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Composing classical and quantum relaxations of CSP and structure isomorphism

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Abstract

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This thesis investigates relaxations of two key decision problems in computer science: the constraint satisfaction problem (CSP) and the structure isomorphism problem. CSP asks whether a homomorphism exists between two relational structures, while structure isomorphism seeks an isomorphism between them. By adopting a categorical approach, we add to a growing list of results showing that relaxations of these problems, studied in universal algebra, finite model theory, and quantum information can all be captured abstractly using (co)monads. In particular, we will show that distribution minion monads capture many important minion tests from universal algebra, and that the quantum monad can be used to capture quantum homomorphisms and isomorphisms from quantum information. These results add to the already well-established connection between game comonads and Spoiler-Duplicator games from finite model theory.

A key focus is the exploration of distributive laws, the existence of which enables the combination of comonads and monads, potentially leading to novel relaxations of CSP and structure isomorphism that integrate ideas from different domains. We establish a no-go theorem, showing that game comonads like the Ehrenfeucht-Fraïssé and pebbling comonads do not distribute over many important distribution minion monads, such as the basic linear programming and arc-consistency monads. On the positive side, we identify sufficient conditions under which a distributive law between game comonads and distribution minion monads exists. Neither our no-go theorems nor our positive results apply to the quantum monad. Thus, it remains an open question whether any game comonads distribute over the quantum monad.

In addition, we prove several novel results related to quantum advantage. Firstly, we compare two existing definitions of quantum graph homomorphisms, and show that there are graphs where quantum advantage manifests under one definition but not the other. Secondly, we extend existing characterisations of tensor product strategies for the non-local CSP and structure isomorphism games to the more general class of commuting operator strategies. Finally, we introduce quantum versions of one-sided Spoiler-Duplicator games and show that quantum advantage is possible in such games. This result suggests that there is scope for adapting further tools from finite model theory to improve our understanding of quantum homomorphisms and quantum isomorphisms.

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List of mathematical symbols

$\mathbf{Set}$	Category of sets and functions.
$\mathbb{E}_k^{\odot}$	Ehrenfeucht-Fraïssé functor in $\mathbf{R}^{\odot}(\sigma)$.
$\mathbb{P}_k^{\odot}$	Pebbling functor in $\mathbf{R}^{\odot}(\sigma)$.
$\mathbb{Q}_{\boldsymbol{H}}^{\odot}$	Quantum functor in $\mathbf{R}(\sigma)$.
$\mathbf{R}^{\odot}(\sigma)$	Category of commeasurable structures.
$\mathcal{D}_{\mathbf{S}}$	Distribution functor over a semiring $\boldsymbol{S}$ in $\mathbf{Set}$.
$\mathbb{D}_{\mathbf{S}}$	Distribution minion functor over a semiring $\boldsymbol{S}$ in $\mathbf{R}(\sigma)$.
$\mathbb{E}_k$	Ehrenfeucht-Fraïssé functor in $\mathbf{R}(\sigma)$.
$\odot$	Commeasurability or joint measurability relation.
$\mathbb{P}_k$	Pebbling functor in $\mathbf{R}(\sigma)$.
$\mathsf{Proj}(\boldsymbol{H})$	Set of projectors on a Hilbert space $\mathbf{H}$.
$Q_{\boldsymbol{H}}$	Quantum functor in $\mathbf{Set}$.
$\mathbb{Q}_{\boldsymbol{H}}$	Quantum functor in $\mathbf{R}(\sigma)$.
$\mathbf{RGraph}$	Category of reflexive graphs.
$\mathbf{R}(\sigma)$	Category of relational structures.
$\sim$	Adjacency relation for graphs.
$\sqsubseteq$	prefix relation for sequences.

$\mathsf{supp}(\varphi)$	Support of a multiset or distribution φ.
$\uparrow$	Comparability relation for sequences.

1

Introduction

1.1 CSP and structure isomorphism

The fundamental structures of computation – graphs, strings, databases, programs, etc. – all find a convenient mathematical representation in terms of (finite) logical structures, known as relational structures. It is not surprising then that many of the most important challenges and achievements of theoretical computer science can be phrased in terms of the properties of these structures, and how they interact with one another. In this thesis, we will focus on two decision problems about relational structures that are of central importance in theoretical computer science. The first is known as the constraint satisfaction problem[1] (CSP). In modern terms, it can be phrased as follows.

Constraint Satisfaction Problem: Given as input a pair of relational structures $\mathcal{X}, \mathcal{Y}$ over the same signature, decide if there exists a homomorphism $\mathcal{X} \to \mathcal{Y}$.

The importance of CSP is underscored by the fact that instances of it arise naturally in practically every area of modern science. In database theory for example, CSPs capture the problem of query evaluation, where determining whether a query can be satisfied involves finding a homomorphism between the query structure and a database. In bioinformatics, CSPs arise in

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[1] This is also known as the structure homomorphism problem. We shall use the two terms interchangeably throughout the thesis.

protein folding, where the challenge is to determine a protein's 3D structure while satisfying constraints related to chemical bonds and angles. In robotics, the path planning problem, in which a robot aims to navigate from some starting position to a target position while avoiding obstacles can be formulated as a CSP. Thus, it is clear that designing efficient algorithms for solving CSPs will be important in all of these fields. Unfortunately, in the most general case, CSP is known to be NP-complete. For this reason, the design of efficient approximation algorithms, and the identification of tractable subclasses of CSP are important research areas.

The second problem is known as structure isomorphism.

Structure Isomorphism: Given as input a pair of relational structures $\mathcal{X}, \mathcal{Y}$ over the same signature, decide if there exists an isomorphism $\mathcal{X} \cong \mathcal{Y}$.

The study of structure isomorphism is of central importance in computational complexity theory. It is clear that this problem is in NP, however, structure isomorphism is neither known to be solvable in polynomial time nor to be NP-complete. Babai's recent breakthrough [Bab16] places the problem in quasi-polynomial time which lends some support to the idea that structure isomorphism might be an NP-intermediate problem. Much like CSPs the study of approximation algorithms and tractable subclasses of structure isomorphism is an active area of research.

We intend to study these approximations or *relaxations* of CSP and structure isomorphism. The perspective we adopt is categorical, and places issues of compositionality at the forefront. From this point of view, it is readily apparent that the two problems are about identifying the existence of morphisms and isomorphisms in the category of relational structures over a fixed signature σ. We denote this category as $\mathbf{R}(\sigma)$. Moreover, as we will discuss further shortly, any monad or comonad over $\mathbf{R}(\sigma)$ can be seen as providing a categorical abstraction for some relaxation of CSP and structure isomorphism through its Kleisli category. In recent years a growing body of work has identified connections between these (co)monadic relaxations and well-known relaxations of CSP and structure isomorphism that are studied in diverse areas of computer science. We will explore and expand upon three of these connections in this thesis.

1. The game comonads program started in [ADW17; AS21] has revealed a link between comonads on $\mathbf{R}(\sigma)$ and Spoiler-Duplicator games in finite model theory.
2. The (graded) quantum monad in [ABdSZ17] provides a categorical abstraction for a non-local CSP game studied in quantum information.
3. Distribution minion monads introduced in [Con22][Chapter 7] are related to the algebraic approach for studying CSPs[2].

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[2] We shall see in chapter 5 that the graded quantum monad can also be viewed as a distribution minion monad.

The identification of such connections opens up the possibility of using the powerful mathematical arsenal of category theory to gain further insights into each of these relaxations. This approach has already proved useful, leading for example to novel results in the study of homomorphism preservation theorems [AR24b], and homomorphism indistinguishability [DJR21]. One of the major questions that we will tackle can be stated informally as follows:

Question 1.1. *(Informal) Is it possible to combine different monads and comonads which represent relaxations of CSP and structure isomorphism? Moreover, would such a combination lead to novel relaxations which are simultaneously monadic and comonadic in nature?*

This question was originally asked for the specific case of the quantum monad and game comonads in [ABdSZ17]. There the authors point out that combining these constructions would allow us to bring together ideas from quantum information and finite model theory, and that this could be the starting point of an interesting form of "quantum finite model theory". As we shall see in the course of this thesis answering question 1.1 turns out to be remarkably subtle. Mathematically, we know from the work of Beck [Bec69] that the existence of a special kind of natural transformation known as a distributive law between the (co)monads involved provides a sufficient condition under which a combination exists. The caveat is that such distributive laws are not always guaranteed to exist, and even when they do exist they can be quite difficult to construct. In fact, one of our main results in chapter 3 will be to show that for many (co)monads of interest, no distributive law can exist. Interestingly, this "no-go result" does not rule out the possibility of combining game comonads with the quantum monad.

Before diving into the mathematical particulars let us spend the remainder of this introduction looking at two relaxations of CSP in more detail, one which arises in quantum information, and another which arises in (finite) model theory. We can then state question 1.1 more formally for these particular relaxations. The hope is that these examples will provide the reader with some early intuition for the themes we explore in future chapters.

1.2 An example from quantum information

Ever since the pioneering works of Bell [Bel64] and Kochen-Specker [KS67] in the 1960s physicists have known that a system which operates according to the laws of quantum physics is capable of producing correlations that no system operating according to classical physics can. Today, this phenomenon is referred to as quantum advantage and is being extensively studied with the hope that it will allow us to build devices such as quantum computers that can perform information processing tasks which are out of reach for classical computers.

An important theoretical tool for understanding quantum advantage is the framework of non-local games. In these games, two collaborating players (usually called Alice and Bob) are placed in separate rooms so that they are unable to communicate with each other. A referee then asks each of them some questions and compares their answers. The two players win the game if the referee is satisfied that their answers agree with some pre-defined winning conditions. The remarkable feature of these games is that if the players are allowed to share an entangled quantum state between them, they can sometimes win games that they would have otherwise lost. In these cases we say that the players have a quantum winning strategy but no classical winning strategy. The study of non-local games has led to tremendous advances in our understanding of quantum theory, and has even helped solve a major open problem in mathematics, known as the Connes embedding question [JNV+21].

Over the past few decades, it has become clear that many standard decision problems studied in theoretical computer science, which are special instances of CSP, can be rephrased in the language of non-local games. This prompted the authors of [ABdSZ17] to recast CSP as a non-local game. We will look at this game in much more detail in chapter 5. For the purposes of this introduction, we aim to convey some intuition for the game by describing its operation in the special case where the relational structures in question are graphs.

Example 1.2. *Consider graphs $\mathcal{X}, \mathcal{Y}$. The $(\mathcal{X}, \mathcal{Y})$-*CSP *game is played as follows:*

1. *Verifier sends Alice an edge (x_1, x_2) of $\mathcal{X}$, and Bob a vertex $x \in X$.*
2. *Alice returns a pair (y_1, y_2) of vertices from $\mathcal{Y}$ and Bob returns a vertex y of $\mathcal{Y}$.*
3. *Alice and Bob win the game if:*
   (a) *(y_1, y_2) is an edge of $\mathcal{Y}$.*
   (b) *$x = x_1 \Rightarrow y = y_1$ and $x = x_2 \Rightarrow y = y_2$.*

We say that Alice and Bob have a perfect strategy in the above game if they can always win the game regardless of what questions the verifier decides to ask them. If only classical strategies are allowed then Alice and Bob have a perfect strategy precisely when CSP$(\mathcal{X}, \mathcal{Y})$ is solvable. In other words, they win the game perfectly if and only if there exists a graph homomorphism $\mathcal{X} \to \mathcal{Y}$. However, when the players are allowed to use quantum resources, they can sometimes win this game perfectly even if $\mathcal{X} \not\to \mathcal{Y}$. Based on these facts, we can say that there exists a *quantum homomorphism*[3] from $\mathcal{X}$ to $\mathcal{Y}$ and write $\mathcal{X} \xrightarrow{q} \mathcal{Y}$ whenever Alice and Bob have a perfect quantum strategy in the $(\mathcal{X}, \mathcal{Y})$-CSP game. Clearly, every classical homomorphism is also a quantum homomorphism. This is why we are justified to say that quantum homomorphisms are a relaxation of classical homomorphisms.

A further important step taken in [ABdSZ17] was the introduction of a graded monad on $\mathbf{R}(\sigma)$,

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[3] We note that this definition of quantum homomorphism of graphs is subtly different than the earlier one introduced in [MR16b]. One of our results in chapter 5 will be to show that while both definitions are sensible, they do give rise to provably different classes of graphs which admit quantum advantage.

which they refer to as the quantum monad. We denote the functor component of this monad by $\mathbb{Q}_H$. As with any monad $\mathbb{Q}_H$ has a Kleisli category associated to it. The objects of this category are the objects of $\mathbf{R}(\sigma)$ while morphisms between objects $\mathcal{X}$ and $\mathcal{Y}$ are maps in $\mathbf{R}(\sigma)$ of the form $\mathcal{X} \to \mathbb{Q}_H\mathcal{Y}$. The authors of [ABdSZ17] prove the following theorem which shows that such Kleisli morphisms are precisely quantum homomorphisms.

Theorem 1.3. *([ABdSZ17]) For graphs $\mathcal{X}$ and $\mathcal{Y}$ there is a one-to-one correspondence between:*

1. $\mathcal{X} \xrightarrow{q} \mathcal{Y}$.
2. $\mathcal{X} \to \mathbb{Q}_H\mathcal{Y}$.

In this sense, $\mathbb{Q}_H$ provides a categorical abstraction which perfectly captures the notion of quantum homomorphism. One of our main contributions in chapter 5 will be to show how the related notion of quantum structure isomorphism can also be captured using the quantum monad.

1.3 An example from finite model theory

(Finite) model theory is the field of study which studies the expressive power of logics over (finite) models. Ideas from this field have contributed to many areas of computer science, such as the study of databases, automata, and computational complexity [HHI+01; Lib04]. Spoiler-Duplicator games are one of the central tools of model theory. Intuitively, these are two-player games where one of the players (Spoiler) is trying to show that there is no homomorphism between two structures, while the other (Duplicator) tries to show that a homomorphism does exist. The importance of these games in finite model theory is that they are intimately connected with certain fragments of first-order logic. An example of a Spoiler-Duplicator game is the k-round one-sided Ehrenfeucht-Fraïssé (EF) game. Again, let us describe how this game works on graphs $\mathcal{X}$ and $\mathcal{Y}$.

Example 1.4. *The k-round one-sided EF game from $\mathcal{X}$ to $\mathcal{Y}$ is played as follows:*

The game proceeds in a series of rounds. At the start of each round, Spoiler chooses a node of $\mathcal{X}$. Duplicator then chooses a node in the other graph. After i rounds their choices create sequences $[x_1, \dots, x_i]$ and $[y_1, \dots, y_i]$. Duplicator wins a play of length i if $\{(x_j, y_j) | 1 \leq j \leq i\}$ is a partial homomorphism. We say that Duplicator wins the k-round game iff they have a strategy to win all plays of up to length k, regardless of how Spoiler plays.

Whenever a graph homomorphism $\mathcal{X} \to \mathcal{Y}$ exists, Duplicator has a winning strategy in this game where they simply play according to that graph homomorphism. However, even when $\mathcal{X} \not\to \mathcal{Y}$ it may still be possible for Duplicator to win the above game. Thus, we are justified to refer to perfect Duplicator strategies as a relaxation of the classical notion of graph homomorphism.

The following theorem shows that Duplicator's strategies in this game are connected to the existential positive fragment of first-order logic (i.e. the fragment of first-order logic where negation and universal quantification are not allowed) with quantifier rank $\leq k$. We denote this fragment by $\exists\mathcal{L}_k$.

Theorem 1.5. *For graphs $\mathcal{X}, \mathcal{Y}$ the following are equivalent:*

- *For every sentence ρ of $\exists\mathcal{L}_k$: $\mathcal{X} \models \rho \Rightarrow \mathcal{Y} \models \rho$.*
- *Duplicator has a winning strategy in the one-way EF game from $\mathcal{X}$ to $\mathcal{Y}$.*

In [AS21] Abramsky and Shah introduce a comonad $\mathbb{E}_k$ on $\mathbf{R}(\sigma)$ which they refer to as the EF comonad. Just as every monad has its associated Kleisli category, $\mathbb{E}_k$ is associated with a coKleisli category where morphisms are maps of the form $\mathbb{E}_k\mathcal{X} \to \mathcal{Y}$ in the base category $\mathbf{R}(\sigma)$. Abramsky and Shah were able to show that coKleisli maps of this form are precisely Duplicator winning strategies for the one-sided EF game. Using this result the previous theorem can now be extended to a triangle of equivalences between logic, games, and categorical semantics.

Theorem 1.6. *For graphs $\mathcal{X}, \mathcal{Y}$ the following are equivalent:*

- *For every sentence ρ of $\exists\mathcal{L}_k$: $\mathcal{X} \models \rho \Rightarrow \mathcal{Y} \models \rho$.*
- *Duplicator has a winning strategy in the one-way EF game from $\mathcal{X}$ to $\mathcal{Y}$.*
- *$\mathbb{E}_k\mathcal{X} \to \mathcal{Y}$.*

In this sense, $\mathbb{E}_k$ provides a categorical abstraction which perfectly captures the one-sided EF game and its corresponding logic fragment. As with the case of the quantum monad, there are also relaxations of structure isomorphism that can be captured by $\mathbb{E}_k$.

1.4 BiKleisli categories

We have now seen how a certain relaxation of CSP, known as a quantum homomorphism is captured via Kleisli morphisms of the form $\mathcal{X} \to \mathbb{Q}_H\mathcal{Y}$. Dually, we have seen that another relaxation, arising from Duplicator strategies in the one-sided EF game is captured via coKleisli morphisms of the form $\mathbb{E}_k\mathcal{X} \to \mathcal{Y}$. Consider now what it would mean to have a morphism $\mathbb{E}_k\mathcal{X} \to \mathbb{Q}_H\mathcal{Y}$. There are two natural ways of interpreting such a map. One way is to think of it as a quantum homomorphism from the structure $\mathbb{E}_k\mathcal{X}$ to the structure $\mathcal{Y}$. The second is to think of it as a perfect Duplicator strategy in the one-sided EF game between the structures $\mathcal{X}$ and $\mathbb{Q}_H\mathcal{Y}$. These maps thus have the flavour of both a Kleilsi morphism and a coKleilsi morphism. For this reason, we refer to them as *biKleisli morphisms*. There are many interesting questions we would like to ask about such maps. For instance, does the existence of a morphism $\mathbb{E}_k\mathcal{X} \to \mathbb{Q}_H\mathcal{Y}$ correspond to perfect Duplicator strategies in a quantum version of the one-sided EF game? Moreover, is there a quantum version of the logic $\exists\mathcal{L}_k$ which is related to such strategies? Could these games aid in the design of approximation algorithms for detecting quantum homomorphisms, in the same way that classical EF games are related to approximation algorithms for classical homomorphisms?

Categorical tools may help us to answer such questions, however, there is an important technicality that needs to be overcome. As we shall see in the next chapter the axioms that monads and comonads satisfy guarantee that Kleisli and coKleisli morphisms always form a category. BiKleisli morphisms om the other hand do not necessarily compose with one another. Creating a *biKleisli category* is thus a non-trivial task, and one which is not always possible. The existence of a special kind of natural transformation $\lambda : \mathbb{E}_k\mathbb{Q}_H \to \mathbb{Q}_H\mathbb{E}_k$ known as a distributive law provides a sufficient (but not necessary) condition under which a biKleisli category exists. Hence, we can finally state a more formal version of question 1.1 as follows:

Question 1.7. *Does there exist a distributive law* $\lambda : \mathbb{E}_k\mathbb{Q}_H \to \mathbb{Q}_H\mathbb{E}_k$ *of the comonad* $\mathbb{E}_k$ *over the (graded) monad* $\mathbb{Q}_H$*?*

Answering this question and its variations for other (co)monads has been one of the main driving forces behind this thesis. While we do not yet have a complete picture our attempts at finding a solution have led to several novel insights into quantum advantage, as well as an improved understanding of the categorical landscape for relaxations of CSP and structure isomorphism. These new results will be explained in detail in the remaining chapters.

1.5 Structure, contributions, and collaborations

The remainder of this thesis is organised as follows:

- Chapter 2 provides background material and introduces monads, comonads, and distributive laws in detail. There are a few original results which are my own work. I have attempted to present the material in a manner that highlights tacit connections between different areas. The concept of a distribution minion monad in particular may be of interest to researchers in CSP and category theory. I also believe that this chapter may have some pedagogical value, as an accessible introduction to mixed distributive laws is not, to the best of my knowledge, available in any published work.
- Chapter 3 explores the question of when distributive laws of comonads over monads exist. In this chapter, we restrict our attention purely to “classical” monads. Questions

involving the quantum monad will be explored in the second half of the thesis. The no-go results mentioned in the chapter are based on joint work with Nihil Shah which has appeared in [KS24]. Some of these results have also appeared in Nihil Shah's DPhil thesis [Sha24]. To avoid duplication between the two documents I have omitted proofs of these results. The last section of this chapter extends a positive result from [Sha24] from the category of sets, to the category of relational structures. This result is my own work, although it has benefited from numerous discussion with Nihil Shah.

- Chapter 4 provides background on quantum information and non-local games. There are no novel contributions.

- Chapter 5 expands the results of [ABdSZ17] in several directions. We begin by comparing the two notions of quantum homomorphism introduced in [MR16b] and [ABdSZ17]. Answering an open question of [ABdSZ17] we show that there exists a pair of graphs where a quantum homomorphism exists under the definition given in [MR16b] but not under the definition in [ABdSZ17]. Our second contribution is to extend the characterisation of quantum homomorphisms in [ABdSZ17] to the commuting operator framework of quantum mechanics. Finally, we explore how the related notion of quantum isomorphism of relational structures fits into the quantum monad framework. This will involve refining the quantum monad and introducing a "monad" with a partial multiplication operation. This construction is motivated by the importance of partial algebraic structures in quantum information, which was first noticed in the seminal work of Kochen and Specker [KS67]. We also point out how the quantum monad can be viewed as a (partial) distribution minion monad. The material of this chapter is almost entirely my own work. The main exception to this is the usage of specification structures in the final section of the chapter as a means of extracting a total category from partial categories which was an idea suggested by and worked out with Samson Abrasmky.

- In chapter 6 we define and begin to explore quantum Spoiler-Duplicator games. We analyse quantum versions of the one-sided EF and pebble games. Through an argument involving the Mermin-Peres magic square, we show that Duplicator can have a quantum winning strategy in such games even when no classical winning strategy exists. Our result shows that one can treat quantum versions of Spoiler-Duplicator games as "tests" for quantum CSPs, in a manner analogous to how the classical games are linked to well-known algorithmic tests for CSPs (e.g. strong k-consistency). As we have briefly mentioned already, Duplicator's winning strategies in these quantum games coincide with the existence of biKleilsi maps $\mathbb{E}_k\mathcal{X} \to \mathbb{Q}_H\mathcal{Y}$. Thus, in the final part of the chapter, we ask if such games can be composed by attempting to answer question 1.7. Interestingly, our no-go theorems from chapter 3 do not apply to this case and we shall prove several results that come close to providing a positive answer. Unfortunately, we shall argue that each of these results has some major limitation. Thus, providing an answer to question 1.7 remains open (and seemingly quite challenging!). The material on distributive laws in this chapter is based on joint work with Nihil Shah. The remainder of the results are my own work.

2

(Co)monads and distributive laws

2.1 Mathematical preliminaries

2.1.1 Abstract algebra

An *algebraic structure* is a set (known as a *carrier set*) equipped with one or more operations that satisfy a list of axioms. These structures are used to generalize and study algebraic concepts systematically. Below we describe some algebraic structures that are relevant to our discussion.

A *monoid* $\boldsymbol{M} = (M, \cdot, e)$ is a set M equipped with a binary operation $\cdot : M \times M \to M$ and an identity element $e \in M$ satisfying:

- **Associativity:** $\forall a, b, c \in M,\ (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Identity:** $\forall a \in M,\ e \cdot a = a \cdot e = a$.

Note that we use boldface letters to distinguish an algebraic structure from its carrier set. A *commutative* or *abelian* monoid must additionally satisfy:

- **Commutativity**: $\forall a, b \in M,\ a \cdot b = b.a$.

A *group* $\boldsymbol{G} = (G, \cdot, e)$ is a monoid where every element $a \in G$ has an inverse $a^{-1} \in G$ such that:

- **Inverses**: $\forall a \in G,\ a \cdot a^{-1} = a^{-1} \cdot a = e$.

A *semiring* $\boldsymbol{S} = (S, +, \cdot, 0, 1)$ is a set S equipped with two binary operations $+$ and $\cdot$ and two distinguished elements $0, 1$ such that $(S, +, 0)$ is a commutative monoid and $(S, \cdot, 1)$ is a monoid. This data must satisfy:

- **Distributivity**: $\forall a, b, c \in S,\ a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.
- **Absorbing Element**: $\forall a \in S,\ 0 \cdot a = a \cdot 0 = 0$.

A *ring* $\boldsymbol{R}$ is a semiring such that $(S, +, 0)$ is an abelian group. A ring is commutative if $\cdot$ is commutative. A *field* $\boldsymbol{F}$ is a commutative ring where $0 \neq 1$ and all non-zero elements are invertible.

2.1.2 Category theory

A (locally small) *category* $\mathbf{C}$ consists of:

- A collection of objects, denoted $\mathrm{Ob}(\mathbf{C})$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathbf{C})$, a set of morphisms (or arrows) $\mathrm{Hom}_{\mathbf{C}}(X, Y)$.
- For each object $X \in \mathrm{Ob}(\mathbf{C})$, an identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathbf{C}}(X, X)$.
- For each triple of objects $X, Y, Z \in \mathrm{Ob}(\mathbf{C})$, a composition law:

$$\circ : \mathrm{Hom}_{\mathbf{C}}(Y, Z) \times \mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{C}}(X, Z)$$

  such that for all $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathbf{C}}(Y, Z)$, and $h \in \mathrm{Hom}_{\mathbf{C}}(Z, W)$, the following axioms hold:

  - **Associativity:** $(h \circ g) \circ f = h \circ (g \circ f)$
  - **Identity:** $\mathrm{id}_Y \circ f = f$ and $f \circ \mathrm{id}_X = f$

Example 2.1. *The category* $\mathbf{Set}$ *has sets as objects and functions between two sets as morphisms.*

A morphism f from X to Y will be denoted by $f : X \to Y$. Such a morphism is said to be:

- An *isomorphism* if there exists an inverse morphism $f^{-1} : Y \to X$ such that:

$$f \circ f^{-1} = \mathrm{id}_Y \quad \text{and} \quad f^{-1} \circ f = \mathrm{id}_X$$

- *Monic* or a *monomorphism* if for any two morphisms $g, h : Z \to X$, we have:

$$f \circ g = f \circ h \Rightarrow g = h$$

- *Epic* or an *epimorphism* if for any two morphisms $g, h : Y \to Z$, we have:

$$g \circ f = h \circ f \Rightarrow g = h$$

- A *split monomorphism* if there exists a morphism $r : Y \to X$ such that $r \circ f = \mathrm{id}_X$.
- A *split epimorphism* if there exists a morphism $s : Y \to X$ such that $f \circ s = \mathrm{id}_Y$.
- An *endomorphism* if the domain X and codomain Y are the same object A.
- An *automorphism* if it is an endomorphism $f : A \to A$ that is also an isomorphism.

Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* $F : \mathbf{C} \to \mathbf{D}$ consists of:

- An assignment of an object $F(X) \in \mathrm{Ob}(\mathbf{D})$ to each object $X \in \mathrm{Ob}(\mathbf{C})$.
- An assignment of a morphism $F(f) \in \mathrm{Hom}_{\mathbf{D}}(F(X), F(Y))$ to each morphism $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$,

such that the following properties hold:

- **Preservation of identities:** For every object $X \in \mathrm{Ob}(\mathbf{C})$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
- **Preservation of composition:** For all morphisms $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathbf{C}}(Y, Z)$, $F(g \circ f) = F(g) \circ F(f)$.

Functors can be composed. Given functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{C} \to \mathbf{D}$ their composition written as $G \circ F$ is defined on objects as $G \circ F(X) = G(F(X))$ and on morphisms as $G \circ F(f) = G(F(f))$. In this way, we obtain a category $\mathbf{Cat}$ where objects are categories and morphisms are functors.

Diagrammatic reasoning plays an important role in category theory. A *diagram* in a category $\mathbf{C}$ is a directed acyclic multigraph whose nodes are labelled by elements of $\mathrm{Ob}(\mathbf{C})$ and where an edge between X and Y is labelled by an element of $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ such that:

1. For each object X in the diagram, id_X is in the diagram (although generally not drawn).
2. For each pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ in the diagram, the composite $g \circ f$ is in the diagram (although generally not drawn).

A diagram is commutative if any two paths with common source and target, at least one of which has length greater than one, are equal.

Example 2.2. *The following diagram commutes iff* $g \circ f = h$*.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\
& Z
\end{tikzcd}$$

Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and let $F, G : \mathbf{C} \to \mathbf{D}$ be functors. A *natural transformation* $\alpha : F \to G$ consists of a family of morphisms $\{\alpha_X : F(X) \to G(X)\}_{X \in \mathrm{Ob}(\mathbf{C})}$ in $\mathbf{D}$ such that for every morphism $f : X \to Y$ in $\mathbf{C}$, the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"'] & G(Y)
\end{tikzcd}$$

The morphism α_X is known as the component of the natural transformation α at the object X. If every component of α is an isomorphism then α is said to be a natural isomorphism.

Functors and natural transformations can be combined using an operation known as *whiskering*. If $\alpha : F \to G$ is a natural transformation between functors $F, G : \mathbf{C} \to \mathbf{D}$, and $H : \mathbf{D} \to \mathbf{E}$ is another functor, then we obtain a natural transformation $H\alpha : H \circ F \to H \circ G$ by defining

$$(H\alpha)_X = H(\alpha_X).$$

Moreover, if $K : \mathbf{B} \to \mathbf{C}$ is another functor we obtain a natural transformation $\alpha K : F \circ K \to G \circ K$ by setting

$$(\alpha K)_X = \alpha_{KX}.$$

Natural transformations can be composed in two different ways. If $\alpha : F \to G$ and $\beta : G \to H$ are natural transformations between functors $F, G, H : \mathbf{C} \to \mathbf{D}$, then their vertical composition $\beta \circ \alpha : F \to H$ is given by

$$(\beta \circ \alpha)_X = \beta_X \circ \alpha_X.$$

Given $F, G : \mathbf{C} \to \mathbf{D}$, $H, J : \mathbf{D} \to \mathbf{E}$, $\alpha : F \to G$, and $\gamma : H \to J$ the horizontal composition $\gamma \star \alpha : HF \to JG$ is given by

$$\gamma \star \alpha = \gamma G \circ H\alpha = J\alpha \circ \gamma F.$$

2.1.2.1 Monoidal categories

Monoidal categories will be mentioned briefly in chapter 5. Even though most of the thesis can be understood without knowledge of this topic we include some background for the sake of completeness.

A *monoidal category* $(\mathbf{C}, \otimes, I)$ is a category $\mathbf{C}$ equipped with a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, an object I (called the *unit object*), and natural isomorphisms:

$$\text{(associator) } \alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z),$$

$$\text{(left and right unitors) } \lambda_X : I \otimes X \cong X, \quad \rho_X : X \otimes I \cong X,$$

for all objects $X, Y, Z \in \mathbf{C}$, satisfying the following coherence conditions:

- **Pentagon Identity**: The associator must satisfy:

$$\alpha_{W,X,Y\otimes Z} \circ \alpha_{W\otimes X,Y,Z} = (\mathrm{id}_W \otimes \alpha_{X,Y,Z}) \circ \alpha_{W,X\otimes Y,Z} \circ (\alpha_{W,X,Y} \otimes \mathrm{id}_Z).$$

- **Triangle Identity**: The left and right unitors must satisfy:

$$(\rho_X \otimes id_Y) = (\mathrm{id}_X \otimes \lambda_Y) \circ \alpha_{X,I,Y}.$$

A monoidal category $\mathbf{C}$ is said to be *braided* if there is a natural isomorphism (called the *braiding*)

$$\beta_{X,Y} : X \otimes Y \cong Y \otimes X,$$

for all objects $X, Y \in \mathbf{C}$, satisfying the following hexagon identities:

- **Hexagon Identity 1**:

$$\alpha_{Y,Z,X} \circ \beta_{X,Y\otimes Z} \circ \alpha_{X,Y,Z} = (\mathrm{id}_Y \otimes \beta_{X,Z}) \circ \alpha_{Y,X,Z} \circ (\beta_{X,Y} \otimes \mathrm{id}_Z).$$

- **Hexagon Identity 2**:

$$\alpha^{-1}_{Z,X,Y} \circ \beta_{X\otimes Y,Z} \circ \alpha^{-1}_{X,Y,Z} = (\beta_{X,Z} \otimes \mathrm{id}_Y) \circ \alpha^{-1}_{X,Z,Y} \circ (\mathrm{id}_X \otimes \beta_{Y,Z}).$$

A braided monoidal category is *symmetric* if the braiding satisfies $\beta_{Y,X} \circ \beta_{X,Y} = \mathrm{id}_{X\otimes Y}$ for all objects $X, Y \in \mathbf{C}$.

A *(lax) monoidal functor* between two monoidal categories $(\mathbf{C}, \otimes, I_{\mathbf{C}})$ and $(\mathbf{D}, \bullet, I_{\mathbf{D}})$ is a functor $F : \mathbf{C} \to \mathbf{D}$ together with a natural isomorphism

$$\phi_{X,Y} : F(X) \bullet F(Y) \cong F(X \otimes Y),$$

and a morphism $\phi_0 : I \to F(I)$, such that the following conditions hold:

- **Associativity:**

$$\phi_{X,Y\otimes Z} \circ (\mathrm{id}_X \bullet \phi_{Y,Z}) \circ \alpha_{F(X),F(Y),F(Z)} = F(\alpha_{X,Y,Z}) \circ \phi_{X\otimes Y,Z} \circ (\phi_{X,Y} \bullet \mathrm{id}_Z).$$

- **Unit:**

$$\rho_{F(X)} = F(\rho_X) \circ \phi_{X,I_{\mathbf{C}}} \circ (\mathrm{id}_{F(X)} \bullet \phi_0),$$
$$\lambda_{F(Y)} = F(\lambda_Y) \circ \phi_{I_{\mathbf{C}},Y} \circ (\phi_0 \bullet \mathrm{id}_{F(Y)}),$$

A monoidal functor is said to be:

- *Strong* if the maps $\phi_{X,Y}$ are invertible.
- *Strict* if the maps $\phi_{X,Y}$ are identities.
- *Braided* if $\mathbf{C}$ and $\mathbf{D}$ are braided monoidal categories and the following additional condition is satisfied:
$$\phi_{Y,X} \circ \beta_{FX,FY} = F(\beta_{X,Y}) \circ \phi_{X,Y}$$
- *Symmetric* if it is braided and $\mathbf{C}, \mathbf{D}$ are symmetric monoidal categories.

Given two monoidal functors (F, ϕ^F, ϕ_0^F) and (G, ϕ^G, ϕ_0^G) from $(\mathbf{C}, \otimes, I_{\mathbf{C}})$ to $(\mathbf{D}, \bullet, I_{\mathbf{D}})$ a *monoidal natural transformation* $\zeta : F \to G$ is a natural transformation such that for all objects $X, Y \in \mathbf{C}$, the following conditions hold:

- $\phi^G_{X,Y} \circ (\zeta_X \bullet \zeta_Y) = \zeta_{X\otimes Y} \circ \phi^F_{X,Y}$.
- $\phi^G_0 = \zeta_{I_{\mathbf{C}}} \circ \phi^F_0$.

2.1.2.2 Adjunctions

Adjunctions describe a weak form of equivalence between two categories. They can be defined in many different ways. Here we recall their definition in terms of unit and counit natural transformations.

Let $\mathbf{C}$ and $\mathbf{D}$ be categories. An *adjunction* between $\mathbf{C}$ and $\mathbf{D}$ consists of:

- A functor $F : \mathbf{D} \to \mathbf{C}$.
- A functor $G : \mathbf{C} \to \mathbf{D}$.
- A natural transformation $\varepsilon : \mathrm{id}_{\mathbf{D}} \to G \circ F$ called the *unit*.
- A natural transformation $\eta : F \circ G \to \mathrm{id}_{\mathbf{C}}$ called the *counit*.

These must satisfy the following:

$$\begin{aligned} \varepsilon F \circ F\eta &= \mathrm{Id}_F, \\ G\varepsilon \circ \eta G &= \mathrm{Id}_G. \end{aligned}$$

We write $(F, G, \eta, \varepsilon)$, or more simpyly $F \dashv G$ to denote this adjunction. F is said to be *left-adjoint* to G, and G is said to be *right-adjoint* to F. An adjunction is said to be reflective if the counit ε is a natural isomorphism. Dually, it is said to be coreflective if the unit η is a natural isomorphism.

2.1.2.3 Monads and comonads

A monad (M, η, μ) in a category $\mathbf{C}$ consists of:

- A functor $M : \mathbf{C} \to \mathbf{C}$.
- A natural transformation (unit) $\eta : \mathrm{id}_{\mathbf{C}} \to M$.
- A natural transformation (multiplication) $\mu : M^2 \to M$,

such that the following axioms hold:

- **Associativity:**

$$\begin{tikzcd} M^3 \arrow[r, "M\mu"] \arrow[d, "\mu M"'] & M^2 \arrow[d, "\mu"] \\ M^2 \arrow[r, "\mu"'] & M \end{tikzcd}$$

- **Unit laws:**

$$\begin{tikzcd} M \arrow[r, "M\eta"] \arrow[dr, "\mathrm{id}_M"'] & M^2 \arrow[d, "\mu"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\eta M"] \arrow[dr, "\mathrm{id}_M"'] & M^2 \arrow[d, "\mu"] \\ & M \end{tikzcd}$$

Example 2.3. *The powerset monad* $(\mathcal{P}, \eta, \mu)$ *is defined by the following components:*

- $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ *is the powerset functor defined by*
  - $\mathcal{P}(X)$ *is the set of all subsets of* X.
  - $\mathcal{P}(f)(S) = \{f(x) \mid x \in S\}, \quad$ *for all* $S \in \mathcal{P}(X)$.
- $\eta_X(x) = \{x\}$.
- μ_X *takes a union of sets.*

An alternative but equivalent definition of a monad is given by its *Kleisli form*. This consists of:

- An object map $M : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C})$.
- Arrows $\eta_X : X \to MX$ for every $X \in \mathrm{Ob}(\mathbf{C})$.
- An extension operation $(\_)^*$ which takes any arrow $f : X \to MY$ to an arrow $f^* : MX \to MY$.

These must satisfy the following equations:

$$\eta_X^* = \mathrm{id}_{MX}; \qquad f^* \circ \eta_X = f; \qquad (g^* \circ f)^* = g^* \circ f^*.$$

Given a monad in Kleisli form we can extend M to a functor by setting $Mf = (\eta_X \circ f)^*$. We can also define a multiplication operation μ by setting $\mu_X = \mathrm{id}_{MX}^*$. (M, η, μ) is then a monad under our original definition. Conversely, given a monad (M, η, μ) we can define the extension operation by

Comonads are the dual construction to monads. A comonad (W, ε, δ) in a category $\mathbf{C}$ consists of:

- A functor $W : \mathbf{C} \to \mathbf{C}$,
- A natural transformation (counit) $\varepsilon : W \to \mathrm{id}_{\mathbf{C}}$,
- A natural transformation (comultiplication) $\delta : W \to W^2$,

such that the following diagrams commute:

- **Coassociativity:**

$$\begin{tikzcd}
W \arrow[r, "\delta"] \arrow[d, "\delta"'] & W^2 \arrow[d, "W\delta"] \\
W^2 \arrow[r, "\delta W"'] & W^3
\end{tikzcd}$$

- **Counit laws:**

$$\begin{tikzcd}
W \arrow[r, "\delta"] \arrow[dr, "\mathrm{id}_W"'] & W^2 \arrow[d, "W\varepsilon"] \\
& W
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "\delta"] \arrow[dr, "\mathrm{id}_W"'] & W^2 \arrow[d, "\varepsilon W"] \\
& W
\end{tikzcd}$$

Example 2.4. *The prefix list comonad* $(L^+, \varepsilon, \delta)$ *is defined by the following components:*

- $L^+ : \mathsf{Set} \to \mathsf{Set}$ *is the prefix list functor defined by*
  - $L^+(X)$ *is the set of all non-empty lists containing elements of* X.
  - $L^+(f)([x_1, \ldots, x_n]) = [f(x_1), \ldots, f(x_n)]$
- $\varepsilon_X([x_1, \ldots, x_n]) = x_n$.
- $\delta_X([x_1, \ldots, x_n]) = [[x_1], [x_1, x_2], \ldots, [x_1, \ldots, x_n]]$.

As exemplified by the definitions above we will make use of colours throughout this thesis to aid readability. Informally, notions related to monads will be written with a teal colour while notions related to comonads will be written with a purple colour.

A *monad morphism from* (M, η^M, μ^M) *to* $(M', \eta^{M'}, \mu^{M'})$ is a natural transformation $\rho\colon M \to M'$ such that

- $\rho \circ \mu^M = \mu^{M'} \circ (\rho \star \rho)$.
- $\rho \circ \eta^M = \eta^{M'}$.

There is a category $\mathbf{Mon}(\mathbf{C})$ of monads and monads maps on $\mathbf{C}$. The category $\mathbf{Mon}(\mathbf{C})$ has a notion of subobject. A monad M is a *submonad of* M' if there exists a monic monad map $\iota\colon M \to M'$. In many cases, we will not need to consider the multiplication μ^M of a monad, and instead consider pointed endofunctors (M, η) where M is an endofunctor and $\eta\colon \mathrm{id}_{\mathbf{C}} \to M$ is a unit natural transformation. A map between two pointed endofunctors $\rho\colon M \to M'$ is a natural transformation satisfying $\rho \circ \eta^M = \eta^{M'}$.

There are two important categories related to any monad. The first of these is called the Kleisli category $\mathbf{Kl}(M)$. It is defined as follows:

- **Objects:** Same as $\mathbf{C}$.

- **Morphisms:** For $X, Y \in \mathbf{C}$, a morphism in $\mathbf{Kl}(M)$ is a morphism $f : X \to MY$ in $\mathbf{C}$.
- **Composition:** For morphisms $f : X \to MY$ and $g : Y \to MZ$ in $\mathbf{Kl}(M)$, their composition is given by

$$g \circ f : X \xrightarrow{f} MY \xrightarrow{Mg} MMZ \xrightarrow{\mu_Z} MZ.$$

- **Identity:** The identity morphism on an object X is given by $\eta_X : X \to MX$.

The second category is the Eilenberg-Moore category $\mathbf{EM}(M)$, also known as the category of algebras for M. It is a standard fact that $\mathbf{Kl}(M)$ is equivalent to the subcategory of free algebras of $\mathbf{EM}(M)$. Eilenberg-Moore categories do not play a major role in this thesis so we omit a definition.

Unsurprisingly we have the dual concept of a coKleisli category $\mathbf{coKl}(W)$ for a comonad. This is defined as follows:

- **Objects:** Same as $\mathbf{C}$.
- **Morphisms:** For $X, Y \in \mathbf{C}$, a morphism in $\mathbf{coKl}(W)$ is a morphism $f : WX \to Y$ in $\mathbf{C}$.
- **Composition:** For morphisms $f : WX \to Y$ and $g : WY \to Z$ in $\mathbf{coKl}(W)$, their composition is given by

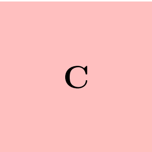
$$g \circ f : WX \xrightarrow{\delta_X} WWX \xrightarrow{Wf} WY \xrightarrow{g} Z$$

- **Identity:** The identity morphism on an object X is given by $\varepsilon_X : WX \to X$.

2.1.2.4 String diagrams

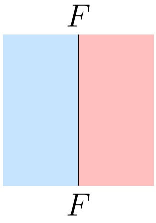
We shall make use of string diagrams to reason about categories, functors, natural transformations, monads, and adjunctions in some of our proofs. As such we include a brief primer on this topic. For more details the reader is referred to [HM23; HM16; Mar14].

In the string diagram formalism, categories are represented by continuous regions of space. We can colour such regions for extra intuition. For example a category $\mathbf{C}$ is represented as:

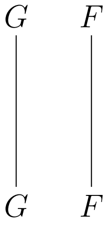


We will usually not explicitly write down what category a colour corresponds to.

A functor $F : \mathbf{C} \to \mathbf{D}$ is represented as an edge or *wire* which separates two regions:

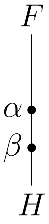


Moving forward we will also omit colours from our diagrams. It will always be clear what categories we are working with from the type information involved. The composition $G \circ F$ of F with a second functor $G : \mathbf{D} \to \mathbf{E}$ is represented as:

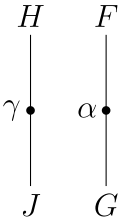


Given $F, G : \mathbf{C} \to \mathbf{D}$, a natural transformations $\alpha : F \to G$ is given by a node:

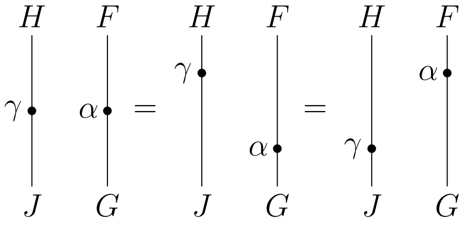
Given $H : \mathbf{C} \to \mathbf{D}$ and $\beta : G \to H$, α and β can be composed vertically:



Given $F, G : \mathbf{C} \to \mathbf{D}$, $H, J : \mathbf{D} \to \mathbf{E}$, $\alpha : F \to G$, and $\gamma : H \to J$, we can compose α and γ horizontally:

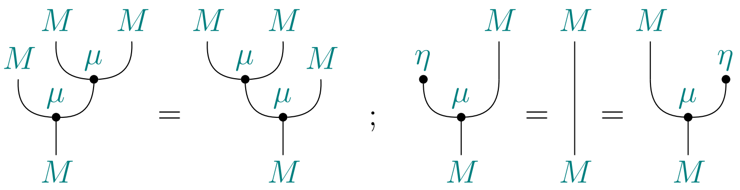


Perhaps the most useful property of string diagrams is the fact that the following sliding equations hold:

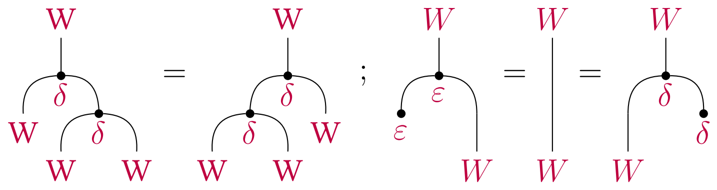


These equations allow us to move natural transformations up and down like beads on a wire, allowing us to eliminate unnecessary bookkeeping details involved with many algebraic proofs:

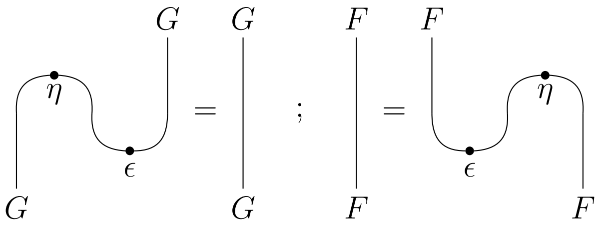
The associativity and unit laws for a monad (M, η, μ) state the following:



Dualities are particularly pleasing in string diagrams. To obtain the counit and coassociativity axiom for a comonad we simply flip the diagrams upside down:



Finally, an adjunction $(F, G, \eta, \varepsilon)$ must satisfy the following “yanking” equations:



2.2 Multiset and distribution monads

We now introduce some monads on the category **Set**. Following the influential work of Moggi [Mog91] we view such monads as providing semantics for effectful computation. We shall be particularly concerned with monads which model various forms of nondeterministic effects, e.g. possibilistic, probabilistic, or quantum. Such nondeterministic computations are modelled abstractly as Kleisli morphisms $X \to MY$ for a suitable monad M. For instance, the Kleisli morphisms of the powerset monad $\wp$ from example 2.3 model possibilistic nondeterminism. A second example is provided by the probability distribution monad. As its name suggests Kleisli morphisms of this monad model probabilistic computations.

Example 2.5. *The probability distribution monad* $(\mathcal{D}, \eta, \mu)$ *is defined by the following components:*

- $\mathcal{D} : \mathbf{Set} \to \mathbf{Set}$ *is the distribution functor defined by:*

- $\mathcal{D}(X)$ *is the set of all functions of type* $\varphi\colon X \to \mathbb{R}$ *where for all but finitely many* $x \in X$, $\varphi(x) = 0$. *These must satisfy the normalisation condition* $\sum_{x \in X} \varphi(x) = 1$.
  - $\mathcal{D}(f)$ *maps* φ *to* $\lambda y.\Sigma_{x \in f^{-1}(y)} \varphi(x)$.
- $\eta_X(x) = \Delta_x$ *where* $\Delta_x(x) = 1$ *and* $\Delta_x(y) = 0$ *for* $y \neq x$.
- $\mu_X(\varphi) = \lambda x.\Sigma_{\psi \in \mathcal{D}(X)} \varphi(\psi) \cdot \psi(x)$.

Given any semiring, one can construct a variant of the distribution monad described above[1].

Example 2.6. *The distribution monad* $(\mathcal{D}_{\mathbf{S}}, \eta^{\mathcal{D}}, \mu^{\mathcal{D}})$ *over a semiring* $\mathbf{S} = (S, +_{\mathbf{S}}, \cdot_{\mathbf{S}}, 0_{\mathbf{S}}, 1_{\mathbf{S}})$ *is defined exactly as* $(\mathcal{D}, \eta, \mu)$ *but with* $(+, \cdot, 0, 1)$ *replaced with* $(+_{\mathbf{S}}, \cdot_{\mathbf{S}}, 0_{\mathbf{S}}, 1_{\mathbf{S}})$.

$\mathcal{D}_{\mathbf{S}}$ is a submonad of another monad known as the multiset monad.

Example 2.7. *The multiset monad* $(\mathcal{M}_{\mathbf{S}}, \eta, \mu)$ *over a semiring* $\mathbf{S} = (S, +_{\mathbf{S}}, \cdot_{\mathbf{S}}, 0_{\mathbf{S}}, 1_{\mathbf{S}})$ *is defined exactly as* $(\mathcal{D}_{\mathbf{S}}, \eta^{\mathcal{D}}, \mu^{\mathcal{D}})$ *but with the normalisation condition removed.*

We define the support of a function $\varphi \in \mathcal{M}_{\mathbf{S}}(X)$ as the set $\mathsf{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0_{\mathbf{S}}\}$. Elements $\varphi \in \mathcal{M}_{\mathbf{S}}(X)$ can be written as finite formal sums $\varphi = \Sigma s_i.x_i$ with $x_i \in \mathsf{supp}(\varphi)$ and $\varphi(x_i) = s_i$.

The constructions $\mathcal{D}_{\mathbf{S}}$ and $\mathcal{M}_{\mathbf{S}}$ cover a remarkably broad range of well-studied monads. Almost every $\mathbf{Set}$ monad we consider in this thesis arises as an instance of this construction. Let us list some examples.

Example 2.8. *The ordinary probability distribution monad of example 2.5 is recovered as* $\mathcal{D}_{\mathbf{S}} = \mathcal{D}_{\mathbb{R}_{\geq 0}}$ *for the semiring of non-negative real numbers* $(\mathbb{R}_{\geq 0}, 0, 1, +, *)$.

Example 2.9. *The ordinary multiset or 'bag' monad* $\mathcal{M}$ *is recovered as* $\mathcal{M}_{\mathbf{S}} = \mathcal{M}_{\mathbb{N}}$ *for the semiring of natural numbers* $(\mathbb{N}^{+}, 0, 1, +, *)$.

Example 2.10. *Given a ring* R, *the free* R*-module and* R*-convex space monad are recovered as* $\mathcal{M}_{\mathbf{S}}$ *and* $\mathcal{D}_{\mathbf{S}}$ *for ring* $\mathbf{S} = R$ *considered as a semiring.*

Example 2.11. *The multiset* $\mathcal{M}_{\mathbf{S}}$ *and distribution* $\mathcal{D}_{\mathbf{S}}$ *monads for the min-plus semiring* $(\mathbb{R} \cup \{\infty\}, \infty, 0, \min, +)$ *are useful for modelling spaces in tropical geometry and logical queries with confidence scores [GHNW22].*

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[1] In the literature it is often additionally required that the semiring in question is commutative. However, this requirement is not necessary to obtain a monad. If a non-commutative semiring is used the outcome is simply that the resulting monad will also be non-commutative.

Example 2.12. *When* $\mathbf{S} = (\mathbb{B}, \bot, \top, \vee, \wedge)$ *is the Boolean semiring,* $\mathcal{M}_{\mathbf{S}}$ *and* $\mathcal{D}_{\mathbf{S}}$ *are isomorphic to the finite powerset monad* $\mathcal{P}_f$ *and finite non-empty powerset monad* $\mathcal{P}_f^+$*.*

In the case where the addition operation of $\mathbf{S}$ can be extended to an arbitrary sum operation which also distributes over multiplication, $\mathbf{S}$ is a complete semiring. As this sum operation is well-defined for infinite subsets, we can remove the finite support restriction on the functions in $\mathcal{M}_{\mathbf{S}}(X)$ and $\mathcal{D}_{\mathbf{S}}(X)$ to obtain analogous monads $\mathcal{M}_{\mathbf{S}}^{\infty}$ and $\mathcal{D}_{\mathbf{S}}^{\infty}$.

Example 2.13. *When* $\mathbf{S}$ *is the Boolean semiring* $\mathbb{B}$*,* $\mathcal{M}_{\mathbf{S}}^{\infty}$ *and* $\mathcal{D}_{\mathbf{S}}^{\infty}$ *are isomorphic to the full powerset monad* $\mathcal{P}$ *and the full non-empty powerset monad* $\mathcal{P}^+$*.*

2.3 Directed container comonads

We now turn our attention to comonads over $\mathbf{Set}$. Just as monads model effectful computation, comonads give semantics for contextual computation where inputs are paired with a context in a larger data structure [Orc14; UV08]. Contextual computations from inputs of type A to outputs of type B are modelled as morphisms of type $WA \to B$ in the coKleisli category of a comonad W. Typically this comonad models contexts in a data structure, e.g. prefixes of a list, indices of a list, nodes in a tree.

All of the $\mathbf{Set}$ comonads we consider in this work have the property that their underlying functor is a *container* [AAG03]. Intuitively, a container captures data structures which have a set of shapes and addressed positions for the data stored within those shapes.

Definition 2.14. A *container* $S \lhd P$ is a set S and a functor $P \colon S \to \mathbf{Set}$ where we consider S as a discrete category, i.e. P defines an S-indexed family of sets. We consider S to be a set of shapes, and for each $s \in S$, $P(s)$ is the set of positions associated with the shape s. The *induced endofunctor on* $S \lhd P$ is $[S \lhd P] \colon \mathbf{Set} \to \mathbf{Set}$ where

- for a set X, $[S \lhd P]X = \{(s, l) \mid s \in S, l \colon P(s) \to X\}$; and
- for a function $g \colon X \to Y$, $[S \lhd P]g \colon [S \lhd P]X \to [S \lhd P]Y$ is defined as $(s, l) \mapsto (s, g \circ l)$.

We abuse terminology and say an endofunctor $F \colon \mathbf{Set} \to \mathbf{Set}$ is a container if $F \cong [S \lhd P]$ for some set S and functor $P \colon S \to \mathbf{Set}$. Note that containers are equivalent to polynomial functors [Koc09], i.e. $[S \lhd P] = \sum_{s \in S} (\cdot)^{P(s)}$.

Example 2.15. *For a fixed set* S*, the product by* S *endofunctor* $S \times (\cdot) \colon \mathbf{Set} \to \mathbf{Set}$ *is a container such that for all* $s \in S$*,* $P(s)$ *is a singleton* $\{\top\}$*. Dually, for a set* T*, the exponentiation by* T *endofunctor* $(\cdot)^T$ *is a container such that* $S = \{\bot\}$ *is a singleton and* $P(\bot) = T$*.*

Example 2.16. *The list endofunctor L is a container where $S = \mathbb{N}$, $P(n) = [n]$. The non-empty list endofunctor L^+ is defined similarly where $S = \mathbb{N}^+$ is the set of positive integers. Additionally, we can define for every $k > 0$, the non-empty lists of length $\leq k$ endofunctor L^+_k as the container with $S = \{1, \ldots, k\}$ and for every $i \in S$, $P(i) = \{1, \ldots, i\}$.*

Example 2.17. *The rooted binary tree endofunctor $B \colon \mathbf{Set} \to \mathbf{Set}$ is a container where S is the set of full unlabelled binary trees and $P(s)$ is the set of internal nodes in tree $s \in S$. The rooted non-empty binary tree endofunctor $B^+ \colon \mathbf{Set} \to \mathbf{Set}$ is obtained by removing the empty tree from S.*

Ahman, Chapman, and Uustalu showed that comonads whose underlying endofunctor is a container are equivalent to containers which are equipped with additional 'directed' structure [ACU14]. Intuitively, directed containers are containers where each position has an associated 'sub container' and every such 'sub container' has a root.

Definition 2.18. A *directed container* consists of

- a container $F = [S \triangleleft P]$;
- for each shape $s \in S$, a root position $\mathsf{o}_s \in P(s)$;
- for each position $p \in P(s)$, a subshape $s \downarrow p \in S$;
- for each position $p' \in P(s \downarrow p)$ in the subshape $s \downarrow p$, a translation into a position $p \oplus_s p' \in P(s)$ in the global shape $s \in S$;

satisfying the equations

$$s \downarrow \mathsf{o}_s = s \quad s \downarrow (p \oplus_s p') = (s \downarrow p) \downarrow p' \tag{2.19}$$

$$p \oplus_s \mathsf{o}_{s \downarrow p} = p = \mathsf{o}_s \oplus_s p \tag{2.20}$$

$$(p \oplus_s p') \oplus_s p'' = p \oplus_s (p' \oplus_{s \downarrow p} p''). \tag{2.21}$$

A directed container $D = ([S \triangleleft P], \mathsf{o}, \downarrow, \oplus)$, has an *induced comonad* $([S \triangleleft P], \varepsilon, \delta)$ where:

- The counit has components defined as

$$\varepsilon_X(s, l \colon P(s) \to X) = l(\mathsf{o}_s)$$

- The comultiplication has components defined as

$$\delta_X(s, l \colon P(s) \to X) = (s, \lambda p.(s \downarrow p, \lambda q.l(p \oplus_s q)))$$

We will abuse terminology and say a comonad (W, ε, δ) is a directed container if it is isomorphic to the induced comonad of a directed container.

Example 2.22. *Extending example 2.15, the coreader comonad* $(S \times (\cdot), \varepsilon, \delta)$ *on a fixed set* S *has counit with components defined as* $\varepsilon(s, x) = x$ *and comultiplication with components defined as* $\delta(s, x) = (s, (s, x))$*. The identity comonad is recovered as the case where* S *is a singleton. The coreader comonad on set* S *is isomorphic to the induced comonad of a directed container on* $[S \lhd P]$ *where for every* $s \in S$*,* $P(s)$ *is the singleton set* $\{\top\}$*. As* $P(s)$ *is trivial for every* $s \in S$*, the directed container structure is such that* $\mathsf{o}_s = \top$*,* $s \downarrow \top = s$*, and* $\top \oplus_s \top = \top$*.*

Example 2.23. *The cowriter comonad* $((\cdot)^T, \varepsilon, \delta)$ *on a fixed monoid* $(T, e, *)$ *has counit defined as* $\varepsilon(f) = f(e)$ *and comultiplication defined as* $\delta(f) = \lambda m \lambda n. f(m * n)$*. For a monoid* $(T, e, *)$*, the cowriter comonad is isomorphic to the induced comonad on the directed container* $([\{\bot\} \lhd P], \mathsf{o}, \downarrow, \oplus)$ *where* $P(\bot) = T$*,* $\mathsf{o}_\bot = e$*, for every* $m \in M$*,* $\bot \downarrow m = \bot$*, and* $\oplus_\bot = *$*.*

Example 2.24. *The list container* L *from Example 2.16 cannot be extended to a directed container as* $P(0) = \varnothing$ *and so there is no possible root* $\mathsf{o}_0 \in P(0)$*. However, the non-empty list container* L^+ *is commonly extended to a directed container in two non-isomorphic ways. The first way induces the prefix list comonad with counit defined as* $\varepsilon([x_1, \ldots, x_n]) = x_n$ *and comultiplication defined as* $\delta([x_1, \ldots, x_n]) = [[x_1], [x_1, x_2], \ldots, [x_1, \ldots, x_n]]$*. Note these definitions restrict to non-empty lists of length* $\leq k$ *yielding a comonad* $(L^+_k, \varepsilon, \delta)$*.*

In this case, for every shape $n \in \mathbb{N}^+$*,* $\mathsf{o}_n = n$*, for every position* $i \in P(n) = \{1, \ldots, n\}$*,* $n \downarrow i = i$*, and for every* $i \in P(n)$*,* $j \in P(n \downarrow i)$*,* $i \oplus_n j = j \in P(n)$*. The suffix list comonad is isomorphic to the prefix list comonad via the 'reverse' natural transformation. The second non-isomorphic way induces the cyclic list comonad, see e.g. [Orc14, Example E.24] for details.*

Example 2.25. *The non-empty binary tree container* B^+ *can be extended to a directed container [AU19]. This induces the comonad where the counit sends an* X*-labelled tree* t *to the label at* t*'s root. The comultiplication sends an* X*-labelled tree* t *to the* $B^+(X)$*-labelled tree* t' *where node* v *of* t' *is replaced with the* X*-labelled subtree* t_v *of* t *rooted at node* v*.*

Example 2.26. *For every comonad* (W, ε, δ)*, the composed functor* $W \circ (S \times (\cdot))$ *has a comonad structure. In the case of a set* $S = [\mathsf{k}]$ *for some* $k \in \mathbb{N}$ *and* $W = L^+$ *is the prefix list comonad, we obtain a pebble list comonad* $(L^+[k], \varepsilon, \delta)$*. This comonad has counit with components defined as* $\varepsilon([(p_1, a_1), \ldots, (p_n, a_n)]) = a_n$ *and comultiplication* $\delta([(p_1, a_1), \ldots, (p_n, a_n)] = [(p_1, s_1), \ldots, (p_n, s_n)]$ *where for all* $i \in [\mathsf{n}]$*,* $s_i = [(p_1, a_1), \ldots, (p_i, a_i)]$*. This comonad is the pebbling comonad of [ADW17] over* **Set***.*

2.4 The category of relational structures

We now wish to introduce (co)monads that are defined on the category of relational structures over a fixed signature σ, which we denote by $\mathbf{R}(\sigma)$. Before doing so let us recall some facts about this category and fix our notation.

A finite relational signature σ is a set of relation symbols $\sigma = \{R_1, \ldots, R_n\}$ where each symbol $R \in \sigma$ has an associated non-negative integer $ar(R)$ called its arity. A relational structure $\mathcal{X}$ with signature σ (also known as a σ-structure) is then a tuple

$$(X, R_1^{\mathcal{X}}, \ldots, R_n^{\mathcal{X}})$$

where X is the underlying set of $\mathcal{X}$ known as its *universe* and for each $R \in \sigma$, $R^{\mathcal{X}} \subseteq X^{ar(R)}$ is a relation of arity $ar(R)$. We shall use the calligraphic font to distinguish a relational structure $\mathcal{X}$ from its universe X. A relational structure is said to be finite if its universe is finite. (Finite) relational structures are a useful abstraction which model many important data structures in computer science. For instance, if σ consists of a single relational symbol of arity 2 then σ-structures are precisely directed graphs. By varying the symbols in our signature[2] we can capture other data structures such as multi-graphs, coloured graphs, strings, finite transition systems, and much more. Relational structures thus provide us with a uniform language to simultaneously prove results about all these data structures.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two relational structures over the same signature σ. A function $f : X \to Y$ is said to *preserve* a relation $R \in \sigma$ with arity k if

$$\forall \boldsymbol{x} = (x_1, \ldots, x_k) \in X^k : \boldsymbol{x} \in R^{\mathcal{X}} \Rightarrow \boldsymbol{f(x)} = (f(x_1), \ldots, f(x_k)) \in R^{\mathcal{Y}}$$

Where we have adopted the convention that boldface lowercase letters denote tuples $\boldsymbol{x} = (x_1, \ldots, x_k) \in X^k$. Moreover, whenever we use the notation x_i we are implicitly implying that x_i is the element at position i of a tuple $\boldsymbol{x}$. We say that f *reflects* a relation if the above holds with the implication reversed. f is said to be a homomorphism if it preserves every $R \in \sigma$. If the homomorphism f is moreover bijective and reflects every relation it is called an isomorphism. A partial homomorphism is a partial function $g \subseteq X \times Y$ which preserves all relations. A partial homomorphism that is injective and reflects all relations is a partial isomorphism. The category $\mathbf{R}(\sigma)$ has σ-structures as objects and σ-structure homomorphisms as morphisms. Note that $\mathbf{Set}$ is isomorphic to $\mathbf{R}(\sigma)$ whenever σ is the empty signature.

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[2] We may also occasionally have to impose additional properties such as symmetry or irreflexivity on our relations to capture data structures such as undirected graphs and acyclic graphs.

2.5 Distribution minion monads

We now describe a class of monads over $\mathbf{R}(\sigma)$ introduced in [Con22] which are related to the algebraic approach for studying CSPs. Functorial constructions have appeared in the CSP literature for many years (e.g. in [CDG13; BBKO21; BGWZ20]). Although they are usually not presented in a categorical manner [Con22] showed that some of these constructions are isomorphic to a variant of the functor $\mathcal{D}_{\mathbb{S}}$ defined on $\mathbf{R}(\sigma)$. Our aim here is to state this correspondence more generally, by appealing to an algebraic construction known as a *minion*. This construction has proved to be an invaluable tool in modern CSP research since it provides an elegant abstraction that uniformly captures some of the main results of the field. From our perspective, we are most interested in the concept of a minion test, which provides a method for studying relaxations of CSPs. Let us make these ideas more precise.

Definition 2.27. A minion $\mathfrak{M}$ is a family of non-empty sets $\mathfrak{M}^{(l)}$ for $l \in \mathbb{N}^+$, equipped with operations $(.)_{\pi} : \mathfrak{M}^{(l)} \to \mathfrak{M}^{(l')}$ (known as *minor operations*) for each pair of numbers $l, l' \in \mathbb{N}^+$ and each map $\pi : [l] \to [l']$. These must satisfy:

- $m_{/\mathrm{id}} = m$ for each $l \in \mathbb{N}^+$, $m \in \mathfrak{M}^{(l)}$ where id is the identity map on $[l]$.
- $(m_{/\pi})_{/\pi'} = m_{/\pi' \circ \pi}$ for each $l, l', l'' \in \mathbb{N}, m \in \mathfrak{M}^{(l)}, \pi : [l] \to [l'], \pi' : [l'] \to [l'']$.

More simply, a minion can be seen as a functor from the skeleton category of non-empty finite sets to the category of non-empty sets. These form a category and a *minion homomorphism* is then a natural transformation between such functors.

Example 2.28. *A* polymorphism *of a relational structure $\mathcal{X} \in \mathbf{R}(\sigma)$ is a morphism $\mathcal{X}^l \to \mathcal{X}$. We denote the set of all polymorphisms with domain $\mathcal{X}^l$ as $\mathsf{Pol}(\mathcal{X})^{(l)}$. The disjoint union of these sets forms the* polymorphism minion $\mathsf{Pol}(\mathcal{X})$.

Associated to any minion is a free structure construction.

Definition 2.29. Let $\mathfrak{M}$ be a minion and take $\mathcal{X} \in \mathbf{R}(\sigma)$. The free structure $F_{\mathfrak{M}}(\mathcal{X})$ is a relational structure defined as:

- The universe of $F_{\mathfrak{M}}(\mathcal{X})$ is the set $\mathfrak{M}^{(n)}$ where $n = |X|$.
- For each symbol $R \in \sigma$, let c be the cardinality of $R^{\mathcal{X}}$. A tuple $\boldsymbol{m} = (m_1, \ldots, m_{ar(R)})$ of elements of $\mathfrak{M}^{(n)}$ belongs to $R^{F_{\mathfrak{M}}(\mathcal{X})}$ if and only if there exists some $m' \in \mathfrak{M}^{(k)}$ such that $m_i = m'_{/\pi_i}$ for each $i \in [ar(R)]$, where $\pi_i : [c] \to [n]$ is the function mapping each tuple $\boldsymbol{x} \in R^{\mathcal{X}}$ to its i-th entry.

Minion tests are then formulated as follows.

Definition 2.30. Let $\mathfrak{M}$ be a minion, a minion test is the function $\text{Test}_{\mathfrak{M}} : \text{Ob}(\mathbf{R}(\sigma))^2 \to \{Yes, No\}$ defined by:

$$(\mathcal{X}, \mathcal{Y}) \mapsto \begin{cases} Yes \text{ if } \mathcal{X} \to F_{\mathfrak{M}}(\mathcal{Y}) \\ No \text{ otherwise.} \end{cases}$$

It is always the case that $\mathcal{X} \to \mathcal{Y}$ implies $\text{Test}_{\mathfrak{M}}(\mathcal{X}, \mathcal{Y}) = Yes$. We say that $\text{Test}_{\mathfrak{M}}$ solves an instance of CSP if the opposite also holds i.e. whenever $\text{Test}_{\mathfrak{M}}(\mathcal{X}, \mathcal{Y}) = Yes$ it holds that $\mathcal{X} \to \mathcal{Y}$.

From the above definition, it is clear that every minion gives rise via its test to an approximation for CSP. These tests can themselves be viewed as decision problems. In many cases, efficient polynomial time algorithms for solving minion tests are known, and coincide with tractable approximation algorithms for CSP. We now introduce the important class of linear minions.

Example 2.31. *A minion $\mathfrak{M}_{\mathbf{S}}$ is linear if there exists a semiring $\mathbf{S}$ and a number $d \in \mathbb{N}^+ \cup \{\aleph_0\}$ (known as the depth of the minion) such that:*

- *The elements $\mathfrak{M}_{\mathbf{S}}^{(l)}$ are $l \times d$ matrices whose entries belong to S, for all $l \in \mathbb{N}^+$.*
- *given $l, l' \in \mathbb{N}^+$, $\pi : [l] \to [l']$ and $m \in \mathfrak{M}^{(l)}$, $m_{/\pi} = PM$, where P is the $l \times l'$ matrix such that for all $i \in [l']$ and $j \in [l]$, the (i, j)-th entry of P is $1_{\mathbf{S}}$ if $\pi(j) = i$ and $0_{\mathbf{S}}$ otherwise.*

In this thesis, we shall further restrict attention to a subclass of linear minions, which we call *distribution minions.*

Example 2.32. *A distribution minion $\mathfrak{D}_{\mathbf{S}}$ over a semiring $\mathbf{S}$ is a linear minion of depth 1 satisfying for all $l \in \mathbb{N}^+$ and for all $\boldsymbol{v} \in \mathfrak{D}^{(L)} : \sum_{v \in \boldsymbol{v}} v = 1_{\mathbf{S}}$.*

It turns out that many well-known approximations algorithms for CSPs arise as distribution minion tests. By this we mean that these algorithms say that the CSP problem is solvable on inputs $\mathcal{X}, \mathcal{Y}$ precisely when $\text{Test}_{\mathfrak{M}}(\mathcal{X}, \mathcal{Y}) = Yes$. Let us list some examples.

Example 2.33. *The basic linear programming (BLP) relaxation arises from $\mathfrak{D}_{\mathbb{R}_{\geq 0}}$ for the semiring of non-negative real numbers [BGWZ20].*

Example 2.34. *The affine integer programming relaxation arises from $\mathfrak{D}_{\mathbb{Z}}$ for the (semi)ring of integers [BGWZ20].*

Example 2.35. *The arc consistency algorithm arises from $\mathfrak{D}_{\mathbb{B}}$ for the boolean semiring [CDG13].*

Next, we explain the extension of distribution monads to $\mathbf{R}(\sigma)$ that was introduced in [Con22]. We refer to these as distribution minion monads.

Example 2.36. *The distribution minion monad* $(\mathbb{D}_{\mathbf{S}}, \eta, \mu)$ *over a semiring* $\mathbf{S} = (S, +_{\mathbf{S}}, \cdot_{\mathbf{S}}, 0_{\mathbf{S}}, 1_{\mathbf{S}})$ *is a monad defined by the following components:*

- $\mathbb{D}_{\mathbf{S}} : \mathbf{R}(\sigma) \to \mathbf{R}(\sigma)$ *is the distribution minion functor defined by:*
    - $\mathbb{D}_{\mathbf{S}}(\mathcal{X})$ *has universe* $\mathbb{D}_{\mathbf{S}}(X) = \mathcal{D}_{\mathbf{S}}(X)$.
    - $\mathbb{D}_{\mathbf{S}}(f) = \mathcal{D}_{\mathbf{S}}(f)$
    - $R^{\mathbb{D}_{\mathbf{S}}\mathcal{X}} = \{(\sum_{\boldsymbol{x} \in R^{\mathcal{X}}} \gamma(\boldsymbol{x}).x_1, \dots, \sum_{\boldsymbol{x} \in R^{\mathcal{X}}} \gamma(\boldsymbol{x}).x_m) \mid \gamma\colon R^{\mathcal{X}} \to S, \sum_{\boldsymbol{x} \in R^{\mathcal{X}}} \gamma(\boldsymbol{x}) = 1\}$[a].
- $\eta_{\mathcal{X}}^{\mathbb{D}_{\mathbf{S}}}(x) = \eta_X^{\mathcal{D}_{\mathbf{S}}}(x)$.
- $\mu_{\mathcal{X}}^{\mathbb{D}_{\mathbf{S}}}(x) = \mu_X^{\mathcal{D}_{\mathbf{S}}}(x)$.

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[a] Note that any distribution $\gamma : R^X \to S$ witnessing $(\varphi_1, \dots, \varphi_m) \in R^{\mathbb{D}_{\mathbf{S}}X}$ can be extended to a distribution γ^* on the set X^m by setting $\gamma(\boldsymbol{x}) = 0$ whenever $\boldsymbol{x} \notin R^{\mathcal{X}}$. It is useful to observe that each φ_i is the marginal of the distribution γ^* at position i.

Note that "at the level of the underlying sets" $(\mathbb{D}_{\mathbf{S}}, \eta, \mu)$ acts exactly the same way as $(\mathcal{D}_{\mathbf{S}}, \eta^{\mathcal{D}}, \mu^{\mathcal{D}})$. This will be a common pattern for all the (co)monads on $\mathbf{R}(\sigma)$ that we study.

Our terminology is justified by the following result.

Theorem 2.37. $F_{\mathfrak{D}_{\mathbf{S}}}(\mathcal{X}) \cong \mathbb{D}_{\mathbf{S}}(\mathcal{X})$.

Proof. As per remark 3.1 in [CŽ23] $F_{\mathfrak{D}_{\mathbf{S}}}(\mathcal{X})$ has the following description:

- The elements of its domain are vectors of length $|X|$ with entries in $\mathbf{S}$ summing to $1_{\mathbf{S}}$.
- For $R \in \sigma$ of arity r, the elements of $R^{F_{\mathfrak{D}_{\mathbf{S}}}(\mathcal{X})}$ are tuples $(P_1 Q, \dots, P_r Q)$, where $Q \in \mathfrak{D}_{\mathbf{S}}^{(|R^{\mathcal{X}}|)}$ is a vector of length $|R^{\mathcal{X}}|$ having entries in $\mathbf{S}$ and, for $i \in [r]$, P_i is the $|X| \times |R^{\mathcal{X}}|$ matrix whose $(x, \boldsymbol{x})$-th entry is $1_{\mathbf{S}}$ if $\boldsymbol{x}_i = x$, and $0_{\mathbf{S}}$ otherwise.

From this description, it is clear that each element of the universe is a vector $\boldsymbol{v}$ which can be written as a formal sum satisfying $\sum_{v \in \boldsymbol{v}} v = 1$. This is in bijective correspondence with $\mathbb{D}_{\mathbf{S}}(X)$. Moreover, the relations are readily seen to be in bijective correspondence as well. For this, note that functions $\gamma : R^{\mathcal{X}} \to \mathbf{S}$ satisfying $\sum_{\boldsymbol{x} \in R^{\mathcal{X}}} \gamma(\boldsymbol{x}) = 1_{\mathbf{S}}$ are in bijective correspondence with the vectors Q defined above. From there we can see that the vector $P_i Q$ corresponds to the formal sum $\sum_{\boldsymbol{x} \in R^{\mathcal{X}}} \gamma(\boldsymbol{x}).\boldsymbol{x}_i$.

$\square$

This theorem unifies and generalises propositions 7.11 and 7.14 in [Con22]. It also simplifies the proof of theorem 7.16 in the same document. As an immediate corollary, we obtain a connection between minion tests and Kleisli morphisms.

Corollary 2.38. $\mathsf{Test}(\mathcal{X}, \mathcal{Y})_{\mathfrak{D}_{\mathbf{S}}} = Yes$ *iff* $\mathcal{X} \to \mathbb{D}_{\mathbf{S}}\mathcal{Y}$.

This result shows that distribution minion monads provide a categorical abstraction for the corresponding distribution minions, which in turn opens up the possibility of using categorical tools to gain further insights into the properties of such minions.

Remark 2.39. *There are several examples of interesting linear minions which are not distribution minions. These include the minion for the combined BLP and AIP algorithm of [BGWZ20], as well as the hierarchies of minions introduced in [CŽ23] which correspond to the Sherali-Adams relaxations hierarchy of relaxations for CSP. We leave it as future work to see if the free structures of these minions admit characterisations similar to the one we have provided for distribution minions.*

2.6 Game comonads

We now introduce the class of game comonads first defined in [ADW17; AS21] and describe their connection to finite model theory, the field which studies the behaviour of logics on finite models. Unlike classical model theory which finds its origins in mathematics and the study of infinite objects, finite model theory grew out of computer science. Because objects held or manipulated by a computer, such as graphs, strings, or programs are always finite, one must study finite structures to understand computation.

Game comonads provide a categorical abstraction for Spoiler-Duplicator games, one of the central tools of finite model theory. Intuitively, these are two-player games where one of the players (Spoiler) is trying to show that two structures are different, while the other (Duplicator) tries to show that they are the same. In each game, Duplicator has a winning strategy whenever the two structures are equivalent with respect to a certain logic fragment associated with the game. Duplicator's winning strategies in each of these games can be captured abstractly in the Kleisli category of a comonad $\mathbb{G}_k$ associated with the game. The index k can be viewed as a resource parameter bounding the amount of access the players have to the underlying structures. coKleisli morphisms $\mathbb{G}_k\mathcal{X} \to \mathcal{Y}$ are in correspondence with Duplicator's winning strategies for a one-sided variant of the games, while coKleisli isomorphisms $\mathcal{A} \cong_{\mathbb{G}_k} \mathcal{B}$ correspond to a bijective variant of the game. For the sake of brevity, we will focus throughout this thesis on only two game comonads. The first is known as the Ehrenfeucht-Fraisse comonad $\mathbb{E}_k$, and the second is the pebbling comonad $\mathbb{P}_k$. Let us now give a more formal description of these constructions. We assume some basic understanding of first-order logic in what follows.

Remark 2.40. *We stress that the game comonads program goes well beyond the two classes of games we consider here. For example, there are comonads for modal logic [AS21], hybrid logic [AM22], guarded logic [AM21], and monadic second-order logic [JMS24]. I believe that most of the results of this thesis should apply to these other game comonads as well. An axiomatic account of game comonads is also available in[AR23]. Moreover, interesting connections have been made with the theory of homomorphism indistinguishability [DJR21; Reg22] and with the study of homomorphism preservation theorems [AR24b]. We refer to [AR24a] for a survey of game comonads.*

2.6.1 Ehrenfeucht-Fraïssé comonad

We describe the original Spoiler-Duplicator game introduced by Ehrenfeucht [Ehr61] and inspired by earlier work published in Fraïssé's thesis.

Definition 2.41. The back-and-forth Ehrenfeucht-Fraïssé (EF) game between $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ is played as follows:

At the start of each round, Spoiler chooses an element of either $\mathcal{X}$ or $\mathcal{Y}$. Duplicator then chooses an element in the other structure. After k rounds their choices create sequences $[x_1, \dots, x_k]$ and $[y_1, \dots, y_k]$. Duplicator wins a play of the game if $\{(x_i, y_i) | 1 \leq i \leq k\}$ is a partial isomorphism. We say that Duplicator wins the k-round game if they have a strategy for winning plays up to length k, regardless of how Spoiler plays.

The Ehrenfeucht-Fraïssé theorem links the k-round version of this game to the expressive power of $\mathcal{L}_k$, the fragment of first-order logic where sentences have quantifier rank at most k i.e. where the depth of nesting of the quantifiers $\exists$ and $\forall$ is at most k.

Theorem 2.42. *For relational structures $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ the following are equivalent:*

- $\mathcal{X} \cong^{\mathcal{L}_k} \mathcal{Y}$.
- *Duplicator has a winning strategy in the back-and-forth EF game between $\mathcal{X}$ and $\mathcal{Y}$.*

Theorems of this kind linking Spoiler-Duplicator games to logic are found throughout finite model theory. In our work, we will be more concerned with the *one-sided* and *bijective* variants of the EF game. These are defined as follows.

Example 2.43. *The one-sided Ehrenfeucht-Fraïssé (EF) game from $\mathcal{X} \in \mathbf{R}(\sigma)$ to $\mathcal{Y} \in \mathbf{R}(\sigma)$ is played as follows:*

At the start of each round, Spoiler chooses an element of $\mathcal{X}$. Duplicator then chooses an element *in $\mathcal{Y}$. After k rounds their choices create sequences $[x_1, \ldots, x_k]$ and $[y_1, \ldots, y_k]$. Duplicator wins a play of length k if $\{(x_i, y_i)|1 \leq i \leq k\}$ is a partial homomorphism.*

Example 2.44. *The bijective Ehrenfeucht-Fraïssé (EF) game from $\mathcal{X} \in \mathbf{R}(\sigma)$ to $\mathcal{Y} \in \mathbf{R}(\sigma)$ is played as follows:*

If $|\mathcal{X}| \neq |\mathcal{Y}|$ then Spoiler wins. Otherwise, at each round i Duplicator chooses a bijection $f : \mathcal{X} \to \mathcal{Y}$, and Spoiler chooses an element $x_i \in \mathcal{X}$. This determines the choice $y_i = f(x_i)$ for Duplicator. Duplicator wins a play of length k if $\{(x_i, y_i)|1 \leq i \leq k\}$ is a partial isomorphism.

These games are related to the existential-positive fragment and counting quantifier extension of $\mathcal{L}_k$ respectively. We denote these fragments by $\exists\mathcal{L}_k$ and $(\#)\mathcal{L}_k$. In the existential positive fragment, sentences cannot make use of the $\forall$ quantifier or the $\neg$ operation. The counting quantifier extension of $\mathcal{L}_k$ is obtained by adding additional quantifiers $\exists^{\geq m}$ and $\forall^{\leq m}$ to $\mathcal{L}_k$. These are to be read as *"there exists at least m"* and *"for all but m"*.

Theorem 2.45. *For relational structures $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ the following are equivalent:*

- $\mathcal{X} \cong^{\exists\mathcal{L}_k} \mathcal{Y}$.
- *Duplicator has a winning strategy in the one-way EF game from $\mathcal{X}$ to $\mathcal{Y}$ and the one-sided EF game from $\mathcal{Y}$ to $\mathcal{X}$.*

Theorem 2.46. *For relational structures $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ the following are equivalent:*

- $\mathcal{X} \cong^{(\#)\mathcal{L}_k} \mathcal{Y}$.
- *Duplicator has a winning strategy in the bijective EF game between $\mathcal{X}$ and $\mathcal{Y}$.*

Before describing the comonad which captures EF games we need one final piece of notation. Let s and s' be two lists of elements with entries from the same set. We shall write $s \sqsubseteq s'$ to mean that s is a prefix of s'. Moreover, we say that s and s' are prefix comparable, written $s \uparrow s'$ whenever $s \sqsubseteq s'$ or $s' \sqsubseteq s$. The EF comonad is then defined as follows:

Example 2.47. *The EF comonad $(\mathbb{E}_k, \varepsilon, \delta)$ is a comonad defined by the following components:*

- $\mathbb{E}_k : \mathbf{R}(\sigma) \to \mathbf{R}(\sigma)$ *is the EF functor defined by:*
  - $\mathbb{E}_k(\mathcal{X})$ *has universe* $\mathbb{E}_k(X) = L_k^+(X)$.
  - $\mathbb{E}_k(f) = L_k^+(f)$.
  - $R^{\mathbb{E}_k\mathcal{X}}$ *consists of tuples* $(s_1, \ldots s_n)$ *satisfying:*
    - $s_i \uparrow s_j$ *for all* $i, j \in [n]$.

    - $(\varepsilon(s_1), \ldots \varepsilon(s_n)) \in R^{\mathcal{X}}$.
- $\varepsilon_{\mathcal{X}} = \varepsilon_{\mathcal{X}}^{L_k^+}$.
- $\delta_{\mathcal{X}} = \delta_{\mathcal{X}}^{L_k^+}$.

It turns out that coKleisli maps for the EF comonad correspond to perfect Duplicator strategies in the one-way EF game. We thus extend our previous correspondence to the following theorem which shows a triangle of equivalences between logic, games, and categorical semantics.

Theorem 2.48. *For relational structures $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ the following are equivalent:*

- $\mathcal{X} \cong^{\exists\mathcal{L}_k} \mathcal{Y}$.
- *Duplicator has a winning strategy in the one-way EF game from $\mathcal{X}$ to $\mathcal{Y}$ and the one-sided EF game from $\mathcal{Y}$ to $\mathcal{X}$.*
- $\mathbb{E}_k\mathcal{X} \to \mathcal{Y}$ *and* $\mathbb{E}_k\mathcal{Y} \to \mathcal{X}$.

Similarly, for coKleisli isomorphisms we have the following result.

Theorem 2.49. *For relational structures $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ the following are equivalent:*

- $\mathcal{X} \cong^{(\#)\mathcal{L}_k} \mathcal{Y}$.
- *Duplicator has a winning strategy in the bijective EF game between $\mathcal{X}$ and $\mathcal{Y}$.*
- $\mathcal{X} \cong_{\mathbb{E}_k} \mathcal{Y}$.

Remark 2.50. *[AS21] also introduces an intermediate back-and-forth form of Kleisli equivalence which characterises the back-and-forth EF game. We omit a description of this as it does not play a major role in the thesis.*

Finally, we note that there is a surprising and beautiful connection between the coalgebras of $\mathbb{E}_k$ and tree-depth, a combinatorial invariant of finite structures which is used extensively in parameterised algorithms and complexity theory [FG06; CFK+15].

Definition 2.51. A *forest cover* for a graph $G = (V, \sim)$ is a forest $(F, \leq)$ such that $V \subseteq F$ and if $v \sim v'$ then $v \leq v'$ or $v' \leq v$. The *tree-depth* $\mathsf{td}(G)$ is defined to be $\mathsf{min}_F \mathsf{ht}(\mathsf{F})$ where F ranges over forest covers of G.

It turns out that $\mathbb{E}_k$-coalgebras are precisely forest covers. We recall that the Gaifman graph of a relational structure $\mathcal{X}$ is the graph whose vertices are given by the universe X of $\mathcal{X}$ and where $x \sim x'$ whenever x and x' appear together in some tuple of some relation in $\mathcal{X}$.

Theorem 2.52. *There is a bijective correspondence between:*

- $\mathbb{E}_k$*-coalgebras* $\alpha : \mathcal{X} \to \mathbb{E}_k\mathcal{X}$.
- *Forest covers of the Gaifman graph of* $\mathcal{X}$ *of height* $\leq k$.

Moreover, we can characterise tree-depth through the concept of a coalgebra number.

Definition 2.53. The $\mathbb{E}_k$-coalgebra number of a structure $\mathcal{X} \in \mathbf{R}(\sigma)$, written as $\zeta^{\mathbb{E}_k}(\mathcal{X})$ is the least k such that there exists a $\mathbb{E}_k$-coalgebra on $\mathcal{X}$.

Theorem 2.54. $\zeta^{\mathbb{E}_k}(\mathcal{X})$ *is equal to the tree-depth of* $\mathcal{X}$.

It is worth mentioning that a tacit awareness of connections between tree-depth, $\mathcal{L}_k$, and EF games seems to be common knowledge among finite model theorists. Part of the appeal of game comonads is that they provide a rigorous mathematical framework which formalises this tacit knowledge.

2.6.2 Pebbling comonad

We describe pebble games, a class of Spoiler-Duplicator games that play a prominent role in computer science.

Example 2.55. *The back-and-forth* k*-pebble game between* $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ *is played as follows:*

Each player has access to k *pebbles. At the start of each round, Spoiler picks one of their pebbles* $p \in [k]$ *and places it on an element of either* $\mathcal{X}$ *or* $\mathcal{Y}$*. Duplicator then picks their corresponding pebble and places it on an element of the other structure. Note that Spoiler can choose to remove a previously placed pebble and move it to a new element. In this way, at the end of each round, the placement of pebbles determines a window of size* $\leq k$ *onto the structures. We say that Duplicator wins the game if they have a strategy which allows them to play indefinitely while ensuring that the two windows always represent a partial isomorphism between the structures.*

The one-way and bijective variants of the k-pebble game are defined in a manner similar to the EF game. Next, we describe the pebbling comonad $\mathbb{P}_k$.

Example 2.56. *The pebbling comonad* $(\mathbb{P}_k, \varepsilon, \delta)$ *is a comonad defined by the following components:*

- $\mathbb{P}_k : \mathbf{R}(\sigma) \to \mathbf{R}(\sigma)$ *is the pebbling functor defined by:*
  - $\mathbb{P}_k(\mathcal{X})$ *has universe* $\mathbb{P}_k(X) = L^+[k](X)$.

  - $\mathbb{P}_k(f) = L^+[k](f)$.
  - $R^{\mathbb{P}_k \mathcal{X}}$ *consists of tuples* $(s_1, \dots s_n)$ *satisfying:*
    - $s_i \uparrow s_j$ *for all* $i, j \in [n]$.
    - $(\varepsilon(s_1), \dots \varepsilon(s_n)) \in R^{\mathcal{X}}$.
    - *The pebble index of the last move in each* s_i *does not appear in the suffix of* s_i *in* s_j *for any* s_j. *We refer to this as the* active pebble condition.
- $\varepsilon_{\mathcal{X}} = \varepsilon^{L^+[k]}_{\mathcal{X}}$.
- $\delta_{\mathcal{X}} = \delta^{L^+[k]}_{\mathcal{X}}$.

Just as in the case of $\mathbb{E}_k$, we have theorems linking logic, games, and categorical semantics for $\mathbb{P}_k$[3]. Writing $\mathcal{L}^k$ for the k-variable fragment of first-order logic we have:

Theorem 2.57. *For relational structures* $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ *the following are equivalent:*

- $\mathcal{X} \cong^{\exists \mathcal{L}^k} \mathcal{Y}$.
- *Duplicator has a winning strategy in the one-way* k*-pebble game from* $\mathcal{X}$ *to* $\mathcal{Y}$ *and one-way* k*-pebble game from* $\mathcal{Y}$ *to* $\mathcal{X}$.
- $\mathbb{P}_k \mathcal{X} \to \mathcal{Y}$ *and* $\mathbb{P}_k \mathcal{Y} \to \mathcal{X}$.

Theorem 2.58. *For relational structures* $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ *the following are equivalent:*

- $\mathcal{X} \cong^{(\#)\mathcal{L}^k} \mathcal{Y}$.
- *Duplicator has a winning strategy in the bijective* k*-pebble game between* $\mathcal{X}$ *and* $\mathcal{Y}$.
- $\mathcal{X} \cong_{\mathbb{P}_k} \mathcal{Y}$.

The coalgebra number of $\mathbb{P}_k$ coincides with *tree-width* which is perhaps the most well-studied and important combinatorial invariant used in parameterised complexity theory.

Definition 2.59. A k-pebble forest cover for a graph $G = (V, \sim)$ is a forest cover $(V, \leq)$ together with a pebbling function $p : V \to [k]$ such that if $v \sim v'$ with $v \leq v'$, then for all $w \in (v, v'], p(v) \neq p(w)$. Let n be the minimum number such that G has a k-pebble forest cover. The *tree-width* tw of G is equal to $n - 1$[a].

[a] The -1 is to ensure that trees have treewidth equal to 1.

---

[3] In fact in this case the correspondence extends even further, and reveals a connection with two extremely well-studied approximation algorithms for CSP and structure isomorphism. These are the k-consistency [ABD07] and k-Weisfeiler-Leman[Kie20] algorithms respectively.

By analogy to the case of $\mathbb{E}_k$, we have:

Theorem 2.60. *There is a bijective correspondence between:*

- $\mathbb{P}_k$*-coalgebras* $\alpha : \mathcal{X} \to \mathbb{P}_k\mathcal{X}$.
- k*-pebble forest covers of the Gaifman graph of* $\mathcal{X}$.

Theorem 2.61. $\zeta^{\mathbb{P}_k}(\mathcal{X}) = 1 + \mathsf{tw}(\mathcal{X})$.

2.7 Distributive laws

So far we have argued that monads model effectful computation through Kleisli maps of the form $X \to MY$. Dually, we have seen that comonads model contextual computations through their coKleilsi categories.

A natural question now arises. Given a monad M and a comonad W can we compose computations which have both a contextual and a nondeterministic flavour to them? In [PW02] Power and Watanabe demonstrated that the existence of a distributive law gives a sufficient (but not necessary) condition for obtaining a biKleisli category where morphisms of type $WA \to MB$ are composed. Such a law consists of a natural transformation $\kappa\colon WM \to MW$ satisfying the four axioms in the definition below.

Definition 2.62. A *mixed distributive law of a comonad* (W, ε, δ) *over a monad* (M, η, μ) is a natural transformation $\kappa\colon WM \to MW$ satisfying the following diagrams:

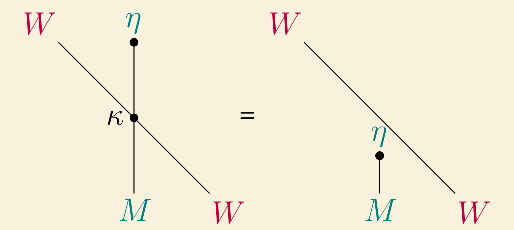
$$\begin{tikzcd}
 & W \arrow[dl, "W\eta"'] \arrow[dr, "\eta W"] & \\
WM \arrow[rr, "\kappa"'] & & MW
\end{tikzcd} \tag{2.63}$$

$$\begin{tikzcd}
WM \arrow[rr, "\kappa"'] \arrow[dr, "\varepsilon M"'] & & MW \arrow[dl, "M\varepsilon"] \\
 & M &
\end{tikzcd} \tag{2.64}$$

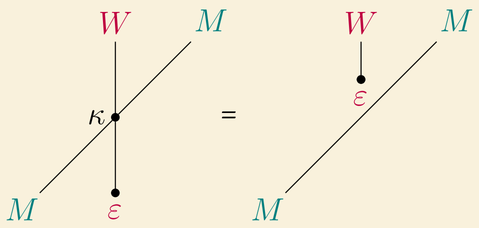
$$\begin{tikzcd}
WMM \arrow[r, "\kappa M"] \arrow[d, "W\mu"'] & MWM \arrow[r, "M\kappa"] & MMW \arrow[d, "\mu W"] \\
WM \arrow[rr, "\kappa"'] & & MW
\end{tikzcd} \tag{2.65}$$

$$\begin{tikzcd}
WM \arrow[rr, "\kappa"'] \arrow[d, "\delta M"] & & MW \arrow[d, "M\delta"'] \\
WWM \arrow[r, "W\kappa"] & WMW \arrow[r, "\kappa W"] & MWW
\end{tikzcd} \tag{2.66}$$

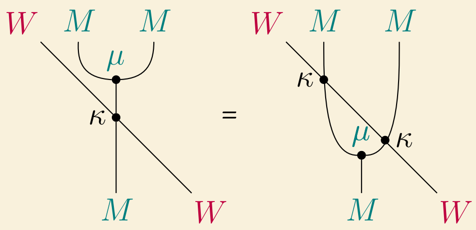
Or in terms of string diagrams:



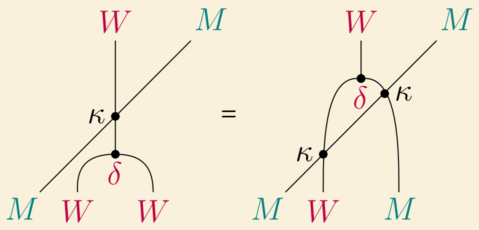
(2.63)



(2.64)



(2.65)



(2.66)

Remark 2.67. *This definition should not be confused with that of a monad over comonad distributive law which is given by a natural transformation* $MW \to WM$ *satisfying four conditions that are dual to the ones described above. The latter has been useful in structural operational semantics [TP97]. Both of these concepts are sometimes referred to as mixed distributive laws to differentiate them from the related notions of monad over monad and comonad* *over comonad distributive laws.*

An example of a distributive law is to take W to be the prefix list comonad L^+ and M to be the partiality monad, i.e. the Haskell 'Maybe' monad [Orc14]. Intuitively, this law takes a non-empty list of potentially undefined values and returns either (1) undefined if the last value of the list is undefined, or (2) the list of those values that are defined. In a sense, the partiality monad is a rudimentary form of nondeterminism–either a value is determined or undetermined. As we will show in the next chapter, for certain other nondeterministic monads M no such distributive law can exist.

We call (2.63), (2.64), (2.65), (2.66) the unit, counit, multiplication, and comultiplication axioms respectively. Note that a given natural transformation can satisfy each axiom independently of the others. Thus, one can consider relaxations of the above definition where some of the diagrams are excluded. Many of these relaxations have been studied in existing literature, e.g. see [Jac17]. One relaxation that will be important in our work is the case where κ is only required to satisfy the unit axiom. We refer to this case as a *pointed law*.

Definition 2.68. A *pointed law of a functor* W *over the pointed endofunctor* (M, η) is a natural transformation $\kappa\colon WM \to MW$ satisfying (2.63).

Note that there is no requirement in this definition for W to have a comonad structure, we only require that it is an endofunctor and M does not need to have an associated multiplication operation. If we do require compatibility with the multiplication in (M, η, μ), we obtain the definition for a Kleisli law [Jac17]. A *Kleisli law of an endofunctor* W *over the monad* (M, η, μ) is a natural transformation $\kappa\colon WM \to MW$ satisfying (2.63) and (2.65).

In fact, Kleisli laws can be defined more generally when one has two monads defined on different base categories together with a functor between these categories. We can think of such laws as a generalised notion of morphism between monads on different categories.

Definition 2.69. Let (M, η^M, μ^M), $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}})$ be monads defined on categories $\mathbf{C}, \mathbf{D}$, and let $U : \mathbf{D} \to \mathbf{C}$ be a functor. A Kleisli law is a natural transformation $\lambda : U\mathbb{M} \to MU$ satisfying the following axioms:

$$\lambda \circ U\eta^{\mathbb{M}} = \eta^M U$$
$$\lambda \circ U\mu^{\mathbb{M}} = \mu^M U \circ M\lambda \circ \lambda\mathbb{M}$$

We also require the dual concept of a coKleisli law.

Definition 2.70. Let $(W, \varepsilon^W, \delta^W)$, $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}})$ be comonads defined on categories $\mathbf{C}, \mathbf{D}$, and let $U : \mathbf{D} \to \mathbf{C}$ be a functor. A coKleisli law is a natural transformation $\lambda : WU \to U\mathbb{W}$ satisfying the following axioms:

$$U\varepsilon^{\mathbb{W}} \circ \lambda = \varepsilon^W U$$
$$U\delta^{\mathbb{W}} \circ \lambda = \lambda\mathbb{W} \circ W\lambda \circ \delta^W U$$

2.7.1 BiKleisli categories

We now describe biKleisli categories in more detail.

Definition 2.71. Let $\kappa : WM \to MW$ be a mixed distributive law of (W, ε, δ) over (M, η, μ). Then:

- There is a comonad $(\bar{W}, \varepsilon^{\bar{W}}, \delta^{\bar{W}})$ on $\mathbf{Kl}(M)$ where:
  - $\bar{W}X = WX$,
  - given $f : X \to MY$, $\bar{W}f = \kappa_Y \circ Wf$,
  - $\varepsilon^{\bar{W}}_X = \eta_X \circ \varepsilon_X$,
  - $\delta^{\bar{W}}_X = \eta_{W^2X} \circ \delta_X$.
- There is a monad $(\bar{M}, \eta^{\bar{M}}, \mu^{\bar{M}})$ on $\mathbf{coKl}(W)$ where:
  - $\bar{M}X = MX$
  - given $f : WX \to Y$, $\bar{M}f = Mf \circ \kappa_X$.
  - $\eta^{\bar{M}}_X = \eta_X \circ \varepsilon_X$,
  - $\mu^{\bar{M}}_X = \mu_X \circ \varepsilon_{M^2X}$.

Moreover, the (co)Kleisli categories of these constructions coincide, resulting in a so-called biKleisli category $\mathbf{biKl}(W, M)$ where:

- $\mathbf{biKl}(W, M)_0 = \mathbf{C}_0$
- A morphism $X \to Y$ is a morphism of type $WA \to MB$ in $\mathbf{C}$.
- The identity morphism on X is given by $\eta_X \circ \varepsilon_X$.
- The composition of $f : WX \to MY$ with $g : WY \to MZ$ is given by $\mu_Z \circ Mg \circ \kappa_Y \circ Wf \circ \delta_X$.

Thus we have $\mathbf{coKl}(\bar{W}) \cong \mathbf{Kl}(\bar{M}) \cong \mathbf{biKl}(W, M)$.

To conclude this chapter let us discuss why having a biKleisli category would be desirable with a concrete example. Take the comonad $\mathbb{E}_k$ and the monad $\mathbb{D}_{\mathbf{S}}$ over the semiring of real numbers. We know that coKleisli maps $\mathbb{E}_k\mathcal{X} \to \mathcal{Y}$ correspond to perfect Duplicator strategies in the one-way EF game. Dually, we know that Kleisli maps $\mathcal{X} \to \mathbb{D}_{\mathbf{S}}\mathcal{Y}$ correspond to the BLP approximation algorithm for CSP. There are thus two ways of interpreting a morphism $\mathbb{E}_k\mathcal{X} \to \mathbb{D}_{\mathbf{S}}\mathcal{Y}$. We can either view this as a Duplicator strategy in a one-way Spoiler-Duplicator game where the structure $\mathcal{Y}$ is replaced with $\mathbb{D}_{\mathbf{S}}\mathcal{Y}$ or as the BLP approximation where the structure $\mathcal{X}$ is replaced with $\mathbb{E}_k\mathcal{X}$. The existence of a biKleisli category would imply that there is non-trivial compositional structure between such maps, which respects the structures of both $\mathbb{E}_k$ and $\mathbb{D}_{\mathbf{S}}$. This leads to a host of interesting questions worth exploring. For example:

- A Spoiler-Duplicator game with target structure $\mathbb{D}_{\mathbf{S}}\mathcal{Y}$ can be viewed as a game where Duplicator responds to each choice that Spoiler makes in a probabilistic manner. Is it possible to identify a suitable probabilistic variant of the logic $\mathcal{L}_k$ which corresponds to this game?
- Consider the lifted comonad $\bar{E}_k$ which can be defined over $\mathbf{Kl}(\mathbb{D}_{\mathbf{S}})$. What do its coalgebras correspond to? Given the probabilistic nature of $\mathbb{D}_{\mathbf{S}}$, a reasonable guess would be that these coalgebras are a probabilistic variant of forest covers, and that the coalgebra number of $\bar{E}_k$ gives rise to a probabilistic notion of tree-depth. Would these notions be of interest in parameterised complexity?
- Any Kleisli morphism gives rise to a biKleisli morphism via the sequence of maps $\mathbb{E}_k\mathcal{X} \xrightarrow{\varepsilon} \mathcal{X} \to \mathbb{D}_{\mathbf{S}}\mathcal{Y}$. A similar fact holds for coKleisli morphisms via the sequence $\mathbb{E}_k\mathcal{X} \to \mathcal{Y} \xrightarrow{\eta} \mathbb{D}_{\mathbf{S}}\mathcal{Y}$. By analogy with the concept of minion testing, we can ask, when does the converse of each of these statements hold? Moreover, are there efficient algorithms for detecting the existence of biKleisli morphisms? This question is perhaps less interesting in the case where $\mathbf{S}$ is the semiring of real numbers since the BLP algorithm is already tractable. However, there are other semirings for which determining the existence of a Kleisli map $\mathcal{X} \to \mathbb{D}_{\mathbf{S}}\mathcal{Y}$ is not efficient[4]. In these cases, it would be interesting to obtain tractable approximation algorithms for detecting Kleisli maps.

If categorical tools are to help answer such questions we must first ask if there exists a distributive law of $\mathbb{E}_k$ over $\mathbb{D}_{\mathbf{S}}$. We look at this topic in the next chapter.

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[4] We shall see in chapter 5 that the partial semiring of projectors on a Hilbert space $\boldsymbol{H}$ has this property.

3

When do distributive laws exist?

We concluded the last chapter by highlighting the desirable properties of a distributive law between game comonads and distribution minion monads. The natural question to ask next is:

Question 3.1. *For which game comonads $\mathbb{G}_k$ and which distribution minion monads $\mathbb{D}_{\mathbf{S}}$ does there exist a comonad over monad distributive law $\kappa : \mathbb{G}_k \mathbb{D}_{\mathbf{S}} \to \mathbb{D}_{\mathbf{S}} \mathbb{G}_k$?*

Answering this question and some of its variants is the goal of this chapter. We begin by explaining the no-go theorems from [KS24] which rule out the existence of a distributive law between many of the (co)monads we have introduced thus far. Even though we are mainly interested in (co)monads over $\mathbf{R}(\sigma)$ these no-go results are more general, and apply to suitable (co)monads in any category which admits **Set** as a reflective or coreflective subcategory. In the final part of the chapter we prove a positive result involving a restricted family of semirings which we call *single measurement semirings.*

We note that the study of comonad over monad distributive laws has received little attention in the literature so far. However, there is a very well-studied and related concept known as a monad over monad distributive law. In this case the existence of a suitable natural transformation $TS \to ST$ provides a sufficient condition for when two monads on functors S and T determine a monad structure on the composed functor $S \circ T$. Many general-purpose techniques were developed to come up with such laws (e.g. [BHKR15; MM07; MM08; Par20; DPS18]). Recent research has also revealed conditions under which no laws can exist. In particular, a result in [VW06] attributed to Plotkin showed that the powerset monad does not distribute over the distribution monad. Zwart and Marsden [ZM22] vastly generalise this result to obtain sufficient conditions for the non-existence of a distributive law between pairs of monads. Our no-go result is partly inspired by their work. However, our techniques are quite different from theirs. This is because their results rely heavily on well-established algebraic presentations of monads. Since the analogous notion of coalgebraic presentations of comonads is an active area of research [DS21] we cannot readily adapt their tools to our setting.

3.1 No-go theorems in Set

3.1.1 Containers over powerset

The first result of [KS24] was to show that if a container $F\colon \mathbf{Set} \to \mathbf{Set}$ has a pointed law κ over $(\mathcal{P}, \eta)$, then κ sends elements in a container with non-empty sets at every position to the set of all possible ways to sample at each position. As a corollary, we showed that there exists a unique pointed law of a container F over $(\mathcal{P}^+, \eta^+)$. As every distributive law is also a pointed law, this theorem prunes the space of possible distributive laws, and is, therefore, an important building block for our no-go theorems.

Theorem 3.2 ([KS24]). *If $F = [S \lhd P]$ is a container and there exists a pointed law $\kappa\colon F\mathcal{P} \to \mathcal{P}F$, then for all sets X and elements $(s, l\colon P(s) \to \mathcal{P}^+(X)) \in F(\mathcal{P}^+(X)) \subseteq F(\mathcal{P}(X))$,*

$$\kappa_X(s,l) = \{(s, j\colon P(s) \to X) \mid \forall p \in P(s), j(p) \in l(p)\}. \tag{3.3}$$

The uniqueness theorem is proved in three stages using a 'Plotkin-style' argument. This style of argument involves, at each stage, chasing specific elements along naturality squares for cleverly chosen functions. We then draw conclusions either from the direct image or pre-image of the element under a component of the pointed law. The first two stages involve demonstrating that equation (3.3) holds for all elements (s, l) which satisfy the following pairwise disjoint condition:

(PD) For all $p, q \in P(s)$, if $p \neq q$, then $l(p) \cap l(q) = \varnothing$.

The first stage uses the unit axiom and involves chasing the κ-naturality square for a 'collapse' function c. To convey some intuition we sketch the proof for the case where $F = L^+$. For full details the reader is referred to [KS24] or Nihil Shah's DPhil thesis [Sha24]. Consider $\mathbf{L} = [X_1, \ldots, X_n]$ in $L^+\mathcal{P}^+(X)$. For $X_i = \{x_i\}$ singletons, $\kappa_X([\{x_1\}, \ldots, \{x_n\}) = \{[x_1, \ldots, x_n]\}$ follows directly from the unit axiom. More generally, as each X_i is in $\mathcal{P}^+X$, we can choose some $y_i \in X_i$. We consider a "collapse the X_i" function $c\colon X \to X$ which maps every $x_i \in X$ to y_i and is the identity otherwise. By the (PD) condition, c is indeed a total function, i.e. single-valued. Chasing the κ-naturality square of c and utilizing the unit axiom allows us to conclude that $\kappa_X(\mathbf{L}) \subseteq \{[x_1, \ldots, x_n] \mid \forall i \in [\mathsf{n}], x_i \in c^{-1}(y_i) = X_i\}$. Intuitively, this argument generalises to containers as the way we defined the collapse function c only depended on the set at each position.

Lemma 3.4 (Collapse, [KS24])**.** *Suppose F and κ satisfy the hypotheses of Theorem 3.2, then for all sets X and $(s, l) \in F(\mathcal{P}^+(X))$ satisfying (PD), $\varnothing \neq \kappa_X(s, l) \subseteq \{(s, j) \mid \forall p \in P(s), j(p) \in l(p)\}$.*

The second stage involves chasing the κ-naturality square of a 'swap' bijection b to obtain the opposite inclusion. In the case where $F = L^+$ and $\mathbf{L} = [X_1, \ldots, X_n] \in F(\mathcal{P}^+(X))$ satisfying (PD), by Lemma 3.4, there must exist $[z_1, \ldots, z_n] \in \kappa_X(\mathbf{L})$ with $z_i \in X_i$. For every $[x_1, \ldots, x_n]$ with $x_i \in X_i$, consider the bijection $b\colon X \to X$ which for each $i \in [\mathsf{n}]$ swaps z_i and x_i. Chasing the κ-naturality square of b and utilizing (PD) allows us to conclude that $\kappa_X(\mathbf{L}) \supseteq \{[x_1, \ldots, x_n] \mid \forall i \in [\mathsf{n}], x_i \in X_i\}$.

Lemma 3.5 (Swap, [KS24])**.** *Suppose F and κ satisfy the hypotheses of Theorem 3.2, then for all sets X and $(s, l) \in F(\mathcal{P}^+(X))$ satisfying (PD), $\kappa_X(s, l) \supseteq \{(s, j) \mid \forall p \in P(s), j(p) \in l(p)\}$.*

The final stage of the proof of Theorem 3.2 is a 'relabel' argument which demonstrates that the condition (PD) does not constitute a loss of generality. In the case where $F = L^+$ and $\mathbf{L} = [X_1, \ldots, X_n] \in F(\mathcal{P}^+(X))$ (the X_i are not necessarily pairwise-disjoint), we first consider the set $Y = [\mathsf{n}] \times X$. There is a list $\mathbf{L}' = [X'_1, \ldots, X'_n] \in F(\mathcal{P}(Y))$ where $X'_i = \{(i, x_i) \mid x_i \in X_i\} \in \mathcal{P}(Y)$ satisfying (PD). By construction, $F(\mathcal{P}(\pi_2))(\mathbf{L}') = \mathbf{L}$ where $\pi_2\colon Y \to X$ is the projection onto the second component. Since $\mathbf{L}'$ satisfies (PD), we can use Lemma 3.4, Lemma 3.5 and the naturality square of π_2 to compute that

$$\begin{aligned}
\kappa_X(\mathbf{L}) &= \kappa_X(F\mathcal{P}\pi_2(\mathbf{L}')) \\
&= \mathcal{P}F\pi_2(\kappa_Y(\mathbf{L}')) \\
&= \mathcal{P}F\pi_2(\{[(1, x_1), \ldots, (n, x_n)] \mid \forall i \in [\mathsf{n}], x_i \in X_i\}) \\
&= \{[x_1, \ldots, x_n] \mid \forall i \in [\mathsf{n}], x_i \in X_i\}.
\end{aligned}$$

Generalising this argument to an arbitrary container completes the proof of Theorem 3.2. It is easy to check that $\kappa^+\colon F\mathcal{P}^+ \to \mathcal{P}^+F$ with components having the same elementwise definition as κ in Equation (3.3) satisfies diagram (2.63) and is natural yielding the following consequence from the proof of Theorem 3.2.

Corollary 3.6. *For every container* $F = [S \triangleleft P]$*, there exists a unique pointed law* $\kappa^+ \colon F\mathcal{P}^+ \to \mathcal{P}^+F$ *of* F *over* $(\mathcal{P}^+, \eta^+)$*, where* $\kappa^+(s, l) = \kappa(s, l)$ *defined in* (3.3).

Remark 3.7. *Equation* (3.3) *is sometimes referred to as the Jacobs law [BKS23], though its definition appears in Barr [Bar70]. For every weak-pullback preserving functor* $T \colon \mathbf{Set} \to \mathbf{Set}$*, the Jacobs law determines the unique 'monotone' Kleisli law* $\kappa \colon T\mathcal{P}^+ \to \mathcal{P}^+T$ *[Jac04; Rut98; CKVW98]. As containers are weak-pullback preserving, it might ostensibly appear that Corollary 3.6 is a consequence of this fact. However by restricting to containers, Corollary 3.6 strengthens this consequence by showing that* κ^+ *is the unique law of* F *over* $(\mathcal{P}^+, \eta^+, \mu^+)$ sine conditione *rather than merely the unique* monotone *law. There are weak-pullback preserving functors* T *which are not containers, such as the powerset endofunctor, for which the Jacobs law is not the only Kleisli law of* T *over monad* $(\mathcal{P}^+, \eta^+, \mu^+)$*, e.g. see [Goy21, Example 2.14].*

Since it may be difficult to parse the equation (3.3) for an arbitrary container, the following examples provide additional intuition for how this transformation works.

Example 3.8. *Given a set* S*, for the product endofunctor* $F = S \times (\cdot)$ *of example 2.15, the pointed laws* $\kappa \colon F\mathcal{P} \to \mathcal{P}F$ *and* $\kappa^+ \colon F\mathcal{P}^+ \to \mathcal{P}^+F$ *have components satisfying* $\kappa_X(s, Y) = \kappa^+_X(s, Y) = \{(s, y) \mid y \in Y \subseteq X\}$ *for all* $Y \neq \varnothing$.

Example 3.9. *For the infinite streams endofunctor* $L^\infty = (-)^{\mathbb{N}}$*, a special case of example 2.15, the pointed law* $\kappa \colon L^\infty\mathcal{P} \to \mathcal{P}L^\infty$ *has components satisfying* $\kappa_X((X_1, X_2, \dots)) = \{(x_1, x_2, \dots) \mid x_i \in X_i\}$ *for all streams* $(X_1, X_2, \dots)$ *such that every* $X_i \neq \varnothing$.

Example 3.10. *For the non-empty list container of example 2.16, the pointed law* $\kappa \colon L^+\mathcal{P} \to \mathcal{P}L^+$ *has components satisfying* $\kappa_X([X_1, \dots, X_n]) = \{[x_1, \dots, x_n] \mid x_i \in X_i\}$ *for all lists* $[X_1, \dots, X_n]$ *such that every* $X_i \neq \varnothing$.

As each of these examples illustrates, the action of κ on a container with non-empty sets is to output the set of all containers which sample an element from each position. This allows us to easily compute the (possibly infinite) cardinality of the subset $\kappa(s, l) \in \mathcal{P}(F(X))$.

Lemma 3.11 ([KS24])**.** *If* $F = [S \triangleleft P]$ *is a container and* $\kappa \colon F\mathcal{P} \to \mathcal{P}F$ *satisfies equation* (3.3) *for* $(s, l) \in F\mathcal{P}^+(X) \subseteq F\mathcal{P}(X)$*, then* $|\kappa_X(s, l)| = \prod_{p \in P(s)} |l(p)|$.

Remark 3.12. *The proof of Theorem 3.2 also applies to pointed laws* $\kappa \colon F\mathcal{P}_f \to \mathcal{P}_fF$ *and* $\kappa \colon F\mathcal{P}^+_f \to \mathcal{P}^+_fF$ *of containers* F *over finite powerset* $(\mathcal{P}_f, \eta)$ *and finite non-empty powerset* $(\mathcal{P}^+_f, \eta)$*, respectively. However, unlike with the full non-empty powerset, the analogue of*

Corollary 3.6 for $(\mathcal{P}_f^+, \eta)$ *does not always hold. For containers* $[S \lhd P]$ *where there exists an* $s \in S$ *such that* $P(s)$ *is infinite, a pointed law* $\kappa\colon F\mathcal{P}_f^+ \to \mathcal{P}_f^+ F$ *satisfying equation (3.3) does not exist as the set* $\kappa_X(s, l)$ *would be necessarily infinite.*

3.1.2 Directed containers over powerset

In this section, we investigate, given a directed container (W, ε, δ), when does the pointed law $\kappa\colon W\mathcal{P} \to \mathcal{P}W$ of W over $(\mathcal{P}, \eta)$ extend to a distributive law of (W, ε, δ) over $(\mathcal{P}, \eta, \mu)$.

For instance, the pointed law $\kappa^+\colon S{\times}\mathcal{P}^+(\cdot) \to \mathcal{P}^+(S \times (\cdot))$ described in example 3.8 does extend to a comonad-monad distributive law of the coreader comonad of example 2.22 over the non-empty powerset monad. Moreover, it follows from the counit axiom (2.64) of Definition 2.62 that any element $(s, \varnothing) \in S{\times}\mathcal{P}(X)$ must be mapped by a distributive law $\kappa\colon S{\times}\mathcal{P}(\cdot) \to \mathcal{P}(S \times (\cdot))$ to $\varnothing \in \mathcal{P}(S{\times}X)$. By Theorem 3.2, it must be the case that κ is equal to κ^+ for elements (s, Y) with $Y \neq \varnothing$, obtaining a t result.

Proposition 3.13 ([KS24])**.** *For a fixed set* S*, the coreader comonad* $W = (S \times (\cdot), \varepsilon, \delta)$ *has a unique distributive law* $\kappa\colon W\mathcal{P} \to \mathcal{P}W$ *over the powerset monad* $(\mathcal{P}, \eta, \mu)$ *and a unique distributive law* $\kappa^+\colon W\mathcal{P}^+ \to \mathcal{P}^+W$ *over the non-empty powerset monad* $(\mathcal{P}^+, \eta^+, \mu^+)$*.*

This distributive law appears as [PW02, Example 7.6]. However, as we will see at the end of this section, the coreader comonad is the only directed container where a distributive law over the powerset monad exists. To illustrate the issue, we first show that the pointed law $\kappa\colon L^+\mathcal{P} \to \mathcal{P}L^+$ in Theorem 3.2 does not extend to a comonad-monad distributive law of the the prefix list comonad $(L^+, \varepsilon, \delta)$ over the powerset monad $(\mathcal{P}, \eta, \mu)$.

Theorem 3.14. *There is no distributive law* $\kappa\colon L^+\mathcal{P} \to \mathcal{P}L^+$ *of* $(L^+, \varepsilon, \delta)$ *over* $(\mathcal{P}, \eta, \mu)$*.*

Proof. Suppose for contradiction there exists a distributive law $\kappa\colon L^+\mathcal{P} \to \mathcal{P}L^+$. As κ must satisfy the unit axiom (2.63), κ is a pointed law of L^+ over $(\mathcal{P}, \eta)$. Theorem 3.2 implies that for lists which contain only non-empty sets, the components of κ satisfy equation (3.3).

Considering the list $\mathbf{L} = [\{a, b\}, \{b\}] \in L^+(\mathcal{P}(X))$ for $X = \{a, b\}$, we obtain the following inequality contradicting the comultiplication axiom:

$$\begin{aligned}
\mathcal{P}\delta_X \circ \kappa_X(\mathbf{L}) &= \{[[a], [a, b]], [[b], [b, b]]\} \\
&\neq \{[[a], [a, b]], [[b], [a, b]], [[a], [b, b]], [[b], [b, b]]\} \\
&= \kappa_{L^+X} \circ L^+\kappa_X \circ \delta_{\mathcal{P}X}(\mathbf{L})
\end{aligned}$$

□

Interestingly, chasing the list $\mathbf{L}' = [\{b\}, \{a, b\}]$ rather than $\mathbf{L}$ would have shown that $\mathcal{P}\delta_X \circ \kappa_X(\mathbf{L}') = \kappa_{L^+X} \circ L^+\kappa_X \circ \delta_{\mathcal{P}X}(\mathbf{L}')$. The difference between these two cases is that the set $\{a, b\}$ is placed in a root position in $\mathbf{L}'$ whereas $\{a, b\}$ is in a non-root position in $\mathbf{L}$. This is the key reason why chasing $\mathbf{L}'$ seemingly verifies the comultiplication axiom, but chasing $\mathbf{L}$ falsifies the comultiplication axiom. Indeed, the existence of a non-root position in a shape $s \in S$ of a directed container is the key property that forbids the pointed law $\kappa\colon W\mathcal{P} \to \mathcal{P}W$ satisfying equation (3.3) from extending to a distributive law of (W, ε, δ) over $(\mathcal{P}, \eta, \mu)$.

We use $\mathcal{C}_W$ to denote the class of directed containers (W, ε, δ) with $W = [S \lhd P]$ such that there exists an $s \in S$ with $|P(s)| \geq 2$.

Since every distributive law $\kappa\colon W\mathcal{P} \to \mathcal{P}W$ must satisfy the unit axiom, then by Theorem 3.2, for elements $(s, l) \in W\mathcal{P}^+(X) \subseteq W\mathcal{P}(X)$, κ satisfies equation (3.3). However, a simple diagram chase of κ applied to a specific $(s, l) \in W\mathcal{P}^+(X)$ for $W \in \mathcal{C}_W$ demonstrates that κ cannot satisfy diagram (2.66). Let $(\wp, \eta, \mu)$ be either either the powerset monad $(\mathcal{P}, \eta, \mu)$, non-empty powerset monad $(\mathcal{P}^+, \eta^+, \mu^+)$, finite powerset monad $(\mathcal{P}_f, \eta, \mu)$, or finite non-empty powerset monad $(\mathcal{P}_f^+, \eta^+, \mu^+)$.

Theorem 3.15 ([KS24])**.** *If $(W, \varepsilon, \delta) \in \mathcal{C}_W$, then there is no distributive law $\kappa\colon W\wp \to \wp W$ of (W, ε, δ) over $(\wp, \eta, \mu)$.*

It is worth noting that the above proof only involves the unit and comultiplication axioms of a distributive law. Thus, we have proven a stronger statement.

Theorem 3.16 ([KS24])**.** *If $(W, \varepsilon, \delta) \in \mathcal{C}_W$, then there is no natural transformation $\kappa\colon W\wp \to \wp W$ which simultaneously satisfies the unit and comultiplication axioms.*

If a directed container W is not in $\mathcal{C}_W$, then by the existence of a root $\mathsf{o}_s \in P(s)$ for every $s \in S$, we have that $|P(s)|$ is non-empty and $|P(s)| = 1$. However, directed containers which satisfy this condition are isomorphic to the coreader comonads described in Example 2.22. This allows us to phrase Theorem 3.15 positively and characterise coreader comonads in terms of distributive laws.

Theorem 3.17 ([KS24])**.** *Let (W, ε, δ) be a directed container with $W = [S \lhd P]$.*

$W = S \times (\cdot)$ is the coreader comonad on S if and only if there exists a distributive law $\kappa\colon W\mathcal{P}^+ \to \mathcal{P}^+W$.

3.1.3 Directed containers over uniform choice monads

In this section, we widen the scope of Theorem 3.15 by showing no distributive law of the form $\rho\colon WM \to MW$ exists for any comonad $W \in \mathcal{C}_W$ and any monad, in fact pointed endofunctor, $M\colon \mathbf{Set} \to \mathbf{Set}$ which has meaningful notion of 'uniform distribution of size ≥ 2'. To formally define this class of monads M, we take inspiration from the equational presentations of monads which arise from universal algebra. From this perspective, we define, given an endofunctor $M\colon \mathbf{Set} \to \mathbf{Set}$, a *$n$-ary term for M* as a natural transformation $\beta\colon \mathbf{Id}_{\mathbf{Set}} \times \cdots \times \mathbf{Id}_{\mathbf{Set}} \to M$ where the domain of β is the endofunctor on $\mathbf{Set}$ mapping a set X to its n-th power $X^n = X \times \cdots \times X$. Beyond this algebraic portion of the definition, we also need to restrict to monads M which have a meaningful notion of support, i.e. there exists a natural transformation $\mathsf{supp}\colon M \to \mathcal{P}$. With these notions in place, we can now define what it means for any pointed endofunctor (M, η) to have a 'uniform distribution'.

Definition 3.18. Given a pointed endofunctor (M, η) with a natural transformation $\mathsf{supp}\colon M \to \mathcal{P}$, a n-ary term $\beta\colon \mathbf{Id}_{\mathbf{Set}} \times \cdots \times \mathbf{Id}_{\mathbf{Set}} \to M$ for M is a *n-uniform choice term* if

1. *β is idempotent:* For all $X \in \mathbf{Set}$ and $x \in X$,

$$\beta(x, \ldots, x) = \eta(x);$$

2. *β is commutative:* For all $X \in \mathbf{Set}$, $x_1, \ldots, x_n \in X$, and permutations $\pi\colon [\mathsf{n}] \to [\mathsf{n}]$,

$$\beta(x_1, \ldots, x_n) = \beta(x_{\pi(1)}, \ldots, x_{\pi(n)});$$

3. *supp preserves β:* For all $X \in \mathbf{Set}$ and $x_1, \ldots, x_n \in X$,

$$\mathsf{supp}(\beta(x_1, \ldots, x_n)) = \{x_1, \ldots, x_n\}.$$

We will say (M, η) is a *n-uniform choice pointed endofunctor* if there exists a natural transformation $\mathsf{supp}\colon M \to \mathcal{P}$ and a n-uniform choice term β for (M, η). We will say a monad (M, η, μ) is a *n-uniform choice monad* if (M, η) is a n-uniform choice pointed endofunctor. Since the the powerset monad $(\mathcal{P}, \eta, \mu)$ has support $\mathrm{id}_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{P}$ and a n-uniform choice term $\beta^{\mathcal{P}}(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$ for every $n > 0$, the terminology for condition 3 in Definition 3.18 is justified. In fact, for every n-uniform choice monad M, it follows that $\mathsf{supp}\colon M \to \mathcal{P}$ is a pointed endofunctor morphism:

$$\mathsf{supp}(\eta(x)) = \mathsf{supp}(\beta(x, \ldots, x)) = \{x\} = \eta^{\mathcal{P}}(x). \tag{3.19}$$

Moreover, every pointed endofunctor (M, η) such that the natural transformation $\mathsf{supp}\colon M \to \mathcal{P}$ is a pointed endofunctor morphism is a 1-uniform choice endofunctor where $\beta = \eta$.

For a n-uniform choice monad (M, η, μ) with n-uniform choice term β and $X \in \mathbf{Set}_0$, we define the set of uniform terms as

$$U_\beta(X) := \{\beta(x_1, \dots, x_n) \mid x_1, \dots, x_n \in X\} \subseteq M(X).$$

The set $U_\beta(X)$ generalises the set of uniform samplings of multisets with n elements. To illustrate, take M to be the discrete probability distribution monad and define β as

$$\beta(x_1, \dots, x_n) = \sum_{i \in [\mathsf{n}]} \frac{1}{n} x_i.$$

In this case, the set $U_\beta(X)$ is precisely the uniform samplings on multisets of X of cardinality n. In particular, if the $x_1, \dots, x_n$ are distinct, then $\beta(x_1, \dots, x_n)$ is the uniform distribution on $\{x_1, \dots, x_n\}$.

We proceed by first proving a generalisation of Theorem 3.2, demonstrating that every pointed law $\rho\colon FM \to MF$ satisfies an analogue of equation (3.3) on supports when restricted to uniform distributions.

Theorem 3.20 ([KS24]). *If $F = [S \triangleleft P]$ is a container, (M, η, μ) is a n-uniform choice monad, and there exists a pointed law $\rho\colon FM \to MF$, then for all sets X and elements $(s, l\colon P(s) \to U_\beta(X)) \in F(U_\beta(X)) \subseteq F(M(X))$,*

$$\begin{aligned}&\mathsf{supp}(\rho_X(s, l)) \\ &= \{(s, j\colon P(s) \to X) \mid \forall p \in P(s), j(p) \in \mathsf{supp}(l(p))\}.\end{aligned} \tag{3.21}$$

The definition of a choice monad has been carefully chosen to allow us to prove this theorem via an argument that mirrors the collapse-swap-relabel argument. Condition 1 that β is idempotent is key to demonstrating the 'collapse' argument. Condition 2 that β is commutative is key to demonstrating that the 'swap' argument. Again, we refer to [KS24] for full details. Using this result we can further generalise theorem 3.15 to cover the larger class of choice monads. Inspired by this argument, we chase an element $(s, l) \in W(U_\beta(X)) \subseteq W(M(X))$ where $l(v)$ is a uniform distribution for a $v \neq \mathsf{o}_s \in P(s)$ such that $|\mathsf{supp}(l(v))| \geq 2$. We define our class $\mathcal{C}_M$ to be those uniform choice monads that allow us to build this counterexample.

We use $\mathcal{C}_M$ to denote the class of n-uniform choice monads (M, η^M, μ^M) where $n \geq 2$.

Theorem 3.22. *If* $(W, \varepsilon, \delta) \in \mathcal{C}_W$ *and* $(M, \eta, \mu) \in \mathcal{C}_M$*, then there is no distributive law* $\rho \colon WM \to MW$ *of* (W, ε, δ) *over* (M, η, μ)*.*

Once again, we have proven a stronger result.

Theorem 3.23. *If* $(W, \varepsilon, \delta) \in \mathcal{C}_W$ *and* $(M, \eta, \mu) \in \mathcal{C}_M$*, then there is no natural transformation* $\rho \colon WM \to MW$ *which simultaneously satisfies the unit and comultiplication axioms.*

To illustrate the generality and limitations of Theorem 3.22, we recall the following examples from the discussion after Definition 3.18.

Example 3.24. *The ordinary discrete probability distribution monad* $\mathcal{D}$ *of Example 2.5 is in* $\mathcal{C}_M$*. This monad has a natural transformation* $\mathsf{supp} \colon \mathcal{D} \to \mathcal{P}$ *which maps a probability distribution to its underlying support, i.e.* $\mathsf{supp}(\varphi) = \{x \mid \varphi(x) \neq 0\}$*, and a* 2*-uniform choice term* $\beta \colon \mathbf{Id}_{\mathbf{Set}} \times \mathbf{Id}_{\mathbf{Set}} \to \mathcal{D}$ *defined as*

$$\beta(x_1, x_2) = \frac{1}{2}x_1 + \frac{1}{2}x_2.$$

Example 3.25. *Every variation of the powerset* $\wp$ *monad, e.g. full, finite, non-empty, has a transformation* $\mathsf{supp} \colon \wp \to \mathcal{P}$ *given by the inclusion into the full powerset functor. Moreover, for every* $n > 0$*,* $\wp$ *has a* n*-uniform choice term* $\beta \colon \mathbf{Id}_{\mathbf{Set}} \times \cdots \times \mathbf{Id}_{\mathbf{Set}} \to \wp$ *defined as*

$$\beta(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}.$$

Thus, we recover Theorem 3.15 as an application of Theorem 3.22.

Both these examples are part of a wider class of distribution and multiset monads over semirings $\mathbf{S}$ which fall under the scope of Theorem 3.22. In particular, Example 3.24 is the case where $\mathbf{S}$ is the semiring of non-negative reals $(\mathbb{R}_{\geq 0}, 0, 1, +, *)$ and Example 3.25 is the case where $\mathbf{S}$ is the Boolean semining $(\{0, 1\}, 0, 1, \vee, \wedge)$.

We enumerate necessary and sufficient conditions for when a distribution $\mathcal{D}_{\mathbf{S}}$ or multiset monad $\mathcal{M}_{\mathbf{S}}$ over a semiring $\mathbf{S}$ is in the class of monads $\mathcal{C}_M$. To state these conditions, recall that the initial object in the category of semirings is the semiring of natural numbers. Therefore, for every semiring $\mathbf{S}$, there is a unique semiring morphism $\top^{\mathbf{S}} \colon \mathbb{N} \to \mathbf{S}$ where $\top^{\mathbf{S}}(n)$ is mapped to the n-fold sum of $1_{\mathbf{S}}$.

(S1) $\mathbf{S}$ *is zero-sumfree*: If $a + b = 0_{\mathbf{S}}$, then $a = 0_{\mathbf{S}}$ and $b = 0_{\mathbf{S}}$.

(S2) $\mathbf{S}$ *has a natural non-trivial unit* $n_{\mathbf{S}}$*:* There exists some $n \geq 2$ such that $n_{\mathbf{S}} = \top^{\mathbf{S}}(n)$ is a unit. i.e. there exists $t \in \mathbf{S}$ such that $n_{\mathbf{S}}t = tn_{\mathbf{S}} = 1_{\mathbf{S}}$. If such a $t \in \mathbf{S}$ exists, then t is unique and so we can denote t as $\frac{1}{n_{\mathbf{S}}}$.

We prove the following lemmas which connect these conditions on $\mathbf{S}$ to the Definition 3.18 of n-uniform choice monad.

Lemma 3.26. *Let $M = \mathcal{D}_{\mathbf{S}}$ or $M = \mathcal{M}_{\mathbf{S}}$ for some semiring $\mathbf{S}$. $\mathbf{S}$ is zero-sumfree, i.e. satisfies Condition (S1) if and only if supp^M is a natural transformation.*

Lemma 3.27. *Let $M = \mathcal{D}_{\mathbf{S}}$ or $M = \mathcal{M}_{\mathbf{S}}$. $\mathbf{S}$ has a natural non-trivial unit $n_{\mathbf{S}}$, i.e. satisfies Condition (S2) if and only if there exists an n-ary idempotent and commutative open term $\beta\colon \mathbf{Id}_{\mathbf{Set}} \times \cdots \times \mathbf{Id}_{\mathbf{Set}} \to M$ which is uniquely defined as*

$$\beta(x_1, \dots, x_n) = \sum_{i \in [\mathsf{n}]} \frac{1}{n_{\mathbf{S}}} x_i.$$

In particular, if $\mathbf{S}$ is zero-sumfree, then β is a n-uniform choice term.

Theorem 3.28. *$\mathbf{S}$ satisfies conditions (S1)- (S2) if and only if $\mathcal{D}_{\mathbf{S}} \in \mathcal{C}_M$ and $\mathcal{M}_{\mathbf{S}} \in \mathcal{C}_M$. If $\mathbf{S}$ is also complete, then $\mathcal{D}^{\infty}_{\mathbf{S}} \in \mathcal{C}_M$ and $\mathcal{M}^{\infty}_{\mathbf{S}} \in \mathcal{C}_M$.*

As a corollary, we obtain an instance of Theorem 3.22 for multiset and distribution monads.

Corollary 3.29. *If $(W, \varepsilon, \delta) \in \mathcal{C}_W$ and $\mathbf{S}$ satisfies conditions (S1)- (S2), then there is no distributive law $\rho\colon WM \to MW$ where $M = \mathcal{D}_{\mathbf{S}}$ or $M = \mathcal{M}_{\mathbf{S}}$.*

Example 3.30. *The multiset $\mathcal{M}_{\mathbf{S}}$ and distribution $\mathcal{D}_{\mathbf{S}}$ monads over the (sub)-semiring of non-negative rationals of the form $\frac{n}{3^k}$ for $n, k \in \mathbb{N}$ are in $\mathcal{C}_M$. In this case, since 2 is not a unit, there are no uniform distributions $\varphi \in \mathcal{D}_{\mathbf{S}}(X)$ such that $|\mathsf{supp}(\varphi)| = 2$ However, for every $k \in \mathbb{N}$, 3^k is a unit and so $\mathcal{D}_{\mathbf{S}}(X)$ has uniform distributions φ such that $|\mathsf{supp}(\varphi)| = 3^k$.*

Example 3.31 (Non-Example)**.** *For every ring R, the free R-module monad $\mathcal{M}_R$ and distribution monad $\mathcal{D}_R$ of Example 2.10 are not zero-sum-free and therefore not in $\mathcal{C}_M$.*

Example 3.32 (Non-Example)**.** *There are semirings which are zero-sum-free, but fail to satisfy the condition (S2). For instance, the semiring of naturals $(\mathbb{N}, 0, 1, +, *)$ does not satisfy condition (S2). Therefore, although $\mathcal{C}_M$ has many multiset monads over other semirings $\mathbf{S}$, $\mathcal{C}_M$ does not contain the ordinary multiset/bag monad $\mathcal{M} = \mathcal{M}_{\mathbb{N}}$. Moreover, $\mathcal{D}_{\mathbb{N}}$ over the semiring $(\mathbb{N}, 0, 1, +, *)$ of natural numbers only has distributions which are singletons and is, in fact, isomorphic to the identity monad.*

Example 3.33 (Non-Example)**.** *For a example of a zero-sum-free semiring* $\mathbf{S}$ *which does not satisfy condition (S2), but where* $\mathcal{D}_{\mathbf{S}}$ *is not the identity monad, consider* $\mathbf{S} = \mathbb{N}[x, y]/(x+y = 1)$*. This semiring is the quotient of the free commutative semiring on the set* $\{x, y\}$ *by the equation* $x + y = 1_{\mathbf{S}}$*. The additional equation* $x + y = 1_{\mathbf{S}}$ *ensures that* $\mathcal{D}_{\mathbf{S}}$ *is not the identity monad by allowing for non-singleton distributions, i.e.* $x.a + y.b \in \mathcal{D}_{\mathbf{S}}(\{a, b\})$*. Neither* x *nor* y *are inverses to* $\top^{\mathbf{S}}(n)$ *for some* $n \geq 2$*. Thus,* $\mathbf{S}$ *does not satisfy condition (S2) and* $\mathcal{D}_{\mathbf{S}} \notin \mathcal{C}_M$*.*

3.2 No-go theorems in $\mathcal{R}(\sigma)$

In this section, we extend our results to the category of relational structures, $\mathbf{R}(\sigma)$. To achieve this, we prove a two-part *transfer theorem* which dictates conditions under which the existence of a mixed distributive law in $\mathbf{R}(\sigma)$ implies the existence of a mixed distributive law in **Set**.

For the remainder of this section let us assume we are working in the following setup:

1. There exists categories $\mathbf{C}, \mathbf{D}$ with a coreflective adjunction $L : \mathbf{C} \to \mathbf{D} \dashv U : \mathbf{D} \to C$ between them. We write α, β for the unit and counit of this adjunction.
2. $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}})$, $(W, \varepsilon^{W}, \delta^{W})$ are comonads over $\mathbf{D}, \mathbf{C}$ respectively.
3. $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}})$, (M, η^{M}, μ^{M}) are monads over $\mathbf{D}, \mathbf{C}$ respectively.

We are now ready to prove our transfer theorems.

Theorem 3.34. *Assume the following:*

1. *There exists a coKleisli law* $w : WU \to U\mathbb{W}$*.*
2. *There exists a Kleisli law* $m : U\mathbb{M} \to MU$*.*
3. $\rho \colon \mathbb{W}\mathbb{M} \to \mathbb{M}\mathbb{W}$ *and* $\rho' \colon WM \to MW$ *are natural transformations satisfying the following "Yang-Baxter" condition:*

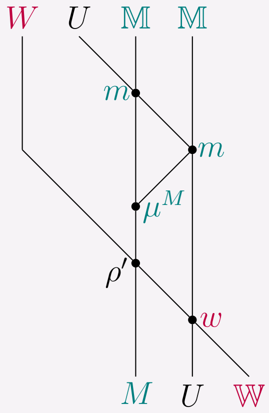
$$Mw \circ \rho' U \circ Wm = m\mathbb{W} \circ U\rho \circ w\mathbb{M}$$

Then we have:

1. *If* ρ *is a distributive law,* m *is epic, and* w *is monic, then* ρ' *is a distributive law.*
2. *If* ρ' *is a distributive law,* m *is monic, and* w *is epic, then* ρ *is a distributive law.*

Proof. We prove the (co)multiplication, and (co)unit axioms for the first part of the theorem. The proof for the second part of the theorem is analogous.

- Multiplication:



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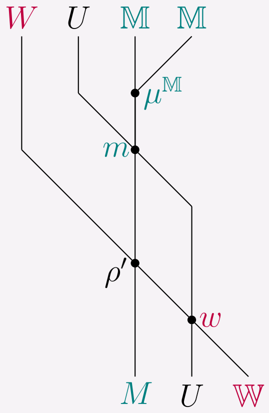

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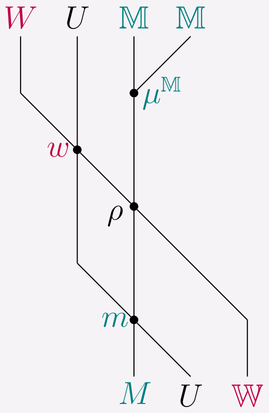

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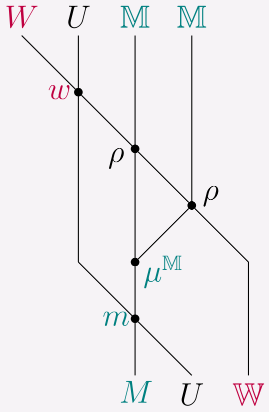

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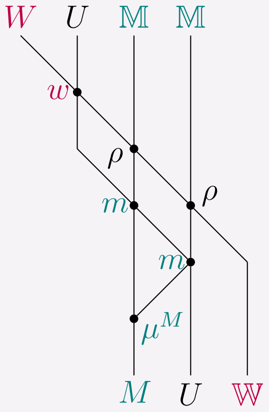

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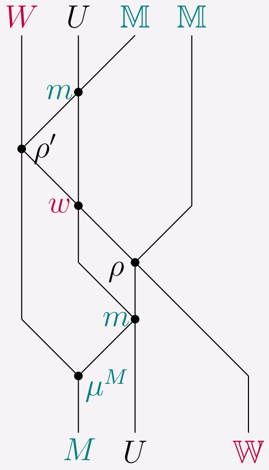

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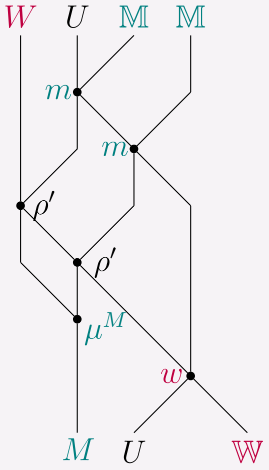
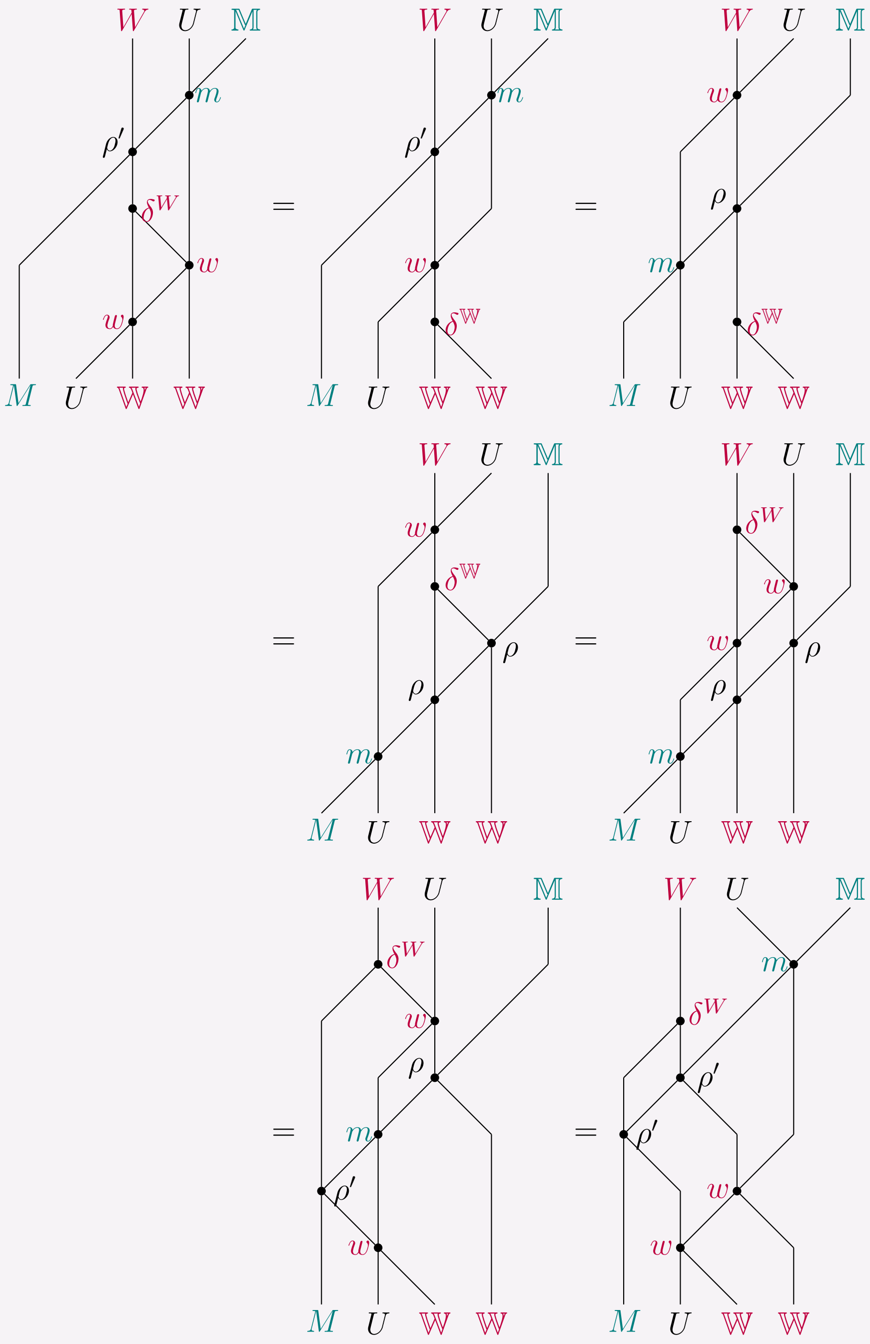


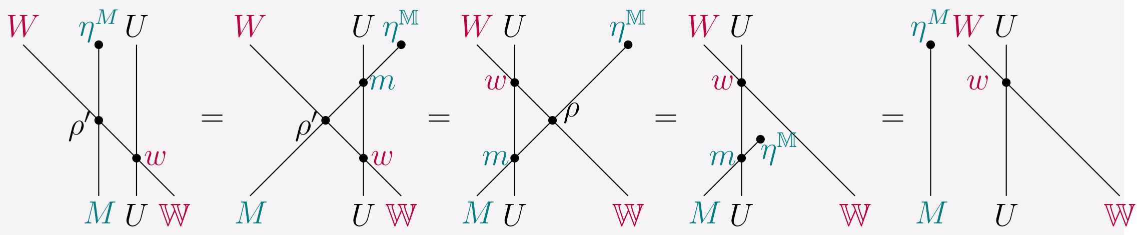
The result follows from m being epic and w being monic.

- Comultiplication:



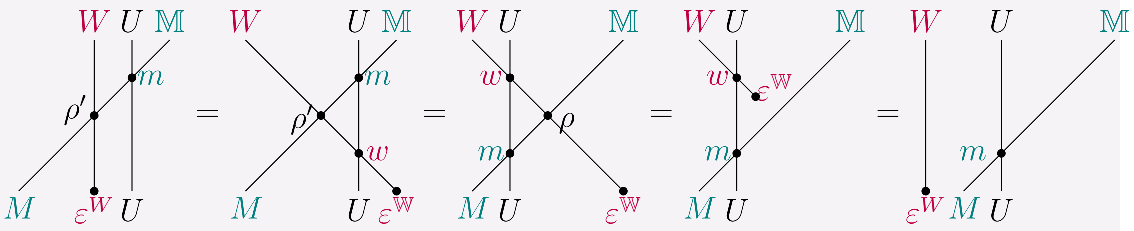
The result follows from m being epic and w being monic.

- Unit:



The result follows from w being monic.

- Counit:



The result follows from m being epic.

$\square$

Remark 3.35. *This theorem can be seen as a generalised comonad-monad variant of [MM07, Theorem 3.1.3], which considers transfer theorems for monad-monad distributive laws defined on the same category. We can recover the theorem of [MM07] by simply considering the case where the functor U above is the identity.*

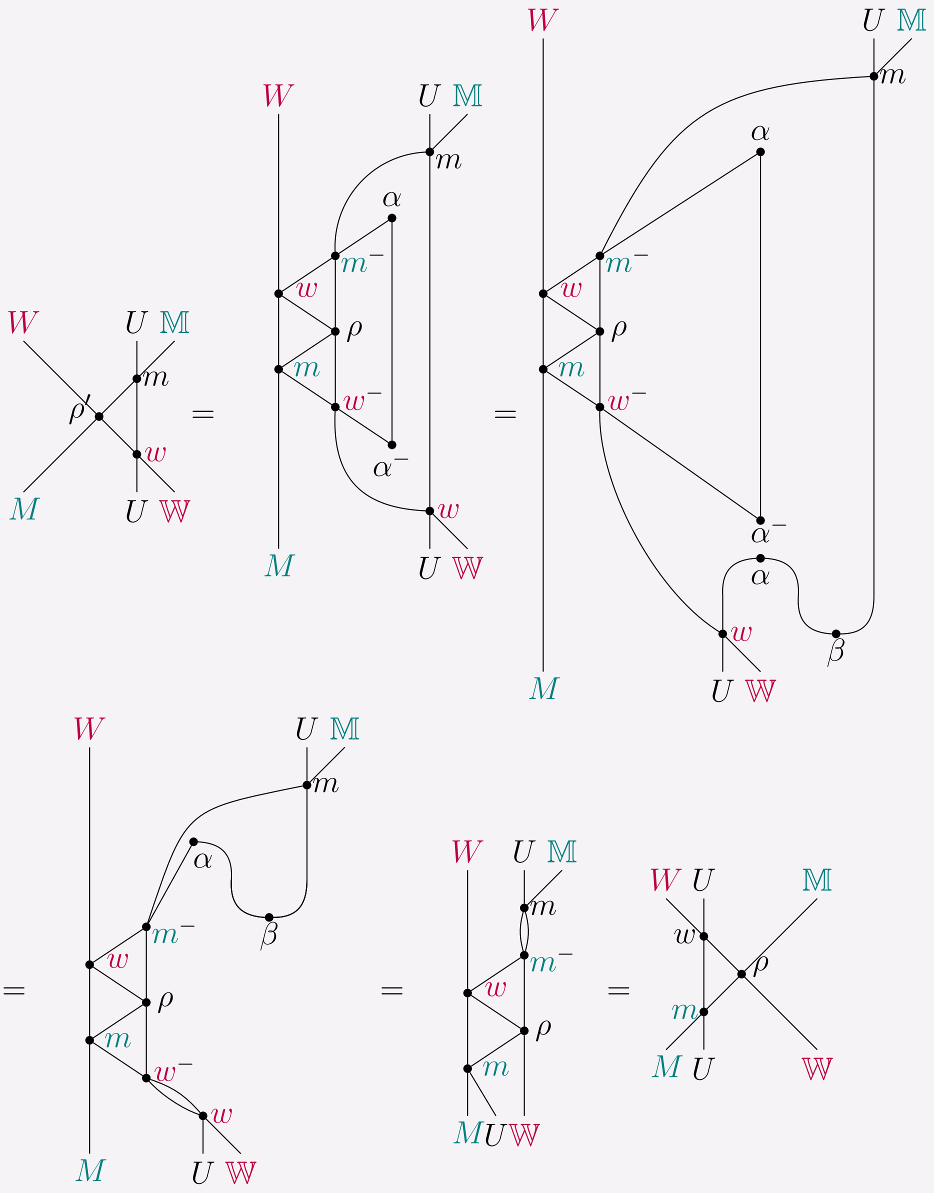
Theorem 3.36. *Assume the following:*

1. *There exists a split epic natural transformation $w : WU \to U\mathbb{W}$. We write w^- for the section of w.*
2. *There exists a split monic natural transformation $m : U\mathbb{M} \to MU$. We write m^- for the retraction of m.*
3. *$\rho\colon \mathbb{W}\mathbb{M} \to \mathbb{M}\mathbb{W}$ is a natural transformation.*

Then, ρ together with the natural transformation $\rho' : WM \to MW$ defined as $\rho' = MW\alpha^{-1} \circ Mw^- L \circ m\mathbb{W}L \circ U\rho L \circ w\mathbb{M}L \circ Wm^- L \circ WM\alpha$, satisfy the following "Yang-Baxter" equation:

$$Mw \circ \rho' U \circ Wm = m\mathbb{W} \circ U\rho \circ w\mathbb{M}$$

Proof.



□

Let us now consider what happens when we enforce the condition $\mathfrak{C} = \mathbf{Set}$. We show that we can combine the two theorems above with our earlier results to obtain sufficient conditions under which no distributive law exists between a class of comonads $\mathcal{C}_{\mathbb{W}}(\mathbf{D})$ and a class of monads $\mathcal{C}_{\mathbb{M}}(\mathbf{D})$ defined on $\mathbf{D}$.

$\mathcal{C}_{\mathbb{W}}(\mathbf{D})$ denotes the class of comonads $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}})$ on $\mathbf{D}$ with an isomorphic coKleisli law $w : WU \to U\mathbb{W}$ for some $W \in \mathcal{C}_W$.

$\mathcal{C}_{\mathbb{M}}(\mathbf{D})$ denotes the class of monads $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}})$ on $\mathbf{D}$ with an isomorphic Kleisli law $m : U\mathbb{M} \to MU$ for some $M \in \mathcal{C}_M$.

Theorem 3.37. *If* $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}}) \in \mathcal{C}_{\mathbb{W}}(\mathbf{D})$ *and* $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}}) \in \mathcal{C}_{\mathbb{M}}(\mathbf{D})$*, then there is no distributive law* $\rho\colon \mathbb{W}\mathbb{M} \to \mathbb{M}\mathbb{W}$ *of* (W, ε, δ) *over* (M, η, μ)*.*

Proof. Assume for the sake of contradiction that a distributive law ρ does exist. We can use Theorem 3.36 to construct a natural transformation $\rho'\colon WM \to MW$ which satisfies the Yang-Baxter condition. It follows from Theorem 3.34 that ρ is a distributive law of (W, ε, δ) over (M, η, μ). However, we know by Theorem 3.22 that no such law exists. ☐

Note that it is necessary for w and m to be isomorphisms for the above argument to work. This is because any monic and split epic morphism must be an isomorphism (similarly for any epic and split monic morphism).

As before, our techniques have proven a stronger statement.

Theorem 3.38. *If* $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}}) \in \mathcal{C}_{\mathbb{W}}(\mathbf{D})$ *and* $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}}) \in \mathcal{C}_{\mathbb{M}}(\mathbf{D})$*, then there is no natural transformation* $\rho\colon \mathbb{W}\mathbb{M} \to \mathbb{M}\mathbb{W}$ *which simultaneously satisfies the unit and comultiplication axioms.*

The above result is our most general no-go theorem. All earlier results are covered by the special case where $\mathbf{D} = \mathbf{Set}$. By varying $\mathbf{D}$ we can observe the non-existence of distributive laws between further comonads, as exemplified below.

Corollary 3.39. *There is no distributive law of* $(\mathbb{P}_k, \varepsilon, \delta)$ *over* $(\mathbb{D}_{\mathbb{R}+}, \eta, \mu)$*.*

Proof. There exists a forgetful functor $U\colon \mathbf{R}(\sigma) \to \mathbf{Set}$ which sends a relational structure to its underlying universe. This functor has a left adjoint $L\colon \mathbf{Set} \to \mathbf{R}(\sigma)$ which sends a set to a relational structure with empty relations. The adjunction $L \dashv U$ is coreflective. Now consider the natural transformations $w\colon U\mathbb{P}_k \to L^+[k]U$ and $m\colon U\mathcal{D} \to L^+[k]U$ whose components, as set functions, are the identity. It is easy to verify that w is an isomorphic coKleisli law and that m is an isomorphic kleisli law. The claim then follows from theorem 3.37. ☐

3.3 Single measurement semiring

The no-go theorems we have proven thus far in this chapter show that for a broad range of game comonads $\mathbb{G}_k$ and distribution minion monads $\mathbb{D}_{\mathbf{S}}$ there is no distributive law of the form $\kappa : \mathbb{G}_k\mathbb{D}_{\mathbf{S}} \to \mathbb{D}_{\mathbf{S}}\mathbb{G}_k$. In fact the only example we have encountered where a distributive law does exist is when $\mathbf{S} = \mathbb{N}$ is the semiring of natural numbers. This example is not very interesting since $\mathbb{D}_{\mathbf{S}}$ in this case is isomorphic to the identity monad. One thus wonders: Are there any semirings $\mathbf{S}$ for which a distributive law exists and where $\mathbb{D}_{\mathbf{S}}$ is not isomorphic to the identity? The goal of this section is to answer this question affirmatively. For this purpose we introduce the following family of semirings which have appeared in [Sha24].

Example 3.40. *The single measurement semiring* [a] *over the set* $\{x_1, \dots, x_n\}$ *is given by* $\mathbf{S} = \mathbb{N}[x_1, \dots, x_n]/(x_1 + \ldots + x_n = 1, x_i^2 = x_i, \forall i \neq j : x_i x_j = 0)$. *We refer to the condition* $x_i x_j = 0$ *as orthogonality, and the condition* $x_i^2 = 0$ *as idempotence*[b].

---

[a] The naming is due to the similarity with the definition of projection valued measurements (PVMs) in quantum information (See chapter 4.).

[b] In fact the orthogonality condition implies the idempotence condition since $x_i = x_i(1) = x_i(x_1 + \cdots + x_n) = x_i x_1 + \cdots + x_i x_n = x_i x_i$. However, since this condition appears explicitly in the definition given in [Sha24] we have restated it here.

For the remainder of this section we restrict to semirings which are instances of the above construction. [Sha24] shows the following:

Theorem 3.41. *There exists a comonad over monad distributive law* $\kappa : L_k^+ \mathcal{D}_\mathbf{S} \to \mathcal{D}_\mathbf{S} L_k^+$.

The transformation κ used to prove the above result is given by

$$\kappa_{\mathcal{X}}[\varphi_1, \dots, \varphi_n] = \sum_{[x_1, \dots, x_n] \in \mathbb{E}_k(X)} \boldsymbol{\varphi}(\boldsymbol{x}).[x_1, \dots, x_n],$$

where employing the notation inherited from [ABdSZ17], $\boldsymbol{\varphi}(\boldsymbol{x}) = (\prod_{i \in [n]} \varphi_i(x_i))$. Thus, $\boldsymbol{\varphi}$ represents the joint distribution of the distributions $\{\varphi_1, \dots, \varphi_n\}$.

Our aim here is to extend theorem 3.41 in two directions. Firstly, we generalise from the category **Set** to the category $\mathbf{R}(\sigma)$:

Theorem 3.42. . *There exists a comonad over monad distributive law* $\kappa : \mathbb{E}_k \mathbb{D}_\mathbf{S} \to \mathbb{D}_\mathbf{S} \mathbb{E}_k$.

Proof. Since the action of $\mathbb{E}_k$ and $\mathbb{D}_\mathbf{S}$ on the underlying universe of relational structures is precisely the same as L_k^+ and $\mathcal{D}_\mathbf{S}$, the fact that the components of κ, viewed as functions, satisfy the naturality, (co)unit, and comultiplication diagrams follows immediately from theorem 3.41. Therefore, the only additional point to check is that each component of κ is a valid morphism in $\mathbf{R}(\sigma)$ (i.e. preserves relations). This will be shown in lemma 3.45 below. ☐

Secondly, we shall prove an analogue of this result for the pebbling comonad $\mathbb{P}_k$.

Theorem 3.43. . *There exists a comonad over monad distributive law* $\kappa : \mathbb{P}_k \mathbb{D}_\mathbf{S} \to \mathbb{D}_\mathbf{S} \mathbb{P}_k$.

Proof. The action of $\mathbb{P}_k$ on the underlying universe of relational structures is precisely the same as $L^+[k]$. Therefore, proving that there exists some κ which is a distributive law of the form $L^+[k]\mathbb{D}_{\mathbf{S}} \to \mathbb{D}_{\mathbf{S}}L^+[k]$ is enough to show that the (co)unit, (co)multiplication, and naturality axioms are satisfied. This will be done in theorem 3.47. The components of the transformation κ are then shown to preserve relations in lemma 3.48. This makes them valid morphisms in $\mathbf{R}(\sigma)$ completing the proof. ☐

To complete the proof of theorem 3.42 we make use of the following property of relations in $\mathbb{D}_{\mathbf{S}}$.

Lemma 3.44. $(\varphi_1, \ldots \varphi_n) \in R^{\mathbb{D}_{\mathbf{S}}\mathcal{X}}$ *iff* $\forall \boldsymbol{x} \in X^n : \boldsymbol{x} \notin R^{\mathcal{X}} \Rightarrow \boldsymbol{\varphi}(\boldsymbol{x}) = 0$.

Proof. $(\Rightarrow)$: Assume we have $(\varphi_1, \ldots \varphi_n) \in R^{\mathbb{D}_{\mathbf{S}}\mathcal{X}}$. Then from the definition of $\mathbb{D}_{\mathbf{S}}$, there must exist a distribution $\gamma : R^{\mathcal{X}} \to \mathbf{S}$ which witnesses this fact. Recall that γ can be extended to a distribution $\gamma^* : X^n \to S$ by setting $\gamma^*(\boldsymbol{x}) = 0$ whenever $\boldsymbol{x} \notin R^{\mathcal{X}}$. Moreover, each φ_i must be the marginal of γ^* at position i. Hence γ^* is the joint distribution of all the φ_i. So we obtain for $\boldsymbol{x} \notin R^{\mathcal{X}}$

$$\boldsymbol{\varphi}(\boldsymbol{x}) = \gamma^*(\boldsymbol{x}) = 0.$$

$(\Leftarrow)$: Assume we have $\varphi_1, \ldots, \varphi_n$ each belonging to the set $\mathcal{D}_{\mathbf{S}}X$ satisfying $\boldsymbol{x} \notin R^{\mathcal{X}} \Rightarrow \boldsymbol{\varphi}(\boldsymbol{x}) = 0$. This condition implies that the restriction of $\boldsymbol{\varphi}$ to the domain $R^{\mathcal{X}}$ is still a valid distribution. Let $\gamma : R^{\mathcal{X}} \to \mathbf{S}$ be this distribution. As each φ_i is the marginal of extending γ to the domain X^n, γ is a witness for the fact that $(\varphi_1, \ldots \varphi_n)$ belongs to $R^{\mathbb{D}_{\mathbf{S}}\mathcal{X}}$. ☐

Next, we verify that each component of κ is a valid morphism.

Lemma 3.45. *For every* $\mathcal{X} \in \mathbf{R}(\sigma)$, $\kappa_{\mathbf{X}} \colon \mathbb{E}_k\mathbb{D}_{\mathbf{S}}\mathcal{X} \to \mathbb{D}_{\mathbf{S}}\mathbb{E}_k\mathcal{X}$ *is an* $\mathbf{R}(\sigma)$*-morphism.*

Proof. Suppose $R \in \sigma$ is a relation symbol of arity n. Take $(l_1 = [\varphi^1_1, \ldots, \varphi^1_{m_1}], \ldots, l_n = [\varphi^n_1, \ldots, \varphi^n_{m_n}]) \in R^{\mathbb{E}_k\mathbb{D}_{\mathbf{S}}\mathcal{X}}$.

We must show that $(\kappa_X l_1 = \boldsymbol{\varphi^1}, \ldots, \kappa_X l_n = \boldsymbol{\varphi^n}) \in R^{\mathbb{D}_{\mathbf{S}}\mathbb{E}_k\mathcal{X}}$. Via lemma 3.44 it suffices to show that $\boldsymbol{\varphi^1(s_1)}.\boldsymbol{\varphi^2(s_2)}., \ldots, .\boldsymbol{\varphi^n(s_n)} = 0$ whenever $(s_1, \ldots, s_n) \notin R^{\mathbb{E}_k\mathcal{X}}$. There are two cases to consider:

- $s_i \not\uparrow s_j$ for some i, j: Assume without loss of generality that $|s_i| >= |s_j| = k$. From the definition of relations for $\mathbb{E}_k$, we must have for any $r \leq k$ $\varphi^i_r = \varphi^j_r$. Take an index $o \leq k$ such that $s_i[o] \neq s_j[o]$. It follows that one of the terms in the product

$\boldsymbol{\varphi^i(s_i)}$ is $\varphi^i_o(s_i[o])$ and that one of the terms in $\boldsymbol{\varphi^j(s_j)}$ is $\varphi^j_o(s_j[o]) = \varphi^i_o(s_j[o])$. From orthogonality of distributions it follows that $\varphi^i_o(s_i[o]).\varphi^i_o(s_j[o]) = 0$. Both of these terms appear in the product $\boldsymbol{\varphi^1(s_1)}.,\dots,.\boldsymbol{\varphi^n(s_n)}$. We can use commutativity to place the two terms next to each other. This makes the entire expression equal to 0.

- $(\varepsilon s_1,\dots,\varepsilon s_n) \notin R^{\mathcal{X}}$: From the definition of relations in $\mathbb{E}_k$ we have $(\varphi^1_{m_1},\dots,\varphi^n_{m_n}) \in R^{\mathbb{D}\mathsf{s}\mathcal{X}}$. Via lemma 3.44 we can deduce that $\varphi^1_{m_1}(\varepsilon s_1).,\dots,.\varphi^n_{m_n}(\varepsilon s_n) = 0$. All of these terms exist in the product $\gamma(s_1,\dots,s_n) = \boldsymbol{\varphi^1(s_1)}.\boldsymbol{\varphi^2(s_2)}.,\dots,.\boldsymbol{\varphi^n(s_n)}$, therefore using commutativity we can place them all next to each other, which shows that the entire expression equals 0.

☐

Now we turn our attention to the pebbling comonad. We could prove theorem 3.43 via similar arguments to theorem 3.42. Instead of doing so, however, we shall opt for a simpler proof which makes use of the idea of an *iterated distributive law* introduced in [Che11]. The original theorem of [Che11] defines iterated distributive laws between three monads. We require a variant of this result involving two comonads and a monad.

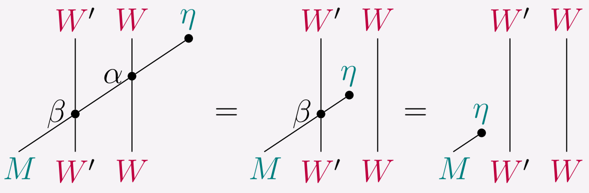
Theorem 3.46. *Assume the following:*

- *W, W' are comonads on a category $\mathbf{C}$, and M is a graded monad on $\mathbf{C}$.*
- *There exists comonad-graded monad distributive laws $\alpha : WM \to MW$ and $\beta : W'M \to MW'$, as well as a comonad-comonad distributive law $\gamma : W'W \to WW'$.*
- *The following "Yang-Baxter" equation is satisfied: $\alpha W' \circ W\beta \circ \gamma M = M\gamma \circ \beta W \circ W'\alpha$.*

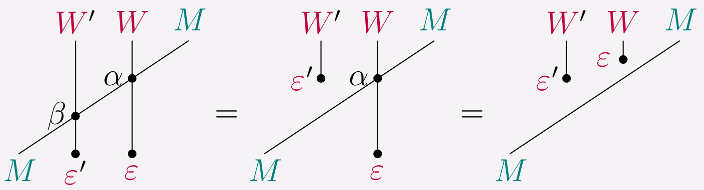
Then, the family of natural transformation $\kappa = \beta W \circ W'\alpha : W'WM \to MW'W$ is a distributive law of the composite comonad $W'W$ over the graded monad M.

Proof. We prove the unit, counit, multiplication, and comultiplication axioms.

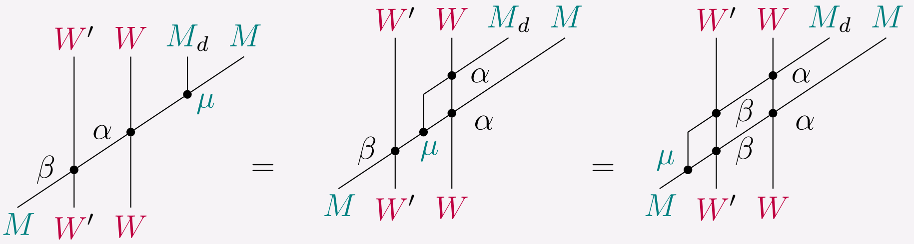
- Unit:



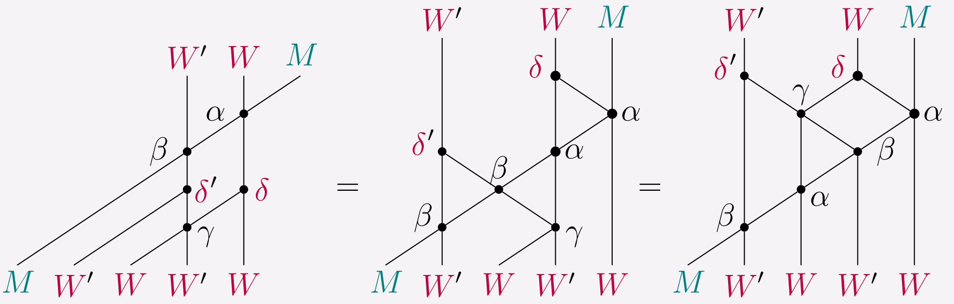
- Counit:



- Multiplication:



- Comultiplication: Note that the comultiplication of the composite comonad $W'W$ is given by the natural transformation $W'\gamma W \circ \delta'\delta$.



☐

Using iterated distributive laws we shall first prove the existence of a distributive law "at the level of sets" between the comonad $L^+[k]$ and the monad $\mathcal{D}_\mathbf{S}$. For this consider the natural transformation $\kappa : L^+[k]\mathcal{D}_\mathbf{S} \to \mathcal{D}_\mathbf{S} L^+[k]$ given by

$$\kappa_X[(p_1,\varphi_1),\ldots,(p_n,\varphi_n)] = \sum_{[(p_1,x_1),\ldots,(p_n,x_n)]\in L^+[k](X)} \boldsymbol{\varphi}(\boldsymbol{x}).[(p_1,x_1),\ldots,(p_n,x_n)]$$

.

Theorem 3.47. *κ is a distributive law of the comonad $L^+[k]$ over the monad $\mathcal{D}_\mathbf{S}$.*

Proof. Recall that $L^+[k]$ is a composite of two comonads:

$$L^+[k] = L^+_k \circ [k] \times (.)$$

Where $[k] \times (.)$ is the coreader comonad for $[k]$. We wish to apply theorem 3.46 in the case where:

- $W = [k] \times (.)$.
- $W' = L_k^+$.
- $M = \mathcal{D}_\mathbf{S}$.

This will show the existence of a distributive law between the composite comonad $L^+[k] = L_k^+ \circ [k] \times (.)$ and $\mathbb{D}_\mathbf{S}$.

Thus, to complete the proof we must define appropriate transformations

- $\alpha : [k] \times (.)\mathcal{D}_\mathbf{S} \to \mathcal{D}_\mathbf{S}[k] \times (.)$;
- $\beta : L_k^+\mathcal{D}_\mathbf{S} \to \mathcal{D}_\mathbf{S}L_k^+$,
- $\gamma : L_k^+[k] \times (.) \to [k] \times (.)L_k^+$

which satisfy the conditions of theorem 3.46. For β we consider the natural transformation used in the proof of 3.42. This is a distributive law because $L_k^+ = \mathbb{E}_k$ on **Set**. We must also construct α and γ which are valid distributive laws. For the case of α we consider the transformation

$$\alpha_X(i, \varphi) = \sum_{x \in X} \varphi(x).(i, x).$$

We verify naturality and the distributive law axioms for α:

- **Naturality**: For a function $f : X \to Y$ we have

$$\begin{aligned}
&\alpha_Y \circ [k] \times (.)\mathcal{D}_\mathbf{S}(f)(i, \varphi) = \\
&\alpha_Y(i, \sum_{x \in X} \varphi(x).f(x)) = \\
&\sum_{x \in X} \varphi(x).(i, f(x))
\end{aligned}$$

  and

$$\begin{aligned}
&\mathcal{D}_\mathbf{S}[k] \times (.)(f) \circ \alpha_X(i, \varphi) = \\
&\mathcal{D}_\mathbf{S}[k] \times (.)(f) \sum_{x \in X} \varphi(x).(i, x) =
\end{aligned}$$

$$\sum_{x \in X} \varphi(x).(i, f(x))$$

- **Unit**:

$$\alpha \circ [k] \times (.)\eta(i, x) = \alpha(i, 1.x) = 1.(i, x) = \eta[k] \times (.)(i, x)$$

- **Counit**:

$$\mathcal{D}_{\mathbf{S}}\varepsilon \circ \alpha(i, \varphi) = \mathcal{D}_{\mathbf{S}}\varepsilon \sum_{x \in X} \varphi(x).(i, x) = \varphi = \varepsilon\mathcal{D}_{\mathbf{S}}(i, \varphi)$$

- **Multiplication**:

$$\begin{aligned}
&\alpha \circ [k] \times (.)\mu(i, \Psi) = \\
&\alpha(i, \sum_{x \in X} \sum_{\varphi \in \mathcal{D}_{\mathbf{S}}(X)} \Psi(\varphi).\varphi(x).x) = \\
&\sum_{x \in X} \sum_{\varphi \in \mathcal{D}_{\mathbf{S}}(X)} \Psi(\varphi).\varphi(x).(i, x)
\end{aligned}$$

$$\begin{aligned}
&\mu[k] \times (.) \circ \mathcal{D}_{\mathbf{S}}\alpha \circ \alpha\mathcal{D}_{\mathbf{S}}(i, \Psi) = \\
&\mu[k] \times (.) \circ \mathcal{D}_{\mathbf{S}}\alpha \sum_{\varphi \in \mathcal{D}_{\mathbf{S}}(X)} \Psi(\varphi).(i, \varphi) = \\
&\mu[k] \times (.) \sum_{\varphi \in \mathcal{D}_{\mathbf{S}}(X)} \Psi(\varphi). \sum_{x \in X} \varphi(x).(i, x) = \\
&\sum_{x \in X} \sum_{\varphi \in \mathcal{D}_{\mathbf{S}}(X)} \Psi(\varphi).\varphi(x).(i, x)
\end{aligned}$$

- **Comultiplication**:

$$\mathcal{D}_{\mathbf{S}}\delta \circ \alpha(i, \varphi) = \mathcal{D}_{\mathbf{S}}\delta \sum_{x \in X} \varphi(x).(i, x) = \sum_{x \in X} \varphi(x).(i, (i, x))$$

$$\begin{aligned}
&\alpha[k] \times (.) \circ [k] \times (.)\alpha \circ \delta\mathcal{D}_{\mathbf{S}}(i, \varphi) = \\
&\alpha[k] \times (.) \circ [k] \times (.)\alpha(i, (i, \varphi)) = \\
&\alpha[k] \times (.)(i, \sum_{x \in X} \varphi(x).(i, x)) = \\
&\sum_{x \in X} \varphi(x).(i, (i, x))
\end{aligned}$$

Note that we can show $\kappa = \beta([k] \times (.)) \circ L_k^+\alpha$ as follows:

$$\begin{aligned}
&\beta_X([k] \times (.)) \circ L_k^+\alpha_X[(p_1, \varphi_1), \ldots, (p_n, \varphi_n)] = \\
&\beta_X([k] \times (.))[\sum_{x \in X} \varphi_1(x).(p_1, x), \ldots, \sum_{x \in X} \varphi_n(x).(p_n, x)] = \\
&\sum_{[(p_1,x_1),\ldots,(p_n,x_n)] \in L_k^+ \circ [k] \times (.)(X)} \boldsymbol{\varphi}(\boldsymbol{x}).[(p_1, x_1), \ldots, (p_n, x_n)] = \\
&\kappa_X[(p_1, x_1), \ldots, (p_n, x_n)]
\end{aligned}$$

For γ we consider the transformation

$$\gamma_X[(p_1, x_1), \ldots, (p_n, x_n)] = (p_n, [x_1, \ldots, x_n]).$$

Again we verify naturality and the distributive law axioms:

- **Naturality**: For a function $f : X \to Y$ we have:

$$\begin{aligned}
&\gamma_Y \circ L_k^+[k] \times (.)(f)[(p_1, x_1), \ldots, (p_n, x_n)] = \\
&\gamma_Y[(p_1, f(x_1)), \ldots, (p_n, f(x_n))] = \\
&(p_n, [f(x_1), \ldots, f(x_n)])
\end{aligned}$$

  And

$$\begin{aligned}
&[k] \times (.)L_k^+(f) \circ \gamma_X[(p_1, x_1), \ldots, (p_n, x_n)] = \\
&[k] \times (.)L_k^+(f)(p_n, [x_1, \ldots, x_n]) =
\end{aligned}$$

$$(p_n,[f(x_1),\dots,f(x_n)])$$

- **Counit for $[k]\times(.)$**: We have

$$\varepsilon L_k^+\circ\gamma[(p_1,x_1),\dots,(p_n,x_n)]=$$
$$\varepsilon L_k^+(p_n,[x_1,\dots,x_n])=[x_1,\dots,x_n]=L_k^+\varepsilon[(p_1,x_1),\dots,(p_n,x_n)]$$

- **Counit for $L_k^+(f)$**:

$$[k]\times(.)\varepsilon\circ\gamma[(p_1,x_1),\dots,(p_n,x_n)]=$$
$$[k]\times(.)\varepsilon(p_n,[x_1,\dots,x_n])=(p_n,x_n)=\varepsilon[k]\times(.)[(p_1,x_1),\dots,(p_n,x_n)]$$

- **Comultiplication for $[k]\times(.)$**:

$$\delta L_k^+\circ\gamma[(p_1,x_1),\dots,(p_n,x_n)]=$$
$$\delta L_k^+(p_n,[x_1,\dots,x_n])=$$
$$(p_n,(p_n,[x_1,\dots,x_n]))$$

$$[k]\times(.)\gamma\circ\gamma[k]\times(.)\circ L_k^+\delta[(p_1,x_1),\dots,(p_n,x_n)]=$$
$$[k]\times(.)\gamma\circ\gamma[k]\times(.)[(p_1,(p_1,x_1)),\dots,(p_n,(p_n,x_n))]=$$
$$[k]\times(.)\gamma(p_n,[(p_1,x_1),\dots,(p_n,x_n)])=$$
$$=(p_n,(p_n,[x_1,\dots,x_n]))$$

- **Comultiplication for $L_k^+(f)$**:

$$[k]\times(.)\delta\circ\gamma[(p_1,x_1),\dots,(p_n,x_n)]=$$
$$[k]\times(.)\delta(p_n,[x_1,\dots,x_n])=$$
$$(p_n,[[x_1],\dots,[x_1,\dots x_n]])$$

$$
\begin{aligned}
&\gamma L_k^+ \circ L_k^+\gamma \circ \delta[k] \times (.)[(p_1, x_1), \dots, (p_n, x_n)] = \\
&\gamma L_k^+ \circ L_k^+\gamma[[(p_1, x_1)], \dots, [(p_1, x_1), \dots, (p_n, x_n)]] = \\
&\gamma L_k^+[(p_1, [x_1]), \dots, (p_n, [x_1, \dots, x_n])] = \\
&(p_n, [[x_1], \dots, [x_1, \dots x_n]])
\end{aligned}
$$

Finally, to apply theorem 3.46 it remains to verify the Yang-Baxter equation. We have

$$
\begin{aligned}
&\alpha_X L_k^+ \circ ([k] \times (.))\beta_X \circ \gamma_X \mathcal{D}_{\mathbf{S}}[(p_1, \varphi_1), \dots, (p_n, \varphi_n)] = \\
&\alpha_X L_k^+ \circ ([k] \times (.))\beta_X (p_n, [\varphi_1, \dots, \varphi_n]) = \\
&\alpha_X L_k^+ (p_n, \sum_{[x_1, \dots, x_n] \in L_k^+(X)} \boldsymbol{\varphi}(\boldsymbol{x}).[x_1, \dots, x_n]) = \\
&\sum_{[x_1, \dots, x_n] \in L_k^+(X)} \boldsymbol{\varphi}(\boldsymbol{x})(p_n, [x_1, \dots, x_n])
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{D}_{\mathbf{S}}\gamma_X \circ \beta_X([k] \times (.)) \circ L_k^+\alpha_X[(p_1, \varphi_1), \dots, (p_n, \varphi_n)] = \\
&\mathcal{D}_{\mathbf{S}}\gamma_X \sum_{[(p_1, x_1), \dots, (p_n, x_n)] \in L_k^+ \circ [k] \times (.)(X)} \boldsymbol{\varphi}(\boldsymbol{x}).[(p_1, x_1), \dots, (p_n, x_n)] = \\
&\sum_{[(p_1, x_1), \dots, (p_n, x_n)] \in L_k^+ \circ [k] \times (.)(X)} \boldsymbol{\varphi}(\boldsymbol{x}).(p_n, [x_1, \dots, x_n]) = \\
&\sum_{[x_1, \dots, x_n] \in L_k^+(X)} \boldsymbol{\varphi}(\boldsymbol{x})(p_n, [x_1, \dots, x_n])
\end{aligned}
$$

As required. □

Now note that the same transformation κ can be viewed as a map of the form $\mathbb{P}_k \mathbb{D}_{\mathbf{S}} \to \mathbb{D}_{\mathbf{S}} \mathbb{P}_k$:

$$
\kappa_X[(p_1, \varphi_1), \dots, (p_n, \varphi_n)] = \sum_{[(p_1, x_1), \dots, (p_n, x_n)] \in \mathbb{P}_k(X)} \boldsymbol{\varphi}(\boldsymbol{x}).[(p_1, x_1), \dots, (p_n, x_n)]
$$

.

To extend the previous theorem to a distributive law in $\mathbf{R}(\sigma)$ all that is left is to show that the components of κ are valid homomorphisms.

Lemma 3.48. *$\kappa_X : \mathbb{P}_k \mathbb{D}_{\mathbf{S}} \mathcal{X} \to \mathbb{D}_{\mathbf{S}} \mathbb{P}_k \mathcal{X}$ is a homomorphism in $\mathbf{R}(\sigma)$.*

Proof. We proceed similarly to lemma 3.45. Suppose $R \in \sigma$ is a relation symbol of arity n. Take $(l_1 = [(p^1_1, \varphi^1_1), \dots, (p^1_{m_1}, \varphi^1_{m_1}]), \dots, (l_n = [(p^n_1, \varphi^n_1), \dots, p^n_{m_n}, \varphi^n_{m_n})]) \in R^{\mathbb{P}_k \mathbb{D}_{\mathbf{S}} \mathcal{X}}$. We must show that $(\kappa_X l_1 = \boldsymbol{\varphi^1}, \dots, \kappa_X l_n = \boldsymbol{\varphi^n}) \in R^{\mathbb{D}_{\mathbf{S}} \mathbb{E}_k \mathcal{X}}$. Via lemma 3.44 it suffices to show that $\boldsymbol{\varphi^1(s_1)}.\boldsymbol{\varphi^2(s_2)}., \dots, .\boldsymbol{\varphi^n(s_n)} = 0$ whenever $(s_1, \dots, s_n) \notin R^{\mathbb{P}_k \mathcal{X}}$. This time there are three cases to consider:

- $s_i \not\uparrow s_j$ for some i, j: The proof is verbatim the same as lemma 3.45 (replacing $\mathbb{E}_k$ with $\mathbb{P}_k$ everywhere).
- $(\varepsilon s_1, \dots, \varepsilon s_n) \notin R^{\mathcal{X}}$: The proof is again the same as lemma 3.45.
- There exists s_i and s_j such that the pebble index of the last move of s_i appears in its suffix in s_j: Let us make use of some additional notation. For $s = [(p^s_1, x_1), \dots, (p^s_a, x_a)] \in \mathbb{P}_k \mathcal{X}$ and $t = [(p^t_1, y_1), \dots, (p^t_b, y_b)] \in \mathbb{P}_k \mathcal{Y}$ we write $s \mathrel{\hat{=}} t$ whenever $a = b$ and $p^s_k = p^t_k$ for all k. Now, if we have both $s_i \mathrel{\hat{=}} l_i$ and $s_j \mathrel{\hat{=}} l_j$ it must be the case that the pebble index of the last move in l_i appears in its suffix in l_j. This contradicts the fact that l_i and l_j appear in the same tuple of $R^{\mathbb{P}_k \mathbb{D}_{\mathbf{S}} \mathcal{X}}$. Therefore, we have either $s_i \not\mathrel{\hat{=}} l_i$ or $s_j \not\mathrel{\hat{=}} l_j$. From the way κ is defined this implies that either $\boldsymbol{\varphi^i(s_i)} = 0$ or $\boldsymbol{\varphi^j(s_j)} = 0$. In either case we obtain $\boldsymbol{\varphi^1(s_1)}.\boldsymbol{\varphi^2(s_2)}., \dots, .\boldsymbol{\varphi^n(s_n)} = 0$ as required.

□

Remark 3.49. *It is worth noting that we can define a variant of the coreader comonad $[k] \times (.)$ on $\mathbf{R}(\sigma)$ whereby $((i_1, x_1), \dots (i_n, x_n)) \in R^{[k] \times (.) \mathcal{X}}$ iff $(x_1, \dots, x_n) \in R^{\mathcal{X}}$. For this variant, it is almost the case that $\mathbb{P}_k = \mathbb{E}_k \circ [k] \times (.)$. The only reason this equality does not hold is the active pebble condition in the definition of $\mathbb{P}_k$. If we consider an alternative comonad $\mathbb{P}^*_k$ which is defined as in $\mathbb{P}_k$ but with the active pebble condition removed we could prove directly using iterated distributive laws that a distributive law of $\mathbb{P}^*_k$ over $\mathbb{D}_{\mathbf{S}}$ exists. Thus, the active condition is the only reason lemma 3.48 requires a more direct proof.*

3.4 Discussion

In this chapter, we derived both negative and positive results for when it is possible to construct a mixed distributive law of a comonad over a monad.

For our negative results, we first considered the class of comonads $\mathcal{C}_W \subseteq \mathbf{Com}(\mathbf{Set})$ and the class of monads $\mathcal{C}_M \subseteq \mathbf{Mon}(\mathbf{Set})$. The class of comonads $\mathcal{C}_W$ is all directed containers except the coreader comonad. The class of monads $\mathcal{C}_M$ is any monad which admits a sensible notion of "uniform distribution". We then proved a transfer theorem which allowed us to extend our results to classes of (co)monads defined over any category $\mathbf{D}$ which admits $\mathbf{Set}$ as a (co)reflective subcategory. This transfer theorem may be of independent interest to researchers working on (co)monad theory. Overall, our negative results show the non-existence of mixed distributive laws between a large number of (co)monads used in areas such as probability theory, programming languages, finite model theory, and universal algebra. As such, we hope that they will be of relevance to researchers working in these areas. For our positive result, we showed that the game comonads $\mathbb{E}_k$ and $\mathbb{P}_k$ distribute over any distribution minion monad $\mathbb{D}_{\mathbf{S}}$ over a commutative semiring satisfying orthogonality and idempotence of distributions. There are many avenues for future work. We list some of them below:

- **Necessary and sufficient conditions on $\mathbb{D}_{\mathbf{S}}$**: We have shown that for some semirings $\mathbf{S}$ it is impossible to construct a distributive law $\mathbb{E}_k\mathbb{D}_{\mathbf{S}} \to \mathbb{D}_{\mathbf{S}}\mathbb{E}_k$ while for other semirings such a law does exist. A natural follow-up question is to identify the necessary and sufficient conditions on $\mathbf{S}$ under which such laws exist.

- **Axiomatic account**: [ZM22] determined axioms for when two algebraic theories do not yield a composite algebraic theory. Since algebraic theories correspond to finitary monads over $\mathbf{Set}$, the axioms provide a framework for determining the non-existence of monad-monad distributive laws. Do similar (co)alegebraic axioms exist for determining the non-existence of comonad-monad distributive laws? This would involve formulating and working with coalgebraic presentations of directed containers and algebraic presentations of monads.

- **Extension to other categories**: We believe that the transfer theorems derived in section 3.2 have applications beyond what we have considered here. Take for instance the Vietoris monad $\mathcal{V}$, and Radon monad $\mathcal{R}$ [Goy21] defined on the category of compact Hausdorff spaces. These can be seen as topological analogues of $\mathcal{P}$, and $\mathcal{D}_{\mathbb{R}_{\geq 0}}$ respectively. Can we use a monad-monad variant of our transfer theorem to prove a no-go theorem between these monads? Such a result would make use of the well-known fact that there is no distributive law $\lambda\colon \mathcal{P}\mathcal{D}_{\mathbb{R}_{\geq 0}} \to \mathcal{D}_{\mathbb{R}_{\geq 0}}\mathcal{P}$ [ZM22].

  Another interesting direction to pursue is the extension of our comonad-monad no-go results to other categories. For instance, quasi-Borel spaces $\mathbf{QBS}$ introduced in [HKSY17] have recently gained much attention in the context of probabilistic programming. The monad of probability measures on $\mathbf{QBS}$ generalises the well-known Giry monad [Gir82], and acts as an analogue of $\mathcal{D}_{\mathbf{S}}$ over this category. Comonads in $\mathbf{QBS}$ are not well-studied, thus it would be interesting to see if any of the directed containers we have considered admit analogues in $\mathbf{QBS}$ and whether our no-go theorems extend to this category.

- **Weak distributive laws**: There is substantive literature [Gar19; GP20; GPA21; Goy22; Goy21] showing that one can recover many of the desirable properties of a composite monad by constructing a natural transformation which only satisfies a subset of the monad-monad distributive law axioms. In the case of comonad-monad laws, we ask if similar relaxations can be used to recover features of a biKleisli category.

It is worth remarking however that for overcoming the particular no-go results discussed in this chapter this line of inquiry may be less practically useful than in the monad-monad case. This is because the natural analogue of the definition of a weak distributive law in our setting would be to drop the requirement that the counit axiom is satisfied. However, as we have discussed our no-go theorems show that it is impossible to simultaneously satisfy the unit and comultiplication axioms.

- **Monad over comonad distributive laws**: For some of the (co)monads covered by our no-go theorems it is possible to construct valid monad over comonad distributive laws. For instance, even though there is no distributive law of the form $\kappa\colon L^{\infty}\mathcal{P}^{+} \to \mathcal{P}^{+}L^{\infty}$, it is easy to check that $\gamma\colon \mathcal{P}^{+}L^{\infty} \to L^{\infty}\mathcal{P}^{+}$ as defined below is a distributive law.

$$\begin{aligned}\gamma_X\{(a_1, a_2, \ldots), (b_1, b_2, \ldots), \ldots, (z_1, z_2, \ldots)\}\\ = (\{a_1, b_1, \ldots, z_1\}, \{a_2, b_2, \ldots, z_2\}, \ldots)\end{aligned}$$

Such laws give rise to categories of bialgebras, a construction which has applications in mathematics and computer science (see e.g. [TP97; LMW15; Kli11]). The following is an open question:

Question 3.50. *For which $M \in \mathcal{C}_M$ and $W \in \mathcal{C}_W$ does there exist a distributive law $\gamma\colon MW \to WM$?*

4

Background on quantum information

So far in this thesis, we have focused on relaxations of CSP and structure isomorphism that arise in classical computer science. As we discussed in the introduction, however, we are also interested in studying relaxations that arise in the study of quantum information. Before introducing these in more detail we shall spend this short chapter describing the necessary background material from quantum information. We begin with a brief review of the basics of quantum theory. After this, we describe the abstract framework of non-local games which plays a major role in the next chapter.

4.1 Linear algebra

Linear algebra studies vector spaces and mappings between them. We shall require some basic definitions from this field.

Given a semiring S, a *left S-semimodule* is an abelian monoid $(M, +_M, 0_M)$ equipped with a scalar multiplication $\cdot_{S,M} : S \times M \to M$ such that:

- $\forall r, s \in S,\ \forall m \in M,\ (r + s) \cdot m = (r \cdot m) + (s \cdot m)$.
- $\forall r \in S,\ \forall m_1, m_2 \in M,\ r \cdot (m_1 + m_2) = (r \cdot m_1) + (r \cdot m_2)$.
- $\forall r, s \in S,\ \forall m \in M,\ (r \cdot s) \cdot m = r \cdot (s \cdot m)$.

- $\forall m \in M,\ 1 \cdot m = m$
- $\forall r \in S, \forall m \in M,\ 0 \cdot m = 0 = r.0$

A *right* $\boldsymbol{S}$*-semimodule* is defined similarly. If S is commutative the two concepts coincide and are called an $\boldsymbol{S}$*-semimodule.*

A *left* $\boldsymbol{R}$*-module* over a ring $\boldsymbol{R}$ is a left $\boldsymbol{R}$-semimodule $(M, +, \cdot)$ where $\boldsymbol{R}$ is a ring. A *vector space* over a field $\boldsymbol{F}$ is an $\boldsymbol{F}$-module $(V, +, \cdot)$. Unless otherwise specified all vector spaces we consider will be over the field of complex numbers $\mathbb{C}$.

A *normed vector space* is a vector space $\boldsymbol{V}$ equipped with a norm $\|\cdot\|$, which is a function $\|\cdot\| : V \to [0, \infty)$ satisfying:

- **Positive Definiteness**: $\|v\| \geq 0$ for all $v \in V$, and $\|v\| = 0$ if and only if $v = 0$.
- **Homogeneity (or Scalability)**: $\|\alpha v\| = |\alpha| \|v\|$ for all $\alpha \in F$ and $v \in V$.
- **Triangle Inequality**: $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

A sequence $(v_n)_{n \in \mathbb{N}}$ in a normed vector space V is called a *Cauchy sequence* if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\|v_n - v_m\| < \epsilon$ for all $n, m \geq N$. In other words, the elements of the sequence get arbitrarily close to each other as the sequence progresses. A space is complete if every Cauchy sequence converges to a limit within the space. A *Banach space* is a normed vector space that is complete. A *Hilbert space* is a Banach space that is also equipped with an inner product $\langle \cdot, \cdot \rangle : \boldsymbol{H} \times \boldsymbol{H} \to \boldsymbol{F}$ ($\boldsymbol{F}$ is $\mathbb{R}$ or $\mathbb{C}$) that satisfies:

- **Conjugate Symmetry**: $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in H$.
- **Linearity in the First Argument**: $\langle \alpha v + \beta w, u \rangle = \alpha \langle v, u \rangle + \beta \langle w, u \rangle$ for all $u, v, w \in H$ and $\alpha, \beta \in F$.
- **Positive Definiteness**: $\langle v, v \rangle \geq 0$ for all $v \in H$, and $\langle v, v \rangle = 0$ if and only if $v = 0$.

The norm induced by the inner product is given by $\|v\| = \sqrt{\langle v, v \rangle}$.

An *(associative)* $\boldsymbol{F}$*-algebra* $\boldsymbol{A}$ over a field $\boldsymbol{F}$ is a vector space over F equipped with a bilinear multiplication operation $\cdot : \boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{A}$ that satisfies:

- **Associativity**: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in A$.
- **Distributivity**: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in A$.
- **Compatibility with Scalars**: $(\alpha a) \cdot b = \alpha(a \cdot b)$ and $a \cdot (\alpha b) = \alpha(a \cdot b)$ for all $\alpha \in F$ and $a, b \in A$.

A *normed* $\boldsymbol{F}$*-algebra* is an $\boldsymbol{F}$-algebra $\boldsymbol{A}$ that is also a normed vector space, with the norm satisfying

- **Submultiplicativity**: $\forall a, b \in A, \|a \cdot b\| \leq \|a\|\|b\|$.

A *Banach algebra* is a normed $\boldsymbol{F}$-algebra that is also a Banach space. An *involution* on an $\boldsymbol{F}$-algebra $\boldsymbol{A}$ is a map $* : \boldsymbol{A} \rightarrow \boldsymbol{A}$ such that for all $a, b \in A$ and $\alpha \in F$:

- $(a^*)^* = a$,
- $(a + b)^* = a^* + b^*$,
- $(\alpha a)^* = \overline{\alpha} a^*$, where $\overline{\alpha}$ is the complex conjugate of α,
- $(a \cdot b)^* = b^* \cdot a^*$.

A *C^*-algebra* is a Banach algebra $\boldsymbol{A}$ equipped with an involution $*$ such that for all $a \in A$:

$$\|a^* a\| = \|a\|^2.$$

A C^*-algebra is unital if it has a multiplicative identity i. A tracial state on a unital C^*-algebra $\mathbf{A}$ is a linear functional $s : A \rightarrow \mathbb{C}$ such that $s(i) = 1, s(a^* a) \geq 0$ for all $a \in a$, and $s(ab) = s(ba)$ for all $a, b \in A$. The tracial state s is faithful whenever we have $s(a^* a) = 0 \iff a = 0$. An element $a \in \mathbf{A}$ is a projection if $a = a^* = aa$. If a and b are projections then $s(ab) = 0 \iff ab = 0$.

Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be vector spaces over the same field $\boldsymbol{F}$. A *linear map* or *vector space homomorphism* is given by a mapping $M : U \rightarrow V$ satisfying:

- **Additive preservation:** $\forall u, v \in U, M(u + v) = \alpha(u) + M(v)$
- **Multiplicative preservation:**$\forall s \in \boldsymbol{F}, u \in U : M(su) = sM(u)$.

When working in finite dimensions all our vector spaces are of the form $\boldsymbol{V} = \mathbb{C}^d$ for some positive integer d. In this case linear maps $F : \mathbb{C}^d \rightarrow \mathbb{C}^e$ can be identified with $d \times e$ matrices with complex entries.

For vector spaces $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ we say that a map $h : U \times V \rightarrow W$ is *bilinear* if:

- For all $u \in U : h(u, \_)$ is a linear map $\boldsymbol{V} \rightarrow \boldsymbol{W}$.
- For all $v \in V : h(\_, v)$ is a linear map $\boldsymbol{U} \rightarrow \boldsymbol{W}$.

The tensor product of $\boldsymbol{U}$ and $\boldsymbol{V}$, denoted by $\boldsymbol{U} \otimes \boldsymbol{V}$ is the unique (up to isomorphism) vector space for which there exists a bijective correspondence between:

- Linear maps $\boldsymbol{U} \otimes \boldsymbol{V} \rightarrow \boldsymbol{W}$;
- Bilinear maps $\boldsymbol{U} \times \boldsymbol{V} \rightarrow \boldsymbol{W}$.

If $\boldsymbol{U}$ and $\boldsymbol{V}$ are are normed we say that a linear map is *bounded* if there exists a real number $m > 0$ such that:

$$\forall u \in U, \|M(u)\|_V \leq m\|u\|_U$$

The smallest m satisfying this condition is called the operator norm of M and is denoted by $\|M\|$. If $\boldsymbol{U}, \boldsymbol{V}$ are Hilbert spaces every linear map has a *(hermitian) adjoint* $M^\dagger : \boldsymbol{V} \to \boldsymbol{U}$ defined as the unique linear map satisfying:

$$\forall u \in U, v \in V, \langle Mu, v\rangle_V = \langle u, M^\dagger v\rangle_U$$

.

A linear map is *unitary* if $MM^\dagger = M^\dagger M = I$.

The term *linear operator* is sometimes used as a synonym for linear map. In this thesis however, a linear operator always refers to an endomorphism of vector spaces i.e. a linear map of type $\boldsymbol{V} \to \boldsymbol{V}$. Now fix a Hilbert space $\boldsymbol{H}$. We write $I_{\boldsymbol{H}}$ for the unique identity operator. We may drop the subscript $\boldsymbol{H}$ when the Hilbert space in question is clear. A bounded linear operator N on $\boldsymbol{H}$ is said to be:

- *Positive semi-definite* if $\forall u \in H, \langle Nu, u\rangle \geq \mathbf{0}$. In this case, we write $N \geq \mathbf{0}$. Likewise, the term *positive definite* is used if we replace the $\geq$ condition with $>$.
- Hermitian if $N = N^\dagger$.
- An (orthogonal) projector if $N = N^2 = N^\dagger$.

We write $\mathcal{B}(\mathbf{H})$ for the C^*-algebra of all bounded linear operators in $\mathbf{H}$. Moreover, we write $\mathcal{B}^+(\mathbf{H})$ for the set of positive bounded operators on $\mathbf{H}$ and $\mathsf{Proj}(\mathbf{H})$ for the set of projectors on $\mathbf{H}$. The closed subpsace onto which any $P \in \mathsf{Proj}(\mathbf{H})$ maps the entire space $\mathbf{H}$ is known as the *range* of P. We refer to two projectors M, N as being orthogonal to each other if $MN = \mathbf{0}$. We write $M \leq N$ iff $N - M$ is positive semi-definite. M and N *commute* iff $MN - NM = \mathbf{0}$. The product of two projectors is a projector iff they commute. For any family of projectors $\{P_i\}_i$ it holds that $\sum_i P_i \leq I_{\boldsymbol{H}}$ iff $P_iP_j = \mathbf{0}$ whenever $i \neq j$.

4.2 Quantum theory basics

We associate to any isolated physical system a Hilbert space $\mathbf{H}$ known as its *state space*. A *pure quantum state* is a vector of unit norm in $\mathbf{H}$. The intuition is that each such vector represents a possible state of the physical system. More generally, quantum theory allows for the state of a system to be described by so-called *mixed quantum states* which, informally, correspond to probabilistic mixtures of pure quantum states[1]. Mixed states do not play a major role in this thesis so we omit a detailed description of their behaviour.

To observe a physical property of a system we must perform a measurement. We shall be concerned with two mathematical formalisms for describing measurements for describing different types of measurements. The first is known as a *projection valued measurement* (PVM). A PVM is a function $P : O \to \mathsf{Proj}(\mathbf{H})$ which satisfies $\sum_{o \in O} p(o) = I_{\mathbf{H}}$. Here, the elements of O represent the possible outcomes of a measurement. We shall also consider *positive operator-valued measurements* (POVMs). A POVM is a function $P : O \to \mathcal{B}^+(\mathbf{H})$ which satisfies $\sum_{o \in O} p(o) = I_{\mathbf{H}}$. Note that $\mathsf{Proj}(\mathbf{H})$ is contained within $\mathcal{B}^+(\boldsymbol{H})$. Therefore, it is clear that every PVM is a POVM. We can physically interpret a PVM as representing an ideal quantum measurement. POVMs instead represent measurements which may be disturbed by factors such as noise and uncertainty in our measurement apparatus. PVMs and POVMs are sometimes referred to as sharp and unsharp observables respectively.

The outcome of a measurement in quantum theory, even on pure states, is probabilistic. If a POVM $P : O \to \mathcal{B}^+(\mathbf{H})$ is performed on a pure state $\psi \in \mathbf{H}$ outcome $o \in O$ will be observed with probability $\psi^\dagger p(o) \psi$. A set of POVMs $\{P_i : O_i \to \mathcal{B}^+(\mathbf{H})\}_i \in [n]$ is *commeasurable*[2] iff there exists a POVM $\mathbf{P} : \prod_{i \in [n]} O_i \to \mathcal{B}^+(\mathbf{H})$ which marginalises to each of the P_i i.e. satisfies the following equation:

$$P_i(o_i) = \sum_{o_1 \in O_1, \ldots o_{i-1} \in O_{i-1}, o_{i+1} \in O_{i+1}, \ldots o_n \in O_n} \mathbf{P}(o_1, \ldots, o_n)$$

.

Informally, commeasurable POVMs are those measurements which can be performed together as part of the same experimental setup without affecting each other's outcomes. A remarkable feature of quantum theory (one which is in fact necessary for quantum advantage [UMG14]) is that not all observations are commeasurable. We will write $P_1 \odot P_2$ whenever P_1 and P_2 are commeasurable. We remark that it is possible to have POVMs P_1, P_2, P_3 which are pairwise commeasurable but not triplewise commeasurable [HFR14]. Two POVMs are said to commute iff every element of their image commutes. If we restrict our attention to PVMs then commeasurability has particularly nice characteristics summarised by the following lemma (see e.g. [HRS08]).

Lemma 4.1. *For a set of PVMs $\{P_i\}_i$ the following are equivalent:*

1. *The P_i are pairwise commuting.*

---

[1] Note that different distributions over pure states can give rise to the same mixed state.

[2] Also known as *compatible* or *jointly measurable*.

2. *There exists a unique joint measurement* $\boldsymbol{P}$ *which is projective and defined as* $\boldsymbol{P}(o_1, \ldots, o_n) = \prod_{i \in [n]} P_i(o_i)$.

4.2.1 The algebra of projectors

A single projector P can be used to describe a PVM $M : \mathbf{2} \to \mathsf{Proj}(\mathbf{H})$ with only two measurement outcomes. This is achieved by setting $M(1) = P$ and $M(0) = I - P$. For this reason, Von Neumann [Von32] noted that a projector can be thought of as the quantum analogue of the classical notion of proposition i.e. a question that has a yes (true) or no (false) answer. This observation was the starting point for the study of quantum logic. As part of this study, considerable attention has been devoted to understanding the various algebraic structures that can be placed on the set of projectors over a Hilbert space. While quantum logic itself is not a major focus of ours having an understanding of some of these algebraic properties will be beneficial for us. One structure that can be placed on projectors is that of a complete orthomodular lattice $(\mathsf{Proj}(\boldsymbol{H}), \vee, \wedge, \perp)$ where the operations are defined as:

- $P_1 \vee P_2$ is given by the projector onto the closure of the direct sum of their ranges.
- $P_1 \wedge P_2$ is given by the projector onto the intersection of their ranges.
- $P^\perp = I - P$.

Following Birkhoff and Neumann [BN36] this is the traditional algebraic structure used to study quantum logic. However, for our purposes, it is more interesting to view projectors as being equipped with the structure of a partial boolean algebra. We recall that a partial boolean algebra is given by a set A together with a reflexive, symmetric, binary relation $\odot$ on A which we refer to as "commeasurability", constants 0, and 1, a total unary operation $\neg$, and partial binary operations $\wedge$ and $\vee$ with domain $\odot$. Moreover, the following "extensionality" property must be satisfied: Every set of pairwise commeasurable elements must be contained in a set of pairwise commeasurable elements which form a total boolean algebra. Under this definition we have the partial boolean algebra $(\mathsf{Proj}(\boldsymbol{H}), \odot, \vee, \wedge, \neg)$ whose operations are given by:

- $P_1 \odot P_2$ iff P_1 and P_2 commute.
- $P_1 \vee P_2 = P_1 + P_2 - P_1 P_2$.
- $P_1 \wedge P_2 = P_1 P_2$.
- $\neg P = I - P$.

This leads to an alternative flavour of quantum logic, advocated for by Kochen and Specker [KS67] (see also [AB21]) whereby logical operations are only defined on the domain where they are physically meaningful. As we shall see this partiality plays a crucial role in our characterisation of quantum isomorphisms in the next chapter. In the same way that every boolean algebra forms a semiring the structure $(\mathsf{Proj}(\mathbf{H}), \vee, \wedge, \odot)$ forms a partial commutative semiring. For our purposes it will be helpful to restrict the domain of the addition operation even further. If P_1 and P_2 are not only commuting but also orthogonal (i.e. $P_1 P_2 = 0$) then it is the case that $P_1 \vee P_2 = P_1 + P_2$. Thus we can consider a partial semiring structure $(\mathsf{Proj}(\mathbf{H}), +, \cdot, \odot^{+}, \odot^{\cdot})$ where multiplication and addition still coincide with $\wedge$ and $\vee$ as defined above, but where they are no longer defined on precisely the same domain. In particular, the domain of addition, represented by the binary relation $\odot^{+}$ is restricted to orthogonal projectors. The domain of multiplication, represented by $\odot^{\cdot}$ is still given by all commuting projectors.

4.3 Non-local games

Non-locality and its generalisation in the form of contextuality are important non-classical features of quantum mechanics. These phenomena have been extensively studied as a source of quantum advantage beginning with the pioneering works of Bell [Bel64; Bel66] and Kochen and Specker [KS67]. By now several elegant mathematical frameworks exist which allow us to abstractly study contextuality [AB11; CSW14; AFLS15; Spe05]. For our purposes here we will not need to use any of these frameworks in detail. It will instead suffice to define the concept of a non-local game directly.

A (bipartite) non-local game is a cooperative game played between a verifier and two players, usually referred to as Alice and Bob. These games provide a very useful abstract framework through which non-locality can be studied. Formally, following [CHTW04] a non-local game is defined as $G = (X_a, X_b, Y_a, Y_b, \pi, V)$ where X_a, X_b are finite input sets, Y_a, Y_b are finite output sets, π is a probability distribution on $X_a \times X_b$, and $V : X_a \times X_b \times Y_a \times Y_b \to \mathbb{R}$ is called the payoff function. For our purposes, it suffices to think of V as a $\{0, 1\}$ valued boolean predicate. We will also assume that π has full support. The game proceeds according to the following protocol:

1. The verifier samples a pair $(x_a, x_b) \in X_a \times X_b$ using π and sends x_a to Alice and x_b to Bob.
2. Without communicating, Alice and Bob respond with $y_a \in Y_a$ and $y_b \in Y_b$ respectively.
3. The players win the game if $V(x_a, x_b, y_a, y_b) = 1$.

We focus exclusively on cases where the game consists of only one round of the above protocol.

4.3.1 Strategies

The goal of Alice and Bob is to maximise their winning probability. To achieve this goal they are allowed to agree on a fixed strategy S for the game beforehand. A *correlation*[3] $p(y_a, y_b|x_a, x_b)$ represents the joint conditional probability of Alice and Bob responding with y_a and y_b on inputs x_a and x_b respectively.

There are several classes of strategies that Alice and Bob can employ in a non-local game. We recount some important ones below.

1. A *deterministic classical strategy* $S_c = (f_a, f_b)$ is defined by two functions $f_a : X_a \to Y_a$ and $f_b : X_b \to Y_b$ which map inputs to outputs for each of Alice and Bob respectively. Hence we have:

$$p(f_a(x_a), f_b(x_b)|x_a, x_b) = 1$$

2. A *quantum tensor strategy* $S_* = (\psi, E, F)$ consists of a shared state $\psi \in \boldsymbol{H_A} \otimes \boldsymbol{H_B}$ for finite dimensional complex Hilbert spaces $\boldsymbol{H_A}, \boldsymbol{H_B}$, and two sets, $E = \{E_x\}$ and $F = \{F_x\}$ where $E_x = \{E_{xy} : y \in Y_a\}$ for each $x \in X_a$ and $F_x = \{F_{xy} : y \in Y_b\}$ for each $x \in X_b$ are POVMs over $\boldsymbol{H_A}$ and $\boldsymbol{H_B}$ respectively. Upon receiving $x_a \in X_a$ and $x_b \in X_b$ Alice and Bob perform the measurements E_{x_a} and F_{x_b} respectively and return the results. Hence we have:

$$p(y_a, y_b|x_a, x_b) = \psi^\dagger (E_{x_a y_a} \otimes F_{x_b y_b})\psi$$

3. A *quantum commuting strategy* $S_{co} = (\psi, E, F)$ consists of a shared state $\psi \in \mathcal{X}$ for some potentially infinite dimensional Hilbert space $\mathcal{X}$, and two sets of POVMs E and F, representing Alice and Bobs measurements as above. Note that in these strategies all of Alice's measurement operators must commute with all of Bob's measurement operators. Then we have:

$$p(y_a, y_b|x_a, x_b) = \psi^\dagger E_{x_a y_a} F_{x_b y_b} \psi$$

4. A *Non-signalling strategy* S_{ns} is any strategy in which Alice's output is independent of Bob's input and vice-versa. This requirement is the well-known *no-signalling principle*. Formally, the following two families of equations hold:

$$\sum_{y_b} p(y_a, y_b|x_a, x_b) = \sum_{y_b} p_{S_{ns}}(y_a, y_b|x_a, x'_b) \forall x_a, y_a, x_b, x'_b$$

---

[3] Also known as a behaviour or empirical model.

$$\sum_{y_a} p(y_a, y_b|x_a, x_b) = \sum_{y_a} p_{S_{ns}}(y_a, y_b|x'_a, x_b) \forall x_a, x'_a, x_b, y_b$$

For $t \in \{c, *, co, ns\}$ we will often refer to a t-strategy for a non-local game. Here t describes the type of strategy being used, c for classical, $*$ for tensor, co for commuting, and ns for non-signalling. We denote the probability of winning a non-local game with a t-strategy S as $\omega^t(G, S)$. The maximum probability of winning a game using each type of strategy is known as its value. This is defined as $\omega^t(G) := \sup \omega^t(G, S)$. We focus almost exclusively on perfect strategies, where the game is won with probability 1. In this case, the following equation must hold:

$$p(y_a, y_b|x_a, x_b) = 0 \text{ when } V(x_a, x_b, y_a, y_b) = 0. \tag{4.2}$$

Remark 4.3. *Notice that in general one should also consider probabilistic classical strategies for non-local games where players are allowed to use shared randomness. However, since we are only focusing on perfect strategies we can use a standard argument to show that whenever a probabilistic perfect strategy exists a deterministic perfect strategy must also exist. This is because correlations belonging to any probabilistic strategy can be written as a convex combination* $p = \sum_i \lambda_i p_i$*, where* λ_i*'s encode shared randomness and* p_i *are correlations arising from a deterministic strategy. Since the correlation* p *belongs to a perfect strategy it must satisfy equation 4.2. But this is only possible if every individual* p_i *satisfies this condition.*

Any c-strategy is a $*$-strategy since we can simply ignore entanglement in the latter case. Remarkably, it can be shown that in some cases a perfect $*$-strategy for a non-local game exists even when a perfect c-strategy cannot exist. In such cases, we say that a non-local game exhibits *pseudotelepathy*. Some prominent examples of this phenomenon are the Mermin-Peres magic square [Mer90; Per90] and Mermin's magic pentagram games [Mer93]. Moreover, it can be straightforwardly shown that any $*$-strategy is a co-strategy. A non-trivial result of Tsirelson shows that if we restrict to finite-dimensional Hilbert spaces these two classes of strategies coincide [Tsi06; SW08]. A recent breakthrough result [JNV+21] shows that (allowing for infinite-dimensional Hilbert spaces) co-strategies form a strictly larger set than $*$-strategies. ns strategies are the most general class of strategies that we can consider without breaking the no-signalling principle, and therefore the most general class of strategies allowed in non-local games. We shall not discuss no-signalling strategies any further in this thesis.

4.3.2 Synchronous games

A non-local game is said to be *synchronous* if Alice and Bob have the same question set X, the same answer set Y, and the following equation is satisfied:

$$V(y, y', x, x) = 0 \quad \forall x \in X, y \neq y' \in Y$$

This means that given the same inputs Alice and Bob must always respond with the same outputs to win the game. To aid readability we write $G_s = (X, Y, V)$ to denote synchronous games. Perfect strategies for these games have nice properties which make them easier to analyse. We shall make use of some of these properties in the next chapter. In particular variations of the following lemmas have appeared throughout the literature (see e.g. [AMR+19; OP16]).

Lemma 4.4. *Let* $G_s = (X, Y, V)$ *be a synchronous game. If there exists a perfect* $*$*-strategy for this game then there exists a perfect* $*$*-strategy* $(\psi, \{E_x\}_x \in X, \{F_y\}_y \in Y)$ *where:*

- *The state* ψ *is a maximally entangled state* $\psi = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} e_i \otimes e_i$*;*
- *The POVMs* E_x *and* F_y *are projective;*
- $E_{x,y} = F_{y,x}^T$ *for all* $x \in X, y \in Y$*;*
- $p(y, y'|x, x') = 0$ *iff* $E_{x,y} E_{x',y'} = 0$.

Lemma 4.5. *Let* $G_s = (X, Y, V)$ *be a synchronous game. There exists perfect* co*-strategy for this game iff there exists a unital* C^**-algebra* $\mathbf{A}$ *which admits a faithful tracial state* $s : \mathbf{A} \to \mathbb{C}$*, and projections* $E_{xy} \in \mathbf{A}$ *for* $x \in X$ *and* $y \in Y$ *satisfying:*

- $\sum_{y \in Y} E_{xy} = I$*;*
- $E_{xy} E_{x'y'} = 0$ *if* $V(x, x', y, y') = 0$.

4.3.3 Interactive proof systems

Although it is not a central focus of this thesis we briefly mention the connections between non-local games and interactive proof systems. This will allow us to highlight some complexity theoretic considerations in the ensuing chapters. An interactive proof system is an abstract model in which a trusted but computationally limited *verifier* exchanges messages with a potentially non-trusted, all-powerful *prover*. The verifier's goal is to determine if a given input string (representing an instance of a computational problem) belongs to a language. Any interactive proof system must satisfy the following soundness and completeness criteria:

- Soundness: No prover should be able to convince the verifier of a false statement, except with some small probability.
- Completeness: The prover can convince the verifier of a true statement with high probability.

Depending on the exact setup, one can define several complexity classes using interactive proof systems. For instance, if we allow for only a single message to be exchanged between the prover and the verifier, and we require that the verifier runs in deterministic polynomial time, then the set of languages recognisable by our proof system is precisely the complexity class NP. If we allow for polynomially many rounds of messages, and require that the verifier behaves like a BPP machine we call the corresponding complexity class IP. If we additionally allow for the messages to contain quantum states, and for the verifier to behave like a BQP machine, we call the corresponding complexity class QIP. A series of famous results show that QIP $=$ IP $=$ PSPACE [LFKN92; Sha92; JJUW10]. Interactive proof systems can be generalised to a multi-prover setting. In this case, we have at least two independent, non-communicating provers and a single verifier. The generalisations of IP and QIP to multiple proves are called MIP and QMIP respectively. It is known that MIP $=$ NEXP [BFL91] . We write MIP$[k, n]$ for the class of problems solvable in the multi-prover interactive proof model with k provers and n rounds of communication. One can show that MIP$[2, 1] =$ MIP [FL92], so we already get the full power of MIP with just two provers and a single round of communication. Thus, we always restrict our analysis to this case.

There is a connection between non-local games and interactive proofs when one thinks of Alice and Bob playing the role of the provers, and the referee acting as the verifier. More precisely MIP can be reformulated as the following class of problems:

Definition 4.6. MIP is the class of languages $L \in \{0, 1\}^*$ for which there exists a uniform family of non-local games $\{G_x\}$ such that:

$$x \in L \Rightarrow \omega^c(G_x) = 1$$

$$x \in L \Rightarrow \omega^c(G_x) < \frac{1}{2}$$

This reformulation allows us to define quantum and non-signalling variants of MIP:

Definition 4.7. For $t \in \{c, *, co, ns\}$ MIP^t is the class of languages $L \in \{0, 1\}^*$ for which there exists a uniform family of non-local games $\{G_x\}$ such that:

$$x \in L \Rightarrow \omega^t(G_x) = 1$$

$$x \in L \Rightarrow \omega^t(G_x) < \frac{1}{2}$$

We can also consider the so-called *zero-gap* versions of these classes which are more closely related to perfect strategies for non-local games:

Definition 4.8. For $t \in \{c, *, co, ns\}$ the MIP_0^t consists of languages $L \in \{0,1\}^*$ for which there exists a uniform family of non-local games $\{G_x\}$ such that:

$$x \in L \Rightarrow \omega^t(G_x) = 1$$
$$x \in L \Rightarrow \omega^t(G_x) < 1$$

Studying the different variants of MIP^t has been one of the most fruitful research programs in theoretical computer science. In the classical setting, this work led to the famous PCP theorem [ALM+98]. In the quantum setting, MIP^* was gradually shown to contain larger and larger complexity classes [IV12; NW19], culminating in the breakthrough proof that $\mathsf{MIP}^* = \mathsf{RE}$ [JNV+21], the class of recursively enumerable problems[4]. Figure 4.1 summarises known relationships between the complexity classes we have discussed (The results we have not explicitly stated come from [Slo19b; Slo19a; KRR14; CS19; MNY20; RUV13].

A more detailed explanation of non-local games is available in recent lecture notes by Vern Paulsen [Pau22]. Those interested in learning more about the connections with interactive proof systems should find the recent course by Thomas Vidick interesting [Vid19].

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[4] This result is not only significant in computer science. It also solves Tsirelson's problem [SW08; Tsi06] and provides a negative answer to Conne's Embedding Conjecture, which was an important open question in the study of operator algebras.

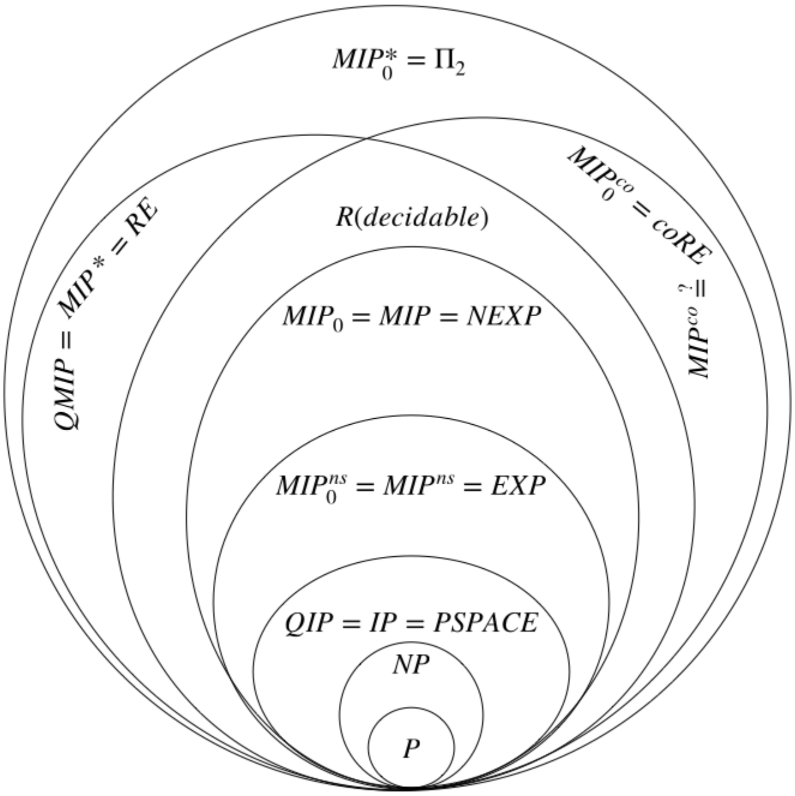

Figure 4.1: Interactive proof complexity classes

5

Quantum CSP and quantum isomorphism

Over the past several decades it has become apparent that many well-known CSPs can be reformulated in terms of the existence of perfect classical strategies for suitable non-local games. As a motivating example let us consider the k-*colouring* problem.

> k-colouring: Given an input graph G decide if there is an assignment of k colours to the vertices of G in such a way that adjacent vertices are assigned different colours.

This problem is equivalent to asking if there exists a graph homomorphism $G \to K_k$ where K_k is the complete graph on k vertices. Thus, the k-colouring problem is a CSP. Now consider the following non-local game:

Example 5.1. *Given a graph $G = (V, E)$, the k-colouring game for G is played as follows:*

- *Verifier sends a vertex $x_a \in V$ to Alice and a vertex $x_b \in V$ to Bob.*
- *Alice responds with a colour $y_a \in [k]$ and Bob responds with a colour $y_b \in [k]$.*
- *Alice and Bob win the game if:*
  1. $x_a = x_b \Rightarrow y_a = y_b$.
  2. $x_a \sim x_b \Rightarrow y_a \neq y_b$.

It is known that classical perfect strategies in this game correspond to k-colourings. The

k-colouring problem can thus be rephrased as follows.

k-colouring: Given an input graph G decide if there exist a perfect classical strategy in the k-colouring game for G.

A natural question now arises. Are there situations where the k-colouring game can be won with a perfect $*$-strategy even though there exists no perfect c-strategy? This leads us to formulate the following relaxation of k-colouring.

tensor k-colouring: Given an input graph G decide if there exist a perfect $*$-strategy in the k-colouring game for G.

Similar relaxations can also be defined in terms of commuting and non-signalling strategies. These have been studied fairly extensively in the literature on non-local games (see e.g. [GW02; CHTW04; AHKS06; CMN+07; MSS13; PSS+16]). By now it is well-established that suitable instances of the k-colouring game exhibit pseudo-telepathy.

In [MR16b] Mančinska and Roberson took the important step of generalising the k-colouring game to the following graph homomorphism game.

Example 5.2. *Given graphs $G = (V_g, E_g)$ and $H = (V_h, E_h)$, the (G, H)-homomorphism game is played as follows:*

- *Verifier sends a vertex $x_a \in V_g$ to Alice and a vertex $x_b \in V_g$ to Bob.*
- *Alice responds with a vertex $y_a \in V_h$ and Bob responds with a vertex $y_b \in V_h$.*
- *Alice and Bob win the game if:*
    1. $x_a = x_b \Rightarrow y_a = y_b$.
    2. $x_a \sim x_b \Rightarrow y_a \sim y_b$.

Mančinska and Roberson used the above game to systematically study quantum variants of standard graph parameters such as the clique number, independence number, and odd girth of a graph. While their analysis focused on the tensor product framework, these results were later extended to the commuting operator framework in [OP16]. Many papers have since expanded upon these results, leading to the emergence of a burgeoning interdisciplinary area of research known as quantum graph theory.

As graphs are just a particular instance of a relational structure, the graph homomorphism game can be generalised even further to a CSP game. Such a game was introduced by Abramsky et al. in [ABdSZ17]. A remarkable result of this paper characterises perfect tensor strategies for the CSP game in terms of Kleisli maps for a graded monad on the category of relational structures, which they refer to as the quantum monad.

In this chapter, we solidify and extend the results of [ABdSZ17] by answering several open questions mentioned in their work. Our first contribution is a comparison of two definitions of quantum graph homomorphisms provided in [MR16b] and [ABdSZ17]. We identify a pair of graphs for which there exists a quantum homomorphism in the sense of [MR16b] but not in the sense of [ABdSZ17]. This shows that the two definitions are different. Our second contribution is to extend the results [ABdSZ17] which only holds in the tensor product framework to the commuting operator framework. Finally, we study a non-local structure isomorphism game which generalises the well-studied graph isomorphism game introduced in [AMR+19]. In this direction, we show how the construction of the quantum monad can be refined to provide categorical semantics for tensor and commuting strategies in this game. This results in a category where morphisms coincide with quantum homomorphisms and isomorphisms coincide with quantum isomorphisms. Along the way, we shall also highlight a connection between the quantum monad and distribution minion monads.

5.1 CSP game

Let us begin by describing the CSP game[1] introduced in [ABdSZ17]. For simplicity, we focus throughout this chapter on the case where our signature σ consists of a single relation R of arity k.

Example 5.3. *Consider relational structures $\mathcal{X}, \mathcal{Y}$. The $(\mathcal{X}, \mathcal{Y})$-*homomorphism *game is played as follows:*

1. *Verifier sends Alice a tuple $\boldsymbol{x} \in X^k$, and Bob an element $x \in X$.*
2. *Alice returns a tuple $\boldsymbol{y} \in Y^k$, and Bob returns an element $y \in Y$.*
3. *Alice and Bob win the game if:*
    - *(a) $\boldsymbol{x} \in R^{\mathcal{X}} \Rightarrow \boldsymbol{y} \in R^{\mathcal{Y}}$.*
    - *(b) $\forall i \in [k] : x = \boldsymbol{x}_i \Rightarrow y = \boldsymbol{y}_i$.*

For $t \in \{c, *, co, ns\}$ we write $\mathcal{X} \xrightarrow{t} \mathcal{Y}$ and say that there exists a t-*homomorphism* between $\mathcal{X}$ and $\mathcal{Y}$ whenever Alice and Bob have a perfect t-strategy in the $(\mathcal{X}, \mathcal{Y})$-homomorphism game. A simple observation is that perfect classical strategies correspond to homomorphisms.

Proposition 5.4 ([ABdSZ17])**.** $\mathcal{X} \xrightarrow{c} \mathcal{Y} \iff \mathcal{X} \to \mathcal{Y}$.

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[1] Referred to as the relational structure homomorphism game in [ABdSZ17]

5.1.1 Tensor strategies

We now recall the characterisation of $*$-strategies for the CSP game that was derived in [ABdSZ17]. Our first observation is that if a perfect $*$-strategy for the CSP game exists, then there must exist a (possibly different) perfect $*$-strategy which exhibits some very strong properties.

Theorem 5.5. *([ABdSZ17][Theorem 5]) The existence of a perfect $*$-strategy in the $(\mathcal{X}, \mathcal{Y})$-CSP game implies the existence of a perfect $*$-strategy $(\psi', \{E_{\boldsymbol{x}}\}_{\boldsymbol{x} \in R^{\mathcal{X}}}, \{F_x\}_{x \in \mathcal{X}})$ which satisfies:*

- *The POVMs $E^i_{\boldsymbol{x}}$ and F_x are projective where we define $E^i_{\boldsymbol{x},y} := \sum_{\boldsymbol{y}_i = y} E_{\boldsymbol{x},\boldsymbol{y}}$.*
- *The state ψ is a maximally entangled state $\psi = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} e_i \otimes e_i$.*
- *$x = \boldsymbol{x_i} \Rightarrow E^i_{\boldsymbol{x},y} = F^T_{x,y}$.*
- *If $\boldsymbol{x} \in R^{\mathcal{X}}$ and $\boldsymbol{y} \notin R^{\mathcal{Y}}$ then $E_{\boldsymbol{x},\boldsymbol{y}=0}$.*

The authors then note that this theorem shows that all the information determining the strategies is contained in Alice's operators E. Hence, they define projectors $P_{x,y} := E^i_{\boldsymbol{x},y}$ whenever $x = \boldsymbol{x}_i$. These projectors are well-defined since we have $E^i_{\boldsymbol{x},y} = P_{x,y} = E^j_{\boldsymbol{x'},y}$ whenever $x = \boldsymbol{x}_i = \boldsymbol{x}'_j$. Moreover, for each $\boldsymbol{x} \in R^{\mathcal{X}}$ we have PVMs $\{P_{\boldsymbol{x}_i}\}$ which are jointly measurable by the PVM $E_{\boldsymbol{x}}$. Thus, $E_{\boldsymbol{x}}$ is equal to the PVM $P_{\boldsymbol{x}}$ given by:

$$P_{\boldsymbol{x},\boldsymbol{y}} = P_{\boldsymbol{x}_1,\boldsymbol{y}_1}, \dots, P_{\boldsymbol{x}_k,\boldsymbol{y}_k}.$$

Based on the above observations the authors define the notion of a $*$-homomorphism[2] as follows:

Definition 5.6. Let $\boldsymbol{H}_d$ be a finite-dimensional Hilbert space of dimension d. A $*$-homomorphism from $\mathcal{X} \in \mathbf{R}(\sigma)$ to $\mathcal{Y} \in \mathbf{R}(\sigma)$ in $\boldsymbol{H}_d$ is given by a family of projectors $\{P_{x,y}\}_{x \in X, y \in Y}$ in $\mathsf{Proj}(\boldsymbol{H}_d)$ satisfying:

1. For all $x \in X : \sum_{y \in Y} P_{x,y} = I$.
2. For all x, x' adjacent in the Gaifman graph of $\mathcal{X}$ and all $y, y' \in Y$ we have $P_{x,y} \odot P_{x'y'}$.
3. If $\boldsymbol{x} \in R^{\mathcal{X}}$ and $\boldsymbol{y} \notin R^{\mathcal{Y}}$ $P_{\boldsymbol{x},\boldsymbol{y}} = 0$.

This definition is justified by the following theorem:

---

[2] This should not be confused with the well-established notion of $*$-homomorphism between two C^*-algebras.

Theorem 5.7 ([ABdSZ17], Theorem 7)**.** *The following are equivalent:*

1. $\mathcal{X} \xrightarrow{*} \mathcal{Y}$
2. *There exists a $*$-homomorphisms from $\mathcal{X}$ to $\mathcal{Y}$.*

5.1.2 Comparing two definitions of tensor homomorphism for graphs

In this section, we restrict our attention to the homomorphism game on graphs. As we have seen already there are two alternative ways of defining the game. The first was described in example 5.2. We refer to this as the MR graph homomorphism game. The second is the restriction of the CSP game from example 5.3 to graphs. This will simply be referred to as the graph homomorphisms game. Each of these games gives rise to a notion of quantum graph homomorphism which we now define.

Definition 5.8. An MR quantum graph homomorphism over a finite-dimensional Hilbert space $\boldsymbol{H}$ from a graph G to a graph H is given by a family of projectors $\{P_{g,h}\}_{g\in V(g),h\in V(h)}$ satisfying:

1. For all $g \in V(g) : \sum_{h\in V(h)} P_{g,h} = I$.
2. For all $g, g' \in V(g)$ and $h, h' \in V(h)$ satisfying $g \sim g'$ and $h \not\sim h'$: $P_{g,h}.P_{g',h'} = 0$.

Definition 5.9. A quantum graph homomorphism is given by a family of projectors P satisfying conditions (1) and (2) above, as well as the following additional condition:

3. For all $g, g' \in V(g)$ and $h, h' \in V(h)$ satisfying $g \sim g'$: $P_{g,h} \odot P_{g',h'}$.

Every quantum homomorphism is an MR quantum homomorphism. We now set out to answer the following question from [ABdSZ17].

Question 5.10. *Does the existence of an MR quantum homomorphism from G to H imply the existence of a quantum homomorphism from G to H?*

We answer this question in the negative by explicitly constructing graphs G and H such that an MR quantum homomorphism from G to H exists but where a quantum homomorphism does not exist. Our separation will be achieved by studying the k-colouring game which is a special case of the graph homomorphism game. Restricting the above definitions to this specific case we have:

Definition 5.11. An MR quantum k-colouring of a graph G is given by a family of projectors $\{v_i\}_{v\in V(G),i\in[k]}$ satisfying:

1. $\forall v \in V(G), \sum_{i\in[k]} v_i = I$.
2. $v_i w_i = 0$ for all $v \sim w$ and all $i \in [k]$.

Definition 5.12. A quantum k-colouring of a graph G is given by a family of projectors $\{v_i\}_{v\in V(G), i\in[k]}$ satisfying conditions 1 and 2 from the previous definition and the additional condition

3. $v_i \odot w_j$ for all $v \sim w$ and all $i, j \in [k]$.

We now consider the case where $G = G_{14}$ is the 14 vertex graph from [MR16a]. We shall describe the construction of G_{14} in more detail shortly. Let us first note however that the following results were already shown in [MR16a].

Proposition 5.13. *([MR16a]) There exists no classical 4-colouring of G_{14}.*

Theorem 5.14. *([MR16a]) There exists an MR quantum 4-colouring of G_{14}.*

Thus, this graph provides an example of quantum advantage in the MR graph colouring game. In fact, in [MR16a] they conjecture that G_{14} is the smallest graph with this property. A recent preprint claims a computational proof of this conjecture [Lal23]. Our goal in this section is to prove the following theorem, which separates MR quantum homomorphisms from quantum homomorphisms.

Theorem 5.15. *There exists no quantum 4-colouring of G_{14}.*

Let us now describe the construction of G_{14} given in [MR16a]. Consider the following family of vectors in 3 dimensions:

$$\begin{aligned} V := &\left\{ A = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right), B = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right), C = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) \right\} \\ &\cup \left\{ Q = \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right), R = \left(\begin{smallmatrix}1\\-1\\0\end{smallmatrix}\right), N = \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right), P = \left(\begin{smallmatrix}1\\0\\-1\end{smallmatrix}\right), L = \left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right), M = \left(\begin{smallmatrix}0\\1\\-1\end{smallmatrix}\right) \right\} \\ &\cup \left\{ W = \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right), Y = \left(\begin{smallmatrix}1\\1\\-1\end{smallmatrix}\right), X = \left(\begin{smallmatrix}1\\-1\\1\end{smallmatrix}\right), Z = \left(\begin{smallmatrix}-1\\1\\1\end{smallmatrix}\right) \right\}. \end{aligned}$$

We define a graph G_{13} from this set by assigning a node to every vertex and an edge between any two nodes which represent orthogonal vectors. This graph is drawn in figure 5.1.

The graph G_{14} is obtained by adding an apex node to G_{13} which we denote by Ω. The following fact about the symmetries of G_{13} was noted in [MR16a].

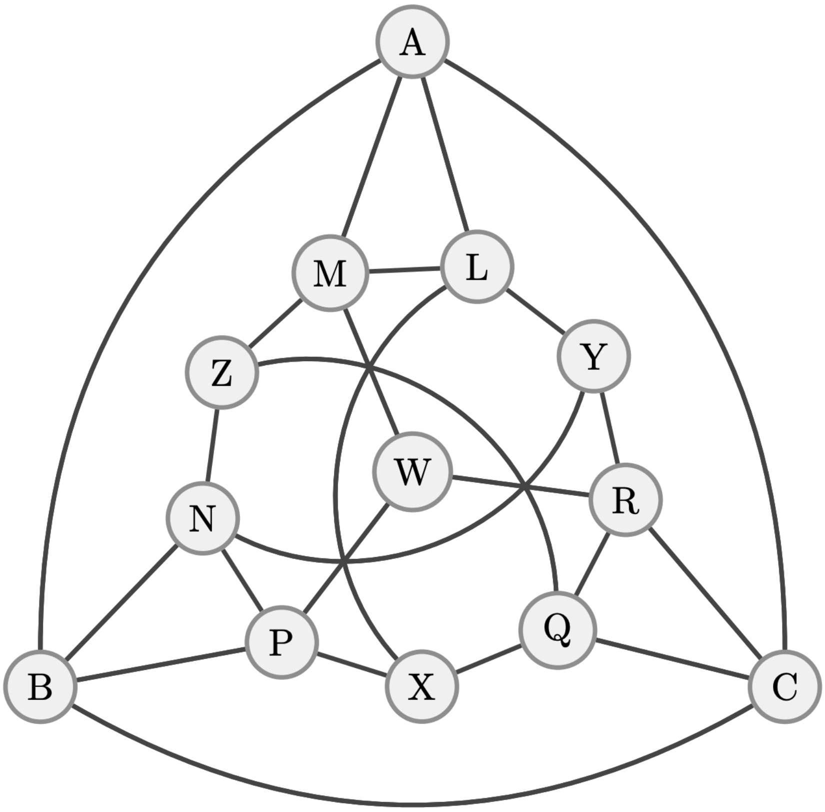

Figure 5.1: The graph G_{13} (Taken from [MR16a].).

Fact 5.16. *Any ordered pair of distinct vertices of* $\{W, X, Y, Z\}$ *can be mapped to any other such pair by an automorphism of* G_{13}.

We note that the same fact holds for G_{14} since is obtained by simply adding an apex vector to G_{13}.

Fact 5.17. *Any ordered pair of distinct vertices of* $\{W, X, Y, Z\}$ *can be mapped to any other such pair by an automorphism of* G_{14}.

We now proceed to prove our main theorem. While the proof may appear somewhat involved it is worth noting that it follows a very similar structure to the argument used in [MR16a] to show that the graph G_{13} is not MR quantum 3-colourable. We also adopt a useful notation from this paper to aid the readability of our proofs. Namely, throughout the proofs we may write $[P_1 P_2 \dots P_n]$ whenever the product of several projectors is guaranteed to equal zero. We now state several lemmas.

Lemma 5.18. *Let* G *be a graph and suppose* S *is a clique of size* c *in* G. *Then in any quantum* c*-colouring of* G

$$\forall i \in [c] : \sum_{v \in S} v_i = I.$$

Proof. This was shown for the case of MR quantum c-colourings in [MR16a]. As every quantum c-colouring is an MR quantum c-colouring their proof can be reused. $\square$

The remainder of our lemmas are specific to G_{14}.

Lemma 5.19. *Let u and v be vertices of G_{14} other than Ω. Take $i, j, k \in [4]$ with $i \neq j \neq k$. In any quantum 4-colouring of G_{14} we have*

$$u_i v_j u_k = 0$$

Proof. First note that any choice of u, v will share some common neighbour which is not Ω. Let w be this common neighbour. Take $l \in [4]$ which is not equal to i, j, k. Using this and the fact that Ω is adjacent to every other node of G_{14} we obtain

$$\begin{aligned}
u_i v_j u_k &= u_i v_j u_k(\Omega_i + \Omega_j + \Omega_k + \Omega_l) \\
&= [u_i \Omega_i] v_j u_k + u_i [v_j \Omega_j] u_k + u_i v_j [u_k \Omega_k] + u_i v_j u_k \Omega_l \\
&= u_i v_j u_k \Omega_l = u_i v_j u_k \Omega_l (w_i + w_j + w_k + w_l) \\
&= [u_i w_i] v_j u_k \Omega_l + u_i [v_j w_j] u_k \Omega_l + u_i v_j [u_k w_k] \Omega_l + u_i v_j u_k [\Omega_l w_l] = 0
\end{aligned}$$

$\square$

Lemma 5.20. *In any quantum 4-colouring of G_{14} we have*

$$X_i A_j W_i = X_i C_j W_i = 0 \text{ for } i \neq j \text{ and}$$

$$\forall i \in [4] : X_i W_i = X_i A_i W_i = X_i C_i W_i.$$

Proof. A, L, M, Ω form a clique of size 4. By lemma 5.18 we have $(A_i + L_i + M_i + \Omega_i) = I$. Since $X \sim L$ and $M \sim W$ we have

$$\begin{aligned}
X_i A_j W_i &= X_i A_j (A_i + L_i + M_i + \Omega_i) W_i \\
&= X_i [A_j A_i] W_i + [X_i L_i] A_j W_i + X_i A_j [M_i W_i] + X_i A_j [\Omega_i W_i] = 0
\end{aligned}$$

Note that we have liberally made use of condition 3 in the definition of quantum k-colourings. The same argument using the clique C, Q, R, Ω shows that $X_iC_jW_i = 0$. Now making use of these facts we have

$$X_iW_i = X_i(A_1 + A_2 + A_3 + A_4)W_i = X_iA_iW_i$$

The proof of $X_iW_i = X_iC_iW_i$ is similar. □

Lemma 5.21. *Take any pair of vertices u, v of G_{14} belonging to the subset $\{W, X, Y, Z\}$. For any $i \in [4]$ we have*

$$u_iv_i = 0.$$

Proof. We prove that $X_1W_1 = 0$. By symmetry of the colours and the symmetries of the graph (as described in fact 5.17) this suffices to prove the entire statement.

By using $B_1 + B_2 + B_3 + B_4 = I$ we have

$$X_1W_1 = B_1X_1W_1 + B_2X_1W_1 + B_3X_1W_1 + B_4X_1W_1$$

We proceed to show that each summand is equal to zero. Let us start by considering $B_2X_1W_1$. Since A, B, C, Ω form a clique we have $A_3 + B_3 + C_3 + \Omega_3 = I$ from lemma 5.18. Using this we obtain

$$\begin{aligned} B_2X_1W_1 &= B_2X_1(A_3 + B_3 + C_3 + \Omega_3)W_1 \\ &= B_2[X_1A_3W_1] + [B_2X_1B_3]W_1 + B_2[X_1C_3W_1] + B_2X_1\Omega_3W_1 \end{aligned}$$

By lemmas 5.19 and 5.20 the first three summands above are zero. Hence we have

$$B_2X_1W_1 = B_2X_1\Omega_3W_1$$

Now we use lemma 5.18 again and obtain $A_4 + B_4 + C_4 + \Omega_4 = I$. Then we get

$$B_2X_1W_1 = B_2X_1(A_4 + B_4 + C_4 + \Omega_4)\Omega_3W_1$$

$$= B_2\Omega_3[X_1A_4W_1] + [B_2X_1B_4]\Omega_3W_1 + B2\Omega_3[X_1C_4W_1] + B_2X_1[\Omega_4\Omega_3]W_1 = 0.$$

Note that we have used condition 3 in the definition of quantum k-colourings to rearrange commuting terms in the summands. Again by lemmas 5.19 and 5.20 all summands are 0. Thus we have shown that $B_2X_1W_1 = 0$. Via very similar arguments we can show the following:

$$B_2X_1W_1 = B_3X_1W_1 = B_4X_1W_1 = X_1W_1B_2 = X_1W_1B_3 = X_1W_1B_4 = 0.$$

Thus it remains to show that the first term of the sum is also zero. We have

$$\begin{aligned} B_1X_1W_1 &= B_1X_1W_1(B_1 + B_2 + B_3 + B_4) \\ &= B_1X_1W_1B_1 + B_1[X_1W_1B_2] + B_1[X_1W_1B_3] + B_1[X_1W_1B_4] = B_1X_1W_1B_1. \end{aligned}$$

Hence it suffices to show that $B_1X_1W_1B_1 = 0$. Next we use lemma 5.20 to show

$$\begin{aligned} B_1X_1W_1B_1 &= B_1X_1C_1W_1B_1 = B_1(I - X_2 - X_3 - X_4)C_1(I - W_2 - W_3 - W_4)B_1 \\ &= [B_1C_1]B_1 - B_1X_2[C_1B_1] - B_1X_3[C_1B_1] - B_1X_4[C_1B_1] \\ &- [B_1C_1]W_2B_1 - [B_1C_1]W_3B_1 - [B_1C_1]W_4B_1 \\ &+ B_1[X_2C_1W_2]B_1 + B_1X_2C_1W_3B_1 + B_1X_2C_1W_4B_1 \\ &+ B_1X_3C_1W_2B_1 + B_1[X_3C_1W_3]B_1 + B_1X_3C_1W_4B_1 \\ &+ B_1X_4C_1W_2B_1 + B_1X_4C_1W_3B_1 + B_1[X_4C_1W_4]B_1 \\ &= B_1X_2C_1W_3B_1 + B_1X_2C_1W_4B_1 + B_1X_3C_1W_2B_1 \\ &+ B_1X_3C_1W_4 + B_1X_4C_1W_2B_1 + B_1X_4C_1W_3B_1 \end{aligned}$$

It remains to show that these six summands are all zero. We prove this for the first term, the other five proofs are similar. Using lemma 5.18 and condition 2 in the definition of quantum k-colourings we have

$$\begin{aligned} C_1 &= I - A_1 - B_1 - \Omega_1 = (I - A_1)(I - B_1)(I - C_1) \\ &= (A_2 + A_3 + A_4)(B_2 + B_3 + B_4)(\Omega_2 + \Omega_3 + \Omega_4) \end{aligned}$$

$$= A_2B_3\Omega_4 + A_2B_4\Omega_3 + A_3B_2\Omega_4 + A_3B_4\Omega_2 + A_4B_2\Omega_3 + A_4B_3\Omega_2$$

Next, we use condition 3 in the definition of quantum k-colourings to obtain

$$\begin{aligned}
&B_1X_2C_1W_3B_1\\
&= B_1X_2(A_2B_3\Omega_4 + A_2B_4\Omega_3 + A_3B_2\Omega_4 + A_3B_4\Omega_2 + A_4B_2\Omega_3 + A_4B_3\Omega_2)W_3B_1\\
&= [B_1X_2B_3]A_2\Omega_4W_3B_1 + [B_1X_2B_4]A_2\Omega_3W_3B_1\\
&+ B_1X_2A_3\Omega_4[B_2W_3B_1] + [B_1X_2B_4]A_3\Omega_2W_3B_1\\
&+ B_1X_2A_4\Omega_3[B_2W_3B_1] + [B_1X_2B_3]A_4\Omega_2W_3B_1 = 0
\end{aligned}$$

This shows that $X_1W_1 = 0$ and concludes the proof.

□

We are now finally ready to prove our main theorem.

Proof of theorem 5.15. Assume a 4 colouring exists. By lemma 5.21 and the fact that Ω is adjacent to every vertex of the graph we have that for all $i \in [4]$ the operators $W_i, X_i, Y_i, Z_i, \Omega_i$ are all mutually orthogonal. From the properties of projectors, this implies $W_i + X_i + Y_i + Z_i + \Omega_i \leq I$. Using this we derive the contradiction

$$5I = \sum_{v\in\{W,X,Y,Z,\Omega\}} \sum_{i\in[4]} v_i = \sum_{i\in[4]} \sum_{v\in\{W,X,Y,Z,\Omega\}} v_i \leq 4I.$$

□

Remark 5.22. *In light of this result, one may ask if there exists an alternative definition of $*$-homomorphisms between relational structures which, when restricted to graphs coincides exactly with the definition of [MR16b]. Indeed, we have already observed that if we simply drop the additional third condition in the definition of $*$-homomorphisms then the two notions do coincide. The trouble with doing so however is that such $*$-homomorphisms would no longer correspond to perfect quantum strategies in the CSP game.*

We conjecture that it is possible to define an alternative k-player version of the non-local CSP game whose perfect quantum strategies do coincide with this alternative definition of $*$-homomorphism. Let us briefly mention how this game would work.

Example 5.23. *Let σ be a relational signature where the maximum arity of any relation is k. Consider relational structures $\mathcal{X}, \mathcal{Y}$. The k-player $(\mathcal{X}, \mathcal{Y})$-*CSP *game is played as follows:*

1. *Verifier sends each player an element $x_i \in X$.*
2. *Each player responds with $y_i \in Y$.*
3. *The players win if:*
   (a) $\boldsymbol{x} \in R^{\mathcal{X}} \Rightarrow \boldsymbol{y} \in R^{\mathcal{Y}}$.
   (b) $\forall i, j \in [k]\colon x_i = x_j \Rightarrow y_i = y_j$.

Note that it is crucial to have at least k players in this game for classical strategies to coincide with classical structure homomorphisms. We leave it as future work to prove this conjecture. It is worth mentioning that such a proof may be quite difficult since most of the sophisticated machinery developed for analysing non-local games only applies in the two-player case.

5.1.3 Commuting operator strategies

We now aim to analyse commuting operator strategies in the CSP game. For the case of the MR graph homomorphism game, such strategies have been explored in [PT15; PSS+16; OP16]. The following result is known:

Theorem 5.24. *Let G and H be graphs. Then $G \overset{co}{\to} H$ iff there exists a unital C^*-algebra $\mathbf{A}$ which admits a faithful tracial state, and projections $P_{g,h} \in A$ for $g \in V(g)$ and $h \in V(h)$ such that:*

1. $\sum_{h \in V(h)} P_{g,h} = I$.
2. $P_{g,h} P_{g',h'} = 0$ *if* $g \sim g', h \not\sim h'$.

To derive these results the authors rely heavily on the fact that the MR graph homomorphism game is synchronous. We wish to derive an analogue of the above result for the CSP game. Unfortunately, this game is not synchronous so we cannot immediately make use of lemma 4.5. Our first step is thus to define a synchronous version of the CSP game and show that its perfect strategies coincide with the original game of [ABdSZ17].

Example 5.25. *Consider relational structures $\mathcal{X}, \mathcal{Y}$. The* synchronous $(\mathcal{X}, \mathcal{Y})$-homomorphism *game is played as follows:*

1. *Verifier sends Alice and bob tuples $\boldsymbol{x^a}, \boldsymbol{x^b} \in X^k$ respectively.*
2. *Alice and Bob return tuples $\boldsymbol{y^a}, \boldsymbol{y^b} \in Y^k$ respectively.*

3. *Alice and Bob win the game if:*
   - $\boldsymbol{x}^{\boldsymbol{a}}_i = \boldsymbol{x}^{\boldsymbol{b}}_j \Rightarrow \boldsymbol{y}^{\boldsymbol{a}}_i = \boldsymbol{y}^{\boldsymbol{b}}_j$.
   - $\boldsymbol{x}^{\boldsymbol{a}} \in R^{\mathcal{X}} \Rightarrow \boldsymbol{y}^{\boldsymbol{a}} \in R^{\mathcal{Y}}$ *and* $\boldsymbol{x}^{\boldsymbol{b}} \in R^{\mathcal{X}} \Rightarrow \boldsymbol{y}^{\boldsymbol{b}} \in R^{\mathcal{Y}}$.

Theorem 5.26. *For $t \in \{c, *, co\}$ there exists a perfect t-strategy in the $(\mathcal{X}, \mathcal{Y})$-CSP game iff there exists a perfect t-strategy in the synchronous $(\mathcal{X}, \mathcal{Y})$-CSP game.*

Proof.

- **Classical case**: ($\rightarrow$) We know via proposition 5.4 that there must be a classical homomorphism $f : \mathcal{X} \rightarrow \mathcal{Y}$. Define Alice and Bob's deterministic functions as $f^a(\boldsymbol{x}) = f^b(\boldsymbol{x}) = (f(x_1), \dots, f(x_k))$. This satisfies the winning conditions because f is a function and because it preserves relations.

  ($\leftarrow$): We first construct a function $f : X \rightarrow Y$. Recall that a deterministic classical strategy is given by a pair of functions $f^a, f^b : X^k \rightarrow Y^k$. Because of the synchronous nature of the game, we must have $f^a = f^b$. Now consider the case where Alice and Bob both receive the same input $\boldsymbol{x}^{\boldsymbol{a}} = \boldsymbol{x}^{\boldsymbol{b}} = (x, \dots, x) \in X^k$. To satisfy the winning condition $x^a_i = x^b_j \Rightarrow y^a_i = y^b_j$ they must respond with tuples $f^a(\boldsymbol{x}^{\boldsymbol{a}}) = f^b(\boldsymbol{x}^{\boldsymbol{b}}) = (y, \dots, y) \in Y^k$. Therefore we define our function by setting $f(x) = y$.

  It remains to verify that this function preserves relations. We first show that for $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k$ we have $f^a(\boldsymbol{x}) = (f(x_1), \dots, f(x_k))$. To see this consider the case where we give $\boldsymbol{x}^{\boldsymbol{a}} = (x_1, \dots, x_k)$ and $\boldsymbol{x}^{\boldsymbol{b}} = (x_i, \dots, x_i) \in X^k$ as input to Alice and Bob respectively. Since we have $x^a_i = x^b_i = x_i$ we must have $y_i = y^a_i = y^b_i = f(x_i)$. Now take $\boldsymbol{x} \in R^X$. The winning condition of the game implies that $f^a(\boldsymbol{x}) = (f(x_1), \dots, f(x_k)) \in R^Y$ as required.

- **Tensor case:** ($\rightarrow$) Assume we have a $*$-homomorphism. We define a strategy $(\psi, \{E_{\boldsymbol{x}}\}_{\boldsymbol{x} \in R^{\mathcal{X}}}, \{F_{\boldsymbol{x}}\}_{\boldsymbol{x} \in R^{\mathcal{X}}})$ in the synchronous game by taking ψ to be the maximally entangled state and setting $E_{\boldsymbol{x},\boldsymbol{y}} = F^T_{\boldsymbol{x},\boldsymbol{y}} = P_{x_1,y_1} \cdots P_{x_k,y_k}$. Then lemma 4.4 shows that $p(\boldsymbol{y}^{\boldsymbol{a}}, \boldsymbol{y}^{\boldsymbol{b}} | \boldsymbol{x}^{\boldsymbol{a}}, \boldsymbol{x}^{\boldsymbol{b}}) = 0 \iff E_{\boldsymbol{x}^{\boldsymbol{a}},\boldsymbol{y}^{\boldsymbol{a}}} E_{\boldsymbol{x}^{\boldsymbol{b}},\boldsymbol{y}^{\boldsymbol{b}}} = 0$. Now assume that the first winning condition fails meaning we have $i, j \in [k]$ such that $x^a_i = x^b_j$ and $y^a_i \neq y^b_j$. We have

  $$E_{\boldsymbol{x}^{\boldsymbol{a}},\boldsymbol{y}^{\boldsymbol{a}}} E_{\boldsymbol{x}^{\boldsymbol{b}},\boldsymbol{y}^{\boldsymbol{b}}} = P_{x^a_1,y^a_1}, \dots P_{x^a_k,y^a_k}.P_{x^b_1,y^b_1}, \dots P_{x^b_k,y^b_k}.$$

  Using commutativity we can place the projectors $P_{x^a_i,y^a_i}$ and $P_{x^b_j,y^b_j}$ next to each other in the above product. As $x^a_i = x^b_j$ these are different projectors belonging to the same PVM so $P_{x^a_i,y^a_i} P_{x^b_j,y^b_j} = 0$. Hence $E_{\boldsymbol{x}^{\boldsymbol{a}},\boldsymbol{y}^{\boldsymbol{a}}} E_{\boldsymbol{x}^{\boldsymbol{b}},\boldsymbol{y}^{\boldsymbol{b}}} = 0$.

  Finally, assume the second winning condition fails, then from condition 3 in the definition of $*$-homomorphisms we have either:

- $\boldsymbol{x^a} \in R^{\mathcal{X}}$ and $\boldsymbol{y^a} \notin R^{\mathcal{Y}}$. In this case $E_{\boldsymbol{x^a},\boldsymbol{y^a}} E_{\boldsymbol{x^b},\boldsymbol{y^b}} = 0.E_{\boldsymbol{x^b},\boldsymbol{y^b}} = 0$.
- $\boldsymbol{x^b} \in R^{\mathcal{X}}$ and $\boldsymbol{y^b} \notin R^{\mathcal{Y}}$. In this case $E_{\boldsymbol{x^a},\boldsymbol{y^a}} E_{\boldsymbol{x^b},\boldsymbol{y^b}} = E_{\boldsymbol{x^a},\boldsymbol{y^a}}.0 = 0$.

($\leftarrow$): Assume a perfect strategy $(\psi, \{E_{\boldsymbol{x}}\}_{\boldsymbol{x} \in R^{\mathcal{X}}}, \{F_{\boldsymbol{x}}\}_{\boldsymbol{x} \in R^{\mathcal{X}}})$ in the form of lemma 4.4 exists in the synchronous game. As before we have a family of projectors $\{P_{x,y}\}_{x \in X, y \in Y}$ where $P_{x,y} = E^i_{\boldsymbol{x},y}$ whenever $x = \boldsymbol{x}_i$. We claim that this is a $*$-homomorphism. To begin with, we must show that these projectors are well-defined by proving that $E^i_{\boldsymbol{x},y} = E^j_{\boldsymbol{x'},y}$ whenever $x_i = x'_j$. For this we note that when $x_i = x'_j$ and $y \neq y'$ we have $E^i_{\boldsymbol{x},y} E^j_{\boldsymbol{x'},y'} = 0$:

$$E^i_{\boldsymbol{x},y} E^j_{\boldsymbol{x'},y'} = \Big(\sum_{y_i = y} E_{\boldsymbol{x},\boldsymbol{y}}\Big)\Big(\sum_{y'_j = y'} E_{\boldsymbol{x'},\boldsymbol{y'}}\Big) = \sum_{y_i = y} \sum_{y'_j = y'} E_{\boldsymbol{x},\boldsymbol{y}} E_{\boldsymbol{x'},\boldsymbol{y'}} = 0,$$

where the last equality follows from the failure of the winning condition $x_i = x'_j \Rightarrow y_i = y'_j$. Now assume we have $\boldsymbol{x}, \boldsymbol{x'}$ such that $x_i = x'_j$, we get:

$$E^i_{\boldsymbol{x},y} = E^i_{\boldsymbol{x},y}.\Big(\sum_{y'} E^j_{\boldsymbol{x'},y'}\Big) = \sum_{y'} E^i_{\boldsymbol{x},y}.E^j_{\boldsymbol{x'},y'} = E^i_{\boldsymbol{x},y}.E^j_{\boldsymbol{x'},y} = \sum_{y'} E^i_{\boldsymbol{x},y'}.E^j_{\boldsymbol{x'},y} = E^j_{\boldsymbol{x'},y}.$$

Thus, we again have for $\boldsymbol{x} \in R^{\mathcal{X}}$ jointly measurable PVMs $\{P_{x_i}\}$. This means that the first two conditions in the definition of $*$-homomorphisms are satisfied. For the final condition assume $\boldsymbol{x} \in R^{\mathcal{X}}$ and $\boldsymbol{y} \notin R^{\mathcal{Y}}$. Then we have

$$P_{\boldsymbol{x},\boldsymbol{y}} = E_{\boldsymbol{x},\boldsymbol{y}} = E_{\boldsymbol{x},\boldsymbol{y}} E_{\boldsymbol{x},\boldsymbol{y}} = 0,$$

where the last equality follows via the failure of the winning condition $\boldsymbol{x^a} \in R^{\mathcal{X}} \Rightarrow \boldsymbol{y^a} \in R^{\mathcal{Y}}$ and lemma 4.4. ☐

Thus, any result that applies to perfect strategies in the synchronous CSP game also applies to the original game.

Remark 5.27. *An import of this result is that the sophisticated machinery that has been developed for dealing with synchronous non-local correlations can also be used to study correlations which have suitable "reductions" to synchronous correlations. It may be possible to formalise this idea more precisely in the language of simulations and resource theories as studied for example in [ABM17; ABKM19; BKM23]. It is also worth noting that our CSP game generalises the well-studied class of binary constraint system (BCS) games. These games were also originally* *stated in a non-synchronous formulation [CM14] but were later shown to have an equivalent synchronous characterisation [KPS18]. Our synchronous game coincides with the synchronous BCS game of [KPS18] when the CSP in question is a BCS.*

We can now prove a commuting operator framework analogue of theorem 5.7.

Definition 5.28. Let $\mathbf{A}$ be a unital C^*-algebra which admits a faithful tracial state $s : \mathbf{A} \to \mathbb{C}$. A co-homomorphism from $\mathcal{X} \in \mathbf{R}(\sigma)$ to $\mathcal{Y} \in \mathbf{R}(\sigma)$ is given by a family of projectors $\{P_{x,y}\}_{x \in X, y \in Y}$ in $\mathbf{A}$ satisfying:

1. For all $x \in X : \sum_{y \in Y} P_{x,y} = I$.
2. For all x, x' adjacent in the Gaifman graph of $\mathcal{X}$ and all $y, y' \in Y$ we have $P_{x,y} \odot P_{x'y'}$.
3. If $\boldsymbol{x} \in R^{\mathcal{X}}$ and $\boldsymbol{y} \notin R^{\mathcal{Y}}$ $P_{\boldsymbol{x},\boldsymbol{y}} = 0$.

Theorem 5.29. *The following are equivalent:*

- $\mathcal{X} \overset{co}{\to} \mathcal{Y}$.
- *There exists a* co*-homomorphism from* $\mathcal{X}$ *to* $\mathcal{Y}$.

Proof. $(\rightarrow)$: We know via lemma 4.4 that there exists a family of projectors $E_{\boldsymbol{x},\boldsymbol{y}} \in \mathbf{A}$ such that $E_{\boldsymbol{x},\boldsymbol{y}} E_{\boldsymbol{x}',\boldsymbol{y}'} = 0$. Define $P_{x,y} = E^i_{\boldsymbol{x},y} = \sum_{\boldsymbol{y}_i = y} E_{\boldsymbol{x},y}$. These are well-defined projectors and satisfy conditions (1)-(3) via exactly the same argument used in the proof of theorem 5.26.

$(\leftarrow)$: Define $E_{\boldsymbol{x},\boldsymbol{y}} = P_{x_1,y_1} \dots P_{x_k,y_k}$. Then for each $x \in \mathcal{X}$ $\{E_{\boldsymbol{x}}\}$ is a well-defined PVM thanks to the joint measurability condition (2) and the normalisation condition (1). Based on lemma 4.5 it suffices to show that $E_{\boldsymbol{x},\boldsymbol{y}} E_{\boldsymbol{x}',\boldsymbol{y}'} = 0$ whenever the winning conditions of the game are not satisfied. Again this follows via exactly the same argument as in theorem 5.26. ☐

5.1.4 Representations of the CSP game

We now mention a theorem of [OP16] which consolidates various results for perfect strategies in the MR graph homomorphism game in the language of C^*-algebras. We first require the definition of a representation for the graph homomorphism game.

Definition 5.30. Let G and H be graphs. A representation of the MR quantum graph homomorphism game over a Hilbert space $\boldsymbol{H}$ is given by a family of projectors $\{P_{g,h}\}_{g \in V(g), h \in V(h)}$ satisfying:

1. For all $g \in V(g) : \sum_{h \in V(h)} P_{g,h} = I$.

2. For all $g, g' \in V(g)$ and $h, h' \in V(h)$ satisfying $g \sim g'$ and $h \not\sim h'$: $P_{g,h} P_{g',h'} = 0$.

In the remainder of this document we refer to such a representation simply as an *MR quantum graph homomorphism* in $\boldsymbol{H}$.

We write $\mathbf{A}(G, H)$ for the universal C^*-algebra generated by such sets of projections. Note that a $\mathbf{A}(G, H)$ may fail to exist if there is no MR quantum homomorphism between G and H in any Hilbert space. The following was shown in [OP16].

Theorem 5.31 ([OP16]). *The following facts are true:*

- $G \overset{co}{\to} H$ *iff there exists a tracial state on* $\mathbf{A}(G, H)$.
- $G \overset{*}{\to} H$ *iff* $\mathbf{A}(G, H)$ *has a finite-dimensional representation.*
- $G \overset{c}{\to} H$ *iff* $\mathbf{A}(G, H)$ *has a one-dimensional representation.*

Based on our discussion thus far in this chapter, we have shown a variant of this theorem for the CSP game.

Definition 5.32. A quantum homomorphism from $\mathcal{X} \in \mathbf{R}(\sigma)$ to $\mathcal{Y} \in \mathbf{R}(\sigma)$ in $\boldsymbol{H}$ is given by a family of projectors $\{P_{x,y}\}_{x \in X, y \in Y}$ in $\mathsf{Proj}(\boldsymbol{H})$ satisfying:

1. For all $x \in X : \sum_{y \in Y} P_{x,y} = I_d$
2. For all x, x' adjacent in the Gaifman graph of $\mathcal{X}$ and all $y, y' \in Y$ we have $P_{x,y} \odot P_{x'y'}$.
3. If $\boldsymbol{x} \in R^{\mathcal{X}}$ and $\boldsymbol{y} \notin R^{\mathcal{Y}}$ $P_{\boldsymbol{x},\boldsymbol{y}} = 0$.

Again we write $\mathbf{A}(\mathcal{X}, \mathcal{Y})$ for the universal C^*-algebra generated by such sets of projections. Note that $\mathbf{A}(\mathcal{X}, \mathcal{Y})$ may fail to exist if there is no quantum homomorphism between $\mathcal{X}$ and $\mathcal{Y}$ in any Hilbert space.

Theorem 5.33. *The following facts are true:*

- $G \overset{co}{\to} H$ *iff there exists a tracial state on* $\mathbf{A}(\mathcal{X}, \mathcal{Y})$.
- $G \overset{*}{\to} H$ *iff* $\mathbf{A}(\mathcal{X}, \mathcal{Y})$ *has a finite-dimensional representation.*
- $G \overset{c}{\to} H$ *iff* $\mathbf{A}(\mathcal{X}, \mathcal{Y})$ *has a one-dimensional representation.*

5.2 Structure isomorphism game

A non-local graph isomorphism game has been introduced in [AMR+19] and studied extensively since (see e.g. [MR20; LMR20; MRV18; MRV19; NRSZ24]). This game can be generalised to arbitrary relational structures. An open question mentioned in [ABdSZ17] is to figure out how this game fits into the quantum monad framework. This is the question which we address in this section. Let us begin by explaining the structure isomorphism game. As with the CSP game, this game can be formulated in both a synchronous and non-synchronous fashion. To simplify some proofs we focus on the synchronous formulation.

Example 5.34. *Consider relational structures $\mathcal{X}, \mathcal{Y}$. The $(\mathcal{X}, \mathcal{Y})$-*isomorphism *game is played as follows:*

1. *Verifier sends Alice and bob tuples $\boldsymbol{c^a}, \boldsymbol{c^b} \in X^k \cup Y^k$ respectively.*
2. *Alice and Bob return tuples $\boldsymbol{d^a}, \boldsymbol{d^b} \in X^k \cup Y^k$ respectively.*
3. *Alice and Bob win the game if:*
   - $\boldsymbol{c^a} \in R^{\mathcal{X}} \iff \boldsymbol{d^a} \in R^{\mathcal{Y}}$ *and* $\boldsymbol{c^b} \in R^{\mathcal{X}} \iff \boldsymbol{d^b} \in R^{\mathcal{Y}}$.

   Assuming this condition is met we define $\boldsymbol{x^a}$ to be the unique tuple in $R^{\mathcal{X}}$ among $\boldsymbol{c^a}$ and $\boldsymbol{d^a}$, and we define $\boldsymbol{y^a}, \boldsymbol{x^b}, \boldsymbol{y^b}$ similarly. To win, the following condition must also be satisfied:
   - $\forall i, j \in [k] : x^a_i = x^b_j \iff y^a_i = y^b_j$.

We write $\mathcal{X} \overset{t}{\cong} \mathcal{Y}$ whenever a perfect t-strategy exists in this game for $t \in \{c, *, co, ns\}$. Notice that unlike in the case of the homomorphism game Alice and Bob do not necessarily receive inputs from the same structure. Moreover, the winning conditions of the game dictate that to win Alice and Bob's responses must be from whichever structure their input was not from.

5.2.1 Classical strategies

As before we can verify that perfect classical strategies are isomorphisms.

Proposition 5.35. $\mathcal{X} \overset{c}{\cong} \mathcal{Y} \iff \mathcal{X} \cong \mathcal{Y}$.

Proof. ($\Rightarrow$): We first construct a function $f : X \to Y$. Recall that a deterministic classical strategy is given by a pair of functions $f^a, f^b : A^k \cup B^k \to A^k \cup B^k$. Because of the synchronous nature of the game, we must have $f^a = f^b$. Now consider the case where Alice and Bob both receive the same input $\boldsymbol{x^a} = \boldsymbol{x^b} = (x, \ldots, x) \in X^k$. To satisfy the winning condition $x^a_i = x^b_j \iff y^a_i = y^b_j$ they must respond with tuples $f^a(\boldsymbol{x^a}) = f^b(\boldsymbol{x^b}) = (y, \ldots, y) \in Y^k$. Therefore we define our function by setting $f(x) = y$. We now claim that this is an isomorphism. To verify this we must show that f is bijective, preserves relations, and reflects relations.

- **bijectivity**: Consider the function $f^{-1} : Y \to X$ defined in the same way as f where we provide $\boldsymbol{y^a} = \boldsymbol{y^b} = (y, \dots, y) \in Y^k$ as inputs to Alice and Bob. We claim that $f^{-1} \circ f(x) = x$. To see this consider what would happen if we gave as input $\boldsymbol{x^a} = (x, \dots, x)$ to Alice and $f^a(\boldsymbol{x^a}) = (y, \dots, y)$ as input to Bob. In this case we have $f(x) = y$ and in order to satisfy $x_i^a = x_j^b \iff y_i^a = y_j^b$ we must have that $f^{-1}(y) = x$. Similar reasoning shows that $f \circ f^{-1}(y) = y$.

- **preservation and reflection**: We first show that for $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k$ we have $f^a(\boldsymbol{x}) = (f(x_1), \dots, f(x_k))$. To see this consider the case where we give $\boldsymbol{x^a} = (x_1, \dots, x_k)$ and $\boldsymbol{x^b} = (x_i, \dots, x_i) \in X^k$ as input to Alice and Bob respectively. Since we have $x_i^a = x_i^b = x_i$ we must have $y_i = y_i^a = y_i^b = f(x_i)$.

  Now take $\boldsymbol{x} \in R^X$. The winning condition of the game implies that $f^a(\boldsymbol{x}) = (f(x_1), \dots, f(x_k)) \in R^Y$ as required. To prove that relations are reflected we simply repeat the same argument noting that for $\boldsymbol{y} = (y_1, \dots, y_k) \in Y^k$ we have $f^a(\boldsymbol{x}) = (f^{-1}(y_1), \dots, f^{-1}(x_k))$.

($\Leftarrow$): Let $f : X \to Y$ be an isomorphism. Consider the strategy given by $f^a(c_1, \dots, c_k) = f^b(c_1, \dots, c_k) = \begin{cases} (f(c_1), \dots, f(c_k)) \text{ if } (c_1, \dots, c_k) \in X^k \\ (f^{-1}(c_1), \dots, f^{-1}(c_k)) \text{ if } (c_1, \dots, c_k) \in Y^k. \end{cases}$. The first winning condition is satisfied via the preservation and reflection of relations. The second winning condition is satisfied because f is a bijection.

□

5.2.2 Tensor strategies

We now turn our attention to $*$-strategies in the isomorphism game. Our goal is to characterise such strategies in a manner analogous to what was done for the homomorphism game in [ABdSZ17]. We begin by showing that strategies can be assumed to have a special form.

Theorem 5.36. *The existence of a perfect $*$-strategy in the $(\mathcal{X}, \mathcal{Y})$-isomorphism game implies the existence of a perfect $*$-strategy $(\psi', \{E_{\boldsymbol{x}}\}_{\boldsymbol{x} \in X^k \cup Y^k}, \{F_{\boldsymbol{x}}\}_{\boldsymbol{x} \in X^k \cup Y^k})$ which satisfies:*

1. *$E^i_{\boldsymbol{x}}$ and $F^i_{\boldsymbol{x}}$ are PVMs where we define $E^i_{\boldsymbol{x},y} := \sum_{\boldsymbol{y}_i = y} E_{\boldsymbol{x},y}$, and similarly for $F^i_{\boldsymbol{x}}$;*
2. *The state ψ is a maximally entangled state $\psi = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} e_i \otimes e_i$;*
3. *$E^i_{\boldsymbol{x},y} = F^{i^T}_{\boldsymbol{x},y}$;*
4. *$E_{\boldsymbol{x},\boldsymbol{y}} = E_{\boldsymbol{y},\boldsymbol{x}}$ for all $\boldsymbol{x}, \boldsymbol{y} \in X^k \cup Y^k$;*
5. *If $(\boldsymbol{x} \in R^{\mathcal{X}} \iff \boldsymbol{y} \in R^{\mathcal{Y}})$ does not hold then $E_{\boldsymbol{x},\boldsymbol{y}} = 0$;*

6. *if* $x_i = x'_j$ *then* $E^i_{\boldsymbol{x},y} = E^j_{\boldsymbol{x'},y}$.

Proof. We use lemma 4.4 to prove these claims.

1. From lemma 4.4 we know that each of the $E_{\boldsymbol{x},\boldsymbol{y}}$ are projectors that sum to I. Therefore they are orthogonal. As $E^i_{\boldsymbol{x}}$ is a sum of orthogonal projectors it is a projector. We also have $\sum_{y \in Y} E_{xy} = \sum_{y \in Y} E_{xy} \sum_{\boldsymbol{y}_i = y} E_{\boldsymbol{x},y} = \sum_{\boldsymbol{y} \in Y^k} E_{\boldsymbol{x},\boldsymbol{y}} = I$.
2. This follows immediately from lemma 4.4.
3. From lemma 4.4 we have $E_{\boldsymbol{x},\boldsymbol{y}} = F^T_{\boldsymbol{x},\boldsymbol{y}}$. So we get

$$E^i_{\boldsymbol{x},y} = \sum_{\boldsymbol{y}_i = y} E_{\boldsymbol{x},\boldsymbol{y}} = \sum_{\boldsymbol{y}_i = y} F^T_{\boldsymbol{x},\boldsymbol{y}} = (\sum_{\boldsymbol{y}_i = y} F_{\boldsymbol{x},\boldsymbol{y}})^T = F^{iT}_{\boldsymbol{x},y}$$

   .

4. We first note that $p(\boldsymbol{y}, \boldsymbol{x'} | \boldsymbol{x}, \boldsymbol{y}) = 0$ whenever $\boldsymbol{x} \neq \boldsymbol{x'}$. This is because we must have some $i \in [k]$ for which $x_i \neq x'_i$. However, as we clearly have $y_i = y_i$ the winning condition $x_i = x_i^{'} \iff y_i = y_i$ is not satisfied. From lemma 4.4 we can deduce that $E_{\boldsymbol{x},\boldsymbol{y}} E_{\boldsymbol{y},\boldsymbol{x'}} = 0$ whenever $\boldsymbol{x} \neq \boldsymbol{x'}$. We then get

$$E_{\boldsymbol{x},\boldsymbol{y}} = E_{\boldsymbol{x},\boldsymbol{y}} \sum_{\boldsymbol{x'} \in X^k} E_{\boldsymbol{y},\boldsymbol{x}} = E_{\boldsymbol{x},\boldsymbol{y}} E_{\boldsymbol{y},\boldsymbol{x}}.$$

   Similarly, whenever $\boldsymbol{y} \neq \boldsymbol{y'}$ we have

$$E_{\boldsymbol{y},\boldsymbol{x}} = \sum_{\boldsymbol{y'} \in Y^k} E_{\boldsymbol{x},\boldsymbol{y'}} E_{\boldsymbol{y},\boldsymbol{x}} = E_{\boldsymbol{x},\boldsymbol{y}} E_{\boldsymbol{y},\boldsymbol{x}}.$$

5. Take $\boldsymbol{x} \in X^k$ and $\boldsymbol{y} \in Y^k$ such that $\boldsymbol{x} \in R^{\mathcal{X}} \iff \boldsymbol{y} \in R^{\mathcal{Y}}$ does not hold. Because of the winning conditions, if either of these tuples is given as input to Alice and Alice responds with the other tuple, then regardless of what input we give to Bob and what output Bob produces we end up with 4 tuples such that $V(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x'}, \boldsymbol{y'}) = 0$. Therefore from lemma 4.4 we can conclude that for any $x' \in X^k$ and $y' \in Y^k$ $E_{\boldsymbol{x},\boldsymbol{y}} E_{\boldsymbol{x'},\boldsymbol{y'}} = 0$. Making use of the argument used in proving item (4) above we get:

$$E_{\boldsymbol{x},\boldsymbol{y}} = E_{\boldsymbol{x},\boldsymbol{y}} E_{\boldsymbol{y},\boldsymbol{x}} = 0.$$

6. This follows via exactly the same argument used in theorem 5.26.

□

Notice that by using lemma 4.4 as a black-box we have avoided the need to reprove some of the intermediate lemmas in the corresponding proof of theorem 5.5 in [ABdSZ17]. Now let us once again define projectors $P_{x,y} = E^i_{\boldsymbol{x},y}$ whenever $x = x_i$ for some $i \in [k]$. Item (6) in theorem 5.36 shows that these are well-defined projectors. As in the homomorphism case we then have for each $\boldsymbol{x} \in R^{\mathcal{X}}$ PVMs $\{P_{x_i}\}$ which are jointly measurable. Hence $E_{\boldsymbol{x}}$ is equivalent to the PVM $P_{\boldsymbol{x}}$ given by

$$P_{\boldsymbol{x},\boldsymbol{y}} = P_{x_1,y_1}\cdot, \ldots, .P_{x_k,y_k}.$$

We can now define a $*$-isomorphism as follows.

Definition 5.37. Let $\boldsymbol{H}_d$ be a finite-dimensional Hilbert space of dimension d. A $*$-isomorphismm in $\boldsymbol{H}_d$ between $\mathcal{X} \in \mathbf{R}(\sigma)$ and $\mathcal{Y} \in \mathbf{R}(\sigma)$ is given by a family of projectors $\{P_{x,y}\}_{x \in X, y \in Y}$ in $\mathsf{Proj}(\boldsymbol{H}_d)$ satisfying:

1. For all $x \in X : \sum_{y \in Y} P_{x,y} = I$;
2. For all $y \in Y : \sum_{x \in X} P_{x,y} = I$;
3. For all x, x' adjacent in the Gaifman graph of $\mathcal{X}$ and all $y, y' \in \mathcal{Y}$ we have $P_{x,y} \odot P_{x'y'}$.
4. For all y, y' adjacent in the Gaifman graph of $\mathcal{Y}$ and all $x, x' \in \mathcal{X}$ we have $P_{x,y} \odot P_{x'y'}$.
5. If $\boldsymbol{x} \in R^{\mathcal{X}} \iff \boldsymbol{y} \in R^{\mathcal{Y}}$ does not hold $P_{\boldsymbol{x},\boldsymbol{y}} = 0$.

A helpful alternative definition is to note that a $*$-isomorphism is given by a family of projectors $\{P_{x,y}\}_{x \in X, y \in Y}$ such that $\{P_{x,y}\}$ is a $*$-homomorphism from $\mathcal{X}$ to $\mathcal{Y}$ and $\{P_{y,x}\}$ is a $*$-homomorphism from $\mathcal{Y}$ to $\mathcal{X}$.

To justify this definition we prove the following:

Theorem 5.38. *The following are equivalent:*

1. $\mathcal{X} \overset{*}{\cong} \mathcal{Y}$
2. *There exists a $*$-isomorphism between $\mathcal{X}$ and $\mathcal{Y}$.*

Proof. (1) $\Rightarrow$ (2): This follows from theorem 5.36 and the ensuing discussion on joint measurability of PVMs. (2) $\Rightarrow$ (1): Consider the strategy $(\psi, \{P_{\boldsymbol{x},\boldsymbol{y}}\}, \{P^T_{\boldsymbol{x},\boldsymbol{y}}\})$ where ψ is the maximally entangled state. This is a valid strategy because of conditions (1)-(4) in the definition of $*$-isomorphisms. Via the last item of lemma 4.4 we see that item (5) implies that $p(\boldsymbol{y}, \boldsymbol{y}'|\boldsymbol{x}, \boldsymbol{x}') = 0$ whenever the winning conditions of the game are not satisfied. ☐

5.2.3 Commuting operator strategies

As in the case of the homomorphism game, we can provide a characterisation of perfect *co*-strategies as follows. We omit a detailed proof as this again makes use of lemma 4.5 and then follows exactly the same style of argument as theorems 5.38.

Definition 5.39. Let $\mathbf{A}$ be a unital C^*-algebra which admits a faithful tracial state $s : \mathbf{A} \to \mathbb{C}$. A *co*-isomorphism between $\mathcal{X} \in \mathbf{R}(\sigma)$ to $\mathcal{Y} \in \mathbf{R}(\sigma)$ is given by a family of projectors $\{P_{x,y}\}_{x \in X, y \in Y}$ in $\mathbf{A}$ satisfying:

1. For all $x \in X : \sum_{y \in Y} P_{x,y} = I$;
2. For all $y \in Y : \sum_{x \in X} P_{x,y} = I$;
3. For all x, x' adjacent in the Gaifman graph of $\mathcal{X}$ and all $y, y' \in \mathcal{Y}$ we have $P_{x,y} \odot P_{x'y'}$.
4. For all y, y' adjacent in the Gaifman graph of $\mathcal{Y}$ and all $x, x' \in \mathcal{X}$ we have $P_{x,y} \odot P_{x'y'}$.
5. If $\boldsymbol{x} \in R^{\mathcal{X}} \iff \boldsymbol{y} \in R^{\mathcal{Y}}$ does not hold $P_{\boldsymbol{x},\boldsymbol{y}} = 0$.

Theorem 5.40. *The following are equivalent:*

- $\mathcal{X} \overset{co}{\cong} \mathcal{Y}$.
- *There exists a* co*-isomorphism between* $\mathcal{X}$ *and* $\mathcal{Y}$.

5.2.4 Representations of the isomorphism game

Finally, to end this section we characterise perfect strategies in the isomorphism game using C^*-algebras, just as we did for the CSP game.

Definition 5.41. A quantum isomorphism from $\mathcal{X} \in \mathbf{R}(\sigma)$ to $\mathcal{Y} \in \mathbf{R}(\sigma)$ in $\boldsymbol{H}$ is given by a family of projectors $\{P_{x,y}\}_{x \in X, y \in Y}$ in $\mathsf{Proj}(\boldsymbol{H})$ satisfying:

1. For all $x \in X : \sum_{y \in Y} P_{x,y} = I$;
2. For all $y \in Y : \sum_{x \in X} P_{x,y} = I$;
3. For all x, x' adjacent in the Gaifman graph of $\mathcal{X}$ and all $y, y' \in \mathcal{Y}$ we have $P_{x,y} \odot P_{x'y'}$.
4. For all y, y' adjacent in the Gaifman graph of $\mathcal{Y}$ and all $x, x' \in \mathcal{X}$ we have $P_{x,y} \odot P_{x'y'}$.
5. If $\boldsymbol{x} \in R^{\mathcal{X}} \iff \boldsymbol{y} \in R^{\mathcal{Y}}$ does not hold $P_{\boldsymbol{x},\boldsymbol{y}} = 0$.

Again we write $\mathbf{A}(\mathcal{X}, \mathcal{Y})$ for the universal C^*-algebra generated by such sets of projections. Note that $\mathbf{A}(\mathcal{X}, \mathcal{Y})$ may fail to exist if there is no quantum isomorphism between $\mathcal{X}$ and $\mathcal{Y}$ in any Hilbert space.

Theorem 5.42. *The following facts hold:*

- $G \overset{co}{\cong} H$ *iff there exists a tracial state on* $\mathbf{A}(\mathcal{X},\mathcal{Y})$.
- $G \overset{*}{\cong} H$ *iff* $\mathbf{A}(\mathcal{X},\mathcal{Y})$ *has a finite-dimensional representation.*
- $G \overset{c}{\cong} H$ *iff* $\mathbf{A}(\mathcal{X},\mathcal{Y})$ *has a one-dimensional representation.*

5.3 Categorical characterisations

In this section, we explain how quantum homomorphisms and isomorphisms can be captured categorically in the Kleisli category of a suitable generalisation of the concept of a monad.

5.3.1 The graded quantum monad

We start by describing the graded quantum monad introduced in [ABdSZ17]. Let us define graded monads.

Definition 5.43. Given a monoid $(N,\cdot,1)$ an N-*graded monad* $(\{M_n\}_{n\in N},\{\mu^{n,n'}\}_{n,n'\in N},\eta)$ over a category $\mathbf{C}$ is a family of endofunctors $M_n : \mathbf{C}\to\mathbf{C}$, $n\in N$, a natural transformation $\eta : \mathrm{id}_{\mathbf{C}} \to M_1$ (the *unit*), and a family of natural transformations $\mu^{n,k} : M_nM_k \to M_{n.k}$, where $n,k\in N$ (the *graded multiplication*). These must satisfy the following commutativity conditions:

$$\begin{tikzcd}
M_nM_kM_m \arrow[r, "M_n\mu^{k,m}"] \arrow[d, "\mu^{n,k}_{M_m}"'] & M_nM_{k.m} \arrow[d, "\mu^{n,k.m}"] \\
M_{n.k}M_m \arrow[r, "\mu^{n.k,m}"'] & M_{n.k.m}
\end{tikzcd}
\qquad
\begin{tikzcd}
M_n \arrow[r, "\eta_{M_n}"] \arrow[d, "M_n\eta"'] \arrow[dr, "\mathrm{id}_{M_n}"] & M_1M_n \arrow[d, "\mu^{1.n}"] \\
M_nM_1 \arrow[r, "\mu^{n.1}"] & M_n
\end{tikzcd}$$

More generally, a graded monad is a (lax) monoidal functor $(\mathbf{M},\otimes,I)\to([\mathbf{C},\mathbf{C}],\circ,\mathrm{id}_{\mathbf{C}})$ from a monoidal category to the category of endofunctors over $\mathbf{C}$ [FKM16].

Now let $\mathbb{N}^+$ denote the monoid of natural numbers under the usual multiplication operation. The graded quantum monad of [ABdSZ17] is defined as follows.

Example 5.44. *The (finite-dimensional) graded quantum* $(\{Q_d\}_{d\in\mathbb{N}^+},\{\mu^{d,d'}\}_{d,d'\in\mathbb{N}^+},\eta)$ *is an* $\mathbb{N}^+$ *graded monad defined by the following components:*

- $\mathbb{Q}_d : \mathbf{Set}\to\mathbf{Set}$ *is the functor defined by:*

  - *$\mathbb{Q}_d(X)$ is the set of all functions of the form $\varphi : X \to \mathsf{Proj}(\mathbb{C}^d)$ where for all but finitely many $x \in X$, $\varphi(x) = 0$. These must satisfy the normalisation condition $\sum_{x \in X} \varphi(x) = I_d$.*
  - *$\mathbb{Q}_d(f)$ maps φ to $\lambda y.\Sigma_{x \in f^{-1}(y)} \varphi(x)$.*
  - *$(\varphi_1, \dots, \varphi_n) \in R^{\mathbb{Q}_d \mathcal{X}}$ iff the following conditions are satisfied:*
    - *$\forall i, j \in [k], x, x' \in \mathcal{X} : \varphi_i(x) \odot \varphi_j(x')$.*
    - *$\forall \boldsymbol{x} \in X^k, \boldsymbol{x} \notin R^{\mathcal{X}} : \boldsymbol{\varphi}(\boldsymbol{x}) = 0$ where $\boldsymbol{\varphi}(\boldsymbol{x}) = \varphi_1(x_1), \dots \varphi_k(x_k)$.*
- *$\eta_X(x) = \Delta_x$ where $\Delta_x(x) = I_1$ and $\Delta_x(y) = 0$ for $y \neq x$.*
- *$\mu_X(\varphi) = \lambda x.\Sigma_{\psi \in \mathbb{Q}_d(X)} \varphi(\psi) \otimes \psi(x)$.*

As with the case of distribution minion monads, we can also define a **Set** version of the graded quantum monad which we denote by Q_d. This construction can be seen as an analogue of the distribution monad $\mathcal{D}_{\mathbb{S}}$ which is not strictly speaking defined over a semiring but rather a family of graded, semiring-like structures given by the data $(\bigsqcup_{d \in \mathbb{N}^+} \mathsf{Proj}(\mathbf{C}^d), \{\mathbf{0}_d\}, \{I_d\}, +, \otimes)$.

The following result links the graded quantum monad to quantum homomorphisms.

Theorem 5.45. *([ABdSZ17]) The following are equivalent for $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$:*

1. *$\mathcal{X} \xrightarrow{*} \mathcal{Y}$.*
2. *$\mathcal{X} \to \mathbb{Q}_d \mathcal{Y}$.*

We now point out that one can define an alternative version of the quantum monad which can be used to study representations of graph homomorphisms over any Hilbert space, not just those which are finite-dimensional. For this purpose, consider the monoidal category $(\mathbf{Hilb}^{\mathrm{iso}}, \otimes, \mathbb{C})$, which has Hilbert spaces as objects, unitary maps (which are isomorphisms in **Hilb**) as morphisms, the tensor product of Hilbert spaces as its monoidal product, and the Hilbert space $\mathbb{C}$ as its unit object.

Example 5.46. *There exists a $(\mathbf{Hilb}^{\mathrm{iso}}, \otimes, \mathbb{C})$-graded quantum monad $(\{\mathbb{Q}_{\boldsymbol{H}}\}_{\boldsymbol{H} \in \mathbb{H}}, \{\mu^{\boldsymbol{H}, \boldsymbol{H'}}\}_{\boldsymbol{H}, \boldsymbol{H'} \in \mathbb{H}}, \eta)$ which is defined in the same way as $\mathbb{Q}_d$, where we replace the unit 1 with the Hilbert space $\mathbb{C}$ and the multiplication operation . with the tensor product $\otimes$. Moreover, since a graded monad is a lax monoidal functor we must also construct a natural transformation $\alpha^U : \mathbb{Q}_{\boldsymbol{H}} \to \mathbb{Q}_{\mathbf{H'}}$ whenever there exists a unitary map $U : \mathbf{H} \to \mathbf{H}'$. This is defined as $\alpha^U_X \varphi = \Sigma_{x \in X} U \varphi(x) U^\dagger$.*

Proposition 5.47. *$(\{\mathbb{Q}_{\boldsymbol{H}}\}_{\boldsymbol{H} \in \mathbb{H}}, \{\mu^{\boldsymbol{H}, \boldsymbol{H'}}\}_{\boldsymbol{H}, \boldsymbol{H'} \in \mathbb{H}}, \eta)$ is a graded monad.*

Proof. Checking the associativity and identity axioms is almost equivalent to the proof of the finite-dimensional case given in [ABdSZ17], we simply replace the data from the monoid $\mathbb{N}^+$ with the data from the monoidal category $\mathbf{Hilb}^{\mathbf{iso}}$. The additional point to check is that the α^U are valid natural transformations. For this there are two things to check:

- For any unitary $U : \mathbf{H} \to \mathbf{H}'$, whenever we have $\varphi \in \mathbb{Q}_{\boldsymbol{H}} X$ we must have $\alpha^U_X \varphi \in \mathbb{Q}_{\mathbf{H}'} X$. To verify this note that $\forall x \in X$ we have

  $$U\varphi(x)U^\dagger U\varphi(x)U^\dagger = U\varphi(x)I\varphi(x)U^\dagger = U\varphi(x)\varphi(x)U^\dagger = U\varphi(x)U^\dagger$$

  and
  $$(U\varphi(x)U^\dagger)^\dagger = (U^\dagger)^\dagger(U\varphi(x))^\dagger = U\rho(x)^\dagger U^\dagger = U\varphi(x)U^\dagger.$$

  In other words $\alpha^U_X\varphi(x)$ is a projector for all $x \in X$. We also have

  $$\sum_{x\in X} U\rho(x)U^\dagger = U\sum_{x\in X}\rho(x)U^\dagger = UIU^\dagger = UU^\dagger = I.$$

  Which means that the projectors $\{U\rho(X)U^\dagger\}$ form a PVM as required. Finally, given $(\varphi_1, \ldots, \varphi_n) \in R^{\mathbb{Q}_{\boldsymbol{H}} \mathcal{X}}$, we must have $(\alpha^U_X\varphi_1, \ldots, \alpha^U_X\varphi_n) \in R^{\mathbb{Q}_{\mathbf{H}'} \mathcal{X}}$. This holds as a consequence of the following two facts:

  $$\begin{aligned}\varphi_i(x) \odot \varphi_j(x') \Rightarrow (\alpha^U_X\varphi_i)(x)(\alpha^U_X(\varphi_j)(x') &= U\varphi_i(x)U^\dagger U\varphi_j(x')U^\dagger \\ &= U\varphi_i(x)\varphi_j(x')U^\dagger \\ &= U\varphi_j(x')\varphi_i(x)U^\dagger \\ &= U\varphi_j(x')U^\dagger U\varphi_i(x)U^\dagger \\ &= (\alpha^U_X(\varphi_j)(x')(\alpha^U_X\varphi_i)(x).\end{aligned}$$

  and

  $$\begin{aligned}&\varphi_1(x_1)\cdots\varphi(x_n) = 0 \\ &\Rightarrow (\alpha^U_X\varphi_1)(x_1)\cdots(\alpha^U_X\varphi_n)(x_n) = U\varphi_1(x_1)U^\dagger U\varphi_2(x_2)U^\dagger \cdots U\varphi_n(x_n)U^\dagger \\ &= U\varphi_1(x_1)\varphi_2(x_2)\cdots\varphi_n(x_n)U^\dagger = U0U^\dagger = 0.\end{aligned}$$

- We must also check naturality. This condition is satisfied since for any $f : X \to Y$ we have

$$
\begin{aligned}
(\mathbb{Q}_{\mathbf{H}'} f \circ \alpha^U_X(\varphi))(y) &= \sum_{x \in f^{-1}(y)} (\alpha^U_X \varphi)(x) \\
&= \sum_{x \in f^{-1}(y)} U\varphi(x)U^\dagger = (U \sum_{x \in f^{-1}(y)} \varphi(x)U^\dagger) \\
&= U(\mathbb{Q}_{\boldsymbol{H}} f(\varphi))(y)U^\dagger) = (\alpha^U_Y \circ \mathbb{Q}_{\boldsymbol{H}} f(\varphi))(y)
\end{aligned}
$$

□

We then have the following analogue of our previous theorem:

Theorem 5.48. *The following are equivalent for $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$:*

- *There exists a quantum homomorphism from $\mathcal{X}$ to $\mathcal{Y}$ in the Hilbert space $\boldsymbol{H}$.*
- $\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}} \mathcal{Y}$.

It follows from this discussion that any result involving the quantum monad we derive in the remainder of the thesis applies to arbitrary Hilbert spaces $\boldsymbol{H}$ not just finite-dimensional ones.

5.3.2 The quantum minion

Very recently a linear minion referred to as the quantum minion was introduced in [Cia24] and used to prove several results about quantum advantage in the CSP game. In this section, we explore connections between this minion and the quantum monad. Our starting point is to note that distribution minions of example 2.32 can be defined over partial semirings.

Example 5.49. *A* partial distribution minion $\mathfrak{D}_{\mathbf{S}}$ *over a partial semiring* $\mathbf{S}$ *is a linear minion of depth 1 satisfying for all $l \in \mathbb{N}$ and for all $\boldsymbol{v} \in \mathfrak{D}^{(l)} : \sum_{v \in \boldsymbol{v}} v = 1_{\mathbf{S}}$.*

The quantum minion of [Cia24] is then the partial distribution minion over the partial semiring $\mathsf{Proj}(\boldsymbol{H})$ which was described in the previous chapter. We write $\mathcal{Q}_{\boldsymbol{H}}$ for this minion and $F_{\mathcal{Q}_{\boldsymbol{H}}}$ for its free structure. The following was shown in [Cia24].

Proposition 5.50. *([Cia24][Proposition 7]) The following are equivalent for σ-structures $\mathcal{X}, \mathcal{Y}$:*

- *There exists a quantum homomorphism from $\mathcal{X}$ to $\mathcal{Y}$.*
- $\mathcal{X} \to F_{\mathcal{Q}_{\boldsymbol{H}}} \mathcal{Y}$.

Our next result is implicit in the proof of the above proposition, although it is not explicitly stated in [Cia24].

Proposition 5.51. $F_{Q_H}\mathcal{X} \cong \mathbb{Q}_H\mathcal{X}$.

Proof. The universes of the two structures are readily seen to be the same via the same argument as in theorem 2.37. Proving that the relations also coincide is more involved. However, this is essentially the main content of the proof of proposition 5.50 in [Cia24]. We also note that the proof strategy used in that paper is very similar to our lemma 3.44. □

This result shows that the quantum monad fits into the framework of distribution minion monads described in chapter 2. Note however that we have had to relax our assumptions so that $\mathbf{S}$ is a partial rather than a total semiring.

5.3.3 The partial quantum monad

So far in this chapter, we have introduced suitable definitions for quantum homomorphisms and quantum isomorphisms of relational structures. Moreover, we have described a graded quantum monad $\mathbb{Q}_H$ whose Kleisli morphisms correspond precisely to quantum homomorphisms. Given these results, one might expect that Kleisli isomorphisms of $\mathbb{Q}_H$ coincide with quantum isomorphisms. Our next result shows that this is not the case.

Proposition 5.52. *The following are equivalent:*

1. $\mathcal{X} \cong \mathcal{Y}$
2. $\mathcal{X} \overset{\mathbf{Kl}(\mathbb{Q}_H)}{\cong} \mathcal{Y}$

Proof. $(1) \Rightarrow (2)$ is straightforward and holds for any monad: Let $f : \mathcal{X} \to \mathcal{Y}$ be an isomorphism in $\mathbf{R}(\sigma)$. Then we have morphisms $g : \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{\eta} \mathbb{Q}_{\mathbb{C}}\mathcal{Y}$ and $g^{-1} : \mathcal{Y} \xrightarrow{f^{-1}} \mathcal{X} \xrightarrow{\eta} \mathbb{Q}_{\mathbb{C}}\mathcal{X}$. We have

$$\begin{aligned} g^{-1} \circ g &= \mu^{\mathbb{C},\mathbb{C}}_{\mathcal{X}} \circ \mathbb{Q}_{\mathbb{C}}\eta_{\mathcal{X}} \circ \mathbb{Q}_{\mathbb{C}} f^{-1} \circ \eta_{\mathcal{Y}} \circ f \\ &= \mathrm{id}_{\mathbb{Q}_{\mathbb{C}}\mathcal{X}} \circ \mathbb{Q}_{\mathbb{C}} f^{-1} \circ \eta_{\mathcal{Y}} \circ f && \text{(Unit axiom)} \\ &= \mathbb{Q}_{\mathbb{C}} f^{-1} \circ \mathbb{Q}_{\mathbb{C}} f \circ \eta_{\mathcal{X}} && \text{(Naturality of } \eta\text{)} \\ &= \eta_{\mathcal{X}} \end{aligned}$$

A dual argument shows that $g^{-1} \circ g = \eta_{\mathcal{Y}}$ as required.

$(2) \Rightarrow (1)$: Note that the only projectors in $\mathbb{C}$ are 1 and 0. Therefore we can easily observe that the functor $\mathbb{Q}_{\mathbb{C}}$ is isomorphic to the identity functor and that η is a natural isomorphism. Let $f : \mathcal{X} \to \mathbb{Q}_H\mathcal{Y}$ be a Kleisli isomorphism. We must have $f^{-1} : \mathcal{Y} \to \mathbb{Q}_{H'}\mathcal{X}$ such that $f \circ f^{-1} = \eta_{\mathcal{X}}$. We know that $f^{-1} \circ f : \mathcal{X} \to \mathbb{Q}_{H \otimes H'}$ and that $\eta_{\mathcal{X}} : \mathcal{X} \to \mathbb{Q}_{\mathbb{C}}$. From this, we can conclude that $H = H' = \mathbb{C}$ since this is the only way $H \otimes H' = \mathbb{C}$. Therefore we can construct the isomorphism $\mathcal{X} \xrightarrow{f} \mathbb{Q}_{\mathbb{C}}\mathcal{Y} \xrightarrow{\eta_{\mathcal{Y}}^{-1}} \mathcal{Y}$ in $\mathbf{R}(\sigma)$. ☐

Therefore, even though Kleisli morphisms of $\mathbb{Q}_{\boldsymbol{H}}$ correspond to quantum homomorphisms we see that Kleisli isomorphisms coincide with classical isomorphisms. The following theorem from [Con22] shows that a similar result holds more generally for any distribution minion monad $\mathbb{D}_{\mathbf{S}}$ which satisfies certain properties.

Theorem 5.53. *For any semiring* $\mathbf{S}$ *whose non-zero elements are closed under both addition and multiplication, the following are equivalent:*

- $X \cong Y$
- $X \overset{\mathbf{Kl}(\mathbb{D}_{\mathbf{S}})}{\cong} Y$

We now describe a method for overcoming this limiting result in the case of the functor $\mathbb{Q}_{\boldsymbol{H}}$. Recall that the set $\mathsf{Proj}(\boldsymbol{H})$ admits the structure of a partial semiring whereby the operation $+$ is only defined when two projectors are orthogonal and .. Crucially, in this structure, two non-zero projectors P_1 and P_2 can multiply to zero. Thus, the non-zero elements of $\mathsf{Proj}(\boldsymbol{H})$ are not closed under matrix multiplication and the assumptions of theorem 5.53 are not satisfied by this partial semiring. Based on this observation we now define an alternative partial notion of Kleisli composition for the functor $\mathbb{Q}_{\boldsymbol{H}}$ where we use the matrix product instead of the Kronecker product. To explain this in more detail recall from [ABdSZ17] that given two Kleisli morphisms $f : \mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$ and $g : \mathcal{Y} \to \mathbb{Q}_{\boldsymbol{H'}}\mathcal{Z}$ their composition $g \circ f : \mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}\otimes\boldsymbol{H'}}\mathcal{Z}$ can be explicitly described by the formula

$$(g \circ f)(x)(c) = \sum_{y \in Y} f(x)(y) \otimes g(y)(c).$$

Now consider instead two morphisms $\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$ and $\mathcal{Y} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Z}$. If these morphisms satisfy the condition that for all $x \in \mathcal{X}, y \in \mathcal{Y}, z \in \mathcal{Z} : f(x)(y) \odot g(y)(z)$ we can define a composite morphism $g \bullet f : \mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Z}$ given by

$$(g \bullet f)(x)(z) = \sum_{y \in Y} f(x)(y).g(y)(z).[3]$$

Moreover, for any Hilbert space $\boldsymbol{H}$ there is a natural notion of Kleisli identity map given by

---

[3] Note that the requirement $f(x)(y) \odot g(y)(z)$ is more strict than asking for $\sum_{y \in Y} f(x)(y) \cdot g(y)(z)$ to be a well-defined projector. This is because the sum of several matrices can be a projector even if the individual matrices themselves are not.

$\eta^{\boldsymbol{H}}_{\mathcal{X}} : \mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{X}$ which sends $x \in X$ to $\delta_x \in \mathbb{Q}_{\boldsymbol{H}}X$ where for $x \neq x'$: $\begin{cases} \delta_x(x) = I_d \\ \delta_x(x') = \mathbf{0}_d \end{cases}$. It is straightforward to verify that whenever the composition of morphisms under $\bullet$ is well-defined the associativity and unitality axioms of a category are satisfied. Thus we almost have a category, except that composition operation $\bullet$ is a partial rather than total operation. These observations prompt us to define the concept of a partial category.

Definition 5.54. A *partial category* $\mathbf{C}$ consists of:

- A collection of objects, denoted $\mathrm{Ob}(\mathbf{C})$.
- For each pair of objects $X, Y \in \mathrm{Ob}(\mathbf{C})$, a set of morphisms (or arrows) $\mathrm{Hom}_{\mathbf{C}}(X, Y)$.
- For each object $X \in \mathrm{Ob}(\mathbf{C})$, an identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathbf{C}}(X, X)$.
- For each triple of objects $X, Y, Z \in \mathrm{Ob}(\mathbf{C})$, a composition law:

$$\bullet : \mathrm{Hom}_{\mathbf{C}}(Y, Z) \times \mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{C}}(X, Z)$$

  Given by a partial operation $\bullet$ where for all $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathbf{C}}(Y, Z)$, and $h \in \mathrm{Hom}_{\mathbf{C}}(Z, W)$, the following axioms hold :

  - **Associativity:** $(h \bullet g) \bullet f$ is well-defined iff $h \bullet (g \bullet f)$ is well-defined. Moreover, when this is the case $(h \bullet g) \bullet f = h \bullet (g \bullet f)$
  - **Identity:** $\mathrm{id}_Y \bullet f$ and $g \bullet \mathrm{id}_Y$ are always well-defined. Moreover, $\mathrm{id}_Y \bullet f = f$ and $g \bullet \mathrm{id}_Y = g$.

This can equivalently be seen as a category enriched in the category of sets and partial functions.

The partial category we have described above then arises as the Kleisli category of a partial version of the quantum monad which we now describe.

Example 5.55. *The partial quantum monad* $(\mathbb{Q}_{\boldsymbol{H}}, \eta, \mu)$ *is a partial monad :*

- $\mathbb{Q}_{\boldsymbol{H}}$ *is defined as previously.*
- $\eta_X(x) = \Delta_x$ *where* $\Delta_x(x) = I_{\boldsymbol{H}}$ *and* $\Delta_x(y) = 0_{\boldsymbol{H}}$ *for* $y \neq x$.
- $\mu_X(\varphi)$ *is defined whenever* $\forall \psi \in \mathbb{Q}_{\boldsymbol{H}}(X), \forall x \in X : \varphi(\psi) \odot \psi(x)$. *In this case we have* $\mu_X(\varphi) = \lambda x.\Sigma_{\psi \in \mathbb{Q}_{\boldsymbol{H}}(X)} \varphi(\psi).\psi(x)$.

Note that μ here fails to be a natural transformation since the morphism $\mu_{\mathcal{X}}$ assigned to each object is a partial rather than a total homomorphism. Nevertheless, whenever compositions are well-defined this operation does satisfy the naturality axiom. From this point onwards we shall write $\mathbf{Kl}(\overset{\bullet}{\mathbb{Q}}_{\boldsymbol{H}})$ and $\mathbf{Kl}(\overset{\otimes}{\mathbb{Q}}_{\boldsymbol{H}})$ to distinguish the (partial) kleilsi categories of the partial and graded version of the quantum monad respectively.

We can now explore what a suitable notion of isomorphism in $\mathbf{Kl}(\overset{\bullet}{\mathbb{Q}}_{\boldsymbol{H}})$ would look like. In a Kleisli category, the identity map is given by the monad unit. Thus, let us write $\mathcal{X} \overset{\mathbf{Kl}(\mathbb{Q}_{\boldsymbol{H}})^{\bullet}}{\cong} \mathcal{Y}$ whenever there exists a pair of Kleisli morphisms $f : \mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$ and $g : \mathcal{Y} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{X}$ satisfying:

- $\forall x \in \mathcal{X}, y \in \mathcal{Y} f(x)(y) \odot g(y)(x)$,
- $g \bullet f = \eta_{\mathcal{X}}$,
- $f \bullet g = \eta_{\mathcal{Y}}$.

Our next theorem shows that such isomorphisms are precisely quantum isomorphisms.

Theorem 5.56. *The following are equivalent:*

1. *There exists a quantum isomorphism $\mathcal{X} \overset{q}{\cong} \mathcal{Y}$ in $\boldsymbol{H}$.*
2. $\mathcal{X} \overset{\mathbf{Kl}(\overset{\bullet}{\mathbb{Q}}_{\boldsymbol{H}})}{\cong} \mathcal{Y}$.

Proof. We recall that one way of characterising a quantum isomorphism is as the existence of a pair of quantum homomorphisms $f : \mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$ and $g : \mathcal{Y} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$ such that $f(x)(y) = g(y)(x)$.

(1) ⇒ (2): Take f and g to be quantum homomorphisms which witness a quantum isomorphism in the sense described above. We have

$$(g \bullet f)(x)(x) = \sum_{y \in Y} f(x)(y).g(y)(x) = \sum_{y \in Y} f(x)(y).f(x)(y) = \sum_{y \in Y} f(x)(y) = I_d.$$

Therefore $(g \bullet f)(x) = \delta_x$ and $g \bullet f = \eta_X$ as required. A symmetric argument shows that $(f \bullet g) = \eta_Y$.

(1) ⇐ (2): Let f and g be Kleisli morphisms witnessing $\mathcal{X} \overset{\mathbf{Kl}(\overset{\bullet}{\mathbb{Q}}_{\boldsymbol{H}})}{\cong} \mathcal{Y}$. We show that $\forall x, y : f(x)(y) = g(y)(x)$. We begin by observing that

$$(g \bullet f)(x)(x) = \sum_{y' \in Y} f(x)(y').g(y')(x) = I_d.$$

Because $f(x)$ is a PVM we have $f(x)(y).f(x)(y') = 0$ whenever $y \neq y'$. Thus, if we multiply both sides of the above equation by $f(x)(y)$ on the left we obtain the equality

$$f(x)(y) = f(x)(y).f(x)(y).g(y)(x) = f(x)(y).g(y)(x).$$

A symmetric argument involving $(f \bullet g)$ results in the equality $g(y)(x) = g(y)(x).f(x)(y)$. Since $f(x)(y)$ and $g(y)(x)$ commute we conclude with

$$f(x)(y) = f(x)(y).g(y)(x) = g(y)(x).f(x)(y) = g(y)(x).$$

$\square$

This theorem shows that by replacing the Kronecker product with matrix multiplication we can overcome the collapse of Kleisli isomorphisms for $\mathbb{Q}_H$ to classical isomorphism.

5.3.3.1 Overcoming partiality

Our final result in this chapter is a method for overcoming the partiality inherent in $\mathbf{Kl}(\overset{\bullet}{\mathbb{Q}}_H)$ to derive a (total) category where morphisms are quantum homomorphisms and isomorphisms are quantum isomorphisms. The idea is to change the objects of $\mathbf{Kl}(\overset{\bullet}{\mathbb{Q}}_H)$ so that they contain more fine-grained type information. In this way, we will be able to limit the morphisms in and out of each object to precisely those morphisms which are composable via $\bullet$. Our approach to formalising this idea is based on a refinement of the notion of specification structures introduced in [AGN96].

Definition 5.57. Let $\mathbf{C}$ be a partial category. A specification structure S over $\mathbf{C}$ is defined by the following data:

- A set PA of properties over A for each object A in $\mathrm{Ob}(\mathbf{C})$.
- A relation $R_{A,B} \subseteq PA \times \mathrm{hom}_{\mathbf{C}}(A, B) \times PB$ for each pair of objects A, B in $\mathrm{Ob}(\mathbf{C})$.

Given $f : A \to B, g : B \to C, p_A \in PA, p_B \in PB, p_C \in PC$, this relation is required to satisfy the following conditions:

- **Skip:** $(p_A, \mathrm{id}_A, p_A) \in R_{A,A}$.
- **Sequential Composition:** $(p_A, f, p_B) \in R_{A,B}$ and $(p_B, g, p_C) \in R_{BC} \Rightarrow (p_A, g \bullet f, p_C) \in R_{A,C}$.
- **Totality:** $(p_A, f, p_B) \in R_{A,B}$ and $(p_B, g, p_C) \in R_{BC} \Rightarrow (g, f) \in \bullet_{\mathbf{C}}$.

Note that our definition of a specification structure is not the same as in [AGN96]. In particular, since they assume that $\mathbf{C}$ is a category they do not need to enforce a totality condition.

Given $\mathbf{C}$ and S as above one can define a (total) category $\mathbf{C}_S$. The objects are pairs (A, p_A) with $A \in \mathrm{Ob}(\mathbf{C})$ and $p_A \in PA$. A morphism $f : (A, p_A) \to (B, p_B)$ is a morphism $f : A \to B$ such that $(p_A, f, p_B) \in R_{A,B}$. The composition of f and g is simply $g \bullet_{\mathbf{C}} f$. Our next result verifies that this construction indeed yields a valid category.

Proposition 5.58. *Let S be a specification structure on a partial category $\mathbf{C}$. $\mathbf{C}_S$ is a category.*

Proof. The identity morphisms for an object (A, p_A) is simply given by id_A. From the skip condition, it is clear that this is a valid morphism in $\mathbf{C}_S$. Since all morphisms in $\mathbf{C}_S$ arise from morphisms in $\mathbf{C}$ it is clear that the identity and associativity conditions are satisfied. The last thing to check is that the composition of morphisms is a total binary operation. This follows from the totality and sequential composition conditions. □

We now describe the specification structure that shall be placed on $\mathbf{Kl}(\dot{\mathbb{Q}}_{\boldsymbol{H}})$. For this purpose, it will be helpful to think of a morphism $\mathcal{X} \to \mathcal{Y}$ for finite structures $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ as $|Y| \times |X|$ column stochastic matrices with entries in $\mathsf{Proj}(\boldsymbol{H})$. Column stochastic means that each column of the corresponding matrix must sum to I (and hence represents a PVM). In this case, the composition operation $\bullet$ is simply the usual notion of matrix multiplication with addition and multiplication given by the corresponding operations in the partial semiring $\mathsf{Proj}(\boldsymbol{H})$. This is by analogy to how morphisms in $\mathbf{Kl}(\mathcal{D})$ can be thought of as column stochastic matrices over the real numbers. Moreover, a PVM with outcomes in X can be thought of as a vector indexed by elements of X whose entries are projectors summing to I. Given a morphism $f : \mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}} \mathcal{Y}$ let us use capital letters F to clarify that we are thinking of it as a matrix in this form. Likewise, given a PVM $m : X \to \mathsf{Proj}(\boldsymbol{H})$ we write $\vec{m}$ to clarify that we are thinking of it as a vector.

Theorem 5.59. *The following data describes a specification structure S on $\mathbf{Kl}(\dot{\mathbb{Q}}_{\boldsymbol{H}})$.*

- *An element $p_{\mathcal{X}}$ of $P\mathcal{X}$ consists of a set of PVMs with outcomes in X. Moreover, $p_{\mathcal{X}}$ must include all delta distributions (i.e. $\forall x \in X : \eta(x) \in p_{\mathcal{X}}$).*
- *Given $p_{\mathcal{X}} \in P\mathcal{X}, p_{\mathcal{Y}} \in P\mathcal{Y}$, and a morphism $f : \mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}} \mathcal{Y}$ we have $(p_{\mathcal{X}}, f, p_{\mathcal{Y}}) \in R_{\mathcal{X},\mathcal{Y}}$ iff $\forall m \in p_{\mathcal{X}} : F\vec{m} \in p_{\mathcal{Y}}$.*

Proof. We verify the conditions one at a time.

- **Skip:** The matrix corresponding to id_A is a diagonal matrix I_A where every entry equals I. Thus, for all $m \in p_\mathcal{X}$ we have $I_a\vec{m} = \vec{m} \in p_\mathcal{X}$ as required.

- **Sequential composition:** Our assumptions are:

  1. $\forall m \in p_\mathcal{X} : F\vec{m} \in p_\mathcal{Y}$
  2. $\forall n \in p_\mathcal{Y} : G\vec{n} \in p_\mathcal{Z}$

  From (1) we have $(F\vec{m}) \in p_\mathcal{Y}$. Hence by (2) we must have $(G \bullet F)\vec{m} = GF\vec{m} \in p_\mathcal{Z}$ as required.

- **Totality:** Our assumptions are the same as in the sequential composition case. To verify totality we must show that every column of $G \bullet F = GF$ is a valid PVM. Recall that $\forall x \in X : \eta(x) \in p_\mathcal{X}$. By (1) this means that $\forall a \in \mathcal{X} : F\eta(x) = \vec{f}(x) \in p_\mathcal{Y}$. Hence, every column of F must be in $p_\mathcal{Y}$. By (2) we must have $\forall x \in \mathcal{X} : G\vec{f}(x) = \in p_\mathcal{Z}$. Since every element of $p_\mathcal{Z}$ is a PVM we have the desired result.

□

Thus, we now have a category $\mathbf{Kl}(\overset{\bullet}{\mathbb{Q}}_{\boldsymbol{H}})_S$ whose morphisms correspond to quantum homomorphisms and whose isomorphisms correspond to quantum isomorphisms.

As a final point, it is important to make sure that in passing from $\mathbf{Kl}(\overset{\bullet}{\mathbb{Q}}_{\boldsymbol{H}})$ to $\mathbf{Kl}(\overset{\bullet}{\mathbb{Q}}_{\boldsymbol{H}})_S$ we do not lose any quantum homomorphisms. This means we need to make sure that whenever a quantum homomorphism exists between $\mathcal{X}$ and $\mathcal{Y}$ there exist properties $p_\mathcal{X}$ and $p_\mathcal{Y}$ such that there is a morphism $(\mathcal{X}, p_\mathcal{X}) \to (\mathcal{Y}, p_\mathcal{Y})$.

Proposition 5.60. *For relational structures $\mathcal{X}$ and $\mathcal{Y}$ the following are equivalent:*

1. *There exists a morphism $f : \mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$.*
2. *There exists properties $p_\mathcal{X} \in P\mathcal{X}$ and $p_\mathcal{Y} \in P\mathcal{Y}$ such that $(p_\mathcal{X}, f, p_\mathcal{Y}) \in R_{\mathcal{X},\mathcal{Y}}$*

Proof. (2) $\Rightarrow$ (1) is trivial. To see why (1) $\Rightarrow$ (2) take $p_\mathcal{X} = \{\eta(x) | x \in X\}$ and $p_\mathcal{Y} = \{\eta(y) | y \in Y\} \cup \{f(x) | x \in X\}$. Now for arbitrary $\eta(x) \in p_\mathcal{X}$ we have $F\eta(x) = \vec{f}(x) \in p_\mathcal{Y}$ as required. □

5.4 Discussion

In this chapter we explored the framework of quantum CSPs that was first studied in [ABdSZ17]. We answered several open questions about quantum homomorphisms mentioned in this paper and showed how the notion of quantum isomorphism of relational structures can fit into the categorical framework of the quantum monad $\mathbb{Q}_{\boldsymbol{H}}$. To achieve this we considered $\mathbb{Q}_{\boldsymbol{H}}$ not as a graded monad but as a monad with a partial multiplication operation which is intimately connected with the partial semiring structure of the set $\mathsf{Proj}(\boldsymbol{H})$. We then showed that this monad gives rise to a partial Kleisli category where morphisms are quantum homomorphisms and isomorphisms are quantum isomorphisms. Finally, we used the idea of a specification structure, which is inspired by Hoare logic [Hoa69] to turn this partial Kleisli category into a total category. We outline some directions for future work:

- **Partial semiring semantics**: Perhaps the most interesting aspect of our results is the usage of the partial semiring structure of $\mathsf{Proj}(\boldsymbol{H})$ in our characterisation of quantum isomorphisms. Semirings play a prominent role in many areas of logic in computer science. In automata theory for example it has long been recognised that semirings provide semantics for weighted automata [KS85; Moh09; DKV09]. In database theory, they can be used to define concepts such as probabilistic or incomplete databases [GKT07]. Semiring valued CSPs have also been studied in [BMR+99]. Motivated by these applications there has recently been much attention devoted to studying semiring valued semantics for first-order logic where logical statements can take values in a commutative semiring rather than just being treated as true or false [GHNW22; Mrk20]. An interesting question is to ask what would happen if we extend this idea to define partial semiring valued semantics for first-order logic. We could then explore the semantics given by the partial semiring $\mathsf{Proj}(\boldsymbol{H})$. This would result in a natural first-order variant of the propositional version of quantum logic introduced by Kochen and Specker [KS67]. Would such a logic be helpful for "quantising" standard concepts from automata theory, databases, or CSPs? Moreover, can our partial quantum monad provide a unified categorical abstraction for such results in the same way that the monads $\mathcal{D}_{\mathbf{S}}$ do for total semirings [Jac17]?
- **Complexity at bounded dimensions:** Recall from our discussion about interactive proof systems that due to the seminal work of Slofstra [Slo19a; Slo19b] we know that identifying the existence of a quantum homomorphism $\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$ or a quantum isomorphism $\mathcal{X} \overset{\mathbf{Kl}(\dot{\mathbb{Q}}_{\boldsymbol{H}})}{\cong} \mathcal{Y}$ is an undecidable problem if there is no bound placed on the dimension of the Hilbert space $\boldsymbol{H}$. However, if we fix a finite-dimensional Hilbert space $\mathbb{C}^d$ both of these problems can be phrased in terms of solving a system of linear equations over the elements of $\mathsf{Proj}(\mathbb{C}^d)$. Solving such a system of equations is a decision problem that belongs to the complexity class $\exists\mathbb{R}$, which stands for the existential theory of reals [SCM24]. This complexity class can be seen as a real-valued analogue of NP which has recently received much attention in fields such as computational geometry and game theory. There is also an interesting connection between this class and quantum logic. Namely, it was shown in [HZ16] that for fixed dimension ≥ 3 the problems of weak and strong[4] satisfiability for Birkhoff and Neumann quantum logic is complete for $\exists\mathbb{R}$. Once again, this is a problem that is known to be undecidable in the unbounded case as

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[4] We refer to [HZ16] for a description of what "weak" and "strong" means in this context.

a corollary of Slofstra's results [Fri20b]. The result of [HZ16] can arguably be treated as a quantum version of the seminal Cook-Levin theorem [Coo71; Lev73] which shows that boolean satisfiability is NP-complete. These results hint at a potential deep connection between quantum information and $\exists\mathbb{R}$. In light of this, we ask the following questions:

Question 5.61. *Are the weak and strong satisfiability problems for Kochen and Specker quantum logic (at fixed dimension ≥ 3) $\exists\mathbb{R}$-complete?*

Question 5.62. *Is determining the existence of a quantum homomorphism (at fixed dimension $\geq$ 3) $\exists\mathbb{R}$-complete?*

Moreover, when it comes to quantum isomorphism it would be interesting to explore if the potential NP-intermediate status of classical structure isomorphism carries over to the quantum setting.

Question 5.63. *Is determining the existence of a quantum isomorphism (at fixed dimension $\geq$ 3) $\exists\mathbb{R}$-intermediate?*

- **Parameterised monads:** We would like to explore connections between our partial quantum monad $\mathbb{Q}_H$ and the notion of parameterised monad introduced by Atkey in [Atk09]. Much like specification structures parameterised monads are inspired by the idea of Hoare logic. The Kleisli category of a parameterised monad over $\mathbf{R}(\sigma)$ has objects which are pairs $(\mathcal{X}, c)$ where $\mathcal{X} \in \mathbf{R}(\sigma)$ and c is an object from a parameterising category $\mathbf{C}$. The composition of morphisms is then "guarded" by suitable pre and post-conditions related to this parameterising category. There is some similarity between this construction and our category $\mathbf{Kl}(\dot{\mathbb{Q}}_H)_\mathbf{S}$. The main difference appears to be that in our case the parameterising category $\mathbf{C}$ depends on the object $\mathcal{X}$. Thus it would be interesting to see if one can refine the definition of parameterised monads to define a parameterised quantum monad whose Kleisli category is equivalent to $\mathbf{Kl}(\dot{\mathbb{Q}}_H)_\mathbf{S}$.

- **Markov categories:** Markov categories have recently gained attention as a categorical abstraction for probability theory [Fri20a]. A well-known fact is that the Kleisli category of any monad M defined on a category with finite products is a Markov category if:

  1. The monad is commutative.
  2. $M(1) \cong 1$ where 1 is the terminal object.

  The functor $\mathbb{Q}_H$ satisfies condition (2). Moreover, we have shown that commutativity is satisfied whenever the necessary morphisms can be composed. Thus we ask if there is a way of viewing $\mathbf{Kl}(\dot{\mathbb{Q}}_H)$ as a form of "partial" Markov category[5]

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[5] It is worth noting that the term partial markov category has been used in [LR23]. However, their usage of the

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term appears quite different to ours, given that the categories they consider are in fact total categories.

6

Quantum Spoiler-Duplicator games

In this chapter, we use our characterisation of quantum homomorphism to define and begin to explore quantum versions of Spoiler-Duplicator games. Let us begin by defining partial quantum homomorphisms.

Definition 6.1. Let $\boldsymbol{H}$ be a Hilbert space. A partial quantum homomorphism in $\boldsymbol{H}$ from $\mathcal{X} \in \mathbf{R}(\sigma)$ to $\mathcal{Y} \in \mathbf{R}(\sigma)$ is given by a set $S \subseteq X$ and a family of projectors $\{P_{s,y}\}_{s \in S, y \in Y}$ in $\mathsf{Proj}(\boldsymbol{H})$ satisfying:

1. For all $x \in S : \sum_{y \in Y} P_{x,y} = I$
2. For all $x, x' \in S$ adjacent in the Gaifman graph of the substructure given by S and all $y, y' \in Y$ we have $P_{x,y} \odot P_{x'y'}$.
3. If $\boldsymbol{x} \in R^{\mathcal{X}}$ and $\boldsymbol{y} \notin R^{\mathcal{Y}}$ $P_{\boldsymbol{x},\boldsymbol{y}} = 0$.

Equivalently, a partial quantum homomorphism is a partial homomorphism $\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$.

Based on this definition we can describe "quantum" versions of the one-sided EF and pebbling games as follows:

Example 6.2. *Let $\boldsymbol{H}$ be a Hilbert space. The one-sided quantum Ehrenfeucht-Fraïssé (EF) game in $\boldsymbol{H}$ from $\mathcal{X} \in \mathbf{R}(\sigma)$ to $\mathcal{Y} \in \mathbf{R}(\sigma)$ is played as follows:*

At the start of round i, Spoiler chooses an element of $x_i \in \mathcal{X}$. Duplicator responds with a PVM $\{P_{x_i,y}\}_{y \in Y}$. After k rounds their choices create sequences $[x_1, \ldots, x_k]$ and $[\{P_{x_1}\}, \ldots, \{P_{x_k}\}]$. Duplicator wins a play of length k if this is a partial quantum homomorphism.

Example 6.3. *Let $\boldsymbol{H}$ be a Hilbert space. The one-side k-pebble game from $\mathcal{X} \in \mathbf{R}(\sigma)$ to $\mathcal{Y} \in \mathbf{R}(\sigma)$ is played as follows:*

Each player has access to k pebbles. At the start of round i, Spoiler picks one of their pebbles $p_i \in [k]$ and places it on an element $x_i \in X$. Duplicator then picks their corresponding pebble p_i and associates it to a PVM $\{P_{x_i,y}\}_{y \in Y}$. After k rounds their choices create sequences $[(p_1, x_1), \ldots, (p_k, x_k)]$ and $[(p_1, \{P_{x_1}\}), \ldots, (p_k, \{P_{x_k}\})]$. The current placing of a pebble $p \in [k]$ in each sequence is the last element with first component p. Duplicator wins a play of length k if this is a partial quantum homomorphism. We say that Duplicator wins the game if they have a strategy which allows them to play indefinitely while ensuring that they always win.

We say that Duplicator has a quantum winning strategy in a Spoiler-Duplicator game if they can win the quantum version of the game.

The goal of this chapter is to explore such games and their compositional properties. As a starting point, we have the following results which are almost immediate from the corresponding definitions and theorems 2.48, 2.57.

Proposition 6.4. *For finite relational structures $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ the following are equivalent:*

- *Duplicator has a perfect quantum strategy in the one-sided k-round quantum EF game from $\mathcal{X}$ to $\mathcal{Y}$ in a Hilbert space $\boldsymbol{H}$.*
- *Duplicator has a perfect strategy in the one-sided k-round EF game from $\mathcal{X}$ to $\mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$.*
- *There exists a biKleisli morphism $\mathbb{E}_k\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$.*

Proposition 6.5. *For finite relational structures $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ the following are equivalent:*

- *Duplicator has a perfect quantum strategy in the one-sided k-pebble game from $\mathcal{X}$ to $\mathcal{Y}$ in a Hilbert space $\boldsymbol{H}$.*
- *Duplicator has a perfect strategy in the one-sided k-pebble game from $\mathcal{X}$ to $\mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$.*
- *There exists a biKleisli morphism $\mathbb{P}_k\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$.*

Based on these results we can argue that biKleisli morphisms $\mathbb{G}_k\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$ provide us with a categorical abstraction for one-sided Quantum Spoiler-Duplicator games, just as coKleilsi morphisms $\mathbb{G}_k\mathcal{X} \to \mathcal{Y}$ provided us with a categorical abstraction for the classical games. As with the other (co)monads we have explored in previous chapters we will ask whether these biKleisli morphisms form a category in the second half of this chapter. In the case of the quantum monad $\mathbb{Q}_{\boldsymbol{H}}$ this question turns out to be even more nuanced than other distribution minion monads. We shall explore several ideas that come close to providing a positive answer to this question. Unfortunately, we shall argue that each of these approaches falls short of providing a satisfactory answer to the question. Before looking at these categorical questions we spend the first part of this chapter justifying the study of quantum Spoiler-Duplicator games by showing that quantum advantage is possible in such games.

6.1 Quantum advantage

We say that quantum advantage exists in a Spoiler-Duplicator game if Duplicator can win the quantum game when they cannot win the classical game. Let us begin this section with a simple argument showing that such games exist.

Theorem 6.6. *There exists finite relational structures $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ and a $k \in \mathbb{N}^+$ satisfying:*

- *Duplicator has a perfect quantum strategy in the one-sided EF game from $\mathcal{X}$ to $\mathcal{Y}$ in some Hilbert space $\boldsymbol{H}$.*
- *Duplicator has no perfect strategy in the one-sided EF game from $\mathcal{X}$ to $\mathcal{Y}$.*

Theorem 6.7. *There exists finite relational structures $\mathcal{X}, \mathcal{Y} \in \mathbf{R}(\sigma)$ and a $k \in \mathbb{N}^+$ satisfying:*

- *Duplicator has a perfect quantum strategy in the one-sided k-pebble game from $\mathcal{X}$ to $\mathcal{Y}$ in some Hilbert space $\boldsymbol{H}$.*
- *Duplicator has no perfect strategy in the one-sided k-pebble game from $\mathcal{X}$ to $\mathcal{Y}$.*

Proof. Take any structures $\mathcal{X}$ and $\mathcal{Y}$ which admit a quantum homomorphism in some Hilbert space $\boldsymbol{H}$ but no classical homomorphism i.e. where there exists a morphism $f : \mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$ but no morphism $\mathcal{X} \to \mathcal{Y}$. For any $k \geq |X|$ there is no morphism $\mathbb{E}_k\mathcal{X} \to \mathcal{Y}$ and no morphism $\mathbb{P}_k\mathcal{X} \to \mathcal{Y}$ since Spoiler can cover the entirety of structure $\mathcal{X}$ in k-rounds. Therefore, by theorems 2.48 and 2.57 there is no classical strategy for Duplicator in the k-round EF game or the k-pebble game. However, there are biKleisli morphism $\mathbb{E}_k\mathcal{X} \xrightarrow{\varepsilon} \mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$ and $\mathbb{P}_k\mathcal{X} \xrightarrow{\varepsilon} \mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$ where Duplicator simply plays according to the quantum homomorphism f. □

This result while interesting, is not surprising because Duplicator is using the already-established fact that quantum homomorphisms may exist even when classical homomorphisms do not. A much more important question is whether quantum advantage can exist even if there is no quantum homomorphism between $\mathcal{X}$ and $\mathcal{Y}$. Answering this question is the main goal of this section. Formally we ask:

Question 6.8. *Are there structures $\mathcal{X}, \mathcal{Y}, \in \mathbf{R}(\sigma)$, $k \in \mathbb{N}^{+}$, and a Hilbert space $\boldsymbol{H}$ such that:*

- *Duplicator has a perfect quantum strategy in the one-sided k-round EF (k-pebble) game from $\mathcal{X}$ to $\mathcal{Y}$ in $\boldsymbol{H}$.*
- *Duplicator has no perfect strategy in the one-sided k-round EF (k-pebble) game from $\mathcal{X}$ to $\mathcal{Y}$.*
- *There is no quantum homomorphism from $\mathcal{X}$ to $\mathcal{Y}$.*

We shall provide a positive answer to this question in theorems 6.14 and 6.7 below. In our proof we will make use of the Mermin-Peres magic square game [Mer90; Per90]. Let us recall the details of this game in the language of CSPs. Consider a signature σ with two ternary symbols $R_{0,3}$ and $R_{1,3}$. There exists a structure $Z_2 \in \mathbf{R}(\sigma)$ with universe {0, 1} and where:

$$R_{0,3}^{Z_2} = \{(i, j, k) | i \oplus j \oplus k = 0\}$$

$$R_{1,3}^{Z_2} = \{(i, j, k) | i \oplus j \oplus k = 1\}$$

.

Now consider a second structure $\mathcal{X} \in \mathbf{R}(\sigma)$ with universe $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ and where:

$$R_{0,3}^{\mathcal{X}} = \{(x_1, x_2, x_3), (x_4, x_5, x_6), (x_7, x_8, x_9), (x_1, x_4, x_7), (x_2, x_5, x_8)\},$$

$$R_{1,3}^{\mathcal{X}} = \{(x_3, x_6, x_9)\}.$$

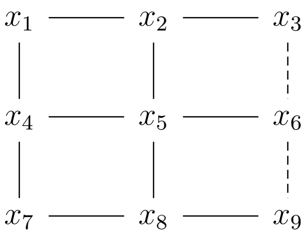
For any $\mathcal{Y} \in \mathbf{R}(\sigma)$ determining the existence of a homomorphism $\mathcal{Y} \to Z_2$ corresponds to the solvability of a binary constraint system.

Definition 6.9. A BCS consists of n binary variables $x_1, x_2, \ldots, x_n$, and m constraints, $c_1, c_2, \ldots, c_m$, where each c_j is a binary-valued function of a subset of the variables.

It is often helpful to write these constraints as equations. For example, the BCS corresponding to $\mathcal{X}$ has the following six equations.

$$\begin{array}{lll} x_1 \oplus x_2 \oplus x_3 = 0 & x_4 \oplus x_5 \oplus x_6 = 0 & x_7 \oplus x_8 \oplus x_9 = 0 \\ x_1 \oplus x_4 \oplus x_7 = 0 & x_2 \oplus x_5 \oplus x_8 = 0 & x_3 \oplus x_6 \oplus x_9 = 1 \end{array}$$

We can also represent a BCS by a graph where nodes are labelled by variables, and where variables appearing along the same straight line are part of the same constraint. A solid line means that a constraint must sum up to 0, while a dotted line means that it must sum up to 1. The BCS corresponding to $\mathcal{X}$ has the following graph.



This is why we refer to this BCS as the magic square. We now introduce the notion of quantum satisfiability for a BCS.

Definition 6.10. A quantum satisfying assignment to a BCS in a Hilbert space $\boldsymbol{H}$ is an assignment of sharp observables (i.e. self-adjoint operators) $X_1, \dots X_n$ to $x_1, \dots x_n$ which satisfy the following properties:

- Each X_i is a binary observable with eigenvalues in $\{+1, -1\}$ (i.e. $X_i^2 = I$).
- X_i commutes with X_j whenever x_i and x_j appear in the same constraint.
- For $b \in \{0, 1\}$ whenever there exists a constraint $x_1^c \oplus x_2^c \dots \oplus x_m^c = -1^b$ we have $X_1^c.X_2^c \dots X_m^c = -1^b I$.

The following are simple observations that were pointed out in [ABdSZ17].

Fact 6.11. *The following are equivalent:*

- *There exists a morphism*[a]
- *There exists a classical satisfying assignment to the BCS represented by* $\mathcal{Y}$ $\mathcal{Y} \to Z_2$.

---

[a] Note that strictly speaking if we wish to add constraints with more or less than three variables to our BCS we must add further relation symbols to our signature. For instance, to capture constraints involving two variables we would add relations $R_{0,2}$ and $R_{1,2}$ to the signature.

Fact 6.12. *The following are equivalent:*

- *There exists a morphism* $\mathcal{Y} \to \mathbb{Q}_{\boldsymbol{H}} Z_2$.
- *There exists a quantum satisfying assignment to the BCS corresponding to* $\mathcal{Y}$ *in* $\boldsymbol{H}$.

Turning our attention back to the magic square BCS, we can see that it is not classically satisfiable by summing over the left and right-hand sides of all the equations. A seminal result of Mermin and Peres [Mer90; Per90] shows that a quantum satisfying assignment does exist for the magic square game in the Hilbert space $\mathbb{C}^4$. It is given by the following assignment of operators to the variables of $\mathcal{X}$:

$$\begin{tikzcd}
\sigma_Z \otimes \sigma_I \arrow[r, dash] \arrow[d, dash] & \sigma_I \otimes \sigma_Z \arrow[r, dash] \arrow[d, dash] & \sigma_Z \otimes \sigma_Z \arrow[d, dash, dashed] \\
\sigma_I \otimes \sigma_X \arrow[r, dash] \arrow[d, dash] & \sigma_X \otimes \sigma_I \arrow[r, dash] \arrow[d, dash] & \sigma_X \otimes \sigma_X \arrow[d, dash, dashed] \\
\sigma_Z \otimes \sigma_X \arrow[r, dash] & \sigma_X \otimes \sigma_Z \arrow[r, dash] & \sigma_Y \otimes \sigma_Y
\end{tikzcd}$$

Where $\sigma_X, \sigma_Y, \sigma_Z, \sigma_I$ are the well-known Pauli matrices given by

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Our discussion so far shows the following:

Theorem 6.13. *for $\mathbb{G}_k \in \{\mathbb{E}_k, \mathbb{P}_k\}$ we have the following*

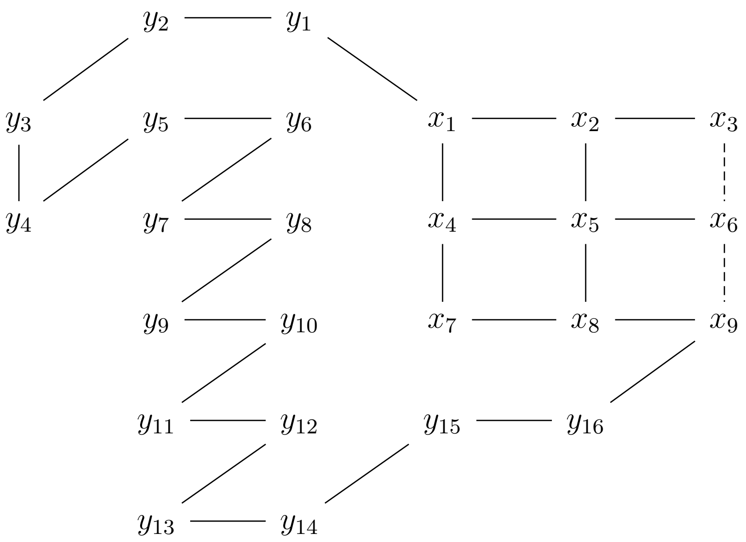
- $\mathcal{X} \not\to Z_2$.
- $\mathcal{X} \to \mathbb{Q}_H Z_2$.
- $\mathbb{G}_9\mathcal{X} \not\to Z_2$
- $\mathbb{G}_9\mathcal{X} \to \mathbb{Q}_{\mathbb{C}^4} Z_2$.

Proof. The first two items have already been shown in our discussion above. The second two items follow via the same proof as in theorems 6.6 and 6.7. ☐

Now, let us consider an alternative structure $\mathcal{X}^+$ where $X^+ = X \cup \{y_1, \dots, y_{16}\}$ and the following constraints are added those in $\mathcal{X}$:

$$\begin{aligned}
x_1 \oplus y_1 &= 0 \\
y_1 \oplus y_2 &= 0 \\
y_2 \oplus y_3 &= 0 \\
&\dots \\
y_{15} \oplus y_{16} &= 0 \\
y_{16} \oplus x_9 &= 0
\end{aligned}$$

The BCS for $\mathcal{X}^+$ is graphically given by:



We now prove the following theorem which provides a positive answer to question 6.8 for the EF game.

Theorem 6.14. *The following claims are true:*

1. $\mathcal{X}^+ \not\to Z_2$.
2. $\mathcal{X}^+ \not\to \mathbb{Q}_H Z_2$.
3. $\mathbb{E}_9 \mathcal{X}^+ \not\to Z_2$
4. $\mathbb{E}_9 \mathcal{X}^+ \to \mathbb{Q}_{\mathbb{C}^4} Z_2$.

Proof of theorem 6.14.

1. This simply follows from the fact that the subset of constraints corresponding to those of $\mathcal{X}$ are not classically satisfiable.
2. Consider any constraint of the form $a \oplus b = 0$. In a quantum satisfying assignment, we must assign operators A and B to the variables a and b respectively such that $AB = I$. Using the fact that $B^2 = I$ we have

$$AB = I \Rightarrow ABB = B \Rightarrow A = B.$$

Using the above in any quantum satisfying assignment we must have

$$X_1 = Y_1 = Y_2 = \ldots = Y_{16} = X_9$$

.

We now wish to derive a contradiction by showing $X_1 \neq X_9$. We propose two alternative proofs. The first is to note that the perfect quantum strategy for the magic square game is a self-test [ŠB20; WBMS16]. This means that the perfect quantum strategy for the game is unique up to local isometry. Thus, in any such strategy, we must have $X_1 \neq X_9$. Since $\mathcal{X}^+$ contains the magic square game as a substructure the same fact is true here. Therefore we have a contradiction.

Since we have not introduced self-testing in detail in this thesis let us also provide a more direct proof via the substitution method attributed to Speelman [CM14]. The idea here is to derive a contradiction $I = -I$. We first show that $X_2 = -X_6$.

$$\begin{aligned}
&X_1X_2X_3 = I && \\
&X_2X_1X_3 = I && \text{(commutativity from } x_1 \oplus x_2 \oplus x_3 = 0\text{)}\\
&X_2X_1X_9X_6 = -I && (X_3 = -X_6X_9 \text{ from } x_3 \oplus x_6 \oplus x_9 = 1)\\
&X_2X_1X_1X_6 = -I && (X_1 = X_9)\\
&X_2X_6 = -I && (X_1^2 = I)\\
&X_2X_6X_6 = -X_6 && \\
&X_2 = -X_6 && (X_6^2 = I)
\end{aligned}$$

A very similar argument shows $X_4 = X_8$. Next we show $X_3X_5X_7 = -I$.

$$\begin{aligned}
&X_3X_5X_7 = X_3X_4X_6X_7 && (X_5 = X_4X_6 \text{ from } x_4 \oplus x_5 \oplus x_6 = 0)\\
&X_3X_5X_7 = -X_3X_8X_2X_7 && (X_2 = -X_6 \text{ and } X_4 = X_8)\\
&X_3X_5X_7 = -X_3X_2X_8X_7 && \text{(commutativity from } x_2 \oplus x_5 \oplus x_8 = 0\text{)}\\
&X_3X_5X_7 = -X_3X_2X_8X_8X_9 && (X_7 = X_8X_9 \text{ from } x_7 \oplus x_8 \oplus x_9 = 0)\\
&X_3X_5X_7 = -X_3X_2X_9 && (X_8^2 = I)\\
&X_3X_5X_7 = -X_3X_2X_1 && (X_1 = X_9)\\
&X_3X_5X_7 = -I && \text{(from } x_1 \oplus x_2 \oplus x_3 = 0\text{)}
\end{aligned}$$

Next, we obtain a contradiction.

$$
\begin{aligned}
&X_1X_3X_2 = I && \text{(from } x_1 \oplus x_2 \oplus x_3 = 0\text{)}\\
&X_1X_3X_5X_8 = I && (X_2 = X_5X_8 \text{ from } x_2 \oplus x_5 \oplus x_8 = 0)\\
&X_1X_3X_5X_7X_9 = I && (X_8 = X_7X_9 \text{ from } x_7 \oplus x_8 \oplus x_9 = 0)\\
&X_1(-I)X_9 = I && (X_3X_5X_7 = -I)\\
&-X_1X_1 = I && (X_1 = X_9)\\
&-I = I && (X_1^2 = I)
\end{aligned}
$$

3. We describe a winning strategy for Spoiler. Consider the play of the game where Spoiler picks the elements $[x_1, \ldots, x_9]$. The substructure determined by this play is $\mathcal{X}$. As there is no homomorphism from $\mathcal{X} \to Z_2$ Duplicator cannot respond with a partial homomorphism for this substructure.

4. We describe a winning strategy for Duplicator. In this strategy whenever Spoiler picks an element $x_i \in \mathcal{X}^+$ for $i \in [9]$ Duplicator simply replies with the operator X_i from the winning strategy for the magic square game. When Spoiler picks an element $y_j \in \mathcal{X}^+$ for $j \in [8]$ Duplicator responds according to the following case analysis:

   - If no element that is adjacent to y_j in the graphical representation of the BCS has previously been picked by Spoiler, the response is X_1.
   - Otherwise Duplicator responds with the same operator that they responded with when the adjacent element was picked.

   Similarly if Spoiler picks an element $y_j \in \mathcal{X}^+$ for $j \in [9, 16]$ Duplicator performs the following case analysis:

   - If no element that is adjacent to y_j in the graphical representation of the BCS has previously been picked by Spoiler, the response is X_9.
   - Otherwise Duplicator responds with the same operator that they responded with when the adjacent element was picked.

   This strategy ensures that when Spoiler plays are contained within the set $\{x_1, \ldots, x_9\}$ Duplicator responds according to the winning quantum satisfying assignment from the magic square and thus satisfies all constraints involving the picked variables.

   Moreover, if Spoiler activates any of the additional constraints of the form $a \oplus b$ Duplicator's strategy guarantees that for the operators assigned to a and b, we have $A = B$.

The final point to note is that there is no contradiction in the strategy because in 9 rounds Spoiler never has enough time to force the equality $X_1 = Y_1 \dots Y_{16} = X_9$. The closest they can get to achieving this goal is with a play similar to $[y_9, y_8, y_7, y_6, y_5, y_4, y_3, y_2, y_1]$ to which Duplicator would respond with $[X_9, X_9, X_9, X_9, X_9, X_9, X_9, X_9, X_9]$. If Spoiler had one more move they could then play x_1 and Duplicator's response of X_1 would lead to failure of the constraint $x_1 \oplus y_1 = 0$. Thus, Spoiler requires 10 rounds to beat the strategy we have described for Duplicator.

□

Observe that this theorem actually proves a stronger statement than what was asked in question 6.8. We have identified a game where a quantum homomorphism does not exist in any Hilbert space, but where Duplicator can nevertheless exhibit quantum advantage in the one-sided 9-round EF game. We note that it is likely possible to show a similar result with a substructure of $\mathcal{X}^+$, however, this would result in a more complicated proof. It would be an interesting direction for future work to identify the smallest structure which exhibits this form of quantum advantage.

Now we turn our attention to the k-pebble game. Our first observation is that the analogue of the above theorem does not hold for 9 pebbles.

Proposition 6.15. *The followinga facts are true:*

1. $\mathcal{X}^+ \not\to Z_2$.
2. $\mathcal{X}^+ \not\to \mathbb{Q}_{\boldsymbol{H}} Z_2$.
3. $\mathbb{P}_9 M \not\to Z_2$
4. $\mathbb{P}_9 M \not\to \mathbb{Q}_{\mathbb{C}^4} Z_2$.

Proof. The first two items are the same as in theorem 6.6.

3. We describe a winning strategy for Spoiler. They place their 9 pebbles on elements $\{x_1, \dots, x_9\}$. Then all the constraints from the magic square game are activated and there is no classical response Duplicator can give.
4. We describe a winning strategy for Spoiler. Spoiler first uses three of their pebbles to force the equality $X_1 = X_9$. This is achieved by placing their first pebble on x_1. Then a second pebble is placed on y_1. This forces $X_1 = Y_1$ in Duplicator's responses. Then a third pebble is placed on y_2 which forces $X_1 = Y_1 = Y_2$. Now the pebble placed on y_1 is removed and placed on y_3 which forces $X_1 = Y_2 = Y_3$. Then the pebble on y_2

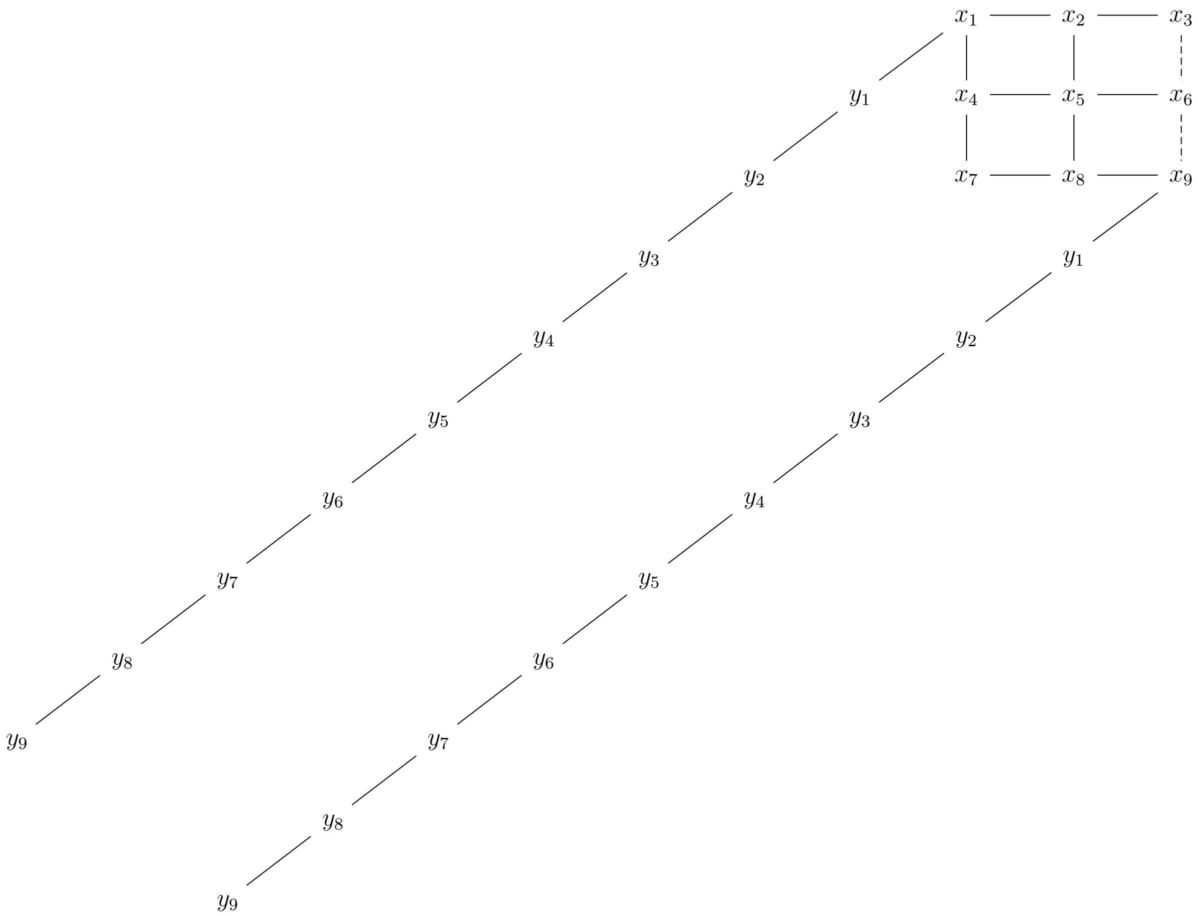
is removed and placed on y_4 and this process is continued until the three pebbles are placed on x_1, y_{16}, x_9. This forces the equality $X_1 = Y_{16} = X_9$. From this point, Spoiler keeps the two pebbles on x_1 and x_9 and uses the remaining 7 pebbles on the elements $x_2, \ldots, x_8$ in any order. Since no strategy wins the magic square game with $X_1 = X_9$ Duplicator cannot satisfy all the constraints.

□

Thus to solve of question 6.8 for the pebble game we consider yet another structure $\mathcal{X}^*$ with universe $X^* = X \cup \{y_1, \ldots, y_8\}$ where the following contraints are added to those in $\mathcal{X}$:

$$x_1 \oplus y_1 + \ldots \oplus y_9 = 0$$
$$x_9 \oplus y_1 + \ldots \oplus y_9 = 0$$

Graphically, the BCS for $\mathcal{X}^*$ is as follows[1]:



Now with regards to $\mathcal{X}^*$ we can prove the following theorem, which provides a positive answer to question 6.8 in the case of $\mathbb{P}_k$.

---

[1] Note that with some abuse of notation we are drawing multiple nodes for some of the variables in $\mathbf{X}^*$.

Theorem 6.16. *The following claims are true:*

1. $\mathcal{X}^* \not\to Z_2$.
2. $\mathcal{X}^* \not\to \mathbb{Q}_{\boldsymbol{H}} Z_2$.
3. $\mathbb{P}_9 \mathcal{X}^* \not\to Z_2$
4. $\mathbb{P}_9 \mathcal{X}^* \to \mathbb{Q}_{\mathbb{C}^4} Z_2$.

Proof.

1. This is the same statement as in theorem 6.14.
2. We show that in any quantum satisfying assignment, we have $X_1 = X_9$. The remainder of the proof is then as in theorem 6.6. Note that $y_1, \dots y_9$ appear in the same constraint therefore the operators assigned to them in a quantum satisfying assignment must be jointly measurable. Therefore $Y_1 \dots Y_9$ is itself a binary observable with eigenvalues $\{+1, -1\}$. Hence we have $(Y_1 \dots Y_9)^2 = I$. Using this fact we obtain:

   $$\begin{aligned} &X_1.Y_1 \dots Y_9 = I \\ &X_1.(Y_1 \dots Y_9).(Y_1 \dots Y_9) = Y_1 \dots Y_9 \\ &X_1 = (Y_1 \dots Y_9) \end{aligned}$$

   And similarly:

   $$\begin{aligned} &X_9.Y_1 \dots Y_9 = I \\ &X_9.(Y_1 \dots Y_9).(Y_1 \dots Y_9) = Y_1 \dots Y_9 \\ &X_9 = (Y_1 \dots Y_9) \end{aligned}$$

3. Consider a strategy where Spoiler places 9 pebbles on $\{x_1, \dots, x_9\}$ in any order. The substructure determined by this is $\mathcal{X}$. As there is no morphism $\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}} Z_2$ Duplicator cannot form a partial quantum homomorphism.
4. We describe a perfect quantum strategy for Duplicator. Whenever Spoiler places a pebble on an element in $\{x_1, \dots, x_9\}$ Duplicator responds with the operator X_i from the winning strategy for the magic square game.

   If Spoiler places a pebble on $\{y_1, \dots, y_8, y_9\}$ Duplicator simply responds with the identity operator. This is a valid strategy since with only 9 pebbles Spoiler's window can never cover enough of the structure $\mathcal{X}^*$ to activate either of the two constraints involving the y_i. Thus there is no way Spoiler can enforce the equality $X_1 = X_9$.

□

6.2 Attempts at a biKleisli category

We now ask if there is a distributive law of the form $\mathbb{G}_k\mathbb{Q}_H \to \mathbb{Q}_H\mathbb{G}_k$ which results in a biKleisli category where Duplicator's winning strategies in quantum games can be composed. To save some space we restrict our attention in this section to $\mathbb{E}_k$ and the graded monad viewpoint on the functor $\mathbb{Q}_H$. These capture the essence of all our arguments and can straightforwardly be adapted to $\mathbb{P}_k$ and the partial monad viewpoint on $\mathbb{Q}_H$.

6.2.1 Graded distributive laws

Since $\mathbb{Q}_H$ is a graded monad we must first clarify what it means for a comonad to distribute over a graded monad.

Definition 6.17. Given a monoid $\mathbf{N} = (N, ., 1)$. A mixed graded distributive law of a comonad (W, ε, δ) over a $\mathbf{N}$-graded monad $(\{M_n\}_{n\in N}, \{\mu^{n,n'}\}_{n,n'\in N}, \eta)$ is a family of natural transformations $\kappa^n : W \circ M_n \to M_n \circ W$, satisfying the following equations:

$$\kappa^1 \circ W\eta = \eta W \tag{unit}$$

$$M_n\epsilon \circ \kappa^n = \epsilon M_n \tag{counit}$$

$$\kappa^{n.n'} \circ W\mu^{n,n'} = \mu^{n,n'}W \circ M_n\kappa^n \circ \kappa^{n'}M_{n'} \tag{multiplication}$$

$$M_n\delta \circ \kappa^n = \kappa^n W \circ W\kappa^n \circ \delta M_n \tag{comultiplication}$$

Remark 6.18. *Distributive laws of graded comonads over graded monads are defined in [GKO+16]. Our definition coincides with theirs in the case where the grading placed on the comonad is trivial, and the grading on the monad is* $\mathbf{N}$ *(considered with the equality preorder).*

Given a graded distributive law we can define a graded biKleisli category.

Definition 6.19. Let $\kappa : WM_d \to M_dW$ be a graded mixed distributive law of (W, ε, δ) over $(\{M_d\}_d, \eta, \{\mu^{d,d'}\}_{d,d'})$. Then:

- There is a comonad $(\bar{W}, \varepsilon^{\bar{W}}, \delta^{\bar{W}})$ on $\mathbf{Kl}(M_d)$ where:
  - $\bar{W}X = WX$
  - given $f : X \to M_dY$, $\bar{W}f = \kappa_Y \circ Wf$
  - $\varepsilon^{\bar{W}}_X = \eta_X \circ \varepsilon_X$
  - $\delta^{\bar{W}}_X = \eta_{W^2X} \circ \delta_X$

- There is a graded monad $(\{\bar{M}_d\}_d, \eta^{\bar{M}}, \{\mu^{\bar{M}^{d,d'}}\}_{d,d'})$ on $\mathbf{coKl}(W)$ where:
  - $\bar{M}_d X = M_d X$
  - given $f : WX \to Y$, $\bar{M}_d f = M_d f \circ \kappa_X$
  - $\eta^{\bar{M}}_X = \eta_X \circ \varepsilon_X$
  - $\mu^{\bar{M}^{d,d'}}_X = \mu^{d,d'}_X \circ \varepsilon_{M_d M_{d'} X}$.

Moreover, the (co)Kleisli categories of these constructions coincide, resulting in a so-called biKleisli category $\mathbf{biKl}(W, M_d)$ where:

- $\mathbf{biKl}(W, M_d)_0 = \mathbf{C}_0$
- A morphism $X \to Y$ is a morphism of type $WA \to M_d B$ in $\mathbf{C}$.
- The identity morphism on X is given by $\eta_X \circ \varepsilon_X$.
- The composition of $f : WX \to M_d Y$ with $g : WY \to M_{d'} Z$ is given by $\mu^{d,d'}_Z \circ M_d g \circ \kappa_Y \circ Wf \circ \delta_X$.

Thus we have $\mathbf{coKl}(\bar{W}) \cong \mathbf{Kl}(\bar{M}_d) \cong \mathbf{biKl}(W, M_d)$.

6.2.2 A distributive law at the level of sets

Let us point out the first peculiar feature of the graded quantum monad. In contrast to the no-go results mentioned in chapter 3 we shall show that at the level of sets, there does exist a valid graded distributive law of the non-empty list comonad L^+_k over the graded version of the quantum monad $Q_{\boldsymbol{H}}$.

Theorem 6.20. *There exists a graded distributive law of the comonad* $(L^+_k, \varepsilon, \delta)$ *over the graded quantum monad* $(Q_{\boldsymbol{H}}, \eta, \mu)$.

At first glance, it appears that we should be able to prove that no such law exists by adapting our no-go result from chapter 3. However, the fact that we are now dealing with a family of natural transformations rather than a single transformation turns out to be significant. As long as no coherence axioms are broken we have the flexibility of defining κ^n to behave in drastically different ways depending on the value of n. This extra flexibility allows us to construct a valid distributive law. In particular, consider the following family of natural transformations:

$$\kappa^{\mathbb{C}}[(I_{\mathbb{C}} x_1), \ldots, (I_{\mathbb{C}} x_n)] = I_{\mathbb{C}}[x_1, \ldots, x_n]$$

$$\kappa^{\boldsymbol{H}}[\varphi_1, \dots, \varphi_n] = \sum_{x \in \mathsf{supp}(\varphi_n)} \varphi_n(x).[x].$$

We shall prove that this forms a valid distributive law.

Proof of theorem 6.20. We check the four mixed graded distributive law axioms.

- Unit:

$$\begin{aligned}
&\kappa^{\mathbb{C}} \circ L_k^+ \eta[x_1, \dots, x_n] \\
&= \kappa^{\mathbb{C}}[I_{\mathbb{C}} x_1, \dots, I_{\mathbb{C}} x_n] \\
&= I_{\mathbb{C}}[x_1, \dots, x_n] \\
&= \eta L_k^+[x_1, \dots, x_n]
\end{aligned}$$

- Counit ($H = \mathbb{C}$ case):

$$\begin{aligned}
&Q_{\mathbb{C}} \varepsilon \circ \kappa^{\mathbb{C}}[I_{\mathbb{C}} x_1, \dots, I_{\mathbb{C}} x_n] \\
&= Q_{\mathbb{C}} \varepsilon I_{\mathbb{C}}[x_1, \dots, x_n] \\
&= I_{\mathbb{C}} x_n \\
&= \varepsilon Q_{\mathbb{C}}[I_{\mathbb{C}} x_1, \dots, I_{\mathbb{C}} x_n]
\end{aligned}$$

- Counit (remaining cases):

$$\begin{aligned}
&Q_{\boldsymbol{H}} \varepsilon \circ \kappa^{\boldsymbol{H}}[\varphi_1, \dots, \varphi_n] \\
&= Q_{\boldsymbol{H}} \varepsilon \sum_{x \in \mathsf{supp}(\varphi_n)} \varphi_n(x).[x] \\
&= \varphi_n \\
&= \varepsilon Q_{\boldsymbol{H}}[\varphi_1, \dots, \varphi_n]
\end{aligned}$$

- Comultiplication ($H = \mathbb{C}$ case):

$$\begin{aligned}
&Q_{\mathbb{C}} \delta \circ \kappa^{\mathbb{C}}[I_{\mathbb{C}} x_1, \dots, I_{\mathbb{C}} x_n] \\
&= Q_{\mathbb{C}} \delta I_{\mathbb{C}}[x_1, \dots, x_n] \\
&= I_{\mathbb{C}}[[x_1], \dots, [x_1, \dots, x_n]]
\end{aligned}$$

$$\kappa^{\mathbb{C}} L_k^+ \circ L_k^+ \kappa^{\mathbb{C}} \circ \ \delta Q_{\mathbb{C}}[I_{\mathbb{C}} x_1, \dots, I_{\mathbb{C}} x_n]$$

$$
\begin{aligned}
&= \kappa^{\mathbb{C}} L_k^+ \circ L_k^+ \kappa^{\mathbb{C}}[[I_{\mathbb{C}}x_1], \ldots, [I_{\mathbb{C}}x_1, \ldots, I_{\mathbb{C}}x_n]] \\
&= \kappa^{\mathbb{C}} L_k^+ [I_{\mathbb{C}}[x_1], \ldots, I_{\mathbb{C}}[x_1, \ldots, x_n]] \\
&= I_{\mathbb{C}}[[x_1], \ldots, [x_1, \ldots, x_n]]
\end{aligned}
$$

- Comultiplication (remaining cases):

$$
\begin{aligned}
& Q_{\boldsymbol{H}}\delta \circ \kappa^{\boldsymbol{H}}[\varphi_1, \ldots, \varphi_n] \\
&= Q_{\boldsymbol{H}}\delta \sum_{x \in \mathsf{supp}(\varphi_n)} \varphi_n(x).[x] \\
&= \sum_{x \in \mathsf{supp}(\varphi_n)} \varphi_n(x).[[x]]
\end{aligned}
$$

$$
\begin{aligned}
& \kappa^{\boldsymbol{H}} L_k^+ \circ L_k^+ \kappa^{\boldsymbol{H}} \circ \; \delta Q_{\boldsymbol{H}}[\varphi_1, \ldots, \varphi_n] \\
&= \kappa^{\boldsymbol{H}} L_k^+ \circ L_k^+ \kappa^{\boldsymbol{H}}[[\varphi_1], \ldots, [\varphi_1, \ldots, \varphi_n]] \\
&= \kappa^{\boldsymbol{H}} L_k^+[\sum_{x \in \mathsf{supp}(\varphi_1)} \varphi_1(x).[x], \ldots, \sum_{x \in \mathsf{supp}(\varphi_n)} \varphi_n(x).[x]] \\
&= \sum_{x \in \mathsf{supp}(\varphi_n)} \varphi_n(x).[[x]]
\end{aligned}
$$

- Multiplication ($H, H' = \mathbb{C}$ case):

$$
\begin{aligned}
& \kappa^{\mathbb{C}} \circ L_k^+ \mu^{\mathbb{C},\mathbb{C}}[I_{\mathbb{C}}(I_{\mathbb{C}}x_1), \ldots I_{\mathbb{C}}(I_{\mathbb{C}}x_n)] \\
&= \kappa^{\mathbb{C}}[I_{\mathbb{C}}x_1, \ldots, I_{\mathbb{C}}x_n] \\
&= I_{\mathbb{C}}[x_1, \ldots, x_n]
\end{aligned}
$$

$$
\begin{aligned}
& \mu^{\mathbb{C},\mathbb{C}} L_k^+ \circ Q_{\mathbb{C}}\kappa^{\mathbb{C}} \circ \kappa^{\mathbb{C}} Q_{\mathbb{C}}[I_{\mathbb{C}}(I_{\mathbb{C}}x_1), \ldots I_{\mathbb{C}}(I_{\mathbb{C}}x_n)] \\
&= \mu^{\mathbb{C},\mathbb{C}} L_k^+ \circ Q_{\mathbb{C}}\kappa^{\mathbb{C}} I_{\mathbb{C}}[I_{\mathbb{C}}x_1, \ldots I_{\mathbb{C}}x_n] \\
&= \mu^{\mathbb{C},\mathbb{C}} L_k^+ I_{\mathbb{C}}(I_{\mathbb{C}}[x_1, \ldots x_n]) \\
&= I_{\mathbb{C}}[x_1, \ldots x_n]
\end{aligned}
$$

- Multiplication ($H = \mathbb{C}$ case):

$$
\begin{aligned}
& \kappa^{\boldsymbol{H'}} \circ L_k^+ \mu^{\mathbb{C},\boldsymbol{H'}}[I_{\mathbb{C}}(\varphi_1), \ldots, I_{\mathbb{C}}(\varphi_n)] \\
&= \kappa^{\boldsymbol{H'}}[\varphi_1, \ldots, \varphi_n]
\end{aligned}
$$

$$= \sum_{x \in \mathsf{supp}(\varphi_n)} \varphi_n(x).[x]$$

$$\begin{aligned}
&\mu^{\mathbb{C},\boldsymbol{H'}} L_k^+ \circ Q_{\mathbb{C}} \kappa^{\boldsymbol{H'}} \circ \kappa^{\mathbb{C}} Q_{\boldsymbol{H'}} [I_{\mathbb{C}}(\varphi_1), \dots, I_{\mathbb{C}}(\varphi_n)] \\
&= \mu^{\mathbb{C},\boldsymbol{H'}} L_k^+ \circ Q_{\mathbb{C}} \kappa^{\boldsymbol{H'}} I_{\mathbb{C}}[\varphi_1, \dots, \varphi_n] \\
&= \mu^{\mathbb{C},\boldsymbol{H'}} L_k^+ I_{\mathbb{C}}[\sum_{x \in \mathsf{supp}(\varphi_n)} \varphi_n(x).[x]] \\
&= \sum_{x \in \mathsf{supp}(\varphi_n)} \varphi_n(x).[x]
\end{aligned}$$

- Multiplication ($H' = \mathbb{C}$ case):

$$\begin{aligned}
&\kappa^{\boldsymbol{H}} \circ L_k^+ \mu^{\boldsymbol{H},\mathbb{C}} [\sum_{i=1}^{m_1} P_i^1(I_{\mathbb{C}} x_i^1), \dots \sum_{i=1}^{m_n} P_i^n(I_{\mathbb{C}} x_i^n)] \\
&= \kappa^{\boldsymbol{H}} \circ [\sum_{i=1}^{m_1} (P_i^1 \otimes I_{\mathbb{C}}) x_i^1, \dots \sum_{i=1}^{m_n} (P_i^n \otimes I_{\mathbb{C}}) x_i^n] \\
&= \sum_{i=1}^{m_n} (P_i^n \otimes I_{\mathbb{C}})[x_i^n]
\end{aligned}$$

$$\begin{aligned}
&\mu^{\boldsymbol{H},\mathbb{C}} L_k^+ \circ Q_{\boldsymbol{H}} \kappa^{\mathbb{C}} \circ \kappa^{\boldsymbol{H}} Q_{\mathbb{C}} [\sum_{i=1}^{m_1} P_i^1(I_{\mathbb{C}} x_i^1), \dots \sum_{i=1}^{m_n} P_i^n(I_{\mathbb{C}}, x_i^n)] \\
&= \mu^{\boldsymbol{H},\mathbb{C}} L_k^+ \circ Q_{\boldsymbol{H}} \kappa^{\mathbb{C}} \sum_{i=1}^{m_n} P_i^n[(I_{\mathbb{C}}, x_i^n)] \\
&= \mu^{\boldsymbol{H},\mathbb{C}} L_k^+ \sum_{i=1}^{m_n} P_i^n(I_{\mathbb{C}}[x_i^n]) \\
&= \sum_{i=1}^{m_n} P_i^n[x_i^n]
\end{aligned}$$

- Multiplication (remaining cases):

$$\begin{aligned}
&\kappa^{\boldsymbol{H} \otimes \boldsymbol{H'}} \circ L_k^+ \mu^{\boldsymbol{H},\boldsymbol{H'}} [\sum_{i=1}^{m_1} P_i^1(\sum_{j=1}^{l_{1,i}} R_j^{1,i} x_j^{1,i}), \dots, \sum_{i=1}^{m_n} P_i^n(\sum_{j=1}^{l_{n,i}} R_j^{n,i} x_j^{n,i})] \\
&= \kappa^{\boldsymbol{H} \otimes \boldsymbol{H'}} [\sum_{i=1}^{m_1} \sum_{j=1}^{l_{1,i}} (P_i^1 \otimes R_j^{1,i}) x_j^{1,i}, \dots, \sum_{i=1}^{m_n} \sum_{j=1}^{l_{n,i}} (P_i^n \otimes R_j^{n,i}) x_j^{n,i}]
\end{aligned}$$

$$= \sum_{i=1}^{m_n} \sum_{j=1}^{l_{n,i}} (P_i^n \otimes R_j^{n,i})[x_j^{n,i}]$$

$$\mu^{\mathbf{H},\mathbf{H}'} L_k^+ \circ Q_{\mathbf{H}} \kappa^{\mathbf{H}'} \circ \kappa^{\mathbf{H}} Q_{\mathbf{H}'} [\sum_{i=1}^{m_1} P_i^1 (\sum_{j=1}^{l_{1,i}} R_j^{1,i} x_j^{1,i}), \dots, \sum_{i=1}^{m_n} P_i^n (\sum_{j=1}^{l_{n,i}} R_j^{n,i} x_j^{n,i})]$$

$$= \mu^{\mathbf{H},\mathbf{H}'} L_k^+ \circ Q_{\mathbf{H}} \kappa^{\mathbf{H}'} \sum_{i=1}^{m_n} P_i^n [\sum_{j=1}^{l_{n,i}} R_j^{n,i} x_j^{n,i}]$$

$$= \mu^{\mathbf{H},\mathbf{H}'} \sum_{i=1}^{m_n} P_i^n (\sum_{j=1}^{l_{n,i}} R_j^{n,i} [x_j^{n,i}])$$

$$= (\sum_{i=1}^{m_n} P_i^n \otimes \sum_{j=1}^{l_{n,i}} R_j^{n,i})[x_j^{n,i}]$$

□

Based on this result we can form biKleisli categories with morphisms of the form $L_k^+ X \to Q_{\mathbf{H}} Y$. However, the shortcoming of this approach is highlighted by our next result, which shows that the components of the transformations we have considered are not valid morphisms in $\mathbf{R}(\sigma)$.

Theorem 6.21. *The family of transformations* $\kappa^{\mathbf{H}} : \mathbb{E}_k \mathbb{Q}_{\mathbf{H}} \to \mathbb{Q}_{\mathbf{H}} \mathbb{E}_k$ *defined componentwise as in the previous theorem is not a graded distributive law of* $\mathbb{E}_k$ *over* $\mathbb{Q}_{\mathbf{H}}$ *as long as* σ *contains a relation of arity* ≥ 2.

Proof. We show that for any $\mathbf{H} \neq \mathbb{C}$, $\kappa_{\mathcal{X}}^{\mathbf{H}}$ fails to be a morphism in $\mathbf{R}(\sigma)$. We prove this for the case where σ consists of a single binary relation. A similar proof works for any σ which has a relation R of arity ≥ 2.

Take $([Ia, Ib], [Ia]) \in R^{\mathbb{E}_k \mathbb{Q}_{\mathbf{H}} \mathcal{X}}$. We show that $(I[b], I[a]) \notin R^{\mathbb{Q}_{\mathbf{H}} \mathbb{E}_k \mathcal{X}}$. To see this, first note that due to condition (1) in the definition of $R^{\mathbb{E}_k \mathcal{X}}$ (as described in example 2.47) we have $([b], [a]) \notin R^{\mathbb{E}_k \mathcal{X}}$. It then follows from Condition (2) of the definition of $R^{\mathbb{Q}_d \mathcal{X}}$ (as described in example 5.44) that for a tuple $(\varphi_1, \varphi_2) \in R^{\mathbb{Q}_d \mathbb{E}_k \mathcal{X}}$ we must have $\varphi_1([b]).\varphi_2([a]) = 0$. This is clearly not the case for $(I[b], I[a])$. □

Thus, if we are to find a distributive law in $\mathbf{R}(\sigma)$ more sophisticated transformations must be considered.

6.2.3 An almost correct distributive law in $\mathbf{R}(\sigma)$

Let us now recall the positive result derived in theorem 3.42. This shows the existence of a distributive law $\kappa : \mathbb{E}_k\mathbb{D}_{\mathbf{S}} \to \mathbb{D}_{\mathbf{S}}\mathbb{E}_k$ for any distribution minion monad $\mathbb{D}_{\mathbf{S}}$ whose underlying semiring is commutative and satisfies the conditions of idempotence and orthogonality of distributions. As we have pointed out in chapter 5 the quantum monad $\mathbb{Q}_{\boldsymbol{H}}$ can be viewed as a distribution minion monad over the partial commutative semiring $\mathsf{Proj}(\boldsymbol{H})$. Moreover, as distributions in this case coincide with PVMs the conditions of orthogonality and idempotence are also satisfied by $\mathsf{Proj}(\boldsymbol{H})$. Thus, one might expect that transformations similar to the ones we used to prove theorems 3.42 can also be used to prove the existence of a distributive law $\kappa^{\boldsymbol{H}} : \mathbb{E}_k\mathbb{Q}_{\boldsymbol{H}} \to \mathbb{Q}_{\boldsymbol{H}}\mathbb{E}_k$. There is but one problem with this idea, which arises from the partiality of $\mathsf{Proj}(\boldsymbol{H})$. To explain this in more detail let us consider the transformation $\kappa^{\boldsymbol{H}}$ given by

$$\kappa^{\boldsymbol{H}}_{\mathcal{X}}[\varphi_1, \dots, \varphi_n] = \sum_{[x_1,\dots,x_n] \in \mathbb{E}_k(X)} \boldsymbol{\varphi}(\boldsymbol{x}).[x_1, \dots, x_n].$$

While it is true that this function, whenever well-defined, satisfies the axioms of a distributive law, the problem is that in the case of $\mathbb{Q}_{\boldsymbol{H}}$ partiality means that this transformation is only a valid function whenever the additional condition $\varphi_i \odot \varphi_j$ is satisfied for all $i, j \in [n]$. These considerations led us to introduce the concept of commeasurable structures in joint work with Nihil Shah.

6.2.4 Commeasurable structures

We now describe commeasurable structures, and show that such structures form a category $\mathbf{R}^{\odot}(\sigma)$ which admits $\mathbf{R}(\sigma)$ as a coreflective subcategory. We then define variants of $\mathbb{E}_k$ and $\mathbb{Q}_{\boldsymbol{H}}$ on this category which we denote by $E_k^{\odot}$ and $\mathbb{Q}_{\boldsymbol{H}}^{\odot}$ and discuss how their (co)Kleisli maps correspond to perfect Duplicator strategies in commeasurable variants of Spoiler-Duplicator games and non-local CSP games respectively. We shall show that valid distributive laws exist between $\mathbb{E}_k^{\odot}$ and $\mathbb{Q}_{\boldsymbol{H}}^{\odot}$ and analyse the biKleisli categories that these laws give rise to.

A reflexive graph is a graph where each vertex v has a distinguished self-loop $e\colon v \to v$. In the spirit of [KS67] we will refer to the edge relation of reflexive graphs as commeasurability. We write $\odot$ for this commeasurability relation. Reflexive graphs together with graph homomorphisms form a category **RGraph**.

Commeasurable structures are generalisations of relational structures where the underlying universe can be thought of as a reflexive graph rather than a set. Two elements $x_1 \in \mathcal{X}^{\odot}$ and $x_2 \in \mathcal{X}^{\odot}$ can only appear in the same relation if $x_1 \odot x_2$. Commeasurable structures form a category $\mathbf{R}^{\odot}(\sigma)$, whose morphisms must preserve $\odot$ in addition to R. Importantly, there exists a forgetful functor $U\colon \mathbf{R}^{\odot}(\sigma) \to \mathbf{R}(\sigma)$ which simply forgets the commeasurability relationship.

This functor has a right adjoint $R\colon \mathbf{R}(\sigma) \to \mathbf{R}^{\odot}(\sigma)$ which sends a relational structure $\mathcal{X}$ to the commeasurable structure which has the same relations as $\mathcal{X}$, and whose universe is the full reflexive graph over the elements of $\mathcal{X}$. We can verify that $UR = \mathbf{Id}_{\mathbf{R}(\sigma)}$. Thus, $\mathbf{R}(\sigma)$ is a reflective subcategory of $\mathbf{R}^{\odot}(\sigma)$. Moreover, there is another functor $G : \mathbf{R}(\sigma) \to \mathbf{R}^{\odot}(\sigma)$ which sends a relational structure $\mathcal{X}$ to the commeasurable structure which has the same relations as $\mathcal{X}$ and where the universe of $(G\mathcal{X})$ is given by the Gaifman graph of $\mathcal{X}$. This functor is the left adjoint of U and we again have that $UG = \mathbf{Id}_{\mathbf{R}(\sigma)}$. This makes $\mathbf{R}(\sigma)$ a coreflective subcategory of $\mathbf{R}^{\odot}(\sigma)$. Since $\mathbf{R}(\sigma)$ is both reflective and coreflective it is known as a bireflective subcategory of $\mathbf{R}^{\odot}(\sigma)$.

We note that the commeasurable structure homomorphism problem remains NP-complete.

Theorem 6.22. *For $\mathcal{X}, \mathcal{Y} \in \mathbf{R}^{\odot}(\sigma)$ determining the existence of a homomorphism $\mathcal{X} \to \mathcal{Y}$ is NP-complete.*

Proof.

- Hardness: Since $UR = \mathbf{Id}$ the mapping is given by the functor $R : \mathbf{R}(\sigma) \to \mathbf{R}^{\odot}(\sigma)$ acts as a polynomial time reduction from the relational structure homomorphism problem to the commeasurable structure homomorphism problem.
- Membership: We can simply treat the relation $\odot$ as an additional relation that needs to be preserved by the underlying function of a homomorphism. From this one can adapt the standard proof showing that relational structure homomorphism is in NP to show that commeasurable structure homomorphism is in NP. ☐

6.2.4.1 Commeasurable EF comonad

We can define a commeasurable variant of $\mathbb{E}_k$ as follows:

Example 6.23. *The commeasurable EF functor $\mathbb{E}_k : \mathbf{R}^{\odot}(\sigma) \to \mathbf{R}^{\odot}(\sigma)$ is given by:*

- *The universe $\mathbb{E}_k^{\odot}(X^{\odot})$ is the set of all sequences $[\bar{x}_1, \ldots, x_n]$ of length $\leq k$ where $x_i \odot x_j$ holds for all $i, j \in [n]$.*
- $\mathbb{E}_k^{\odot}(f)([x_1, \ldots, x_n]) = [f(x_1), \ldots, f(x_n)]$.
- *for $s_1, s_2 \in \mathbb{E}_k^{\odot}\mathcal{X}^{\odot}$ we have $s_1 \odot s_2$ iff $s_1 \uparrow s_2$.*
- *$R^{\mathbb{E}_k \mathcal{X}^{\odot}}$ consists of tuples $(s_1, \ldots s_n)$ satisfying $(\varepsilon(s_1), \ldots \varepsilon(s_n)) \in R^{\mathcal{X}^{\odot}}$.*

This functor together with the counit and comultiplication maps we have previously considered for $\mathbb{E}_k$ forms a comonad. We also describe a commeasurable version of the EF game.

Example 6.24. *The one-sided commeasurable (EF) game in from $\mathcal{X}^{\odot} \in \mathbf{R}^{\odot}(\sigma)$ to $\mathcal{Y}^{\odot} \in \mathbf{R}^{\odot}(\sigma)$ is played in exactly the same way as the one-sided EF game for relational structures with one difference: The elements that Spoiler and Duplicator choose at each round must be commeasurable with all their previous choices.*

It is straightforward to see that as in the case of $\mathbb{E}_k$, there is a correspondence between $\mathbb{E}_k^{\odot}$ and commeasurable games.

Theorem 6.25. *For $\mathcal{X}^{\odot}, \mathcal{Y}^{\odot} \in \mathbf{R}^{\odot}(\sigma)$ the following are equivalent:*

- *Duplicator has a winning strategy in the commeasurable one-sided EF game from $\mathcal{X}^{\odot}$ to $\mathcal{Y}^{\odot}$.*
- $\mathbb{E}_k^{\odot}\mathcal{X}^{\odot} \to \mathcal{Y}^{\odot}$.

6.2.4.2 Commeasurable quantum monad

We can also define a commeasurable variant of $\mathbb{Q}_{\boldsymbol{H}}$.

Example 6.26. *The commesurable quantum functor $\mathbb{Q}_{\boldsymbol{H}}^{\odot} : \mathbf{R}^{\odot}(\sigma) \to \mathbf{R}^{\odot}(\sigma)$ is given by:*

- $\mathbb{Q}_{\boldsymbol{H}}^{\odot}\mathcal{X}$ *is the set of all PVMs in* $\boldsymbol{H}$ *with outcomes in* $\mathbf{X}$.
- $\mathbb{Q}_{\boldsymbol{H}}^{\odot} f$ *is defined as for* $\mathbb{Q}_{\boldsymbol{H}}$.
- *for* $\psi_1, \psi_2 \in \mathbb{Q}_{\boldsymbol{H}}^{\odot}\mathcal{X}$ *we have* $\psi_1 \odot \psi_2$ *iff:*
  - ψ_1 *and* ψ_2 *are jointly measurable PVMs.*
  - *Whenever* $a_1 \odot a_2$ *does not hold we have* $\psi_1(a_1)\psi_2(a_2) = 0$.
- $R^{\mathbb{Q}_{\boldsymbol{H}}^{\odot}\mathcal{X}}$ *is the set of all tuples* $(p_1, \ldots, p_n)$ *such that for all* $\boldsymbol{x} \in A^k$, *if* $x \notin R^{\mathcal{A}}$, *then* $\boldsymbol{p}(\boldsymbol{x}) = \mathbf{0}$.

Again, together with the unit and multiplication maps we considered for $\mathbb{Q}_{\boldsymbol{H}}$ in the previous chapter, $\mathbb{Q}_{\boldsymbol{H}}^{\odot}$ can be given the structure of a graded monad. Moreover, we can also describe a version of the non-local CSP game for commeasurable structures.

Example 6.27. *The (synchrounous) commeasurable $(\mathcal{X}, \mathcal{Y})$-CSP game is played as follows:*

1. *Verifier sends Alice and bob tuples* $\boldsymbol{x^a}, \boldsymbol{x^b} \in X^k$ *respectively.*
2. *Alice and Bob return tuples* $\boldsymbol{y^a}, \boldsymbol{y^b} \in Y^k$ *respectively.*
3. *Alice and Bob win the game if:*

- $x_i^a = x_j^b \Rightarrow y_i^a = y_j^b$.
- $x_i^a \odot x_j^b \Rightarrow y_i^a \odot y_j^b$.
- $\boldsymbol{x^a} \in R^{\mathcal{X}} \Rightarrow \boldsymbol{y^a} \in R^{\mathcal{Y}}$ *and* $\boldsymbol{x^b} \in R^{\mathcal{X}} \Rightarrow \boldsymbol{y^b} \in R^{\mathcal{Y}}$.

And we can also define quantum homomorphisms for commeasurable structures.

Definition 6.28. A commeasurable quantum homomorphism from $\mathcal{X}^{\odot} \in \mathbf{R}^{\odot}(\sigma)$ to $\mathcal{Y}^{\odot} \in \mathbf{R}^{\odot}(\sigma)$ in $\boldsymbol{H}$ is given by a family of projectors $\{P_{x,y}\}_{x \in X, y \in Y}$ in $\mathsf{Proj}(\boldsymbol{H})$ satisfying:

1. For all $x \in X : \sum_{y \in Y} P_{x,y} = I$
2. For all $x \odot x'$ in $\mathcal{X}^{\odot}$ and all $y, y' \in Y$ we have $P_{x,y} \odot P_{x'y'}$.
3. If $x \in$ ‘$R^{\mathcal{X}}$ and $y \notin R^{\mathcal{Y}}$ $P_{\boldsymbol{x},\boldsymbol{y}} = 0$.

A simple adaptation of our results from chapter 5 shows that different classes of commeasurable quantum homomorphisms correspond to perfect strategies in the commeasurable CSP game in exactly the same way that quantum homomorphisms correspond to perfect strategies in the CSP game. Finally, we have:

Theorem 6.29. *The following are equivalent:*

- *There exists a commeasurable quantum homomorphism from $\mathcal{X}^{\odot}$ to $\mathcal{Y}^{\odot}$ in $\boldsymbol{H}$.*
- $\mathcal{X}^{\odot} \to \mathbb{Q}_{\boldsymbol{H}}^{\odot} \mathcal{Y}^{\odot}$.

6.2.4.3 Distributive law

We now show that the family of natural transformations $\kappa^{\boldsymbol{H}} : \mathbb{E}_k^{\odot} \mathbb{Q}_{\boldsymbol{H}}^{\odot} \to \mathbb{Q}_{\boldsymbol{H}}^{\odot} \mathbb{E}_k^{\odot}$ given below is a valid distributive law.

$$\kappa_{\mathcal{X}^{\odot}}^{\boldsymbol{H}}[\varphi_1, \ldots, \varphi_n] = \sum_{[x_1, \ldots, x_n] \in \mathbb{E}_k^{\odot}(X)} \boldsymbol{\varphi(x)}.[x_1, \ldots, x_n].$$

Theorem 6.30. $\kappa^{\boldsymbol{H}}$ *is a distributive law of the comonad* $\mathbb{E}_k^{\odot}$ *over the graded monad* $\mathbb{Q}_{\boldsymbol{H}}^{\odot}$

To prove this theorem we shall make use of the following theorem that has been proved in Nihil Shah's DPhil Thesis. Noting that $\mathbf{Rgraph} \cong \mathbf{R}^{\odot}(\sigma)$ when σ is the empty signature, we write $E_k^{\odot}$ and $Q_{\boldsymbol{H}}^{\odot}$ for the version of $\mathbb{E}_k^{\odot}$ and $\mathbb{Q}_{\boldsymbol{H}}^{\odot}$ defined on $\mathbf{Rgraph}$. Then we have:

Theorem 6.31. $\kappa^{\boldsymbol{H}}$ *is a distributive law of the comonad* $E_k^{\odot}$ *over the graded monad* $Q_{\boldsymbol{H}}^{\odot}$

Proof of theorem 6.30. The fact that $\kappa_X^{\boldsymbol{H}}$ (as a function) satisfies naturality and the distributive law axioms follows immediately from theorem 6.31. The fact that $\odot$ is preserved also follows from this.

Thus it only remains to verify that relations are preserved. This is true via exactly the same argument used in theorem 3.42. The only point to be careful about is that the partiality of the semiring $\mathsf{Proj}(\boldsymbol{H})$ does not cause any issues with the proof. This is indeed not an issue as $\mathbb{E}_k^{\odot}$ enforces that all PVMs appearing in the same sequence $[\varphi_1, \dots, \varphi_n] \in \mathbb{E}_k^{\odot}\mathbb{Q}_{\boldsymbol{H}}^{\odot}\mathcal{X}$ are jointly measurable. This guarantees that all the projectors whose commutativity is taken for granted in proving theorem 3.42 do commute. ☐

Thus we obtain a biKleisli category with morphisms $\mathbb{E}_k^{\odot}\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}^{\odot}\mathcal{Y}$.

6.2.4.4 Commeasurable strategies in $\mathbf{R}(\sigma)$

Let us now return to our original motivating setting of $\mathbf{R}(\sigma)$. As we have mentioned this category is a reflexive subcategory of $\mathbf{R}^{\odot}(\sigma)$. We can now ask if the fact that we have a way of composing morphisms $\mathbb{E}_k^{\odot}\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}^{\odot}\mathcal{Y}$ in this larger category allows us to compose morphisms of the form $\mathbb{E}_k\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y} \in \mathbf{R}(\sigma)$.

Let us first observe that we have a way of capturing coKleisli morphisms of $\mathbb{E}_k$ in the larger category $\mathbf{R}^{\odot}(\sigma)$.

Proposition 6.32. *The following are equivalent:*

- $\mathbb{E}_k\mathcal{X} \to \mathcal{Y}$.
- $\mathbb{E}_k^{\odot}R\mathcal{X} \to R\mathcal{Y}$.

Proof. $R\mathcal{X}$ and $R\mathcal{Y}$ have the same relations as $\mathcal{X}$ and $\mathcal{Y}$ respectively and from the definition of R we know that they both have complete commeasurability graphs. Thus in this case the commeasurable EF game and the regular EF game coincide. ☐

We also have a way of capturing Kleisli morphisms of $\mathbb{Q}_{\boldsymbol{H}}$.

Proposition 6.33. *The following are equivalent:*

- $\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$.
- $G\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}^{\odot}R\mathcal{Y}$.

Proof. Looking at the definitions of a quantum homomorphism and a commeasurable quantum homomorphism we observe that the difference between the two definitions is that in the first definition, two elements x_1 and x_2 are assigned to jointly measurable PVMs when they are adjacent in the Gaifman graph of $\mathcal{X}$ while in the second definition x_1 and x_2 are assigned to jointly measurable PVMs whenever $x_1 \odot x_2$. Because the commeasurability graph of $G\mathcal{X}$ is defined to be the Gaifman graph of $\mathcal{X}$ these two relations are precisely the same. $\square$

At this point, we run into a problem. Observe that when viewed in $\mathbf{R}^{\odot}(\sigma)$ there is a mismatch between the codomain of coKleisli maps for $\mathbb{E}_k$ where we are applying the functor R to elements $\mathcal{Y} \in \mathbf{R}(\sigma)$ and the domain of Kleisli maps for $\mathbb{Q}_{\boldsymbol{H}}$ where we are applying the functor G to elements $\mathcal{X} \in \mathbf{R}(\sigma)$. This makes it impossible to use biKleisli composition for $\mathbb{E}_k^{\odot}$ and $\mathbb{Q}_{\boldsymbol{H}}^{\odot}$ to compose the original morphisms in $\mathbf{R}(\sigma)$. Nevertheless, we can identify a subclass of strategies for Duplicator which are composable.

Definition 6.34. A Duplicator strategy in the quantum one-sided EF game is called *commeasurable* if, within a single play of the game, all of their choices of PVMs are jointly measurable.

If we restrict biKleisli morphisms $\mathbb{E}_k\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$ to only those morphisms which are commeasurable, then the natural transformation $\kappa^{\boldsymbol{H}} : \mathbb{E}_k\mathbb{Q}_{\boldsymbol{H}} \to \mathbb{Q}_{\boldsymbol{H}}\mathbb{E}_k$ becomes well-defined. Moreover, via the same argument used in theorem 6.30 all of the distributive law axioms will be satisfied. Thus we obtain a biKleisli category whose objects are the objects of $\mathbf{R}(\sigma)$ and where morphisms are maps of the form $\mathbb{E}_k\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$ which satisfy the commeasurability condition in the definition above.

6.2.4.5 Lack of quantum advantage

We have now shown that commeasurable versions of one-sided quantum EF games can be composed together in a biKleisli category with morphisms of the form $\mathbb{E}_k^{\odot}\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}^{\odot}\mathcal{Y}$. We have also seen that in $\mathbf{R}(\sigma)$, biKleisli composition is possible as long as Duplicator is restricted to using commeasurable strategies. Could this biKleisli category then provide a setting in which one can study quantum advantage? In this section, we prove that this is not the case through an application of Vorob'ev's theorem [Vor62]. Let us begin by stating this theorem in our language. Recall that an induced cycle in a graph is a cycle where no two-non adjacent vertices have an edge between them.

Theorem 6.35 ([Vor62; XC19]). *Assume we have a morphism $f : \mathcal{X}^{\odot} \to \mathbb{Q}_{\boldsymbol{H}}^{\odot}\mathcal{Y}^{\odot}$ but no morphism $g : \mathcal{X}^{\odot} \to \mathcal{Y}^{\odot}$. Then the underlying reflexive graph of $\mathcal{X}^{\odot}$ contains an induced cycle of size 4 or more.*

It turns out that structures of the form $\mathbb{E}_k^{\odot}\mathcal{X}^{\odot}$ do not satisfy this requirement.

Proposition 6.36. *The underlying reflexive graph of* $\mathbb{E}_k^{\odot}\mathcal{X}^{\odot}$ *never contains an induced cycle of size 4 or more.*

Proof. We proceed by contradiction. Assume an induced cycle of length n exists and is given by the unique sequences $s_1, \ldots, s_n$. Then it must be the case that $s_1 \uparrow s_2$ and $s_1 \uparrow s_n$ but we cannot have $s_n \uparrow s_2$. This is only possible if we have both $s_1 \leq s_n$ and $s_1 \leq s_2$. A similar argument involving s_1, s_2, s_3 shows that $s_2 \leq s_1$ and $s_2 \leq s_3$. Thus, we have both $s_1 \leq s_2$ and $s_2 \leq s_1$. Hence, $s_1 = s_2$ which contradicts their uniqueness. ☐

The main theorem of this section is the following.

Theorem 6.37. *The following are equivalent:*

1. *There exists a biKleisli morphism* $\mathbb{E}_k^{\odot}\mathcal{X}^{\odot} \to \mathbb{Q}_{\boldsymbol{H}}^{\odot}\mathcal{Y}^{\odot}$.
2. *There exists a coKleisli morphism* $\mathbb{E}_k^{\odot}\mathcal{X}^{\odot} \to \mathcal{Y}^{\odot}$.

Proof. $(2) \Rightarrow (1)$ is true since any classical strategy is a quantum strategy. $(1) \Rightarrow (2)$ is a corollary of theorem 6.35 and propsition 6.36. ☐

It is worth mentioning that the existence of an induced cycle is not just a necessarry condition for contextuality but also a sufficient one [XC19]. Thus, an informally stated corollary of our result is that the only truly non-classical strategies that Duplicator can employ are precisely the non-commeasurable ones. Because of this one could argue that failure of composition (at least in the sense explored in this section) is precisely what allows for quantum advantage in Spoiler-Duplicator games. It remains to be seen if this barrier to compositionality can be overcome, perhaps through enforcing clever compatibility constraints on the strategies involved, in a similar manner to the techniques employed in chapter 5 in order to capture quantum isomorphism categorically.

6.3 Discussion

In this chapter, we introduced and studied quantum versions of one-sided Spoiler-Duplicator games. Our main contribution was to show that quantum advantage is possible in these games. This means that Duplicator can sometimes win the quantum version of the games even when they have no classical winning strategy. We then strengthened this result further and showed that quantum advantage is still possible even if there is no quantum homomorphism between the structures involved. As we discussed in chapter 2 classical Spoiler-Duplicator games are connected to many central logical, algorithmic, and combinatorial results in computer science. The natural direction for future work then is to see which of these connections can be "quantised" via our quantum games. Let us list some open questions in more detail:

- **Quantum advantage in other Spoiler-Duplicator games**: Throughout this chapter, we focused only on one-sided EF and pebble games. We can of course consider quantum versions of other Spoiler-Duplicator games, such as the back-and-forth or bijective variants of each game. In the case of EF, for instance, the back-and-forth quantum game is played as follows:

  Example 6.38. *Let $\boldsymbol{H}$ be a Hilbert space. The back-and-forth quantum Ehrenfeucht-Fraïssé (EF) game in $\boldsymbol{H}$ from $\mathcal{X} \in \mathbf{R}(\sigma)$ to $\mathcal{Y} \in \mathbf{R}(\sigma)$ is played as follows:*

  At the start of round i, Spoiler chooses an element of $x_i \in \mathbf{X}$ or $\bar{y}_i \in \mathbf{Y}$. Duplicator responds with a PVM $\{P_{x_i,y}\}_{y \in Y}$ or $\{P_{y_i,x}\}_{x \in X}$ from the other structure. After k rounds their choices create a sequence of pvms $[\{P_{c_1}\}, \ldots, \{P_{c_k}\}]$ with each $c_i \in X \cup Y$. Duplicator wins a play of length k if this sequence is a partial quantum isomorphism.

  Moreover, $\mathbb{E}_k$ and $\mathbb{P}_k$ are just two instances of the large class of game comonads which have been introduced in [AM22; AM21; CD24; MS22]. Each of these comonads provides categorical semantics for some class of Spoiler-Duplicator games from finite model theory. The natural question is the following:

  Question 6.39. *For which Spoiler-Duplicator games is it possible to exhibit "non-trivial" (In the sense of question 6.8) quantum advantage?*

- **Algorithmic and complexity-theoretic questions**: Despite the fact that classical CSP is an NP-complete problem, identifying the existence of coKleisli maps $\mathbb{E}_k\mathcal{X} \to \mathcal{Y}$ and $\mathbb{P}_k\mathcal{X} \to \mathcal{Y}$ is known to be solvable in polynomial time. In fact, in the case of $\mathbb{P}_k$, this problem is solvable via an extremely well-studied algorithm known as strong k-consistency. Likewise, detecting coKleisli isomorphisms for $\mathbb{P}_k$ is solvable via another important algorithm known as k-Weisfeiler-Leman [Kie20]. In light of these connections, we ask the following questions:

  Question 6.40. *What is the complexity of determining the existence of perfect quantum strategies for Duplicator in different Spoiler-Duplicator games?*

  This question can be asked in both cases where the dimension of the Hilbert space $\boldsymbol{H}$ is bounded and where no such bound is enforced. Both versions of the question would be interesting to tackle.

  A related question is to identify the classes of structures for which quantum Spoiler-Duplicator games provide an exact account of quantum homomorphisms.

Question 6.41. *For what class of structures does* $\mathbb{P}_k\mathcal{A} \to \mathbb{Q}_H\mathcal{B} \iff \mathcal{A} \to \mathbb{Q}_H$ *hold?*

- **Connections with logic**: One of the main appeals of Spoiler-Duplicator games is the fact that they can be used to study the expressive power of fragments of first-order logic. Recall that writing $\mathcal{L}_k$ for the fragment of first-order logic with quantifier rank at most k we have the following triangle of equivalences from [AS21].

  Theorem 6.42. *The following are equivalent:*

  - *Duplicator has a winning strategy in the one-sided k-round EF game from $\mathcal{X}$ to $\mathcal{Y}$.*
  - *There exists a Kleisli morphisms $\mathbb{E}_k\mathcal{X} \to \mathcal{Y}$.*
  - $\mathcal{X} \models^{\mathcal{L}_k} \rho \Rightarrow \mathcal{Y} \models^{\mathcal{L}_k} \rho$.

  Thus, we ask the following:

  Question 6.43. *Does there exist a quantised version of the logic $\mathcal{L}_k$, which we denote by $\mathcal{Q}_k^H$ such that the following are equivalent?*

  - *Duplicator has a perfect winning strategy in the quantum one-sided k-round EF game from $\mathcal{X}$ to $\mathcal{Y}$.*
  - *There exists a biKleisli morphism $\mathbb{E}_k\mathcal{X} \to \mathbb{Q}_H\mathcal{Y}$.*
  - $\mathcal{X} \models^{\mathcal{Q}_k^H} \rho \Rightarrow \mathcal{Y} \models^{\mathcal{Q}_k^H} \rho$.

  Moreover, is there a connection between such a logic and the Kochen-Specker style approach to quantum logic?

  A similar question applies to other game comonads.

- **Homomorphism indistinguishability:** In [Lov67], Lovász showed that two finite structures are isomorphic if and only if they have the same number of homomorphisms from every other finite structure. Recently there has been much interest in characterising relaxations of structure isomorphism via similar "Lovász type theorems" [DJR21; Gro20; GRS22; AKW21]. For instance, the bijective k-pebble game and quantum (commuting) isomorphism both admit such characterisations:

  Theorem 6.44 ([Dvo10; Gro20; DJR21])**.** *The following are equivalent.*

  - *Duplicator wins the bijective k-pebble game between $\mathcal{X}$ and $\mathcal{Y}$.*
  - *$\mathcal{X}$ and $\mathcal{Y}$ admit the same number of homomorphisms from all structures with treewidth $\leq k$.*

Theorem 6.45 ([MR20; NRSZ24]). *The following are equivalent.*

- *There exists a quantum (commuting) isomorphism between $\mathcal{X}$ and $\mathcal{Y}$.*
- *$\mathcal{X}$ and $\mathcal{Y}$ admit the same number of homomorphisms from all planar structures.*

We ask if these two theorems can be combined to characterise a quantum version of the bijective k-pebble game.

Question 6.46. *Are the following equivalent?*

- *Duplicator wins a quantum variant of the bijective k-pebble game between $\mathcal{X}$ and $\mathcal{Y}$.*
- *$\mathcal{X}$ and $\mathcal{Y}$ admit the same number of homomorphisms from all planar structures with treewidth $\leq k$.*

Again, a similar question applies to other game comonads.

- **An operational account of quantum Spoiler-Duplicator games**: In physics, an operational theory is a framework that describes a physical system based on the operations that one can perform on it, e.g. in a laboratory. In the context of quantum information, an operational protocol often describes a series of steps in which parties may prepare certain quantum states and perform operations such as measurements on them. For instance, the non-local games we studied in the previous chapter can be viewed as such a protocol. We ask if there is a similar way of viewing quantum Spoiler-Duplicator games through such protocols.

In the second part of this chapter we discussed how Duplicator's strategies in quantum games can be expressed in terms of the existence of biKleisli maps $\mathbb{G}_k\mathcal{X} \to \mathbb{Q}_{\boldsymbol{H}}\mathcal{Y}$. We then asked if there exists a distributive law between $\mathbb{G}_k$ and $\mathbb{Q}_{\boldsymbol{H}}$ which would result in a biKleisli category which allows for the composition of such strategies. We saw that the no-go theorems of chapter 3 do not apply in this case and explored several ideas for proving a positive answer to this question. Unfortunately, all of these approaches fell short of providing a satisfactory answer. Solving this question remains a very important direction for future work not least because a positive answer could be very helpful in tackling the other open questions mentioned above.

7

Conclusions

We have studied relaxations of two decision problems which are of central practical and theoretical importance in computer science. The first, known as CSP or structure homomorphism asks if there exists a homomorphism between relational structures $\mathcal{X}$ and $\mathcal{Y}$ over the same signature σ. The second known as structure isomorphism asks if an isomorphism exists between such structures. By adopting a compositional viewpoint, we have expanded existing results to show that three well-known classes of relaxations for these problems, which have so far been studied by disjoint communities of researchers using techniques from universal algebra, (finite) model theory, and quantum information, can all be captured through the categorical abstraction of (co)monads. More specifically we have seen that distribution minion monads $\mathbb{D}_{\mathbf{S}}$ provide an abstraction for certain minion tests from the universal algebraic approach, game comonads $\mathbb{G}_k$ provide an abstraction for Spoiler-Duplicator games from finite model theory, and that (partial) quantum monads $\mathbb{Q}_{\boldsymbol{H}}$ provide an abstraction for quantum homomorphisms/isomorphisms from quantum information. In fact, by pointing out a simple corollary of the results in [Cia24] we have also argued that quantum monads can be seen as a special case of distribution minion monads.

Given a comonad W and a monad M the existence of a special natural transformation $\lambda : WM \to MW$ known as a distributive law allows us to combine W and M. For a natural transformation to be a distributive law it must satisfy four coherence conditions related to the (co)unit and (co)multiplication operations of W and M. A question that we explored in detail was to identify pairs of game comonads $\mathbb{G}_k$ and distribution minion monads $\mathbb{D}_{\mathbf{S}}$ for which a distributive law exists. Such a combination could lead to novel relaxations for CSP and structure isomorphism which combine ideas from universal algebra and finite model theory. This question turned out to be remarkably subtle. We have proved a no-go theorem which shows that the Ehrenfeucht-Fraisse comonad $\mathbb{E}_k$ and the pebbling comonad $\mathbb{P}_k$ fail to distribute over many interesting distribution minion monads, such as those corresponding to the BLP and arc-consistency minion tests. Interestingly, our no-go theorem does not apply to quantum monads $\mathbb{Q}_{\boldsymbol{H}}$. In this case, it remains an open question whether a distributive law exists.

Finally, we have introduced quantum versions of Spoiler-Duplicator games and showed that quantum advantage is possible in these games. These early results suggest that it may be possible to quantise ideas from finite model theory to improve our understanding of quantum advantage. There are many other directions for future work which we have mentioned in discussion sections throughout the thesis.

Bibliography

[AAG03] Michael Gordon Abbott, Thorsten Altenkirch, and Neil Ghani. "Categories of containers". In: *6th International Conference on Foundations of Software Science and Computational Structures, FOSSACS 2003 Held as Part of the Joint European Conference on Theory and Practice of Software, ETAPS 2003, Warsaw, Poland, April 7-11, 2003, Proceedings*. Ed. by Andrew D. Gordon. Vol. 2620. Lecture Notes in Computer Science. Springer, 2003, pp. 23–38. DOI: 10.1007/3-540-36576-1_2 (cit. on p. 36).

[AB21] Samson Abramsky and Rui Soares Barbosa. "The logic of contextuality". In: *29th EACSL Annual Conference on Computer Science Logic, CSL 2021, January 25-28, 2021, Ljubljana, Slovenia (Virtual Conference)*. Ed. by Christel Baier and Jean Goubault-Larrecq. Vol. 183. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021, 5:1–5:18. DOI: 10.4230/LIPICS.CSL.2021.5 (cit. on p. 88).

[ABdSZ17] Samson Abramsky, Rui Soares Barbosa, Nadish de Silva, and Octavio Zapata. "The quantum monad on relational structures". In: *42nd International Symposium on Mathematical Foundations of Computer Science (MFCS 2017)*. Ed. by Kim G. Larsen, Hans L. Bodlaender, and Jean-Francois Raskin. Vol. 83. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2017, 35:1–35:19. ISBN: 978-3-95977-046-0. DOI: 10.4230/LIPIcs.MFCS.2017.35 (cit. on pp. 14–17, 20, 71, 98–101, 108, 113–114, 116, 118–120, 123, 128, 137).

[ABKM19] Samson Abramsky, Rui Soares Barbosa, Martti Karvonen, and Shane Mansfield. "A comonadic view of simulation and quantum resources". In: *34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019, Vancouver, BC, Canada, June 24-27, 2019*. IEEE, 2019, pp. 1–12. DOI: 10.1109/LICS.2019.8785677 (cit. on p. 110).

[ABM17] Samson Abramsky, Rui Soares Barbosa, and Shane Mansfield.
"Contextual Fraction as a Measure of Contextuality".
In: *Physical Review Letters* 119 (5 Oct. 2017), p. 050504.
DOI: 10.1103/PhysRevLett.119.050504 (cit. on p. 110).

[AB11] Samson Abramsky and Adam Brandenburger.
"The sheaf-theoretic structure of non-locality and contextuality".
In: *New Journal of Physics* 13.11 (Nov. 2011), p. 113036.
DOI: 10.1088/1367-2630/13/11/113036 (cit. on p. 89).

[ADW17] Samson Abramsky, Anuj Dawar, and Pengming Wang.
"The pebbling comonad in finite model theory".
In: *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017.* IEEE Computer Society, 2017, pp. 1–12.
DOI: 10.1109/LICS.2017.8005129 (cit. on pp. 14, 38, 43).

[AGN96] Samson Abramsky, Simon Gay, and Rajagopal Nagarajan.
"Specification structures and propositions-as-types for concurrency".
In: *Logics for Concurrency: Structure versus Automata.*
Ed. by Faron Moller and Graham Birtwistle.
Berlin, Heidelberg: Springer Berlin Heidelberg, 1996, pp. 5–40.
ISBN: 978-3-540-49675-5. DOI: 10.1007/3-540-60915-6_2 (cit. on p. 126).

[AM21] Samson Abramsky and Dan Marsden.
"Comonadic semantics for guarded fragments".
In: *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021.* IEEE, 2021, pp. 1–13.
DOI: 10.1109/LICS52264.2021.9470594 (cit. on pp. 44, 158).

[AM22] Samson Abramsky and Dan Marsden.
"Comonadic semantics for hybrid logic".
In: *47th International Symposium on Mathematical Foundations of Computer Science, MFCS 2022, August 22-26, 2022, Vienna, Austria.*
Ed. by Stefan Szeider, Robert Ganian, and Alexandra Silva. Vol. 241. LIPIcs.
Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022, 7:1–7:14.
DOI: 10.4230/LIPICS.MFCS.2022.7 (cit. on pp. 44, 158).

[AR23] Samson Abramsky and Luca Reggio.
"Arboreal categories: An axiomatic theory of resources".
In: *Logical Methods in Computer Science* 19.3 (2023).
DOI: 10.46298/LMCS-19(3:14)2023 (cit. on p. 44).

[AR24a] Samson Abramsky and Luca Reggio.
"An invitation to game comonads".

In: *ACM SIGLOG News* 11.3 (Aug. 2024), pp. 5–48.
DOI: `10.1145/3687256.3687260` (cit. on p. 44).

[AR24b] Samson Abramsky and Luca Reggio.
“Arboreal categories and equi-resource homomorphism preservation theorems”.
In: *Annals of Pure and Applied Logic* 175.6 (2024), p. 103423. ISSN: 0168-0072.
DOI: `https://doi.org/10.1016/j.apal.2024.103423` (cit. on pp. 15, 44).

[AS21] Samson Abramsky and Nihil Shah.
“Relating structure and power: Comonadic semantics for computational resources”.
In: *Journal of Logic and Computation* 31.6 (2021), pp. 1390–1428.
DOI: `10.1093/LOGCOM/EXAB048` (cit. on pp. 14, 18, 43–44, 46, 159).

[AFLS15] Antonio Acín, Tobias Fritz, Anthony Leverrier, and Ana Belén Sainz.
“A combinatorial approach to nonlocality and contextuality”.
In: *Communications in Mathematical Physics* 334.2 (2015), pp. 533–628.
DOI: `10.1007/s00220-014-2260-1` (cit. on p. 89).

[ACU14] Danel Ahman, James Chapman, and Tarmo Uustalu.
“When is a container a comonad?”.
In: *Logical Methods in Computer Science* Volume 10, Issue 3 (Sept. 2014).
DOI: `10.2168/lmcs-10(3:14)2014` (cit. on p. 37).

[AU19] Danel Ahman and Tarmo Uustalu.
“Decomposing comonad morphisms”.
In: *8th Conference on Algebra and Coalgebra in Computer Science, CALCO 2019, June 3-6, 2019, London, United Kingdom.*
Ed. by Markus Roggenbach and Ana Sokolova. Vol. 139. LIPIcs.
Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019, 14:1–14:19.
DOI: `10.4230/LIPIcs.CALCO.2019.14` (cit. on p. 38).

[ALM+98] Sanjeev Arora, Carsten Lund, Rajeev Motwani, Madhu Sudan, and Mario Szegedy.
“Proof verification and the hardness of approximation problems”.
In: *Journal of the ACM* 45.3 (May 1998), pp. 501–555. ISSN: 0004-5411.
DOI: `10.1145/278298.278306` (cit. on p. 94).

[Atk09] Robert Atkey.
“Parameterised notions of computation”.
In: *Journal of Functional Programming* 19.3-4 (2009), pp. 335–376.
DOI: `10.1017/S095679680900728X` (cit. on p. 130).

[ABD07] Albert Atserias, Andrei A. Bulatov, and Víctor Dalmau.
“On the power of k -consistency”.
In: *34th International Colloquium on Automata, Languages, and Programming,*

ICALP 2007, July 9-13, 2007, Wroclaw, Poland.
Ed. by Lars Arge, Christian Cachin, Tomasz Jurdzinski, and Andrzej Tarlecki.
Vol. 4596. Lecture Notes in Computer Science. Springer, 2007, pp. 279–290.
DOI: 10.1007/978-3-540-73420-8_26 (cit. on p. 48).

[AKW21] Albert Atserias, Phokion G. Kolaitis, and Wei-Lin Wu.
"On the expressive power of homomorphism counts".
In: *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021.* IEEE, 2021, pp. 1–13.
DOI: 10.1109/LICS52264.2021.9470543 (cit. on p. 159).

[AMR+19] Albert Atserias, Laura Mančinska, David E. Roberson, Robert Šámal, Simone Severini, and Antonios Varvitsiotis.
"Quantum and non-signalling graph isomorphisms".
In: *Journal of Combinatorial Theory, Series B* 136 (2019), pp. 289–328.
ISSN: 0095-8956. DOI: https://doi.org/10.1016/j.jctb.2018.11.002
(cit. on pp. 92, 99, 112).

[AHKS06] David Avis, Jun Hasegawa, Yosuke Kikuchi, and Yuuya Sasaki.
"A quantum protocol to win the graph colouring game on all hadamard graphs".
In: *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences* E89-a.5 (May 2006), pp. 1378–1381. ISSN: 0916-8508.
DOI: 10.1093/ietfec/e89-a.5.1378 (cit. on p. 98).

[Bab16] László Babai.
"Graph isomorphism in quasipolynomial time [extended abstract]".
In: *48th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2016, Cambridge, MA, USA, June 18-21, 2016.* Ed. by Daniel Wichs and Yishay Mansour.
ACM, 2016, pp. 684–697. DOI: 10.1145/2897518.2897542 (cit. on p. 14).

[BFL91] László Babai, Lance Fortnow, and Carsten Lund.
"Non-deterministic exponential time has two-prover interactive protocols".
In: *Computational Complexity* 1 (1991), pp. 3–40. DOI: 10.1007/bf01200056
(cit. on p. 93).

[BKM23] Rui Soares Barbosa, Martti Karvonen, and Shane Mansfield.
"Closing bell: Boxing black box simulations in the resource theory of contextuality".
In: *Samson Abramsky on Logic and Structure in Computer Science and Beyond.*
Ed. by Alessandra Palmigiano and Mehrnoosh Sadrzadeh.
Cham: Springer International Publishing, 2023, pp. 475–529.
ISBN: 978-3-031-24117-8. DOI: 10.1007/978-3-031-24117-8_13 (cit. on p. 110).

[Bar70] Michael Barr.
"Relational algebras".

In: *Reports of the Midwest Category Seminar IV*. Ed. by S. MacLane, H. Applegate, M. Barr, B. Day, E. Dubuc, Phreilambud, A. Pultr, et al.
Berlin, Heidelberg: Springer Berlin Heidelberg, 1970, pp. 39–55.
ISBN: 978-3-540-36292-0. DOI: https://doi.org/10.1007/BFb0060439
(cit. on p. 58).

[BBKO21] Libor Barto, Jakub Bulín, Andrei Krokhin, and Jakub Opršal.
“Algebraic approach to promise constraint satisfaction”.
In: *Journal of the ACM* 68.4 (July 2021). ISSN: 0004-5411. DOI: 10.1145/3457606
(cit. on p. 40).

[Bec69] Jon Beck.
“Distributive laws”.
In: *Seminar on Triples and Categorical Homology Theory*. Ed. by B. Eckmann.
Berlin, Heidelberg: Springer Berlin Heidelberg, 1969, pp. 119–140.
ISBN: 978-3-540-36091-9. DOI: https://doi.org/10.1007/BFb0083084
(cit. on p. 15).

[Bel64] J. S. Bell.
“On the Einstein Podolsky Rosen paradox”.
In: *Physics Physique Fizika* 1 (3 Nov. 1964), pp. 195–200.
DOI: 10.1103/PhysicsPhysiqueFizika.1.195 (cit. on pp. 15, 89).

[Bel66] John S. Bell.
“On the problem of hidden variables in quantum mechanics”.
In: *Reviews of Modern Physics* 38 (3 July 1966), pp. 447–452.
DOI: 10.1103/RevModPhys.38.447 (cit. on p. 89).

[BN36] Garrett Birkhoff and John Von Neumann.
“The logic of quantum mechanics”.
In: *Annals of Mathematics* 37.4 (1936), pp. 823–843. ISSN: 0003486X, 19398980.
DOI: 10.2307/1968621 (cit. on pp. 88, 129).

[BMR+99] Stefano Bistarelli, Ugo Montanari, Francesca Rossi, Thomas Schiex, Gérard Verfaillie, and Hélène Fargier.
“Semiring-based CSPs and valued CSPs: Frameworks, properties, and comparison”.
In: *Constraints* 4 (Sept. 1999), pp. 199–240. DOI: 10.1023/a:1026441215081
(cit. on p. 129).

[BKS23] Mikołaj Bojańczyk, Bartek Klin, and Julian Salamanca.
“Monadic monadic second order logic”.
In: *Samson Abramsky on Logic and Structure in Computer Science and Beyond*. Ed. by Alessandra Palmigiano and Mehrnoosh Sadrzadeh.

Cham: Springer International Publishing, 2023, pp. 701–754.
ISBN: 978-3-031-24117-8. DOI: 10.1007/978-3-031-24117-8_19 (cit. on p. 58).

[BHKR15] Marcello M. Bonsangue, Helle Hvid Hansen, Alexander Kurz, and Jurriaan Rot. "Presenting distributive laws". In: *Logical Methods in Computer Science* Volume 11, Issue 3 (2015). ISSN: 1860-5974. DOI: 10.2168/lmcs-11(3:2)2015 (cit. on p. 56).

[BGWZ20] Joshua Brakensiek, Venkatesan Guruswami, Marcin Wrochna, and Stanislav Zivny. "The power of the combined basic linear programming and affine relaxation for promise constraint satisfaction problems". In: *SIAM Journal on Computing* 49.6 (2020), pp. 1232–1248. DOI: 10.1137/20M1312745 (cit. on pp. 40–41, 43).

[CSW14] Adán Cabello, Simone Severini, and Andreas Winter. "Graph-theoretic approach to quantum correlations". In: *Physical Review Letters* 112.4 (2014), p. 040401. DOI: 10.1103/PhysRevLett.112.040401 (cit. on p. 89).

[CMN+07] Peter J. Cameron, Ashley Montanaro, Michael W. Newman, Simone Severini, and Andreas J. Winter. "On the quantum chromatic number of a graph". In: *Electronic Journal of Combinatorics* 14.1 (2007). DOI: 10.37236/999 (cit. on p. 98).

[CKVW98] A Carboni, GM Kelly, D Verity, and RJ Wood. "A 2-categorical approach to change of base and geometric morphisms II". In: *Theory and Applications of Categories* 4.5 (1998), pp. 82–136 (cit. on p. 58).

[CDG13] Hubie Chen, Victor Dalmau, and Berit Grußien. "Arc consistency and friends". In: *Journal of Logic and Computation* 23.1 (2013), pp. 87–108. DOI: 10.1093/logcom/exr039 (cit. on pp. 40–41).

[Che11] Eugenia Cheng. "Iterated distributive laws". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 150.3 (2011), pp. 459–487. DOI: 10.1017/S0305004110000599 (cit. on p. 73).

[Cia24] Lorenzo Ciardo. "Quantum advantage and CSP complexity". In: *39th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '24. Tallinn, Estonia: Association for Computing Machinery, 2024. ISBN: 9798400706608. DOI: 10.1145/3661814.3662118 (cit. on pp. 121–122, 161).

[CŽ23] Lorenzo Ciardo and Stanislav Živnẏ.
"Hierarchies of minion tests for PCSPs through tensors".
In: *Annual ACM-SIAM Symposium on Discrete Algorithms (SODA) 2023.*
Siam. 2023, pp. 568–580. DOI: 10.1137/1.9781611977554.ch25
(cit. on pp. 42–43).

[CHTW04] R. Cleve, P. Hoyer, B. Toner, and J. Watrous.
"Consequences and limits of nonlocal strategies".
In: *19th IEEE Annual Conference on Computational Complexity, 2004.* 2004,
pp. 236–249. DOI: 10.1109/ccc.2004.1313847 (cit. on pp. 89, 98).

[CM14] Richard Cleve and Rajat Mittal.
"Characterization of binary constraint system games".
In: *41st International Colloquium on Automata, Languages, and Programming -, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings, Part I.*
Ed. by Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias.
Vol. 8572. Lecture Notes in Computer Science. Springer, 2014, pp. 320–331.
DOI: 10.1007/978-3-662-43948-7_27 (cit. on pp. 111, 140).

[CD24] Adam Ó Conghaile and Anuj Dawar.
"Game comonads & generalised quantifiers".
In: *Logical methods in computer science* 20.3 (2024).
DOI: 10.46298/LMCS-20(3:8)2024 (cit. on p. 158).

[Con22] Adam Connolly.
"Game comonads and beyond: compositional constructions for logic and algorithms".
PhD thesis. Apollo - University of Cambridge Repository, 2022.
DOI: 10.17863/CAM.104155 (cit. on pp. 14, 40, 42, 123).

[Coo71] Stephen A. Cook.
"The complexity of theorem-proving procedures".
In: *Third Annual ACM Symposium on Theory of Computing.* Stoc '71.
Shaker Heights, Ohio, USA: Association for Computing Machinery, 1971,
pp. 151–158. ISBN: 9781450374644. DOI: 10.1145/800157.805047 (cit. on p. 130).

[CS19] Matthew Coudron and William Slofstra.
"Complexity lower bounds for computing the approximately-commuting operator value of non-local games to high precision".
In: *34th Computational Complexity Conference, CCC 2019, July 18-20, 2019, New Brunswick, NJ, USA.* Ed. by Amir Shpilka. Vol. 137. LIPIcs.
Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019, 25:1–25:20.
DOI: 10.4230/LIPIcs.CCC.2019.25 (cit. on p. 94).

[CFK+15] Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. "Parameterized algorithms". Springer, 2015. ISBN: 978-3-319-21274-6. DOI: 10.1007/978-3-319-21275-3 (cit. on p. 46).

[DPS18] Fredrik Dahlqvist, Louis Parlant, and Alexandra Silva. "Layer by layer–combining monads". In: *17th International Colloquium on Theoretical Aspects of Computing, (ICTAC 2018)*. Springer. 2018, pp. 153–172. DOI: 10.1007/978-3-030-02508-3_9 (cit. on p. 56).

[DS21] Fredrik Dahlqvist and Todd Schmid. "How to write a coequation ((Co)algebraic pearls)". In: *9th Conference on Algebra and Coalgebra in Computer Science, CALCO 2021, August 31 to September 3, 2021, Salzburg, Austria.* Ed. by Fabio Gadducci and Alexandra Silva. Vol. 211. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021, 13:1–13:25. DOI: 10.4230/LIPICS.CALCO.2021.13 (cit. on p. 56).

[DJR21] Anuj Dawar, Tomas Jakl, and Luca Reggio. "Lovász-type theorems and game comonads". In: *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021.* IEEE, 2021, pp. 1–13. DOI: 10.1109/LICS52264.2021.9470609 (cit. on pp. 15, 44, 159).

[DKV09] Manfred Droste, Werner Kuich, and Heiko Vogler. "Handbook of weighted automata". 1st. Springer Publishing Company, Incorporated, 2009. ISBN: 3642014917. DOI: 10.1007/978-3-642-01492-5 (cit. on p. 129).

[Dvo10] Zdeněk Dvořák. "On recognizing graphs by numbers of Hhmomorphisms". In: *Journal of Graph Theory* 64.4 (Aug. 2010), pp. 330–342. ISSN: 0364-9024. DOI: 10.1002/jgt.20461 (cit. on p. 159).

[Ehr61] Andrzej Ehrenfeucht. "An application of games to the completeness problem for formalized theories". eng. In: *Fundamenta Mathematicae* 49.2 (1961), pp. 129–141 (cit. on p. 44).

[FL92] Uriel Feige and László Lovász. "Two-prover one-round proof systems: their power and their problems (Extended abstract)". In: *24th Annual ACM Symposium on Theory of Computing, Stoc 1992.* Stoc '92.

Victoria, British Columbia, Canada: Association for Computing Machinery, 1992, pp. 733–744. ISBN: 0897915119. DOI: 10.1145/129712.129783 (cit. on p. 93).

[FG06] J. Flum and M. Grohe.
"Parameterized complexity theory (Texts in theoretical computer science. An EATCS series)".
Berlin, Heidelberg: Springer-Verlag, 2006. ISBN: 3540299521.
DOI: 10.1007/3-540-29953-X (cit. on p. 46).

[Fri20a] Tobias Fritz.
"A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics".
In: *Advances in Mathematics* 370 (2020), p. 107239. ISSN: 0001-8708.
DOI: https://doi.org/10.1016/j.aim.2020.107239 (cit. on p. 130).

[Fri20b] Tobias Fritz.
"Quantum logic is undecidable".
In: *Archive for Mathematical Logic* 60.3–4 (Sept. 2020), pp. 329–341.
ISSN: 1432-0665. DOI: 10.1007/s00153-020-00749-0 (cit. on p. 130).

[FKM16] Soichiro Fujii, Shin-ya Katsumata, and Paul-André Melliès.
"Towards a formal theory of graded monads".
In: *19th International Conference on Foundations of Software Science and Computation Structures, FOSSACS 2016, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2016, Eindhoven, The Netherlands, April 2-8, 2016, Proceedings.* Ed. by Bart Jacobs and Christof Löding. Vol. 9634. Lecture Notes in Computer Science. Springer, 2016, pp. 513–530.
DOI: 10.1007/978-3-662-49630-5_30 (cit. on p. 118).

[GKO+16] Marco Gaboardi, Shin-ya Katsumata, Dominic Orchard, Flavien Breuvart, and Tarmo Uustalu.
"Combining effects and coeffects via grading".
In: *21st ACM SIGPLAN International Conference on Functional Programming.* International Conference on Functional Programming, 2016.
Nara, Japan: Association for Computing Machinery, 2016, pp. 476–489.
ISBN: 9781450342193. DOI: 10.1145/2951913.2951939 (cit. on p. 145).

[GW02] V. Galliard and S. Wolf.
"Pseudo-telepathy, entanglement, and graph colorings".
In: *IEEE International Symposium on Information Theory,* 2002, pp. 101–.
DOI: 10.1109/isit.2002.1023373 (cit. on p. 98).

[Gar19] Richard Garner.
"The Vietoris monad and weak distributive laws".

In: *Applied Categorical Structures* 28.2 (Nov. 2019), pp. 339–354.
DOI: 10.1007/s10485-019-09582-w (cit. on p. 81).

[Gir82] Michèle Giry.
"A categorical approach to probability theory".
In: *Categorical aspects of topology and analysis (Ottawa, Ont., 1980).* Vol. 915.
Lecture Notes in Mathematics. Berlin: Springer, 1982, pp. 68–85.
DOI: 10.1007/BFb0092872 (cit. on p. 81).

[Goy21] Alexandre Goy.
"On the compositionality of monads via weak distributive laws".
PhD thesis. Gif-sur-Yvette, France: Université Paris-Saclay, Oct. 2021
(cit. on pp. 58, 81).

[Goy22] Alexandre Goy.
"Weakening and Iterating Laws using String Diagrams".
2022. DOI: 10.48550/arxiv.2205.03640 (cit. on p. 81).

[GP20] Alexandre Goy and Daniela Petrişan.
"Combining probabilistic and non-deterministic choice via weak distributive laws".
In: *35th Annual ACM/IEEE Symposium on Logic in Computer Science.* LICS 2020.
Saarbrücken, Germany: Association for Computing Machinery, 2020,
pp. 454–464. ISBN: 9781450371049. DOI: 10.1145/3373718.3394795
(cit. on p. 81).

[GPA21] Alexandre Goy, Daniela Petrişan, and Marc Aiguier.
"Powerset-like monads weakly distribute over themselves in toposes and compact hausdorff spaces".
In: *48th International Colloquium on Automata, Languages, and Programming (ICALP 2021).* Ed. by Nikhil Bansal, Emanuela Merelli, and James Worrell.
Vol. 198. Leibniz International Proceedings in Informatics (LIPIcs).
Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021,
132:1–132:14. ISBN: 978-3-95977-195-5. DOI: 10.4230/LIPIcs.ICALP.2021.132
(cit. on p. 81).

[GHNW22] Erich Grädel, Hayyan Helal, Matthias Naaf, and Richard Wilke.
"Zero-one laws and almost sure valuations of first-order logic in semiring semantics".
In: *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science.* LICS '22. Haifa, Israel: Association for Computing Machinery, 2022.
ISBN: 9781450393515. DOI: 10.1145/3531130.3533358 (cit. on pp. 35, 129).

[GKT07] Todd J. Green, Grigoris Karvounarakis, and Val Tannen.
"Provenance semirings".

In: *26th ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems*. PODS 2007.
Beijing, China: Association for Computing Machinery, 2007, pp. 31–40.
ISBN: 9781595936851. DOI: 10.1145/1265530.1265535 (cit. on p. 129).

[Gro20] Martin Grohe.
"Counting bounded tree depth homomorphisms".
In: *35th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS 2020.
Saarbrücken, Germany: Association for Computing Machinery, 2020,
pp. 507–520. ISBN: 9781450371049. DOI: 10.1145/3373718.3394739
(cit. on p. 159).

[GRS22] Martin Grohe, Gaurav Rattan, and Tim Seppelt.
"Homomorphism tensors and linear equations".
In: *49th International Colloquium on Automata, Languages, and Programming, ICALP 2022, July 4-8, 2022, Paris, France*.
Ed. by Mikolaj Bojanczyk, Emanuela Merelli, and David P. Woodruff. Vol. 229.
LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022, 70:1–70:20.
DOI: 10.4230/LIPICS.ICALP.2022.70 (cit. on p. 159).

[HHI+01] Joseph Y. Halpern, Robert Harper, Neil Immerman, Phokion G. Kolaitis,
Moshe Y. Vardi, and Victor Vianu.
"On the unusual effectiveness of logic in computer science".
In: *The Bulletin of Symbolic Logic* 7.2 (2001), pp. 213–236.
ISSN: 10798986, 19435894. DOI: 10.2307/2687775 (cit. on p. 17).

[HRS08] Teiko Heinosaari, Daniel Reitzner, and Peter Stano.
"Notes on joint measurability of quantum observables".
In: *Foundations of Physics* 38.12 (Nov. 2008), pp. 1133–1147. ISSN: 1572-9516.
DOI: 10.1007/s10701-008-9256-7 (cit. on p. 87).

[HZ16] Christian Herrmann and Martin Ziegler.
"Computational complexity of quantum satisfiability".
In: *Journal of the ACM* 63.2 (May 2016). ISSN: 0004-5411. DOI: 10.1145/2869073
(cit. on pp. 129–130).

[HFR14] Chris Heunen, Tobias Fritz, and Manuel L. Reyes.
"Quantum theory realizes all joint measurability graphs".
In: *Physical Review A* 89 (3 Mar. 2014), p. 032121.
DOI: 10.1103/PhysRevA.89.032121 (cit. on p. 87).

[HKSY17] Chris Heunen, Ohad Kammar, Sam Staton, and Hongseok Yang.
"A convenient category for higher-order probability theory".
In: *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017*.
IEEE, June 2017. DOI: 10.1109/LICS.2017.8005137 (cit. on p. 81).

[HM16] Ralf Hinze and Dan Marsden.
"Equational reasoning with lollipops, forks, cups, caps, snakes, and speedometers".
In: *Journal of Logical and Algebraic Methods in Programming* 85.5, Part 2 (2016), pp. 931–951. DOI: 10.1016/j.jlamp.2015.12.004 (cit. on p. 32).

[HM23] Ralf Hinze and Dan Marsden.
"Introducing string diagrams: The art of category theory".
Cambridge University Press, 2023. DOI: 10.1017/9781009317825 (cit. on p. 32).

[Hoa69] C. A. R. Hoare.
"An axiomatic basis for computer programming".
In: *Commun. ACM* 12.10 (Oct. 1969), pp. 576–580. ISSN: 0001-0782.
DOI: 10.1145/363235.363259 (cit. on p. 129).

[IV12] Tsuyoshi Ito and Thomas Vidick.
"A multi-prover interactive proof for NEXP sound against entangled provers".
In: *2012 IEEE 53rd Annual Symposium on Foundations of Computer Science.*
IEEE. 2012, pp. 243–252 (cit. on p. 94).

[Jac04] Bart Jacobs.
"Trace Semantics for Coalgebras".
In: *Electronic Notes in Theoretical Computer Science* 106 (2004). Proceedings of the Workshop on Coalgebraic Methods in Computer Science (CMCS), pp. 167–184.
ISSN: 1571-0661. DOI: https://doi.org/10.1016/j.entcs.2004.02.031
(cit. on p. 58).

[Jac17] Bart Jacobs.
"Introduction to coalgebra".
Vol. 59. Cambridge University Press, 2017. DOI: 10.1017/CBO9781316823187
(cit. on pp. 51, 129).

[JJUW10] Rahul Jain, Zhengfeng Ji, Sarvagya Upadhyay, and John Watrous.
"Qip = Pspace".
In: *42nd ACM Symposium on Theory of Computing*. Stoc '10.
Cambridge, Massachusetts, USA: Association for Computing Machinery, 2010, pp. 573–582. ISBN: 9781450300506. DOI: 10.1145/1806689.1806768
(cit. on p. 93).

[JMS24] Tomáš Jakl, Dan Marsden, and Nihil Shah.
"A categorical account of composition methods in logic (extended version)".
2024. arXiv: 2405.06664 [cs.LO] (cit. on p. 44).

[JNV+21] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen.
"MIP* = RE".

In: *Commun. ACM* 64.11 (Oct. 2021), pp. 131–138. ISSN: 0001-0782.
DOI: 10.1145/3485628 (cit. on pp. 16, 91, 94).

[KRR14] Yael Tauman Kalai, Ran Raz, and Ron D. Rothblum.
“How to delegate computations: the power of no-signaling proofs”.
In: *Symposium on Theory of Computing, STOC 2014, New York, NY, USA, May 31 - June 03, 2014*. Ed. by David B. Shmoys. ACM, 2014, pp. 485–494.
DOI: 10.1145/2591796.2591809 (cit. on p. 94).

[KS24] Amin Karamlou and Nihil Shah.
“No go theorems: Directed containers that do not distribute over distribution monads”.
In: *39th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS 2024. Tallinn, Estonia: Association for Computing Machinery, 2024.
ISBN: 9798400706608. DOI: 10.1145/3661814.3662137 (cit. on pp. 20, 55–60, 62).

[Kie20] Sandra Kiefer.
“Power and limits of the Weisfeiler-Leman algorithm”.
PhD thesis. Dissertation, RWTH Aachen University, 2020.
DOI: 10.18154/RWTH-2020-03508 (cit. on pp. 48, 158).

[KPS18] Se-Jin Kim, Vern Paulsen, and Christopher Schafhauser.
“A synchronous game for binary constraint systems”.
In: *Journal of Mathematical Physics* 59.3 (2018). DOI: 10.1063/1.4996867 (cit. on p. 111).

[Kli11] Bartek Klin.
“Bialgebras for structural operational semantics: An introduction”.
In: *Theoretical Computer Science* 412.38 (2011). CMCS Tenth Anniversary Meeting, pp. 5043–5069. ISSN: 0304-3975.
DOI: https://doi.org/10.1016/j.tcs.2011.03.023 (cit. on p. 82).

[KS67] Simon Kochen and E. P. Specker.
“The problem of hidden variables in quantum mechanics”.
In: *Journal of Mathematics and Mechanics* 17.1 (1967), pp. 59–87.
ISSN: 00959057, 19435274 (cit. on pp. 15, 20, 88–89, 129–130, 151).

[Koc09] Joachim Kock.
“Notes on polynomial functors”.
2009 (cit. on p. 36).

[KS85] Werner Kuich and Arto Salomaa, eds.
“Semirings, automata, languages”.
Berlin, Heidelberg: Springer-Verlag, 1985. ISBN: 3540137165.
DOI: 10.1007/978-3-642-69959-7 (cit. on p. 129).

[Lal23] Olivier Lalonde.
"On the quantum chromatic numbers of small graphs".
2023. arXiv: `2311.08194 [quant-ph]` (cit. on p. 102).

[LR23] Elena Di Lavore and Mario Román.
"Evidential decision theory via partial markov categories".
In: *38th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2023, Boston, MA, USA, June 26-29, 2023.* IEEE, 2023, pp. 1–14.
DOI: `10.1109/LICS56636.2023.10175776` (cit. on p. 130).

[Lev73] Leonid Anatolevich Levin.
"Universal sequential search problems".
In: *Problemy peredachi informatsii* 9.3 (1973), pp. 115–116 (cit. on p. 130).

[Lib04] Leonid Libkin.
"Elements of finite model theory".
Vol. 41. Springer, 2004. DOI: `10.1007/978-3-662-07003-1` (cit. on p. 17).

[LMW15] Muriel Livernet, Bachuki Mesablishvili, and Robert Wisbauer.
"Generalised bialgebras and entwined monads and comonads".
In: *Journal of Pure and Applied Algebra* 219.8 (2015), pp. 3263–3278.
ISSN: 0022-4049. DOI: `https://doi.org/10.1016/j.jpaa.2014.10.013`
(cit. on p. 82).

[Lov67] László Lovász.
"Operations with structures".
In: *Acta Mathematica Academiae Scientiarum Hungarica* 18.3-4 (1967),
pp. 321–328. DOI: `10.1007/BF02280291` (cit. on p. 159).

[LFKN92] Carsten Lund, Lance Fortnow, Howard Karloff, and Noam Nisan.
"Algebraic methods for interactive proof systems".
In: *J. ACM* 39.4 (Oct. 1992), pp. 859–868. ISSN: 0004-5411.
DOI: `10.1145/146585.146605` (cit. on p. 93).

[LMR20] Martino Lupini, Laura Mančinska, and David E. Roberson.
"Nonlocal games and quantum permutation groups".
In: *Journal of Functional Analysis* 279.5 (2020), p. 108592. ISSN: 0022-1236.
DOI: `https://doi.org/10.1016/j.jfa.2020.108592` (cit. on p. 112).

[MR16a] Laura Mancinska and David E. Roberson.
"Oddities of quantum colorings".
English. In: *Baltic Journal on Modern Computing* 4.4 (Dec. 2016), pp. 846–859.
ISSN: 2255-8942. DOI: `10.22364/bjmc.2016.4.4.16` (cit. on pp. 102–104).

[MR16b] Laura Mančinska and David E Roberson.
"Quantum homomorphisms".

In: *Journal of Combinatorial Theory, Series B* 118 (2016), pp. 228–267.
DOI: `10.1016/j.jctb.2015.12.009` (cit. on pp. 16, 20, 98–99, 107).

[MR20] Laura Mančinska and David E Roberson.
"Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs".
In: *61st Annual IEEE Symposium on Foundations of Computer Science (FOCS)*.
IEEE. 2020, pp. 661–672. DOI: `10.1109/FOCS46700.2020.00067`
(cit. on pp. 112, 160).

[MSS13] Laura Mančinska, Giannicola Scarpa, and Simone Severini.
"New separations in zero-error channel capacity through projective Kochen–Specker sets and quantum coloring".
In: *IEEE Transactions on Information Theory* 59.6 (2013), pp. 4025–4032.
DOI: `10.1109/tit.2013.2248031` (cit. on p. 98).

[MM07] Ernie Manes and Philip Mulry.
"Monad compositions I: General constructions and recursive distributive laws".
In: *Theory and Applications of Categories* 18.7 (2007), pp. 172–208
(cit. on pp. 56, 68).

[MM08] Ernie Manes and Philip Mulry.
"Monad compositions II: Kleisli strength".
In: *Mathematical Structures in Computer Science* 18.3 (2008), pp. 613–643
(cit. on p. 56).

[Mar14] Daniel Marsden.
"Category theory using string diagrams".
2014. arXiv: `1401.7220 [math.CT]` (cit. on p. 32).

[Mer90] N. David Mermin.
"Simple unified form for the major no-hidden-variables theorems".
In: *Physical Review Letters* 65 (27 Dec. 1990), pp. 3373–3376.
DOI: `10.1103/PhysRevLett.65.3373` (cit. on pp. 91, 136, 138).

[Mer93] N. David Mermin.
"Hidden variables and the two theorems of John Bell".
In: *Reviews of Modern Physics* 65 (3 July 1993), pp. 803–815.
DOI: `10.1103/RevModPhys.65.803` (cit. on p. 91).

[Mog91] Eugenio Moggi.
"Notions of computation and monads".
In: *Information and computation* 93.1 (1991), pp. 55–92.
DOI: `10.1016/0890-5401(91)90052-4` (cit. on p. 34).

[Moh09] Mehryar Mohri.
"Weighted automata algorithms".

In: *Handbook of weighted automata* (2009), pp. 213–254.
DOI: `10.1007/978-3-642-01492-5_6` (cit. on p. 129).

[MS22] Yoàv Montacute and Nihil Shah.
"The pebble-relation comonad in finite model theory".
In: *37th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS 2022. Haifa, Israel: Association for Computing Machinery, 2022. ISBN: 9781450393515.
DOI: `10.1145/3531130.3533335` (cit. on p. 158).

[MNY20] Hamoon Mousavi, Seyed Sajjad Nezhadi, and Henry Yuen.
"On the complexity of zero gap MIP*".
In: *47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany (Virtual Conference).*
Ed. by Artur Czumaj, Anuj Dawar, and Emanuela Merelli. Vol. 168. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020, 87:1–87:12.
DOI: `10.4230/LIPIcs.ICALP.2020.87` (cit. on p. 94).

[Mrk20] Lovro Mrkonjić.
"The model theory of semiring semantics".
2020 (cit. on p. 129).

[MRV18] Benjamin Musto, David Reutter, and Dominic Verdon.
"A compositional approach to quantum functions".
In: *Journal of Mathematical Physics* 59.8 (2018), p. 081706.
DOI: `10.1063/1.5020566` (cit. on p. 112).

[MRV19] Benjamin Musto, David Reutter, and Dominic Verdon.
"The Morita theory of quantum graph isomorphisms".
In: *Communications in Mathematical Physics* 365.2 (2019), pp. 797–845.
DOI: `10.1007/s00220-018-3225-6` (cit. on p. 112).

[NW19] Anand Natarajan and John Wright.
"NEEXP is contained in MIP*".
In: *60th Annual IEEE Symposium on Foundations of Computer Science (FOCS).* IEEE. 2019, pp. 510–518. DOI: `10.1109/FOCS.2019.00039` (cit. on p. 94).

[NRSZ24] Prem Nigam Kar, David E. Roberson, Tim Seppelt, and Peter Zeman.
"NPA hierarchy for quantum isomorphism and homomorphism indistinguishability".
In: *arXiv e-prints*, arXiv:2407.10635 (July 2024), arXiv:2407.10635.
DOI: `10.48550/arXiv.2407.10635`. arXiv: `2407.10635 [quant-ph]`
(cit. on pp. 112, 160).

[Orc14] Dominic Orchard.
"Programming contextual computations".
Tech. rep. UCAM-CL-TR-854.

University of Cambridge, Computer Laboratory, May 2014.
DOI: 10.48456/tr-854 (cit. on pp. 36, 38, 51).

[OP16] Carlos M. Ortiz and Vern I. Paulsen.
"Quantum graph homomorphisms via operator systems".
In: *Linear Algebra and its Applications* 497 (2016), pp. 23–43. ISSN: 0024-3795.
DOI: https://doi.org/10.1016/j.laa.2016.02.019
(cit. on pp. 92, 98, 108, 111–112).

[Par20] Louis Parlant.
"Monad composition via preservation of algebras".
PhD thesis. UCL (University College London), 2020 (cit. on p. 56).

[Pau22] Vern Paulsen.
"Lecture notes on the mathematics of quantum entanglement via nonlocal games".
https://qmath.ku.dk/events/conferences/quantum-entanglement/. 2022
(cit. on p. 94).

[PSS+16] Vern I. Paulsen, Simone Severini, Daniel Stahlke, Ivan G. Todorov, and Andreas Winter.
"Estimating quantum chromatic numbers".
In: *Journal of Functional Analysis* 270.6 (2016), pp. 2188–2222. ISSN: 0022-1236.
DOI: https://doi.org/10.1016/j.jfa.2016.01.010 (cit. on pp. 98, 108).

[PT15] Vern I. Paulsen and Ivan G. Todorov.
"Quantum chromatic numbers via operator systems".
In: *The Quarterly Journal of Mathematics* 66.2 (Mar. 2015), pp. 677–692.
ISSN: 0033-5606. DOI: 10.1093/qmath/hav004. eprint: https://academic.oup.com/qjmath/article-pdf/66/2/677/6868764/hav004.pdf
(cit. on p. 108).

[Per90] Asher Peres.
"Incompatible results of quantum measurements".
In: *Physics Letters A* 151.3 (1990), pp. 107–108. ISSN: 0375-9601.
DOI: https://doi.org/10.1016/0375-9601(90)90172-K
(cit. on pp. 91, 136, 138).

[PW02] John Power and Hiroshi Watanabe.
"Combining a monad and a comonad".
In: *Theoretical Computer Science* 280.1 (2002). Coalgebraic Methods in Computer Science, pp. 137–162. ISSN: 0304-3975.
DOI: https://doi.org/10.1016/S0304-3975(01)00024-X (cit. on pp. 49, 59).

[Reg22] Luca Reggio.
"Polyadic sets and homomorphism counting".

In: *Advances in Mathematics* 410 (2022), p. 108712.
DOI: `10.1016/j.aim.2022.108712` (cit. on p. 44).

[RUV13] Ben W. Reichardt, Falk Unger, and Umesh V. Vazirani.
"A classical leash for a quantum system: command of quantum systems via rigidity of CHSH games".
In: *Innovations in Theoretical Computer Science, ITCS '13, Berkeley, CA, USA, January 9-12, 2013*. Ed. by Robert D. Kleinberg. ACM, 2013, pp. 321–322.
DOI: `10.1145/2422436.2422473` (cit. on p. 94).

[Rut98] J.J.M.M. Rutten.
"Relators and metric bisimulations".
In: *Electronic Notes on Theorertical Computer Science* 11.C (May 1998), pp. 252–258. ISSN: 1571-0661. DOI: `10.1016/s1571-0661(04)00063-5` (cit. on p. 58).

[SCM24] Marcus Schaefer, Jean Cardinal, and Tillmann Miltzow.
"The existential theory of the reals as a complexity class: A compendium".
2024. arXiv: `2407.18006 [cs.CC]` (cit. on p. 129).

[SW08] V. B. Scholz and R. F. Werner.
"Tsirelson's Problem".
2008. arXiv: `0812.4305 [math-ph]` (cit. on pp. 91, 94).

[Sha24] Nihil Shah.
"Arboreal covers over relational structures".
PhD thesis. University of Oxford, 2024 (cit. on pp. 20, 56, 71).

[Sha92] Adi Shamir.
"IP = PSPACE".
In: *Journal of the ACM* 39.4 (Oct. 1992), pp. 869–877. ISSN: 0004-5411.
DOI: `10.1145/146585.146609` (cit. on p. 93).

[Slo19a] William Slofstra.
"The Set Of Quantum Correlations Is Not Closed".
In: *Forum of Mathematics, Pi* 7 (2019), e1. DOI: `10.1017/fmp.2018.3` (cit. on pp. 94, 129).

[Slo19b] William Slofstra.
"Tsirelson's problem and an embedding theorem for groups arising from non-local games".
In: *Journal of the American Mathematical Society* 33.1 (Sept. 2019), pp. 1–56.
ISSN: 1088-6834. DOI: `10.1090/jams/929` (cit. on pp. 94, 129).

[Spe05] R. W. Spekkens.
"Contextuality for preparations, transformations, and unsharp measurements".

In: *Physical Review A* 71 (5 May 2005), p. 052108.
DOI: 10.1103/PhysRevA.71.052108 (cit. on p. 89).

[ŠB20] Ivan Šupić and Joseph Bowles.
"Self-testing of quantum systems: a review".
In: *Quantum* 4 (Sept. 2020), p. 337. ISSN: 2521-327X.
DOI: 10.22331/q-2020-09-30-337 (cit. on p. 140).

[Tsi06] Boris Tsirelson.
"Bell inequalities and operator algebras".
In: *This was posted by Boris Tsirelson as problem* 33 (2006) (cit. on pp. 91, 94).

[TP97] D. Turi and G. Plotkin.
"Towards a mathematical operational semantics".
In: *12th Annual IEEE Symposium on Logic in Computer Science*. 1997, pp. 280–291.
DOI: 10.1109/LICS.1997.614955 (cit. on pp. 50, 82).

[UMG14] Roope Uola, Tobias Moroder, and Otfried Gühne.
"Joint measurability of generalized measurements implies classicality".
In: *Physical Review Letters* 113.16 (Nov. 2014). ISSN: 1079-7114.
DOI: 10.1103/physrevlett.113.160403 (cit. on p. 87).

[UV08] Tarmo Uustalu and Varmo Vene.
"Comonadic notions of computation".
In: *Electronic Notes in Theoretical Computer Science* 203.5 (2008), pp. 263–284.
DOI: 10.1016/j.entcs.2008.05.029 (cit. on p. 36).

[VW06] Daniele Varacca and Glynn Winskel.
"Distributing probability over non-determinism".
In: *Mathematical Structures in Computer Science* 16.1 (2006), pp. 87–113.
DOI: 10.1017/s0960129505005074 (cit. on p. 56).

[Vid19] Thomas Vidick.
"Course FSMP: Interactive proofs with quantum devices".
http://users.cms.caltech.edu/~vidick/teaching/fsmp/. 2019
(cit. on p. 94).

[Von32] John Von Neumann.
"Mathematical foundations of quantum mechanics: New edition".
Vol. 53. Princeton university press, 1932. DOI: 10.2307/j.ctt1wq8zhp
(cit. on p. 88).

[Vor62] N. N. Vorob'ev.
"Consistent Families of Measures and Their Extensions".
In: *Theory of Probability & Its Applications* 7.2 (1962), pp. 147–163.
DOI: 10.1137/1107014. eprint: https://doi.org/10.1137/1107014
(cit. on p. 156).

[WBMS16] Xingyao Wu, Jean-Daniel Bancal, Matthew McKague, and Valerio Scarani. "Device-independent parallel self-testing of two singlets". In: *Physical Review A* 93 (6 June 2016), p. 062121. DOI: 10.1103/PhysRevA.93.062121 (cit. on p. 140).

[XC19] Zhen-Peng Xu and Adán Cabello. "Necessary and sufficient condition for contextuality from incompatibility". In: *Phys. Rev. A* 99 (2 Feb. 2019), p. 020103. DOI: 10.1103/PhysRevA.99.020103 (cit. on pp. 156–157).

[ZM22] Maaike Zwart and Dan Marsden. "No-Go Theorems for Distributive Laws". In: *Logical Methods in Computer Science* Volume 18, Issue 1 (Jan. 2022). DOI: 10.46298/lmcs-18(1:13)2022 (cit. on pp. 56, 81).