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Arrangements, Channel Assignments, and Associated Polynomials

Dominic J. A. Welsh

Mathematical Institute, Oxford OX1 3LB, United Kingdom
E-mail: dwelsh@maths.ox.ac.uk

and

Geoffrey P. Whittle

Mathematics Department, Victoria University, Wellington, New Zealand
E-mail: Geoff.Whittle@vuw.ac.uk

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1. INTRODUCTION

The last decade has seen some massive developments in the theory of subspace arrangements and embeddings. Much of this has been concerned primarily with arrangements over the rational, real, and complex fields though several of the enumerative properties do carry over to arbitrary finite fields. Here we interpret an arrangement in a very general sense, develop a theory of the characteristic (or equivalently Poincaré) polynomial in this general setting, and show how the theory developed has a wide range of unexpected applications.

One of the original motivations for this work was the observations that if $b_k(G; \lambda)$ denotes the number of λ-colourings of a graph G in which exactly k edges are monochromatic or bad, then for all $k \geq 1$,

$$b_k(G; \lambda) = b_k(G'_e; \lambda) - b_k(G''_e; \lambda) + b_{k-1}(G''_e, \lambda).$$

Here G'_e, G''_e have their usual meaning of G delete (respectively, contract) e. We realized that this relation, which we call the *three-term recurrence* is satisfied by several other naturally occurring counting functions. What follows is, in part, an attempt to develop a theory, including recipe type theorems, for families of such functions analogous to the very fruitful theory developed for the well-known Tutte Grothendieck (TG)-invariants.

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Coincidentally, at the same time as we were working on this purely abstract problem we were also considering the highly applicable problem of assigning radio channel frequencies. This is a problem of huge commercial significance, which has received a great deal of interest over the last few years. As we progressed we realized that one of the nicest applications of the pure theory that we had been developing was to a counting problem in this area. We describe this as our first real application in Section 8, though emphasize that historically this was not our original motivation or example. Other applications include counting lattice points, developing chromatic and Tutte type polynomials for hypergraphs, circular colourings, Redei functions, and establishing an intersection theory for subspace arrangements which greatly extends the classical results on the critical problem due to Crapo and Rota [6].

2. ARRANGEMENTS

A *subspace arrangement* $\mathscr{A}$ in $\mathbb{R}^n$ is a finite collection of proper affine subspaces of $\mathbb{R}^n$. If all the subspaces in $\mathscr{A}$ are hyperplanes, that is have dimension $n - 1$, then $\mathscr{A}$ is a *hyperplane arrangement*. The theory of hyperplane arrangements has deep connections with many areas of mathematics, see, for example, Orlik and Terao [14].

In many situations the subspace arrangement $\mathscr{A}$ can be defined over the integers, in other words each affine subspace of $\mathscr{A}$ can be expressed

$$Ax = b,$$

where A is a matrix and b a vector with integer entries. For obvious reasons such an arrangement is called a *rational arrangement*. A subspace arrangement is called *central* if all its members pass through the origin.

More generally, as in Athanasidis [2] we can consider arrangements of subspaces over any field. In particular any rational arrangement can also be regarded as an arrangement over all sufficiently large finite fields F_q.

In this paper we consider an even more general class of arrangement. The reason for this is that first our results and proofs hold in this general situation. Second, and more importantly, the transparency of the general treatment gives greater insight and unifies what seem to have been two parallel schools of researchers who seem to be unaware of the others' activities.

Accordingly we define an *arrangement* $\mathscr{A}$ to be any collection of elements of a geometric lattice L. Classical theory as described earlier is obtained by just regarding L to be the lattice of subspaces of the relevant vector or affine space.

We often denote such an arrangement $\mathscr{A}$ by the pair (E, L) where E is the set of elements of L which make up $\mathscr{A}$.

Associated with such an $\mathscr{A} = (E, L)$ are two natural rank functions on the set E. The *upper rank function*, denoted by μ, is defined for all subsets $\{a_1, \dots, a_k\}$ of E by

$$\mu(A) = r(a_1 \vee a_2 \vee \cdots \vee a_k),$$

where r is the ordinary rank function of the geometric lattice L and $\vee$ and $\wedge$ refer to L. The *lower rank* function denoted by δ, is also defined on all subsets of E, and is given for $A = \{a_1, \dots, a_k\}$ by

$$\delta(A) = r(L) - r(a_1 \wedge a_2 \wedge \cdots \wedge a_k).$$

Note. Here we adopt the convention that an empty join has rank 0, while an empty meet has rank $r(L)$: thus the empty set has both upper and lower rank 0.

A function f: $2^E \to Z$ is *submodular* if for all $A, B \subseteq E$,

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B),$$

and *supermodular* if the reverse inequality holds. It is easy to see that the upper rank function of an arrangement is submodular. However a lower rank function may be neither submodular nor supermodular as the following shows.

EXAMPLE 2.1. Let $\mathscr{A}$ be the arrangement $\{a, b, c, d\}$ of four lines in $\mathbb{R}^2$ given by

$$a \equiv x + y = 0, \qquad b \equiv y = 0, \qquad c \equiv x = 0, \qquad d \equiv x = 1.$$

Since the underlying geometric lattice has rank 3 we have

$$\delta(c) = \delta(d) = 1 \qquad \delta(c, d) = 3,$$

and we see δ is not submodular. Also,

$$\begin{aligned} \delta\{a, b\} &= 3 - r(a \wedge b) = 2, \\ \delta\{c\} &= 1, \\ \delta\{a, b, c\} &= 3 - r(a \wedge b \wedge c) = 2. \end{aligned}$$

But $\{a, b\} \cap \{c\} = \varnothing$ has lower rank 0 showing δ is not supermodular.

However we do know that if f is either the upper or lower rank function of an arrangement then

(i) $f(\phi) = 0$,

(ii) $f(A) \geq f(B)$ whenever $A \supseteq B$.

From the perspective of this paper we regard functions satisfying (i) and (ii) as abstracting properties of upper and lower rank functions of arrangements. With this perspective in mind we define a *configuration* Q to be a pair (E, f) where E is any set and f is an integer-valued set function on E satisfying (i) and (ii).

Two configurations (E_1, f_1) and (E_2, f_2) are *isomorphic* if there is a bijection ψ between E_1 and E_2 such that for any $A \subseteq E_1$, $f_1(A) = f_2(\psi(A))$ where ψ has its obvious interpretation as a map: $2^{E_1} \to 2^{E_2}$.

Clearly, from what has gone earlier, we see that each arrangement $\mathscr{A} = (E, L)$ gives rise to two distinct configurations, namely, (E, μ) and (E, δ), where μ, δ are, respectively, the upper and lower rank functions of $\mathscr{A}$.

We call these the *upper configuration* and *lower configurations* defined by $\mathscr{A}$.

A configuration $Q = (E, f)$ is *upper* (*lower*) *embeddable* in a geometric lattice L if there is an arrangement $\mathscr{A}$ on L such that Q is isomorphic to the upper (respectively, lower) configuration determined by $\mathscr{A}$.

In much of what follows we shall be obtaining results about general configurations, and most of these do not come from arrangements. However, almost all the combinatorial interpretations we present are for the case where the configuration is either the upper or lower configuration of an arrangement.

3. THE CHARACTERISTIC AND COBOUNDARY POLYNOMIALS

Given a configuration $Q = (E, f)$ its *characteristic polynomial* $\chi(Q, \lambda)$ is defined by

$$\chi(Q; \lambda) = \sum_{A \subseteq E} (-1)^{|A|} \lambda^{f(E) - f(A)}.$$

When f is the rank function of a matroid this is equivalent to the classical definition.

While the characteristic polynomial in this general form has already been studied in the literature, as far as we know, the coboundary polynomial which we now define, has only been studied in very special cases of graph colouring and matroids and even then under a variety of names.

The *coboundary polynomial* $B(Q; \lambda, s)$ is a 2-variable polynomial and has a similarly straightforward definition, namely,

$$B(Q; \lambda, s) = \sum_{A \subseteq E} (s - 1)^{|A|} \lambda^{f(E) - f(A)}. \tag{3.1}$$

One relation which we highlight is that when E is the edge set of a connected graph G and f is the rank function of the cycle matroid of G, then

$$\lambda B(Q; \lambda, s) = \sum_{k=0}^{|E|} s^k b_k(\lambda),$$

where $b_k(\lambda)$ is the number of λ-colourings of $V(G)$ in which exactly k edges are *bad*, that is, have endpoints of the same colour. In particular $b_0(\lambda)$ is the chromatic polynomial of G.

To see this, recall from Welsh [18] that when G is a graph, and $b_k(G; \lambda)$ denotes the number of λ-colourings of its vertex set V in which exactly k edges are bad (= monochrome), that is have endpoints the same colour, then

$$B(G; \lambda, s) = \sum_{k=0}^{|E|} b_k(G; \lambda) s^k = k^{k(G)}(s-1)^{V-1} T\left(G; \frac{s+\lambda-1}{s-1}, s\right)$$
$$= \lambda^{k(G)} \sum_{A \subseteq E} \lambda^{r(E)-r(A)}(s-1)^{|A|}.$$

Here r is the usual rank function of G, $k(G)$ is the number of connected components, and T is the Tutte polynomial.

Alternatively, with reparametrization

$$\lambda = Q, \qquad s = \exp(2\beta J),$$

where J is the interaction energy and β is inverse temperature, B is the partition function Z of the Q-state Potts model on G. More precisely, following [18] we can write, for any connected graph G,

$$Z(G; Q, \beta, J) = e^{-2\beta J|E|} B(G; Q, e^{2\beta J}).$$

4. FRAMED CONFIGURATIONS

For the rest of this paper we will be almost exclusively concerned with configurations and arrangements. We will denote such a configuration (E, f) by the single term Q. We call E the *groundset* and f the *rank function*. To emphasize this we will henceforth use r rather than f.

Given $Q = (E, r)$ and $e \in E$, the *deletion* Q'_e and *contraction* Q''_e are defined by $(E \setminus e, r'_e)$ and $(E \setminus e, r''_e)$, respectively, and where r'_e, r''_e are defined for $A \subseteq E$ by

$$r'_e(A) = r(A),$$
$$r''_e(A) = r(A \cup e) - r\{e\}.$$

The basic problem that arises in extending Tutte-type polynomials to configurations is the need to have useful deletion–contraction type recurrences. For example, the coboundary polynomial $B(Q; \lambda, s)$ satisfies

$$B(Q, \lambda, x) = \lambda^{r(E)-r(E\setminus e)}B(Q'_e) + (s-1)B(Q''_e).$$

The problem here is that the term $\lambda^{r(E)-r(E\setminus e)}$ depends on the choice of e. This is true for matroids also but there it causes little difficulty since we need only distinguish the case that e is a coloop or not. For more general configurations the situation is quite different, for we may have elements e for which $r(E\setminus e)$ is significantly lower than $r(E)$.

The above problem is overcome by a simple device that we now describe. We call an element e of a configuration a *spanning element* if $r(e) = r(E)$. A *framed configuration* is one with a distinguished element $\square$ such that $\square$ is a spanning element. Given a configuration Q with ground set E, we can *frame* E by adding an element $\square$ with $(r\,\square) = n$, where $n \geq r(E)$. This gives a *framed configuration* $\hat{Q}$ with rank function $r_{\hat{Q}}(A) = r_Q(A)$ if $\square \notin A$, and $r_{\hat{Q}}(A) = n$, if $\square \in A$. We then say that $\hat{Q}$ is a *framing* of Q. It is easy to verify that the following is true:

PROPOSITION 4.1. *Let Q be a configuration. Then any framing of Q is also a configuration. Moreover, for each $n \geq r(Q)$, there is a unique framing $\hat{Q}$ of Q with $r(\hat{Q}) = n$.*

In particular, for each configuration Q there is a *natural* or *canonical* framing which does not change the rank.

If $Q = (E, f)$ is a framed configuration then its *coboundary polynomial* is defined by

$$B(Q; \lambda, s) = \sum_{A \subseteq E\setminus\square} (s-1)^{|A|}\lambda^{r(E)-r(A)}. \tag{4.1}$$

Thus the only effect of the framing on B is to preserve the surroundings. In particular if Q is an unframed configuration and $Q^{\square}$ denotes its *natural framing* then

$$B(Q; \lambda, s) = B(Q^{\square}; \lambda, s).$$

We call sums of the type (3.1) and (4.1) *states models* for the function in question.

Given a framed configuration $Q = (E, r)$ we can define for any $e \in E\setminus\square$, the *deletion* Q'_e to be the framed configuration on $E\setminus e$ with rank function given by

$$r'(A) = r(A) \qquad A \subseteq E\setminus e.$$

The *contraction* Q''_e is the framed configuration $(E \setminus e, r'')$ with

$$r''(A) = r(A \cup e) - r(e), \qquad A \subseteq E \setminus e.$$

It is routine to check that these are indeed configurations, they are clearly framed. Also contraction–deletion commute. Any configuration obtained by some series of deletions and contractions is a *minor*.

We immediately have, for a framed configuration Q, and $e \neq \square$, that

$$B(Q; \lambda, s) = B(Q'_e) + (s-1) B(Q''_e). \tag{4.2}$$

In other words, provided we do not remove the special element we have an entirely satisfactory recurrence.

At this stage the device of framing may seem somewhat artificial. But framing is not artificial; as well as giving us well behaved Tutte-type recursion, framing gives us natural, and valuable, information about embeddings of configurations.

In a sense, the frame can be thought of as analogous to similar objects in the classical theory of knots and tangles.

For an arrangement $\mathscr{A} = (E, L)$ we have a canonical framing of both of its associated configurations. The *framed* upper and lower configurations are obtained by adding the element $\square$ with $r(\square) = r(L)$.

Henceforth almost all the configurations which arise in this paper will be framed configurations coming from arrangements. The framing will almost always be the canonical or natural framing.

5. THE WHITNEY POLYNOMIAL

The two variables in the Tutte polynomial of a matroid or graph reflect the fact that there are exactly two distinct single-element objects, namely, a loop and a coloop. TG-invariants of matroids are those invariants that can take independent values on these two single-element matroids, but are then completely determined by a deletion–contraction recursion. Configurations are more general and there are an infinite number of single-element structures. Indeed, for each $n > 0$, there is a single-element configuration whose rank is n. It therefore seems reasonable to allow the possibility that values of invariants are independent on distinct single-element framed configurations.

We denote the special framed configuration $E = \square$, $r(\emptyset) = 0$, $r(\square) = k$ by Z_k and as we see, the family $\{Z_k\}$ of single-element framed configurations has a special role in what follows.

We now define the *Whitney polynomial* of a framed configuration Q to be the polynomial in independent variables $\{s, z_0, z_1, \dots, z_i, \dots\}$ given by

$$W(Q; s, z_0, z_1, \dots) = W(Q) = \sum_{A \subseteq E \setminus \square} (s-1)^{|A|} z_{r(E)-r(A)}.$$

Note that putting $z_k = \lambda^k$ we get the coboundary polynomial of Q. The variables z_k correspond to the value on the single-element framed configurations Z_k of rank k as it is clear that $W(Z_k) = z_k$.

A simple example which we refer to as an illustration several times later is

EXAMPLE 5.1. Let Q_0 be the framed configuration with ground set $E = \{a, b, c, \square\}$ where

$$r(a) = r(b) = r(c) = 2, \qquad r(a,b) = r(a,c) = r(b,c) = r(a,b,c) = 3;$$

and where $r(\square) = 3$. Then

$$\begin{aligned} W(Q_0) &= (s-1)^0 z_3 + 3(s-1)z_1 + 3(s-1)^2 z_0 + (s-1)^3 z_0 \\ &= z_3 + 3(s-1)z_1 + (s-1)^2(s+2)z_0, \end{aligned}$$

or, writing it in the form $\Sigma \phi_i s^i$, we have

$$W(Q_0) = (z_3 - 3z_1 + 2z_0) + 3(z_1 - z_0)s + z_0 s^3. \tag{5.1}$$

As we noted above, the Whitney polynomial gives the coboundary polynomial by a simple substitution. It is also the case that the coboundary polynomial B uniquely determines W. To see this, we just replace each power λ^k by the indeterminate z_k in the power series expansion of B. Thus it could be argued that there is no need to introduce W. However as we see, in several of the applications below, the interesting specializations of W occur by making substitutions for the z_k which are not of the form λ^k. For example

$$z_k \to k^\lambda.$$

What does follow immediately from this correspondence and the statement (4.2) is that W satisfies the same recurrence, namely.

The Whitney polynomial of a framed configuration satisfies the following recursion for all $e \neq \square$:

$$W(Q) = W(Q'_e) + (s-1)W(Q''_e).$$

We also show that any invariant of configurations that satisfies a "reasonable" delete–contract recursion is an evaluation of the Whitney polynomial.

Furthermore if we now rewrite the Whitney polynomial in the form

$$W(Q) = \sum \phi_i(Q) s^i,$$

where for each i the polynomial ϕ_i does not involve s, then for $i \geq 0$,

$$\phi_i(Q) = \phi_i(Q'_e) - \phi_i(Q''_e) + \phi_{i-1}(Q''_e).$$

For $i = 0$, this reduces (modulo the natural assumption that ϕ_{-1} is identically zero) to the familiar recurrence for characteristic polynomials. For $i > 0$ it is the three-term recurrence.

When the Whitney polynomial, or one of the evaluations, is written in the form $\Sigma\phi_i s^i$, we say that ϕ_i is the *coefficient* of s^i even when it is a multivariate function.

6. THE THREE-TERM RECURRENCE FOR LATTICES

After obtaining different ad hoc proofs of several of the theorems described later we recognized a common thread. This turned out to be a general counting result for lattices which seems to be new and enables many of the proofs of these theorems to become machinelike. To present this theorem we need to introduce the concept of an arrangement $\mathscr{A} = (E, L)$ in which we allow E to be a family or multiset of elements of L.

We now consider operations on arrangements that correspond to deletion and contraction in the associated upper and lower configurations.

Choose $e \in E$. The *deletion* of e from $\mathscr{A}$, denoted $\mathscr{A}'_e$ is defined by $\mathscr{A}'_e = (E \backslash e, L)$.

We have two types of contraction. The *upper contraction* $\mathscr{A}''_e$ is defined by

$$\mathscr{A}''_e = \left(\{a \vee e : a \in E \backslash e\}, [e, \hat{1}]\right),$$

that is, we replace L by the interval $[e, \hat{1}]$, and replace $a \in E \backslash e$ by $a \vee e$. By an abuse of notation we regard $a \vee e$ as being labelled by a.

The *lower contraction* $\mathscr{A}'''_e$ is defined dually. Thus

$$\mathscr{A}'''_e = (\{a \wedge e : a \in E \backslash e\}, [0, e]).$$

Let $U(\mathscr{A})$ and $D(\mathscr{A})$ denote the framed upper and lower configurations of $\mathscr{A}$, respectively. The following is straightforward to prove.

PROPOSITION 6.1. *For any arrangement* $\mathscr{A} = (E, L)$ *and* $e \in E$, (i) $[U(\mathscr{A})]'_e = U(\mathscr{A}'_e)$, (ii) $[D(\mathscr{A})]'_e = D(\mathscr{A}'_e)$, (iii) $[U(\mathscr{A})]''_e = U(\mathscr{A}''_e)$, (iv) $[D(\mathscr{A})]''_e = D(\mathscr{A}'''_e)$.

That is, deletion in arrangements corresponds to deletion in both upper and lower framed configurations, while upper (respectively, lower) contraction in an arrangement corresponds to contraction in upper (respectively, lower) framed configurations.

A *coloured lattice* $\mathscr{L} = (L, B)$ is a lattice L endowed with a finite family B of elements of L. We call the members of B the *blue* elements. An *interval* of a coloured lattice is an interval $[x, y]$ of L. The blue elements of the interval $[x, y]$ are those blue elements of L that belong to $[x, y]$.

A *coloured arrangement* $\mathscr{A} = (E, \mathscr{L})$ is just a finite family E of elements of a coloured lattice $\mathscr{L}$ of the form (L, B). Note that we have no restriction that the underlying lattice of a coloured arrangement is either geometric or finite. We extend the definitions of $\mathscr{A}_e'$, $\mathscr{A}_e''$, and $\mathscr{A}_e'''$ to coloured arrangements in the obvious way.

A *member* of a coloured arrangement $\mathscr{A} = (E, L)$ is just an element of E, and an element x of L is *above* the element y if $x \geq y$. Given the coloured arrangement $\mathscr{A}$, let $\phi_c(\mathscr{A})$ denote the number of blue elements that are above exactly c members of $\mathscr{A}$.

We are finally in a position to state our three-term theorem.

THEOREM 6.2. *If e is a member of a coloured arrangement $\mathscr{A}$, then*

$$\phi_c(\mathscr{A}) = \phi_c(\mathscr{A}_e') - \phi_c(\mathscr{A}_e'') + \phi_{c-1}(\mathscr{A}_e'').$$

Proof. For positive integer k, let $S_k(\mathscr{A})$ denote the set of blue elements which are above exactly k members of $\mathscr{A}$. Let $e \in \mathscr{A}$ and let x be blue and y a member of $\mathscr{A}$. Clearly x is above both e and y iff it is above $e \vee y$. Hence from the definition of upper contraction, x is above exactly k members of $\mathscr{A}$ including e iff x is above exactly $k - 1$ members of $\mathscr{A}_e''$.

Consider $S_k(\mathscr{A}) \backslash S_k(\mathscr{A}_e')$. An element x belongs to this set iff it is above exactly k members of $\mathscr{A}$ including e. Hence

$$S_k(\mathscr{A}) \backslash S_k(\mathscr{A}_e') = S_{k-1}(\mathscr{A}_e'').$$

Consider $S_k(\mathscr{A}_e') \backslash S_k(\mathscr{A})$. An element x belongs to this set iff it is above k elements of $S_k(A_e')$ and is also above e. That is $x \in S_k(A_e') \backslash S_k(A)$ iff $x \in S_k(\mathscr{A}_e'')$. Hence

$$S_k(\mathscr{A}_e') \backslash S_k(A) = S_k(\mathscr{A}_e'').$$

Combining these observations, elementary set theory gives

$$|S_k(\mathscr{A})| = |S_{k-1}(A_e'')| + |S_k(A_e')| - |S_k(A_e'')|,$$

which completes the proof. ∎

Evidently Theorem 6.2 can be dualized. An element x of a lattice is *below* the element y if $x \leq y$. Let $\varphi_c(\mathscr{L})$ denote the number of blue members of L that are below exactly c members of $\mathscr{A}$. By lattice duality we have

COROLLARY 6.3. *If e is an element of $\mathscr{A}$, then*

$$\varphi_c(\mathscr{A}) = \varphi_c(\mathscr{A}_e) - \varphi_c(\mathscr{A}_e''') + \varphi_{c-1}(\mathscr{A}_e''').$$

The point of Theorem 6.2 and the above corollary is that they provide very general situations in which the three-term recurrence holds. In all of the examples of this paper the three-term recurrence can be established by providing an appropriate lattice-theoretic interpretation and applying Theorem 6.2 or its dual corollary.

7. COUNTING LATTICE POINTS

As one of our primary applications we consider a problem of counting lattice points in rational arrangements. In this case the information is given by an appropriate evaluation of the Whitney polynomial of the lower configuration of the arrangement. We begin by considering affine arrangements regarded as arrangements over appropriate finite fields.

THEOREM 7.1. *Let q be a prime power and $\mathscr{A}$ be an arrangement in $AG(n, q)$. Let Q be the framed lower configuration of $\mathscr{A}$. Then the coefficient of s^j in*

$$W\big(Q, s, z_0 = 0, z_k = q^{k-1}, k \geq 1\big)$$

is the number of points of $AG(n, q)$ which belong to exactly j members of $\mathscr{A}$.

Proof. For each non-negative lattice n, let $\mathscr{L}_n$ denote the coloured lattice whose underlying lattice is the lattice of subspaces of $AG(n, q)$ and whose set of blue elements is the set of points of $AG(n, q)$. Via this, the arrangement $\mathscr{A}$ becomes a coloured arrangement that we also denote by $\mathscr{A}$.

We now proceed by induction on the cardinality of E. It is easily checked that the result holds if $E = \varnothing$. Assume that $E \neq \varnothing$ and that the result holds for arrangements with less than $|E|$ elements.

Let $\phi_c(\mathscr{A})$ denote the number of blue elements that are below exactly c members of $\mathscr{A}$. It is immediate that $\phi_c(\mathscr{A})$ and $\phi_c(\mathscr{A}_e')$ count the number of points of $AG(n, q)$ that are below c members of $\mathscr{A}$ and $\mathscr{A}_e'$, respectively. By definition $\phi_c(\mathscr{A}_e''')$ is the number of blue points of $[\hat{0}, e]$ below c members of $\mathscr{A}_e'''$. This is easily checked to be the number of points of

$AG(m,q)$ that are below exactly c members of $\mathscr{A}_e'''$ where m is the rank of e in $\mathscr{A}$. (This last observation, although easy, is crucial.)

By Corollary 6.3,

$$\phi_c(\mathscr{A}) = \phi_c(\mathscr{A}_e') - \phi_c(\mathscr{A}_e''') + \phi_{c-1}(\mathscr{A}_e''').$$

Let $w_c(Q)$ denote the coefficient of s^c in $W(Q,s,z_0=0,\ z_k=q^{k-1};\ k\geq 1)$. Using Proposition 6.1 and the induction assumption we have $\phi_c(\mathscr{A}_e') = w_c(Q_e')$, $\phi_c(\mathscr{A}_e''') = w_c(Q_e'')$, and $\phi_{c-1}(\mathscr{A}_e''') = w_c(Q_e'')$. Hence

$$\phi_c(\mathscr{A}) = w_c(Q_e') - w_c(Q_e'') + w_{c-1}(Q_e'').$$

It now follows from the three-term recurrence that $\phi_c(\mathscr{L}) = w_c(Q)$ as required. ▮

We immediately obtain

COROLLARY 7.2. *The number of points of $AG(n,q)$ which belong to exactly j members of the subspace arrangement $\mathscr{A}$ depends only on the lower rank function.*

EXAMPLE 7.3. Consider the very small example consisting of the embedding of the two hyperplanes

$$a \equiv x_1 = 0 \qquad \text{and} \qquad b \equiv x_2 = 0,$$

in $AG(3,q)$.

Thus we have the arrangement $\mathscr{A} = (E, L)$ where $E(a, b, \square)$ and L is the lattice of subspaces of $AG(3,q)$.

The Whitney polynomial of the lower configuration of $\mathscr{A}$ is

$$W = \sum_{A \subseteq E \setminus \square} (s-1)^{|A|} z_{\delta(E)-\delta(A)},$$

and here $\delta(E) \equiv \delta(\square) = r(L) = 4$,

$$\begin{aligned} \delta(a) = \delta(b) \quad &= r(E) - r(a) = 1, \\ \delta(a,b) \quad &= r(E) - r(a \wedge b) = 2, \\ \delta(\varnothing) = 0. & \end{aligned}$$

Hence

$$W = z_4 + 2(s-1)z_3 + (s-1)^2 z_2.$$

Applying Theorem 7.1 with $z_k = q^{k-1}$ gives

$$q^3 + 2(s-1)q^2 + (s-1)^2 q.$$

This agrees with the checks that there are exactly

$$q^3 - 2q^2 + q = q(q-1)^2$$

points belonging to neither of these hyperplanes; exactly

$$2q^2 - 2q = 2q(q-1)$$

points belonging to exactly one of these hyperplanes, and exactly q points belong to both.

Note also that a configuration can often be embedded in strikingly different ways. For example, the rearrangements in $AG(n, q)$ consisting of three collinear points on the one hand and three non-collinear points on the other both have the same framed lower rank function.

Now consider rational arrangements. It is noted in Athanasiadis [2] that to determine the rank of a subspace of a rational arrangement one performs Gaussian elimination on linear equations with rational coefficients. Thus, if q is a sufficiently large prime, these computations also hold over F_q.

Now let $\mathscr{A}$ be a rational arrangement and for prime q, identify F_q^n with the set $\{0, 1, \dots, q-1\}^n$. Then any rational arrangement $\mathscr{A}$ in R^n gives rise to an arrangement over the finite field F_q. Thus, $F_q^n \setminus \mathscr{A}$ is the set of all $(x_1, x_2, \dots, x_n) \in F_q^n$ which do not satisfy in F_q the defining equations of any of the subspaces in $\mathscr{A}$. In this way Athanasiadis extends an earlier result of Blass and Sagan [3] by proving

THEOREM 7.4. *Let $\mathscr{A}$ be any rational subspace arrangement in $\mathbb{R}^n$ and let q be a large enough prime. Then*

$$\chi(\mathscr{A}; q) = |F_q^n \setminus \mathscr{A}|.$$

We extend this result as

THEOREM 7.5. *If $\mathscr{A}$ is any rational arrangement over $\mathbb{R}^n$ there exists a prime p such that for all prime $q \geq p$, if a_k denotes the number of points contained in exactly k members of $\mathscr{A}$, regarded as subspaces of F_q^n then*

$$\sum_{k=0} a_k s^k = W\big(Q; s, z_0 = 0, z_j = q^{j-1}, j \geq 1\big),$$

where Q is the framed lower configuration of $\mathscr{A}$.

EXAMPLE 7.6. Consider the arrangement of three lines a, b, c in $\mathbb{R}^2$ given by

$$a \equiv x = 0, \qquad b \equiv y = 0, \qquad c \equiv x = 1.$$

Regarded as an arrangement $\mathscr{A}$ its lower rank function δ is given by

$$\begin{aligned}\delta(E) &= r(L) = 3,\\ \delta(a) &= \delta(b) = \delta(c) = 3 - 2 = 1,\\ \delta(a,b) &= \delta(b,c) = 3 - 1 = 2,\\ \delta(a,c) &= \delta(\emptyset) = 0.\end{aligned}$$

So

$$W = \sum_{A \subseteq E \setminus \square} (s-1)^{|A|} z_{\delta(E) - \delta(A)}$$

is given by

$$W = z_3 + 3(s-1)z_2 + 2(s-1)^2 z_1 + \left[(s-1)^2 + (s-1)^3\right] z_0.$$

Substituting $z_i = q^{i-1}$ and putting $z_0 = 0$ gives

$$W = q^2 + 3(s-1)q + 2(s-1)^2.$$

Hence for any q such that Q is embeddable in F_q^n the number of points of F_q^n which belong to no member of Q is given by

$$q^2 - 3q + 2,$$

and the number which belong to exactly one line is

$$3q - 4,$$

and two two lines 2. ▮

8. THE CHANNEL ASSIGNMENT PROBLEM

This is a problem of huge commercial interest and with a history going back at least to 1973, see Anderson [1]. However, prompted partly by the explosion in communication theory, its importance has increased significantly.

A fairly general version of the problem can be defined as follows. Given a graph G on n vertices each edge (i, j) is assigned an integer $c_{ij} \geq 0$ which represents a threshold of *interference* between the vertices (i, j). More precisely, think of the vertices as *transmitters*, and the problem is to find a colouring w of the vertices with the integers $0, 1, \dots, k$, such that for each edge (i, j) the *colours* (= *frequencies*) $w(i), w(j)$ satisfy the constraint

$$|w(i) - w(j)| \geq c_{ij}, \tag{8.1}$$

and such that k is a minimum.

A second version of this problem, known as the *cyclic version* asks for a q-colouring where the constraint (8.1) is replaced by

$$|w(i) - w(j)|_q \geq c_{ij},$$

and where for positive integer n and $k, j \in \{0, 1, \ldots, n-1\}$ we define

$$|k - j|_n = \begin{cases} |k - j|, & \text{if } |k - j| \leq \frac{1}{2}n, \\ n - |k - j|, & \text{if } |k - j| \geq \frac{1}{2}n. \end{cases}$$

This metric is known as *cyclic distance*. For some reason cyclic channel constraints seem easier to handle, see, for example, the exact results in [8], and we concentrate on this type of constraint here.

Unlike ordinary graph colouring, the number of k-colourings either cyclic or non-cyclic, in a channel problem is *not* given by the evaluation of a polynomial. Hence there is no complete analogue of the chromatic polynomial from ordinary graph colouring. However what we can do is

THEOREM 8.1. *Consider a channel assignment problem specified by the matrix C. There exists an integer $n_0(C)$ and a sequence of polynomials $b_k(C; \lambda)$ such that for any prime $q \geq n_0(C)$, $b_k(C; q)$ counts the number of q-cyclic assignments to C in which exactly k of the constraints are broken (= not satisfied).*

Thus we have a very natural extension of the familiar concepts in the enumeration theory of ordinary graph colouring.

Proof. This is a fairly straightforward consequence of the results of the previous section. First, given the matrix C we define its *associated hyperplane arrangement* $\mathscr{A}(C)$ to be the rational arrangement

$$\{x_i - x_j = 0, \pm 1, \pm 2, \ldots, \pm c_{ij} - 1\},$$

over all (i, j) for which $c_{ij} > 0$.

Now let Q be the lower configuration of $\mathscr{A}(C)$. Applying Theorem 7.5 with q large enough gives that

$$W(Q; s, z_0 = 0, z_j = q^{j-1}, j \geq 1) = \sum s^k a_k(C),$$

where $a_k(C)$ counts the number of points of $AG(n, q)$ lying on exactly k of the hyperplanes of $\mathscr{A}(C)$. But because of the special form of this hyperplane arrangement, with a parallel class of hyperplanes corresponding to each non-zero entry of the constraint matrix, $a_k(C)$ is exactly the number of q-cyclic channel assignments in which exactly k of the constraints are broken. In other words $a_k(C) = b_k(C)$ which proves the theorem. ▮

Examination of the above proof shows that it gives the following explicit expression for the "bad colouring" polynomials $b_k(C; \lambda)$.

COROLLARY 8.2. *Let C be an $n \times n$ channel constraint matrix with associated hyperplane arrangement $(H_i;\ i \in I)$. Then for sufficiently large λ, the bad colouring polynomials $b_k(C, \lambda)$ have generating function*

$$B(\lambda, s) = \sum (s-1)^{|J|} \lambda^{d(J)},$$

where $d(J) = \dim(\bigcap H_i : i \in J)$, and J runs through all subsets of I with $d(J)$ positive and $d(\phi) = n$.

Proof. Just observe the effect of putting $z_0 = 0$, $z_j = \lambda^{j-1}$ for $j \geq 1$ in W. ▮

EXAMPLE 8.3. Consider the toy example where the constraint matrix C is given by

$$C = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

The associated hyperplane arrangement $\mathscr{A}$ is the set of seven planes

$$x_1 - x_2 = 0,$$
$$x_2 - x_3 = 0, \pm 1,$$
$$x_1 - x_3 = 0, \pm 1.$$

Denoting them, in an obvious notation by

$$H_{12}, H_{23}, H_{23}^{+}, H_{23}^{-}, H_{13}, H_{13}^{+}, H_{13}^{-},$$

we see that

$$\delta(E) = r(L) = 4, \qquad \delta(\varnothing) = 0,$$
$$\delta(H_{ij}) = r(E) - r(H_{ij}) = 1,$$
$$\delta(H_{ij}, H_{jk}^{\pm}) = r(E) - r(H_i \cap H_{jk}) = 2 \qquad \text{if } i, j, k \text{ are all different},$$
$$\delta(H_{ij}, H_{ij}^{+}) = r(E) - r(\varnothing) = 4.$$

hence the 2-sets contribute

$$\left[\binom{7}{2} - 6\right](s-1)^2 z_2 + 6(s-1)^2 z_0.$$

Some 3-sets A have $\delta(A) = 2$, the rest have $\delta(A) = 4$. Those with $\delta(A) = 2$ are

$$\{H_{12}, H_{23}, H_{13}\},$$
$$\{H_{12}, H_{23}^{+}, H_{13}^{+}\},$$
$$\{H_{12}, H_{23}^{-}, H_{13}^{-}\},$$

so they contribute $3(s-1)^3 z_2$. So W is given by

$$z_4 + 7(s-1)z_3 + 15(s-1)^2 z_2 + 6(s-1)^2 z_0$$
$$+ 3(s-1)^3 z_2 + \left[\binom{7}{3} - 3\right](s-1)^3 z_0 + \sum_{k=4}^{7}\binom{7}{k}(s-1)^k z_0.$$

Putting $z_k = \lambda^{k-1}$ we get

$$\sum b_k(\lambda)s^k = \lambda^3 + 7(s-1)\lambda^2 + 15(s-1)^2\lambda + 3(s-1)^3\lambda,$$

which gives

$$b_0(\lambda) = \lambda^3 - 7\lambda^2 + 12\lambda,$$
$$b_1(\lambda) = 7\lambda^2 - 30\lambda + 9\lambda = 7\lambda^2 - 21\lambda,$$
$$b_2(\lambda) = 15\lambda - 9\lambda = 6\lambda,$$
$$b_3(\lambda) = 3\lambda.$$

Circular colourings. Given G and integers k, d with $1 \le d \le k$, a (k,d)-*colouring* of G is a colouring ϕ of $V(G)$ with colours $\{0, 1, \dots, k-1\}$ such that if (x, y) is an edge of G then

$$d \le |\phi(x) - \phi(y)| \le k - d. \tag{8.2}$$

The *circular chromatic number* $\chi_c(G)$ is defined to be

$$\chi_c(G) = \inf\left\{\frac{k}{d} : G \text{ has a } (k,d) \text{ colouring}\right\},$$

and it is known that

$$\chi(G) - 1 \le \chi_c(G) \le \chi(G).$$

It is easy to see that the above proofs can be transformed to give a proof of an analogous result about these circular colourings. We state the result without proof.

Suppose we are given G, and $d \geq 1$. Let $b_j(G, d; k)$ denote the number of (k, d)-colourings of G in which exactly j of the edge constraints (8.2) are violated. Then we have the following theorem, its proof follows exactly the same lines as that of Theorem 8.1.

THEOREM 8.4. *Given G and $d \geq 1$, there exists prime $p = p(G, d)$ such that for all prime $q \geq p$, $b_j(G, d; q)$ is a polynomial in q with generating function the Whitney polynomial*

$$W\big(Q(\mathscr{A}); s, z_0 = 0, z_k = q^{k-1} \text{ for } k \geq 1\big).$$

Here $\mathscr{A} = \mathscr{A}(G; d)$ is the rational arrangement

$$\mathscr{A} = \big\{x_i - x_j = 0, \pm 1, \pm (d-1)\big\},$$

where (i, j) runs through all edges (i, j) of G.

The concept of circular chromatic number seems to have been first introduced by Vince [16] under the name *star chromatic number*, and for a good review of the area and its applications see Zhu [23]. Questions of counting circular colourings do not seem to have been considered in the literature though we should draw attention to the paper of de la Harpe and Jaeger [7]. In this they show for a range of colouring problems the existence of single variate counting functions which for sufficiently large integer arguments do turn out to be polynomials. It has also been shown by McDiarmid [13] using a different, ad hoc, method that for the channel problem the numbers $b_k(C, \lambda)$ are given by a polynomial for all sufficiently large integer λ.

We close with one point concerning computation. The sceptical reader may consider that the effort involved in obtaining the sequence $b_k(\lambda)$ via the Whitney function is too great. While accepting that the calculations quickly become horrendous they are *mechanical*—just involving working out ranks and can easily be automated. Compared with the effort needed to calculate say $b_1(\lambda)$ from first principles, even in an example as small as Example 8.3, the savings are huge.

9. HYPERGRAPH COLOURING

Given a hypergraph $H = (V, E)$ an *m-colouring* is a map $\phi: V \to [m]$. The colouring is *proper* if for each edge e the vertices incident with e do not all have the same colour. An edge is *bad* or *monochromatic* in ϕ if each vertex incident with it is assigned the same colour.

The idea of extending the chromatic polynomial from graphs to hypergraphs was initiated by Helgason [9], see also Whittle [20]. Here we obtain interpretations of associated Whitney and coboundary polynomials which completely generalize familiar ideas in graph colouring, and the Potts model.

Given a hypergraph $H = (V, E)$, we define its associated framed *colouring configuration* $Q(H)$, to have ground set $E \cup \square$ and rank function defined by

$$\begin{aligned} r(A) &= |V| - k(H|A), && \text{if } A \subseteq E, \\ &\equiv |V| - 1 && \text{if } \square \in A. \end{aligned}$$

Here $k(H|A)$ denotes the number of connected components of the hypergraph obtained by restricting H to the edges in A *including* isolated vertices.

Note that if H is a graph, then apart from the frame $Q(H)$ is just the usual cycle matroid of the graph.

THEOREM 9.1. *The number of* λ *colourings of a hypergraph* H *with exactly* k *monochromatic edges is given by the coefficient of* s^k *in*

$$W\big(Q(H), s, z_i = \lambda^{i+1}, i \geq 0\big).$$

Proof. Given the hypergraph $H = (V, E)$, let L be the lattice of partitions of its vertex set. We now embed the edge set E in L as follows. Suppose edge e consists of the subset U of V, then we identify e with the partition of V which has one block U and the other blocks the singleton subsets of $V \setminus U$. Thus we have E as an arrangement in the partition lattice. It is easy to check that the upper configuration of this arrangement is precisely the framed colouring configuration $Q(H)$ of H.

Now consider any λ-colouring of V. This defines a partition of V with blocks the colour sets. Thus the set of λ-colourings of V defines a natural family of elements of the partition lattice L. We emphasize that this family is a multiset. For example, in the case of 2-colourings of $\{1, \dots, 5\}$ there would be the two partitions $\{1, 2\}$ black, $\{3, 4, 5\}$ white and $\{1, 2\}$ white, $\{3, 4, 5\}$ black. The key observation is that an edge of H is *bad* in a colouring w iff the partition corresponding to w is *above* the partition corresponding to e in the lattice L.

Now consider the associated coloured arrangement $(E(H), L)$ with blue elements the family of λ-colourings of H. The number of λ-colourings of H with exactly k bad edges is $\phi_k(\mathscr{A})$ and hence by Theorem 6.2 we get that for $k \geq 1$,

$$\phi_k(\mathscr{A}) = \phi_k(\mathscr{A}_e') - \phi_k(\mathscr{A}_e'') + \phi_{k-1}(\mathscr{A}_e'').$$

It is routine to check that $\mathscr{A}_e'$ and $\mathscr{A}_e''$ are the coloured arrangements corresponding to λ-colourings of the hypergraphs H_e' and H_e'', respectively, and hence by induction we get that $\phi_k(\mathscr{A})$ is given by the coefficient of s^k in $W(Q(H); s, z_i = \lambda^{i+1}, i \geq 0)$ as required. ∎

EXAMPLE 9.2. Consider the hypergraph H with $V = \{a, b, c, d\}$ and three edges $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}$. The Whitney polynomial of $Q(H)$ is

$$W = \sum_{A \subseteq E \setminus \square} (s-1)^{|A|} z_{r(E)-r(A)}$$

$$= z_3 + 3(s-1)z_1 + 3(s-1)^2 z_0 + (s-1)^3 z_0.$$

Putting $z_i = \lambda^{i+1}$ gives

$$W\big(s, z_i = \lambda^{i+1} i \geq 0\big) = \lambda^4 + 3(s-1)\lambda^2 + \big[3(s-1)^2 + (s-1)^3\big]\lambda.$$

Comparing this with colourings of the hypergraph, if all three edges are to be bad then all vertices must have same colour. Hence $b_3(H; \lambda) = \lambda$ which is the coefficient of s^3 in W. If exactly one edge is to be bad, say $\{a, b, c\}$ then each of a, b, c must be the same colour but d must be different. Since there are three choices

$$b_1(H; \lambda) = 3(\lambda - 1),$$

as given by the coefficient of s in W.

10. THE CRITICAL PROBLEM

In the classical critical problem we are given a set S of points in a vector space over a finite field. The task is to find the minimum number k so that there exists a k-tuple of hyperplanes $(H_1, \ldots, H_k)$ such that $H_1 \cap \cdots \cap H_k \cap S = \emptyset$. This is the so-called *critical exponent* of S. Since the kernel of a linear functional is a hyperplane (or the whole space for the trivial linear functional) this critical exponent is equal to the minimum number k for which there exists a k-tuple of linear functionals $(\psi_1, \ldots, \psi_k)$ such that

$$\operatorname{Ker}(\psi_1) \cap \cdots \cap \operatorname{Ker}(\psi_k) \cap S = \emptyset.$$

Crapo and Rota [6] show that the number of such k-tuples is enumerated by the characteristic polynomial of the matroid $M(S)$ determined by S. In Whittle [21] this result is generalized to configurations representable over vector spaces. Here we generalize further by proving:

THEOREM 10.1. *Let $\mathscr{A}$ be an arrangement in $V(r, q)$ with framed upper configuration Q. Then the number of λ-tuples of linear functionals $(\psi_1, \ldots, \psi_\lambda)$*

such that

$$\mathrm{Ker}(\psi_1) \cap \mathrm{Ker}(\psi_2) \cap \cdots \cap \mathrm{Ker}(\psi_\lambda)$$

contains exactly k members of $\mathscr{A}$, is given by the coefficient of s^k in

$$W\big(Q; s, z_i = q^{i\lambda}, i \geq 0\big).$$

Proof. We can regard the arrangement $\mathscr{A}$ in $V(r, q)$ as a coloured lattice with the blue elements being the multiset of subspaces of the form $\mathrm{Ker}(\psi_1) \cap \cdots \cap \mathrm{Ker}(\psi_\lambda)$ for each λ-tuple of linear functionals $(\psi_1, \ldots, \psi_\lambda)$. In this setting, all we need is to count the number of blue elements above exactly k members of the arrangement. All one needs to note is that if $r(e) = m$, the blue elements of $\mathscr{A}_e''$ are precisely those obtained by taking intersections of λ-tuples of linear functions in $V(r - m, q)$. The rest of the proof now follows the pattern of that of Theorem 9.1. ▮

EXAMPLE 10.2. Recall our configuration Q_0 from Example 5.1. We can represent Q_0 as the upper configurations of the following distinct arrangements in the vector space $V_3(F_2)$ (equivalently $PG(2, 2)$), namely,

$$\mathscr{A}_1 = \begin{cases} \psi_1(a) \equiv x_1 = 0, \\ \psi_1(b) \equiv x_2 = 0, \\ \psi_1(c) \equiv x_3 = 0, \end{cases}$$

and

$$\mathscr{A}_2 = \begin{cases} \psi_2(a) \equiv x = 0, \\ \psi_2(b) \equiv x_2 = 0, \\ \psi_2(c) \equiv x_1 + x_2 = 0. \end{cases}$$

Clearly the kernels of nontrivial linear functionals are in 1-1 correspondence with lines of $PG(2, 2)$. Consider now the Whitney polynomial of Q_0 given by (5.1). Substituting $z_i = 2^{\lambda i}$ gives the function

$$(2^{3\lambda} - 3.2^{\lambda} + 2) + 3(2^{\lambda} - 1)s + s^3.$$

Putting $\lambda = 1$ gives the polynomial

$$4 + 3s + s^3,$$

which reflects the fact that four of the lines of $PG(2, 2)$ do not contain any of $\{a, b, c\}$, three of them contain exactly one, and the whole of $PG(2, 2)$ (the kernel of the trivial linear functional) contains all three members of $\{a, b, c\}$.

Note also, that the above example is too small to display the inherent power of Theorem 10.1 which is that all these results are *independent* of the embedding.

Finally we point out that a special case of Theorem 10.1, namely, that if ϕ is an embedding of Q in $V(r, q)$ which spans the whole space then the part of $W(Q, s, z_i = q^{i\lambda})$ which is independent of s is the usual characteristic polynomial. In other words, Theorem 10.1 is a substantial extension of the classical Crapo–Rota theorem.

11. INTERSECTION THEORY

Intersection theory in the classical sense is a subject of major importance in algebraic geometry. In 1979 Brylawski extended some of these ideas to embeddings of matroids in certain types of geometric lattices known as upper uniform lattices. Here we extend this theory even further by developing an intersection theory for arrangements in these same lattices.

Partition lattices, Boolean lattices, and lattices of subspaces of a vector space are all highly regular. Moreover, it is clear that to have a sensible notion of intersection theory the class of lattices in which we are operating must also have a considerable amount of regularity. This is captured by calling a lattice L *upper uniform* if, for all a, b with $r(a) = r(b)$, the lattices $[a, \hat{1}], [b, \hat{1}]$ are isomorphic.

In other words a geometric lattice is *upper uniform* if any two upper intervals of the same rank are isomorphic. Examples are projective geometries, partition lattices, and many others.

Let L be an upper uniform geometric lattice. Recall that the *corank* of a flat F is just $r(L) - r(F)$. For $i, k \geq 0$, let w_{ik} denote the number of flats of corank i in any upper interval of L of rank k.

Suppose that $\mathscr{A} = (E, L)$ is an arrangement in some upper uniform geometric lattice L. Let $s_{ij}(\mathscr{A})$ denote the number of flats of L of corank j which are above exactly i members of $\mathscr{A}$. Let $A_k(v) = \sum_{i=0}^{k} w_{ik} v^i$.

THEOREM 11.1. *Let Q denote the framed upper configuration of $\mathscr{A}$. Then*

$$\sum s_{ij} u^i v^j = W(Q; s = u, z_k = A_k(v), k \geq 0).$$

Proof. Choose $j > 0$. Regard $\mathscr{A}$ as a coloured arrangement by setting the blue elements to be the set of flats of L of corank j. Arguing as in the previous proofs we see that

$$W(Q; s = u, z_k = w_{jk}, k \geq 0) = \sum s_{ij} u^i.$$

Set $W_j(Q) = W(Q; s = u, z_k = w_{jk}, k \geq 0)$. Then we readily check that

$$W(Q; s = u, z_k = A_k(v); k \geq 0) = \sum_{j \geq 0} W_j(Q) v^j,$$

and the theorem follows.

COROLLARY 11.2. *The intersection numbers $s_{ij}(\mathscr{A})$ depend only on the framed upper configuration of $\mathscr{A}$.*

In other words:

COROLLARY 11.3. *The intersection numbers are independent of the embedding ψ.*

Notice that this gives the results of Brylawski [4, 5] as a very special case. Note also that whereas the intersection theory of matroids in free matroids is completely trivial, in the case of configurations this situation is genuinely interesting and non-trivial. It amounts to counting stable sets as in the next section.

EXAMPLE 11.4. Consider embeddings of the arrangements $\mathscr{A}_1$ and $\mathscr{A}_2$ with upper configuration Q_0 in $PG(2,2)$ as given in Section 10. For embeddings into $PG(2,2)$, we see that

$$\begin{aligned} A_0(v) &= 1, \\ A_1(v) &= 1 + v, \\ A_2(v) &= 1 + 3v + v^2, \\ A_3(v) &= 1 + 7v + 7v^2 + v^3. \end{aligned}$$

By Theorem 11.1, the intersection polynomial of Q_0 in $PG(2,2)$ is given by

$$\sum s_{ij} u^i v^j = W(Q_0; s = u, z_k = A_k(v), k \geq 0),$$

which works out in this case to

$$u^3 + v^3 + 7v^2 + 4v + 3uv.$$

Thus the term $3uv$ above corresponds to the three lines of $PG(2,2)$ which contain exactly one member of Q_0, the term v^3 corresponds to the empty set which contains nothing, and so on. In light of the difference between the two embeddings of Q_0 it does seem remarkable that the intersection numbers are independent of the embedding.

A geometric lattice is *lower uniform* if all of the rank i lower intervals are isomorphic. Familiar examples are projective and affine geometries and the class of geometric lattices obtained as paving matroids from Steiner systems, see [17]. However, we should note that the partition lattice is upper uniform but not lower uniform.

For a lower uniform geometric lattice L, let x_{jk} denote the number of flats of rank j contained in a lower interval of rank k. Let $\mathscr{A}$ be an arrangement in L. Let $t_{ij}(\mathscr{A})$ denote the number of flats of L of rank j which are below exactly i members of $\mathscr{A}$.

Let $B_k(v) = \sum_{j=0}^{k} x_{jk} v^j$. A dual theorem to Theorem 11.1 is:

THEOREM 11.5. *Let Q denote the framed lower configuration of $\mathscr{A}$. Then*

$$\sum t_{ij} u^i v^j = W(Q; s = u, z_k = B_k(v); k \geq 0).$$

EXAMPLE 11.6. Consider the arrangements $\mathscr{A}_1$ and $\mathscr{A}_2$ embedded in $PG(2,2)$ as in Example 10.1. They have the same upper configuration, namely, Q_0 of Example 5.1.

However, their lower configurations are different. $\mathscr{A}_1$ has lower configuration $Q_1 = (E, \delta_1)$ where $E = \{a, b, c\}$ and

$$\begin{aligned}
\delta_1(a) &= \delta_1(b) = \delta_1(c) &&= r(E) - r(a), \\
& &&= 1, \\
\delta_1(a,b) &= r(E) - r(a \wedge b) = 2, \\
&= \delta_1(a,c) = \delta_1(b,c), \\
\delta_1(a,b,c) &= r(E) - 0 = 3.
\end{aligned}$$

This gives $W(Q_1) = z_3 + 3(s-1)z_2 + 3(s-1)^2 z_1 + (s-1)^3 z_0$.

On the other hand $\mathscr{A}_2$ has lower configuration $Q_2 = (E', \delta_2)$ where

$$E' = \{a', b', c'\},$$

and

$$\begin{aligned}
\delta_2(a') &= \delta_2(b') = \delta_2(c') = 1, \\
\delta_2(a', b') &= \delta_2(a', c') = \delta_2(b', c') \\
&= \delta_2(a', b', c') = 2,
\end{aligned}$$

but $\delta_2(\square) = 3$ since we are framing $\mathscr{A}_2$ as an arrangement in $PG(2,2)$ which has rank 3. So

$$W(Q_2) = z_3 + 3(s-1)z_2 + 3(s-1)^2 z_1 + (s-1)^3 z_1.$$

By projective geometry duality

$$B_k(v) = A_k(v), \qquad k = 0, 1, 2, 3.$$

Hence substituting these in W gives the different intersection polynomials

$$I(Q_1; u, v) = u^3 + v + 3uv + 3u^2v + 3uv^2 + 4v^2 + v^3,$$

$$I(Q_2; u, v) = u^3 + 6uv + u^3v + 3uv^2 + 4v^2 + v^3.$$

Comparing the coefficients of uv shows that there are three points on one line in $\mathscr{A}_1$ and six points on one line in $\mathscr{A}_2$.

12. HYPERGRAPH POLYNOMIALS

We have already met one polynomial of hypergraphs in Section 9. Here we develop an intersection theory for hypergraphs.

Let $H = (V, E)$ be a hypergraph. We can regard H as an arrangement $\mathscr{A}(H)$ in the following way. The underlying lattice of L is the Boolean lattice of subsets of V, which is, of course, geometric and both upper and lower uniform. An edge e of H is identified in L with the subset $V(e)$ of vertices incident with it.

Let $U(H)$ denote the framed upper configuration of $\mathscr{A}(H)$ with rank function μ. Evidently for all $A \subseteq E$,

$$\mu(A) = \left| \bigcup_{a \in A} V(a) \right|,$$

and

$$\mu(\square) = |V|.$$

Notice that $U(H) = (E(H), \mu)$ is quite different from the chromatic configuration $Q(H)$ considered in Section 9. We call $U(H)$ the *Boolean configuration* of H. It is easily seen that $Q(H)$ is the Dilworth truncation of $U(H)$ in the sense of Lovász [12].

Suppose now we apply the intersection theory developed in the last section to the embedding of $U(H)$ in the upper uniform Boolean lattice.

Consider the intersection polynomial of $U(H)$. Using the notation of the previous section, we have, for the free matroid on k-elements,

$$A_k(v) = \sum_{i=0}^{k} \binom{k}{i} v^i.$$

We can now translate Theorem 11.1 to the current situation as

THEOREM 12.1. *For a hypergraph H, let s_{ij} denote the number of subsets of vertices with cardinality $|V| - i$ that contain exactly j edges of H. Then*

$$\sum s_{ij} u^i v^j = W\left(U(H); s = u, z_k = \sum_{i=0}^{k} \binom{k}{i} v^i \right).$$

This gives information of genuine interest about the hypergraph. Recall that a set of vertices in a hypergraph is said to be *stable* if it contains no edge. The coefficient of u^0 in the intersection polynomial is the generating function for stable sets of vertices so that the coefficient of $u^0 v^i$ gives the number of stable sets of cardinality $|V| - i$. More generally the coefficient of v^k is the generating function for the number of "k-defect" stable sets.

EXAMPLE 12.2. We again use our familiar configuration Q_0. We have $Q_0 = U(H)$ where H is just the graph K_3, with edge set $\{a, b, c\}$. Here we have

$$A_0 = 1, \qquad A_1 = 1 + v, \qquad A_2 = 1 + 2v + v^2,$$

and

$$A_3 = 1 + 3v + 3v^2 + v^3.$$

By Theorem 12.1 we obtain the intersection polynomial

$$3v^2 + v^3 + 3uv + u^3.$$

Thus we have one stable set of size 0, three of size 1, three sets of size 2 that contain exactly one edge, and one set of size 3 that contains all three edges.

On the other hand let $D(H)$ denote the framed lower configuration of $\mathscr{A}(H)$ with rank function δ. We have, for $A \subseteq E$,

$$\delta(A) = \left| \bigcap_{a \in A} V(a) \right|,$$

and $\delta(\square) = |V|$. Let t_{ij} denote the number of subsets of vertices of H with cardinality i that are contained in exactly j edges of H. Then we have,

COROLLARY 12.3.

$$\sum t_{ij} u^i v^j = W\left(D(H); s = u, z_k = \sum_{i=0}^{k} \binom{k}{i} v^i, k \geq 0 \right).$$

EXAMPLE 12.4. Revisiting the above example

$$W(D(H)) = z_3 + 3(s-1)z_2 + 3(s-1)^2 z_1 + (s-1)^3,$$

and we leave the reader to check that Corollary 12.3 gives

$$v^3 + 3uv^2 + 3u^2v + u^3,$$

as the generating function of the t_{ij}.

13. REDEI FUNCTIONS OF RELATIONS

Redei functions of particular relations arise in several diverse areas of classical mathematics, see Kung [10, 11]. Here we follow [10] and for $R \subseteq S \times T$ and $e \in S$ define the *deletion* of e from R to be the restriction of R to $(S \backslash e) \times T$, and denote this by R'_e. The *contraction* R''_e of e from E is the restriction of R to $(S \backslash e) \times T \backslash \{e\}^{\perp}$ where the operator $\perp$ is defined on all subsets A of S by

$$A^{\perp} = \{f \in T : eRf\}.$$

For $u = (u_1, \dots, u_\lambda)$ a λ-tuple of elements of T, the *kernel*, $\ker(u)$ is defined by

$$\ker(u) = u_1^{\perp} \cap \cdots \cap u_\lambda^{\perp}.$$

The λ-tuple u *distinguishes* S if $\ker(u) = \phi$. The *Redei function* of R, denoted by $\zeta(R; \lambda)$ is the number of λ-tuples which distinguish S and is shown in [10] to satisfy

$$\zeta(R; \lambda) = \zeta(R'_e; \lambda) - \zeta(R''_e; \lambda).$$

We extend the result of [10] follows. For nonnegative integer k, define

$$\zeta_k(R; \lambda),$$

to be the number of different λ-tuples $\langle u_1, \dots, u_\lambda \rangle$ such that

$$|\ker(u) \cap S| = k.$$

thus our $\zeta_0(R; \lambda)$ is just Kung's $\zeta(R, \lambda)$. We define the *Redei polynomial* of R to be the generating function

$$U(R; s, \lambda) = \sum_{k=0} s^k \zeta_k(R; \lambda).$$

Associated with a relation $R \subseteq S \times T$ is a framed configuration Q_R. The ground set of Q_R is $S \cup \square$. In this configuration we have, for $A \subseteq S$, $r(A) = |\{t \in T : \exists s \in S \text{ such that } (s,t) \in R\}|$ and $r(\square) = |T|$. Note that we can interpret the relation R as a hypergraph H as follows. The vertex set of H is T; the edge set is S; and edge $e \in S$ is incident with $v \in T$ if and only if $(e, v) \in R$. Under this interpretation Q_R is precisely the framed Boolean configuration of H and we obtain the following interpretation of Theorem 12.1.

THEOREM 13.1. *Let $R \subseteq S \times T$ be a relation whose associated framed configuration Q_R has Whitney polynomial*

$$W(Q_R; s, z_0, z_1, \dots).$$

Then the Redei polynomial of R is given by

$$U(R; s, \lambda) = \sum_k s^k \zeta_k(R; \lambda) = W\big(Q_R; s, z_k = k^\lambda, k \geq 0\big).$$

COROLLARY 13.2. *The Redei function $\zeta(R; \lambda)$ is given by*

$$\zeta(R; \lambda) = W\big(Q_R; 0, z_k = k^\lambda, k \geq 0\big).$$

Not surprisingly, we illustrate this with our familiar example Q_0 representing it by

$$S = \{a, b, c\}, \qquad T = \{1, 2, 3\}$$

and

$$R = \{a, 1\}, \{a, 2\}, \{b, 1\}, \{b, 3\}, \{c, 2\}, \{c, 3\}.$$

Then $Q_R = Q_0$ and since $W(Q_0) = z_3 + 3(s-1)z_1 + (s-1)^2(s+2)z_0$, by Theorem 13.1,

$$\sum_k s^k \zeta_k(R; \lambda) = (3^\lambda - 3) + 3s.$$

Thus $\zeta_0(R; \lambda) = 3^\lambda - 3$. This implies that there is no single element of T whose kernel distinguishes S, but there are six pairs of such subsets, and these are easily seen to be the six pairs (x, y) where $x \neq y$.

We have $\zeta_1 = 3$. This accords with the fact that, for $\lambda > 0$, the only λ-tuples of elements of T that distinguish all but one element of S are the three λ-tuples $\{(x, x, x, \dots, x) : x \in T\}$.

14. TWO RECIPE THEOREMS

We saw in Section 5 that for any framed configuration the Whitney polynomial W satisfies

$$W(Q) = W(Q'_e) + (s-1)W(Q''_e). \tag{14.1}$$

We now show that any invariant of a framed configuration that satisfies the three-term recurrence is an evaluation of W.

A class $\mathscr{F}$ of framed configurations is said to be *minor closed* if for all $Q \in \mathscr{F}$ and any $e \in Q$, with $e \neq \square$, then Q'_e and Q''_e belong to $\mathscr{F}$.

By analogy with TG-theory for matroids we say that a function ψ from such a minor closed class $\mathscr{F}$ into a commutative ring R is a *Whitney invariant* with respect to the ring R if the following condition hold.

(C) There are elements $\alpha, \beta \in R$ such that, for all $Q \in \mathscr{F}$, and all $e \in E(Q) - \square$:

$$\psi(Q) = \alpha\psi(Q'_e) + \beta\psi(Q''_e).$$

Just as with TG-invariants of matroids we can obtain a "recipe" theorem which essentially says that the Whitney polynomial evaluates all invariants of framed configurations satisfying a reasonable deletion–contraction recursion.

Before stating the theorem we deal with a technicality. If $\mathscr{F}$ is a minor-closed class of framed configurations, and $Q \in \mathscr{F}$, has rank n, then by deleting all elements of $E(Q) \setminus \Box$ we see that $Z_n \in \mathscr{F}$. It follows that for any natural minor-closed class we would have $Z_n \in \mathscr{F}$ for all $n \geq 0$. However this is not necessary; for example, the class of all configurations whose rank is less than a given positive integer is minor closed.

Consider a Whitney invariant ψ satisfying (C). For $i \geq 0$, set $w_i = \psi(Z_i)$ if $Z_i \in \mathscr{F}$, and $w_i = 0$, otherwise. We say that $\alpha, \beta, w_0, w_1, \dots$ are the *parameters* of ψ.

THEOREM 14.1. (i) *If R is a commutative ring then, $W(Q; s, z_0, z_1, \dots)$ is a Whitney invariant with respect to the ring $R[s, z_0, z_1, \dots]$.*

(ii) *Conversely if ψ is any Whitney invariant with respect to the ring R and ψ has parameters $\alpha, \beta, w_0, w_1, \dots$ with $\alpha \neq 0$, then for all $Q \in \mathscr{F}$,*

$$\psi(Q) = \alpha^{|E|-1} W\left(Q; \frac{\alpha+\beta}{\alpha}, w_0, w_1, \dots\right)$$

where we are interpreting the right-hand side as a formal power series.

Proof. Part (i) follows immediately from (14.1). Consider part (ii). The result clearly holds if $E(Q) = \Box$. Assume that $|E(Q)| = k \geq 2$, and assume, for induction that the result holds for all configurations in $\mathscr{F}$ whose ground set has $k-1$ elements. Then

$$\text{(i)} \quad \alpha^{|E|-1} W\left(Q; \frac{\alpha+\beta}{\alpha}, w_0, w_1, \dots\right)$$
$$= \alpha^{|E|-1} W\left(Q'_e; \frac{\alpha+\beta}{\alpha}, w_0, w_1, \dots\right)$$
$$+ \left(\frac{\alpha+\beta}{\alpha} - 1\right)^{|E|-1} W\left(Q''_e; \frac{\alpha+\beta}{\alpha}, w_0, w_1, \dots\right).$$

By induction the right-hand side of (i) is equal to

$$\text{(ii)} \quad \alpha\psi(Q'_e) + \beta\psi(Q''_e).$$

But, by the definition of ψ, (ii) is equal to $\psi(Q)$, and the theorem follows. ∎

In other words, Theorem 14.1 is saying that any Whitney invariant of a minor-closed class $\mathscr{F}$ with respect to a ring is an evaluation of W.

We are also interested in knowing how other invariants that satisfy the three-term recurrence connect with the Whitney polynomial. The crucial point is that all such invariants are determined by the Whitney polynomial. We make this precise now.

Again let $\mathscr{F}$ be a minor-closed class of framed configurations, and let R be a commutative ring with 0. Let $\Phi = (\phi_0, \phi_1, \dots, \phi_i, \dots)$ be a sequence of functions from $\mathscr{F}$ into R. Then we say Φ is a *Whitney sequence* if,

(i) for all $Q \in \mathscr{F}$ and $e \in E(Q) - \square$ and $i \geq 1$,

$$\phi_i(Q) = \phi_i(Q'_e) - \phi_i(Q''_e) + \phi_{i-1}(Q''_e),$$

(ii) for $k \geq 1$, and all n such that $Z_n \in \mathscr{F}$, $\phi_k(Z_n) = 0$,

(iii) for all $e \in E(Q) - \square$, $\phi_0(Q) = \phi_0(Q'_e) - \phi_0(Q''_e)$.

Let f be a polynomial in a number of variables including s. Then f can be written $f = \Sigma f_i s^i$, where the polynomial f_i does not involve s. We say that f_i is the *coefficient* of s^i and conversely f is the *generating function* of the f_i. With these technicalities in hand we can characterize Whitney sequences.

First note that it is almost obvious that if W is a Whitney polynomial of a minor-closed $\mathscr{F}$ then it is the generating function of some Whitney sequence of invariants of $\mathscr{F}$. The converse is not quite so obvious.

THEOREM 14.2. *Let $\Phi = \{\phi_0, \phi_1, \dots, \phi_i, \dots\}$ be a Whitney sequence on $\mathscr{F}$. Define λ_k by*

$$\lambda_k = \begin{cases} \phi_0(Z_k), & \forall Z_k \in \mathscr{F}, \\ 0 & Z_k \notin \mathscr{F}. \end{cases}$$

Then, for any $Q \in \mathscr{F}$, the Whitney polynomial

$$W(Q; s, \lambda_0, \lambda_1, \dots)$$

is the generating function of the sequence Φ.

Proof. Consider ϕ_0. We have, for any $Q \in \mathscr{F}$,

$$\phi_0(Q) = \phi_0(Q'_e) - \phi_0(Q''_e).$$

Moreover, $\phi_0(Z_i) = \lambda_i$. It now follows by Theorem 14.1 that

$$\phi_0(Q) = W(Q; 0, \lambda_0, \lambda_1, \dots).$$

But $W(Q; 0, \lambda_0, \lambda_1, \dots)$ is exactly the coefficient of s^0 in $W(Q; s, \lambda_0, \lambda_1, \dots)$.

This establishes one base for induction. For another, we note that the theorem clearly holds for configurations in $\mathscr{F}$ with a single-element ground set. Now suppose $Q \in \mathscr{F}$ has at least two elements, and $i > 0$. We make the following inductive assumption. If $Q_0 \in \mathscr{F}$ is such that $|E(Q_0)| \leq |E(Q)|$, and j is a positive integer with $j \leq i$, and at least one of these inequalities is strict, then the coefficient of s^j in $W(Q_0; s, \lambda_0, \lambda_1, \dots)$ is equal to $\phi_j(Q_0)$.

Choose an element e of Q. Write

$$W(Q; s, \lambda_0, \lambda_1, \dots) = \sum_k w_k(Q) s^k.$$

It follows from the fact that that the three-term recurrence holds for Whitney polynomials that

$$w_i(Q) = w_i(Q'_e) - w_i(Q''_e) - w_{i-1}(Q''_e).$$

By the induction assumption,

$$w_i(Q'_e) = \phi_i(Q'_e), w_i(Q''_e) = \phi_i(Q''_e),$$

and $w_{i-1}(Q''_e) = \phi_{i-1}(Q''_e)$. Also, by definition,

$$\phi_i(Q) = \phi_i(Q'_e) - \phi_i(Q''_e) + \phi_{i-1}(Q''_e).$$

Hence $w_i(Q) = \phi_i(Q)$, and the theorem now follows by induction. ∎

15. CONCLUSION

Ideally we would like to understand better (or reinterpret) the polynomials $b_k(\lambda)$ for general increasing set functions f. In particular, one guiding principle in our work was the following belief

"*If f is such that $b_0(\lambda)$ counts a natural class of objects then for all positive integer k, $b_k(\lambda)$ counts a "defect version" of these objects.*"

Finally we should mention that the examples given in the earlier sections are just illustrations. It is clear that the techniques can be applied to give similar results about counting any class of objects which can be defined via arrangements interpreted in the very general sense which we use here.

REFERENCES

1. L. G. Anderson, A simulation study of some dynamic channel assignment algorithms in a high capacity mobile telecommunications system, *IEEE Trans. Commun.* **21** (1973), 1294–1301.
2. C. A. Athanasiadis, Characteristic polynomials of subspace arrangements and finite fields, *Adv. in Math.* **122** (1996), 193–233.
3. A. Blass and B. E. Sagan, Characteristic and Ehrhart polynomials, *J. Algebraic Combin.* **7** (1998), 115–126.
4. T. H. Brylawski, Intersection theory for embeddings of matroids into uniform geometries, *Stud. Appl. Math.* **61** (1979), 211–244.
5. T. H. Brylawski, Intersection theory for graphs, *J. Combin. Theory Ser. B* **30** (1981), 233–246.
6. H. H. Crapo and G.-C. Rota, "On the Foundations of Combinatorial Theory: Combinatorial Geometries," M.I.T. Press, Cambridge, MA, 1970.
7. P. de la Harpe and F. Jaeger, Chromatic invariants for finite graphs: Theme and polynomial variations, *Linear Algebra Appl.* **226–228** (1995), 687–722.
8. J. van den Heuvel, R. Leese, and M. A. Shepherd, Graph labeling and radio channel assignment, *J. Graph Theory* **29** (1998), 263–283.
9. T. Helgason, Aspects of the theory of hypermatroids, *in* "Hypergraph Seminar" (C. Berge, D. K. Ray-Chaudhuri, Eds.), Lecture Notes in Mathematics, Vol. 411, pp. 191–214, Springer-Verlag, Berlin/New York, 1974.
10. J. P. S. Kung, The Redei function of a relation, *J. Combin. Theory Ser. A* **29** (1980), 287–296.
11. J. P. S. Kung, Critical problems, *Contemp. Math.* **197** (1996), 1–128.
12. L. Lovász, Flats in matroids and geometric graphs, *in* "Combinatorial Surveys" (P. Cameron, Ed.), pp. 45–86, Academic Press, London, 1977.
13. C. J. H. McDiarmid, "Counting and Constraint Matrices," January, 1999, private communication.
14. P. Orlik and H. Terao, "Arrangements of Hyperplanes," Grundlehren, Vol. 300, Springer-Verlag, New York, 1992.
15. J. G. Oxley, "Matroid Theory," Oxford Univ. Press, London, 1992.
16. A. Vince, Star chromatic number, *J. Graph Theory* **12** (1988), 551–559.
17. D. J. A. Welsh, "Matroid Theory," London Mathematical Society Monographs, No. 8, Academic Press, New York, 1976.
18. D. J. A. Welsh, "Complexity, Knots, Colourings and Counting," London Mathematical Society Lecture Note Series, Vol. 196, Cambridge Univ. Press, Cambridge, U.K., 1993.
19. G. P. Whittle, Characteristic polynomials of weighted lattices, *Adv. in Math.* **99** (1993), 125–151.
20. G. P. Whittle, A geometric theory of hypergraph colouring, *Aequationes Math.* **43** (1992), 45–58.
21. G. P. Whittle, The critical problem for polymatroids, *Oxford Quart. J. Math.* **45** (1994), 117–125.
22. T. Zaslavsky, The Möbius function and the characteristic polynomial, *in* "Combinatorial Geometries" (N. White, Ed.), Chap. 7, Cambridge Univ. Press, Cambridge, U.K., 1987.
23. X. Zhu, Circular chromatic number; a survey, Nov. 1997, preprint.