\langle t_1, \cdots, t_r\rangle$, where $t_i v$ corresponds to $x_i \otimes v$ for some $\mathcal{O}$-basis $\{x_1, \cdots, x_r\}$ for $\mathcal{L}/\mathcal{H}$. Thus $M\langle t_1, \cdots, t_r\rangle$*

carries the structure of a $\widehat{U(\mathcal{L})}_K$ *module.*

Moreover, if $r = 1$*, so* $\mathcal{L} = \mathcal{H} \oplus \mathcal{O}x$ *for some* $x \in \mathcal{L}$*, then we can choose this isomorphism* $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M \cong M\langle t\rangle$ *such that:*

(i) x *acts by* t *on* $M\langle t\rangle$.

(ii) If $y, z \in \mathcal{H}$ *act on* M *by scalars* $\beta_y, \beta_z \in \mathcal{O}$*,* $[x, z] = 0$ *and* $[y, x] = \alpha z$ *for some* $\alpha \in K$*, then* y *acts on* $M\langle t\rangle$ *by* $\alpha\beta_z\frac{d}{dt} + \beta_y$.

(iii) If $\alpha, \beta_z \neq 0$ *and* M *is irreducible over* $\widehat{U(\mathcal{H})}_K$*, then* $M\langle t\rangle$ *is irreducible over* $\widehat{U(\mathcal{L})}_K$.

Proof. Let $\{x_1, \cdots, x_d\}$ be an $\mathcal{O}$-basis for $\mathcal{L}$ such that $\{x_{r+1}, \cdots, x_d\}$ is a basis for $\mathcal{H}$. Then by Lemma 2.8.3, writing $\underline{x}^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, we have: $\widehat{U(\mathcal{L})}_K = \{\sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha \underline{x}^\alpha : \lambda_\alpha \in \mathcal{O}, \lambda_\alpha \to 0 \text{ as } |\alpha| \to \infty\}$.

Define a map:

$$\Theta : \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M \to M\langle t_1, \cdots, t_r\rangle, \sum_{\alpha\in\mathbb{N}^d} \lambda_\alpha \underline{x}^\alpha \otimes v \mapsto \sum_{\beta\in\mathbb{N}^r} t_1^{\beta_1} \cdots t_r^{\beta_r}\Big(\sum_{\gamma\in\mathbb{N}^{d-r}} \lambda_{(\beta,\gamma)}\underline{x}^\gamma v\Big).$$

Note that here (β, γ) refers to the d-tuple whose first r terms are the terms of β, and the last $d - r$ terms are the terms of γ. It is straightforward but technical to show that this is a well defined K-linear map, so we need to prove that it is an isomorphism.

Firstly, $M = \widehat{U(\mathcal{H})}_K v_1 + \cdots + \widehat{U(\mathcal{H})}_K v_t$, so any element of $M\langle t_1, \cdots, t_r\rangle$ will have the form $\sum_{\beta\in\mathbb{N}^r} t_1^{\beta_1} \cdots t_r^{\beta_r}(a_{1,\beta}v_1 + \cdots + a_{t,\beta}v_t)$ for some $a_{i,\beta} \in \widehat{U(\mathcal{H})}_K$. This is the image of $\sum_{\beta\in\mathbb{N}^r} x_1^{\beta_1} \cdots x_r^{\beta_r} a_{1,\beta} \otimes v_1 + \cdots + \sum_{\beta\in\mathbb{N}^r} x_1^{\beta_1} \cdots x_r^{\beta_r} a_{t,\beta} \otimes v_t$, so Θ is surjective.

Furthermore, if $\sum_{\beta\in\mathbb{N}^r} t_1^{\beta_1}\cdots t_r^{\beta_r}(a_{1,\beta}v_1+\cdots+a_{t,\beta}v_t)=0$ then $a_{1,\beta}v_1+\cdots+a_{t,\beta}v_t=0$ for all β. Since $\widehat{U(\mathcal{L})}_K\otimes_{\widehat{U(\mathcal{H})}_K} M$ is finitely generated, it is complete by Proposition 2.8.4, thus

$$\begin{aligned}\sum_{\beta\in\mathbb{N}^r} x_1^{\beta_1}\cdots x_r^{\beta_r}a_{1,\beta}\otimes v_1+\cdots+\sum_{\beta\in\mathbb{N}^r} x_1^{\beta_1}\cdots x_r^{\beta_r}a_{t,\beta}\otimes v_t\\ =\sum_{\beta\in\mathbb{N}^r} x_1^{\beta_1}\cdots x_r^{\beta_r}\otimes(a_{1,\beta}v_1+\cdots+a_{t,\beta}v_t)=0,\end{aligned}\tag{5.3}$$

hence Θ is injective.

Hence Θ is an isomorphism of K-vector spaces, and $\Theta(x_i\otimes v)=t_iv$ for all $i\leq r$, $v\in M$. So clearly we can define an action of $\widehat{U(\mathcal{L})}_K$ on $M\langle t_1,\cdots,t_r\rangle$ making Θ into an isomorphism of $\widehat{U(\mathcal{L})}_K$-modules.

(i) Since $\Theta(x^nv)=t^nv$ for all $v\in M$, $n\in\mathbb{N}$, it is clear that the action of x on $M\langle t\rangle$ is given by multiplication by t.

(ii) Since y and z act by scalars on M, their action on $M\langle t\rangle$ is determined entirely by their action on the powers of t.

Since $[x,z]=0$, it follows that z commutes with all powers of x, and hence $z\cdot t^nv=z\Theta(x^n\otimes v)=\Theta(x^n\otimes z\cdot v)=\Theta(\beta_z(x^n\otimes v))=\beta_zt^nv$, so z acts on $M\langle t\rangle$ via β_z.

Clearly $y\cdot v=\beta_yv$, so we will assume that for some $n\geq 0$, $y\cdot t^nv=n\alpha\beta_zt^{n-1}v+\beta_yt^nv$ and show that $y\cdot t^{n+1}v=(n+1)\alpha\beta_zt^nv+\beta_yt^{n+1}v$, and it will follow using induction that y acts by $\alpha\beta_z\frac{d}{dt}+\beta_y$:

$$\begin{aligned}y\cdot t^{n+1}v&=y\Theta(x^{n+1}\otimes v)=\Theta(yx^{n+1}\otimes v)=\Theta(([y,x]x^n+xyx^n)\otimes v)\\&=\alpha z\Theta(x^n\otimes v)+xy\Theta(x^n\otimes v)=\alpha zt^nv+xyt^nv\\&=\alpha\beta_zt^nv+xn\alpha\beta_zt^{n-1}v+x\beta_yt^nv=(n+1)\alpha\beta_zt^nv+\beta_yt^{n+1}v\end{aligned}\tag{5.4}$$

(iii) Let $\partial := \frac{d}{dt}$, and let $\rho : \widehat{U(\mathcal{L})}_K \to \text{End}_K(M\langle t\rangle)$ be the action, then by part (ii), $\rho(y) = \alpha\beta_z\partial + \beta_y$, so $\partial = (\alpha\beta_z)^{-1}(\rho(y) - \beta_y) = \rho((\alpha\beta_z)^{-1}(y - \beta_y)) \in \text{im}(\rho)$.

Hence for each $n \in \mathbb{N}$, $\partial^{[n]} = \frac{1}{n!}\partial^n \in \text{im}(\rho)$.

So, suppose that $0 \neq T \leq M\langle t\rangle$ is a submodule, i.e there exists $\sum\limits_{m\geq 0} t^m s_m \in T$, $s_m \in M$, s_m not all zero, $s_m \to 0$ as $m \to \infty$.

Then since $\partial^{[n]}(T) \subseteq T$ for all n, it follows that $\sum\limits_{m\geq n} \binom{m}{n} t^{m-n} s_m \in T$, hence we may assume that $s_0 \neq 0$.

Set $s := s_0 \in M\backslash\{0\}$, and define a sequence of elements in T by $r_0 := \sum\limits_{m\geq 0} t^m s_m$, and for $i > 0$, $r_i := r_{i-1} - t^i\partial^{[i]}(r_{i-1})$.

Now, if $r_i = s + \sum\limits_{m>i} t^m s_{i,m}$, then $t^i\partial^{[i]}(r_i) = \sum\limits_{m>i} t^m \binom{m}{i} s_{i,m}$, so

$$r_{i+1} = r_i - t^i\partial^{[i]}(r_i) = s + \sum_{m>i+1} t^m(s_{i,m} - \binom{m}{i} s_{i,m}).$$

So inductively, we get that for each $i \in \mathbb{N}$, $r_i = s + \sum\limits_{m>i} s_{i,m} t^m$ for some $s_{i,m} \in M$ with $v(s_{i,m}) \geq v(s_{i-1,m})$. It follows easily that $r_i \to s$ in $M\langle t\rangle$ as $i \to \infty$.

But since $M\langle t\rangle$ is finitely generated over $\widehat{U(\mathcal{L})}_K$, which is Noetherian, it follows that T is finitely generated over $\widehat{U(\mathcal{L})}_K$, and hence T is π-adically complete by Proposition 2.8.4. Therefore, since each $r_i \in T$, this means that $s \in T$. So $0 \neq s \in M \cap T$, and thus $M \cap T \neq 0$. But since M is irreducible, $M \cap T = 0$ or M, hence $M \cap T = M$.

It follows that $M\langle t\rangle = \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{L}')}_K} M \subseteq T$, and hence $T = M\langle t\rangle$. Since our choice of T was arbitrary, this implies that $M\langle t\rangle$ is irreducible as required. $\square$

Now we prove the main theorem of this section, which is the affinoid analogue of [16, Theorem 6.1.1]. This proof follows a strategy of induction on the dimension of $\mathfrak{g}$ that we will often use throughout this chapter.

Theorem 5.3.3. *Suppose that $\mathfrak{g}$ is nilpotent, $\lambda \in \mathcal{L}^*$, and let $\mathcal{B}_\lambda = \mathfrak{b}_\lambda \cap \mathcal{L}$ be a standard polarisation of $\mathcal{L}$ at λ with respect to some chain of ideals. Then the affinoid Dixmier module $\widehat{D(\lambda)}_{\mathfrak{b}_\lambda}$ of $\mathcal{L}$ at λ with respect to $\mathfrak{b}_\lambda$ is irreducible as a $\widehat{U(\mathcal{L})}_K$-module.*

Proof. Let $n = \dim_K \mathfrak{g}$, and first suppose that $n = 1$. Then $\mathfrak{g}$ is abelian, so λ is a character of $\mathfrak{g}$, and $\widehat{D(\lambda)} = K$, which is clearly irreducible. So we will assume that the result holds for all $m < n$ and apply induction:

Suppose first that there exists a non-zero ideal $\mathfrak{a} \trianglelefteq \mathfrak{g}$ such that $\lambda(\mathfrak{a}) = 0$. Let $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{a}$, then since $\mathfrak{a} \neq 0$, $\dim_K \mathfrak{g}_1 < n$, so we may apply the inductive hypothesis to $\mathfrak{g}_1$.

Let $\mathcal{A} := \mathcal{L} \cap \mathfrak{a}$, and let $\mathcal{L}_1 := \frac{\mathcal{L}}{\mathcal{A}}$, then $\mathcal{A}, \mathcal{L}_1$ are lattices in $\mathfrak{a}$, $\mathfrak{g}_1$ respectively. Let λ_1 be the linear form of $\mathfrak{g}_1$ induced by λ, and clearly $\lambda_1(\mathcal{L}_1) \subseteq \mathcal{O}$. Then using Lemma 5.2.3, we see that $\mathfrak{a} \subseteq \mathfrak{b}_\lambda$ and that $\mathfrak{b}_1 := \mathfrak{b}_\lambda/\mathfrak{a}$ is a standard polarisation of $\mathfrak{g}_1$ at λ_1.

Let $\mathcal{B}_1 := \mathfrak{b}_1 \cap \mathcal{L}_1$, a Lie lattice in $\mathfrak{b}_1$, and let $M := \widehat{D(\lambda_1)}_{\mathfrak{b}_1} = \widehat{U(\mathcal{L}_1)}_K \otimes_{\widehat{U(\mathcal{B}_1)}_K} K_{\lambda_1}$, then by the inductive hypothesis, M is irreducible over $\widehat{U(\mathcal{L}_1)}_K$.

Using the surjection $\widehat{U(\mathcal{L})}_K \twoheadrightarrow \widehat{U(\mathcal{L}_1)}_K$ given by Lemma 5.3.1(i), we see that M has the structure of an irreducible $\widehat{U(\mathcal{L})}_K$-module, and using the fact that $\mathcal{B}_1 = \frac{\mathcal{B}_\lambda}{\mathcal{A}}$ and Lemma 5.3.1(ii) we see that $M \cong \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B}_\lambda)}_K} K_\lambda = \widehat{D(\lambda)}_{\mathfrak{b}_\lambda}$, and hence $\widehat{D(\lambda)}_{\mathfrak{b}_\lambda}$ is irreducible as required.

So from now on, we may assume that $\lambda(\mathfrak{a}) \neq 0$ for all non-zero ideals $\mathfrak{a}$ of $\mathfrak{g}$.

Using Proposition 5.2.2, we can find a reducing quadruple $(x, y, z, \mathfrak{g}')$ such that $\mathfrak{b}_\lambda \subseteq \mathfrak{g}'$, and $\mathfrak{b}_\lambda$ is a standard polarisation of $\mathfrak{g}'$ at $\mu := \lambda|_{\mathfrak{g}'}$. Setting $\mathcal{L}' := \mathcal{L} \cap \mathfrak{g}'$, since $\dim_K \mathfrak{g}' = n - 1 < n$, we see that $\widehat{D(\mu)}_{\mathfrak{b}_\lambda} = \widehat{U(\mathcal{L}')}_K \otimes_{\widehat{U(\mathcal{B}_\lambda)}_K} K_\lambda$ is irreducible over $\widehat{U(\mathcal{L}')}_K$ by induction.

Now, since $\widehat{U(\mathcal{L})}_K = \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{L}')}_K} \widehat{U(\mathcal{L}')}_K$, it follows that $\widehat{D(\lambda)}_{\mathfrak{b}_\lambda} = \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{L}')}_K} \widehat{D(\mu)}_{\mathfrak{b}_\lambda}$. Therefore, setting $M := \widehat{D(\mu)}_{\mathfrak{b}_\lambda}$, we have that $\widehat{D(\lambda)}_{\mathfrak{b}_\lambda} \cong M\langle t \rangle$ as a K-vector space by Proposition 5.3.2(i), where x acts on $M\langle t \rangle$ by t.

Also, y, z are central in $\mathfrak{g}'$ and $[y, x] = \alpha z$ for some $0 \neq \alpha \in K$. Therefore, since y and z act by scalars $\lambda(y)$, $\lambda(z)$ respectively, we see using Proposition 5.3.2(ii) that y acts on $M\langle t \rangle$ by $\alpha\lambda(z)\frac{d}{dt} + \lambda(y)$. So, finally, since Kz is an ideal of $\mathfrak{g}$, $\lambda(z) \neq 0$, so since $\alpha, \lambda(z) \neq 0$, it follows from Proposition 5.3.2(iii) that $\widehat{D(\lambda)}_{\mathfrak{b}_\lambda} = M\langle t \rangle$ is irreducible over $\widehat{U(\mathcal{L})}_K$ as required. □

5.4 Dixmier Annihilators

The aim of this chapter is to describe all primitive ideals of $\widehat{U(\mathcal{L})}_K$ as the annihilator ideals of affinoid Dixmier modules, so in this section we will begin to examine these ideals. From now on, we will always assume that $\mathfrak{g}$ is nilpotent.

The first result we need is analogous to [16, Proposition 5.1.7]:

Lemma 5.4.1. *Let $\mathfrak{h} \leq \mathfrak{g}$ be a subalgebra, let $\mathcal{H} := \mathfrak{h} \cap \mathcal{L}$, and let M be a finitely generated $\widehat{U(\mathcal{H})}_K$-module, with $J := \operatorname{Ann}_{\widehat{U(\mathcal{H})}_K} M$. Then $\operatorname{Ann}_{\widehat{U(\mathcal{L})}_K}\left(\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M\right)$ is the largest two-sided ideal in $\widehat{U(\mathcal{L})}_K$ contained in $\widehat{U(\mathcal{L})}_K J$. It follows that if M, N are finitely generated $\widehat{U(\mathcal{H})}_K$-modules such that $\operatorname{Ann}_{\widehat{U(\mathcal{H})}_K} M = \operatorname{Ann}_{\widehat{U(\mathcal{H})}_K} N$ then*

$$\operatorname{Ann}_{\widehat{U(\mathcal{L})}_K}(\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M) = \operatorname{Ann}_{\widehat{U(\mathcal{L})}_K}(\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} N).$$

Proof. Fix an $\mathcal{O}$-basis $\{x_1, \cdots, x_d\}$ for $\mathcal{L}$ such that $\{x_1, \cdots, x_r\}$ is a basis for $\mathcal{H}$. Then by Lemma 2.8.3, every element $r \in \widehat{U(\mathcal{L})}_K$ can be written as $r = \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, for some $\lambda_\alpha \in K$ converging to zero as $\alpha \to \infty$, i.e. $r = \sum_{\alpha \in \mathbb{N}^r} \underline{x}^\alpha s_\alpha$ for some $s_\alpha \in \widehat{U(\mathcal{H})}_K$ such that $s_\alpha \to 0$ as $\alpha \to \infty$.

Using Proposition 5.3.2, we see that $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M$ is isomorphic as a K-vector space to the Tate module $M\langle t_1, \cdots, t_r \rangle = \{ \sum_{\alpha \in \mathbb{N}^r} t_1^{\alpha_1} \cdots t_r^{\alpha_r} s_\alpha : s_\alpha \in M, s_\alpha \to 0$ as $|\alpha| \to \infty\}$ via an isomorphism Ψ sending $\underline{x}^\alpha \otimes m$ to $t_1^{\alpha_1} \cdots t_r^{\alpha_r} m$. It is clear that the set of all elements in $\widehat{U(\mathcal{L})}_K$ that annihilate the set M inside $M\langle t_1, \cdots, t_r \rangle$ on the left contains the left ideal $\widehat{U(\mathcal{L})}_K J$.

Moreover, if $rM = 0$ for some $r = \sum_{\alpha \in \mathbb{N}^r} \underline{x}^\alpha s_\alpha \in \widehat{U(\mathcal{L})}_K$, then for all $m \in M$:

$0 = rm = \sum_{\alpha \in \mathbb{N}^r} \underline{x}^\alpha s_\alpha m = \Psi^{-1}(\sum_{\alpha \in \mathbb{N}^r} t_1^{\alpha_1} \cdots t_r^{\alpha_r} s_\alpha m)$, and hence $s_\alpha m = 0$ for all $\alpha \in \mathbb{N}^r$.

Thus $s_\alpha \in J$ for all α, and hence $r \in \widehat{U(\mathcal{L})}_K J$. Therefore the right ideal $\widehat{U(\mathcal{L})}_K J$ is the set of all elements of $\widehat{U(\mathcal{L})}_K$ that annihilate the set M.

It follows that if $r\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M = 0$, then $r\widehat{U(\mathcal{L})}_K$ annihilates M, so the two-sided ideal generated by r is contained in $\widehat{U(\mathcal{L})}_K J$. Since our choice of r was arbitrary, it follows that the annihilator of $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M$ is contained in $\widehat{U(\mathcal{L})}_K J$.

Furthermore, if $I \subseteq \widehat{U(\mathcal{L})}_K J$ is a two-sided ideal of $\widehat{U(\mathcal{L})}_K$, then I annihilates M, so since $I\widehat{U(\mathcal{L})}_K = \widehat{U(\mathcal{L})}_K I$, it must also annihilate the submodule generated by M inside $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M$, which is clearly the whole module, and the result follows. □

The next result will be essential to several of the proofs in this chapter, since it allows us to safely pass to and from a reducing quadruple.

Theorem 5.4.2. *Suppose that* $\mathfrak{g}$ *has a reducing quadruple* $(x, y, z, \mathfrak{g}')$ *with* $x, y, z \in \mathcal{L}$*, and let* $\mathcal{L}' := \mathcal{L} \cap \mathfrak{g}'$*. Then if* I *is a two-sided ideal of* $\widehat{U(\mathcal{L})}_K$ *such that* $z + I$ *is not a zero divisor in* $\widehat{U(\mathcal{L})}_K/I$*, then* I *is controlled by* $\mathcal{L}'$*, i.e.:*

$$I = \left(I \cap \widehat{U(\mathcal{L}')}_K\right)\widehat{U(\mathcal{L})}_K = \widehat{U(\mathcal{L})}_K\left(I \cap \widehat{U(\mathcal{L}')}_K\right).$$

Proof. Using Lemma 2.8.3, we see that every element of $\widehat{U(\mathcal{L})}_K$ can be written as $g(x)$ for some Tate power series g with coefficients in $\widehat{U(\mathcal{L}')}_K$. We will prove that if $g(x) \in I$ then the coefficients of g all lie in I, and the result follows.

It will suffice to show that if $g(x) = c_0 + c_1x + c_2x^2 + \cdots \in I$ then the formal derivative $g'(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots$ also lies in I. Then using an argument similar to the proof of Proposition 5.3.2(iii), we can construct a sequence of elements in I converging to c_0. By closure of I in $\widehat{U(\mathcal{L})}_K$, it follows that $c_0 \in I$, so repeating the argument for $\frac{1}{n!}g^{(n)}(x)$ for each x, it follows that all coefficients of $g(x)$ lie in I as required.

To prove that $g'(x)$ lies in I, consider the action of y on $\widehat{U(\mathcal{L})}_K/I$:

Since y is central in $\mathcal{L}'$, y commutes with everything in $\widehat{U(\mathcal{L}')}_K$. Also, since $[x, y] = \alpha z$, clearly $y \cdot x = xy - \alpha z$, and an easy induction shows that $y \cdot x^n = x^n y - n\alpha x^{n-1}z$. So if l_y is the left action of y on $\widehat{U(\mathcal{L})}_K/I$, r_y is the right action, then $l_y - r_y = -\alpha z\frac{d}{dx}$. Therefore, since z is not a zero divisor modulo I, and $\alpha \neq 0$, it follows that if $g(x) \in I$ then $\frac{d}{dx}(g(x)) \in I$ as required. □

For us to unambiguously describe the annihilator of a Dixmier module, we want to ensure that this does not depend on the choice of polarisation, as [16, Theorem 6.1.4] shows in the classical case. The following lemma establishes this in a special case, and we will follow it by a proof in full generality:

Lemma 5.4.3. *Suppose that $\mathcal{L}$ has an $\mathcal{O}$-basis $\{x, y, z\}$ such that z is central and $[x, y] = \alpha z$ for some $0 \neq \alpha \in \mathcal{O}$. Then for any $\beta \in \mathcal{O}\backslash 0$, the ideal $(z - \beta)\widehat{U(\mathcal{L})}_K$ is a maximal two-sided ideal of $\widehat{U(\mathcal{L})}_K$.*

Proof. Let I be a proper ideal of $\widehat{U(\mathcal{L})}_K$ containing $z - \beta$. Then since $\beta \neq 0$, $z + I$ is not a zero divisor in $\widehat{U(\mathcal{L})}_K/I$. So setting $\mathfrak{g}' := \text{Span}_K\{y, z\}$, since $(x, y, z, \mathfrak{g}')$ is a reducing quadruple, it follows from Theorem 5.4.2 that I is controlled by $\mathcal{L}' = \mathfrak{g}' \cap \mathcal{L}$. Therefore, if we can prove that $I \cap \widehat{U(\mathcal{L}')}_K = (z - \beta)\widehat{U(\mathcal{L}')}_K$ then it follows that $I = (z - \beta)\widehat{U(\mathcal{L})}_K$.

Let $\bar{y}$ be the image of y in $\widehat{U(\mathcal{L}')}_K/(z-\beta)\widehat{U(\mathcal{L}')}_K$, then given $r \in \widehat{U(\mathcal{L}')}_K$, by Lemma 2.8.3, the image of r in $\widehat{U(\mathcal{L}')}_K/(z - \beta)\widehat{U(\mathcal{L}')}_K$ has the form $\bar{r} = \sum_{n\geq 0} \lambda_n \bar{y}^n$ for some $\lambda_n \in K$, $\lambda_n \to 0$ as $n \to \infty$.

Since $[x, y] = \alpha z$, we have that $x \cdot \bar{y} = \bar{y}x + \alpha\beta$, and an easy induction shows that $x \cdot \bar{y}^n = \bar{y}^n x + \alpha\beta\bar{y}^{n-1}$ for all n, i.e. if l_x and r_x are the respective left and right actions of x on $\widehat{U(\mathcal{L}')}_K/(z - \beta)$, then $l_x - r_x = \alpha\beta\frac{d}{d\bar{y}}$. Since $\alpha, \beta \neq 0$ and I is a two-sided ideal, $\frac{d}{d\bar{y}}$ preserves $I \cap \widehat{U(\mathcal{L}')}_K/(z - \beta)$.

So if $g(\bar{y}) = \lambda_0 + \lambda_1\bar{y} + \lambda_2\bar{y}^2 + \cdots \in I \cap \widehat{U(\mathcal{L}')}_K/(z - \beta)\widehat{U(\mathcal{L}')}_K$, it follows that $\frac{1}{n!}g^{(n)}(\bar{y}) \in I$ for all $n \in \mathbb{N}$, and using an argument similar to the proof of Proposition 5.3.2(iii), we can construct a sequence of elements in $I\cap\widehat{U(\mathcal{L}')}_K$ converging to $\lambda_0 \in K$.

By closure of $I \cap \widehat{U(\mathcal{L})}_K$, this implies that $\lambda_0 \in I$, and hence $\lambda_0 = 0$, and it follows after replacing $g(\bar{y})$ by $\frac{1}{n!}g^{(n)}(\bar{y})$ that $\lambda_n = 0$ for all n, i.e. $g(\bar{y}) = 0$. Therefore $I \cap \widehat{U(\mathcal{L}')}_K = (z - \beta)\widehat{U(\mathcal{L}')}_K$, and $I = (z - \beta)\widehat{U(\mathcal{L})}_K$. So since our choice of I was arbitrary, $(z - \beta)\widehat{U(\mathcal{L})}_K$ is maximal as required. $\square$

Theorem 5.4.4. *Suppose $\mathfrak{g}$ is nilpotent, and let $\lambda \in \mathcal{L}^*$. Then for any polarisations $\mathfrak{b}_1$,$\mathfrak{b}_2$ of $\mathfrak{g}$ at λ, $\text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda)}_{\mathfrak{b}_1} = \text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda)}_{\mathfrak{b}_2}$.*

Proof. If $\mathfrak{g}$ is abelian then $\mathfrak{b}_1 = \mathfrak{b}_2 = \mathfrak{g}$ so the statement is obvious. Since all nilpotent Lie algebras of dimension 1 and 2 are abelian, we may assume that $\dim_K \mathfrak{g} \geq 3$.

If $\mathfrak{g}$ is non-abelian and $\dim_K \mathfrak{g} = 3$ then it is straightforward to show that $\mathcal{L}$ has basis $\{x, y, z\}$ with z central and $[x, y] = \alpha z$ for some $\alpha \in \mathcal{O} \backslash 0$. If $\lambda(z) = 0$ then λ is a character of $\mathfrak{g}$, so $\mathfrak{g}$ is the only polarisation and the statement is trivially true. If $\lambda(z) \neq 0$, then for any polarisation $\mathfrak{b}$, z acts on $\widehat{D(\lambda)}_{\mathfrak{b}}$ by $\lambda(z)$, and so the $\widehat{U(\mathcal{L})_K}$-annihilator must contain $(z - \lambda(z))$, which is a maximal ideal by Lemma 5.4.3, hence this must be the annihilator in all cases as we require.

So from now on, we may assume that $n = \dim_K \mathfrak{g} \geq 4$ and we will proceed by induction on n:

Suppose first that there exists a non-zero ideal $\mathfrak{a} \trianglelefteq \mathfrak{g}$ such that $\lambda(\mathfrak{a}) = 0$, so λ induces a linear form λ_0 of $\mathfrak{g}_0 := \mathfrak{g}/\mathfrak{a}$. Setting $\mathcal{A} := \mathfrak{a} \cap \mathcal{L}$, $\mathcal{L}_0 := \frac{\mathcal{L}}{\mathcal{A}}$, it is clear that $\mathcal{L}_0$ is a Lie lattice in $\mathfrak{g}_0$ and $\lambda_0(\mathcal{L}_0) \subseteq \mathcal{O}$.

Note that $\mathfrak{a} \subseteq \mathfrak{b}_1, \mathfrak{b}_2$ by Lemma 5.1.2, so set $\mathfrak{b}_{i,0} := \mathfrak{b}_i/\mathfrak{a}$ for $i = 1, 2$, and $\mathfrak{b}_{1,0}$,$\mathfrak{b}_{2,0}$ are polarisations of $\mathfrak{g}_0$ at λ_0.

Since $\dim_K \mathfrak{g}_0 < n$, it follows from induction that $\text{Ann}_{\widehat{U(\mathcal{L}_0)_K}} \widehat{D(\lambda_0)}_{\mathfrak{b}_{1,0}} = \text{Ann}_{\widehat{U(\mathcal{L}_0)_K}} \widehat{D(\lambda_0)}_{\mathfrak{b}_{2,0}}$.

Using Lemma 5.3.1, we see that for $i = 1, 2$, $\widehat{D(\lambda_0)}_{\mathfrak{b}_{i,0}} = \widehat{U(\mathcal{L}_0)_K} \otimes_{\widehat{U(\mathcal{B}_{i,0})_K}} K_\lambda$ is naturally a $\widehat{U(\mathcal{L})_K}$-module, and that it is isomorphic to $\widehat{U(\mathcal{L})_K} \otimes_{\widehat{U(\mathcal{B}_i)_K}} K_\lambda = \widehat{D(\lambda)}_{\mathfrak{b}_i}$.

If $\phi : \widehat{U(\mathcal{L})_K} \twoheadrightarrow \widehat{U(\mathcal{L}_0)_K}$ is the natural surjection, then it is clear that $\text{Ann}_{\widehat{U(\mathcal{L}_0)_K}} \widehat{D(\lambda_0)}_{\mathfrak{b}_{i,0}} = \frac{\text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda_0)}_{\mathfrak{b}_{i,0}}}{\ker(\phi)}$ for $i = 1, 2$. Thus $\text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda_0)}_{\mathfrak{b}_{1,0}} = \text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda_0)}_{\mathfrak{b}_{2,0}}$, and hence:

$\text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda)}_{\mathfrak{b}_1} = \text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda_0)}_{\mathfrak{b}_{1,0}} = \text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda_0)}_{\mathfrak{b}_{2,0}} = \text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda)}_{\mathfrak{b}_2}$ as required.

So from now on, we may assume that $\lambda(\mathfrak{a}) \neq 0$ for all non-zero ideals $\mathfrak{a}$ of $\mathfrak{g}$. Using Proposition 5.2.2, we see that this means that there exists a reducing quadruple $(x, y, z, \mathfrak{g}')$ for $\mathfrak{g}$. Since we are assuming $\dim_K \mathfrak{g} > 3$, we may apply Proposition 5.2.5 to get that for each $i = 1, 2$ there exists a proper subalgebra $\mathfrak{h}_i$ of $\mathfrak{g}$ containing $\mathfrak{b}_i$, and a polarisation $\mathfrak{b}_i'$ of $\mathfrak{g}$ at λ contained in $\mathfrak{h}_i$ and $\mathfrak{g}'$.

By induction, since $\dim_K \mathfrak{g}' < n$, we get that $\text{Ann}_{\widehat{U(\mathcal{L}')_K}} \widehat{D(\lambda|_{\mathfrak{g}'})}_{\mathfrak{b}_1'} = \text{Ann}_{\widehat{U(\mathcal{L}')_K}} \widehat{D(\lambda|_{\mathfrak{g}'})}_{\mathfrak{b}_2'}$, so by Lemma 5.4.1, $\text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda)}_{\mathfrak{b}_1'} = \text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda)}_{\mathfrak{b}_2'}$.

Similarly, since $\mathfrak{h}_1, \mathfrak{h}_2$ are proper subalgebras of $\mathfrak{g}$, we also have that $\text{Ann}_{\widehat{U(\mathcal{H}_i)_K}} \widehat{D(\lambda|_{\mathfrak{h}_i})}_{\mathfrak{b}_i} = \text{Ann}_{\widehat{U(\mathcal{H}_i)_K}} \widehat{D(\lambda|_{\mathfrak{h}_i})}_{\mathfrak{b}_i'}$ for $i = 1, 2$, and applying Lemma 5.4.1 again gives that $\text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda)}_{\mathfrak{b}_i} = \text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda)}_{\mathfrak{b}_i'}$, and it follows that $\text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda)}_{\mathfrak{b}_1} = \text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda)}_{\mathfrak{b}_2}$ as required. $\square$

Now that we have established that the annihilator of a Dixmier module does not depend on the choice of polarisation, we can unambiguously make the following definition:

Definition 5.4.5. *Let F/K be a finite extension, and let $\lambda \in \mathcal{L}_F^* := (\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_F)^*$ be a linear form. Define the* Dixmier annihilator *in $\widehat{U(\mathcal{L})_K}$ associated to λ to be the two sided ideal $I(\lambda) := \text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda)}_F$ (or $I(\lambda)_F$ if it is unclear what the base field is).*

Note: This definition makes sense because there is a natural embedding $\widehat{U(\mathcal{L})_K} \to \widehat{U(\mathcal{L}_F)_F}$ for any finite extension F/K. Using Theorem 5.3.3, we see that $I(\lambda)$ is a primitive ideal of $\widehat{U(\mathcal{L})_K}$ whenever $F = K$.

5.5 Locally closed ideals in $\widehat{U(\mathcal{L})}_K$

Now we will study some ring theoretic properties of the affinoid enveloping algebra. For the moment, we assume only that $\mathcal{L}$ is an $\mathcal{O}$-Lie lattice in a K-Lie algebra $\mathfrak{g}$.

Proposition 5.5.1. *Let P be a prime ideal of $\widehat{U(\mathcal{L})}_K$, and let J be a two-sided ideal of $R := \widehat{U(\mathcal{L})}_K/P$. Then if $J \neq 0$ there exists an element $a \in J$ such that a^n does not converge to 0 as $n \to \infty$.*

Proof. Let w be the π-adic filtration on $\widehat{U(\mathcal{L})}_K$ corresponding to the lattice $\widehat{U(\mathcal{L})}$, and let $\bar{w}$ be the quotient filtration on $R := \widehat{U(\mathcal{L})}_K/P$. Then since R is complete with respect to $\bar{w}$ and $\mathrm{gr}_{\bar{w}}\ R \cong \frac{\mathrm{gr}_w\ \widehat{U(\mathcal{L})}_K}{\mathrm{gr}_w\ P}$, it follows from [28, Ch II Theorem 2.2.1] that $\bar{w}$ is a Zariskian filtration on R.

Also, since $\mathrm{gr}_w\ \widehat{U(\mathcal{L})}_K \cong U(\mathcal{L}/\pi\mathcal{L})[t, t^{-1}]$ by [6, Lemma 3.1], and it is well known that $U(\mathcal{L}/\pi\mathcal{L})$ is finitely generated over its centre, it follows that $\mathrm{gr}_{\bar{w}}\ R$ is also finitely generated over its centre.

Furthermore, $t = \mathrm{gr}(\pi)$ is central of positive degree in $\mathrm{gr}_{\bar{w}}\ R$, and it is non-nilpotent, so it follows that $\mathrm{gr}_{\bar{w}}\ R$ is finitely generated over a central Noetherian subring whose positive part is non-nilpotent. So we can apply Theorem 4.1.3 to find a non-commutative valuation v on the ring of quotients $Q(R)$ such that the inclusion $(R, \bar{w}) \to (Q(R), v)$ is continuous.

Therefore, we may define the growth rate function $\rho : Q(R) \to \mathbb{R} \cup \{\infty\}, q \mapsto \lim\limits_{n\to\infty} \frac{v(q^n)}{n}$ associated to v, as in Definition 3.6.1. Suppose that for every element $a \in J$, $a^n \to 0$ as $n \to \infty$, we will prove that this implies that $J = 0$. Choose an arbitrary $a \in J$, and let $m := \lceil \rho(a) \rceil$.

If we assume that $m < \infty$, then set $b := \pi^{-(m+1)}a$, and since π is central in R, and hence v-regular, we see using Lemma 3.6.2(v) that $\rho(b) = \rho(a) - (m+1)v(\pi) \leq \rho(a) - (\rho(a)+1)v(\pi) < 0$, and hence b^n does not converge to 0 as $n \to \infty$ – contradiction since $b \in J$.

Therefore $m = \rho(a) = \infty$, and since $Q(R)$ is simple and Artinian, it follows from Lemma 3.6.2(iv) that a is nilpotent.

Since our choice of a was arbitrary, this means that every element of J is nilpotent, and using [16, Lemma 3.1.14] it follows that J is a nilpotent ideal of R. Since R is prime, this means that $J = 0$ as required. □

The following result is the affinoid version of [16, Proposition 3.1.15]:

Theorem 5.5.2. *Let $\mathcal{L}$ be a Lie lattice in $\mathfrak{g}$ and let I be a two sided ideal of $\widehat{U(\mathcal{L})}_K$. Then I is semiprime if and only if I is an intersection of primitive ideals.*

Proof. Clearly if I is an intersection of primitive ideals, then it is semiprime, so it remains only to prove the converse, i.e. that if I is semiprime then it is the intersection of primitive ideals.

Since semiprime ideals arise as an intersection of primes, we can assume that I is prime in $\widehat{U(\mathcal{L})}_K$, and we will show that $J(\widehat{U(\mathcal{L})}_K/I) = 0$, from which the result follows.

Assume for contradiction that $J := J(\widehat{U(\mathcal{L})}_K/I) \neq 0$, then since I is prime it follows from Proposition 5.5.1 that we can choose an element $a \in J$ such that a^n does not converge to zero as $n \to \infty$.

Let $R := \frac{\widehat{U(\mathcal{L})}_K}{I}$, and let $C := R\langle X \rangle$ be the Tate algebra in one variable over R, as defined in Appendix D. Then if we set $\mathfrak{g}_0 := \mathfrak{g} \times K$, $\mathcal{L}_0 := \mathcal{L} \times \mathcal{O}$, it is clear that $\mathcal{L}_0$

is a Lie lattice in $\mathfrak{g}_0$ and it follows from Lemma 2.8.3 that $C \cong \widehat{U(\mathcal{L}_0)}_K / I\widehat{U(\mathcal{L}_0)}_K$.

Consider the element $1 - aX \in C$. If this element is a unit, its inverse must have the form $a_0 + a_1X + a_2X^2 + \cdots$ for some $a_i \in \widehat{U(\mathcal{L})}_K/I$ with $a_n \to 0$ as $n \to \infty$. But since $1 = (1 - aX)(a_0 + a_1X + a_2X^2 + \cdots)$, it follows that $a_0 = 1$, $a_1 = a$, $a_2 = a^2$, $\cdots$, $a_n = a^n$, and hence $a^n \to 0$ as $n \to \infty$ – contradiction.

Therefore $1 - aX$ is not a unit in C, so there exists a maximal left ideal of C containing $1 - aX$, i.e. there exists an irreducible C-module M and an element $0 \neq m \in M$ such that $(1 - aX)m = 0$.

Now, X does not act by zero on M, otherwise $1 - aX$ would act by 1, and we would have $(1 - aX)m = m \neq 0$. So using Schur's Lemma, the action of X is invertible, and using [35, Theorem 6.4.6] we see that the action of X^{-1} is algebraic over K, i.e. there exists $f(t) = a_0 + a_1t + \cdots + a_nt^n$ for some $a_i \in K$ such that $f(X^{-1}) = 0$, and we may assume that $a_0 \neq 0$. So let $g(t) := a_0^{-1}f(t) = 1 + b_1t + \cdots + b_nt^n$.

Since $aXm = m$, we have that $am = X^{-1}aXm = X^{-1}m$, hence $a^rm = X^{-r}m$ for all $r \in \mathbb{N}$, and thus $g(a)m = g(X^{-1})m = 0$.

But $g(a) = 1 + (b_1 + b_2a + \cdots + b_na^{n-1})a$, so since $a \in J(\widehat{U(\mathcal{L})}_K/I)$, this means that $g(a)$ is a unit in $\widehat{U(\mathcal{L})}_K/I$. Therefore, since $m \neq 0$, $g(a)m \neq 0$ – contradiction.

Therefore $J(\widehat{U(\mathcal{L})}_K/I) = 0$ as we require. $\square$

Now, since this proposition gives us that all prime ideals in the affinoid enveloping algebra arise as an intersection of primitive ideals, we can now focus on a particular class of primitives:

Definition 5.5.3. *Let R be a ring, and let P be a prime ideal of R. We say that P is* locally closed *if P is not equal to the intersection of all prime ideals properly containing it.*

It is not difficult to prove that a prime ideal P is locally closed in R if and only if the singleton $\{P\}$ is locally closed in *Spec* R with the Zariski topology.

Using Theorem 5.5.2, we see that if P is a locally closed prime ideal in $\widehat{U(\mathcal{L})}_K$, then P must be primitive. The most obvious examples of locally closed ideals are maximal ideals, since the intersection of all prime ideals properly containing them is empty.

In this section, we want to prove that all locally closed ideals arise as Dixmier annihilators. We first need the following technical results:

Lemma 5.5.4. *Suppose that $\mathfrak{g}$ is nilpotent and $\dim_K \mathfrak{g} > 1$. Let I be an ideal of $\widehat{U(\mathcal{L})}_K$. Suppose further that $Z(\widehat{U(\mathcal{L})}_K/I) = K$, and $I \cap \mathfrak{g} = 0$. Then $\mathfrak{g}$ has a reducing quadruple $(x, y, z, \mathfrak{g}')$.*

Proof. Firstly, suppose $u, v \in \mathfrak{g}$ are central, then $u + I, v + I \in Z(\widehat{U(\mathcal{L})}_K/I) = K$, hence they are K-linearly dependent. So there exist non-zero $\alpha, \beta \in K$ such that $\alpha u + \beta v \in I \cap \mathfrak{g} = 0$, hence u, v are K-linearly dependent in $\mathfrak{g}$. Since $\mathfrak{g}$ is nilpotent, $Z(\mathfrak{g}) \neq 0$, so it follows that $Z(\mathfrak{g})$ has dimension 1, i.e. $Z(\mathfrak{g}) = Kz$ for some $z \in Z(\mathfrak{g})$.

So, since $\dim_K \mathfrak{g} > 1$, $\mathfrak{g}$ is non-abelian, and again using nilpotence of $\mathfrak{g}$, we can find $y \in \mathfrak{g}$ such that y is not central, but $[y, \mathfrak{g}] \subseteq Z(\mathfrak{g}) = Kz$.

Let $\mathfrak{g}' := \ker(\text{ad}(y))$. Then $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$, and since $\text{ad}(y) : \mathfrak{g} \to Kz$ is non-zero, $\mathfrak{g}'$ must have codimension 1 in $\mathfrak{g}$, so $\mathfrak{g} = \mathfrak{g}' \oplus Kx$, and it is clear that $(x, y, z, \mathfrak{g}')$ is a reducing quadruple for $\mathfrak{g}$. $\square$

Proposition 5.5.5. *Let I be a two sided ideal of $\widehat{U(\mathcal{L})}_K$ such that $F = Z(\widehat{U(\mathcal{L})}_K/I)$ is a finite field extension of K. Then $U(\widehat{\mathcal{L}\otimes_{\mathcal{O}}\mathcal{O}_F})_F \cong \widehat{U(\mathcal{L})}_K \otimes_K F$, and there exists a surjection $U(\widehat{\mathcal{L}\otimes_{\mathcal{O}}\mathcal{O}_F})_F \to \widehat{U(\mathcal{L})}_K/I$ of F-algebras with kernel containing $I \otimes_K F$.*

Proof. To see that $U(\widehat{\mathcal{L}\otimes_{\mathcal{O}}\mathcal{O}_F})_F \cong \widehat{U(\mathcal{L})}_K \otimes_K F$, note that:

$$U(\mathcal{L}) \otimes_{\mathcal{O}} \mathcal{O}_F \to U(\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_F)$$

$$a \otimes \alpha \mapsto \alpha a$$

is an isomorphism of $\mathcal{O}_F$ algebras, whose inverse is the natural extension of the map $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_F \to U(\mathcal{L}) \otimes_{\mathcal{O}} \mathcal{O}_F, u \otimes \alpha \mapsto u \otimes \alpha$ to $U(\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_F)$. These maps preserve the π-adic filtration, hence they induce an isomorphism $U(\widehat{\mathcal{L}\otimes_{\mathcal{O}}\mathcal{O}_F}) \cong \widehat{U(\mathcal{L})} \otimes_{\mathcal{O}} \mathcal{O}_F$, and the result follows.

Since $F = Z(\widehat{U(\mathcal{L})}_K/I) \subseteq \widehat{U(\mathcal{L})}_K/I$, it is clear that $\widehat{U(\mathcal{L})}_K/I$ is an F-algebra, and the map $\widehat{U(\mathcal{L})}_K \otimes_K F \to \widehat{U(\mathcal{L})}_K/I, r \otimes (\alpha + I) \mapsto \alpha r + I$ is clearly a surjection of F-algebras sending $I \otimes F$ to 0 as required. □

Proposition 5.5.6. *Let $\mathfrak{g}$ be a nilpotent K-Lie algebra, and let $\mathcal{L}$ be an $\mathcal{O}$-Lie lattice in $\mathfrak{g}$ such that every locally closed prime ideal in $\widehat{U(\mathcal{L})}_K$ has the form $I(\lambda)_F$ for some finite extension F/K and some $\lambda \in \mathcal{L}_F^*$. Then given any prime ideal P in $\widehat{U(\mathcal{L})}_K$, P arises as an intersection of Dixmier annihilators.*

Proof. Note that since gr $\widehat{U(\mathcal{L})}_K \cong U(\mathcal{L}/\pi\mathcal{L})[t, t^{-1}]$ has finite left and right Krull dimension, it follows from [28, Ch.I Theorem 7.1.3] that $\widehat{U(\mathcal{L})}_K$ has finite left and right Krull dimension. Therefore, using [29, Lemma 6.4.5], it follows that $\widehat{U(\mathcal{L})}_K$ has finite *classical Krull dimension*, i.e. there is a finite upper bound on the length of chains of prime ideals in $\widehat{U(\mathcal{L})}_K$.

So, given a prime ideal P of R, define the *dimension* $\dim(P)$ of P to be the largest integer $n \geq 0$ such that there exists a chain of prime ideals $P = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$

of $\widehat{U(\mathcal{L})}_K$. We will proceed by induction on $\dim(P)$.

If $\dim(P) = 0$, then P is maximal, and hence locally closed, so by the hypothesis, $P = I(\lambda)_F$ for some finite extension F, $\lambda \in \mathcal{L}_F^*$ as required. Now suppose the result holds whenever $\dim(P) < n$.

If $\dim(P) = n$ then for every prime ideal Q of $\widehat{U(\mathcal{L})}_K$ with $P \subsetneq Q$, Q arises as an intersection of Dixmier annihilators by the inductive hypothesis.

If P is locally closed, then P is a Dixmier annihilator by hypotheis, otherwise P is equal to the intersection of all prime ideals properly containing it, and hence it is an intersection of Dixmier annihilators as required. □

Now we can proceed to prove the main theorem of this section, but first we need a small lemma:

Lemma 5.5.7. *Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$ nilpotent, let $\mathcal{A} := \mathfrak{a} \cap \mathcal{L}$ and let $\mathcal{L}_0 := \mathcal{L}/\mathcal{A}$. Let P be a prime ideal of $\widehat{U(\mathcal{L})}_K$, containing $\mathfrak{a}$, such that the image $P_0 \trianglelefteq \widehat{U(\mathcal{L}_0)}_K$ of P under the surjection $\widehat{U(\mathcal{L})}_K \to \widehat{U(\mathcal{L}_0)}_K$ is a Dixmier annihilator. Then P is a Dixmier annihilator.*

Proof. We know that $P_0 = Ann_{\widehat{U(\mathcal{L}_0)}_K} \widehat{D(\mu)}_F$ for some finite extension F/K, $\mu \in (\mathcal{L}/\mathcal{A})_F^*$. Clearly μ is induced from a linear form λ of $\mathfrak{g} \otimes_K F$ such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$ and $\lambda(\mathfrak{a}) = 0$. We will prove that $P = Ann_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_F$.

Choose a polarisation $\mathfrak{b}$ of $\mathfrak{g} \otimes_K F$ at λ, and since the annihilator is independent of the choice of polarisation by Theorem 5.4.4, we may assume that $\mathfrak{a} \subseteq \mathfrak{b}$, i.e. $\mathfrak{b}/\mathfrak{a}$ is a polarisation of $\mathfrak{g}/\mathfrak{a}$ at μ. Using Lemma 5.3.1(*iii*), we see that $\widehat{D(\lambda)}_F = \widehat{U(\mathcal{L})}_F \otimes_{\widehat{U(\mathcal{B})}_F} F \cong \widehat{U(\mathcal{L}/\mathcal{A})}_F \otimes_{\widehat{U(\mathcal{B}/\mathcal{A})}_F} F = \widehat{D(\mu)}_F$.

Using Lemma 5.3.1(i), we know that $\widehat{U(\mathcal{L})}/\mathfrak{a}\widehat{U(\mathcal{L})} \cong \widehat{U(\mathcal{L}/\mathcal{A})}_K$, and hence $P_0 = P/\mathfrak{a}\widehat{U(\mathcal{L})}_K$. Therefore, since $P_0 = \text{Ann}_{\widehat{U(\mathcal{L}_0)}_K} \widehat{D(\mu)}_F$, and hence $P\widehat{D(\mu)}_F = 0$, it follows that $P\widehat{D(\lambda)}_F = 0$, i.e. $P \subseteq \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_F$.

Moreover, if $x\widehat{D(\lambda)}_F = 0$ then $x\widehat{D(\mu)}_F = 0$ so $x + \mathfrak{a}\widehat{U(\mathcal{L})}_K \in P_0$ and hence $x \in P$. Therefore $P = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_F$ as required. □

Theorem 5.5.8. *Let $\mathfrak{g}$ be a nilpotent K-lie algebra, with $\mathcal{O}$-Lie lattice $\mathcal{L}$, and let P be a locally closed prime ideal of $\widehat{U(\mathcal{L})}_K$. Then P is a Dixmier annihilator.*

Proof. We will use induction on $n = \dim_K \mathfrak{g}$:

First suppose that $n = 1$, and hence $\widehat{U(\mathcal{L})}_K \cong K\langle u\rangle$ by Lemma 2.8.3. So if P is a locally closed ideal, then it is primitive, and hence maximal since $\widehat{U(\mathcal{L})}_K$ is commutative. So let $F := \widehat{U(\mathcal{L})}_K/P$, then F is a field.

Furthermore, using [11, Corollary 2.2.12], we see that F is a finite extension of K, so define $\lambda : \mathfrak{g} \to F, x \mapsto x+P$, and clearly this map is K-linear. Also, $\widehat{U(\mathcal{L})}/P\cap\widehat{U(\mathcal{L})} = \mathcal{O}\langle u\rangle/P \cap \mathcal{O}\langle u\rangle$ is a lattice in $F = K\langle u\rangle/P$. Thus $\widehat{U(\mathcal{L})}/P \cap \widehat{U(\mathcal{L})} \subseteq \mathcal{O}_F$ so clearly $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$.

So $\widehat{D(\lambda)}_F = F$, where $x \in \widehat{U(\mathcal{L})}_K$ acts by zero if and only if $\lambda(x) = 0$, i.e. if and only if $x \in P$, so $P = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_F = I(\lambda)_F$ as required.

So now suppose that the result holds whenever $\dim_K \mathfrak{g} < n$.

Again, suppose that P is a locally closed ideal of $\widehat{U(\mathcal{L})}_K$, and let $\mathfrak{a} := P\cap\mathfrak{g}$, $\mathcal{A} := \mathfrak{a}\cap\mathcal{L}$. Clearly $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, contained in P, and $\mathcal{A}$ is a Lie lattice in $\mathfrak{a}$. We will suppose first that $\mathfrak{a} \neq 0$.

Let P_0 be the image of P under the surjection $\widehat{U(\mathcal{L})}_K \to \widehat{U(\mathcal{L}/\mathcal{A})}_K$, then P_0 is a locally closed ideal of $\widehat{U(\mathcal{L}/\mathcal{A})}_K$. Since $\dim_K \mathfrak{g}/\mathfrak{a} < n$, it follows from induction that P_0 is a Dixmier annihilator. Therefore, using Lemma 5.5.7, P is a Dixmier annihilator as required.

So from now on we may assume that $\mathfrak{a} = P \cap \mathfrak{g} = 0$.

Since we know by Theorem 5.5.2 that P is primitive, it follows from [35, Theorem 6.4.6] that $F = Z(\widehat{U(\mathcal{L})}_K/P)$ is an algebraic field extension of K. Since the centre of $\widehat{U(\mathcal{L})}_K/P$ is closed and $\widehat{U(\mathcal{L})}_K/P$ is complete, it follows that F is complete, so it must in fact be a finite extension of K.

We will assume for now that $F = K$, so applying Lemma 5.5.4, we see that $\mathfrak{g}$ has a reducing quadruple $(x, y, z, \mathfrak{g}')$. So let $\mathcal{L}' := \mathfrak{g}' \cap \mathcal{L}$, then since $z \notin P$, it is clear that $z + P \in Z(\widehat{U(\mathcal{L})}_K/P) = K$ is not a zero divisor, so using Theorem 5.4.2, we see that P is controlled by $\mathcal{L}'$, i.e. $P = \widehat{U(\mathcal{L})}_K(P \cap \widehat{U(\mathcal{L}')}_K)$.

Let $Q := P \cap \widehat{U(\mathcal{L}')}_K$, then Q is a semiprime ideal of $\widehat{U(\mathcal{L}')}_K$ by Proposition 2.10.2, so since all locally closed prime ideals in $\widehat{U(\mathcal{L}')}_K$ are Dixmier annihilators by induction, it follows from Proposition 5.5.6 that all semiprime ideals arise as an intersection of Dixmier annihilators, i.e. there exist finite extensions F_j/K, $\mu_j \in (\mathcal{L}')^*_{F_j}$, as j ranges over some indexing set X, and

$$Q = \bigcap_{j \in X} I(\mu_j)_{F_j}.$$

Since $z \notin Q$ and $Z(\widehat{U(\mathcal{L})}_K/P) = K$, there exists $0 \neq \beta \in K$ such that $z - \beta \in Q$. Therefore $z - \beta \in I(\mu_j)_{F_j}$ for each j. Since $\beta \neq 0$, this means that $z \notin I(\mu_j)_{F_j}$, i.e. $\mu_j(z) \neq 0$.

Now, it is clear that $(x\otimes 1, y\otimes 1, z\otimes 1, \mathfrak{g}'\otimes_K F_j)$ is a reducing quadruple for $\mathfrak{g}\otimes_K F_j$, so applying Lemma 5.2.4 gives that if $\mathfrak{b}$ is a polarisation of $\mathfrak{g}'\otimes_K F_j$ at μ_j and λ_j is an extension of μ_j to $\mathfrak{g}\otimes_K F_j$, then $\mathfrak{b}$ is a polarisation of $\mathfrak{g}\otimes_K F_j$ at λ_j.

Therefore, $\widehat{D(\lambda_j)}_{F_j}\cong\widehat{U(\mathcal{L})}_{F_j}\otimes_{\widehat{U(\mathcal{L}')}_{F_j}}\widehat{D(\mu_j)}_{F_j}$, so by Lemma 5.4.1, $I(\lambda_j)_{F_j}=\text{Ann}_{\widehat{U(\mathcal{L})}_{F_j}}\widehat{D(\lambda_j)}_{F_j}$ is the largest two-sided ideal of $\widehat{U(\mathcal{L})}_{F_j}$ contained in $\widehat{U(\mathcal{L})}_{F_j}\text{Ann}_{\widehat{U(\mathcal{L}')}_{F}}\widehat{D(\mu_j)}_{F_j}$.

But $P=\widehat{U(\mathcal{L})}_K Q\subseteq\widehat{U(\mathcal{L})}_K I(\mu_j)_{F_j}$, and by Proposition 5.5.5, $\widehat{U(\mathcal{L})}_{F_j}=\widehat{U(\mathcal{L})}_K\otimes_K F_j$, hence $P\otimes_K F_j\subseteq\widehat{U(\mathcal{L})}_{F_j}I(\mu_j)_{F_j}\subseteq\widehat{U(\mathcal{L})}_{F_j}\text{Ann}_{\widehat{U(\mathcal{L}')}_{F_j}}\widehat{D(\mu_j)}_{F_j}$.

Thus $P\otimes_K F_j\subseteq\text{Ann}_{\widehat{U(\mathcal{L})}_{F_j}}\widehat{D(\lambda_j)}_{F_j}$ and $P\subseteq\text{Ann}_{\widehat{U(\mathcal{L})}_K}\widehat{D(\lambda_j)}_{F_j}=I(\lambda_j)_{F_j}$.

Furthermore, given $r\in\bigcap_{j\in X}I(\lambda_j)_{F_j}$, we have that $r=\sum_{i\geq 0}x^i r_i$ for some $r_i\in\widehat{U(\mathcal{L}')}_K$ by Lemma 2.8.3, with $r_i\to 0$ as $i\to\infty$. Then since each $I(\lambda_j)_{F_j}$ is a prime ideal of $\widehat{U(\mathcal{L})}_K$, and $z\notin I(\lambda_j)_{F_j}$, it follows from Theorem 5.4.2 that each r_i lies in $I(\lambda_j)_{F_j}$ for every j.

This means that $r_i\widehat{D(\lambda_j)}_{F_j}=0$ for all i,j, so $r_i\widehat{D(\mu_j)}_{F_j}=0$ and thus $r_i\in\bigcap_{j\in X}I(\mu_j)_{F_j}=Q$ for every i. Therefore $r\in\widehat{U(\mathcal{L})}_K Q=P$. Since our choice of r was arbitrary, it follows that:

$$P=\bigcap_{j\in X}I(\lambda_j)_{F_j}.$$

Since P is locally closed and each $I(\lambda_j)_{F_j}$ is a prime ideal of $\widehat{U(\mathcal{L})}_K$ containing P, it follows that $P=I(\lambda_j)_{F_j}$ for some $j\in X$ as we require.

Finally, take P to be a general locally closed prime ideal. Then $F=Z(\widehat{U(\mathcal{L})}_K/P)$ is a finite extension of K, so let $\mathfrak{g}_0:=\mathfrak{g}\otimes_K F$, $\mathcal{L}_0:=\mathcal{L}\otimes_{\mathcal{O}}\mathcal{O}_F$. Then $\dim_F\mathfrak{g}_0=\dim_K\mathfrak{g}=n$, $\mathcal{L}_0$ is a Lie lattice in $\mathfrak{g}_0$, and by Proposition 5.5.5, there exists a surjection of F-algebras $\widehat{U(\mathcal{L}_0)}_F=\widehat{U(\mathcal{L})}_K\otimes_K F\twoheadrightarrow\widehat{U(\mathcal{L})}_K/P$ whose kernel contains $P\otimes_K F$.

Let J be this kernel.

Then J is a locally closed prime ideal of $\widehat{U(\mathcal{L}_0)}_F$ and $\widehat{U(\mathcal{L}_0)}_F/J \cong \widehat{U(\mathcal{L})}_K/P$. But $Z(\widehat{U(\mathcal{L})}_F/J) \cong Z(\widehat{U(\mathcal{L})}_K/P) = F$ so it follows from the above discussion that $J = \operatorname{Ann}_{\widehat{U(\mathcal{L}_0)}_F} \widehat{D(\lambda)}_{F'}$ for some finite extension F'/F and some linear form λ of $\mathfrak{g}_0 \otimes_F F'$ such that $\lambda(\mathcal{L}_0) \subseteq \mathcal{O}_{F'}$.

It is clear that $J \cap \widehat{U(\mathcal{L})}_K = P$, and hence $P = \operatorname{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{F'} = I(\lambda)_{F'}$ as required. $\square$

Remark: In the previous proof, and also in the proof of Theorem 5.5.2, we cite a result from [35], a doctoral thesis which is still in preparation and has not yet been examined. However, a very similar version of this result can be found in [6, Corollary 8.6], which still allows us to prove all the results in this section, provided we make the additional assumption that $[\mathcal{L}, \mathcal{L}] \subseteq p\mathcal{L}$. Since we need to pass to a sublattice $p^n\mathcal{L}$ to prove Theorem D anyway, this assumption would change nothing in the statement or the proof.

Corollary 5.5.9. *Let $\mathcal{L}$ be a Lie lattice in $\mathfrak{g}$ nilpotent. Then given a prime ideal P of $\widehat{U(\mathcal{L})}_K$, P arises as an intersection of Dixmier annihilators.*

Proof. This is immediate from Theorem 5.5.8 and Proposition 5.5.6. $\square$

5.6 The Coadjoint action

For the rest of this chapter, we will always assume that $\mathfrak{g}$ is nilpotent. In Appendix C.2, we defined the adjoint group of $\mathfrak{g}$, $\mathbb{G} := \{\exp(\operatorname{ad}(u)) : u \in \mathfrak{g}\} \subseteq \operatorname{Aut}(\mathfrak{g})$, and recall that this acts on the linear dual $\mathfrak{g}^*$, with orbits termed *coadjoint orbits*.

In the classical case, using [16, Proposition 6.2.3], we see that if two linear forms lie in the same coadjoint orbit, then their Dixmier annihilators are the same. And the following result shows that the same is true in the affinoid case when considering the action of the adjoint group $\mathbb{G}_0$ associated to the lattice $p\mathcal{L}$:

Lemma 5.6.1. *If $\lambda, \mu \in \mathcal{L}^*$ lie in the same coadjoint orbit of $\mathbb{G}_0$, then $I(\lambda) = I(\mu)$.*

Proof. Suppose that $\mu = g \cdot \lambda$ where $g = \exp(\mathrm{ad}(u))$ for some $u \in p\mathcal{L}$. Then the K-linear map $\mathrm{ad}(u) : \widehat{U(\mathcal{L})}_K \to \widehat{U(\mathcal{L})}_K$ sends $\widehat{U(\mathcal{L})}$ to $p\widehat{U(\mathcal{L})}$, and thus the sequence $\mathrm{ad}(u)^n$ converges to 0 as $n \to \infty$. Therefore the series $\exp(\mathrm{ad}(u)) = \sum_{n \geq 0} \frac{1}{n!} \mathrm{ad}(u)^n$ converges to a continuous K-algebra automorphism of $\widehat{U(\mathcal{L})}_K$, and clearly this restricts to g on $\mathfrak{g}$.

If $r \in \mathrm{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\mu)}$ then for any basis $\{u_1, \cdots, u_d\}$ for $\mathcal{L}$, we can write r as a Tate power series $\sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha u_1^{\alpha_1} \cdots u_d^{\alpha_d}$, where $\lambda_\alpha \in K \to 0$ as $\alpha \to \infty$.

Since $\widehat{D(\mu)} = \widehat{D(g \cdot \lambda)}$, it follows that for any $v \in \mathfrak{g}$, the action of v on $\widehat{D(\mu)}$ coincides with the action of $g^{-1} \cdot v$ on $\widehat{D(\lambda)}$. In particular, since $r\widehat{D(\mu)} = 0$, it follows that:

$$g^{-1}(r) = \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha g^{-1}(u_1)^{\alpha_1} \cdots g^{-1}(u_d)^{\alpha_d} \in \mathrm{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}.$$

Therefore $r \in g(\mathrm{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)})$, but it is clear that $g = \exp(\mathrm{ad}(u))$ preserves two-sided ideals of $\widehat{U(\mathcal{L})}_K$, and therefore $r \in \mathrm{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}$, and hence $\mathrm{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\mu)} \subseteq \mathrm{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}$.

After replacing g by g^{-1}, we similarly obtain that $\mathrm{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)} \subseteq \mathrm{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\mu)}$ as required. □

Unfortunately, the proof of this does not carry across to when λ, μ lie in the same $\mathbb{G}$-coadjoint orbit, since we cannot always extend elements of $\mathbb{G}$ to automorphisms of $\widehat{U(\mathcal{L})}_K$. Despite this drawback, we will explore in the rest of this section how we can still relate affinoid Dixmier annihilators in this case. Firstly, we will show how our Dixmier annihilator does not depend on the choice of the defining field extension:

Proposition 5.6.2. *Let F/K be a finite extension, and let $\lambda : \mathfrak{g} \to K$ be K-linear. Then there exists a polarisation $\mathfrak{b}$ of $\mathfrak{g}$ at λ such that $\mathfrak{b} \otimes_K F$ is a polarisation for $\mathfrak{g} \otimes_K F$ at the extension $\lambda_F : \mathfrak{g} \otimes_K F \to F$.*

Proof. Using induction on $\dim(\mathfrak{g})$. If $\dim(\mathfrak{g}) = 1$ then it is obvious, because $\mathfrak{g}$ and $\mathfrak{g} \otimes F$ are the only polarisations. So suppose the result holds whenever $\dim(\mathfrak{g}) < n$.

If $\lambda(\mathfrak{a}) = 0$ for some non-zero ideal $\mathfrak{a}$ of $\mathfrak{g}$, then using induction we may choose a polarisation $\mathfrak{a} \subseteq \mathfrak{b}$ such that $\frac{\mathfrak{b}}{\mathfrak{a}} \otimes_K F = \frac{\mathfrak{b} \otimes_K F}{\mathfrak{a} \otimes_K F}$ is a polarisation for $\frac{\mathfrak{g} \otimes_K F}{\mathfrak{a} \otimes_K F}$ at λ_F. Hence $\mathfrak{b} \otimes_K F$ is a polarisation for $\mathfrak{g} \otimes_K F$ at λ_F.

So from now on, we may assume that $\lambda(\mathfrak{a}) = 0$ for all non-zero ideals $\mathfrak{a}$ of $\mathfrak{g}$. Then it follows from Proposition 5.2.2 that $\mathfrak{g}$ has a reducing quadruple $(x, y, z, \mathfrak{g}')$, and $\lambda(z) \neq 0$. Clearly $(x \otimes 1, y \otimes 1, z \otimes 1, \mathfrak{g}' \otimes_K F)$ is a reducing quadruple for $\mathfrak{g} \otimes_K F$.

Let $\mathfrak{b}$ be a polarisation for $\mathfrak{g}'$ at $\lambda|_{\mathfrak{g}'}$ such that $\mathfrak{b} \otimes_K F$ is a polarisation for $\mathfrak{g}' \otimes_K F$. Then using Lemma 5.2.4 we see that $\mathfrak{b}$ is a polarisation for $\mathfrak{g}$ at λ, and $\mathfrak{b} \otimes_K F$ is a polarisation for $\mathfrak{g} \otimes_K F$ at λ_F as required. □

Corollary 5.6.3. *Let F/K be a finite extension, and let $\lambda : \mathfrak{g} \to F$ be K-linear such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$. Then for any finite extension L/F, $I(\lambda)_F = I(\lambda)_L$.*

Proof. This is immediate from Proposition 5.6.2 and Theorem 5.4.4. □

So, from now on, we will assume that λ and μ take values in K. We will now assume further that they lie in the same coadjoint orbit, i.e. $\mu = \exp(\mathrm{ad}(u)) \cdot \lambda$ for some $u \in \mathfrak{g}$. Let $a := \exp(\mathrm{ad}(u)) \in \mathbb{G}$, and fix a natural number $N \in \mathbb{N}$ such that $u \in p^{-N}\mathcal{L}$. Also let c be the nilpotency class of $\mathfrak{g}$, i.e. c is minimal such that $\mathrm{ad}(\mathfrak{g})^c = 0$.

Since a is a Lie automorphism of $\mathfrak{g}$, it follows that $a\mathcal{L}$ is an $\mathcal{O}$-Lie lattice in $\mathfrak{g}$, hence there exists a natural number $n \in \mathbb{N}$ such that $p^n\mathcal{L} \subseteq a\mathcal{L}$ and $p^n a\mathcal{L} \subseteq \mathcal{L}$.

Lemma 5.6.4. *For any $n \geq cN + v_p(c!)$, $p^n\mathcal{L} \subseteq a\mathcal{L}$ and $p^n a\mathcal{L} \subseteq \mathcal{L}$.*

Proof. Since $a = \exp(\mathrm{ad}(u))$, where $u = p^{-N}v$ for some $v \in \mathcal{L}$, it follows that for all $w \in \mathcal{L}$:

$$a(w) = w + p^{-N}[v,w] + \frac{1}{2}p^{-2N}[v,[v,w]] + \cdots + \frac{1}{c!}p^{-cN}(ad(v))^c(w). \tag{5.5}$$

But for each $0 \leq i \leq c$, $(ad(v))^i(w) \in \mathcal{L}$, $v_p(\frac{1}{i!}p^{-iN}) = -iN - v_p(i!) \geq -cN - v_p(c!) \geq -n$, so $\frac{1}{i!}p^{-iN}(ad(v))^i(w) \in p^{-n}\mathcal{L}$. Hence $a\mathcal{L} \subseteq p^{-n}\mathcal{L}$, and $p^n a\mathcal{L} \subseteq \mathcal{L}$.

Also, a is an isomorphism, and $a^{-1} = \exp(ad(-u))$, with $-u \in p^{-N}\mathcal{L}$. Therefore, since $a^{-1} : a\mathcal{L} \to \mathcal{L}$ is a Lie-isomorphism, it follows from the above discussion that $p^n\mathcal{L} \subseteq a\mathcal{L}$. □

It is clear that since $a : \mathcal{L} \to a\mathcal{L}$ is a continuous isomorphism of $\mathcal{O}$-Lie lattices, it extends to a continuous isomorphism $a : \widehat{U(\mathcal{L})}_K \to \widehat{U(a\mathcal{L})}_K$ of K-algebras. Moreover, for any $n \in \mathbb{N}$, a induces an isomorphism $a : \widehat{U(p^n\mathcal{L})}_K \to \widehat{U(p^n a\mathcal{L})}_K$, and thus using Lemma 5.6.4, for $n \geq cN + v_p(c!)$, there is an injective K-algebra homomorphism $a : \widehat{U(p^n\mathcal{L})}_K \to \widehat{U(\mathcal{L})}_K$.

Proposition 5.6.5. *Given $n \in \mathbb{N}$ such that $n \geq cN + v_p(c!)$, if I is a two-sided ideal of $\widehat{U(\mathcal{L})}_K$, then $a : \widehat{U(p^n\mathcal{L})}_K \to \widehat{U(\mathcal{L})}_K$ maps $I \cap \widehat{U(p^n\mathcal{L})}_K$ into I.*

Proof. Consider the sequence of continuous $\mathcal{O}$-linear maps $a_i := \sum\limits_{0 \leq j \leq i} \frac{1}{j!}(ad(u))^j$: $\widehat{U(p^n\mathcal{L})}_K \to \widehat{U(\mathcal{L})}_K$. Clearly each of these sends $I \cap \widehat{U(p^n\mathcal{L})}_K$ into I.

We will show that a_i converges pointwise to a as $i \to \infty$, and it will follow from closure of ideals in $\widehat{U(\mathcal{L})}_K$ that $a(I \cap \widehat{U(p^n\mathcal{L})}_K) \subseteq I$.

Let $\delta := \mathrm{ad}(u)$, and let v be the p-adic filtration on $\widehat{U(\mathcal{L})}_K$ induced from $\widehat{U(\mathcal{L})}$. Then for all $u \in \mathcal{L}$, $v(\delta(u)) \geq v(u) - N$. Since δ is a derivation, a standard inductive argument shows that for all $x_1, \cdots, x_r \in \widehat{U(\mathcal{L})}_K$:

$$\sum_{0\leq j\leq i} \frac{1}{j!}\delta^j(x_1x_2\cdots x_r) = \sum_{0\leq j\leq i}\left(\sum_{j_1+\cdots+j_r=j}\left(\prod_{1\leq m\leq r}\frac{1}{j_m!}\delta^{j_m}(x_j)\right)\right). \tag{5.6}$$

So, if $x \in \widehat{U(\mathcal{L})}_K$, then $x = \sum\limits_{(r,u)} \lambda_u u_1 \cdots u_r$, where the sum is taken over all $r \geq 0$, $u = u_1 \cdots u_r$ for $u_i \in \mathcal{L}$, and $v_p(\lambda_u) - nr \to \infty$ as $r \to \infty$. Therefore, fixing $t \in \mathbb{N}$, we have:

$$(a-a_t)(x) = \sum_{(r,u)}\lambda_u(a-a_t)(u_1\cdots u_r) = \sum_{(r,u)}\lambda_u\left(\sum_{j>t}\left(\sum_{j_1+\cdots+j_r=j}\left(\prod_{1\leq m\leq r}\frac{1}{j_m!}\delta^{j_m}(u_m)\right)\right)\right). \tag{5.7}$$

Note that $\delta^c(u) = 0$ for all $u \in \mathcal{L}$. So for each $r \geq 0$, let $A_r := \{\alpha \in [c]^r : |\alpha| > t\}$, where $[c] = \{0, \cdots, c-1\}$.

Then A_r is a finite set and $(a - a_t)(x) = \sum\limits_{(r,u)} \lambda_u \left(\sum\limits_{\alpha\in A_r}\prod\limits_{1\leq m\leq r} \frac{1}{\alpha_m!}\delta^{\alpha_m}(u_m)\right)$, where $\sum\limits_{\alpha\in A_r}\prod\limits_{1\leq m\leq r} \frac{1}{\alpha_m!}\delta^{\alpha_m}(u_m)$ is a finite sum.

Since $\alpha_m < c$ for all $\alpha \in A_r$, we have that $v_p(\alpha_m!) \leq v_p(c!)$. Also, since $v(\delta(u)) \geq v(u) - N$, it follows that $v(\delta^{\alpha_m}(u)) \geq v(u) - \alpha_m N$. Therefore $v(\frac{1}{\alpha_m!}\delta^{\alpha_m}(u_m)) \geq v(u_m) - \alpha_m N - v_p(c!)$ for all $m \leq r$.

Thus for each pair (r, u):

$$v\Big(\sum_{\alpha\in A_r}\prod_{1\leq m\leq r}\frac{1}{\alpha_m!}\delta^{\alpha_m}(u_m)\Big) \geq v(u_1) + \cdots + v(u_r) - (|\alpha|N + v_p(c!)r) \geq$$
$$-(|\alpha|N + rv_p(c!)) \geq -r(cN + v_p(c!)),$$

where the last inequality follows since $|\alpha| \leq rc$.

Therefore, $v\left(\lambda_u\left(\sum_{\alpha\in A_r}\prod_{1\leq m\leq r}\frac{1}{\alpha_m!}\delta^{\alpha_m}(u_m)\right)\right)\geq v_p(\lambda_u)-r(cN+v_p(c!))\geq v_p(\lambda_u)-nr\to\infty$ as $r\to\infty$.

Moreover, for $r\leq\frac{t}{c}$, $A_r=\emptyset$, so we have:

$$(a-a_t)(x)=\sum_{(r,u),r>\frac{t}{c}}\lambda_u\left(\sum_{\alpha\in A_r}\prod_{1\leq m\leq r}\frac{1}{\alpha_m!}\delta^{\alpha_m}(u_m)\right). \tag{5.8}$$

Therefore, $v((a-a_t)(x))\geq\inf\{v_p(\lambda_u)-nr : u=u_1\cdots u_r \text{ with } r>\frac{t}{c}\}$, and this tends to infinity as $t\to\infty$. Hence $(a-a_t)(x)\to 0$ as $t\to\infty$.

So $a(x)=\lim_{t\to\infty}a_t(x)$, so if $x\in I$ then $a(x)\in I$ as required. □

Now, let $\mathfrak{b}$ be a polarisation for $\mathfrak{g}$ at λ, and let $\mathfrak{b}'$ be a polarisation for $\mathfrak{g}$ at μ. Since $\mu=a\cdot\lambda$, it follows that $a\mathfrak{b}$ is also a polarisation for $\mathfrak{g}$ at μ. Also, it is clear that $a\mathfrak{b}\cap a\mathcal{L}=a(\mathfrak{b}\cap\mathcal{L})$, so let $\mathcal{B}:=\mathfrak{b}\cap\mathcal{L}$ and $\mathcal{B}':=\mathfrak{b}'\cap\mathcal{L}$.

Consider the Dixmier modules $\widehat{D(\lambda)}_{\mathcal{B}}:=\widehat{U(\mathcal{L})}_K\otimes_{\widehat{U(\mathcal{B})}_K}K_\lambda$, $\widehat{D(\mu)}_{a\mathcal{B}}:=\widehat{U(a\mathcal{L})}_K\otimes_{\widehat{U(a\mathcal{B})}_K}K_\mu$, $\widehat{D(\mu)}_{\mathcal{B}'}:=\widehat{U(\mathcal{L})}_K\otimes_{\widehat{U(\mathcal{B}')}_K}K_\mu$.

Then $\widehat{D(\lambda)}_{\mathcal{B}}$ and $\widehat{D(\mu)}_{\mathcal{B}'}$ are $\widehat{U(\mathcal{L})}_K$-modules, topological completions of the $U(\mathfrak{g})$-modules $D(\lambda)_{\mathfrak{b}}$ and $D(\mu)_{\mathfrak{b}'}$ respectively, while $\widehat{D(\mu)}_{a\mathcal{B}}$ is a $\widehat{U(a\mathcal{L})}_K$-module, a topological completion of $D(\mu)_{a\mathfrak{b}}$.

Let $I(\mu):=\operatorname{Ann}_{\widehat{U(\mathcal{L})}_K}\widehat{D(\mu)}_{\mathcal{B}'}\trianglelefteq\widehat{U(\mathcal{L})}_K$, and let $I'(\mu):=\operatorname{Ann}_{\widehat{U(a\mathcal{L})}_K}\widehat{D(\mu)}_{a\mathcal{B}}\trianglelefteq\widehat{U(a\mathcal{L})}_K$.

Lemma 5.6.6. *Given $n\in\mathbb{N}$ such that $p^n\mathcal{L}\subseteq a\mathcal{L}$, $I(\mu)\cap\widehat{U(p^n\mathcal{L})}_K=I'(\mu)\cap\widehat{U(p^n\mathcal{L})}_K$.*

Proof. Let $\mathcal{C}:=a\mathfrak{b}\cap p^n\mathcal{L}$, then the $\widehat{U(p^n\mathcal{L})}_K$-affinoid Dixmier module $\widehat{D(\mu)}_{\mathcal{C}}:=\widehat{U(p^n\mathcal{L})}_K\otimes_{\widehat{U(\mathcal{C})}_K}K_\mu$ embeds densely into $\widehat{D(\mu)}_{a\mathcal{B}}$. So if $x\in\widehat{U(p^n\mathcal{L})}_K$ then $x\widehat{D(\mu)}_{\mathcal{C}}=0$

if and only if $x\widehat{D(\mu)}_{a\mathcal{B}} = 0$.

But since $a\mathfrak{b}$ and $\mathfrak{b}'$ are polarisations of $\mathfrak{g}$ at μ with $a\mathfrak{b}\cap p^n\mathcal{L} = C$ and $\mathfrak{b}'\cap p^n\mathcal{L} = p^n\mathcal{B}'$, we can apply Theorem 5.4.4 to get that $\text{Ann}_{\widehat{U(p^n\mathcal{L})}_K}\widehat{D(\mu)}_{\mathcal{C}} = \text{Ann}_{\widehat{U(p^n\mathcal{L})}_K}\widehat{D(\mu)}_{p^n\mathcal{B}'}$. Therefore, given $x \in \widehat{U(p^n\mathcal{L})}_K$:

$$x\widehat{D(\mu)}_{a\mathcal{B}} = 0 \iff x\widehat{D(\mu)}_{\mathcal{C}} = 0 \iff \widehat{D(\mu)}_{p^n\mathcal{B}'} = 0 \iff x\widehat{D(\mu)}_{\mathcal{B}'} = 0.$$

Therefore $I(\mu) \cap \widehat{U(p^n\mathcal{L})}_K = I'(\mu) \cap \widehat{U(p^n\mathcal{L})}_K$ as required. □

Lemma 5.6.7. $a(I(\lambda)) = I'(\mu)$.

Proof. Consider the map $\Theta : \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B})}_K} K_\lambda \to \widehat{U(a\mathcal{L})}_K \otimes_{\widehat{U(a\mathcal{B})}_K} K_{a\lambda}, x \otimes v \mapsto a(x) \otimes v$.

We will show that Θ is a K-linear isomorphism such that $\Theta(xm) = a(x)\Theta(m)$ for all $x \in \widehat{U(\mathcal{L})}_K$, $m \in \widehat{D(\lambda)}_{\mathcal{B}}$. It will follow from this that $x\widehat{D(\lambda)}_{\mathcal{B}} = 0$ if and only if $a(x)\widehat{D(\mu)}_{\mathcal{B}'} = 0$, and hence $a(I(\lambda)) = I'(\mu)$ as required.

It is clear that Θ is K-linear, and that it has an inverse defined by $x\otimes v \mapsto a^{-1}(x)\otimes v$, hence it is an isomorphism of vector spaces.

Finally, $\Theta(x(y \otimes v)) = \Theta(xy \otimes v) = a(xy) \otimes v = a(x)(a(y) \otimes v) = a(x)\Theta(y \otimes v)$. □

The next theorem is the key step in the proof of Theorem D, and it allows us to compare Dixmier annihilators for $\lambda, \mu \in \mathfrak{g}^*$ in the same $\mathbb{G}$-coadjoint orbit:

Theorem 5.6.8. *Let $\mathfrak{g}$ be a nilpotent K-Lie algebra, with nilpotency class c, and let $\mathcal{L}$ be an $\mathcal{O}$-Lie lattice in $\mathfrak{g}$. Let $\lambda, \mu : \mathfrak{g} \to \overline{K}$ be K-linear maps such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_{\overline{K}}$, and suppose that $\mu = \exp(\text{ad}(u)) \cdot \lambda$ for some $u \in p^{-N}(\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_{\overline{K}})$. Then given $n \in \mathbb{N}$ such that $n \geq Nc + v_p(c!)$, $I(\lambda) \cap \widehat{U(p^{2n}\mathcal{L})}_K = I(\mu) \cap \widehat{U(p^{2n}\mathcal{L})}_K$*

Proof. Firstly, note that u lies in $p^{-N}(\mathcal{L}\otimes_{\mathcal{O}}\mathcal{O}_F)$ for some finite extension F of K, and we may assume further that λ and μ take values in F, possibly after extending F.

Using Corollary 5.6.3, we see that $I(\lambda) = I(\lambda)_F$, i.e. $I(\lambda) = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_F$, and similarly $I(\mu) = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\mu)}_F$, so we may safely assume that $F = K$. So λ and μ are K-linear forms of $\mathfrak{g}$, $u \in p^{-N}\mathcal{L}$, and setting $a := \exp(\text{ad}(u))$, since $n \geq Nc + v(c!)$ we see using Lemma 5.6.4 that $p^n\mathcal{L} \subseteq a\mathcal{L}$ and $p^na\mathcal{L} \subseteq \mathcal{L}$.

We will prove that $I(\mu)\cap\widehat{U(p^{2n}\mathcal{L})}_K \subseteq I(\lambda)$, and after replacing a by $a^{-1} = \exp(\text{ad}(-u))$, it will follow that $I(\lambda) \cap \widehat{U(p^{2n}\mathcal{L})}_K \subseteq I(\mu)$ as required.

By Lemma 5.6.6, we see that $I(\mu)\cap\widehat{U(p^{2n}\mathcal{L})}_K = I'(\mu)\cap\widehat{U(p^{2n}\mathcal{L})}_K$, and using Lemma 5.6.7 we see that $I'(\mu) = a(I(\lambda))$.

Therefore $I'(\mu) \cap \widehat{U(p^{2n}\mathcal{L})}_K = a(I(\lambda)) \cap \widehat{U(p^{2n}\mathcal{L})}_K \subseteq a(I(\lambda)) \cap \widehat{U(p^na\mathcal{L})}_K$
$= a(I(\lambda) \cap \widehat{U(p^n\mathcal{L})}_K)$.

So $I(\mu)\cap\widehat{U(p^{2n}\mathcal{L})}_K \subseteq a(I(\lambda)\cap\widehat{U(p^n\mathcal{L})}_K)$, but since $I(\lambda)$ is a two-sided ideal of $\widehat{U(\mathcal{L})}_K$, it follows from Proposition 5.6.5 that $a(I(\lambda) \cap \widehat{U(p^n\mathcal{L})}_K) \subseteq I(\lambda)$ as required. □

5.7 Proof of Theorem D

Now we are ready to prove the main theorem of this chapter. As in the statement, fix a weakly rational ideal P of $\widehat{U(\mathcal{L})}_K$. Then since P is prime, we see using Corollary 5.5.9 that P arises as an intersection of Dixmier annihilators:

$$P = \bigcap_{j\in X} I(\lambda_j)$$

for some $\lambda_j : \mathfrak{g} \to \overline{K}$ K-linear, such that $\lambda_j(\mathcal{L}) \subseteq \mathcal{O}_{\overline{K}}$ for each j. Since $\widehat{D(\lambda)}$ is a topological completion of $D(\lambda)$, it follows that $I(\lambda_j) \cap U(\mathfrak{g}) = \text{Ann}_{U(\mathfrak{g})} D(\lambda_j)$.

Since P is weakly rational, it follows that $Z\left(U(\mathfrak{g})/P \cap U(\mathfrak{g})\right)$ is a finite field extension of K, and hence $P \cap U(\mathfrak{g})$ is a maximal ideal of $U(\mathfrak{g})$ by [16, Proposition 4.7.4]. Since $P \subseteq I(\lambda_j)$ for each j and $P \cap U(\mathfrak{g})$ is maximal, it follows that $P \cap U(\mathfrak{g}) = I(\lambda_j) \cap U(\mathfrak{g})$ for each j. Therefore $\operatorname{Ann}_{U(\mathfrak{g})} D(\lambda_j) = \operatorname{Ann}_{U(\mathfrak{g})} D(\lambda_k)$ for all $j, k \in X$.

Using [16, Proposition 6.2.3], we see that for any linear forms $\lambda, \mu : \mathfrak{g} \to \overline{K}$, if $\operatorname{Ann}_{U(\mathfrak{g})} D(\lambda) = \operatorname{Ann}_{U(\mathfrak{g})} D(\mu)$ then there exists an element $a \in \mathbb{G}(\overline{K})$ such that $a \cdot \lambda = \mu$. Therefore, for any $j, k \in X$, there exists $a_{j,k} \in \mathbb{G}(\overline{K})$ such that $a_{j,k} \cdot \lambda_j = \lambda_k$, i.e. all λ_j lie in the same coadjoint orbit.

Proposition 5.7.1. *Let $\lambda : \mathfrak{g} \to \overline{K}$ be a K-linear map such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_{\overline{K}}$. Then there exists an integer $N \geq 0$ such that for any linear form $\mu : \mathfrak{g} \to \overline{K}$ in the $\mathbb{G}$-coadjoint orbit of λ with $\mu(\mathcal{L}) \subseteq \mathcal{O}_{\overline{K}}$, $\mu = \exp(\operatorname{ad}(u)) \cdot \lambda$ for some $u \in p^{-N}(\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_{\overline{K}})$.*

Proof. Let Y be the coadjoint orbit in $\mathfrak{g}^* = \operatorname{Hom}_K(\mathfrak{g}, \overline{K})$ containing λ, and recall from Lemma 2.10.1 that there exists an affine algebraic subgroup S of $\mathbb{G}$ such that $\mathbb{G} \cong S \times Y$ as varieties, where the natural morphism $\mathbb{G} \to Y, g \mapsto g \cdot \lambda$ is just the projection to the second factor. Consider the following sequence in the category of $\overline{K}$-varieties defined over K:

$$\operatorname{ad}(\mathfrak{g}) \underset{\exp}{\to} \mathbb{G} \cong S \times Y \to Y. \tag{5.9}$$

Apply the rigid analytification functor as described in Appendix D, and we get the following sequence in the category of rigid spaces over K:

$$\operatorname{ad}(\mathfrak{g})^{an} \underset{\exp}{\to} \mathbb{G}^{an} \cong (S \times Y)^{an} \to Y^{an}. \tag{5.10}$$

and the $\overline{K}$-points on the rigid varieties in (5.10) are precisely the $\overline{K}$-points on the varieties in (5.9).

Let U be the set of all $\mu \in Y^{an}$ such that $\mu(\mathcal{L}) \subseteq \mathcal{O}_{\overline{K}}$. Then U is an affinoid subdomain of Y^{an}, isomorphic to Sp $\widehat{S(\mathfrak{g})}$, where $\widehat{S(\mathfrak{g})}$ is the π-adic completion of the symmetric algebra $S(\mathfrak{g})$ with respect to the lattice $S(\mathcal{L})$. Since exp is an isomorphism, we may take the inverse image $V := \exp^{-1}(1 \times U)$ of $1 \times U$, which will be an affinoid subdomain of $\mathrm{ad}(\mathfrak{g})^{an}$.

But $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}/Z(\mathfrak{g})$ is a union of open discs containing $p^{-n}(\mathcal{L}/Z(\mathcal{L}))$ for $n \in \mathbb{N}$. So since V is affinoid, it follows that V is contained in $p^{-N}(\mathcal{L}/Z(\mathcal{L}) \otimes_{\mathcal{O}} \mathcal{O}_{\overline{K}})$ for some $N \in \mathbb{N}$.

Therefore, for any $\mu \in U \cap Y$, we can choose $u \in p^{-N}(\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_{\overline{K}})$ such that μ is the image of $\mathrm{ad}(u)$ under the composition $\mathrm{ad}(\mathfrak{g}) \to \mathbb{G} \to Y$, i.e. $\mu = \exp(\mathrm{ad}(u)) \cdot \lambda$ as required. □

Now we can finally prove the main theorem of this chapter:

Proof of Theorem D. Let P be a weakly rational ideal of $\widehat{U(\mathcal{L})}_K$, then since P is prime, we see using Corollary 5.5.9 that $P = \bigcap_{j \in X} I(\lambda_j)$ for linear forms $\lambda_j =$ all lying in the same coadjoint orbit. Since $\lambda_j(\mathcal{L}) \subseteq \mathcal{O}_{\overline{K}}$ for each j, it follows from Proposition 5.7.1 that we can choose $N \in \mathbb{N}$, $u_{j,k} \in p^{-N}(\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_{\overline{K}})$ for each $j, k \in X$ such that $\lambda_k = \exp(\mathrm{ad}(u_{j,k})) \cdot \lambda_j$.

Therefore, let c be the nilpotency class of $\mathfrak{g}$, and choose $n \in \mathbb{N}$ with $n \geq 2Nc + 2v(c!)$. Then using Theorem 5.6.8, we see that $I(\lambda_j) \cap \widehat{U(p^n\mathcal{L})}_K = I(\lambda_k) \cap \widehat{U(p^n\mathcal{L})}_K$ for each $j, k \in X$.

Therefore, $P \cap \widehat{U(p^n\mathcal{L})}_K = \bigcap_{j \in X} I(\lambda_j) \cap \widehat{U(p^n\mathcal{L})}_K = I(\lambda_j) \cap \widehat{U(p^n\mathcal{L})}_K$ for any $j \in X$. Hence $P \cap \widehat{U(p^n\mathcal{L})}_K = \mathrm{Ann}_{\widehat{U(p^n\mathcal{L})}_K} \widehat{D(\lambda_j)}$ is a Dixmier annihilator as required. □

Chapter 6

The Nilpotent, Abelian-by-procyclic case in characteristic 0

Throughout this chapter, we will assume that $p > 2$, and fix G a uniform pro-p group, $\mathcal{L} := \frac{1}{p}\log(G)$, and $\mathfrak{g} := \mathcal{L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The aim of this chapter is to study primitive ideals in KG when G is nilpotent, abelian-by-procyclic, and ultimately prove Theorem E.

6.1 Faithful Dixmier Annihilators

Let F/K be a finite extension, and let $\lambda : \mathfrak{g} \to F$ be a $\mathbb{Q}_p$-linear form such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$. Then setting $\mathfrak{g}_F := \mathfrak{g} \otimes_{\mathbb{Q}_p} F$ and $\mathcal{L}_F := \mathcal{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_F$, we choose a polarisation $\mathfrak{b}$ of $\mathfrak{g}_F$ at λ, set $\mathcal{B} := \mathfrak{b} \cap \mathcal{L}_F$, and let $\widehat{D(\lambda)}_F = \widehat{U(\mathcal{L})}_F \otimes_{\widehat{U(\mathcal{B})}_F} F_\lambda$ be the corresponding Dixmier module.

Recall from Theorem 2.9.1 that there is a dense embedding $KG \to \widehat{U(\mathcal{L})}_K$. So since $\widehat{U(\mathcal{L})}_K$ clearly embeds into $\widehat{U(\mathcal{L})}_F$, there is a natural action of KG on $\widehat{D(\lambda)}_F$. Set $P := \mathrm{Ann}_{KG}\,\widehat{D(\lambda)}_F = I(\lambda) \cap KG$ to be the corresponding Dixmier annihilator in KG, and it follows from Theorem 5.4.4 that this does not depend on the choice of polarisation.

Lemma 6.1.1. *Let $P = \mathrm{Ann}_{KG}\,\widehat{D(\lambda)}_F$ be a Dixmier annihilator in KG. Then P is a prime J-ideal of KG.*

Proof. Since $\widehat{D(\lambda)}_F$ is irreducible over $\widehat{U(\mathcal{L})}_F$ by Theorem 5.3.3, it follows that $I = \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\lambda)}_F$ is a primitive ideal of $\widehat{U(\mathcal{L})}_F$, and hence it is prime. It follows immediately that $Q := I \cap \widehat{U(\mathcal{L})}_K = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}$ is a prime ideal of $\widehat{U(\mathcal{L})}_K$.

Now, suppose that A, B are two-sided ideals in KG such that $AB \subseteq P$, then let $\widehat{A}, \widehat{B}$ be the closures of A and B respectively inside $\widehat{U(\mathcal{L})}_K$. Then $\widehat{A}, \widehat{B}$ are two-sided ideals of $\widehat{U(\mathcal{L})}_K$, and $\widehat{A}\widehat{B} \subseteq \widehat{P}$.

Since $\widehat{U(\mathcal{L})}_K$ is complete with respect to the π-adic filtration, which is a Zariskian filtration, it follows from [28, Ch.II Corollary 2.1.5] that Q is closed in $\widehat{U(\mathcal{L})}_K$. Therefore, since $P \subseteq Q$, it follows that $\widehat{P} \subseteq Q$, and hence $\widehat{A}\widehat{B} \subseteq Q$. So since Q is prime, we may assume without loss of generality that $\widehat{A} \subseteq Q$, and hence $A \subseteq \widehat{A} \cap KG \subseteq Q \cap KG = P$. Therefore P is prime.

Finally, since $\log(Z(G)) \subseteq Z(\mathcal{L})$ and $Z(\mathcal{L})$ acts by scalars in F on $\widehat{D(\lambda)}_F$, it follows that $KZ(G)/KZ(G) \cap P$ is a ring extension of K, contained in F, and since P is prime, it is in fact a domain. But F is a finite extension of K, so it follows that $KZ(G)/KZ(G) \cap P$ is a finite field extension of K, and hence P is a prime J-ideal by Definition 2.11.2. □

Lemma 6.1.2. *Let F/K be a finite extension, and let I' a right ideal of FG. Setting $I := I' \cap KG$, we have that if I' is controlled by $U \leq_c G$ then I is controlled by U.*

Proof. We will first suppose that U is open in G. Then given $r \in I$, choose a complete set of coset representatives $\{g_1, \cdots, g_r\}$ for U in G, then $r = \sum_{1 \leq i \leq r} r_i g_i$ for some $r_i \in KU \subseteq FU$.

So since $I = I' \cap KG$ and I' is controlled by U, it follows that $r_i \in I' \cap FU \cap KG = I' \cap KU = I$ for each i, and hence I is controlled by U.

So, let I^χ be the controller subgroup of I, i.e. the intersection of all open subgroups of G controlling I. So since this includes all open subgroups of G controlling I', we have that $I^\chi \subseteq I'^\chi$, hence any closed subgroup controlling I' also controls I. $\square$

So since $P = \text{Ann}_{FG}\widehat{D(\lambda)}_F \cap KG$, it follows from this lemma that if we can prove a control theorem for $\text{Ann}_{FG}\widehat{D(\lambda)}_F$ in FG, the same result will follow for P in KG, and thus we can safely replace F by K. So from now on, we will assume that $\lambda : \mathfrak{g} \to K$ with $\lambda(\mathcal{L}) \subseteq \mathcal{O}$.

Definition 6.1.3. *Define the* λ-scalar ideal *of* $\mathfrak{g}$ *to be the largest ideal of* $\mathfrak{g}$ *contained in* $\ker(\lambda)$*, and denote this ideal by* $\mathfrak{a}_\lambda$*. Also, set* $\mathcal{A}_\lambda := \mathfrak{a}_\lambda \cap \mathcal{L}$*, and define the* λ-scalar subgroup *of* G *to be* $A_\lambda := \exp(p\mathcal{A}_\lambda)$*. This is a closed, isolated, normal subgroup of* G*.*

Note: Using Lemma 5.1.2, we see that for any choice of polarisation $\mathfrak{b}$ of $\mathfrak{g} \otimes_{\mathbb{Q}_p} K$ at λ, $\mathfrak{a}_\lambda \subseteq \mathfrak{b}$.

Lemma 6.1.4. *Suppose* G *is nilpotent, and let* $P = \text{Ann}_{KG}\widehat{D(\lambda)}$*, then* $P^\dagger := \{g \in G : g - 1 \in P\}$ *is equal to the* λ*-scalar subgroup* A_λ*. It follows that* P *is faithful if and only if the restriction of* λ *to* $Z(\mathfrak{g})$ *is injective.*

Proof. Firstly, since $\mathfrak{a}_\lambda \subseteq \mathfrak{b}$ and $\lambda(\mathfrak{a}_\lambda) = 0$, we see that $\mathfrak{a}_\lambda\widehat{U(\mathcal{L})}_K\widehat{D(\lambda)} = \mathfrak{a}_\lambda\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B})}_K} K_\lambda = \widehat{U(\mathcal{L})}_K\mathfrak{a}_\lambda \otimes_{\widehat{U(\mathcal{B})}_K} K_\lambda \subseteq \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B})}_K} \mathfrak{a}_\lambda K_\lambda = 0$.

So since $A_\lambda - 1 \subseteq \mathfrak{a}_\lambda\widehat{U(\mathcal{L})}_K$, it is clear that $A_\lambda - 1\widehat{D(\lambda)} = 0$. Hence $A_\lambda - 1 \subseteq P$, i.e. $A_\lambda \subseteq P^\dagger$.

Now, since $T = P^\dagger$ is a closed, normal subgroup of G, $\mathcal{T} := \frac{1}{p}\log(T)$ is an ideal of $\mathcal{L}$, and it contains $\frac{1}{p}\log(A_\lambda) = \mathcal{A}_\lambda$. Also, since $(T-1)\widehat{D(\lambda)} = 0$, it follows that $\mathcal{T}\widehat{D(\lambda)} = 0$, which is only possible if $\mathcal{T} \subseteq \mathcal{B}$ and $\lambda(\mathcal{T}) = 0$.

Setting $\mathfrak{t} := \mathcal{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $\mathfrak{t}$ is an ideal of $\mathfrak{g}$, $\mathfrak{a}_\lambda \subseteq \mathfrak{t}$ and $\lambda(\mathfrak{t}) = 0$. So by the definition of $\mathfrak{a}_\lambda$, this means that $\mathfrak{a}_\lambda = \mathfrak{t}$, and hence $\mathcal{A}_\lambda = \mathfrak{t} \cap \mathcal{L} = \mathcal{T}$. It follows immediately that

$A_\lambda = T$.

Finally, since G is nilpotent, $\mathcal{L}$ is nilpotent, and thus if $\mathcal{A}_\lambda \neq 0$, then it must have non-trivial intersection with $Z(\mathfrak{g})$. So since P is faithful if and only if $A_\lambda = 1$ (i.e. if and only if $\mathfrak{a}_\lambda = 0$), and any subspace of $Z(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, it follows that P is faithful precisely when nothing in $Z(\mathfrak{g})$ is sent to zero under λ, i.e. $\lambda|_{Z(\mathfrak{g})}$ is injective. $\square$

This lemma is useful to know, because it implies that for any Dixmier annihilator P, $P^\dagger$ is a closed, isolated normal subgroup of G, and hence we can replace G by $G_0 = \frac{G}{P^\dagger}$, which is still a nilpotent, uniform group, and $P_0 = \frac{P}{(P^\dagger-1)KG}$ becomes a faithful Dixmier annihilator.

Now, let us suppose that G is nilpotent and abelian-by-procyclic with principal subgroup $H := C_G(Z_2(G))$ and procyclic element X. Setting $\mathcal{H} = \frac{1}{p}\log(H)$ as the $\mathbb{Z}_p$-Lie algebra of H, we want to examine the action of $\widehat{U(\mathcal{H})_K}$ on $\widehat{D(\lambda)}$.

From now on, we will always assume that $\lambda|_{Z(\mathfrak{g})}$ is injective, and note that since $\mathfrak{h} := \mathcal{H} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an abelian ideal of codimension 1 in $\mathfrak{g}$, this means that $\mathfrak{h}$ is a polarisation for $\mathfrak{g}$ at λ.

Using Proposition 5.3.2, we see that $\widehat{D(\lambda)}$ is isomorphic as a K-vector space to $K\langle x\rangle$, where $x := p\log(X)$.

Notation: In this chapter, for each $u \in \mathfrak{g}$, we write $\text{ad}(u)(v) := [v, u]$ as opposed to the more conventional $\text{ad}(u)(v) = [u, v]$.

Lemma 6.1.5. *Let $\partial := \frac{d}{dx} \in \text{End}_K K\langle x\rangle$. Then for each $u \in \mathfrak{h}$, u acts on $\widehat{D(\lambda)} = K\langle x\rangle$ by $\sum\limits_{n\geq 0}\frac{1}{n!}\lambda(ad(x)^n(u))\partial^n$, which is a finite sum since $\mathfrak{g}$ is nilpotent.*

Proof. It suffices to prove that for each $j \geq 0$, $u \cdot x^j = \sum\limits_{n\geq 0} \frac{1}{n!}\lambda(ad(x)^n(u))\partial^n(x^j) = \sum\limits_{n\geq 0} \binom{j}{n}\lambda(ad(x)^n(u))x^{j-n}$.

We will prove this by induction on j:

For the base case, suppose that $j = 0$. Note that we may take the element $1 \in F\langle x\rangle$ to be the basis vector for F_λ, i.e. for all $u \in \mathcal{H}$, $u \cdot 1 = \lambda(u) \cdot 1$, thus $u \cdot x^0 = u \cdot 1 = \lambda(u) = \sum\limits_{n\geq 0} \binom{0}{n}\lambda(ad(x)^n(u))x^{j-n}$.

So, suppose the result holds for some $j \geq 0$. Then:

$$u \cdot x^{j+1} = ux \cdot x^j = [u, x] \cdot x^j + xu \cdot x^j$$

$$= \sum_{0\leq n\leq j} \binom{j}{n}\lambda(ad(x)^n([u, x]))x^{j-n} + x \sum_{0\leq n\leq j} \binom{j}{n}\lambda(ad(x)^n(u))x^{j-n}$$

$$= \sum_{0\leq n\leq j} \binom{j}{n}\lambda(ad(x)^{n+1}(u))x^{j-n} + \sum_{0\leq n\leq j} \binom{j}{n}\lambda(ad(x)^n(u))x^{j+1-n}$$

$$= \lambda(ad(x)^{j+1}(u)) + \sum_{1\leq n\leq j} (\binom{j}{n} + \binom{j}{n-1})\lambda(ad(x)^n(u))x^{j+1-n} + \lambda(u)x^{j+1}$$

$$= \sum_{0\leq n\leq j+1} \binom{j+1}{n}\lambda(ad(x)^n(u))x^{j+1-n}.$$ □

So, for each $u \in \mathcal{H}$, the action of u on $K\langle x\rangle$ is given by a polynomial f_u in ∂, and $f_u(0) = \lambda(u) \in \mathcal{O}$. This will become very important in our proof of Theorem E.

6.2 Dixmier-Standard groups

In this section, we will define a class of groups for which Conjecture 1.1.3 is satisfied for faithful, primitive ideals, and ultimately prove the corollary to Theorem D stated in the introduction. For now, we assume only that G is nilpotent.

Proposition 6.2.1. *Given a primitive ideal P of KG, there exists $m \in \mathbb{N}$, finite extensions $F_1, \cdots, F_r/K$ and $\mathbb{Q}_p$-linear maps $\lambda_i : \mathfrak{g} \to F_i$ with $\lambda_i(p^m\mathcal{L}) \subseteq \mathcal{O}_{F_i}$ for each $i = 1, \cdots, r$, such that:*

$$P \cap KG^{p^m} = \mathrm{Ann}_{KG^{p^m}} \widehat{D(\lambda_1)}_{F_1} \cap \cdots \cap \mathrm{Ann}_{KG^{p^m}} \widehat{D(\lambda_r)}_{F_r}.$$

Proof. Using Proposition 2.9.2, if P is primitive, then for any sufficiently high $n \in \mathbb{N}$, there is a primitive ideal Q of $D_{p^n} = \widehat{U(p^n\mathcal{L})}_K * \frac{G}{G^{p^n}}$ such that $Q \cap KG = P$, and hence $Q \cap KG^{p^n} = P \cap KG^{p^n}$.

Let $I = Q \cap \widehat{U(p^n\mathcal{L})}_K$, then using Lemma A.1.2 we see that I is a semiprimitive ideal of $\widehat{U(p^n\mathcal{L})}_K$, so choose primitive ideals $J_1, J_2, \cdots, J_r$ of $\widehat{U(p^n\mathcal{L})}_K$ such that $I = J_1 \cap J_2 \cap \cdots \cap J_r$.

Since each J_i is primitive, it follows from Theorem D that there exists $m \geq n$ such that for each i, $J_i \cap \widehat{U(p^m\mathcal{L})}_K = \mathrm{Ann}_{\widehat{U(p^m\mathcal{L})}_K} \widehat{D(\lambda_i)}_{F_i}$, for F_i/K a finite extension, $\lambda_i : \mathfrak{g} \to F_i$ $\mathbb{Q}_p$-linear with $\lambda_i(\mathcal{L}) \subseteq \mathcal{O}_{F_i}$. Thus:

$$P \cap KG^{p^m} = Q \cap KG^{p^m} = I \cap KG^{p^m} = (J_1 \cap \widehat{U(p^m\mathcal{L})}_K) \cap \cdots \cap (J_r \cap \widehat{U(p^m\mathcal{L})}_K) \cap KG^{p^m}$$

is an intersection of Dixmier annihilators as required. □

We want to show that all faithful, primitive ideals of KG are centrally generated, so using the previous proposition we see that it is useful to assume that this condition holds for Dixmier annihilators.

Definition 6.2.2.

- *We say that a nilpotent, uniform pro-p group G is a* weakly Dixmier standard group *if for any finite extension F/K, and any $\mathbb{Q}_p$-linear form $\lambda : \mathfrak{g} \to F$ with $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$ and $\lambda|_{Z(\mathfrak{g})}$ injective, the Dixmier annihilator $P = I(\lambda) \cap KG = \mathrm{Ann}_{KG} \widehat{D(\lambda)}_F$ is controlled by $Z(G)$.*
- *We say that G is a* Dixmier standard group *if G^{p^n} is weakly Dixmier standard for all $n \geq 0$.*

We believe that all nilpotent groups are Dixmier standard, but we will not prove this here. Until the end of this section, we will assume that G is a Dixmier standard group.

Theorem 6.2.3. *Let G be a uniform, Dixmier standard group with centre Z, and let P be a faithful, primitive ideal of KG. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $P \cap KG^{p^n}$ is controlled by Z^{p^n}.*

Proof. Using Proposition 6.2.1, we see that for some $N \in \mathbb{N}$, there are finite extensions $F_1, \cdots, F_r$ and $\mathbb{Q}_p$-linear maps $\lambda_i : \mathfrak{g} \to F_i$ with $\lambda(p^N \mathcal{L}) \subseteq \mathcal{O}_{F_i}$ such that $P \cap KG^{p^N} = \text{Ann}_{KG^{p^N}} \widehat{D(\lambda_1)}_{F_1} \cap \cdots \cap \text{Ann}_{KG^{p^N}} \widehat{D(\lambda_r)}_{F_r}$.

For each $i = 1, \cdots, r$, set $J_i := \text{Ann}_{KG^{p^N}} \widehat{D(\lambda_i)}_{F_i}$ for convenience, and by Lemma 6.1.1, the J_i are prime J-ideals of KG^{p^N}. Thus $P \cap KG^{p^N}$ is semiprime and $J_1, \cdots, J_r$ are the minimal primes above $P \cap KG^{p^N}$, hence they are all G-conjugate by [1, Lemma 5.4(b)].

Also, since P is faithful, $P \cap KG^{p^N}$ is faithful, so $J_1^\dagger \cap \cdots \cap J_r^\dagger = P^\dagger = 1$. But since $J_1^\dagger, \cdots, J_r^\dagger$ are G-conjugate and G is orbitally sound by [1, Proposition 5.9], this means that the subgroup 1 must have finite index in $J_i^\dagger$ for each i, which means that they are finite. But G is torsionfree, so $J_i^\dagger = 1$ for all i, i.e. $J_1, \cdots, J_r$ are faithful.

So since $J_i = \text{Ann}_{KG^{p^N}} \widehat{D(\lambda_i)}_{F_i}$ is faithful, it follows from Lemma 6.1.4 that λ_i is injective when restricted to $Z(\mathfrak{g})$.

So since G is Dixmier standard, G^{p^n} is weakly Dixmier standard for all $n \geq N$, and hence $J_i \cap KG^{p^n}$ is controlled by $Z(G^{p^n})$ for each i, and using [1, Lemma 8.4(a)], $Z(G^{p^n}) = Z(G) \cap G^{p^n} = Z^{p^n}$.

Therefore, setting $B_{i,n} := J_i \cap KG^{p^n} = \text{Ann}_{KG^{p^n}} \widehat{D(\lambda_i)}$, $B_{i,n} = (B_{i,n} \cap KZ^{p^n})KG^{p^n}$ for each i, so using Lemma B.2.1:

$$P \cap KG^{p^n} = B_{1,n} \cap \cdots \cap B_{r,n} = (B_{1,n} \cap KZ^{p^n})KG^{p^n} \cap \cdots \cap (B_{r,n} \cap KZ^{p^n})KG^{p^n}$$

$$= (B_{1,n} \cap \cdots \cap B_{r,n} \cap KZ^{p^n})KG^{p^n} = (P \cap KZ^{p^n})KG^{p^n}$$

Hence $P \cap KG^{p^n}$ is controlled by Z^{p^n} as required. □

So to prove that a primitive ideal P is standard, it remains to extend this result from KG^{p^n} to KG:

Proposition 6.2.4. *Let G be a Dixmier standard group, and let $P_1 \subseteq P_2$ be faithful, primitive ideals of KG. Then there exists $n \in \mathbb{N}$ such that $P_1 \cap KG^{p^n} = P_2 \cap KG^{p^n}$. It follows that if P is a faithful, primitive ideal of KG then P is maximal.*

Proof. Using Theorem 6.2.3, we see that there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n_1 \geq N_1$, $n_2 \geq N_2$, $P_i \cap KG^{p^{n_i}}$ is controlled by $Z(G)^{p^{n_i}}$ for both i. So choose $n \geq \max\{N_1, N_2\}$ and we have that $P_1 \cap KG^{p^n}, P_2 \cap KG^{p^n}$ are controlled by $Z(G)^{p^n}$.

Since P_1 is primitive, it is weakly rational, so using Lemma 2.11.3 we see that $P_1 \cap KZ(G)$ is a maximal ideal of $KZ(G)$. So since $P_1 \cap KZ(G) \subseteq P_2 \cap KZ(G)$, we have that $P_1 \cap KZ(G) = P_2 \cap KZ(G)$, and hence $P_1 \cap KZ(G)^{p^n} = P_2 \cap KZ(G)^{p^n}$. Therefore:

$$P_1 \cap KG^{p^n} = (P_1 \cap KZ(G)^{p^n})KG^{p^n} = (P_2 \cap KZ(G)^{p^n})KG^{p^n} = P_2 \cap KG^{p^n}.$$

Finally, given a faithful, primitive ideal P of KG, let Q be a maximal ideal of KG containing P. Since P and Q are primitive, they are prime J-ideals, so since P is faithful, Q is faithful by Lemma 2.11.4. Thus, by the above, there exists $n \in \mathbb{N}$ such that $P \cap KG^{p^n} = Q \cap KG^{p^n}$ is controlled by $Z(G)^{p^n}$.

But $P \cap KZ(G)$ is prime in $KZ(G)$, so $P \cap KG^{p^n} = (P \cap KZ(G)^{p^n})KG^{p^n}$ is prime in KG^{p^n} by Theorem B.2.6. So since $P \cap KG^{p^n} = Q \cap KG^{p^n}$, it follows from [30, Theorem 16.6(iii)] that $P = Q$, and hence P is maximal. $\square$

Now, we can finally prove Corollary 1.4.5 to Theorem D from the introduction:

Theorem 6.2.5. *Let G be a uniform, Dixmier standard group. Then all faithful, primitive ideals of KG are maximal and standard.*

Proof. Let P be a faithful, primitive ideal of KG, and let $Z = Z(G)$. We know from Proposition 6.2.4 that P is maximal, so we just need to prove that P is controlled by Z.

Using Theorem D, we know that there exists $n \in \mathbb{N}$ such that $P \cap KG^{p^n}$ is controlled by Z^{p^n}, and hence is prime in KG^{p^n} by Theorem B.2.6. So let $J := (P \cap KG^{p^n})KG$, then using Lemma A.1.1 we see that J is a semiprime ideal of KG, and P is minimal prime above J.

Let $Q := P \cap KZ$, then Q is prime in KZ, so QKG is prime in KG by Theorem B.2.6. And since $P \cap KG^{p^n} = (P \cap KZ^{p^n})KG^{p^n}$, we have that:

$$J = (P \cap KG^{p^n})KG = (P \cap KZ^{p^n})KG \subseteq QKG.$$

But clearly $QKG \subseteq P$, so since QKG is prime and P is minimal prime above J, it follows that $P = QKG = (P \cap KZ)KG$, and hence P is controlled by Z as required. $\square$

So to prove Theorem E, it remains to prove that nilpotent, abelian-by-procyclic groups are Dixmier standard.

6.3 Almost-Polynomial maps

In Appendix D, we give an overview of rigid analytic geometry – the p-adic analogue of differential geometry. We will now prove some technical results from rigid geometry that will be essential in the proof of our main theorem. Recall from this appendix the definition of an affinoid K-algebra A, and recall that it carries a complete, Zariskian filtration w_A.

Lemma 6.3.1. *Let $\phi : K\langle u_1, \cdots, u_d\rangle \to A$ be a map of affinoid algebras, and let $a \in A$ lie in the image of ϕ. Then there exists $m \in \mathbb{N}$ such that the image of ϕ inside A contains the affinoid K-subalgebra topologically generated by $\pi^m a$.*

Proof. We know that $a = \phi(r)$ for some $r \in K\langle u_1, \cdots, u_d\rangle$, so choose $m \in \mathbb{N}$ such that $w_{\inf}(\pi^m r) \geq 0$ and $w_A(\pi^m a) \geq 0$.

Then there exist K-algebra maps $\Theta_1 : K\langle X\rangle \to K\langle u_1, \cdots, u_d\rangle$ and $\Theta_2 : K\langle X\rangle \to A$ sending X to $\pi^m r$ and $\pi^m a$ respectively, and it is clear that $\Theta_2 = \phi\Theta_1$. Therefore, the image of ϕ must contain the image of Θ_2, which is precisely the affinoid K-algebra topologically generated by $\pi^m a$ as required. $\square$

Now we will make a special definition:

Definition 6.3.2. *A map $\phi : K\langle u_1, \cdots, u_d\rangle \to K\langle t\rangle$ of K-algebras is called an* almost-polynomial map *if*

- *$\phi(u_i) \in K[t]$ for each i,*
- *$\phi(u_i)(0) \in \mathcal{O}$ for each i,*
- *t is contained in the image of ϕ.*

Using Lemma 6.3.1, we see that if ϕ is an almost-polynomial map then there exist $m \in \mathbb{N}$ such that $\mathrm{im}(\phi)$ contains $K\langle \pi^m t\rangle$.

Now, following [26, 5.1.2], for each non-constant polynomial $g(t) := b_0 + b_1 t + \cdots + b_n t^n \in K[t]$ with $b_0 \in \mathcal{O}$, define

$$\chi(g) := \max_{1 \leq j \leq n} -\tfrac{v_\pi(b_j)}{j}.$$

Lemma 6.3.3. *Let $g(t) \in K[t]$ be a polynomial with $g(0) \in \mathcal{O}$, and let $\beta \in K$ with $v_\pi(\beta) > 0$. Then $\chi(\beta g) < \chi(g)$. It follows that if $f_1(t), \cdots, f_d(t) \in K[t]$ are polynomials with $f_i(0) \in \mathcal{O}$ for each i, then setting $\mu_i := \max_{1 \leq j \leq d} \chi(\beta^i f_j)$ for each $i \geq 0$, we have that $\mu_0 > \mu_1 > \mu_2 > \cdots$.*

Proof. Suppose $g(t) = b_0 + b_1 t + \cdots + b_n t^n$, with $b_0 \in \mathcal{O}$, $b_n \neq 0$. Then by definition;

$$\chi(\beta g) = \max_{1 \leq j \leq n} -\tfrac{v_\pi(\beta b_j)}{j} = \max_{1 \leq j \leq n} -\tfrac{v_\pi(b_j)}{j} - \tfrac{v_\pi(\beta)}{j}.$$

So since $v_\pi(\beta) > 0$, this maximum is strictly less than $\max_{1 \leq j \leq n} -\frac{v_\pi(b_j)}{j} = \chi(g)$.

To prove the second statement, it suffices to prove that $\mu_0 > \mu_1$ and apply induction. So suppose $\mu_1 = \chi(\beta f_i)$ and $\mu_0 = \chi(f_j)$, then we have that $\mu_1 = \chi(\beta f_i) < \chi(f_i) \leq \chi(f_j) = \mu_0$. □

Recall from [26, Theorem 5.1.2] that if we assume that $b_0 \neq 0$, then the set $X(g) := \{\alpha \in \overline{K} : v_\pi(g(\alpha)) \geq 0\}$ is an affinoid subdomain of $\mathbb{A}_K^{1,an} = \overline{K}$, whose G-connected component about 0 is the set $\mathbb{D}^1_{\chi(g)} = \{\alpha \in \overline{K} : v_\pi(\alpha) \geq \chi(g)\}$.

Furthermore, if $b_0 = 0$, it is clear that $\chi(g) = \chi(1 + g)$, and that $X(g) = \{\alpha \in \overline{K} : v_\pi(g(\alpha)) \geq 0\} = \{\alpha \in \overline{K} : v_\pi(1 + g(\alpha)) \geq 0\} = X(1 + g)$, so we reach the same conclusion.

Lemma 6.3.4. *Let $\phi : K\langle u_1, \cdots, u_d\rangle \to K\langle t\rangle$ be an almost-polynomial map, and let $f_i(t) := \phi(u_i) \in K[t]$ for each i. Setting $Y := \{\alpha \in \overline{K} : v_\pi(f_i(\alpha)) \geq 0$ for each $i\}$, we have that:*

i. Y is an affinoid subdomain of $\mathbb{A}_K^{1,an}$, in fact it is a disjoint union of closed discs.

ii. The image of ϕ in $K\langle t\rangle$ is isomorphic to the Tate algebra in one variable over a finite extension of K.

iii. The image of ϕ is contained in the set of all functions in $K\langle t\rangle$ converging on Y.

Proof. Set $A := \mathrm{im}(\phi)$, then since $K\langle t\rangle$ is affinoid, it follows from Lemma 6.3.1 that there exists $m \in \mathbb{N}$ such that A contains $K\langle \pi^m t\rangle$, and we may of course choose m to be arbitrarily large.

If we set $B := K\langle \pi^m t\rangle\langle t_1, \cdots, t_d\rangle/(t_i - f_i(t) : i = 1, \cdots, d)$, then there is a natural surjection from B to A, identical on $K\langle \pi^m t\rangle$, which sends t_i to $f_i(t)$. This gives rise to a closed embedding of affinoid varieties $\mathrm{Sp}\ A \hookrightarrow \mathrm{Sp}\ B$.

i. Since each f_i is a polynomial, it is clear that there exists $N > 0$ such that if $\alpha \in \overline{K}$ and $v_\pi(\alpha) < -N$ then $v_\pi(f_i(\alpha)) < 0$ for all i. So by choosing $m > N$ we may assume that

$$Y = \{\alpha \in \mathbb{D}^1_{-m} : v_\pi(f_i(\alpha)) \geq 0\}.$$

Hence using [11, Lemma 3.3.10(i)] and the proof of [11, Proposition 3.3.11], we see that $Y = \mathrm{Sp}\ B$, and hence Y is an affinoid subdomain of $\mathbb{A}^{1,an}_K$, and B is the set of all analytic functions converging on Y. Moreover, we see using [26, Corollary 5.1.2] that $X(f_i) := \{\alpha \in \overline{K} : v_\pi(f_i(\alpha)) \geq 0\}\}$ is a disjoint union of closed discs for each i, and hence so is $Y = X(f_1) \cap \cdots \cap X(f_d)$ as required.

ii. Since $\mathrm{Sp}\ A \to Y$ is a closed embedding of affinoid varieties, we see that $\mathrm{Sp}\ A$ is a disjoint union of discs and points. But since $A \subseteq K\langle t\rangle$, it follows that A is a domain, and hence $\mathrm{Sp}\ A$ is connected, so it is either a disc or a point.

But since A contains $K\langle \pi^m t\rangle$, it has infinite dimension over K, hence $\mathrm{Sp}\ A$ cannot be a point, so it is a disc, and hence A is the set of all functions in $K\langle t\rangle$ converging on a disc in $\mathbb{A}_K^{1,an}$, i.e. a Tate algebra in one variable over a finite extension of K.

iii. Notice that $K\langle u_1, \cdots, u_d\rangle$ is precisely the set of functions converging on the closed unit disc $\mathbb{D}_0^d$, so it follows that the image of $K\langle u_1, \cdots, u_d\rangle$ under ϕ is contained in the set of functions converging on $\{\alpha \in \overline{K} : (f_1(\alpha), \cdots, f_d(\alpha)) \in \mathbb{D}_0^d\} = \{\alpha \in \overline{K} : v_\pi(f_i(\alpha)) \geq 0 \text{ for all } i\} = Y$ as required. □

The following result will be essential later when proving a control theorem.

Proposition 6.3.5. *Let $\phi : K\langle u_1, \cdots, u_d\rangle \to K\langle t\rangle$ be an almost-polynomial map, and let $f_i(t) := \phi(u_i) \in K[t]$ for each i. Then there exists $k \in \{1, \cdots, d\}$ such that* $\exp(pf_k(t))$ *does not lie in* $\phi(K\langle pu_1, \cdots, pu_d\rangle)$.

Proof. We may assume that K contains an element ω such that $\omega^{p-1} = p$, and note that $v_p(\omega) = \frac{1}{p-1}$. If we prove the result in this case, then it follows generally, since if $K' := K(\sqrt[p-1]{p})$ and we can find k such that $\exp(pf_k)$ does not lie in the image of $K'\langle pu_1, \cdots, pu_d\rangle$, then it will also not lie in the image of $K\langle pu_1, \cdots, pu_d\rangle$.

For each $j \geq 0$ let $Y_j := \{\alpha \in \overline{K} : v_\pi(\omega^j f_i(\alpha)) \geq 0 \text{ for all } i\}$. Then using Lemma 6.3.4(*i*, *iii*), we see that Y_j is an affinoid subdomain of $\mathbb{A}_K^{1,an}$, and that $\phi(K\langle \omega^j u_1, \cdots, \omega^j u_d\rangle)$ is contained in the set of all functions in $K\langle t\rangle$ converging on Y_j.

Moreover, set $\mu_j := \max_{i=1,\cdots,d} \chi(\omega^j f_i)$, and using [26, Theorem 5.1.2] we see that the G-connected component of Y_j about zero is the closed disc $\mathbb{D}^1_{\mu_j}$. So it remains to prove that for some k, $\exp(pf_k) = \exp(\omega^{p-1} f_k)$ does not converge on $\mathbb{D}^1_{\mu_{p-1}}$, and thus does not converge on Y_j and cannot lie in the image of $K\langle pu_1, \cdots, pu_d\rangle = K\langle \omega^{p-1}u_1, \cdots, \omega^{p-1}u_d\rangle$.

Using [24, Example 0.4.1], the disc of convergence for exp is $\{\lambda \in \overline{K} : v_p(\lambda) > \frac{1}{p-1}\}$, so it remains only to find $\alpha \in \mathbb{D}^1_{\mu_{p-1}}$ such that $v_p(pf_k(\alpha)) \leq \frac{1}{p-1}$ for some k, i.e.

$v_p(f_k(\alpha)) \leq \frac{1}{p-1} - 1$.

For each $j \geq 0$, fix $i_j = 1, \cdots, d$ such that $\chi(\omega^j f_{i_j}) = \mu_j$. Using Lemma 6.3.3 we see that $\mu_0 > \mu_1 > \mu_2 > \cdots$, and we know that for each j, the G-connected component of $X(\omega^{j-1} f_{i_{j-1}})$ about zero is $\mathbb{D}^1_{\mu_{j-1}}$, which means in particular that $\mathbb{D}^1_{\mu_j} \not\subseteq X(\omega^{j-1} f_{i_{j-1}})$ since $\mu_{j-1} > \mu_j$, and $\mathbb{D}^1_{\mu_j}$ is G-connected. So for each j, we may choose $\alpha_j \in \mathbb{D}^1_{\mu_j} \backslash X(\omega^{j-1} f_{i_{j-1}})$.

But $X(\omega^{j-1} f_{i_{j-1}}) = \{\alpha \in \overline{K} : v_\pi(f_{i_{j-1}}(\alpha)) \geq -(j-1)\} = \{\alpha \in \overline{K} : v_p(f_{i_{j-1}}(\alpha)) \geq -v_p(\pi)(j-1)\}$, so this means that $v_p(f_{i_{j-1}}(\alpha_j)) < -v_p(\pi)(j-1)$.

So, finally, choose $j = \frac{1}{v_p(\pi)} \geq p - 1$, and let $k := i_{j-1}$. Then

$$-(j-1)v_p(\pi) = v_p(\pi) - 1 \leq \frac{1}{p-1} - 1,$$

so $\alpha_j \in \mathbb{D}^1_{\mu_j} \subseteq \mathbb{D}^1_{\mu_{p-1}}$ and $v_p(f_k(\alpha_j)) = v_p(f_{i_{j-1}}(\alpha_j)) < -(j-1)v_p(\pi) \leq \frac{1}{p-1} - 1$ as required. $\square$

Now, we will return to the action on the affinoid Dixmier module $\widehat{D(\lambda)}$. Since $\widehat{U(\mathcal{H})_K} \cong K\langle u_1, \cdots, u_d \rangle$, we are considering the action of the Tate algebra in d variables on $\widehat{D(\lambda)}$. We will be assuming that $\widehat{D(\lambda)} = K\langle x \rangle$, and using Lemma 6.1.5, we see that the action of each $u \in \text{Span}_{\mathbb{Q}_p}\{u_1, \cdots, u_d\}$ is given by a polynomial in $\partial = \frac{d}{dx}$ with coefficients in K.

Theorem 6.3.6. *Suppose that $K\langle u_1, \cdots, u_d \rangle$ acts on $K\langle t \rangle$ by K-linear endomorphisms, where each u_i acts by $f_i(\partial)$ for some $f_i(x) \in K[x]$ with $f_i(0) \in \mathcal{O}$. Let A denote the image of this action in $End_K K\langle t \rangle$, and we suppose that $\partial \in A$, then:*

i. A is isomorphic to a direct product of rings $A_1 \times \cdots \times A_m$.

ii. For each i, there is a finite extension F_i/K such that $A_i \cong F_i$ or $F_i\langle s_i\rangle$.

iii. Not all the A_i are fields.

iv. If $A_i = F_i\langle s_i\rangle$ then the natural surjection $F_i\langle u_1, \cdots, u_d\rangle \to A_i$ is an almost polynomial map.

Proof. By definition, A is an affinoid K-algebra, so using Lemma 6.3.1 we see that A contains $K\langle \pi^m \partial\rangle$ for some $m \in \mathbb{N}$. Consider the affinoid algebra

$$B := K\langle \pi^m s\rangle\langle \zeta_1, \cdots, \zeta_d\rangle/(\zeta_i - f_i(s) : i = 1, \cdots, d).$$

Then there is a natural map from B to $\text{End}_K K\langle t\rangle$ sending s to ∂ and ζ_i to $f_i(\partial)$ for each i, and it is clear that the image of this map is A. This gives rise to a natural closed embedding of affinoid varieties Sp $A \hookrightarrow$ Sp B.

After choosing m sufficiently high and applying [11, Lemma 3.3.10] and the proof of [11, Proposition 3.3.11], we see that Sp $B \cong \{\alpha \in \overline{K} : v_\pi(f_i(\alpha)) \geq 0 \text{ for all } i\}$ as affinoid varieties, and using Lemma 6.3.4(i), we see that this is a disjoint union of closed discs, so Sp $B = U_1 \sqcup \cdots \sqcup U_r$, for U_i a closed disc in $\mathbb{A}_K^{1,an}$.

Thus the image of the closed embedding Sp $A \to$ Sp B must be a disjoint union of discs and points in $\mathbb{A}_K^{1,an}$, so:

$$\text{Sp } A = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_t$$

where each V_j is a disc or a point. Therefore, since A can be realised as the ring of analytic functions on Sp A, if we set A_i as the ring of analytic functions on V_i, parts i and ii follow immediately.

Also, if all the A_i are finite extensions of K, then A must be finite dimensional over K, but this is impossible since A contains $K\langle \pi^m \partial\rangle$. Hence not all the A_i are fields, so part *iii* follows.

In particular, there exists i such that V_i is not a point, hence it is a disc, and moreover, V_i is a connected component of $\text{Sp } B = \{\alpha \in \overline{K} : f_i(\alpha) \geq 0 \text{ for all } i\}$. So if we choose any $\alpha_i \in V_i \subseteq \overline{K}$, and set s as the coordinate of B, then we may consider s_i to be the image of $\pi^{r_i}(s - \alpha_i)$ under the surjection $B \twoheadrightarrow A_i$, where r_i is the radius of V_i. In other words, the image of s in A_i is $\pi^{-r_i}s_i + \alpha_i$.

Finally, for each $k = 1, \cdots, d$, the image of u_k in A_i is $f_k(\pi^{-r_i}s_i + \alpha_i) =: g_{i,k}(s_i)$, which is clearly a polynomial in s_i, and $g_{i,k}(0) = f_j(\pi^{-r_i}0 + \alpha_i) = f_k(\alpha_i)$. So since $v_\pi(f_k(\alpha_i)) \geq 0$, it follows that $f_k(\alpha_i) \in \mathcal{O}$, and hence $g_{i,k}(0) \in \mathcal{O}$.

Also, since the composition $K\langle u_1, \cdots, u_d\rangle \to A \to A_i$ is surjective, it follows that s_i lies in the image, and hence this is an almost-polynomial map as required. $\square$

6.4 Using the completion D_p

In this section, we will prove a control theorem for kernels of almost-polynomial maps. Throughout, we will assume that K contains a p'th root of unity ζ.

Fix A a free abelian pro-p group of rank d, let $\mathcal{A} := \frac{1}{p}\mathcal{L}_A$ be the associated $\mathbb{Z}_p$-Lie algebra of A, and let $\phi : \widehat{U(\mathcal{A})}_K \to K\langle t\rangle$ be an almost polynomial map.

Consider the crossed product $D_p = D_p(A) = \widehat{U(p\mathcal{A})}_K * \frac{A}{A^p}$ as defined in Chapter 2.9. This is a Banach completion of KA with respect to the extension of the dense embedding $\iota : KA^p \to \widehat{U(p\mathcal{A})}_K$ to KA, and there is a natural map $\tau : D_p \to \widehat{U(\mathcal{A})}_K$. Define $\phi' : D_p \to K\langle t\rangle$ and $\phi_A : KA \to K\langle t\rangle$ making the following diagram commute:

$$\begin{array}{ccc} D_p & \xrightarrow{\tau} & \widehat{U(\mathcal{A})}_K \\ {\scriptstyle\iota}\uparrow & \searrow^{\phi'} & \downarrow{\scriptstyle\phi} \\ KA & \xrightarrow{\phi_A} & K\langle t\rangle \end{array}$$

From now on, set $I = \ker(\phi')$, and let $Q := \ker(\phi_A) = I \cap KA$, and define:

$$U := \{a \in A : \phi(a) \in \phi(\widehat{U(p\mathcal{A})}_K)\}.$$

Proposition 6.4.1. *U is a proper open subgroup of A containing A^p.*

Proof. Since ϕ is a ring homomorphism, it is clear that for all $a, b \in U$, $ab \in U$, and since KA^p is a subalgebra of $\widehat{U(p\mathcal{A})}_K$, it is clear that $A^p \subseteq U$. Therefore, since $\frac{A}{A^p}$ is a finite group, and $\frac{U}{A^p}$ is closed under multiplication, it follows that U is a subgroup of A containing A^p, and hence it it open.

Finally, since ϕ is an almost polynomial map, it follows from Proposition 6.3.5 that there exists $u \in \mathcal{A}$ such that $\exp(p\phi(u)) = \phi(\exp(pu))$ does not lie in the image of $\widehat{U(p\mathcal{A})}_K$ under ϕ. But $a := \exp(pu) \in A$ and hence $a \notin U$. Therefore U is a proper subgroup of G. $\square$

Using this proposition, and Lemma B.1.2, we can fix a $\mathbb{Z}_p$-basis $\{a_1, \cdots, a_d\}$ for A such that $\{a_1, \cdots, a_r, a_{r+1}^p, \cdots, a_d^p\}$ is a $\mathbb{Z}_p$-basis for U, so $a_1, \cdots, a_r \in U$ and $a_{r+1}, \cdots, a_d \notin U$.

Since A is a free abelian pro-p group, we have that $\frac{A}{A^p}$ is a direct product of d copies of the cyclic group of order p, where the i'th copy is generated by the image of a_i in $\frac{A}{A^p}$. Setting $c_i := a_iA^p$, it follows from Lemma 2.9.3 that:

$$D_p = \widehat{U(p\mathcal{A})}_K * \langle c_1 \rangle * \cdots * \langle c_d \rangle \tag{6.1}$$

where $\overline{c_i}^r = \overline{c_i^r}$ for $0 \leq r < p$ and $\overline{c_i}^p = a_i^p$.

From now on, let $S := \phi(\widehat{U(p\mathcal{A})}_K) \subseteq K\langle t \rangle$, and let $B := \widehat{U(p\mathcal{A})}_K * \langle c_1 \rangle * \cdots * \langle c_r \rangle \leq D_p$. Then since $a_1, \cdots, a_r$ lie in U, the image of B under ϕ is S by the definition of U. Furthermore, since $KU = KA^p * \frac{U}{A^p} = KA^p * \langle c_1 \rangle * \cdots * \langle c_r \rangle$, it is clear that $KU \subseteq B$.

Let $J := I \cap B \trianglelefteq B$ be the kernel of the restriction of ϕ' to B, and let $I' := JD_p$, an ideal of D_p contained in I.

Lemma 6.4.2. *I is a prime ideal of D_p, minimal prime above I'.*

Proof. Since $D_p/I \cong \operatorname{im}(\phi') \leq K\langle t \rangle$, it is clear that I is a prime ideal of D_p. And since D_p is a crossed product of B with a finite group, it follows from Lemma A.1.1(ii) that I is minimal prime above $I' = (I \cap B)D_p$. $\square$

We are now ready to prove the key result needed in the proof of Theorem E. First, we need a small result from Galois theory [37].

Lemma 6.4.3. *Let F be a field of characteristic 0, containing a p'th root of unity ζ. Let $r \in F$, and suppose that r has no p'th root in F. Choose a p'th root $\alpha \in \overline{F}$ of r, and let $F' := F(\alpha)$. Then if $\beta \in F'$ and $\beta^p \in F$ then $\beta = c\alpha^m$ for some $c \in F$, $0 \leq m < p$.*

Proof. Since F' is the splitting field for the polynomial $x^p - r$ over F, it is clear that F' is a Galois extension of F. So since $[F' : F] = p$ this means that $Gal(F'/F)$ has order p.

In fact, if we consider the element $\sigma \in Gal(F'/F)$ sending α to $\zeta\alpha$, then $Gal(F'/F)$ is cyclic of order p, generated by σ.

The result is clear if $\beta \in F$, so assume $\beta \notin F$ and $\beta^p \in F$. Then β is a root of the polynomial $x^p - \beta^p \in F[x]$, and hence $\sigma(\beta)$ is also a root. Therefore $\sigma(\beta) = \zeta^m\beta$ for some $0 \leq m < p$, so $\sigma(\alpha^{-m}\beta) = \zeta^{-m}\alpha^{-m}\zeta^m\beta = \alpha^{-m}\beta$.

But since σ generates $Gal(F'/F)$, it follows that $\alpha^{-m}\beta$ is fixed by the Galois group, so since F'/F is a Galois extension, this means that $c := \alpha^{-m}\beta \in F$ and $\beta = c\alpha^m$ as required. $\square$

For clarity, we will introduce/recall the following data:

- $I = \ker(\phi') \trianglelefteq D_p$.

- $Q = I \cap KA \trianglelefteq KA$.

- $U = \{a \in A : \phi(a) \in \phi(\widehat{U(p\mathcal{A})}_K)\} = \langle a_1, \cdots, a_r, a_{r+1}^p, \cdots, a_d^p \rangle$.

- $B = \widehat{U(p\mathcal{A})}_K * \langle c_1 \rangle * \cdots * \langle c_r \rangle \leq D_p$.

- $S = \phi(\widehat{U(p\mathcal{A})}_K) = \phi'(B) \leq K\langle t \rangle$.

- $J = I \cap B \trianglelefteq B$.

- $I' = JD_p \trianglelefteq D_p$.

- $R := D_p/I'$

Proposition 6.4.4. *R is a domain.*

Proof. Since $D_p = B*\langle c_{r+1}\rangle*\cdots*\langle c_d\rangle$ and $I' = JD_p$, it follows from Lemma A.1.1(iii) that $R \cong S * \langle c_{r+1} \rangle * \cdots * \langle c_d \rangle$, where $\overline{c}_i^p = \phi(a_i^p)$ for each i.

Since $\widehat{U(p\mathcal{A})}_K$ is a Tate algebra in d variables, we see using Lemma 6.3.4(ii) that S is isomorphic to a Tate algebra in one variable over a finite extension of K, hence it is an integrally closed domain by [11, Proposition 2.2.15], and using [30, Theorem 4.4] we see that R is a semiprime ring, i.e. it contains no nilpotents.

Therefore, we may consider the usual semisimple Artinian ring of quotients $Q(R)$ of R, which has the form:

$$Q(R) = Q(S) * \langle c_{r+1} \rangle * \cdots * \langle c_d \rangle,$$

where $Q(S)$ is the field of fractions of S. It remains to prove that $Q(R)$ is a field.

Let $T_0 := Q(S)$, and for each $i = 1, \cdots, d-r$, define $T_i := T_{i-1} * \langle c_{r+i} \rangle$, so that $T_{d-r} = Q(R)$.

Clearly T_0 is a field, so we will use induction to show that T_i is a field for each i, so in particular, $Q(R)$ is a field. So assume that for some $j > 0$, $T_0, \cdots, T_{j-1}$ are all fields:

Then since $T_j = T_{j-1} * \langle c_{r+j} \rangle$ where $\bar{c}_{r+j}^p = \phi(a_{r+j}^p) \in S$, it follows that

$$T_j = T_{j-1}[x]/(x^p - \phi(a_{r+j}^p)).$$

So we only need to show that the polynomial $x^p - \phi(a_{r+j}^p) \in T_{j-1}[x]$ is irreducible over the field T_{j-1}.

Since K contains a p'th root of unity, we see using standard Galois theory that this just means we need to show that this polynomial has no root in T_{j-1}, i.e. that there is no $b \in T_{j-1}$ such that $b^p = \phi(a_{r+j}^p)$.

Let us suppose for contradiction that $b_1^p = \phi(a_{r+j}^p)$ for some $b_1 \in T_{j-1} = T_{j-2} * \langle c_{r+j-1} \rangle$. Then since $\phi(a_{r+j}^p) \in S \subseteq T_{j-2}$ and T_{j-2} is a field containing K, it follows from Lemma 6.4.3 that $b_1 = b_2 \bar{c}_{r+j-1}^{k_1}$ for some $b_2 \in T_{j-2}$, $0 \leq k_1 < p$.

Therefore, $b_2^p = \phi((a_{r+j} a_{r+j-1}^{-k_1})^p) \in S$, so applying a second induction, for each $i > 0$, we can find integers $0 \leq k_1, \cdots, k_{i-1} < p$ and $b_i \in T_{j-i}$ such that $b_i^p = \phi((a_{r+j} a_{r+j-1}^{-k_1} a_{r+j-2}^{-k_2} \cdots a_{r+j-i+1}^{-k_{i-1}})^p) \in S$.

Taking $i = j$ we have that $b_j \in T_0 = Q(S)$ and $b_j^p \in S$. So since S is integrally closed, it follows that $b_j \in S \subseteq K\langle t \rangle$.

Now, $(b_j \phi(a_{r+j}^{-1} a_{r+j-1}^{k_1} \cdots a_{r+1}^{k_{j-1}}))^p = 1$, so it follows that there is a p'th root of unity $\zeta \in K$ such that:

$$\zeta b_j = \phi(a_{r+j} a_{r+j-1}^{-k_1} a_{r+j-1}^{-k_2} \cdots a_{r+1}^{-k_{j-1}}).$$

Therefore, since $b_j \in S$, this means that $\phi(a_{r+j}a_{r+j-1}^{-k_1}a_{r+j-1}^{-k_2}\cdots a_{r+1}^{-k_{j-1}}) \in S = \phi(\widehat{U(p\mathcal{A})}_K)$, or in other words $a_{r+j}a_{r+j-1}^{-k_1}a_{r+j-1}^{-k_2}\cdots a_{r+1}^{-k_{j-1}} \in U$ by the definition of U.

This is the required contradiction since $\{a_1, \cdots, a_r, a_{r+1}^p, \cdots, a_d^p\}$ is a $\mathbb{Z}_p$-basis for U, and each k_i is less than p. $\square$

Now we can prove the main result of this section:

Theorem 6.4.5. *Let $\phi : \widehat{U(\mathcal{A})}_K \to K\langle t\rangle$ be an almost-polynomial map. Then the kernel Q of the restriction of this map to KA is controlled by a proper open subgroup of A.*

Proof. If $I = \ker(\phi') \trianglelefteq D_p$, then we see that $R = D_p/(I \cap B)D_p$ is a domain using Proposition 6.4.4. But we know that I is minimal prime above $(I \cap B)D_p$ by Lemma 6.4.2, so it follows that $I = (I \cap B)D_p$.

Now, if $r \in Q = I \cap KA$, then since $KA = KU * \frac{A}{U}$, $r = \sum_{a \in A//U} s_a a$ for some $s_a \in KU \subseteq B$, where $A//U$ denotes a complete set of cost representatives for U in A.

Since $r \in I = (I \cap B)D_p$ it follows that $s_a \in I \cap B \cap KU = Q \cap KU$ for each a, and hence $r \in (Q \cap KU)KA$. But our choice of r was arbitrary, so this means that $Q = (Q \cap KU)KA$, i.e. Q is controlled by U, and U is a proper open subgroup of A by Proposition 6.4.1. $\square$

6.5 Proof of Theorem E

From now on, we will assume that G is a uniform, nilpotent, abelian-by-procyclic group, with principal subgroup H, and let $\mathcal{L}$, $\mathfrak{g}$, $\mathcal{H}$, $\mathfrak{h}$ be the usual $\mathbb{Z}_p$ and $\mathbb{Q}_p$-Lie algebras of these groups.

Proposition 6.5.1. *Let $P = \mathrm{Ann}_{KG}\widehat{D(\lambda)}$ be a faithful, Dixmier annihilator in KG. Then $P \cap KH$ is a prime ideal of KH.*

Proof. This proof is similar to the proof of [1, Proposition 5.5]. Firstly, write $G = H \rtimes \langle X\rangle$, and for each $n \in \mathbb{N}$, let $G_n = H \rtimes \langle X^{p^n}\rangle$. Then G_n is an open, uniform normal subgroup of G containing H, so if we let $\mathcal{L}_n := \frac{1}{p}\log(G_n)$ then $\mathcal{L}_n$ is a powerful Lie-lattice in $\mathfrak{g}$, contained in $\mathcal{L}$.

Clearly $\lambda(\mathcal{L}_n) \subseteq \mathcal{O}$, so we may define the affinoid Dixmier module $\widehat{D(\lambda)}_n$ over $\widehat{U(\mathcal{L}_n)}_K$, and since these both arise as completions of the standard Dixmier module $D(\lambda)$, we have a dense embedding of $\widehat{U(\mathcal{L}_n)}_K$-modules $\widehat{D(\lambda)}_n \hookrightarrow \widehat{D(\lambda)}$. Therefore:

$$\operatorname{Ann}_{\widehat{U(\mathcal{L}_n)}_K} \widehat{D(\lambda)}_n = (\operatorname{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}) \cap \widehat{U(\mathcal{L}_n)}_K. \tag{6.2}$$

Therefore, it follows that $P \cap KG_n = \operatorname{Ann}_{KG_n} \widehat{D(\lambda)_n}$ is a Dixmier annihilator, and hence it is a prime J-ideal of KG_n by Lemma 6.1.1, and clearly it is faithful.

Now, let $Q := P \cap KH$. Then Q is a semiprime ideal of KH, so let $Q_1, \cdots, Q_r$ be the minimal prime ideals of KH above Q, and $Q = Q_1 \cap \cdots \cap Q_r$. Since Q is G-invariant, the conjugation action of G permutes $Q_1, \cdots, Q_r$. So let U be the kernel of the action of G on $\{Q_1, \cdots, Q_r\}$.

Then U is a normal subgroup of G of finite index, hence it is open in G, and clearly it contains H. Since $\frac{G}{H} \cong \mathbb{Z}_p$, it follows that the only open subgroups of G containing H are the subgroups G_n, and hence $U = G_m$ for some $m \in \mathbb{N}$.

Therefore, by the above, $P \cap KU$ is a prime J-ideal of KU, so it follows from Theorem 3.5.3 that $P \cap KU$ is controlled by H, i.e. $P \cap KU = (P \cap KH)KU = QKU$.

But since U fixes each ideal Q_i, it follows that $Q_1KU, Q_2KU, \cdots, Q_rKU$ are two-sided ideals of KU, and using Lemma B.2.1(i), $Q_1KU \cap \cdots \cap Q_rKU = (Q_1 \cap \cdots \cap Q_r)KU = QKU = P \cap KU$. So since $P \cap KU$ is prime, this implies that $P \cap KU = Q_iKU$ for some $i = 1, \cdots, r$.

Therefore, $P \cap KH = Q_i KU \cap KH = Q_i$ by Lemma B.2.1(ii), and hence $P \cap KH$ is prime as required. $\square$

Theorem 6.5.2. *Let G be a uniform, nilpotent, abelian-by-procyclic group. Then G is a weakly Dixmier standard group.*

Proof. Using Definition 6.2.2, we need to prove that if $P = \mathrm{Ann}_{KG} \widehat{D(\lambda)}$ is a Dixmier annihilator, where $\lambda(\mathcal{L}) \subseteq \mathcal{O}$ and $\lambda|_{Z(\mathfrak{g})}$ is injective, then P is controlled by $Z(G)$.

Using Lemma 6.1.2, we may assume that K contains a p'th root of unity, and thus we can apply Theorem 6.4.5. Also, since $\lambda|_{Z(\mathfrak{g})} : Z(\mathfrak{g}) \to K$ is injective, it follows from Lemmas 6.1.1 and 6.1.4 that $P = \mathrm{Ann}_{KG} \widehat{D(\lambda)}$ is a faithful, prime J-ideal of KG, and using Proposition 6.5.1, we see that $Q := P \cap KH$ is a prime ideal of KH.

Using Theorem 3.5.3, we see that P is controlled by $H = C_G(Z_2(G))$. We will examine the action of $\widehat{U(\mathcal{H})_K}$ on $\widehat{D(\lambda)} = K\langle x\rangle$.

Let $A = P^{\chi}$ be the controller subgroup of P. Then A is a closed, normal subgroup of G, contained in H, and since $P \cap KH$ is prime, $Q := P \cap KA = \mathrm{Ann}_{KA} \widehat{D(\lambda)}$ is a prime ideal of KA. Let $\mathcal{A} := \frac{1}{p}\log(A) \leq \mathcal{H}$, and normality of A implies that $\mathcal{A}$ is an ideal of $\mathcal{L}$.

Let us assume that A is not central in G, and hence $\mathcal{A}$ is not central in $\mathcal{L}$. We will show that in this case we can find a proper, closed subgroup U of A which controls P, which will be a contradiction since A is the controller subgroup of P.

Since $\mathcal{A}$ is not central, we see using nilpotence of $\mathcal{L}$ that there is an element $v \in \mathcal{A}$ such that $v \notin Z(\mathfrak{g})$ and $[v, \mathfrak{g}] \subseteq Z(\mathfrak{g})$. Therefore, using Lemma 6.1.5, we see that v acts on $K\langle x\rangle$ by $\lambda(v) + \lambda([v, x])\partial$, where $\partial = \frac{\mathrm{d}}{\mathrm{d}x}$. But since v is not central, $[v, x] \neq 0$, and injectivity of $\lambda|_{Z(\mathfrak{g})}$ implies that $\lambda([v,x]) \neq 0$.

So setting $\Theta : \widehat{U(\mathcal{L})}_K \to \text{End}_K K\langle x\rangle$ as the action, $\partial = \Theta(\lambda([v,x])^{-1}(v - \lambda(v)))$. So if we let $S = \Theta(\widehat{U(\mathcal{A})}_K) \subseteq \text{End}_K K\langle x\rangle$, then $\partial \in S$ and it follows from Theorem 6.3.6 that:

1. $S \cong S_1 \times \cdots \times S_t$ for some commutative domains S_i.

2. We assume without loss of generality that $S_1 \cong F\langle t\rangle$ for some finite extension F/K.

3. The natural surjection $\phi : \widehat{U(\mathcal{A})}_F \to S_1$ is an almost-polynomial map.

Let $I \trianglelefteq KA$ be the kernel of the map ϕ restricted to KA. Clearly $Q \subseteq I$, and using the decomposition $S = S_1 \times \cdots \times S_t$ it is straightforward to show that I is a minimal prime above Q. So since Q is prime, this means that $I = Q$.

Now, let $I' = \ker(\phi|_{FA})$, then clearly $I = I' \cap KA$, so if we show that I' is controlled by a proper, closed subgroup U of A, then U controls I by Lemma 6.1.2. So we may assume that $F = K$.

Since ϕ is an almost-polynomial map, it follows from Theorem 6.4.5 that $I = Q = P \cap KH$ is controlled by a proper open subgroup of A, and hence so is P, which is the desired contradiction. □

Now, we are finally ready to prove our main theorem in characteristic 0. But first we need a small lemma:

Lemma 6.5.3. *Let G be a uniform pro-p group, let N be a closed, normal subgroup of G. Then there exists an open, uniform normal subgroup U of G such that $N \cap U$ is a closed, isolated normal subgroup of U.*

Proof. Recall from [39, Definition 1.6] the definition of the *isolater* $i_G(N)$ of N in G, and recall from [39, Proposition 1.7, Lemma 1.8] that $i_G(N)$ is a closed, isolated normal subgroup of G, and N is open in $i_G(N)$.

Therefore, there exists $n \in \mathbb{N}$ such that if $g \in i_G(N)$ then $g^{p^n} \in N$. So if $g = h^{p^n} \in U := G^{p^n}$ and $g^p = h^{p^{n+1}} \in N \subseteq i_G(N)$, then $h \in i_G(N)$, so $g = h^{p^n} \in N$. Hence $N \cap U$ is isolated in U as required. $\square$

Proof of Theorem E. If G is a nilpotent, abelian-by-procyclic group, then so is G^{p^n} for every $n \in \mathbb{N}$. So using Theorem 6.5.2, it follows that G is a Dixmier-standard group in the sense of Definition 6.2.2. Therefore, applying Theorem 6.2.5 gives us that all faithful, primitive ideals of KG are maximal and standard.

So, let P be a primitive ideal of KG, and we want to prove that P is virtually standard, i.e. that $P \cap KU$ is standard for some open, normal subgroup U of G.

Let $N := P^\dagger = \{g \in G : g - 1 \in P\}$. Then N is a closed, normal subgroup of G, so by Lemma 6.5.3, there exists an open, uniform normal subgroup U of G such that $N \cap U$ is isolated in U. Let $Q := P \cap KU$, then Q is a semiprimitive ideal in KU by Lemma A.1.2, and $Q^\dagger = N \cap U$ is a closed, isolated normal subgroup of U.

Let $U_1 := \frac{U}{Q^\dagger}$, and let $Q_1 := \frac{Q}{(Q^\dagger - 1)KU}$. Then U_1 is a uniform, nilpotent, abelian-by-procyclic group by Lemma 2.2.4, and Q_1 is a faithful semiprimitive ideal of KU_1. Therefore, it follows that Q is a finite intersection of faithful, primitive ideals in KU_1. Since all faithful primitives in KU_1 are maximal and standard, this means that Q_1 is a finite intersection of maximal standard ideals.

Therefore, since Q_1 is a homomorphic image of Q, this means that Q is a finite intersection of maximal, standard ideals. It follows from Definition 1.1.1 that P is a virtually standard prime ideal of KG, so it remains to show that P is maximal.

Using Lemma A.1.1(ii), we see that P is minimal prime above the semiprime ideal $(P \cap KU)KG$. So since $P \cap KU$ is semimaximal in KU, it follows from [30, Theorem 16.6(iii)] that P is maximal in KG as required. □

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Appendices

Appendix A

Ring theory

A.1 Properties of Crossed products

Recall that a *crossed product* of a ring R with a group G is a ring $S = R * G$, free as an R-module, with a basis $\{\bar{g} : g \in G\}$ in bijection with G such that for all $g, h \in G$:

- $\bar{g}R = R\bar{g}$ and
- $\bar{g}R\bar{h}R = \overline{gh}R$.

Lemma A.1.1. *Let R be a Noetherian $\mathbb{Q}$-algebra, F a finite group. Then if P is a prime ideal of a crossed product $S = R * F$, then:*

i. $P \cap R$ is semiprime in R.

ii. $J := (P \cap R) \cdot S$ is semiprime in S, and P is a minimal prime above J.

iii. $S/J = (P/P \cap R) * F$.

Proof. We will prove that $P \cap R$ is an F-prime ideal, i.e. it is F-invariant, and for any F-invariant ideals A, B of R, if $AB \subseteq P \cap R$ then $A \subseteq P \cap R$ or $B \subseteq P \cap R$.

Having established this, part i follows from the fact that all minimal primes above $P \cap R$ form a single F-orbit by [30, Lemma 14.2(ii)], part iii is obvious since $J = \bigoplus_{g \in F} (P \cap R)\bar{g}$, and part ii is part iii together with [29, Proposition 10.5.8] and [30, Theorem 4.4].

So, suppose $A, B \trianglelefteq R$ are F-invariant, i.e. for all $g \in F$, $\bar{g}A = A\bar{g}$ and $\bar{g}B = B\bar{g}$, and suppose that $AB \subseteq P \cap R$. Then AS, BS are two-sided ideals of S, and $(AS)(BS) \subseteq P$. So since P is prime, we can assume without loss of generality that $AS \subseteq P$.

So since $AS = \bigoplus_{g \in F} A\bar{g}$, it follows that $A \subseteq P \cap R$, and hence $P \cap R$ is F-prime as required. □

Lemma A.1.2. *Let R be a Noetherian ring, F a finite group. Then if P is a primitive ideal of a crossed product $R * F$, then $P \cap R$ is semiprimitive.*

Proof. Let $S = R * F$, then since P is primitive, $P = \operatorname{Ann}_S M$ for some irreducible S-module M. Since F is finite, M is finitely generated over R, so since R is Noetherian, we can choose a maximal R-submodule U of M.

For each $g \in F$, gUg^{-1} is a maximal R-submodule of M, so set $M_g := M/gUg^{-1}$, an irreducible R-module, and let $Q_g := \operatorname{Ann}_R M_g$, a primitive ideal of R. Clearly if $r \in P \cap R = \operatorname{Ann}_R M$ then $rM_g = 0$ for all $g \in F$, so $P \cap R \subseteq \bigcap_{g \in F} Q_g$.

Also, $\bigcap_{g \in F} gUg^{-1}$ is an S-submodule, so by simplicity of M, $\bigcap_{g \in F} gUg^{-1} = 0$. So if $r \in \bigcap_{g \in F} Q_g$ then $rM_g = 0$ for all g, so $rM \subseteq gUg^{-1}$ for all g, i.e. $rM \subseteq \bigcap_{g \in F} gUg^{-1} = 0$ and hence $r \in \operatorname{Ann}_R M = P \cap R$. Hence:

$$P \cap R = \bigcap_{g \in F} Q_g$$

Hence $P \cap R$ is semiprimitive as required. □

A.2 Ring filtrations

Definition A.2.1. *A* filtration *of a ring R is a map $w : R \to \mathbb{R} \cup \{\infty\}$ such that for all $x, y \in R$:*

- *$w(x + y) \geq \min\{w(x), w(y)\}$,*
- *$w(xy) \geq w(x) + w(y)$,*

- $w(1) = 0$ *and* $w(0) = \infty$.

The filtration is separated *if* $w(x) = \infty$ *implies that* $x = 0$.

Note: Unless otherwise stated, we will always assume that our filtrations take values in $\mathbb{Z} \cup \{\infty\}$.

If R carries a filtration w, then there is an induced topology on R with the subgroups $F_nR := \{r \in R : w(r) \geq n\}$ forming a basis for the neighbourhoods of the identity. This topology is Hausdorff if and only if the filtration is separated.

Recall from [28, Ch.II Definition 2.1.1] that a filtration is *Zariskian* if $F_1R \subseteq J(F_0R)$ and the *Rees ring* $\widetilde{R} := \underset{n\in\mathbb{Z}}{\oplus} F_nRt^n$ is Noetherian. Zariskian filtrations can only be defined on Noetherian rings, and using [28, Ch.II Theorem 2.1.2] we see that a Zariskian filtration is separated.

If R carries a filtration w, then define the *associated graded ring* of R to be

$$\text{gr } R := \underset{n\in\mathbb{Z}}{\oplus} \frac{F_nR}{F_{n+1}R}.$$

This is a graded ring with multiplication given by $(r + F_{n+1}R) \cdot (s + F_{m+1}R) = (rs + F_{n+m+1}R)$.

Notation: For $r \in R$ with $w(r) = n$ we define $\text{gr}(r) := r + F_{n+1}R \in \text{gr } R$.

Our convention is to say that a filtration w is *positive* if $w(r) \geq 0$ for all $r \in R$.

Definition A.2.2. *If R carries a filtration w and $x \in R\backslash\{0\}$, we say that x is w-*regular *if $w(xy) = w(x) + w(y)$ for all $y \in R$, i.e. gr(x) is not a zero divisor in gr R. If all non-zero x are w-regular we say that w is a* valuation.

*Also, given a central subring S of R, we say that w is S-*linear *if every non-zero element of S is w-regular.*

Note that if x is w-regular and a unit then x^{-1} is w-regular and $w(x^{-1}) = -w(x)$.

Lemma A.2.3. *Suppose w is a separated $\mathbb{Z}_p$-linear filtration on an $\mathbb{Z}_p$-algebra A of characteristic 0, and suppose that $x \in A$ with $w(x-1) > w(p)$. Then $w(x^{p^m}-1) = mw(p) + w(x-1)$ for all $m \in \mathbb{N}$.*

Proof. Using the binomial theorem, it is clear that

$$x^{p^m} - 1 = (x-1+1)^{p^m} - 1 = \sum_{k\geq 1} \binom{p^m}{k}(x-1)^k = p^m(x-1) + \sum_{k\geq 2} \binom{p^m}{k}(x-1)^k$$

Clearly $w(p^m(x-1)) = w(p^m) + w(x-1) = mw(p) + w(x-1)$ since w is $\mathcal{O}$-linear. So it remains to show that $w(\binom{p^m}{k}(x-1)^k) > mw(p) + w(x-1)$ for all $k \geq 2$.

First, note that $w(\binom{p^m}{p^m}(x-1)^{p^m}) = w((x-1)^{p^m}) \geq p^m w(x-1) = (p^m-1)w(x-1) + w(x-1) > (p^m-1)w(p) + w(x-1) \geq mw(p) + w(x-1)$.

So from now on, we may assume that $k < p^m$.

Now, $k = a_0 + a_1 p + \cdots + a_t p^t$ for some integers $0 \leq a_i < p$, and since $k \leq p^m - 1$ we may assume that $t = m-1$. Let $m \geq i \geq 1$ be maximal such that $a_{m-i} \neq 0$, then $p^m = (p-1)p^{m-i} + (p-1)p^{m-i+1} + \cdots + (p-1)p^{m-1} + p^{m-i}$, and:

$$p^m - k = (p-1-a_{m-i})p^{m-i} + (p-1-a_{m-i+1})p^{m-i+1} + \cdots + (p-1-a_{m-1})p^{m-1} + p^{m-i}$$

$$= (p-a_{m-i})p^{m-i} + (p-1-a_{m-i+1})p^{m-i+1} + \cdots + (p-1-a_{m-1})p^{m-1}.$$

It follows that i is equal to the number of carries when $p^m - k$ is added to k in base p. So by Kummer's theorem, $v_p(\binom{p^m}{k}) = i$.

Also, $k = a_{m-i}p^{m-i} + \cdots + a_{m-1}p^{m-1} = p^{m-i}(a_{m-i} + \cdots + a_{m-1}p^{i-1})$, so $k \geq p^{m-i} \geq m - i + 1$ if $i < m$.

Now, $w(\binom{p^m}{k}(x-1)^k) \geq w(\binom{p^m}{k}) + kw(x-1) \geq v_p(\binom{p^m}{k})w(p) + kw(x-1)$

$$= iw(p) + (k-1)w(x-1) + w(x-1) > iw(p) + (k-1)w(p) + w(x-1).$$

So if $i < m$ then this is at least $iw(p) + (m-i)w(p) + w(x-1) = mw(p) + w(x-1)$. Whereas if $i = m$ then since $k > 1$ we have $iw(p) + (k-1)w(p) + w(x-1) = mw(p) + (k-1)w(p) + w(x-1) \geq mw(p) + w(x-1)$ as required. $\square$

Appendix B

Groups and Group algebras

B.1 Uniform pro-p groups and p-valuations

Let G be a group. Recall from [32, Section 23] that we define a *p-valuation* on G to be a map $\omega : G \to \mathbb{R} \cup \{\infty\}$ such that for all $g, h \in G$:

- $\omega(g^{-1}h) \geq \min\{\omega(g), \omega(h)\}$.
- $\omega((g, h)) \geq \omega(g) + \omega(h)$.
- $\omega(g) = \infty$ if and only if $g = 1$.
- $\omega(g^p) = \omega(g) + 1$.
- $\omega(g) > \frac{1}{p-1}$.

We say that G is *p-valued* if it carries a p-valuation ω, and we say that (G, ω) is *p-saturated* if for all $g \in G$, whenever $\omega(g) > \frac{1}{p-1} + 1$ there is an element $h \in G$ such that $g = h^p$.

If G carries a p-valuation ω then there is a natural, ultrametric topology on G, induced by the metric $d(g, h) := c^{-\omega(gh^{-1})}$ for $c > 1$. This topology is naturally Hausdorff and totally disconnected, and we usually assume further that G is complete with respect to ω, in which case we can define p-adic exponentiation, i.e. for all $g \in G$, $\alpha \in \mathbb{Z}_p$, $g^\alpha \in G$ is well defined.

It follows from [25, III 2.2.6] that (G,ω) is compact if and only if G has *finite rank*, i.e. there exists a finite subset $\underline{g} = \{g_1, \cdots, g_d\} \subseteq G$ such that for every $g \in G$ there exist a unique d-tuple $\alpha \in \mathbb{Z}_p^d$ such that $g = \underline{g}^\alpha := g_1^{\alpha_1} \cdots g_d^{\alpha_d}$ and $\omega(g) = \min\{v_p(\alpha_i) + \omega(g_i) : i = 1, \cdots, d\}$. We call $\underline{g}$ an *ordered basis* for (G,ω), and the unique integer d is the *rank* of G.

It follows that G is a pro-p group of finite rank in the sense of [17], and hence G is isometric with $\mathbb{Z}_p^d$. In fact, if G is abelian then $G \cong \mathbb{Z}_p^d$ as groups.

If we suppose that G is a compact, p-adic Lie group, then its group structure completely determines its topology, i.e. there is only one possible topology on G which makes it a profinite topological group. Therefore, if G carries a complete p-valuation ω of finite rank, then the induced topology is the natural topology of G. This prompts the following definition.

Definition B.1.1. *We say that a compact p-adic Lie group G is* p-valuable *if it carries a complete p-valuation ω of finite rank.*

Recall the definition of a *uniform pro-p group* from [17, Definition 4.1], and setting $\epsilon := \begin{cases} 2 & p = 2 \\ 1 & p > 2 \end{cases}$, recall from [17, Theorem 4.5] that G is uniform if and only if G is a torsionfree pro-p group of finite rank, and $(G,G) \subseteq G^{p^\epsilon}$.

If G is a uniform pro-p group, then G carries a p-valuation ω given by $\omega(g) = \min\{n + \epsilon : g \in G^{p^n} \backslash G^{p^{n+1}}\}$, and in fact this is p-saturated. Note that G is complete with respect to ω, and any minimal topological generating set for G is an ordered basis for (G,ω), which implies that (G,ω) has finite rank. Also note that any closed subgroup of a p-valuable group is automatically p-valuable by restriction of the p-valuation.

We use the following result very often:

Lemma B.1.2 ([1, Lemma 4.2]). *Let G be a p-valuable group, and let H be a closed subgroup of G. Then there exists an ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$ for G, $r \leq d$, and $n_1, \cdots, n_s \in \mathbb{N}$ such that $\{g_1^{p^{n_1}}, \cdots, g_r^{p^{n_r}}\}$ is an ordered basis for H. Furthermore, $r = d$ if and only if H is open in G, and H is isolated in G if and only if $n_i = 1$ for all i.*

Recall from [25, III 3.3.1] that if (G, ω) is complete of finite rank, then there exists a canonical, p-saturated group $Sat(G)$ such that G embeds as an open subgroup into $Sat(G)$, and the p-valuation on $Sat(G)$ restricts to ω. Moreover, Sat is a functor from the category of p-valuable groups to the category of p-saturated groups.

B.2 Technical results for KG

In [1, Section 5], a number of technical results were stated and proved for completed group algebras in characteristic p. These results are fundamental for the study of ideals in Iwasawa algebras, so it is important for us to establish them in characteristic zero. We assume only that G is a p-valuable group throughout.

Lemma B.2.1. *Let H be a closed subgroup of G, and let $I_1, \cdots, I_m, J$ be right ideals of KH. Then:*

(i) $I_1KG \cap \cdots \cap I_mKG = (I_1 \cap \cdots \cap I_m)KG$.

(ii) $JKG \cap KH = J$.

Proof. We will prove that KG is faithfully flat over KH. Then part (i) follows from applying the functor $- \otimes_{KH} KG$ to the short exact sequence $0 \to I_1 \cap \cdots \cap I_m \to KH \to \underset{j \leq m}{\oplus} \frac{KH}{I_j} \to 0$, and part (ii) follows from [29, Lemma 7.2.5], taking $R = KH$, $S = KG$ and $M = \frac{KH}{J}$.

Recall that $\mathcal{O}G$ carries the Lazard filtration w, and the restriction of w to $\mathcal{O}H$ is just the Lazard filtration on $\mathcal{O}H$. We will prove that the associated graded gr $\mathcal{O}G$ is faithfully flat over gr $\mathcal{O}H$, and since w is Zariskian, it will follow from [28, Chapter

II Proposition 1.2.2] that $\mathcal{O}G$ is faithfully flat over $\mathcal{O}H$. It follows easily that KG is faithfully flat over KH.

Choose an ordered basis $\{g_1, \cdots, g_d\}$ for G such that $\{g_1^{p^{n_1}}, \cdots, g_s^{p^{n_s}}\}$ is an ordered basis for H. We know that gr $\mathcal{O}G \cong k[t, t_1, \cdots, t_d]$ where $t_i = \text{gr}(g_i - 1)$, and thus gr $\mathcal{O}H = k[t, t_1^{p^{n_1}}, \cdots, t_r^{p^{n_r}}]$.

Now, it is clear that $k[t, t_1, \cdots, t_d]$ is a free $k[t, t_1^{p^{n_1}}, \cdots, t_r^{p^{n_r}}]$-module, and since free modules are faithfully flat, the result follows. □

Using this Lemma, we can now carry over the proofs of every result in [1, Chapter 5], with the exception of [1, Lemma 5.3], whose proof strongly depends on the assumption that the ground field has characteristic p.

Let us first reintroduce some definitions from [1, Section 5]:

Definition B.2.2. (i). *Given a prime ideal P of KG, we say that P is* non-splitting *if for any open normal subgroup U of G controlling P, $P \cap KU$ is prime in KU.*

(ii). *Let $\mathcal{P}$ be a property satisfied by two-sided ideals in KH, for H any compact p-adic Lie group. Then given a right ideal I of KG, we say that I* virtually *satisfies $\mathcal{P}$ if there exists an open subgroup U of G and a two sided ideal J of KU such that J satisfies $\mathcal{P}$ and $I = JKG$.*

In particular, I is virtually non-splitting *if $I = PKU$ for some non-splitting prime ideal P of KU.*

Proposition B.2.3. *Let P be a non-splitting prime ideal of KG, then $P \cap KP^{\chi}$ is prime in KP^{χ}.*

Proof. This is the proof of [1, Proposition 5.5], applied using Lemma B.2.1. □

Now, recall from [1, Definition 5.6] that if R is a ring, $J_1, \cdots, J_r$ are right ideals of R with intersection I, then $I = J_1 \cap \cdots \cap J_r$ is an *essential decomposition* for I if the R-module embedding $\frac{R}{I} \to \frac{R}{J_1} \times \cdots \times \frac{R}{J_r}$ has essential image in the sense of [29, Definition 2.2.1].

Proposition B.2.4. *Let P be a prime ideal of KG, and let $P = I_1 \cap \cdots \cap I_r$ be an essential decomposition for P such that each I_j is virtually prime and $I_1, \cdots, I_r$ form a single G-orbit. If we assume that r is as large as possible then each I_j is virtually non-splitting.*

Proof. This is the proof of [1, Theorem 5.7], applied using Lemma B.2.1. □

Lemma B.2.5. *Let I be a two-sided ideal of KG such that $I = J_1 \cap \cdots \cap J_r$ for some right ideals J_i of KG forming a complete G-orbit via the conjugation action. Then if I is faithful, each J_i is faithful.*

Proof. For any ideal J of KG, let $J^\dagger := \{g \in G : g - 1 \in J\}$. Then clearly $J^\dagger$ is a closed subgroup of G, and if J is a two-sided ideal then it is a normal subgroup. Clearly J is faithful if and only in $J^\dagger = 1$.

It is also clear that $(J_1)^\dagger, \cdots, (J_r)^\dagger$ form a single G-orbit, and $I^\dagger = (J_1)^\dagger \cap \cdots \cap (J_r)^\dagger$.

But since G is p-valuable, it follows from [1, Proposition 5.9] that G is *orbitally sound*, i.e. for any closed subgroup H of G with finitely many G-conjugates, the intersection of these conjugates is open in H. Therefore $I^\dagger$ has finite index in $(J_i)^\dagger$ for each i.

So if I is faithful, then $(J_i)^\dagger$ is a finite subgroup of G. So since G is torsionfree, this means that $(J_i)^\dagger = 1$, and J_i is faithful as required. □

Finally, we prove [1, Theorem 8.6] in characteristic 0:

Theorem B.2.6. *Let P be a prime ideal of $KZ(G)$. Then PKG is a completely prime ideal of KG (i.e. KG/PKG is a domain), and if P is faithful then PKG is faithful.*

Proof. Let $Z := Z(G)$. We will prove that if P is a prime ideal of $\mathcal{O}Z$ with $p \notin P$ then $P\mathcal{O}G$ is completely prime, and it is faithful if P is faithful. The result for the rational Iwasawa algebras follows immediately.

Let Q be the field of fractions of $\mathcal{O}Z/P$. If we let w be the Lazard filtration on $\mathcal{O}Z$, then since w is a Zariskian filtration and the associated graded is a commutative, infinite dimensional k-algebra, it follows from [1, Theorem C] that there exists a valuation v' on Q such that the natural map $\tau : \mathcal{O}G \to Q$ is continuous, and if $w(x) \geq 0$ then $v'(\tau(x)) \geq 0$.

Furthermore, if $v'(\tau(z-1)) = 0$ for some $z \in Z$ then $v'(\tau(z-1)^n) = 0$ for all n since v' is a valuation, which is a contradiction since $(z-1)^n$ converges to zero in $\mathcal{O}G$, and hence in Q by continuity of τ. Therefore $v'(\tau(z-1)) > 0$ for all $z \in Z(G)$, and after choosing an ordered basis $\{z_1, \cdots, z_n\}$ for Z and an integer M such that $Mv'(\tau(z_i - 1)) \geq w(z_i - 1)$ for all i, then we obtain an equivalent valuation $v := Mv'$ on Q such that $v(\tau(x)) \geq w(x)$ for all $x \in \mathcal{O}Z$.

Recall that if we fix an ordered basis $\{g_1, \cdots, g_e\}$ for $\frac{G}{Z}$, then every element of $\mathcal{O}G$ has the form $\sum\limits_{\alpha \in \mathbb{N}^e} \mu_\alpha \underline{c}^\alpha$ for some $\mu_\alpha \in \mathcal{O}Z$ where $c_i = g_i - 1$. Define a map $u : \mathcal{O}G \to \mathbb{Z} \cup \{\infty\}$ via:

$$u : \mathcal{O}G \to \mathbb{Z} \cup \{\infty\}, \sum_{\alpha \in \mathbb{N}^e} \mu_\alpha \underline{c}^\alpha \mapsto \inf\{v(\tau(\mu_\alpha)) + w(\underline{c}^\alpha) : \alpha \in \mathbb{N}^e\}. \qquad \text{(B.1)}$$

Since v is a separated valuation, it is clear that $u(\sum\limits_{\alpha \in \mathbb{N}^e} \mu_\alpha \underline{c}^\alpha) = \infty$ if and only if $\mu_\alpha \in P$ for all α, i.e. if and only if $\sum\limits_{\alpha \in \mathbb{N}^e} \mu_\alpha \underline{c}^\alpha \in P\mathcal{O}G$. Therefore $u^{-1}(\infty) = P\mathcal{O}G$. So following the proof of [1, Theorem 8.6], we will prove that u is a valuation on $\mathcal{O}G$, from which it will follow that $P\mathcal{O}G = u^{-1}(\infty)$ is a completely prime ideal.

Firstly, it is clear from the definition that $u(r+s) \geq \min\{u(r), u(s)\}$, $u(\mu) = v(\tau(\mu))$ and $u(\mu r) = u(\mu) + u(r)$ for all $r, s \in \mathcal{O}G$, $\mu \in \mathcal{O}Z$. It is also clear that if $r_1, r_2, \cdots \in \mathcal{O}G$ with $r_i \to 0$ as $i \to \infty$ then $u(r_1+r_2+\cdots) \geq \inf\{u(r_i) : i \geq 1\}$, therefore to prove that u is a filtration it remains to prove that $u(\underline{c}^\alpha \underline{c}^\beta) \geq u(\underline{c}^\alpha) + u(\underline{(c)}^\beta)$ for all $\alpha, \beta \in \mathbb{N}^r$.

Write $\underline{c}^\alpha \underline{c}^\beta = \sum\limits_{\gamma \in \mathbb{N}^e} \lambda_\gamma^{\alpha,\beta} \underline{c}^\gamma$, then by the definition of the Lazard filtration, $w(\sum\limits_{\gamma \in \mathbb{N}^e} \lambda_\gamma^{\alpha,\beta} \underline{c}^\gamma) = \inf\{w(\lambda_\gamma^{\alpha,\beta}) + w(\underline{c}^\gamma) : \gamma \in \mathbb{N}^d\}$. So since $u(x) \geq w(\tau(x))$ for all $x \in \mathcal{O}Z$, we have:

$$u(\underline{c}^\alpha \underline{c}^\beta) = \inf\{v(\tau(\lambda_\gamma^{\alpha,\beta})) + w(\underline{c}^\gamma) : \gamma \in \mathbb{N}^e\} \geq \inf\{w(\lambda_\gamma^{\alpha,\beta}) + w(\underline{c}^\gamma) : \gamma \in \mathbb{N}^e\} =$$
$$w(\underline{c}^\alpha) + w(\underline{c}^\beta) = u(\underline{c}^\alpha) + u(\underline{c}^\beta).$$

So u is a filtration on $\mathcal{O}G$, and to verify that it is a valuation, we will show that the associated graded $\operatorname{gr}_u \mathcal{O}G$ is a domain. First note that the definition of u gives rise to a natural inclusion of graded rings $\operatorname{gr}_v \mathcal{O}Z/P \to \operatorname{gr}_u \mathcal{O}G$, and this gives rise to an isomorphism of graded rings $\operatorname{gr}_v (\mathcal{O}Z/P)[Y_1, \cdots, Y_e] \to \operatorname{gr}_u \mathcal{O}G$ where Y_i is sent to $\operatorname{gr}(c_i)$. Therefore $\operatorname{gr}_u \mathcal{O}G$ is a domain and u is a valuation as required.

Finally, if P is faithful, then suppose $g \in G$ and $g - 1 \in P\mathcal{O}G$. Then write $g = zg_1^{\alpha_1} \cdots g_e^{\alpha_e}$ for some $z \in Z$, $\alpha_i \in \mathbb{Z}_p$, and it follows that:

$$h - 1 = (z-1) + (z-1)\sum_{0 \neq \gamma \in \mathbb{N}^e} \binom{\alpha}{\gamma} \underline{c}^\alpha + \sum_{0 \neq \gamma \in \mathbb{N}^e} \binom{\alpha}{\gamma} \underline{c}^\alpha.$$

Therefore, we see that $z - 1 \in P$ and hence $z = 1$ since P is faithful. It also follows that for each $0 \neq \gamma \in \mathbb{N}^e$, $\binom{\alpha}{\gamma} \in P$, and hence $\binom{\alpha}{\gamma} = 0$ since $P \cap \mathcal{O} = 0$. This is only possible if $\alpha = (\alpha_1, \cdots, \alpha_e) = 0$, and hence $h = zg_1^{\alpha_1} \cdots g_e^{\alpha_e} = 1$ and $P\mathcal{O}G$ is faithful as we require. $\square$

Appendix C

Lie theory

C.1 The Lie algebra of G

Let G be a p-valuable group, let w be the Lazard filtration on the rational Iwasawa algebra $\mathbb{Q}_pG$ as defined in Chapter 2.3, and let $\widehat{\mathbb{Q}_pG}$ be the completion of $\mathbb{Q}_pG$ with respect to w. For each $g \in G$, the series $\log(g) := \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}(g-1)^n$ converges in $\widehat{\mathbb{Q}_pG}$, so we define $\mathcal{L}_G = \log(G) := \{\log(g) : g \in G\} \subseteq \widehat{\mathbb{Q}_pG}$.

If G is p-saturated of rank d, then $\mathcal{L}_G$ is a free $\mathbb{Z}_p$-Lie subalgebra of $\widehat{\mathbb{Q}_pG}$ of rank d, which we call the *$\mathbb{Z}_p$-Lie algebra of* G. It follows from [25, Proposition IV 3.2.3] that given $g, h \in G$, $\alpha \in \mathbb{Z}_p$, the Lie operations on $\mathcal{L}_G$ are given by:

- $\log(g) + \log(h) = \log(\lim\limits_{n\to\infty}(g^{p^n}h^{p^n})^{p^{-n}})$.
- $\alpha\log(g) = \log(g^\alpha)$.
- $[\log(g), \log(h)] = \log(\lim\limits_{n\to\infty}(g^{p^n}h^{p^n}g^{-p^n}h^{-p^n})^{p^{-2n}})$.

We define the *$\mathbb{Q}_p$-Lie algebra of G* to be $\mathfrak{g}_G := \mathcal{L}_G \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Clearly this is a $\mathbb{Q}_p$-Lie algebra, and $\mathcal{L}_G$ is a $\mathbb{Z}_p$-Lie lattice in $\mathfrak{g}_G$.

Define $\epsilon := \begin{cases} 1 & p > 2 \\ 2 & p = 2 \end{cases}$, and recall from [17, Chapter 9], that a free $\mathbb{Z}_p$-Lie algebra $\mathcal{L}$ of finite rank is *powerful* if $[\mathcal{L}, \mathcal{L}] \subseteq p^\epsilon\mathcal{L}$, and it follows from [17, Theorem 9.10] that a p-saturable group G is uniform if and only if $\mathcal{L}_G$ is powerful.

Also, recall that every p-valuable group G can be embedded as an open subgroup into a p-saturable group $Sat(G)$, thus we define the *$\mathbb{Q}_p$-Lie algebra of G*, $\mathfrak{g}_G$, to be the $\mathbb{Q}_p$-Lie algebra of $Sat(G)$.

Lemma C.1.1. *Let G be a p-valuable group, and let $g, h \in G$ such that h and ghg^{-1} commute. Set $v = \log(h)$, $u = \log(g)$ in $\log(Sat(G))$, then $(g,h) = \exp(\sum\limits_{n\geq 1} \frac{1}{n!}(\mathrm{ad}(u))^n(v))$.*

Proof. $ghg^{-1} = g\exp(v)g^{-1} = \sum\limits_{n\geq 0} \frac{1}{n!} gv^n g^{-1} = \sum\limits_{n\geq 0} \frac{1}{n!}(gvg^{-1})^n = \exp(gvg^{-1})$.

Let l_x, r_x be left and right multiplication by x, then note that $l_x - r_x = ad(x)$, $l_{\exp(x)} = \exp(l_x)$ and $l_{\exp(r_x)} = \exp(r_x)$.

Then $gvg^{-1} = \exp(u)v\exp(u)^{-1} = \exp(u)v\exp(-u) = (l_{\exp(u)} r_{\exp(-u)})(v)$

$= (\exp(l_u)\exp(r_{-u}))(v) = \exp(l_u - r_u)(v) = \exp(ad(u))(v) = \sum_{n\geq 0} \frac{1}{n!}(ad(u))^n(v)$.

Therefore $ghg^{-1} = \exp(gvg^{-1}) = \exp(\sum\limits_{n\geq 0} \frac{1}{n!}(ad(u))^n(v))$.

Finally, $\log((g,h)) = \log((ghg^{-1})h^{-1}) = \log(ghg^{-1}) - \log(h)$ since h and ghg^{-1} commute. Clearly this is equal to $\sum\limits_{n\geq 1} \frac{1}{n!}(ad(u))^n(v)$ as required. □

The following definition allows us to define a category of $\mathbb{Z}_p$-Lie algebras which are the Lie theoretic equivalent of p-saturated groups:

Definition C.1.2. *Let $\mathcal{L}$ be a $\mathbb{Z}_p$-Lie algebra. A map $w : \mathcal{L} \to \mathbb{R}\cup\{\infty\}$ is a* valuation *if for all $u, v \in \mathcal{L}$, $\alpha \in \mathbb{Z}_p$:*

- $w(u+v) \geq \min\{w(u), w(v)\}$,
- $w([u,v]) \geq w(u) + w(v)$,

- $w(\alpha u) = v_p(\alpha) + w(u)$,
- $w(u) = \infty$ *if and only if* $u = 0$,
- $w(u) > \frac{1}{p-1}$.

Furthermore, we say that w is a *saturated valuation* if for any $u \in \mathcal{L}$, $w(u) > \frac{1}{p-1} + 1$ if and only if $u = pv$ for some $v \in \mathcal{L}$.

If G is p-saturated, and w is a valuation on $\mathcal{L}_G$, then it follows from [32, Proposition 32.6] this defines a p-valuation ω on G via $\omega(g) := w(\log(g))$, and w is saturated if and only if (G, ω) is p-saturated.

Moreover, we see using [25, IV 3.2.6] that the category of complete, p-saturated groups of finite rank is isomorphic to the category of saturated $\mathbb{Z}_p$-Lie algebras, via the *transport of structure functors* exp and log. Similarly, the category of uniform pro-p groups is isomorphic to the category of powerful $\mathbb{Z}_p$-Lie algebras.

Finally, recall from [1, Section 4.5] that we define the *degree* of a linear endomorphism σ of a valued Lie algebra $(\mathcal{L}, w)$ as $\deg(\sigma) := \inf\{w(\sigma(u)) - w(u) : u \in \mathcal{L}\}$.

C.2 The Adjoint group functor

The adjoint algebraic group associated to a finite dimensional Lie algebra is a commonly studied object in representation theory, and it can be defined as a group functor, as outlined in [22]. This is usually done over fields of characteristic 0, but we can generalise it:

Let R be a commutative domain containing $\mathbb{Z}$, and let $\mathfrak{h}$ be a nilpotent, torsionfree Lie algebra over R, and we will assume further that $\mathrm{ad}(u)^n(\mathfrak{h}) \subseteq n!\mathfrak{h}$ for each $u \in \mathfrak{h}, n \in \mathbb{N}$. Note that the map $\mathrm{ad}(u)$ is a nilpotent derivation of $\mathfrak{h}$, so we can define:

$$\exp(\mathrm{ad}(u)) := \sum_{n\geq 0}\frac{1}{n!}\,\mathrm{ad}(u)^n : \mathfrak{h}\to\mathfrak{h}. \tag{C.1}$$

Since $\mathrm{ad}(u)$ is a derivation, it follows that $\exp(\mathrm{ad}(u))$ is a Lie-automorphism of $\mathfrak{h}$. Note that our assumptions on R are satisfied if R is any field of characteristic 0, or if R is the ring of integers of a p-adic field and $[\mathfrak{h},\mathfrak{h}]\subseteq p\mathfrak{h}$.

Definition C.2.1. *Define the* adjoint group *of* $\mathfrak{h}$ *to be*

$$\mathbb{H}(\mathfrak{h}) := \{\exp(\mathrm{ad}(u)) \in \mathrm{Aut}(\mathfrak{h}) : u\in\mathfrak{h}\}$$

Then $\mathbb{H}(\mathfrak{h})$ is a subgroup of $\mathrm{Aut}(\mathfrak{h})$. If we let $\mathcal{C}_{\mathfrak{h}}$ be the category of commutative R-algebra domains S such that $\mathfrak{h}_S := \mathfrak{h}\otimes_R S$ is torsionfree over S and $(\mathrm{ad}(u))^n(\mathfrak{h}_S)\subseteq n!\mathfrak{h}_S$ for all $u\in\mathfrak{h}_S$ and $n\in\mathbb{N}$, then we can define the group functor $\mathbb{H}:\mathcal{C}_{\mathfrak{h}}\to\mathrm{Grp}$, $S\mapsto\mathbb{H}(\mathfrak{h}_S)$. We call this the *adjoint algebraic group* of $\mathfrak{h}$, although if R is not a field this need not actually be an algebraic group.

Note: If $R=F$ is a field of characteristic 0, then $\mathcal{C}_{\mathfrak{h}} = F$-alg and $\mathbb{H}$ is an affine algebraic group over F in the sense of [22, Definition I.2.1], and it is unipotent. If we view the space $\mathrm{ad}(\mathfrak{h})\subseteq End_R(\mathfrak{h})$ as an affine variety, then the map $\exp:\mathrm{ad}(\mathfrak{h})\to\mathbb{H}$ is an isomorphism of varieties, with inverse log.

Now, let $\mathfrak{h}^*:\mathcal{C}_{\mathfrak{h}}\to\mathrm{Set}$ be the linear dual of $\mathfrak{h}$, i.e. $\mathfrak{h}^*(S)=\mathrm{Hom}_S(\mathfrak{h}\otimes_R S,S)\cong\mathfrak{h}^*(R)\otimes_R S$. Then there is an action of the group functor $\mathbb{H}$ on $\mathfrak{g}^*$, i.e. a natural transformation $\mathbb{H}\times\mathfrak{h}^*\to\mathfrak{h}^*$, given by $(g\cdot f)(u)=f(g^{-1}u)$. This is the *coadjoint action*, and the orbits of this action in $\mathfrak{h}^*$ are called *coadjoint orbits*.

Again, note that if R is a field of characteristic 0 then $\mathfrak{h}^*$ is an affine variety in the sense of [22, Definition I.1.3], and this is an action of algebraic groups.

Appendix D

Rigid Analytic Geometry

During this thesis, we occasionally exploit geometric techniques, particularly when using the adjoint algebraic group. To this end, we need to explore the non-archimedean equivalent of differential geometry, usually termed *rigid analytic geometry*. We give a brief recap of this subject here.

Recall from [11, Definition 2.2.2] that if R is a ring carrying a complete, separated filtration w, the *Tate algebra* in d variables $t_1, \cdots, t_d$ over R is the algebra:

$$R\langle t_1, \cdots, t_d\rangle := \{ \sum_{\alpha\in\mathbb{N}^d} \lambda_\alpha t_1^{\alpha_1}\cdots t_d^{\alpha_d} : w(\lambda_\alpha) \to \infty \text{ as } \alpha \to \infty\}.$$

In other words, the Tate algebra is the ring of power series with coefficients in R that converge on the unit disc R_+^d, we call these *Tate power series*. This ring carries a separated filtration given by $w_{\inf}(\sum_{\alpha\in\mathbb{N}^d} \lambda_\alpha t_1^{\alpha_1}\cdots t_d^{\alpha_d}) := \inf\{w(\lambda_\alpha) : \alpha \in \mathbb{N}^d\}$.

Normally, R is assumed to be commutative, and in our case, we will usually take $R = K$, in which case the Tate algebra is Noetherian, and the filtration $w_{\inf}$ is Zariskian. Recall from [11, Definition 3.1.1] that we define an affinoid algebra to be any quotient of a Tate algebra over a complete, discretely valued field. Clearly any affinoid algebra A will carry a complete, Zariskian filtration w_A given by the quotient filtration with respect to $w_{\inf}$.

Affinoid algebras play a similar role in rigid geometry as commutative algebras play in standard algebraic geometry. Specifically, if A is an affinoid algebra, we define Sp A to be the space of maximal ideals of A. We call this an *affinoid variety*, and we can realise A as a ring of $\overline{K}$-valued functions on Sp A, where $a(\mathfrak{p}) := a + \mathfrak{p}$. This takes values in $\overline{K}$ since any maximal ideal in the Tate algebra has finite codimension by [11, Corollary 2.2.12].

We say that A is the *ring of analytic functions* on the affinoid variety Sp A. Note that for any ring homomorphism $\phi : A \to B$ between affinoid algebras induces a map $\phi^{\#} : \text{Sp } B \to \text{Sp } A, \mathfrak{q} \mapsto \phi^{-1}(\mathfrak{q})$, continuous with respect to the Zariski topology, and we call this a *morphism of affinoid varieties.* Therefore, we can realise affinoid varieties as a category, equivalent to the category of affinoid algebras, via an equivalence where each variety is sent to its ring of analytic functions.

Now, affinoid varieties are useful in p-adic analysis, since they can indeed be realised as non-archimedean spaces. Recall that for each $\epsilon \in \mathbb{R}$, we define the d-dimensional polydisc of radius ϵ to be the space

$$\mathbb{D}_\epsilon^d := \{\underline{\alpha} \in \overline{K}^d : v_\pi(\alpha_i) \geq \epsilon \text{ for each } i\}.$$

When $\epsilon = 0$ we call this the *unit disc.* We can consider this disc an affinoid space, isomorphic to Sp $K\langle u_1, \cdots, u_d \rangle$, and thus all discs are isomorphic, regardless of the radius.

Note that the Tate algebra $K\langle u_1, \cdots, u_d \rangle$ is precisely the set of power series converging on $\mathbb{D}_0^d$, so we can indeed realise the Tate algebra as the ring of analytic functions on the unit disc. Moreover, for each $n \in \mathbb{N}$, the subalgebra $K\langle \pi^n u_1, \cdots, \pi^n u_d \rangle$ is precisely those functions which converge on $\mathbb{D}_{-n}^d$.

In this thesis, we will not explore the more general theory of *rigid varieties*, which are essentially spaces that locally have the structure of affinoid varieties, since this would require us to introduce the deep theory of G-topologies which is not relevant to our investigation (see [11, Chapter 5] for a detailed introduction to this theory).

However, we will briefly recap the notion of analytification, which is a fully faithful functor from the category of K-schemes to the category of rigid varieties, defined in [11, Definition and Proposition 5.4.3], that associates a scheme X to its *analytification* X^{an}, which is essentially the smallest rigid variety that encompasses the algebraic structure of X.

Example: Let $X = \mathbb{A}_K^n$ be standard affine n-space, i.e. $X = \text{Spec } K[u_1, \cdots, u_n]$. Then X^{an} can be realised as the direct limit of the polydiscs $\mathbb{D}_\epsilon^n$ as $\epsilon \to \infty$, and the ring of analytic functions on this space is the ring $K\{u_1, \cdots, u_d\} = \varprojlim_k K\langle \pi^k u_1, \cdots, \pi^k u_d\rangle$ of rapidly convergent power series.

In fact, since the definition of a polydisc gives that $\overline{K}^n = \bigcup_{\epsilon \in \mathbb{R}} \mathbb{D}_\epsilon^n$, we can actually identify both X and X^{an} with the set $\overline{K}^n$, so we may often interchange the two.