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Strong discrete Morse theory

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The purpose of this work is to develop a version of Forman's discrete Morse theory for simplicial complexes, based on *internal strong collapses*. Classical discrete Morse theory can be viewed as a generalization of Whitehead's collapses, where each Morse function on a simplicial complex K defines a sequence of elementary internal collapses. This reduction guarantees the existence of a CW-complex that is homotopy equivalent to K, with cells corresponding to the critical simplices of the Morse function. However, this approach lacks an explicit combinatorial description of the attaching maps, which limits the reconstruction of the homotopy type of K. By restricting discrete Morse functions to those induced by total orders on the vertices, we develop a *strong discrete Morse theory*, generalizing the strong collapses introduced by Barmak and Minian. We show that, in this setting, the resulting reduced CW-complex is regular, enabling us to recover its homotopy type combinatorially. We also provide an algorithm to compute this reduction and apply it to obtain efficient structures for complexes in the library of triangulations by Benedetti and Lutz.

Keywords: collapses; discrete Morse theory; posets; simplicial complexes; strong homotopy theory

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1. Introduction

The combinatorial analysis of homotopy types dates back to the 1940s with Whitehead's simple homotopy theory [35–38]. In an effort to describe homotopy equivalences through rigid transformations of combinatorial structures, Whitehead introduced collapses and expansions for finite simplicial complexes. These moves were inspired by Tietze transformations for group presentations. While Tietze's theory provides a complete set of transformations between equivalent presentations, Whitehead's simple homotopy theory is more restrictive and fails to capture all homotopy deformations. Nevertheless, it remains a powerful combinatorial

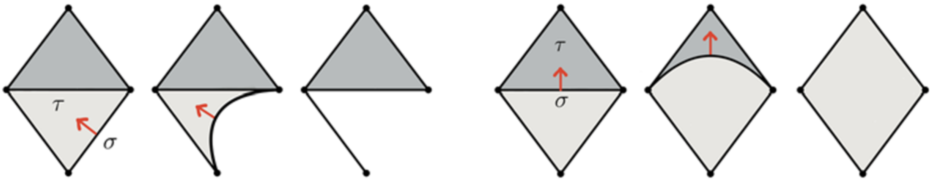


Figure 1. Collapse (left) vs internal collapse (right).

framework, influencing classical problems [1, 25]; such as Zeeman's conjecture [39]; the Andrews–Curtis conjecture [2, 20]; and the Poincaré conjecture [30, 40]; as well as modern computational approaches to analysing complexes and data [4, 5, 31].

Whitehead's elementary deformations are called *collapses* (and their reverse moves, *expansions*), which rely on the concept of a *free face*. If τ is the unique simplex of a simplicial complex K properly containing σ, then both σ and τ can be removed from K while preserving its homotopy type and simplicial structure (see Figure 1, left). However, many simplicial complexes lack free faces (e.g., triangulations of closed manifolds). A more general concept is that of an *internal free face*, leading to an *internal collapse* [16, 21, 23]. Here, σ becomes a free face of τ in a *subcomplex* $L \subseteq K$. To preserve the homotopy type of K after removing σ and τ, the simplices in $K \smallsetminus L$ must be re-attached, often losing the simplicial structure (see Figure 1, right). In general, there is no explicit or computable method to determine such attachments in arbitrary dimensions.

Discrete Morse theory [18, 19, 22], developed by Forman in the 1990s as a combinatorial analogue of smooth Morse theory for manifolds, is a tool for studying the homotopy type of simplicial complexes via real-valued functions defined on them. Concretely, given a simplicial complex K and a discrete Morse function $f\colon K \to \mathbb{R}$, there exists a reduced CW-complex built from the critical simplices of f, denoted by $\mathrm{core}_f(K)$, which is homotopy equivalent to K. Moreover, a recent result establishes a connection between discrete Morse theory and simple homotopy theory: if K has dimension N, then there exists a sequence of collapses and expansions from K to $\mathrm{core}_f(K)$ through intermediate complexes of dimension at most $N+1$ (also called an $(N+1)$-deformation and denoted by $K \diagup\!\!\searrow^{N+1} \mathrm{core}_f(K)$) [16]. This result relies on the fact that discrete Morse functions are equivalent to well-ordered sequences of internal collapses, which encode the deformation from K to $\mathrm{core}_f(K)$ [16, 23]. The construction of the reduced complex $\mathrm{core}_f(K)$, however, often sacrifices the original simplicial and combinatorial structure of K. The resulting attaching maps tend to be more intricate and are not explicitly determined, transferring the topological complexity of the space from many simplices with simple attaching maps to fewer cells (corresponding to the critical simplices) whose attaching maps are more complex. This work focuses on a particular case of discrete Morse theory in which it is possible to provide an explicit and computable method for describing these attaching maps, even in higher dimensions.

In 2012, a strong version of homotopy theory was developed by Barmak and Minian [3]. This approach builds upon simple homotopy theory by introducing *strong collapses* of simplicial complexes. Inspired by vertex reductions on

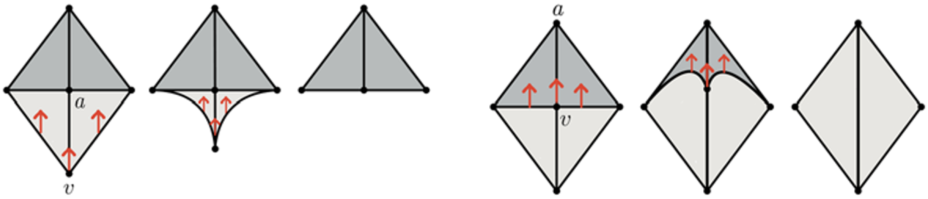


Figure 2. Strong collapse (left) vs internal strong collapse (right).

posets [33], strong collapses involve eliminating vertices v (called *dominated vertices*) whose link is a simplicial cone. In general, removing (the open star of) vertices with a contractible link preserves the homotopy type, as a consequence of the Gluing Theorem [13]. Coned links, however, offer a systematic way to simple collapse K to the subcomplex $K \setminus v$ (see Figure 2, left).

Strong collapses, as collections of simultaneous collapses, are highly efficient. Unlike Whitehead's collapses, strong collapses always yield the same minimal irreducible subcomplex — the *strong core* of K. However, the strong core of K often coincides with K itself, as many simplicial complexes lack dominated vertices. To overcome this limitation, one can consider performing strong collapses *internally*, where domination is restricted to a subcomplex of K (see Figure 2, right). In such cases, a strategy must again be developed to recompute the attaching maps of the remaining cells, potentially compromising the simplicial structure of the reduced complex.

In this work, we establish a method for studying *internal strong collapses* and prove that the reduced CW-complex obtained from any sequence of such collapses is always *regular* (i.e., its attaching maps are homeomorphisms onto their images). As a consequence, the CW-structure of the reduced complex is fully combinatorially determined by the incidences of its cells. More broadly, we develop the theoretical foundations of *strong Morse theory*, a version of discrete Morse theory in which discrete Morse functions correspond to sequences of internal strong collapses. We propose an efficient algorithm for computing the reduced complex $\mathrm{core}_f(K)$ for such discrete Morse functions f. To demonstrate its applicability, we use this method to identify more efficient regular models of the simplicial complexes in the Library of Triangulations [5–7] by Benedetti and Lutz.

1.1. Motivation and related work

Classical discrete Morse theory does not provide an explicit description of the reduced homotopy equivalent CW-complex. A discrete Morse function on a simplicial complex K induces a collection of critical simplices that correspond to the cells in the reduced CW-complex, along with information about their incidences, which is sufficient to determine the homology of the simplicial complex K. However, recovering the homotopy type of K from the reduced CW-complex requires a more refined understanding of the attaching maps. Hence, except in special cases—such as when the homotopy type is fully determined by the list of critical cells (e.g., wedges of spheres)—the algorithmic reconstruction of homotopy types from discrete Morse data remains an open problem.

Some progress has been made toward addressing this challenge. In [16], the author develops a combinatorial framework to describe the attaching maps of the reduced CW-complex for 2-dimensional simplicial complexes equipped with a discrete Morse function with a single critical 0-simplex. The approach makes strong use of the correspondence between 2-complexes and group presentations. However, it does not extend to higher dimensions, as the homotopy types of N-dimensional complexes with $N > 2$ generally cannot be fully captured combinatorially via group presentations. Rather than attempting to recover arbitrary attaching maps, the present work focuses on cases in which discrete Morse functions yield reduced complexes with particularly simple attaching maps—namely, homeomorphisms.

Our central goal is to develop a combinatorial and computable framework for a strong discrete Morse theory that allows for the explicit reconstruction of the reduced CW-complex. By restricting Morse functions to those induced by internal strong collapses, we prove that the resulting reduced complex is always a regular CW-complex, and hence its homotopy type can be entirely determined by the poset of incidence cells. Moreover, as strong Morse theory is a restricted version of classical discrete Morse theory, the reduced complex $(N+1)$-deforms to the original simplicial complex K, where N is the dimension of K. Consequently, this theory also provides a computational method to perform $(N+1)$-dimensional deformations.

In [17], the authors also establish connections between discrete Morse theory and strong collapses, focusing on general discrete Morse functions. Their work introduces a classification of internal collapses into critical and regular pairs, providing criteria to determine whether a sub-sequence of internal collapses is induced by a strong collapse. In contrast, our approach only focuses on the specific class of Morse functions derived from only internal strong collapses.

Our work builds on the main principles of Bestvina–Brady Morse theory [8, 9]. In that framework, Morse functions are piecewise-linear (PL) functions defined on affine complexes, the critical points are the vertices, and changes in the topology of sublevel sets are studied through the *descending links* of vertices. Restricting to simplicial complexes and functions defined on their vertices, our theory refines this perspective by classifying vertices according to the structure of their descending links: either as *descending dominated vertices* (their descending links are simplicial cones, and induce internal strong collapses) or as *strong critical vertices*. While Bestvina–Brady Morse theory studies local changes in the homotopy type of sublevel sets, our approach also aims at the combinatorial and global reconstruction of a reduced complex induced by these local changes. In particular, instead of treating every vertex as critical, our theory identifies a potentially smaller subset of strong critical vertices that form the cells of the final reduced complex. The remaining descending dominated vertices are systematically collapsed, leading to a complete reconstruction of the homotopy type of the original complex.

A complementary perspective is presented in [27, 28], where the authors encode the homotopy type of a simplicial complex endowed with a discrete Morse function into a combinatorially defined *flow category*. The classifying space of the flow category recovers the homotopy type of the original complex. However, these categories are often large and computationally challenging, limiting their application in algorithmic homotopy reconstruction. Indeed, while the computation of the nerve of a category can be exponential, the reconstruction of a regular CW-complex from its face poset is combinatorially direct.

1.2. Main results and outline

Let K be a finite simplicial complex, and let $g\colon V(K) \to \mathbb{R}$ be a real-valued function on its set of vertices. The function g induces a filtration of K by subcomplexes, where each subcomplex consists of all simplices whose vertices lie below a given threshold in g. For each vertex $v \in V(K)$, we consider the first subcomplex in the filtration that contains v, and define its *descending open star* as the collection of simplices in that subcomplex that contain v. A vertex is called *strong critical* if, at the filtration step where it first appears, it is not dominated by any vertex from an earlier subcomplex.

This filtration of K leads to a *strong homotopy* analogue of the classical discrete Morse lemmas (Lemma 2.4). Intervals in the filtration without strong critical vertices correspond to strong collapses, while those with strong critical vertices introduce changes in the homotopy type, determined by the simplices in its descending open stars (Lemma 3.6). This reduction process produces a sequence of internal strong collapses, resulting in a reduced complex, called the *strong internal core* and denoted by $\operatorname{core}_g(K)$. Notably, we prove that this reduced CW-complex is *regular*, that is, its attaching maps are homeomorphisms with its image.

Our main result in strong Morse theory is as follows.

Theorem (Strong Morse Theory) *Let K be a finite simplicial complex, and let $g\colon V(K) \to \mathbb{R}$ be a real-valued function on the vertex set of K. Then K is homotopy equivalent to a* ***regular*** *CW-complex $\operatorname{core}_g(K)$, whose cells are in one-to-one correspondence with the simplices in the descending open stars of the strong critical vertices of g. Moreover, if $\dim(K) = N$, then $K \nearrow\searrow^{N+1} \operatorname{core}_g(K)$.*

We also show that this result is a particular case of classical discrete Morse theory (Theorem 3.9). More precisely, for every function $g\colon V(K) \to \mathbb{R}$, there exists a discrete Morse function $f\colon K \to \mathbb{R}$ such that the critical simplices of f are exactly the simplices in the descending open stars of the strong critical vertices of g. Moreover, the CW-complexes $\operatorname{core}_f(K)$ and $\operatorname{core}_g(K)$ coincide. (By a slight abuse of notation, we denote both by core depending on the context: $\operatorname{core}_f(K)$ denotes the internal core obtained from a discrete Morse function f on simplices, and $\operatorname{core}_g(K)$ denotes the strong internal core obtained from a function g on vertices.)

The combinatorial structure of the strong internal core admits a computational interpretation of internal strong collapses, which in turn leads to an efficient algorithm for their computation (Appendix A, Algorithms A1 and A2). An implementation is publicly available at `https://github.com/ximenafernandez/Strong-Morse-Theory`. To illustrate the effectiveness of this method, we apply it to examples from the *Library of Triangulations* [7], a repository of challenging simplicial complexes frequently used in the study of homotopy types. The results of the reductions are presented in Tables 1 and 2.

Beyond its role in understanding homotopy types, the algorithmic reduction of simplicial complexes also has applications in topological data analysis, where the complexity of persistent homology computations increases rapidly with input size. For instance, computing degree-d persistent homology of the Vietoris–Rips filtration of a point cloud of size N has worst-case complexity $O(N^{3(d+2)})$ [29]. The method introduced in [12] uses strong collapses to simplify Vietoris–Rips filtrations prior to computing persistent homology. Since our results extend algorithmic simplification of complexes via internal strong collapses, they have potential applications in further optimising such filtration-based computations.

The remainder of this manuscript is organized as follows. Section 2 presents discrete Morse theory as a generalization of Whitehead's collapses. In Section 3, we introduce a strong version of discrete Morse theory, based on strong collapses, which enables the recovery of a regular structure in the reduced CW-complex (the strong internal core). Section 4 provides a combinatorial construction of the strong internal core in terms of posets and acyclic matchings. Finally, Section 5 discusses algorithms for the simplification of simplicial complexes and presents experiments on the Library of Triangulations [7]. The Appendix details algorithms for constructing the critical poset and a random strong internal core.

Notation. We use $\simeq$ for homotopy equivalences, $\cong$ for homeomorphisms, and $|\cdot|$ for geometric realizations. The symbol $\sim$ is reserved for relations. The relation $\prec$ denotes the face relation between simplices of consecutive dimension.

2. Discrete Morse theory as generalized collapses

In this section, we revisit discrete Morse theory, framing it as a combinatorial generalization of collapsibility of simplicial complexes, in the context of Whitehead's simple homotopy theory. The classical foundations of simple homotopy theory can be found in Whitehead's seminal works [35, 36, 38], Milnor's influential article [25], and Cohen's textbook [15]. Throughout this manuscript, we will restrict our attention to finite simplicial complexes.

Definition 2.1. *Let K and L be simplicial complexes. There is an* elementary collapse *from K to L if there exist simplices $\sigma, \tau \in K$ such that $L = K \smallsetminus \{\sigma, \tau\}$ and τ is the unique simplex containing σ properly in K. In this case, σ is called a* free face *of τ.*

A collapse*, denoted $K \searrow L$, is a sequence $K = K_n \searrow^{e} K_{n-1} \searrow^{e} \dots \searrow^{e} K_0 = L$ of elementary collapses from K to L. The inverse operation is called an* expansion*, denoted $L \nearrow K$. If $K \searrow L$ we say that L is a* weak core *of K, and if L has no free faces, we say it is* minimal.

*There is an N-*deformation*, denoted $K \nearrow\!\searrow^{N} L$, between K and L if there is a sequence $K = K_n, K_{n-1}, \dots, K_0 = L$ such that $K_i \searrow K_{i-1}$ or $K_i \nearrow K_{i-1}$, and* $\dim(K_i) \leq N$ *for all i. If there exists an N-deformation for some $N \in \mathbb{N}$ between K and L, it is said that they have the same* simple homotopy type.

Computing a minimal weak core of a simplicial complex provides a computational method for reducing its number of simplices while preserving its homotopy type. However, this approach is quite restrictive. For instance, many simplicial complexes lack free faces (e.g., triangulations of closed manifolds or famous examples such as the Dunce Hat [39] and Bing's House [10]). Furthermore, a simplicial complex may have different (non-isomorphic) weak cores. For instance, if D is any triangulation of the Dunce Hat, then $D \times I$ collapses to both D and the singleton $*$. Thus, the order of collapses matters. A more general approach is provided by N-deformations (e.g., the above shows that $D \nearrow\!\!\searrow^{3} *$, even though its minimal weak core is D itself). Nonetheless, the existence of an N-deformation between two simplicial complexes not computable, as this would imply being able to algorithmically determine the contractibility of spaces.

Discrete Morse theory [18, 19], inspired by its smooth counterpart [26], provides a more general combinatorial framework for simplifying the structure of a simplicial complex while preserving its homotopy type. Here, deformations are encoded in terms of discrete Morse functions.

Definition 2.2. *Let K be a simplicial complex. A function $f\colon K \to \mathbb{R}$ defined on the simplices of K is called a* discrete Morse function *if, for every simplex $\sigma \in K$, the set*

$$M(\sigma) = \{\tau \prec \sigma \colon f(\tau) \geq f(\sigma)\} \cup \{\tau \succ \sigma \colon f(\tau) \leq f(\sigma)\},$$

contains at most one element. A simplex $\sigma \in K$ is said to be critical *if $M(\sigma) = \varnothing$.*

To every simplicial complex K, we associate its *face poset* $\mathcal{X}(K)$, which is the poset of simplices of K ordered by the face relation. Each Morse function determines a pairing of simplices $M = \big\{\{\sigma, \tau\} \colon M(\sigma) = \{\tau\}\big\}$. It can be shown that M is a *matching* on the set of simplices of K. Moreover, the quotient of $\mathcal{X}(K)$ obtained by identifying matched simplices inherits a poset structure (see Section 4 for more details). This matching is referred to as an *acyclic matching* Chari [14] proved that acyclic matchings are in correspondence with discrete Morse functions.

The main theorem in discrete Morse theory establishes the connection between critical simplices of discrete Morse functions and the homotopy type of the simplicial complex.

Theorem 2.3 (Forman [16, 18, 19]). *Let K be a finite simplicial complex of dimension N, and let $f\colon K \to \mathbb{R}$ be a discrete Morse function. Then, there is an $(N+1)$-deformation from K to a CW-complex with exactly one cell of dimension d for every critical d-simplex of f.*

We denote by $\mathrm{core}_f(K)$ the CW-complex obtained from the critical simplices of f in Theorem 2.3, and refer to it as the *internal core* of K associated to f. It is important to note that Theorem 2.3 does not provide an explicit description of how the critical cells are attached along their boundaries in $\mathrm{core}_f(K)$. In [16, Prop. 1.3, Thm. 1.6], the author offers a recursive construction of the attaching maps for the critical cells, based on the finiteness of K; however, the construction is only made explicit for 2-dimensional cells.

The deformation from K to $\mathrm{core}_f(K)$ is described locally by the Morse Lemma 2.4. Any discrete Morse function $f\colon K \to \mathbb{R}$ induces a filtration of K by sublevel subcomplexes: for each $\alpha \in \mathbb{R}$, K_α is the smallest subcomplex of K containing all simplices σ with $f(\sigma) \leq \alpha$. The following result describes the evolution of the homotopy type complex along the filtration: intervals that do not contain critical simplices correspond to collapses, while those containing a single critical simplex account for homotopy changes.

Lemma 2.4. (Thm. 3.3 and 3.4, [18]). *Let $f\colon K \to \mathbb{R}$ be a discrete Morse function.*

A. *If $f^{-1}\big((\alpha,\beta]\big)$ contains no critical simplices, then $K_\beta \searrow K_\alpha$.*
B. *If $f^{-1}\big((\alpha,\beta]\big)$ contains exactly one critical k-simplex, then $K_\beta \simeq K_\alpha \cup e^k$, where e^k is a k-dimensional cell.*

From a combinatorial perspective, discrete Morse functions can be viewed as a totally ordered list of internal collapses.

Proposition 2.5. (Internal collapse [16]). *Let K be a finite simplicial complex of dimension N. If a subcomplex $K_0 \subseteq K$ collapses to a subcomplex L_0, then $K \nearrow\searrow^{N+1} L$, where L is a CW-complex obtained from L_0 by attaching one k-cell for each k-simplex in $K \smallsetminus K_0$.*

Proposition 2.5 also admits a formulation in terms of CW-complexes; see [16, Proposition 1.3]. We refer to the deformation from K to L as an *internal collapse*. The collapse takes place within a subcomplex $K_0 \subseteq K$, while the resulting deformation affects the entire complex through the reattachment of cells corresponding to the simplices in $K \smallsetminus K_0$.

Internal collapses arise naturally in discrete Morse theory: for any discrete Morse function $f\colon K \to \mathbb{R}$, the transitions between sublevel complexes as described in the Morse Lemma (Lemma 2.4) correspond to internal collapses. Specifically, if the interval $(\alpha,\beta]$ contains no critical simplices, then $K_\beta \simeq K_\alpha$, and there is an internal collapse from K to a CW-complex obtained from K_α by attaching a k-cell for each k-simplex in $K \smallsetminus K_\beta$. In general, the entire deformation from K to $\mathrm{core}_f(K)$ can be interpreted as a well-ordered sequence of internal collapses.

We summarize the viewpoint of discrete Morse theory as a generalization of classical collapses in the following statement.

Proposition 2.6. *Let K be a finite simplicial complex.*

(1) If $K \searrow L$, then there exists a discrete Morse function $f\colon K \to \mathbb{R}$ such that $\mathrm{core}_f(K) = L$.
(2) If $f\colon K \to \mathbb{R}$ is a discrete Morse function, then there exists a sequence of internal collapses from K to $\mathrm{core}_f(K)$.

3. Strong discrete Morse theory

Building on the strong homotopy theory for simplicial complexes introduced by Barmak and Minian [3], we present a version of discrete Morse theory centred on the concept of *internal strong collapses*. We recall first the notion of strong collapses.

Given a simplicial complex K and a vertex $v \in V(K)$, the *open star* of v, denoted $\mathrm{st}(v,K)$, is the collection of simplices in K that contain v. The subcomplex of simplices disjoint from v is denoted $K \smallsetminus v$. The *link* of v in K, denoted $\mathrm{lk}(v,K)$, is the subcomplex of $K \smallsetminus v$ consisting of simplices σ such that $\sigma \cup \{v\}$ is a simplex of K. A *simplicial cone* over K with apex a (a vertex not in K) is the complex $a * K$ consisting of all simplices of K, the vertex $\{a\}$, and all simplices of the form $\sigma \cup \{a\}$ with $\sigma \in K$. The *closed star* of v, denoted $\overline{\mathrm{st}}(v,K)$, is the subcomplex $v * \mathrm{lk}(v,K)$.

Definition 3.1. (Def. 2.1, [3]) *Let K be a simplicial complex. A vertex v of K is* dominated *by a vertex a if $\operatorname{lk}(v,K)$ is a simplicial cone $a * K_0$, where K_0 is a subcomplex of K. In that case, there is an* elementary strong collapse *from K to $K \smallsetminus v$, denoted $K \searrow K \smallsetminus v$. A* strong collapse *$K \searrow\!\!\searrow L$ from a simplicial complex K to a subcomplex L is a sequence of elementary strong collapses starting at K and ending at L. In that case, we say that L is a* strong core *of K. If L has no dominated vertices, we say that L is a* minimal strong core *of K.* (Note that he strong core is simply called a *core* in [3]. We will use the term *minimal strong core* in this manuscript to avoid confusion with other core constructions. In [3], it is proved that the *mimimal strong core* of a complex is unique modulo homeomorphisms).

If $v \in K$ is dominated by a, then there is an explicit strong deformation retraction $|r_a| \colon |K| \to |K \smallsetminus v|$ induced by the simplicial map

$$r_a(\sigma) = \begin{cases} \sigma & \text{if } v \notin \sigma, \\ \{a\} \cup \sigma \smallsetminus \{v\} & \text{if } v \in \sigma. \end{cases} \tag{1}$$

More generally, if $K \searrow\!\!\searrow L$ via a sequence of elementary strong collapses

$$K = K_0 \searrow\!\!\searrow^{e} K_1 \searrow\!\!\searrow^{e} \ldots \searrow\!\!\searrow^{e} K_n = L,$$

with $K_{i+1} = K_i \smallsetminus v_i$ and v_i dominated by $a_i \in K_i$, then there is a strong deformation retraction $|r| \colon |K| \to |L|$ induced by the composition

$$r := r_{a_n} \circ r_{a_{n-1}} \circ \cdots \circ r_{a_1}. \tag{2}$$

Moreover, if $K \searrow\!\!\searrow L$, then $K \searrow L$ [3, Rmk. 2.4]. That is, Whitehead's notion of collapse is weaker than the notion of strong collapse.

Theorem (Thm. 2.11. [3]). *Any sequence of strong collapses from a simplicial complex K to a subcomplex without dominated vertices yields a unique (up to isomorphism) subcomplex of K. That is, the minimal strong core of a simplicial complex is unique up to isomorphism.*

Recall that a CW-complex K is *regular* if the attaching map $\varphi \colon S^{n-1} \to K^{(n-1)}$ of each open n-cell e^n is a homeomorphism onto its image ∂e^n in the $(n-1)$-skeleton. A regular CW-complex is said to be *combinatorial* if, for each cell e^n, the attaching map — regarded as a cellular map from S^{n-1} endowed with a CW-structure — is a *combinatorial map*, meaning that it maps each open k-cell of S^{n-1} homeomorphically onto an open k-cell of K (see [20, Ch. II]). The geometric realization $|K|$ of any simplicial complex K is canonically a regular combinatorial CW-complex. Given a vertex v in a regular CW-complex K, we define the *open star* $\operatorname{st}(v,K)$ as the union of all open cells in K that contain v as a face. We denote by $\overline{e}$ the closure of a cell $e \in K$, i.e., the smallest subcomplex of K containing e, which consists of e together with all its faces. Similarly, we denote by $K \smallsetminus v$ the subcomplex of K consisting of all cells e such that $v \notin \overline{e}$.

The following result formalizes the notion of an *internal strong collapse*, illustrated in Figure 2.

Theorem 3.3 *Let K be a regular combinatorial CW-complex, and let w be a vertex of K such that the subcomplex $K \smallsetminus w$ is a simplicial complex. Suppose there exists a subcomplex $L \subseteq K \smallsetminus w$ such that $K \smallsetminus w \searrow\!\!\searrow L$ via a sequence of strong collapses.*

Then there exists a regular combinatorial CW-complex

$$Z = L \cup \bigcup_{\ell} \tilde{e}_\ell,$$

that is homotopy equivalent to K, where:

- *L is embedded as a subcomplex of Z,*
- *for each cell $e_\ell \in K$ such that $w \in \overline{e_\ell}$, there is a corresponding ℓ-cell $\tilde{e}_\ell$ in Z not contained in L, and*
- *the attaching map of each $\tilde{e}_\ell$ is induced from that of e_ℓ and the strong deformation retraction $|r|\colon |K \smallsetminus w| \to |L|$ determined by the strong collapse.*

Moreover, if $\dim(K) = N$, then $K \nearrow\!\!\searrow^{N+1} Z$.

Proof. Consider the pushout diagram

$$\begin{tikzcd}
|K \smallsetminus w| \arrow[r, "|r|"] \arrow[d, hook, "i"'] & |L| \arrow[d, "\bar{i}"] \\
|K| \arrow[r, "\bar{r}"] & Z
\end{tikzcd}$$

where i is the inclusion map and $|r|\colon |K \smallsetminus w| \to |L|$ is the strong deformation retraction induced by the strong collapse. Since $|r|$ is a homotopy equivalence and i is a closed cofibration, the Gluing Theorem (see, e.g., [13, 7.5.7, Corollary 2]) implies that $\bar{r}$ is also a homotopy equivalence.

We now define a regular combinatorial CW-structure on Z. Assume first that the collapse $K \smallsetminus w \searrow\!\!\searrow L$ is elementary, i.e., $L = (K \smallsetminus w) \smallsetminus v$ for some vertex v dominated by $a \in K \smallsetminus w$.

The 0-skeleton of Z is defined as $Z^{(0)} = L^{(0)} \cup \{w\}$. We define a map $q\colon K^{(0)} \to Z^{(0)}$ by

$$q(x) = \begin{cases} x & \text{if } x \in L^{(0)} \cup \{w\}, \\ a & \text{if } x = v, \end{cases}$$

which agrees with the retraction r on $K^{(0)} \smallsetminus \{w\}$.

Inductively, suppose that a regular combinatorial CW-structure has been defined on $Z^{(\ell-1)}$, and that the map $q\colon K^{(\ell-1)} \to Z^{(\ell-1)}$ is combinatorial and coincides with the simplicial retraction r on $(K \smallsetminus w)^{(\ell-1)}$. Let e be an ℓ-cell in K with attaching map $\varphi\colon S^{\ell-1} \to K^{(\ell-1)}$.

If $w \notin \overline{e}$, then e belongs entirely to the subcomplex $K \smallsetminus w$, and its image in Z is defined via $q = r$. Since $r\colon K \smallsetminus w \to L$ is simplicial, and the attaching maps in $K \smallsetminus w$ are simplicial, it follows that $q(e)$ inherits a regular combinatorial CW-structure. In particular, if $e \in L$, then q acts as the identity on e.

If $w \in \overline{e}$, we distinguish three cases:

(1) If $v \notin \overline{e}$, then q acts as the identity on e, and the attaching map of e remains unchanged in Z, inheriting the regular combinatorial CW-structure from K.
(2) If $v \in \overline{e}$ but $a \notin \overline{e}$, then r sends v to a in $\partial e \cap (K \smallsetminus w)$, resulting in a relabelling, inducing a bijective identification on the boundary. The attaching map $\tilde{\varphi}\colon S^{\ell-1} \to Z^{(\ell-1)}$ of $\tilde{e} \in Z$ is given by $\tilde{\varphi} := r \circ \varphi$, and the image cell $q(e)$ is obtained from e via this relabelling.
(3) If both v and a belong to $\overline{e}$, then the retraction r modifies the attaching map non-trivially. Since K is regular and combinatorial, the boundary ∂e is a PL $(\ell - 1)$-sphere. Let

$$A := |\overline{\mathrm{st}}(v, K \smallsetminus w)| \cap \partial e, \quad B := |\mathrm{lk}(v, K \smallsetminus w)| \cap \partial e.$$

Then A is a PL $(\ell - 1)$-ball and B a PL $(\ell - 2)$-ball, and the strong collapse induces a retraction $r|_A\colon A \to B$. By [32, Ch. 3], there exist regular neighbourhoods $U \supseteq A$ and $V \supseteq B$ in ∂e and a map $f\colon \partial e \to \partial e$ such that $f|_A = r|_A$, and $f|_{\partial e \setminus U}\colon \partial e \setminus U \to \partial e \setminus V$ is a PL homeomorphism. This induces a homeomorphism

$$\bar{f}\colon \partial e/\sim \longrightarrow \partial e, \quad x \sim r(x) \text{ for } x \in A.$$

If $\varphi\colon S^{\ell-1} \to \partial e$ is the original attaching map of e, then the modified attaching map is given by

$$\tilde{\varphi} := \bar{f}^{-1} \circ \varphi\colon S^{\ell-1} \longrightarrow \partial e/\sim\, \subseteq Z^{(\ell-1)}.$$

Therefore, the cell $\tilde{e}$ is attached regularly, and inherits a combinatorial CW-structure. Define $q(e) = \tilde{e}$.

The general case follows by induction on the number of elementary strong collapses in the sequence

$$K \smallsetminus w = K_0 \searrow\!\searrow K_1 \searrow\!\searrow \cdots \searrow\!\searrow K_n = L.$$

At each step, the same argument applies to the retraction $K_i \to K_{i+1}$, ensuring that the regularity and combinatorial structure are preserved. Since w is not removed in the process, the identification maps never collapse an entire boundary sphere to a point.

Thus, Z is a regular combinatorial CW-complex homotopy equivalent to K. The fact that this is an $(N + 1)$-deformation follows from [16, Prop. 1.3]. ☐

The previous result generalizes to the case where $K_0 \searrow\!\searrow L_0$ and K_0 is obtained from a simplicial complex K by successively removing vertices (together with their open stars). In this case, K_0 is a *full subcomplex* of K, meaning that every simplex of K whose vertices all belong to $V(K_0)$ is itself a simplex in K_0.

THEOREM 3.4 (Strong internal collapse). *Let K be a simplicial complex of dimension N, and let $K_0 \subseteq K$ be a full subcomplex. Suppose K_0 strong collapses to a subcomplex L_0. Then K is $(N + 1)$-homotopy equivalent to a combinatorial regular CW-complex L obtained from L_0 by attaching one k-cell for each k-simplex in $K \smallsetminus K_0$.*

Proof. Follows by induction on the vertices in $V(K) \setminus V(K_0)$, applying Theorem 3.3 at each step. ☐

Under the assumptions of Theorem 3.4, we say that there is a *strong internal collapse* from K to L.

Let K be a finite simplicial complex, and let $g\colon V(K) \to \mathbb{R}$ be a real-valued function on its vertex set. The function g induces a filtration of K by sublevel subcomplexes $\{K_\alpha\}_{\alpha\in\mathbb{R}}$, where each K_α is the subcomplex spanned by the vertices in the sublevel set $g^{-1}\big((-\infty, \alpha]\big)$.

Definition 3.5. *Let $g\colon V(K) \to \mathbb{R}$ be a real-valued function, and let $v \in V(K)$ be a vertex. The* descending open star *of v, denoted $\mathrm{st}^{\downarrow}(v, K)$, is the open star of v in the subcomplex $K_{g(v)}$. Similarly, the* descending link *of v, denoted $\mathrm{lk}^{\downarrow}(v, K)$, is the link of v in $K_{g(v)}$. We say that v is* descending dominated *if it is dominated in $K_{g(v)}$ by a vertex a with $g(a) < g(v)$. Otherwise, v is called a* strong critical vertex.

We have an analogous version of the classical discrete Morse Lemma 2.4 in this setting.

Lemma 3.6. *Let K be a simplicial complex and $g\colon V(K) \to \mathbb{R}$ a real-valued function.*

A. If $g^{-1}\big((\alpha, \beta]\big)$ contains no strong critical vertices, then $K_\beta \searrow\searrow K_\alpha$.
B. If $g^{-1}\big((\alpha, \beta]\big)$ contains a single strong critical vertex w, then K_β is homotopy equivalent to a CW-complex of the form

$$K_\alpha \cup \bigcup_i e_i,$$

where the cells e_i are in one-to-one correspondence with the simplices in $\mathrm{st}^{\downarrow}(w, K)$. This correspondence preserves dimensions, and the attaching maps of the cells are determined as in Theorem 3.3, resulting in a regular CW-complex structure.

Proof. The function g induces a total order $\{v_1, v_2, \dots, v_n\}$ on $V(K)$ such that $g(v_i) \leq g(v_j)$ whenever $i < j$. For each i, let K_i be the subcomplex of K generated by all simplices whose vertices lie in $\{v_1, v_2, \dots, v_i\}$.

For part A., assume that $g^{-1}\big((\alpha, \beta]\big)$ contains no strong critical vertices. Then for each $v_i \in g^{-1}\big((\alpha, \beta]\big)$, the descending link $\mathrm{lk}^{\downarrow}(v_i, K) = \mathrm{lk}(v_i, K_{g(v_i)})$ is a simplicial cone with apex a_i, where $g(a_i) < g(v_i)$. Since $K_i \subseteq K_{g(v_i)}$, we have

$$\mathrm{lk}(v_i, K_i) = \mathrm{lk}(v_i, K_{g(v_i)}) \setminus \{\sigma \in \mathrm{lk}(v_i, K_{g(v_i)})\colon \sigma \ni v_j \text{with } j > i \text{and } g(v_j) = g(v_i)\}.$$

This subcomplex remains a cone with apex a_i. Therefore, v_i is dominated in K_i by a_i, and $K_i \searrow\searrow^e K_i \setminus v_i = K_{i-1}$. Applying this inductively, we obtain a sequence of strong collapses from K_β to K_α.

Part B. follows from part A. and Theorem 3.4. ☐

Building on Theorem 3.4 and Lemma 3.6, we establish the following result, which formalize the relationship between internal strong collapses, regular CW-complexes, and discrete Morse theory.

THEOREM 3.7 (Strong discrete Morse theory). *Let K be a finite simplicial complex, and let $g\colon V(K) \to \mathbb{R}$ be a real-valued function. Then, K is homotopy equivalent to a* regular *CW-complex* $\mathrm{core}_g(K)$*, whose cells are in one-to-one correspondence with the simplices in the descending open star of the strong critical vertices of g.*

Proof. Subdivide the image of g into subintervals, each satisfying the conditions of Lemma 3.6 A. or B., and iteratively apply the corresponding deformations to K. The cells in the resulting CW-complex correspond to the simplices in $\{\mathrm{st}^{\downarrow}(w, K)\colon w \text{is a critical vertex in } K\}$. ☐

We denote by $\mathrm{core}_g(K)$ the regular CW-complex obtained from the critical simplices of $g\colon V(K) \to \mathbb{R}$ in Theorem 3.7, and refer to it as the *strong internal core* of K associated to g. Notice that the strong internal core depends on a choice of dominating vertices for the strong internal collapses, but we omit this information unless necessary.

PROPOSITION 3.8. (Connection to strong collapses). *If $K \searrow\searrow L$, then there exists a function $g\colon V(K) \to \mathbb{R}$ such that* $\mathrm{core}_g(K) = L$*.*

Proof. Let $v_1, v_2, \ldots, v_n \in V(K)$ be the sequence of dominated vertices corresponding to a strong collapse from K to L, i.e., define $K_n = K$ and $K_{i-1} = K_i \setminus v_i$, with each $K_i \searrow\searrow^e K_{i-1}$ and $K_0 = L$. Define a function $g\colon V(K) \to \mathbb{R}$ by

$$g(v) = \begin{cases} i & \text{if } v = v_i, \\ 0 & \text{if } v \in V(L). \end{cases}$$

We claim that the strong internal core, $\mathrm{core}_g(K)$, coincides with L. By construction, each v_i is dominated in K_i by some vertex $a_i \in K_i$. Since v_i is removed at stage i, and $a_i \in K_j$ for some $j < i$, it follows that $g(a_i) < g(v_i)$. Therefore, v_i is descending dominated. On the other hand, any vertex $v \in V(L)$ satisfies $g(v) = 0$, and hence $g^{-1}((-\infty, g(v))) = \varnothing$. Thus, v cannot be descending dominated and is a strong critical vertex.

Hence, the only strong critical vertices of g are those in L, and the strong internal core, $\mathrm{core}_g(K)$, coincides with L. ☐

THEOREM 3.9 (Connection to discrete Morse theory). *Let K be a finite simplicial complex and let $g\colon V(K) \to \mathbb{R}$ be a real-valued map. Then, there exists a discrete Morse function $f\colon K \to \mathbb{R}$ on K such that its critical simplices are the simplices in $\{\mathrm{st}^{\downarrow}(w, K) : w \text{is a strong critical vertex of } g\}$. Moreover,* $\mathrm{core}_g(K) \cong \mathrm{core}_f(K)$*, that is the strong internal core associated to g is precisely the internal core associated to f.*

Proof. Let $C = \{\mathrm{st}^{\downarrow}(w, K) : w \text{is a strong critical vertex of } g\} \subseteq K$. Given the correspondence between discrete Morse functions and acyclic matchings [14], we will construct an acyclic matching M with critical simplices C as follows. For each simplex $\sigma \in K$ containing descending dominated vertices, let

$$v_\sigma = \arg\max\{g(v)\colon v \in \sigma \text{is a descending dominated vertex}\},$$

and let a_σ denote the vertex dominating v_σ. The pairing M is then defined as follows: if $a_\sigma \in \sigma$, we include the pair $(\sigma \setminus \{a_\sigma\}, \sigma)$ in M; otherwise, we include the pair $(\sigma, \sigma \cup \{a_\sigma\})$ in M.

To verify that M is a matching, assume for contradiction that a simplex in K is matched to more than one simplex. Without loss of generality, suppose $(\sigma, \{a\} \cup \sigma) \in M$ with v_σ the associated descending dominated vertex. If there exists another pair $(\{a\} \cup \sigma, \{a', a\} \cup \sigma) \in M$, then a' dominates a vertex $v' \in \sigma$. Since v' must also satisfy $v' = \arg\max\{g(v)\colon v \in \sigma \text{ is descending dominated}\}$ this contradicts the uniqueness of v_σ. Alternatively, if there exists a pair $(\sigma', \{a\} \cup \sigma) \in M$, then $a\sigma = \{a'\} \cup \sigma'$ implies $\sigma = \{a'\} \cup \tau$ and $\sigma' = \{a\} \cup \tau$ for some simplex τ. This forces τ to contain two distinct dominated vertices, v_σ and v', each maximizing $g(v)$, which is again a contradiction. Therefore, M is a valid matching.

To establish that M is acyclic, consider any pair $(\sigma, \{a\} \cup \sigma) \in M$. For any $\tau \succ \sigma$, we claim that $(\tau \smallsetminus \{a'\}, \tau) \notin M$ for any $a' \neq a$. Since $\tau \succ \sigma$, the simplex τ contains v_σ. If v_σ also satisfies $v_\sigma = \arg\max\{g(v)\colon v \in \tau \text{is a descending dominated vertex}\}$, then $(\tau, \{a\} \cup \tau) \in M$, as a dominates v_σ. Alternatively, if there exists another dominated vertex $w \in \tau$ such that $w > v_\sigma$, then $(\tau, \{a_w\} \cup \tau) \in M$, where a_w dominates w. In either case, $(\tau \smallsetminus \{a'\}, \tau) \notin M$ for any $a' \neq a$, ensuring that M is acyclic. This completes the proof. □

REMARK 3.10 (Connection to Bestvina–Brady Morse theory). Let K be a finite simplicial complex. A Bestvina–Brady Morse function $h\colon |K| \to \mathbb{R}$ is given by an injective function $g\colon V(K) \to \mathbb{R}$ that extends linearly over simplices. In this setting, the critical points of h are precisely all the vertices of K, and the changes in homotopy type of sublevel sets occur at critical levels by attaching cones over the descending links of the corresponding vertices. In our framework, we also consider a vertex function g, but further classify each vertex as either *strong critical* or *descending dominated*, depending on whether its descending link forms a simplicial cone. This distinction not only describes the local homotopy behaviour (descending dominated vertices have coned descending links and, hence, do not change the homotopy type of sublevel complexes), but crucially allows a global, purely combinatorial reconstruction of the reduced complex $\mathrm{core}_g(K)$. This leads to an efficient reduction, as changes to the homotopy type are introduced only by the subset of strong critical vertices, while the remaining dominated vertices are systematically collapsed.

Forman's discrete Morse theory fits within the Bestvina–Brady framework, as observed by Zaremsky [24], and can also be interpreted through the lens of strong discrete Morse theory. Specifically, any injective discrete Morse function $f\colon K \to \mathbb{R}$ induces a Bestvina–Brady Morse function $g\colon V(K') \to \mathbb{R}$ such that critical simplices of f correspond to vertices in K' whose descending links are barycentric subdivisions of the boundaries of those simplices — and hence homeomorphic to spheres. Each matched pair (σ, τ), with $\sigma \prec \tau$ and $f(\sigma) > f(\tau)$, satisfies that σ is descending dominated by τ, and $\mathrm{lk}^{\downarrow}(\tau, K')$ is a topological cone on the unique vertex $v \in \tau \setminus \sigma$ (though not necessarily a simplicial one). This suggests a natural extension of our theory to cases where strong critical vertices have descending links that are topological cones.

4. The construction of the internal core

Given a CW-complex K, its face poset $\mathcal{X}(K)$ encodes the incidence relations among its cells: $e \leq e'$ in $\mathcal{X}(K)$ if $\overline{e} \subseteq \overline{e'}$. While this poset does not determine the homotopy type of K in general, in the case of *regular* CW-complexes it serves as a complete combinatorial model: the *order complex* of $\mathcal{X}(K)$, denoted $\mathcal{K}(\mathcal{X}(K))$, is homotopy equivalent to K (see [11]).

Let X be a finite poset. An *matching* on X is a collection M of disjoint pairs $\{x, y\} \subseteq X$ such that the quotient set $X/_\sim$, where $x \sim y$ if $\{x, y\} \in M$, admits a partial order induced by the original order on X (the relation $[x] \leq [y] \in X/_\sim$ if $x \leq y$ in X is antisymmetric).

Discrete Morse functions on a simplicial complex K are in bijection with acyclic matchings on $\mathcal{X}(K)$, where the unmatched elements correspond to the critical simplices [14]. Given an acyclic matching M on $\mathcal{X}(K)$, one can construct auxiliary posets that reflect the structure of the Morse reduced CW-complex. Inspired by Nanda's work on discrete Morse theory and localization of categories [27], we introduce the following definitions.

Definition 4.1. *Let K be a simplicial complex, and let M be an acyclic matching on $\mathcal{X}(K)$. The* localization poset $\mathrm{Loc}_M(K)$ *is the poset obtained by identifying matched simplices in $\mathcal{X}(K)$. The* critical poset $\mathrm{Crit}_M(K)$ *is the subposet of* $\mathrm{Loc}_M(K)$ *consisting of equivalence classes of unmatched (i.e., critical) simplices.*

In general, neither $\mathrm{Crit}_M(K)$ nor $\mathrm{Loc}_M(K)$ determine the homotopy type of K, even when K is regular or simplicial (see Example 4.3). However, when M arises from internal strong collapses, Theorem 3.9 implies that $\mathrm{Crit}_M(K)$ is a finite model of K.

Corollary 4.2. *Let K be a simplicial complex and let $g\colon V(K) \to \mathbb{R}$ be a real-valued function. Then, there exists an acyclic matching M_g on $\mathcal{X}(K)$ such that the critical poset* $\mathrm{Crit}_{M_g}(K)$ *is the face poset of* $\mathrm{core}_g(K)$. *In particular,* $\mathrm{Crit}_{M_g}(K)$ *is a finite model of K.*

For details on the algorithmic construction of the critical poset, see Appendix A, Algorithm A1.

Example 4.3. Let K be the boundary of the 3-simplex with vertices labelled from 1 to 4 (see Figure 3). Consider the acyclic matching $M = \{\{(0), (0, 1)\}, \{(1), (1, 2)\}, \{(2), (2, 3)\}, \{(0, 2), (0, 2, 3)\},$
$\{(0, 3), (0, 2, 3)\}\}$, depicted in Figure 4. The posets $\mathrm{Loc}_M(K)$ and $\mathrm{Crit}_M(K)$ are contractible, and do not reflect the homotopy type of $K \simeq S^2$.

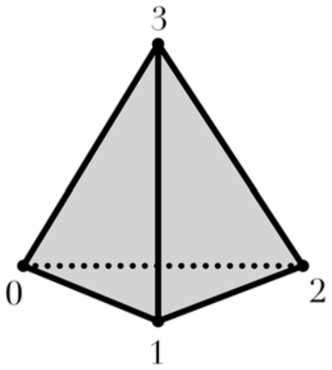


Figure 3. The boundary of a 3-simplex, K, endowed with a function $g\colon V(K) \to \mathbb{R}$ on vertices. By abuse of notation, and due to the injectivity of g, we identify each vertex with its image under g.

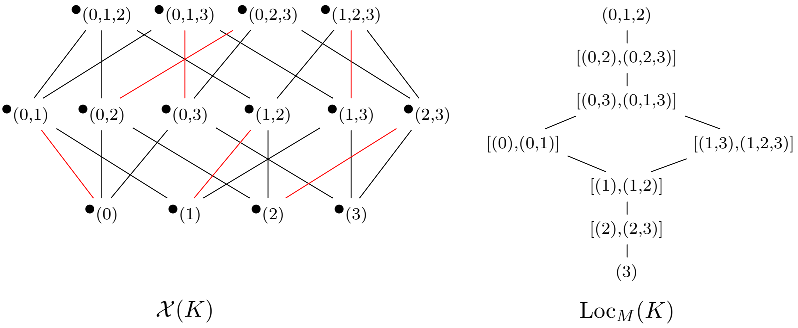


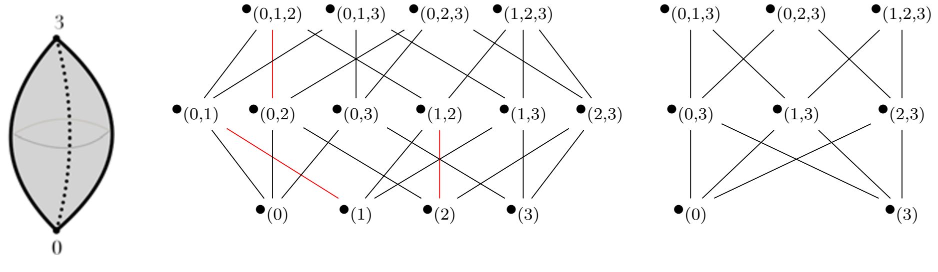
Figure 4. *Left:* The face poset $\mathcal{X}(K)$, for K the boundary of a 3-smplex, with an acyclic matching M depicted in red. *Centre:* The localization poset $\mathrm{Loc}_M(K)$. *Right:* The critical poset $\mathrm{Crit}_M(K)$ (c.f. Fig 3).

On the other hand, consider the function $g\colon V(K) \to \mathbb{R}$ on the vertices depicted in Figure 3. For the associated acyclic matching M_g, the critical poset $\mathrm{Crit}_{M_g}(K)$ recovers the homotopy type of K, and it is the face poset of the strong internal core (see Figure 5).

Remark 4.4. Consider the order-preserving maps

$$\mathcal{Q}\colon \mathcal{X}(K) \to \mathrm{Loc}_M(K) \quad \text{and} \quad \mathcal{J}\colon \mathrm{Crit}_M(K) \to \mathrm{Loc}_M(K),$$

where $\mathcal{Q}$ denotes the quotient map and $\mathcal{J}$ the inclusion map. In future work, we will establish general conditions on the acyclic matching M under which these maps induce homotopy equivalences at the level of simplicial complexes. In particular, we will show that when M is induced by internal strong collapses, both $\mathcal{Q}$ and $\mathcal{J}$ induce homotopy equivalences.



$\mathrm{core}_g(K)$ $\mathcal{X}(K)$ $\mathrm{Crit}_{M_g}(K)$

Figure 5. *Left:* The strong internal core associated to the map $g\colon V(K) \to \mathbb{R}$ on vertices (Fig. 3). *Centre:* The face poset $\mathcal{X}(K)$, with the acyclic matching M_g induced by internal strong collapses, in red. *Right:* The critical poset $\mathrm{Crit}_{M_g}(K)$. Here, $\mathrm{core}_g(K)$ has two 0-cells, three 1-cells and three 2-cells.

5. Algorithmic simplification of simplicial complexes

Let K be a simplicial complex of dimension N. We summarize below the different core notions associated to K:

- **Strong core:** A subcomplex $L \subseteq K$ such that $K \searrow\!\searrow L$. The minimal strong core (i.e. with no dominated vertices) is unique up to isomorphism.
- **Weak core:** A subcomplex $L \subseteq K$ such that $K \searrow L$. A minimal weak core (i.e. with no free faces) is not necessarily unique.
- **Internal core:** Given a discrete Morse function $f\colon K \to \mathbb{R}$, the internal core $\mathrm{core}_f(K)$ is the CW-complex obtained by performing the sequence of internal collapses determined by f. It satisfies $K \nearrow\!\searrow^{N+1} \mathrm{core}_f(K)$. The internal core is not necessarily regular.
- **Strong internal core:** Given a function $g\colon V(K) \to \mathbb{R}$ defined on the vertices of K, the strong internal core, $\mathrm{core}_g(K)$, is obtained by the sequence of internal strong collapses guided by g. It satisfies $K \nearrow\!\searrow^{N+1} \mathrm{core}_g(K)$, and the resulting space is a regular CW-complex.

The diagram below shows implication relations among the core constructions.

$$\begin{tikzcd}
 & \text{Internal Core} & \\
\text{Strong Internal Core} \arrow[ur, "\text{Thm. 3.9}"] & & \text{Weak Core} \arrow[ul, "\text{Prop. 2.6}"'] \\
 & \text{Strong Core} \arrow[ul, "\text{Prop. 3.8}"] \arrow[ur, "\text{[3, Rmk 2.4]}"'] &
\end{tikzcd}$$

For any method of reduction of simplicial complexes, computing the associated core requires the choice of an *ordered* sequence of collapses. While the minimal strong core is unique (up to isomorphism) regardless of the sequence of strong collapses, a minimal weak core does depend on the chosen sequence. Similarly, the internal core of a simplicial complex depends on the discrete Morse function. Note that (strong) discrete Morse theory can be seen as a systematic procedure for performing a sequence of (strong) reductions. These reductions may consist of (strong) internal collapses, or, when no free face (resp. dominated vertex) is available, the removal of a maximal simplex (resp. vertex). To address the dependence on the choice of the reduction sequence, one approach is to explore all possible sequences. However, this can be computationally prohibitive due to the exponential growth in the number of options. Alternatively, random reductions — where reductions are chosen randomly among the available options at each step — provide an efficient strategy to deal with this dependence.

To compute the minimal strong internal core, we iteratively apply a greedy algorithm that performs randomly chosen internal strong collapses whenever a dominated vertex is available. If no dominated vertex exists, we randomly select a vertex to be removed (see Appendix A, Algorithm A2 for further details). As part of this procedure, we construct the acyclic matching associated to the internal strong collapses as stated in Theorem 3.9. Notice that this algorithm can also be applied to study $(N+1)$-deformations, as the strong internal core $(N+1)$-deforms to the original complex. A similar procedure can be applied for the randomized computation of a minimal weak core of a simplicial complex.

In general, there is no combinatorial description of the internal core determined by a general discrete Morse function. Benedetti and Lutz [6] performed a computational analysis of the topology of simplicial complexes using randomized discrete Morse theory, studying the number of critical cells arising from random discrete Morse functions.

We analyse the examples from the *Library of Triangulations* [7] and compare the results of minimals weak core, strong core, and strong internal core, cases where the reduced complex can be completely reconstructed. We perform 10 iterations of the computation of random simple and strong internal cores for the simplicial complexes listed in Table 1, and 100 iterations for those in Table 2. Additionally, we implement a combined reduction strategy, where we first compute a random minimal weak core of the original complex and subsequently compute a random strong internal core of the resulting complex. Tables 1 and 2 present the mean sizes of the different cores and the corresponding computation times. The experiments were conducted on a MacBook Pro with an Apple M3 Pro chip, 12-core CPU, 18-core GPU, and 18 GB of unified memory. The implementation was developed in SageMath [34], and the code is available at `https://github.com/ximenafernandez/Strong-Morse-Theory`.

We observe that in cases where there are neither dominated vertices nor free faces, internal strong collapses allow significant reductions in the size of the complex, with reductions of 30% to 40% in instances as `BH_3`, `BH_4`, `Bernadette_Sphere`, and `non_PL` and 50% to 60% in the large examples `trefoil_bsd` and `Hom_C5_K4`. Notably, in many of these cases, the vector of critical simplices in a (classical) discrete Morse function does not fully determine the homotopy type. For example, the 18-vertex non-PL triangulation of the 5-dimensional sphere S^5 (`non_PL`) admits $(1,0,0,2,2,1)$ as its smallest discrete Morse vector (see [6]). Although this cellular decomposition is highly efficient in its cell count, this information alone is insufficient to reconstruct the reduced CW-complex and verify its homotopy type. By contrast, our strong

Table 1 *Comparison of reduction methods for simplicial complexes, averaged over 10 iterations.* ***Bold*** *values indicate cases where strong internal collapses yield the most significant size reduction. Italicized values indicate cases where strong internal collapses also lead to a significant improvement in execution time compared to other methods*

	Number of simplices/Time per iteration (sec)				
Name example	**Original**	**Minimal strong core**	**Minimal weak core**	**Strong internal core**	**Strong internal core of minimal weak core**
bing	8131	2259	2377.6	**1665.80**	1581.20
	–	185.7559	1532.325	*112.5924*	1586.5839
non_4_2_colorable	5982	5982	5982.00	5838.20	5857.40
	–	2.2379	1.5879	78.1108	79.2957
Hom_C5_K4	6240	6240	6240.00	**4632.40**	4669.60
	–	3.0053	1.6739	139.9162	141.8681
trefoil_bsd	5876	5876	5876.00	**2072.20**	2134.80
	–	3.7306	1.5139	102.2188	110.9461
knot	6203	1639	1946.20	**1184.00**	1271.60
	–	86.4477	2088.4905	*56.3632*	2130.4097

Morse approach yields a regular CW-complex whose structure is fully determined, while still reducing the number of simplices in this example from 2608 to an average of 1690.56, a size reduction of 35% (see Table 2).

In several large cases with available simple collapses, the strong Morse theory approach yields a more efficient model, significantly reducing the original complex size by around 80% and computation times by over 90% compared to simple collapses in examples such as `knot` and `bing`. For the `knot` complex, for instance, applying simple collapses reduces the 6203 simplices to an average of 1946, while the strong internal core method reduces it even further to an average of 1184 simplices. Crucially, the latter reduction is achieved faster, taking an average of only 56 seconds, compared to the 2088 seconds (about 35 minutes) required for simple collapses (see Table 1).

On the other hand, a few cases exhibit simplicial complexes with small weak cores but large strong internal cores (on average). Examples include `B_3_9_18`, `trefoil_arc`, `rudin`, and `triple_trefoil_arc`. In these scenarios, the combined strategy emerges as the optimal choice: preprocessing the simplicial complex by computing its minimal weak core before determining the strong internal core.

Table 2 *Comparison of algorithmic reduction methods for simplicial complexes, averaged over 100 iterations.* ***Bold*** *values highlight cases where significant reduction is achieved via strong internal collapses (compared to other methods). Italicized values indicate cases where significant reduction is only achieved through Whitehead collapses*

	Number of simplices/Time per iteration (sec)				
Name example	**Original**	**Minimal Strong core**	**Minimal weak core**	**Strong internal core**	**Strong internal core of minimal weak core**
Abalone	101 –	101 0.0273	101.00 0.0009	**73.30** 0.0484	70.58 0.0486
BH	131 –	131 0.0185	131.00 0.0015	100.68 0.0839	99.60 0.0824
BH_3	301 –	301 0.0401	301.00 0.0068	**206.32** 0.4438	207.78 0.4473
BH_4	401 –	401 0.0564	401.00 0.0124	**277.06** 0.8285	274.60 0.8371
BH_5	501 –	501 0.0806	501.00 0.0197	**349.08** 1.4011	343.86 1.4009
d2_n8_3torsion	49 –	49 0.0060	49.00 0.0003	38.48 0.0135	38.38 0.0138
d2_n8_4torsion	53 –	53 0.0060	53.00 0.0003	43.66 0.0135	43.98 0.0138
d2_n9_5torsion	65 –	65 0.0073	65.00 0.0005	57.12 0.0193	55.68 0.0204
dunce_hat	49 –	49 0.0056	49.00 0.0003	37.60 0.0135	38.84 0.0129
d2n12g6	122 –	122 0.0140	122.00 0.0013	118.80 0.00602	118.66 0.0612
regular_2_21_23_1	266 –	266 0.0333	266.00 0.0058	259.52 0.2230	259.52 0.2349
rand2_n25_p0.328	1076 –	1076 0.1965	1074.00 0.2955	1069.90 1.3532	1067.18 1.6397

(*Continued*)

Table 2 *(Continued.)*

	Number of simplices/Time per iteration (sec)				
Name example	**Original**	**Minimal Strong core**	**Minimal weak core**	**Strong internal core**	**Strong internal core of minimal weak core**
dunce_hat_in_3_ball	75 –	1 0.0149	1.00 0.0151	1.00 0.0119	1.00 0.0153
Barnette_sphere	92 –	92 0.0130	92.00 0.0005	**40.66** 0.0182	39.58 0.0195
B_3_9_18	103 –	59 0.0220	*1.00* 0.0286	31.38 0.0212	1.00 0.0287
trefoil_arc	193 –	193 0.0281	*1.76* 0.1257	148.68 0.0865	1.62 0.1262
trefoil	250 –	250 0.0369	250.00 0.0031	206.98 0.1238	211.12 0.1298
rudin	215 –	215 0.0329	*1.00* 0.1668	137.74 0.1057	1.00 0.1670
poincare	392 –	392 0.0631	392.00 0.0074	361.80 0.3636	364.24 0.2744
double_trefoil	400 –	400 0.0643	400.00 0.0078	380.20 0.2714	383.08 0.2797
triple_trefoil_arc	449 –	449 0.0746	*172.36* 1.3937	438.62 0.3337	157.60 1.5215
triple_trefoil	536 –	536 0.0922	536.00 0.0141	526.50 0.4367	525.70 0.4467
hyperbolic_dodecahedral_space	718 –	718 0.1313	718.00 0.0241	708.82 23.3319	709.36 74.4473
S_3_50_1033	4232 –	4232 1.5958	4232.00 0.8286	4198.36 16.9282	4295.20 17.73918

(*Continued*)

Table 2 *(Continued.)*

	Number of simplices/Time per iteration (sec)				
Name example	**Original**	**Minimal Strong core**	**Minimal weak core**	**Strong internal core**	**Strong internal core of minimal weak core**
600_cell	2640	2640	2640.00	2343.46	2349.68
	–	0.6912	0.3057	16.5970	16.9279
CP2	255	255	255.00	219.78	219.68
	–	0.0464	0.0020	0.0798	0.0823
RP4	991	991	991.00	942.36	937.34
	–	0.2400	0.0293	0.7976	0.8290
K3_16	1704	1704	1704.00	1691.34	1690.62
	–	0.4895	0.09545	1.61122	1.7076
K3_17	1854	1854	1854.00	1835.32	1836.22
	–	0.6331	0.1108	1.9267	2.0365
RP4_K3_17	2813	2813	2813.00	2757.98	2751.66
	–	0.9971	0.2429	5.2140	5.4425
RP4_11S2xS2	3179	3179	3179.00	3099.04	3105.62
	–	1.2477	0.3071	6.8191	7.148
SU2_SO3	1534	1534	1534.00	1474.70	1471.84
	–	0.5173	0.0453	1.2220	1.2677
non_PL	2608	2608	2608.0	**1690.56**	1539.16
	–	1.1274	0.1274	2.8094	2.6714

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Appendix A. Algorithms

This appendix contains the pseudocode of the main algorithms used in the construction and computation of the critical poset associated with a strong internal core. An implementation in SageMath [34] can be found at https://github.com/ximenafernandez/Strong-Morse-Theory.

Remark A.1. Let K be a finite simplicial complex with V vertices, F simplices, and dimension N. Algorithm 2 compute the strong internal core by repeatedly identifying descending dominated vertices and performing internal strong collapses. A domination test for a vertex v inspects its (descending) open star and thus costs $O(S_v)$ time, where S_v is the number of simplices containing v. Since each k-simplex has $k{+}1 \leq N{+}1$ vertices,

$$\sum_{v \in V(K)} S_v = O(N\,F).$$

Since every simplex belongs to at most $N+1$ vertex stars, the cumulative number of simplex inspections during the reduction is bounded by $O(N\,F)$. The overall runtime depends on the implementation. A naive approach that recomputes all domination tests after each collapse would have a worst-case complexity of $O(V\,N\,F)$. However, in practice, only the neighbourhood of the removed vertex needs updating. An efficient implementation using adjacency lists and a priority

Algorithm 1 *The critical poset*

```
Input: Face poset $\mathcal{X}(K)$ of a simplicial complex $K$, set of critical simplices $C$,
acyclic matching $M$ on $\mathcal{X}(K)$
Output: Critical poset $\mathrm{Crit}_M(K)$
$P \leftarrow$ elements of $\mathcal{X}(K)$
$R \leftarrow$ cover relations of $\mathcal{X}(K)$
Define a map QuotientClass : $P \to$ new elements
for all $(\sigma, \tau) \in M$ do
    $P \leftarrow P \setminus \{\sigma, \tau\}$
    $[\sigma \sim \tau] \leftarrow$ new element representing $\{\sigma, \tau\}$
    $P \leftarrow P \cup \{[\sigma \sim \tau]\}$
    QuotientClass$[\sigma] \leftarrow [\sigma \sim \tau]$
    QuotientClass$[\tau] \leftarrow [\sigma \sim \tau]$
end for
for all $\sigma \in P$ such that QuotientClass$[\sigma]$ is undefined do
    QuotientClass$[\sigma] \leftarrow \sigma$
end for
$R' \leftarrow \emptyset$
for all $(\sigma, \tau) \in R$ do
    if QuotientClass$[\sigma] \neq$ QuotientClass$[\tau]$ then
        $R' \leftarrow R' \cup \big(\mathrm{QuotientClass}[\sigma], \mathrm{QuotientClass}[\tau]\big)$
    end if
end for
$\mathrm{Loc}_M(K) \leftarrow$ poset with elements $P$ and relations $R'$
$\mathrm{Crit}_M(K) \leftarrow$ subposet of $\mathrm{Loc}_M(K)$ induced by $C$
return $\mathrm{Crit}_M(K)$
```

queue of candidate vertices to be tested yields an overall running time of

$$O(N\,F + V \log V).$$

For a fixed, small dimension N, this complexity is essentially linear in the number of simplices.

In contrast, classical discrete Morse algorithms construct an acyclic matching on the face poset $\mathcal{X}(K)$. A basic implementation must inspect all cover relations between comparable simplices when iteratively attempting to form pairs, and test global acyclicity on the current directed graph after each addition. This may require $O(N^2\,F^2)$ time in the worst case. With optimized cycle–detection and data structures, the cost could be reduced to

$$O(N\,F \log F).$$

The strong-internal-collapse approach avoids global searches as reductions depend only on local vertex–link tests, leading to lower asymptotic complexity.

Algorithm 2 *A random strong internal core*

```
Input: $\mathcal{X}(K)$: the face poset of a simplicial complex $K$
Output: The critical poset of a random strong internal core of $K$
  function StrongMorseReduction($X, C, M$)
                ▷ Auxiliary function. Input: $X$: face poset, $M$: acyclic matching, $C$: critical elements
    if $X = \emptyset$ then
      return $((X, C, M))$
    end if
    $S \leftarrow$ Shuffle(MinimalElements($X$))  ▷ Randomized linear extension of minimal elements
    for $v \in S$ do
      if there exists $a \in$ MinimalElements($X$) $\setminus \{v\}$ such that $a$ dominates $v$ then
        for all $\sigma \in \{\sigma \in X : v \leq \sigma\}$ such that $a \nleq \sigma$ do
          $M \leftarrow M \cup \{(\sigma, \{a\} \cup \sigma)\}$
        end for
        $X \leftarrow X \setminus \{\tau \in X : v \leq \tau\}$
        return StrongMorseReduction($X, C, M$)
      end if
    end for
    $v \leftarrow$ First($S$)          ▷ Take first non-dominated minimal element
    $C \leftarrow C \cup \{\sigma \in X : v \leq \sigma\}$
    $X \leftarrow X \setminus \{\sigma \in X : v \leq \sigma\}$
    return StrongMorseReduction($X, C, M$)
  end function
  procedure ComputeInternalStrongCore($\mathcal{X}(K)$)
    $(-, C, M) \leftarrow$ StrongMorseReduction($\mathcal{X}(K), \emptyset, \emptyset$)
    return CriticalPoset($\mathcal{X}(K), C, M$)
  end procedure
```