

MINIMAL GENERATING PAIRS FOR PERMUTATION GROUPS

Thesis

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Doctor of Philosophy

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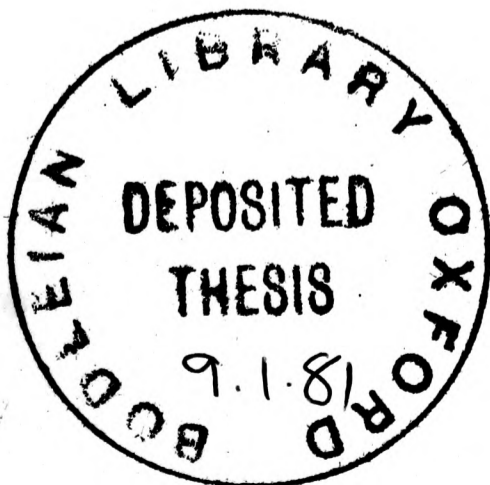
University of Oxford

by

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Oxford



Trinity Term

1980

To my family

MINIMAL GENERATING PAIRS FOR PERMUTATION GROUPS

by

Marston D.E. Conder, Brasenose College, Oxford

A thesis submitted for the degree of Doctor of Philosophy
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ABSTRACT

In this thesis we consider two-element generation of certain permutation groups.

Interest is focussed mainly on the finite alternating and symmetric groups. Specifically, we prove that if k is any integer greater than six, then all but finitely many of the alternating groups A_n can be generated by elements x, y which satisfy

$$x^2 = y^3 = (xy)^k = 1$$

and further, if k is even then the same is true of (all but finitely many of) the symmetric groups S_n .

The case $k=7$ is of particular importance. Any finite group which can be generated by elements x, y satisfying

$$x^2 = y^3 = (xy)^7 = 1$$

is called a Hurwitz group, and gives rise to a compact Riemann surface of which it is a maximal automorphism group. The bulk of the thesis is devoted to showing that all but 64 of the alternating groups are Hurwitz. Also we give a classification of all Hurwitz groups of order less than one million.

An appendix deals with two-element generation of the group associated with the Hungarian 'magic' colour-cube.

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NOTATION

Most of the set-theoretic and group-theoretic notation used in this thesis is standard, as in the books of Gorenstein [5], Johnson [10], and Wielandt [25].

In particular, if G is a group and X is a subset of G , then we make the following abbreviations:

$ X $	order of X
$C_G(X)$	centralizer of X in G
$Z(G)$	centre of G
$\text{Aut}(G)$	automorphism group of G
G'	commutator subgroup of G
G/G'	factor-commutator group of G .

Roman letters tend to represent numbers, elements, and groups, while greek letters are usually used to denote points and sets. If G is a group which acts on a set Ω , and $g \in G$ and $\alpha \in \Omega$, then αg denotes the image of the point α under the element g . If r is any positive real number, then $[r]$ stands for the largest integer which does not exceed r .

Reference is made quite often to the following well-known permutation groups:

S_n	symmetric group of degree n
A_n	alternating group of degree n
$\text{PSL}(2, q)$	projective special linear group of dimension 2 over the field with q elements

Others will be encountered in the introductory chapters.

The bulk of the thesis involves the abstract triangle groups, as follow.

$\Delta(k, l, m)$ is the group with presentation

$$\langle x, y \mid x^k = y^l = (xy)^m = 1 \rangle$$

$\Delta(k, l, m; p)$ is the group with presentation

$$\langle x, y \mid x^k = y^l = (xy)^m = (xy^{-1}xy)^p = 1 \rangle$$

$\Delta^*(k, l, m)$ is the group with presentation

$$\langle x, y, t \mid x^k = y^l = (xy)^m = t^2 = (xt)^2 = (yt)^2 = 1 \rangle$$

(where in each case k, l, m, p are positive integers).

INTRODUCTION

Let G be a finite group.

Suppose there exist elements x and y which together generate G , and are such that x has order k , and y has order ℓ , and their product xy has order m , where $k \leq \ell \leq m$. Then we call (x, y) a (k, ℓ, m) -generating pair for G .

If such a pair exists, then G is a homomorphic image of the triangle group $\Delta(k, \ell, m)$, that is, the group with presentation

$$\Delta(k, \ell, m) = \langle a, b \mid a^k = b^\ell = (ab)^m = 1 \rangle .$$

Suppose further that whenever r, s, t are positive integers such that $r \leq s \leq t$ and G has an (r, s, t) -generating pair, then $k \leq r$ and if $k = r$ then $\ell \leq s$. Then we shall call (x, y) a minimal generating pair for G .

The triangle groups are discussed in [1], [2] and [10].

It is known that $\Delta(k, \ell, m)$ is finite precisely when

$$\frac{1}{k} + \frac{1}{\ell} + \frac{1}{m} > 1 , \text{ and the groups which arise in this case are}$$

as below:

$\Delta(1, m, m)$	$\cong C_m$	cyclic group of order m
$\Delta(2, 2, n)$	$\cong D_{2n}$	dihedral group of order $2n$
$\Delta(2, 3, 3)$	$\cong A_4$	tetrahedral group
$\Delta(2, 3, 4)$	$\cong S_4$	octahedral group
$\Delta(2, 3, 5)$	$\cong A_5$	icosahedral group

If $\frac{1}{k} + \frac{1}{\ell} + \frac{1}{m} = 1$, that is if $(k, \ell, m) = (2, 3, 6)$ or $(2, 4, 4)$ or $(3, 3, 3)$, then the group $\Delta(k, \ell, m)$ is infinite but soluble: the

commutator subgroup is free Abelian of rank two, and the factor-commutator group is cyclic of order m .

In this thesis our interest centres on insoluble permutation groups (other than A_5) and we therefore look for (k, l, m) -generating pairs with $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$. Any finite group G which possesses such a generating pair is representable (cf. [13]) as a group of automorphisms of a compact Riemann surface of genus g , where

$$2g - 2 = |G| \cdot \left(1 - \left(\frac{1}{k} + \frac{1}{l} + \frac{1}{m} \right) \right).$$

From this formula it is evident that $|G| \leq 84(g - 1)$, the maximum bound being attained only if $(k, l, m) = (2, 3, 7)$. The discovery of these facts was first made by Hurwitz, and so we call any finite factor group of $\Delta(2, 3, 7)$ a Hurwitz group.

We conduct a search in Chapter 3 for Hurwitz groups of small order. In particular, we prove that if G is a Hurwitz group whose order is less than one million, then G is Janko's first simple group (of order 175560), or the Hall-Janko group (of order 604800), or G contains a normal subgroup K such that G/K is isomorphic to $PSL(2, q)$ for one of a small number of values of q .

The bulk of the thesis is devoted to finding generators for alternating and symmetric groups. To this end we have developed a diagrammatic argument suggested by Graham Higman.

First we use coset diagrams for $\Delta(2, 3, 7)$ to show that all but 64 of the alternating groups A_n are Hurwitz groups. In fact we determine exactly which of the alternating and symmetric groups

are homomorphic images of the extended triangle group

$$\Delta^*(2,3,7) = \langle x, y, t \mid x^2=y^3=(xy)^7=t^2=(xt)^2=(yt)^2=1 \rangle.$$

The full classification appears in Chapter 4.

It has been known for some time that all finite alternating and symmetric groups except A_6, A_7, A_8, S_5, S_6 and S_8 occur as homomorphic images of the modular group $\Gamma = \langle x, y \mid x^2=y^3=1 \rangle$. Miller [20] proved this in 1901, and more recently I.M.S. Dey and James Wiegold [3] found explicit generators in terms of the degree.

In Chapter 5 we improve their results by showing that for any integer k greater than six, all but finitely many of the groups A_n possess $(2,3,k)$ -generating pairs, the same being true of (all but finitely many of) the symmetric groups S_n if k is even. Specifically, the following result is proved.

THEOREM Let k be any integer greater than six.

Then there exists $N_k > 0$ such that for all integers $n \geq N_k$:

(a) both A_n and S_n can be generated by elements x, y, t which satisfy

$$x^2 = y^3 = (xy)^k = t^2 = (xt)^2 = (yt)^2 = (xyt)^{2m} = 1,$$

and

(b) A_n , and if k is even then also S_n , can be generated by elements x, y which satisfy

$$x^2 = y^3 = (xy)^k = (xy^{-1}xy)^m = 1,$$

where $m = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 360360$.

Finally we give in an appendix (intended as something of a light-hearted supplement) a minimal generating pair for the group associated with the Hungarian 'magic' colour-cube.

Chapter 1 : PRELIMINARY RESULTS

We begin with a lemma which is fundamental to the study of generating pairs for permutation groups.

1.1 LEMMA Suppose g and h are permutations of n points such that g is a product of r disjoint cycles (some of which may have length one), h is a product of s disjoint cycles, and the permutation gh has order 2 and degree $2k$. If g and h generate a transitive group of degree n , then $k \geq r + s - 2$.

Proof Write $(gh)^{-1}$ as $t_1 t_2 \dots t_k$, the product of k disjoint transpositions t_i ($1 \leq i \leq k$).

Let $t_1 = (\alpha\beta)$, say. If α and β lie in different cycles of h , say $(\alpha\gamma_1\gamma_2\dots\gamma_u)$ and $(\beta\delta_1\delta_2\dots\delta_v)$, then these cycles are juxtaposed to form the cycle $(\alpha\gamma_1\gamma_2\dots\gamma_u\beta\delta_1\delta_2\dots\delta_v)$ of the permutation ht_1 ; the other cycles of ht_1 are the same as those of h . If on the other hand α and β lie in the same cycle of h , then that cycle is split to form two cycles of ht_1 , so that ht_1 has $s+1$ cycles altogether.

It follows that we can arrange t_1, t_2, \dots, t_k in such a way that $ht_1 t_2 \dots t_i$ has $s-i$ cycles, for $1 \leq i \leq m$, where m is some integer less than s . Suppose m is the largest integer for which this is possible. Now denote by Λ one of the orbits of the element $ht_1 t_2 \dots t_m$. Then $\Lambda ht_1 t_2 \dots t_m = \Lambda$, and since Λ is the union of some of the orbits of h , we know also that $\Lambda h = \Lambda$.

On the other hand, for each i greater than m the two points moved by the transposition t_i must lie in the same cycle of $ht_1t_2\dots t_m$ (else $ht_1t_2\dots t_mt_i$ would have $s-m-1$ cycles, contradicting the choice of m), and so $\Lambda t_i = \Lambda$. Consequently $\Lambda t_{m+1}t_{m+2}\dots t_k = \Lambda$. But now $\Lambda g^{-1} = \Lambda h(gh)^{-1}$

$$\begin{aligned} &= \Lambda ht_1t_2\dots t_mt_{m+1}t_{m+2}\dots t_k \\ &= \Lambda t_1t_2\dots t_mt_{m+1}t_{m+2}\dots t_k \\ &= \Lambda t_{m+1}t_{m+2}\dots t_k \\ &= \Lambda \end{aligned}$$

and so $\Lambda g = \Lambda h = \Lambda$. Thus Λ is an orbit for the group generated by g and h . By hypothesis, this means Λ has size n , so $ht_1t_2\dots t_m$ is an n -cycle. In particular, $s-m=1$. From this it follows that the permutation $g^{-1} = ht_1t_2\dots t_mt_{m+1}t_{m+2}\dots t_k$ has at most $1 + (k - (s-1))$ cycles, whence $r \leq k - s + 2$.

1.2 COROLLARY If the triangle group $\Delta(2,3,7)$ has a transitive permutation representation of degree n , then

$$\left\lfloor \frac{n}{2} \right\rfloor + 2 \left\lfloor \frac{n}{3} \right\rfloor + 6 \left\lfloor \frac{n}{7} \right\rfloor \geq 2n - 2 .$$

Proof Suppose x and y are permutations which satisfy $x^2 = y^3 = (xy)^7 = 1$ and which generate a transitive group of degree n . Then x moves at most $2 \left\lfloor \frac{n}{2} \right\rfloor$ points. Also xy has at most $\left\lfloor \frac{n}{7} \right\rfloor$ cycles of length 7, and fixes at least $n - 7 \left\lfloor \frac{n}{7} \right\rfloor$ points. Similarly y has at most $\left\lfloor \frac{n}{3} \right\rfloor$ cycles of length 3, and fixes at least $n - 3 \left\lfloor \frac{n}{3} \right\rfloor$ points. Taking $g = xy$ and $h = y^{-1}$ in the lemma, we find

$$\left\lfloor \frac{n}{2} \right\rfloor \geq \left(\left\lfloor \frac{n}{7} \right\rfloor + n - 7 \left\lfloor \frac{n}{7} \right\rfloor \right) + \left(\left\lfloor \frac{n}{3} \right\rfloor + n - 3 \left\lfloor \frac{n}{3} \right\rfloor \right) - 2 ,$$

from which the desired inequality follows.

In fact we can achieve more than this. Let x and y be as above, and denote by e , f and g the numbers of fixed points of x , y and xy respectively. Since $x = (xy)y^{-1}$, the product of two elements of odd order, we know that x must be an even permutation. Hence $\frac{1}{2}(n - e)$ is divisible by 2. By the lemma we have also that the quantity

$$N = \frac{1}{2}(n - e) - \left(\frac{1}{3}(n - g) + g\right) - \left(\frac{1}{3}(n - f) + f\right) + 2$$

is a non-negative integer. But now

$$42N = 21(n - e) - (20n - 28f - 36g) + 84 \quad ;$$

the expression on the right-hand side is divisible by 4; and so N must be even. Putting $N = 2p$, we have

$$(1.3) \quad n = 84(p - 1) + 21e + 28f + 36g$$

which is the genus formula cited in [24] for subgroups of finite index in $\Delta(2, 3, 7)$.

We now turn to some elementary facts about the triangle groups. First let $\Delta^*(2, 3, k)$ be the group with presentation

$$\Delta^*(2, 3, k) = \langle x, y, t \mid x^2 = y^3 = (xy)^k = t^2 = (xt)^2 = (yt)^2 = 1 \rangle$$

(where k is any integer greater than six). It is evident that the subgroup generated by x and y is isomorphic to the group $\Delta(2, 3, k)$; for $txt = x$ and $tyt = y^{-1}$, hence this subgroup is normal of index 2 in $\Delta^*(2, 3, k)$.

1.4 LEMMA

- (a) Any pair of xy , t and xyt generate $\Delta^*(2,3,k)$.
- (b) The element $(xyt)^2$ is conjugate to the commutator $xy^{-1}xy$ in $\Delta^*(2,3,k)$.
- (c) In any permutation representation of $\Delta^*(2,3,k)$ the orbits of xyt are permuted by x .

Proof

(a) We have $[xy, t] = (xy)^{-1}t(xy)t = y^{-1}xxtyt = y^{-1}y^{-1}t^2 = y^{-2} = y$ and $xy[t, xy] = xy[xy, t]^{-1} = (xy)y^{-1} = x$; consequently x and y both lie in the subgroup generated by xy and t . The rest is obvious.

(b) Notice $(xyt)^2 = xytxyt = xty^{-1}xyt = txy^{-1}xyt$.

(c) Let $\Delta(\alpha)$ denote the orbit of a point α under the action of xyt . Then since $(xyt)x(xyt) = xytyt = xyy^{-1}t^2 = x$ we have $\Delta(\alpha x) = \Delta(\alpha)x$ for every point α .

1.5 PROPOSITION Let $\Delta(2,3,7;p)$ denote the group with presentation $\Delta(2,3,7;p) = \langle x, y \mid x^2=y^3=(xy)^7=(xy^{-1}xy)^p=1 \rangle$.

This group is finite whenever $p \leq 8$. In particular,

$\Delta(2,3,7;p)$ is (a) trivial when $p = 1, 2, 3$ or 5

(b) $PSL(2,7)$ when $p = 4$

(c) $PSL(2,13)$ when $p = 6$ or 7

(d) an extension by $PSL(2,7)$ of an elementary Abelian 2-group of order 64 when $p = 8$.

[21].

These assertions are verified in [23], [1] and [12].

Next we give some of the known properties of Hurwitz groups.

1.6 LEMMA Let G be a non-trivial Hurwitz group.

Then (a) G is perfect

(b) G has a normal subgroup K such that G/K is a simple
Hurwitz group

and (c) the order of G is divisible by 84.

Proof (cf. [13])

First we notice that G has no non-trivial Abelian factor groups. For any such factor group would be generated by elements x, y satisfying $x^2 = y^3 = (xy)^7 = xy^{-1}xy = 1$, but these relations imply $x = y = 1$.

In particular if G' denotes the commutator subgroup of G , then the factor group G/G' must be trivial; hence G is perfect.

Now let K be a maximal normal subgroup of G . Then G/K is simple, and also is a factor group of $\Delta(2,3,7)$. It contains elements of orders 2, 3 and 7, hence its order is divisible by 42. But also it is well known (cf. Corollary 14.3.1 of [6]) that the order of a non-Abelian group is divisible by 12 or by the cube of its smallest prime divisor. In this case, 4 must divide the order of G/K , hence 84 divides the order of G .

The following result was achieved by considering the homology groups of the Riemann surface on which a Hurwitz group acts (cf. [21]).

1.7 PROPOSITION (Chih-han Sah)

Let G be a non-trivial Hurwitz group of order at most 8400.

Then G is one of the following:

- (a) $PSL(2,7)$
- (b) $PSL(2,8)$
- (c) $PSL(2,13)$
- (d) an extension by $PSL(2,7)$ of an elementary Abelian 2-group of order 8
- (e) an extension by $PSL(2,7)$ of an elementary Abelian 3-group of order 27
- (or possibly (f) $PSU(3,3)$).

We will find in Chapter 3 that (f) is impossible.

(Indeed, also (e) is impossible; for $PSL(2,7)$ has no faithful representation of degree 3 over the field with 3 elements.)

1.8 PROPOSITION (Macbeath)

The linear fractional group $PSL(2,q)$ is a Hurwitz group if and only if

$$q = 7, \text{ or}$$

$$q = p \text{ for some prime } p \equiv \pm 1 \pmod{7}, \text{ or}$$

$$q = p^3 \text{ for some prime } p \equiv \pm 2 \text{ or } \pm 3 \pmod{7}.$$

In fact Macbeath proved (in [17]) that whenever 7 divides the order of $PSL(2,q)$ then that group contains a non-trivial Hurwitz subgroup, and all such subgroups are isomorphic.

Following (b) of Lemma 1.6 it is natural in a search for Hurwitz groups of small order to look at the non-Abelian simple groups of order less than say one million. These are all known (a reference for this fact may be found in [18]) and they are conveniently listed in [11] along with their character tables. We shall make extensive use of the latter in Chapter 3. We shall also consider the natural permutation representations of certain of these groups: namely that of $\text{PSL}(3,4)$ on the 21 points of a projective plane of order 4, that of the Mathieu group M_{22} on 22 points, and that of the Hall-Janko group J_2 on the 100 cosets of its largest proper subgroup.

The following formula plays a vital role in the techniques which we apply in Chapter 3.

1.9 LEMMA Let $\Omega_1, \Omega_2, \dots, \Omega_k$ denote the conjugacy classes of the finite group G , let x_i be a representative from the class Ω_i (for $1 \leq i \leq k$), and let $\chi_1, \chi_2, \dots, \chi_k$ denote the irreducible complex characters of G . Then for $1 \leq i, j, t \leq k$ the number $a_{ij t}$ of pairs (x, y) with $x \in \Omega_i$ and $y \in \Omega_j$ and $xy = x_t$ is given by

$$a_{ij t} = \frac{|\Omega_i| |\Omega_j|}{|G|} \sum_{s=1}^k \frac{\chi_s(x_i) \chi_s(x_j) \overline{\chi_s(x_t)}}{\chi_s(1)}$$

(where $\overline{\chi_s(x_t)}$ denotes the complex conjugate of $\chi_s(x_t)$).

This is a simple consequence of the orthogonality relations among the characters (cf. Theorem 4.2.12 of [5]).

Finally we need some results which will help us recognise when a transitive permutation group is alternating or symmetric.

1.10 THEOREM . (Jordan)

Let G be a primitive group of degree $n = p+k$, where p is a prime and $k \geq 3$. If G contains an element of order and degree p , then G is alternating or symmetric.

A proof is given in [25].

1.11 COROLLARY Let G be a transitive group of degree n , and let s be the smallest non-trivial divisor of n . If G contains an element of order and degree p , where p is a prime satisfying $\frac{n}{s} < p < n-2$, then G is alternating or symmetric.

Proof Assume G is imprimitive. Then since there can be no more than $\frac{n}{s}$ blocks of imprimitivity, the element of order p must fix each block. But also any block has at most $\frac{n}{s}$ points, so this element fixes every point. We have a contradiction, so G must be primitive, and the theorem applies.

1.12 COROLLARY Let g and h be permutations which generate a transitive subgroup of the symmetric group S_n . Suppose that g contains a cycle of prime length p ,

where (a) $p < n-2$,

and (b) p divides the length of no other cycle of g ,

and (c) the p -cycle contains either a fixed point of h or

the points from a cycle of h .

Then the subgroup generated by g and h is either A_n or S_n .

Proof By the theorem, all we need do is show that the group generated by g and h is primitive. Well, assume the contrary.

Let B be a block of imprimitivity which contains a point α from the p -cycle of g , and a further point β . If β does not lie in the p -cycle, then some power of g fixes β , and therefore fixes B , but moves α to some other point γ from the p -cycle. In any case, B must contain at least two points, and hence all p points, of this cycle of g . In particular, we have $Bg = B$. Also condition (c) tells us that $Bh = B$. It follows that B is fixed by the group $\langle g, h \rangle$; but this contradicts transitivity!

Jordan's theorem has the following generalization. A reference for its proof is given in [25].

1.13 THEOREM (Jordan, Manning, Weiss)

Let p be a prime and G a primitive group of degree $qp+k$ which contains an element of order p and degree qp , but which is neither alternating or symmetric. Then:

from	$q =$	1	2	3	4	4	5	6	7	≥ 8
and	$p \geq$	2	5	5	7	5	7	11	11	$2q - 1$
it follows that	$k \leq$	2	2	3	4	5	6	6	8	$4q - 4$

Chapter 2 : COSET DIAGRAMS AND COMPOSITION

A diagrammatic argument lies at the heart of the results proved in Chapters 4 and 5 on the alternating and symmetric groups.

Specifically, for each integer k greater than six, we construct coset diagrams for the triangle group

$$\Delta(2,3,k) = \langle x, y \mid x^2 = y^3 = (xy)^k = 1 \rangle .$$

A coset diagram with n vertices depicts the transitive permutation representation of $\Delta(2,3,k)$ on the cosets of some subgroup of index n . Fixed points of y are denoted by heavy dots, and 3-cycles of y by triangles whose vertices are permuted anti-clockwise by y ; while any two points which are interchanged by the involution x are joined by an edge. If the diagram possesses a vertical axis of symmetry, then it may be regarded as a coset diagram for the extended triangle group

$$\Delta^*(2,3,k) = \langle x, y, t \mid x^2 = y^3 = (xy)^k = t^2 = (xt)^2 = (yt)^2 = 1 \rangle .$$

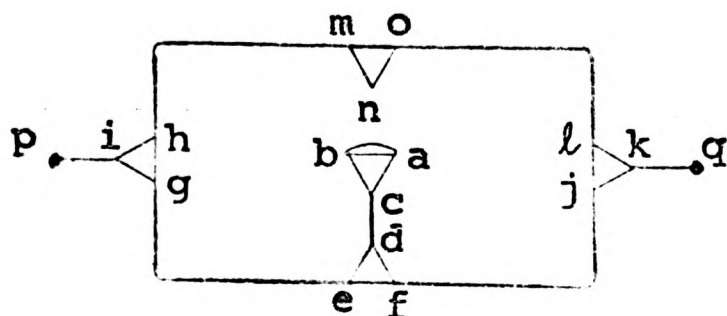
In this case the action of t is given by reflection in the axis of symmetry.

For example, the following diagram depicts a (transitive) permutation representation of $\Delta^*(2,3,8)$ of degree 17, in which

x acts as $(ab)(cd)(eg)(fj)(hm)(ip)(kq)(lo)$,

y acts as $(abc)(def)(ghi)(jkl)(mno)$, and

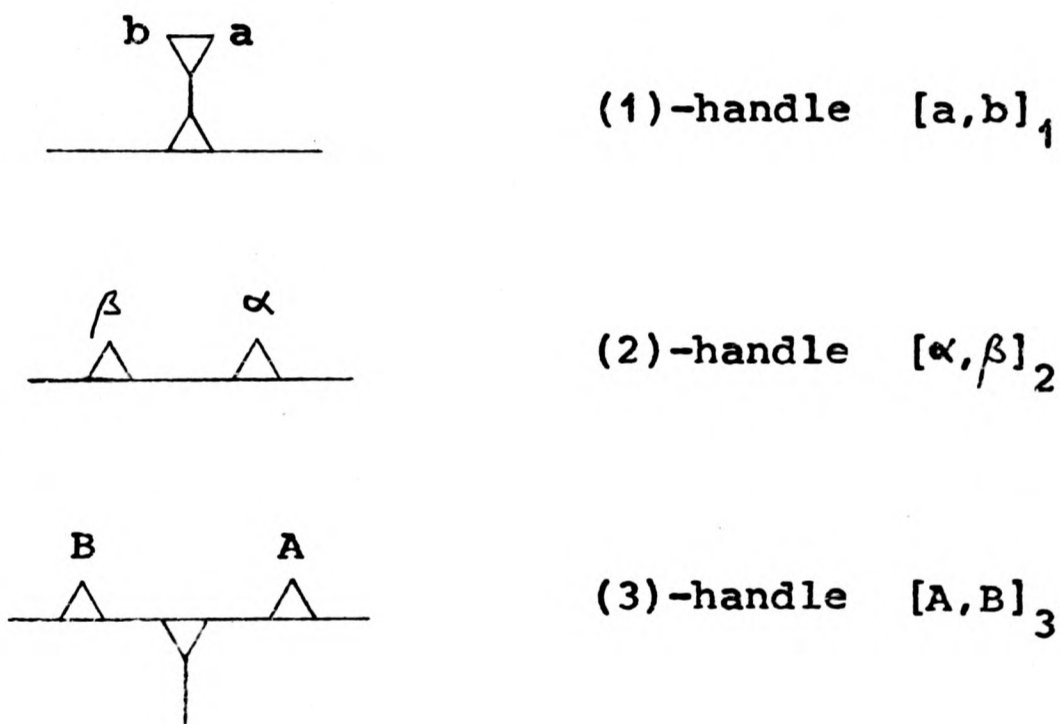
t acts as $(ab)(ef)(gj)(hl)(ik)(mo)(pq)$:



Now let k be a fixed integer greater than six.

Suppose j is any integer which satisfies $1 \leq j \leq \frac{k}{2}$.

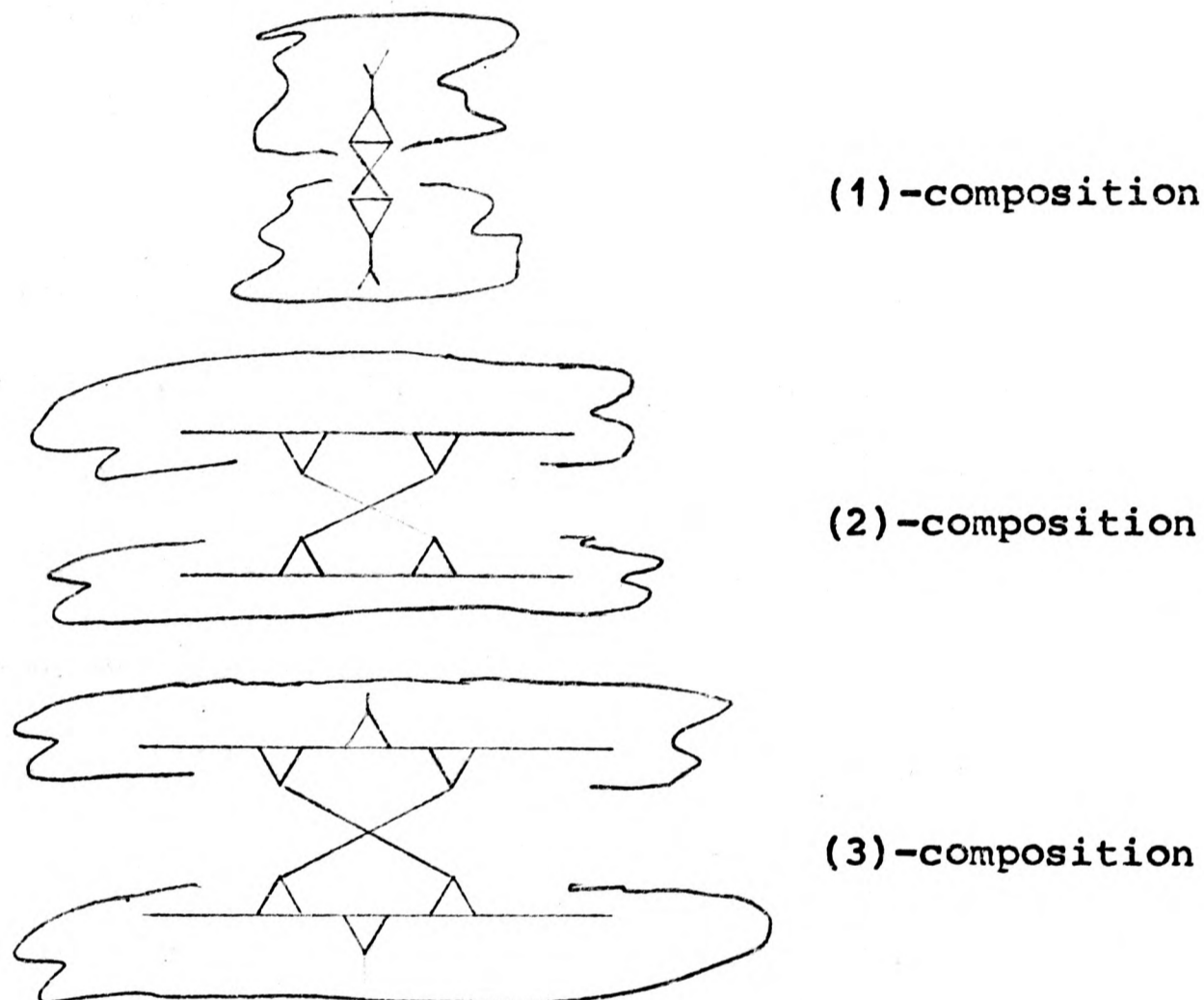
In an arbitrary permutation representation of $\Delta^*(2,3,k)$, any two points λ, μ which are fixed by x and are such that both $(xy)^j$ and t map λ to μ , will be said to form a (j)-handle. We will write such a (j)-handle as $[\lambda, \mu]_j$. When $j = 1, 2$ or 3 , the configurations appear on the central axes of the corresponding diagrams as follows:



We require a method of 'stitching' diagrams together, to give transitive permutation representations of $\Delta^*(2,3,k)$ of larger degree. The method we adopt is called (j)-composition (where j is as above).

Suppose V is a coset diagram for $\Delta^*(2,3,k)$ containing a (j)-handle $[\lambda, \mu]_j$, and W is another containing a (j)-handle $[\sigma, \tau]_j$, where j is the same for both. Then we form a (j)-composite of V and W in the following manner. Place the two diagrams on a common vertical axis of symmetry, one above the

other, and add two x -edges, one joining λ to σ and the other joining μ to τ . We write the resulting diagram as $V(j)W$ or as $W(j)V$. If $j = 1, 2$ or 3 then part of $V(j)W$ will appear as in the appropriate illustration:



It is clear that $V(j)W$ depicts a transitive permutation group of degree $m+n$, where m and n are the numbers of vertices of V and W respectively. This group is generated by elements x, y, t which satisfy $x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1$. Furthermore, if $(\lambda \gamma_1 \dots \gamma_{j-1} \mu \gamma_j \dots \gamma_{r-2})$ and $(\sigma \delta_1 \dots \delta_{j-1} \tau \delta_j \dots \delta_{s-2})$ are cycles of xy in the representations of $\Delta^*(2,3,k)$ depicted by V and W respectively, then in the constructed representation we see that $(\lambda \delta_1 \dots \delta_{j-1} \tau \gamma_j \dots \gamma_{r-2})$ and $(\sigma \gamma_1 \dots \gamma_{j-1} \mu \delta_j \dots \delta_{s-2})$ are cycles of the element xy . All

other cycles of xy are unchanged. In particular, the lengths of cycles of xy remain the same, so xy still has order k . Hence $V(j)W$ is a coset diagram for $\Delta^*(2,3,k)$.

For later use we notice what happens to the cycle structure of the element xyt as a result of this (j) -composition of the diagrams V and W . If before composition the cycle (of xyt) ending μ contains λ , and the cycle ending τ contains σ , then these cycles are juxtaposed to form a single cycle which can be rearranged to end σ , and is split into two cycles, one ending λ and containing τ , and the other ending σ and containing μ . Otherwise the cycles ending μ and τ are juxtaposed to form a single one, which can be rearranged if necessary, and at this stage the cycles ending λ and σ are juxtaposed to form a single one. In both cases, all other cycles of xyt remain as they were before composition.

If a cycle of xyt contains either a fixed point of t or a pair of points which are interchanged by t but which do not form a (j) -handle (for any j), then we shall call that cycle a useful one.

Notice that the points from a useful cycle in any diagram will almost always lie in a useful cycle after composition of that diagram with another.

Before we continue, it should be pointed out that the action of xyt (or, indeed, of any element of the group $\Delta^*(2,3,k)$) can be determined easily from inspection of the coset diagram.

Coset diagrams and an analogous method of composition were used by Stothers [24] to prove the following.

2.1 PROPOSITION The triangle group $\Delta(2,3,7)$ possesses a subgroup of (finite) index n precisely when there exists a quadruple (p, e, f, g) of non-negative integers satisfying the genus formula $n = 84(p-1) + 21e + 28f + 36g$, with the exceptional condition that $n \neq 16$.

We shall call any positive integer n which satisfies the conditions of the proposition an H-number.

Consider for a moment the genus formula. We obtain from it

$$\text{that} \quad \left[\frac{n}{2} \right] \geq 42(p-1) + 10e + 14f + 18g$$

$$\text{and} \quad 2 \left[\frac{n}{3} \right] \geq 56(p-1) + 14e + 18f + 24g$$

$$\text{and} \quad 6 \left[\frac{n}{7} \right] \geq 72(p-1) + 18e + 24f + 30g$$

$$\text{whence} \quad \left[\frac{n}{2} \right] + 2 \left[\frac{n}{3} \right] + 6 \left[\frac{n}{7} \right] \geq 170(p-1) + 42e + 56f + 72g \\ = 2n + 2(p-1),$$

and so $\left[\frac{n}{2} \right] + 2 \left[\frac{n}{3} \right] + 6 \left[\frac{n}{7} \right] \geq 2n - 2$. This is precisely the inequality we obtained in Corollary 1.2.

Those positive integers which are not H-numbers can now be determined easily. We list them overleaf, in columns corresponding to residue classes modulo 14.

2	3	4	5	6		10	11	12	13
16	17	18	19	20	23		25	26	27
	31	32	33	34		38	39	40	41
		46	47	48			53	54	55
	59		61	62			67	68	69
		74	75	76				82	83
			89	90			95		97
			103	104				110	111
				118					125
			131						139
				146					
									167

Similarly we define an H*-number to be any positive integer n for which there exists a permutation representation of the group $\Delta^*(2,3,7)$ of degree n in which the subgroup $\Delta(2,3,7)$ is transitive. It is obvious that any H*-number must be an H-number.

A search for coset diagrams for the groups $\Delta(2,3,7)$ and $\Delta^*(2,3,7)$ led us to the following conclusions.

2.2 PROPOSITION

- (a) $\Delta(2,3,7)$ has no primitive permutation representations of degree 24 or 60.
- (b) $\Delta(2,3,7)$ has only two transitive permutation representations of degree 7, both corresponding to the action of $\text{PSL}(2,7)$ on ^{the cosets of} a subgroup of order 24.
- (c) $\Delta(2,3,7)$ has only one transitive permutation representation of degree 8, namely as $\text{PSL}(2,7)$ acting on the projective line over

the field with 7 elements.

(d) $\Delta(2,3,7)$ has only one transitive permutation representation of degree 9, namely as $\text{PSL}(2,8)$ acting on the projective line over the field with 8 elements.

(e) If $p = 13, 29$ or 43 , then $\Delta(2,3,7)$ has only three primitive permutation representations of degree $p+1$, namely as $\text{PSL}(2,p)$ acting on the projective line over the field with p elements.

(f) If $n = 7, 29, 43, 45, 81$ or 96 , then $\Delta^*(2,3,7)$ has no transitive permutation representation of degree n .

(g) $\Delta^*(2,3,7)$ has no permutation representation of degree 88 in which the subgroup $\Delta(2,3,7)$ is transitive.

(h) If $n = 21$ or 80 , then $\Delta^*(2,3,7)$ has no transitive permutation representation of degree n in which the subgroup $\Delta(2,3,7)$ acts primitively.

(i) If $n = 22, 36, 37, 50, 52$ or 73 , then $\Delta^*(2,3,7)$ has no permutation representation of degree n in which the subgroup $\Delta(2,3,7)$ acts primitively and the involution t acts as an even permutation.

(j) If $n = 15, 51$ or 66 , then $\Delta^*(2,3,7)$ has no permutation representation of degree n in which the subgroup $\Delta(2,3,7)$ acts primitively and the involution t acts as an odd permutation.

Our proof is rather tedious, and does not really warrant a place in this thesis. The results have, however, been verified by Joachim Neubüser, who performed computations on the CYBER 175

at the Rechenzentrum der Rhein.-Westf. Technische Hochschule, Aachen. He used two programs, as below.

- 1.) A program for the determination of low index subgroups of a finitely presented group, which was implemented by Robyn Gallagher in Aachen, developing some predecessors and ultimately proposals by Charles Sims.
- 2.) A program for the identification of permutation groups. This is part of the CAYLEY GROUP-SYSTEM. It was implemented by John Cannon and others in Sydney, using ideas of Sims and Michael Atkinson.

A consequence of results (f) and (g) of the proposition is that each of the integers 7, 29, 43, 45, 81, 88 and 96 is an H-number but not an H*-number.

2.3 COROLLARY If n is a prime, or one of 35, 52, 58 or 124, then any permutation representation of $\Delta^*(2,3,7)$ of degree n , in which the subgroup $\Delta(2,3,7)$ is transitive, must in fact be primitive.

Proof The number of blocks in any system of imprimitivity must be an H*-number, as both $\Delta^*(2,3,7)$ and its subgroup $\Delta(2,3,7)$ permute the blocks transitively. But no non-trivial divisor of n is an H*-number, so the result follows.

Chapter 3 : HURWITZ GROUPS OF SMALL ORDER

Suppose G is a non-trivial Hurwitz group of order less than one million.

Then we know by Lemma 1.6 that G has a normal subgroup K such that G/K is a simple Hurwitz group, whose order must be divisible by 84.

The non-Abelian simple groups of order less than one million are known, and are listed in Appendix 2. In particular, those of order a multiple of 84 are the following:

(3.1)	<u>Alternating</u>	A_7, A_8, A_9
	<u>Classical</u>	$PSL(2, q)$ for $q = 7, 8, 13, 27, 29,$ $41, 43, 49, 64, 71, 83, 97, 113, 125$ $PSU(3, 3)$ $PSL(3, 4)$ $PSU(3, 5)$
	<u>Sporadic</u>	J_1 (Janko) M_{22} (Mathieu) J_2 (Hall-Janko)

We proceed to determine exactly which of these groups are Hurwitz.

The three alternating groups can be eliminated immediately by resorting to Proposition 2.2, parts (b) to (d). Any pair (x,y) of elements of A_9 satisfying $x^2 = y^3 = (xy)^7 = 1$ must generate either the trivial group or a subgroup isomorphic to $PSL(2,7)$ or $PSL(2,8)$.

Next the work of Macbeath (cf. Proposition 1.8) tells us precisely when the linear fractional group $PSL(2,q)$ is a Hurwitz group.

We are left with just six groups to consider. The elimination of four of them is easy, and illustrates the range of techniques which can be applied in this direction.

Using the formula of Lemma 1.9, we find that each of the six groups does contain at least one non-trivial Hurwitz subgroup.

For example, in $PSU(3,3)$, given any element u of order 7, there are precisely seven pairs (x,y) with x of order 2, and y of order 3, and $xy = u$. On the other hand the same is true inside $PSL(2,7)$, which is known (cf. [4]) to be a subgroup of $PSU(3,3)$. Since all cyclic subgroups of order 7 in $PSU(3,3)$ are conjugate (by Sylow's theorem), it follows that any non-trivial Hurwitz subgroup of $PSU(3,3)$ must be isomorphic to $PSL(2,7)$.

In particular, $PSU(3,3)$ itself is not a Hurwitz group. (This refutes the suggestion made by Chih-han Sah in [21].)

(Inc Now consider $PSL(3,4)$, and in particular its doubly transitive

permutation representation on the 21 points of a projective plane of order 4. The permutation character θ corresponding to this action is of the form $\theta = 1 + \chi$, where 1 denotes the trivial character and χ the irreducible complex character of degree 20. From the character table of $\text{PSL}(3,4)$ we learn that any involution fixes 5 points, any element of order 3 fixes 3 points, while any element of order 7 moves all 21 points. If $\text{PSL}(3,4)$ were to have a $(2,3,7)$ -generating pair, then Lemma 1.1 would tell us that $\frac{1}{2}(21 - 5) \geq 3 + (3 + \frac{1}{3}(21 - 3)) - 2$, which is not so. Hence $\text{PSL}(3,4)$ is not a Hurwitz group.

The Mathieu group M_{22} is a triply transitive group of degree 22 (obtained from $\text{PSL}(3,4)$ by transitive extension). From its character table, we find that any involution fixes 6 points, any element of order 3 fixes 4 points, while any element of order 7 moves all but one of the 22 points. Again the inequality from Lemma 1.1 reveals that any Hurwitz subgroup cannot be transitive.

The same argument does not work in the case of $\text{PSU}(3,5)$. Suppose however that (x,y) is a pair of non-trivial elements of this group such that $x^2 = y^3 = (xy)^7 = 1$. Now x and $y^{-1}xy$ come from the same conjugacy class in $\text{PSU}(3,5)$, so there exists at least one pair of conjugates of x whose product is the element $xy^{-1}xy$. An application of the formula of Lemma 1.9 to the character table of $\text{PSU}(3,5)$ yields that $xy^{-1}xy$ cannot have order 10, and must therefore have order less than 9.

(Incidentally, the calculation shows that $\text{PSU}(3,5)$ has no

dihedral subgroups of order greater than 16.) It follows that the subgroup generated by x and y is a homomorphic image of one of the groups (b) to (d) of Proposition 1.5. A comparison of orders shows that in fact $\langle x, y \rangle$ is isomorphic to $\text{PSL}(2, 7)$. In particular, $\text{PSU}(3, 5)$ is not a Hurwitz group.

We now turn to the sporadic groups J_1 and J_2 .

3.2 PROPOSITION Janko's simple group J_1 of order 175560 is a Hurwitz group.

This result was proved by Graham Higman in [8], using techniques similar to those presented here. It was proved independently in [21] by Chih-han Sah, who noticed that the only perfect proper subgroups (other than the trivial one) of J_1 are $\text{PSL}(2, 5)$ and $\text{PSL}(2, 11)$, neither of which is a Hurwitz group.

3.3 PROPOSITION The Hall-Janko group J_2 (the simple group of order 604800) is a Hurwitz group.

Proof We consider the transitive permutation representation of J_2 on the 100 cosets of a subgroup isomorphic to $\text{PSU}(3, 3)$, as discussed in [7].

If θ denotes the permutation character corresponding to this action, then θ is of the form $1 + \psi + \chi$, where 1 denotes the trivial character, ψ the irreducible complex character of degree 36, and χ that of degree 63. Using the formula of Lemma

1.9, we find that there exist elements x and y in J_2 satisfying

$$x^2 = y^3 = (xy)^7 = 1 \quad ,$$

with $\theta(x) = 0$, $\theta(y) = 4$, and $\theta(xy) = 2$.

It turns out that (x,y) is a minimal generating pair for J_2 .

For assume that the subgroup H generated by x and y is proper. Then H has order at most 6048, because J_2 has no permutation character of degree less than 100. Now by Proposition 1.7, H must be isomorphic to $PSL(2,7)$ or $PSL(2,8)$ or $PSL(2,13)$ or a group of order 1344 or 4536. (Recall that we have already shown that $PSU(3,3)$ is not a Hurwitz group.) The order of J_2 is not divisible by 13 nor by 81, hence H cannot be isomorphic to $PSL(2,13)$ nor of order 4536. Next $PSL(2,8)$ contains elements of order 9, but J_2 does not, hence H cannot be isomorphic to $PSL(2,8)$. Also J_2 has no permutation character of degree 450 - this is not obvious but quite easy to check from the character table using known properties (cf. [7] or [8]) of permutation characters - therefore no subgroup of J_2 has order 1344. Thus H must be isomorphic to $PSL(2,7)$.

We now look at the action of H on the 100 points of the coset-space corresponding to the subgroup $PSU(3,3)$. Since the element x acts as an even permutation fixing no points, the length of each orbit of H must be a multiple of 4. Also each orbit length must divide 168 (the order of H), and therefore is one of 4, 8, 12, 24, 28, 56 or 84. Next the 7-element xy fixes only two points, so the orbit lengths 4, 12 and 24 are impossible. Finally the 3-element y fixes exactly four points, so H must act with three orbits, of lengths 8, 8 and 84. But now the orbit of

length 84 corresponds to the action of H on ^{the cosets of} a subgroup of order 2, and since in $\text{PSL}(2,7)$ all elements of order 2 are conjugate, this implies that x fixes a point!

We have obtained a contradiction, hence H must be the whole of J_2 .

Thus we know exactly which of the groups in the list 3.1 are Hurwitz groups.

In fact this knowledge could have been gleaned from computational results stated in a paper [19] by John McKay and Kiang-chuen Young. They give in that paper a print-out (on microfiche) of minimal generating pairs for all the simple groups of order less than one million, other than the groups $\text{PSL}(2,q)$. (It should be noted here that the definition of a minimal generating pair in [19] differs slightly from that which we have adopted.)

The author of this thesis was unaware of the work of McKay and Young until he had completed his own classification of Hurwitz groups of small order.

We now proceed to give this classification.

Suppose again that G is a Hurwitz group of order less than one million, let K be a maximal normal subgroup of G , and now suppose that G/K is isomorphic to J_1 or to J_2 .

If $G/K \cong J_2$, then a comparison of orders shows that K is trivial and therefore $G \cong J_2$.

On the other hand, if $G/K \cong J_1$, then K must have order less than six; in particular, K is Abelian and its automorphism group $\text{Aut}(K)$ has order at most six. Now $K \leq C_G(K) \leq G$, and conjugation of K by elements of G induces an isomorphism from $G/C_G(K)$ onto a subgroup of $\text{Aut}(K)$. Since G/K is simple and of order (much) greater than six, this implies that $G = C_G(K)$, hence $K \leq Z(G)$. Next G is perfect (by Lemma 1.6), so we are in a position to apply the following result, and deduce that G is isomorphic to J_1 .

3.4 LEMMA The Schur multiplier of J_1 is trivial; that is, if R is any finite group containing a normal subgroup X such that $R/X \cong J_1$ and $X \leq Z(R)$ and $R' \geq X$, then X is trivial.

A proof of this is given by Janko in [9].

We obtain from all this the following result.

3.5 THEOREM

Let G be a non-trivial Hurwitz group of order less than one million. Then:

either (a) G is perfect and contains a normal subgroup K such

that G/K is isomorphic to $\text{PSL}(2, q)$, where

$q = 7, 8, 13, 27, 29, 41, 43, 71, 83, 97, 113$ or 125 ,

or (b) G is isomorphic to the simple group of order 175560,

or (c) G is isomorphic to the simple group of order 604800.

It is possible to achieve a more detailed classification, for example by considering the homology groups of the Riemann surface on which the Hurwitz group acts. Some of these surfaces (and groups) are investigated in [14], [15], [16] and [21].

Chapter 4 : ALTERNATING GROUPS AS HURWITZ GROUPS

In this chapter we show that all but 64 of the (finite) alternating groups have minimal $(2,3,7)$ -generating pairs.

Coset diagrams for the group $\Delta^*(2,3,7)$ are first used to show that all but finitely many of the A_n and S_n can be generated by elements x, y, t which satisfy

$$x^2 = y^3 = (xy)^7 = t^2 = (xt)^2 = (yt)^2 = (xyt)^{2m} = 1$$

where $m = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53$.

Only a small number of diagrams are required. It is convenient for us to set aside those basic diagrams which serve as 'building blocks' in our construction of larger ones, so we give them first on pages 31 to 62.

For each we have drawn the diagram, labelling (among others) those fixed points of x which appear in (1)-, (2)- or (3)-handles, we have given the diagram itself a label, and we have written down a specification consisting of the degree of the (transitive) permutation representation it depicts, the number and type of handles, the parity of (the action of) the element t , and the cycle structure of the element xyt . In the latter cycles we have indicated the positions of some of the labelled vertices, in order to facilitate computation of the cycle structure of xyt in composite diagrams.

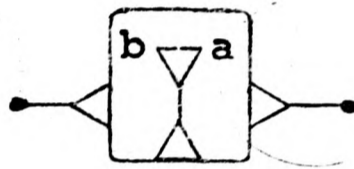
Consider (as an illustration) the specification

$$72 : \underline{1(3)} : \text{odd} : (A9B9) \ 2^2 \ \underline{19^1} \ \underline{29^1}$$

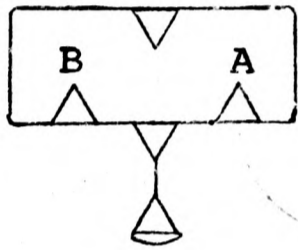
of the diagram on page 51. We see that diagram P_4 depicts a transitive permutation representation of $\Delta^*(2,3,7)$ of degree 72, in which t acts as an odd permutation. The diagram contains a (3)-handle $[A,B]_3$; and the element xyt has five cycles, one of length 20 containing both the points A and B , two of length 2, one of length 19 and one of length 29. Each of the last two cycles contains a pair of points which are interchanged by t , and is therefore a useful cycle (cf. Chapter 2).

We have underlined 34 of the useful cycles of xyt in a selection of the basic diagrams. It is clear that the points from any one of these cycles will remain in a useful cycle after composition of the corresponding diagram with any other.

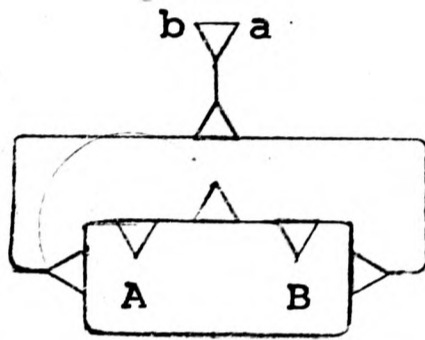
Here then are the basic diagrams.



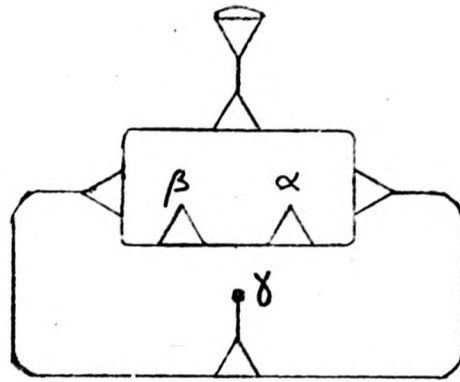
C : 14 : 1(1) : even : a (b12)



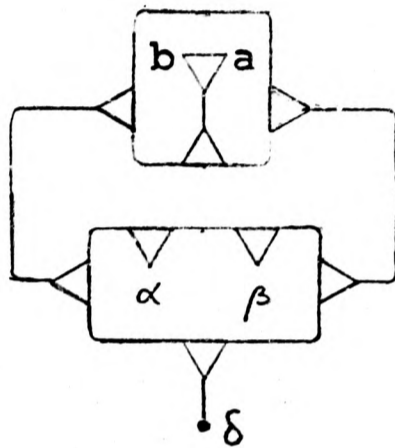
D : 15 : 1(3) : even : (A6) (B2) 5¹



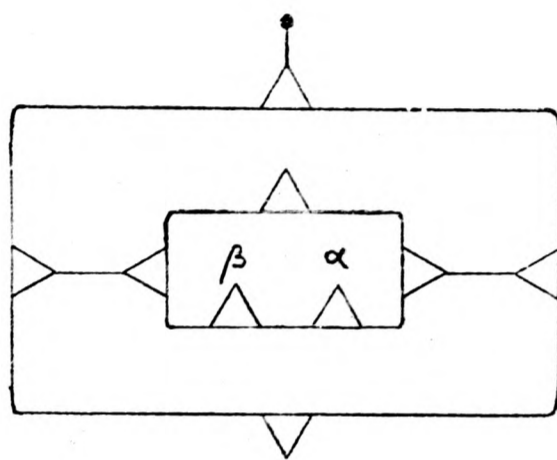
E : 21 : 1(1), 1(3) : odd : a (b3B3) (A3) 8¹



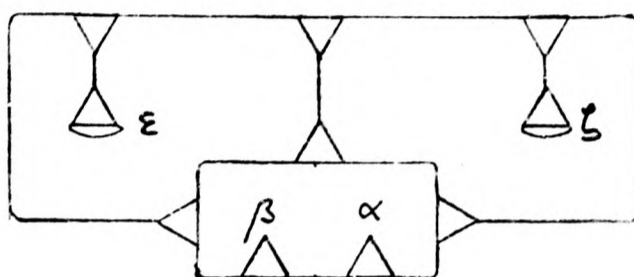
F : 22 : 1(2) : odd : (α10) (β4) 6¹



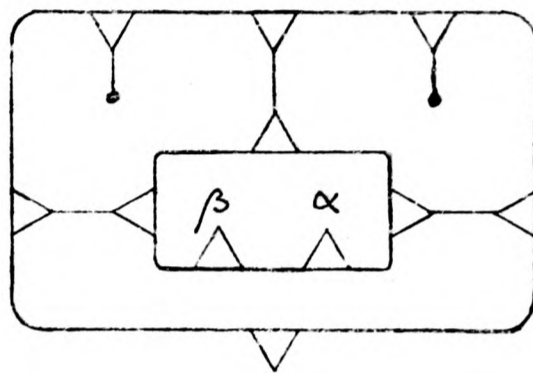
$G_1 : 28 : 1(1), 1(2) : \text{even} : a (b8) (\alpha 8) (\beta 8)$



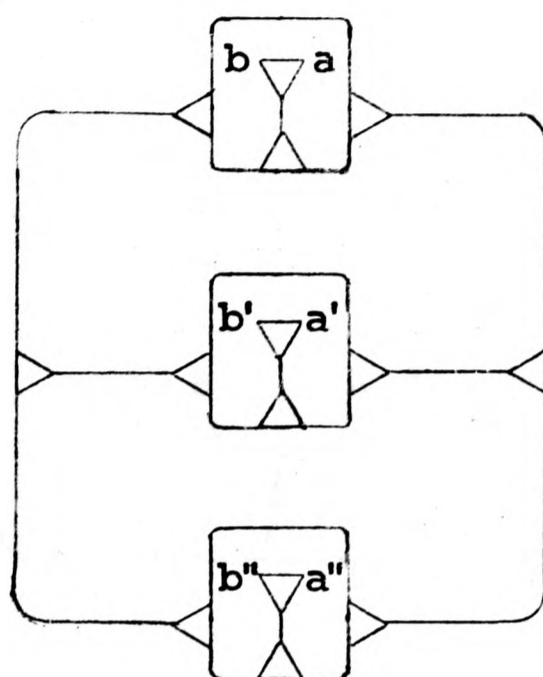
$G_2 : 28 : 1(2) : \text{even} : (\underline{\alpha 7}) (\beta 8) 4^1 \underline{7}^1$



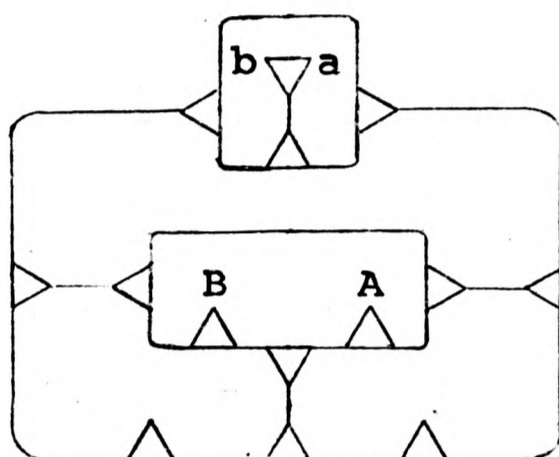
H : 30 : 1(2) : even : (α 14) (β 14)



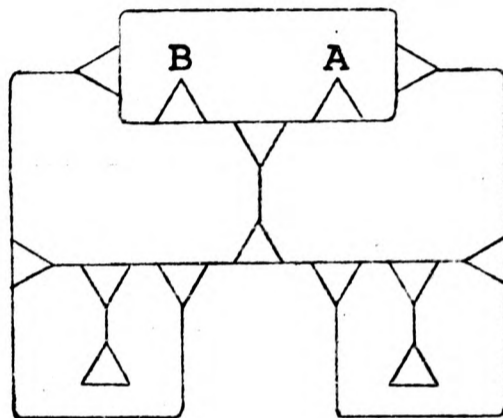
I : 35 : 1(2) : even : $(\alpha 8 \beta 8) \underline{7}^1 10^1$



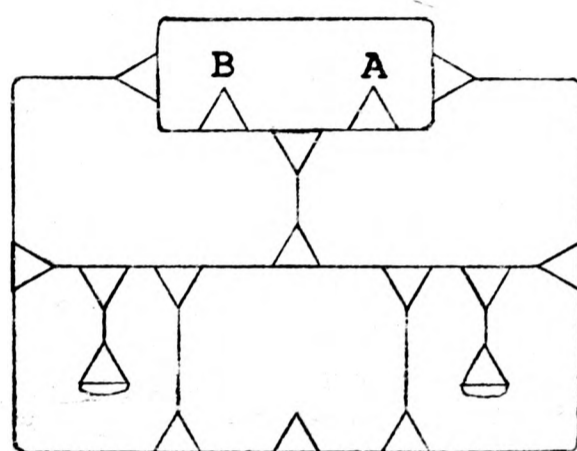
$J_1 : 42 : 3(1) : \text{even} : a (b12) a' (b'12) a'' (b''12)$



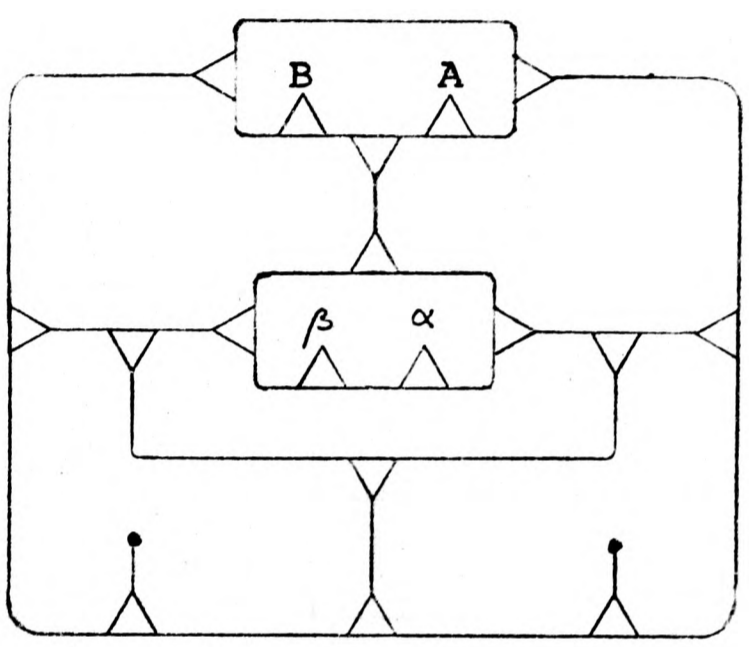
$J_2 : 42 : 1(1), 1(3) : \text{odd} : a \text{ (b9) (A16) (B10) } 3^1$



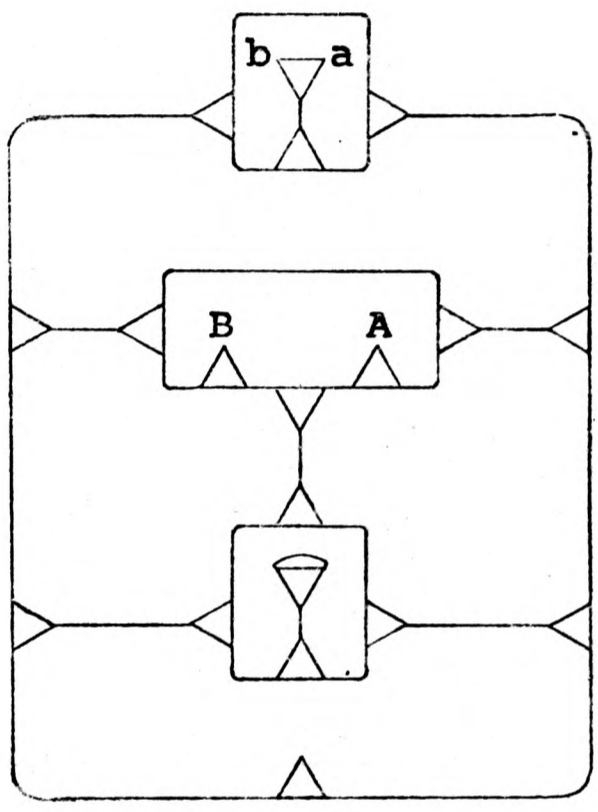
$J_3 : 42 : 1(3) : \text{even} : (A9B9) 2^1 7^1 \underline{13^1}$



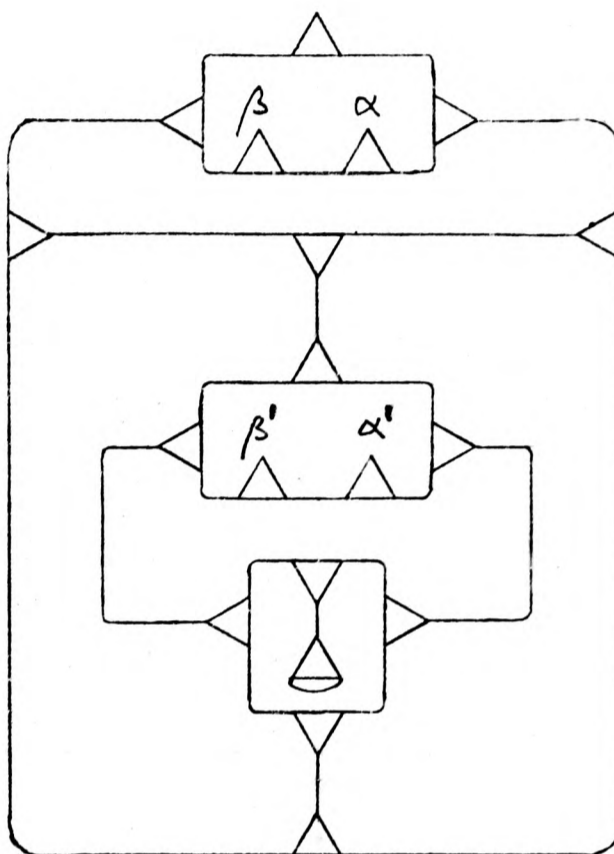
K : 51 : 1(3) : even : (A7B7) 13¹ 22¹



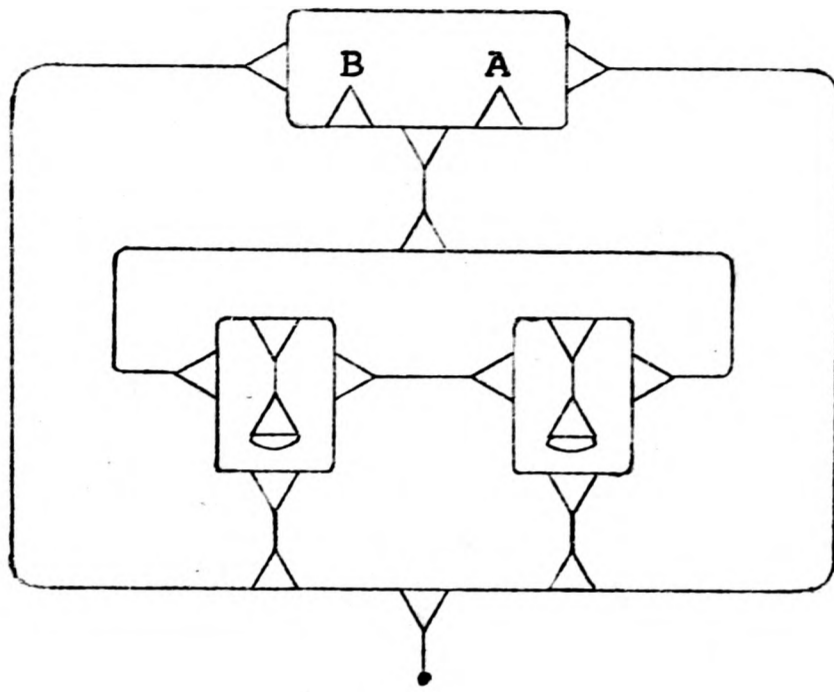
L : 56 : 1(2), 1(3) : even : (α 5A5) (β 18) (B12) 12¹



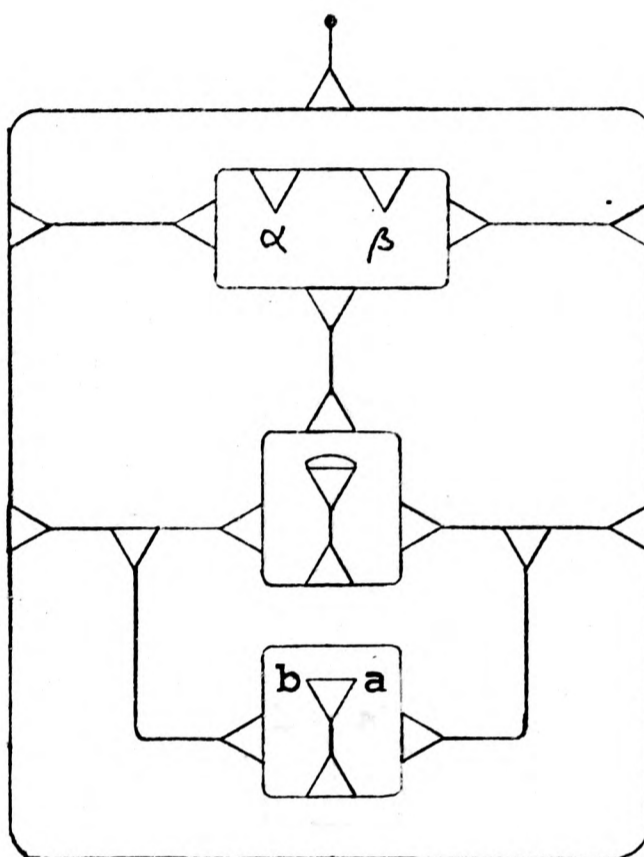
$M_1 : 57 : 1(1), 1(3) : \text{odd} : a \text{ (b16) (A8) (B10) } 6^1 \underline{13}^1$



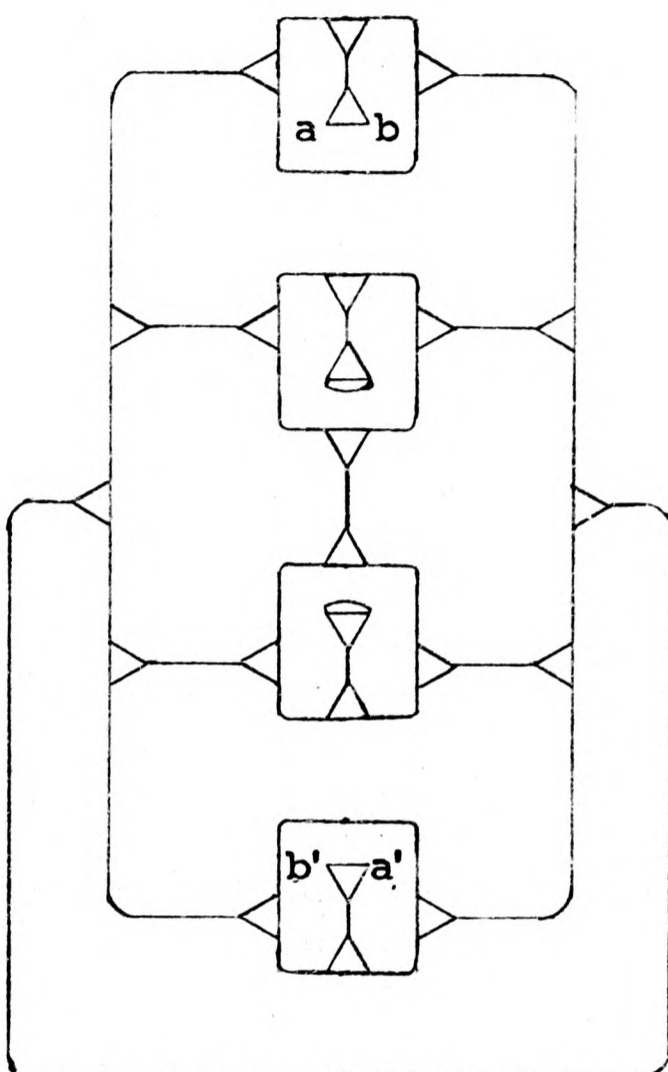
$M_2 : 57 : 2(2) : \text{odd} : (\underline{\alpha 7}) (\underline{\beta 12}) (\alpha' 14) (\beta' 6) 4^1 10^1$



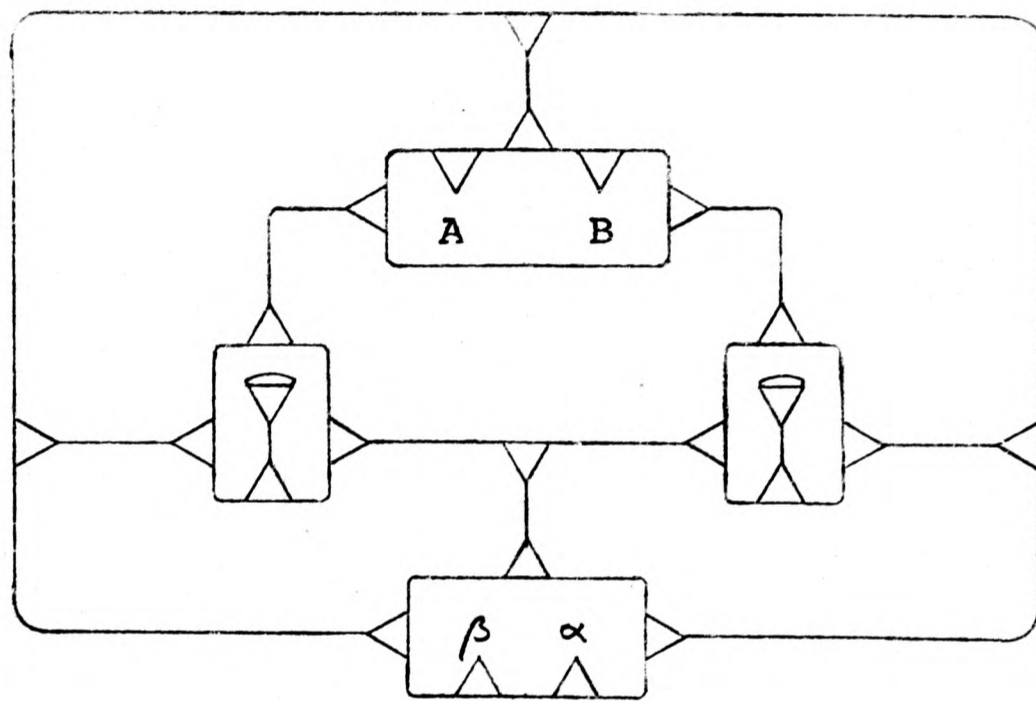
N : 58 : 1 (3) : odd : (A26) (B6) 24¹



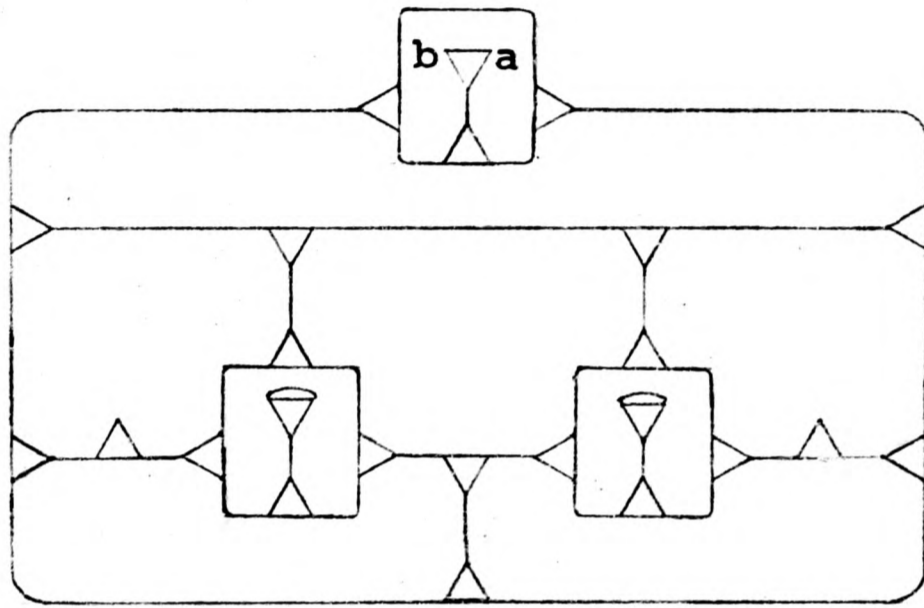
0 : 64 : 1(1), 1(2) : even : a (b10) (α 12) (β 18) 8^1 12^1



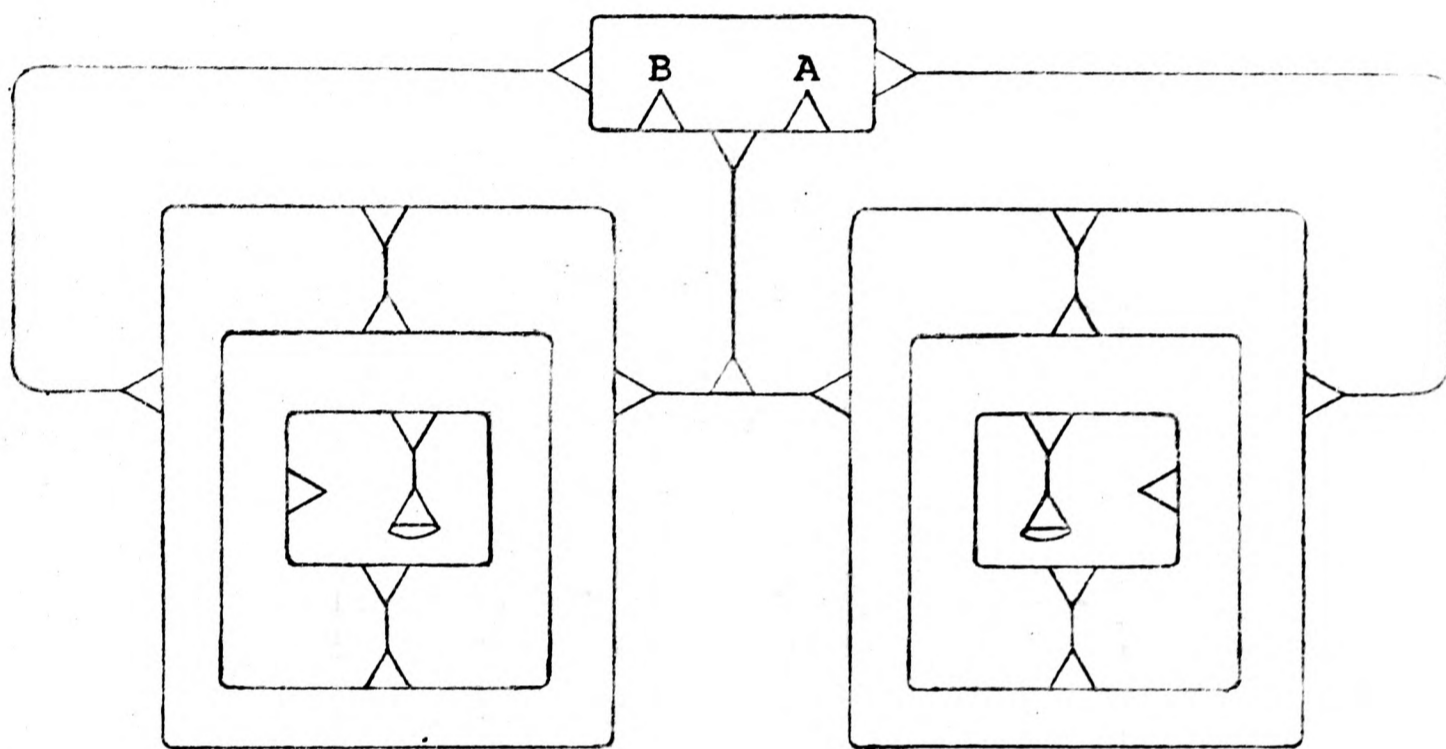
$P_1 : 72 : 2(1) : \text{odd} : a (b'10) a' (b'10) 10^1 16^1 22^1$



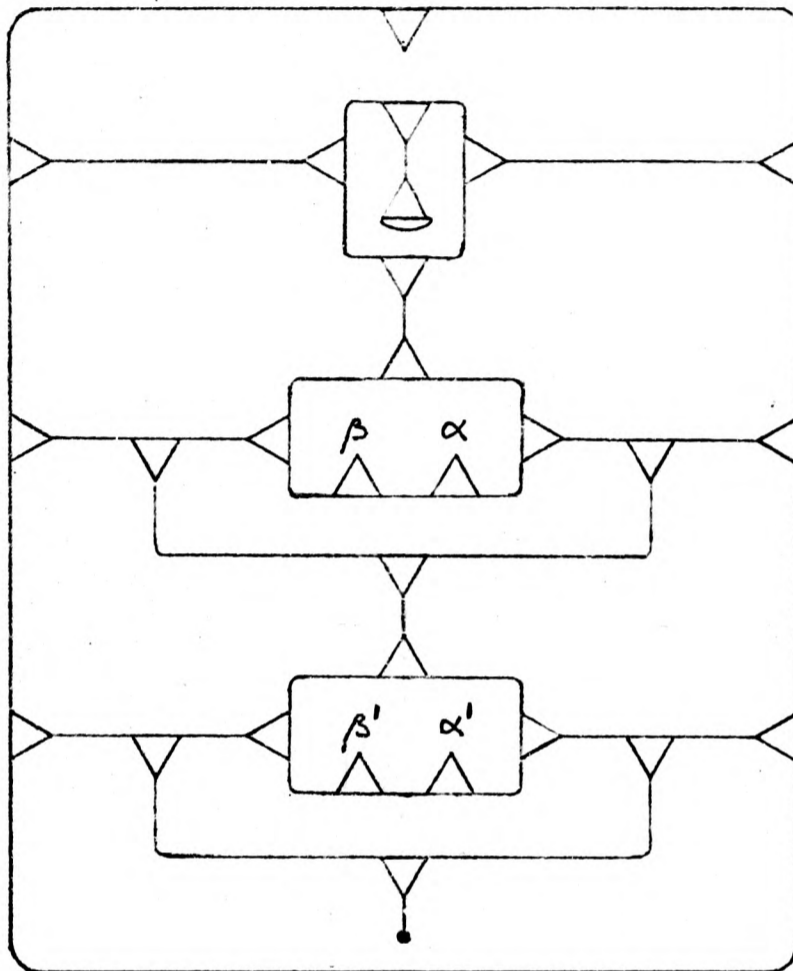
$P_2 : 72 : 1(2), 1(3) : \text{even} : (\alpha 10 \beta 10) (\underline{A10}) (B20) 18^1$



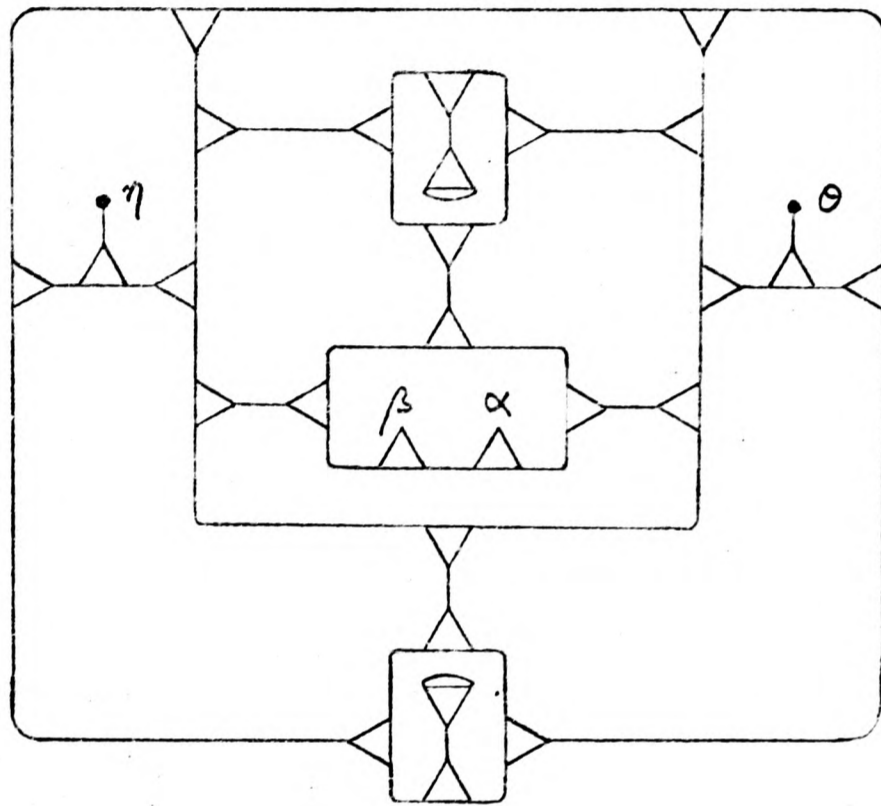
$P_3 : 72 : 1(1) : \text{even} : a (b48) 5^1 \underline{17}^1$



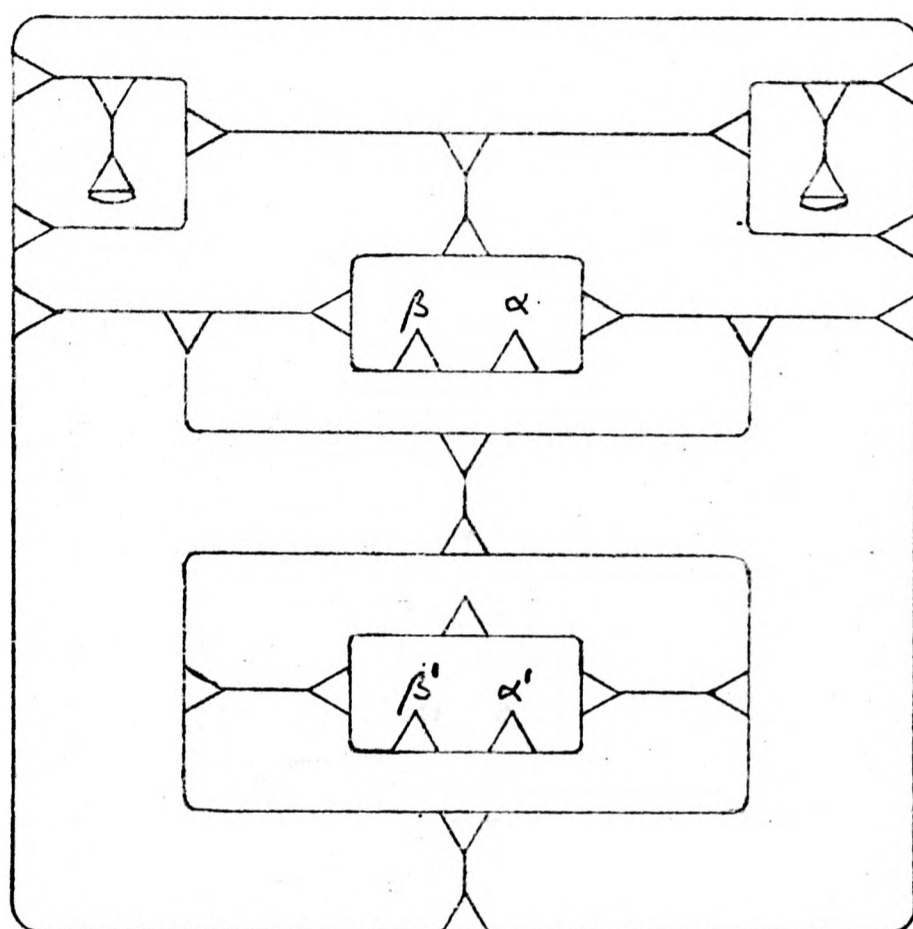
$P_4 : 72 : 1(3) : \text{odd} : (A9B9) 2^2 \underline{19^1} \underline{29^1}$



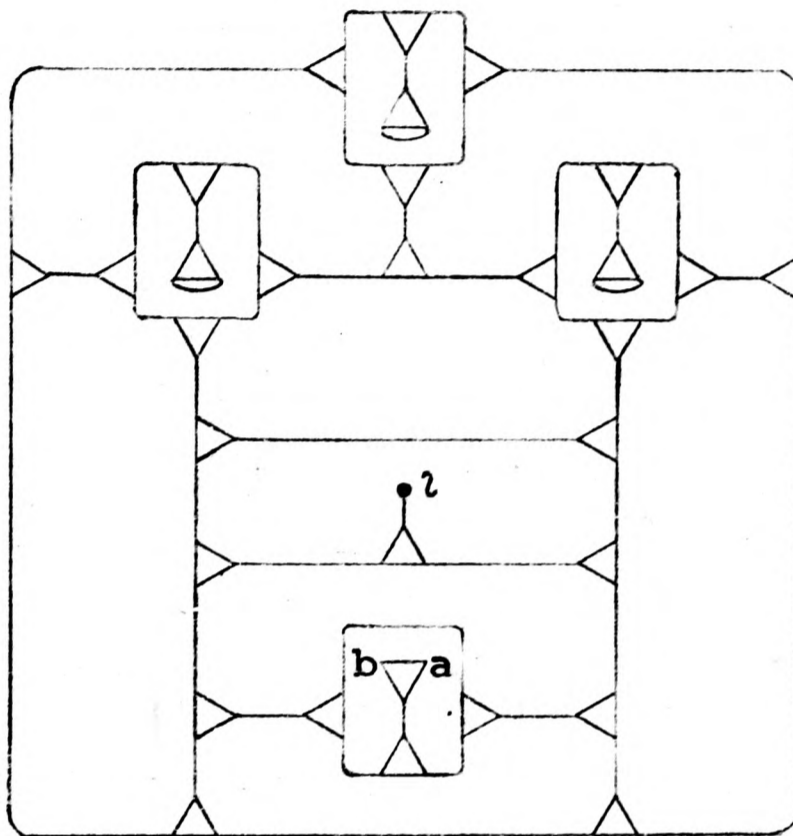
Q : 85 : 2(2) : even : $(\alpha_{12}) (\beta_8 \beta'_8) (\alpha'_{14}) 6^1 8^1 9^1 16^1$



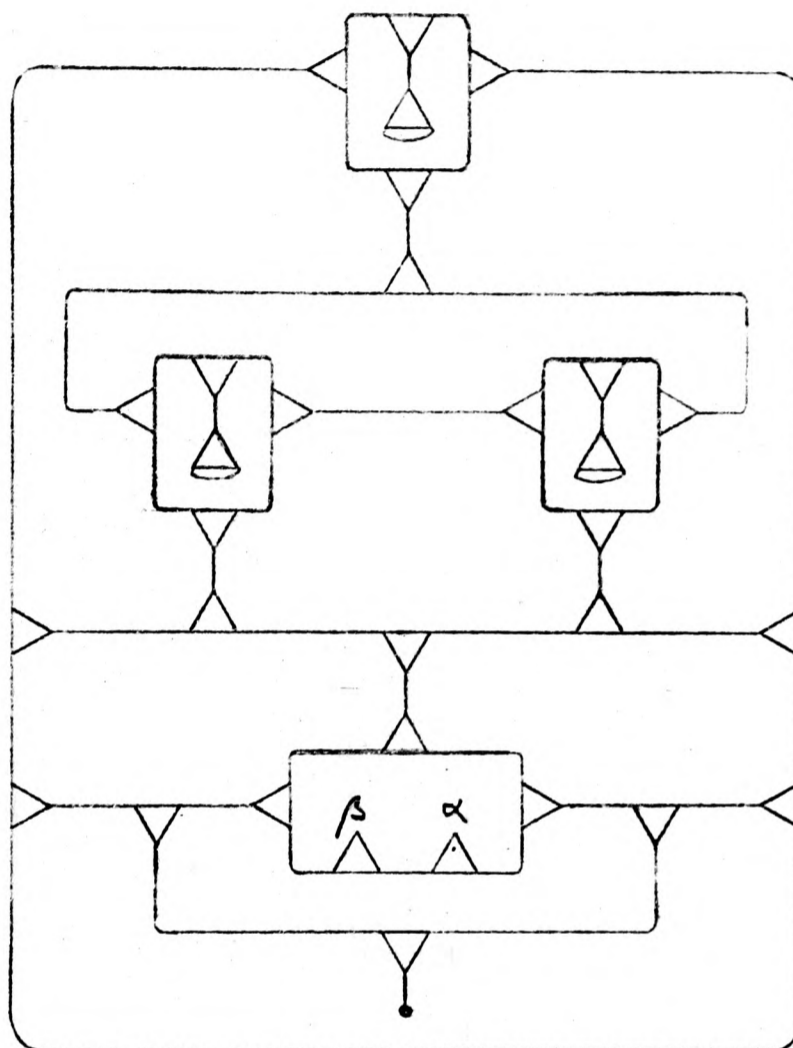
R : 86 : 1(2) : odd : $(\alpha 12)$ $(\beta 22)$ 12^2 26^1



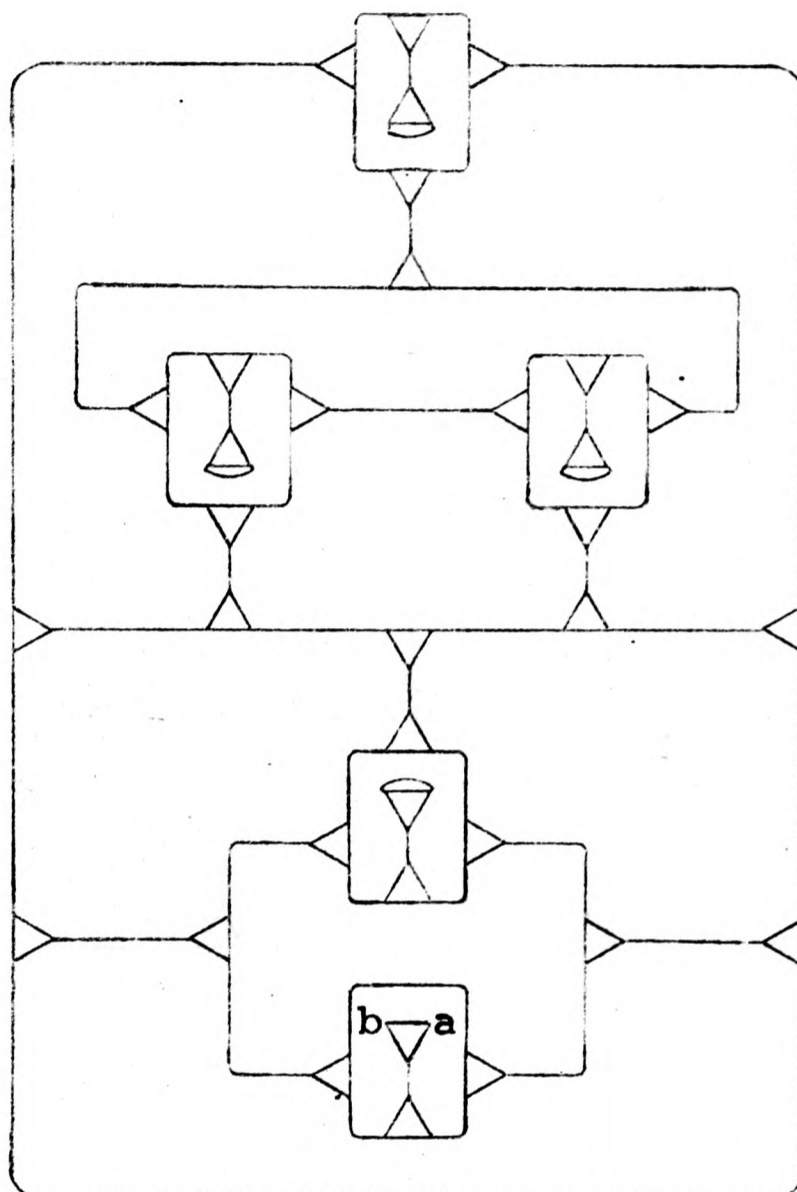
s : 93 : 2(2) : odd : (α 30) (β 8) . (α' 7) (β' 14) 4^1 26^1



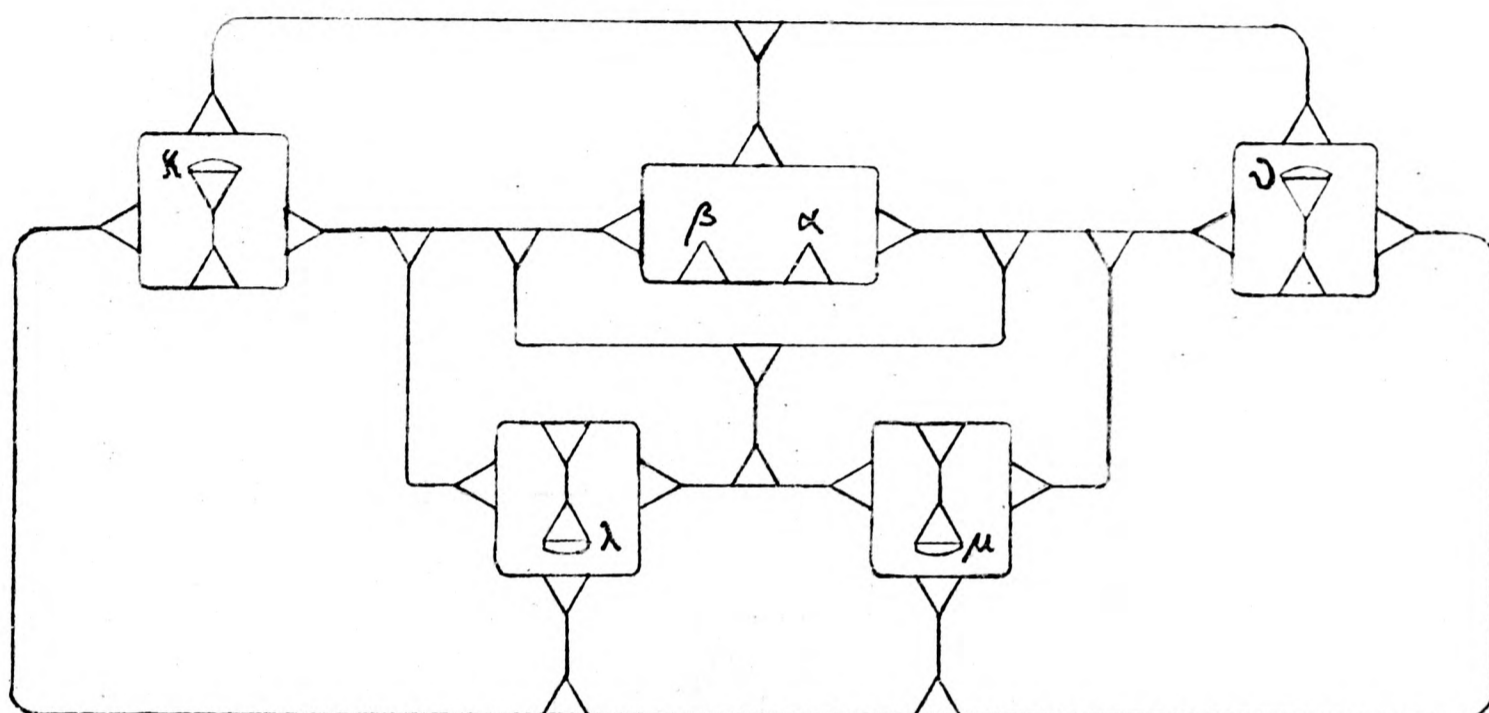
$T_1 : 94 : 1(1) : \text{odd} : a (b28) 14^1 22^1 28^1$



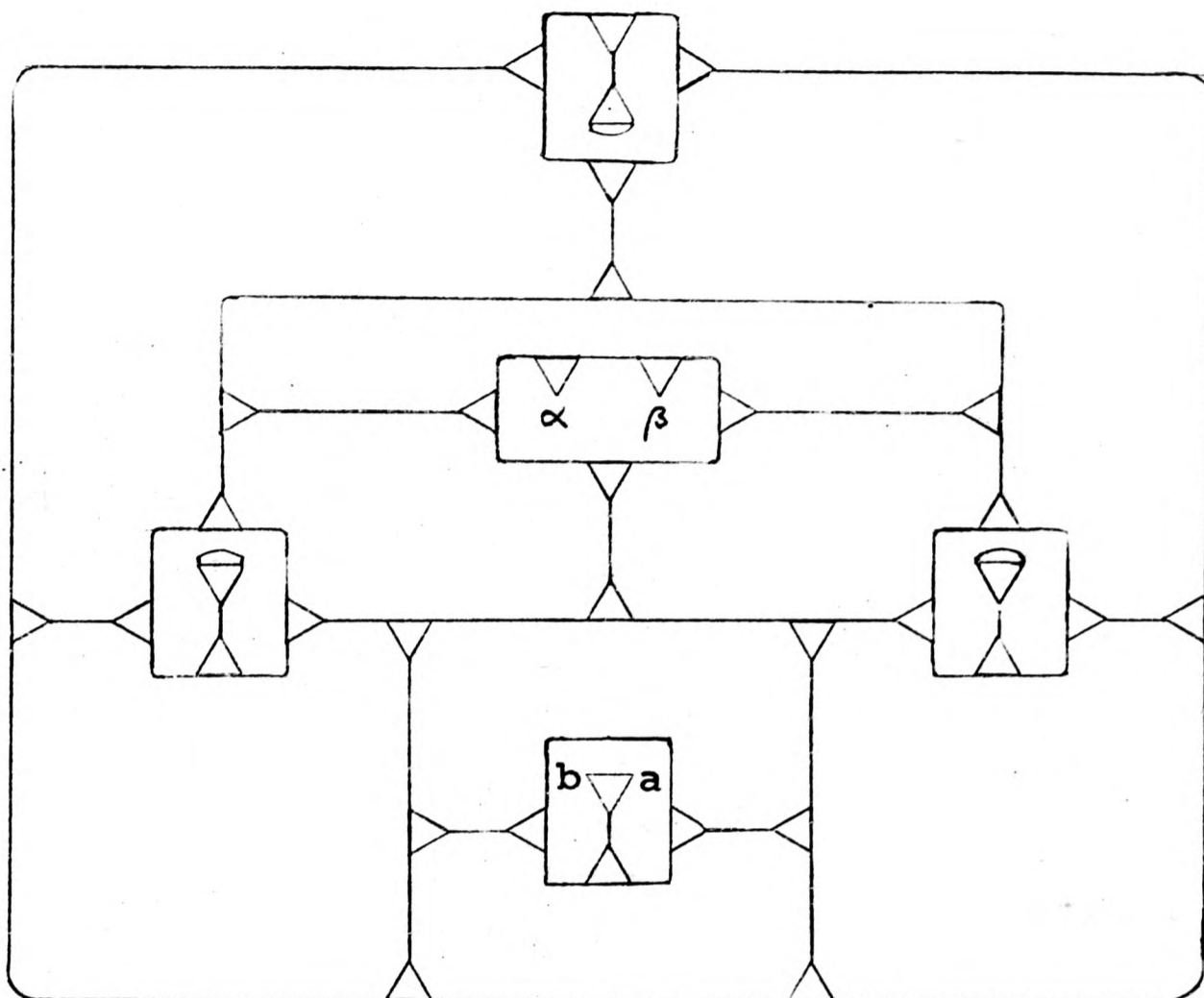
$T_2 : 94 : 1(2) : \text{odd} : (\alpha 17 \beta 17) 8^1 10^1 12^1 28^1$



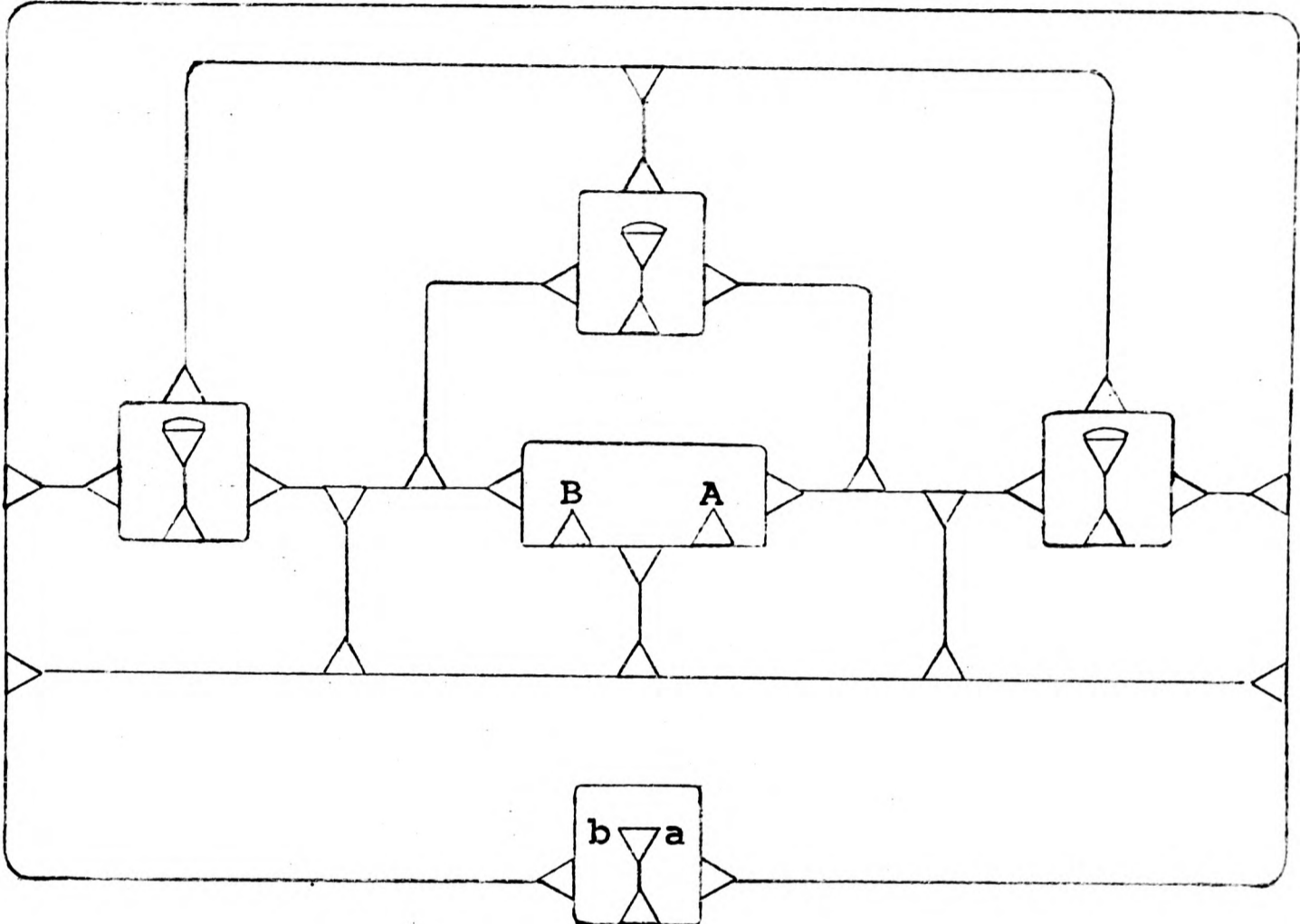
$u_1 : 102 : 1(1) : \text{even} : a \text{ (b10)} 12^1 14^1 28^1 36^1$



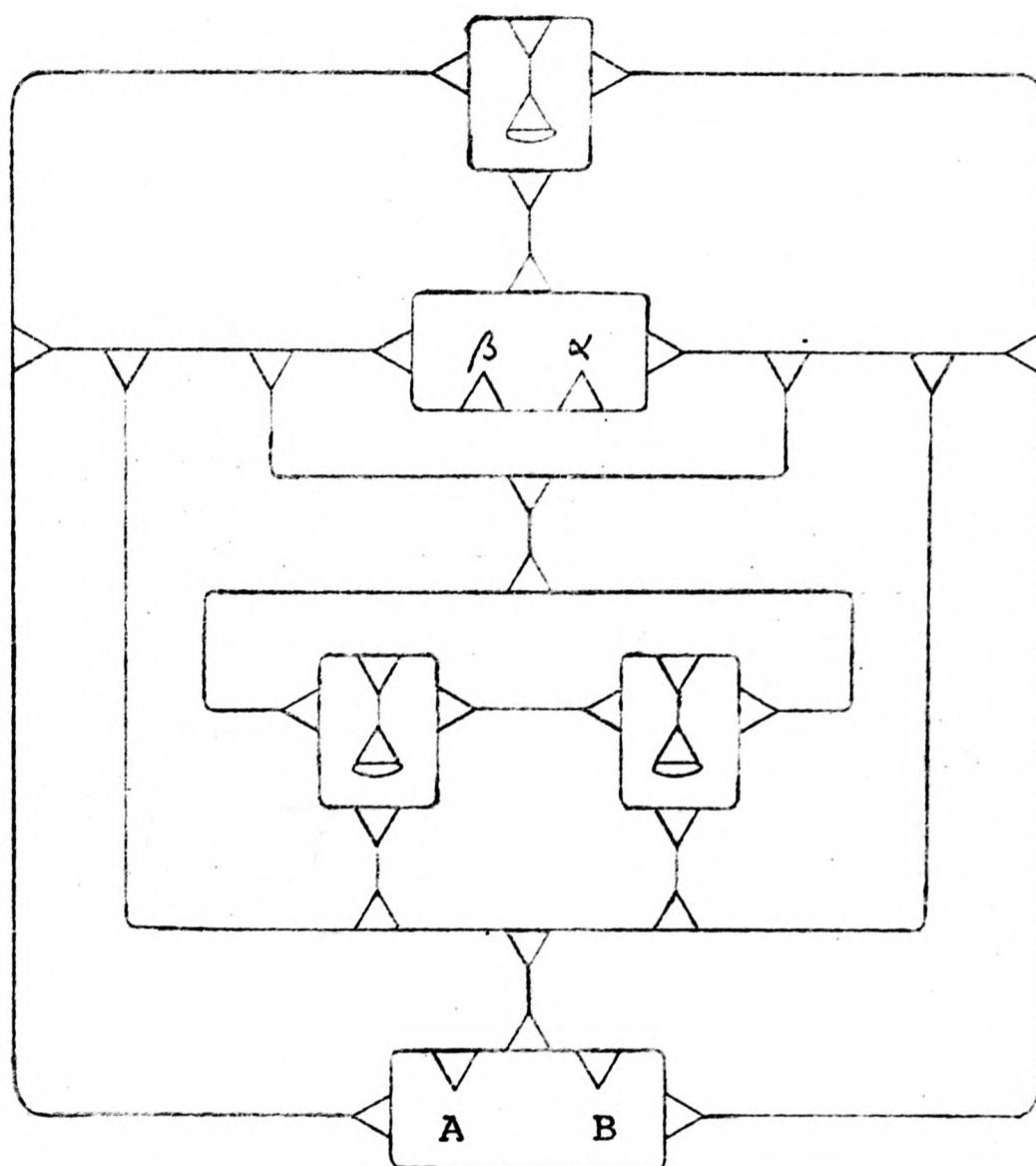
$$U_2 : 102 : 1(2) : \text{odd} : (\alpha 20) \ (\underline{\beta 22}) \ 58^1$$



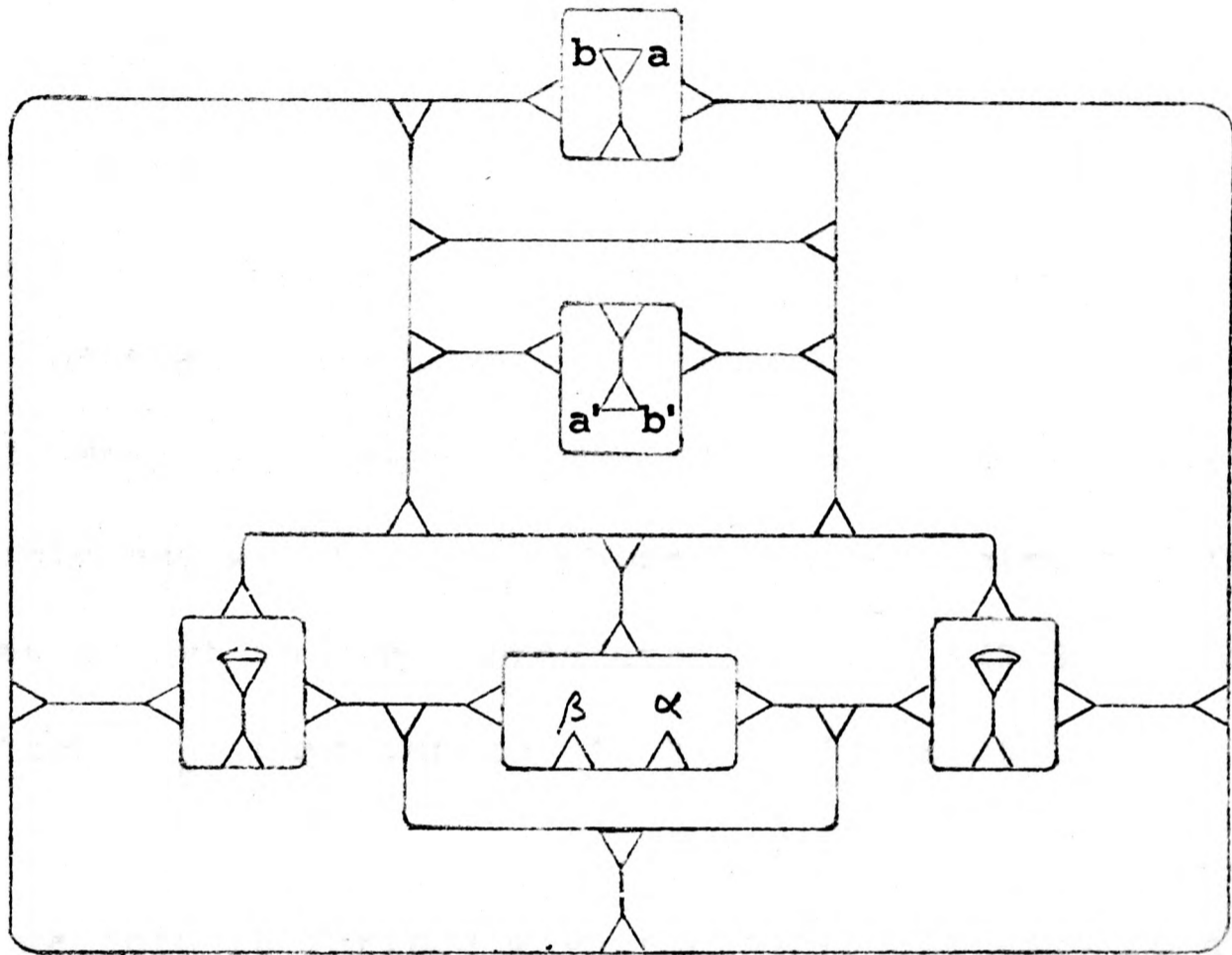
v_1 : 108 : 1(1), 1(2) : even : a (b17 \times 17) (β 18) 12¹ 14¹ 26¹



$v_2 : 108 : 1(1), 1(3) : \text{even} : a (b20) (A38) (B8) 18^1 20^1$



$v_3 : 108 : 1(2), 1(3) : \text{even} : (\underline{\alpha 12}) (\beta 10) (\underline{A 30}) (B 8) 14^1 30^1$



W : 114 : 2(1), 1(2) : odd : a (b14β14) a' (b'22) (α30) 12¹ 16¹

Now we construct from these a family $\{ X_s : s = 0, 1, 2, \dots \}$ of coset diagrams which we will call stocks. The stock X_s is to depict a transitive representation of $\Delta^*(2, 3, 7)$ of degree $14s$, in which t acts as an even permutation, and is to contain a (1)-handle (except, of course, when $s = 0$).

Take as stocks X_0, X_1, X_2 and X_3 respectively the empty diagram, diagram C, diagram G_1 and diagram J_1 . For $s > 3$ construct the stock X_s by taking $\lceil \frac{s}{3} \rceil$ copies of the diagram J_1 , joining them by (1)-composition into a chain, and using $s - 3\lceil \frac{s}{3} \rceil$ of the remaining (1)-handles to adjoin copies of the diagram C. There are several ways of doing this; for our purposes it is irrelevant which way is chosen. In any case, and for any s , all cycles of xyt in the representation of $\Delta^*(2, 3, 7)$ depicted by the stock X_s have length 1, 2, 9, 13 or 26.

Next we join in various ways the basic diagrams to give appendages which can be (1)-composed to each of the stocks in turn.

Overleaf is a list of specifications for fourteen such appendages - one corresponding to each residue class modulo 14. Each diagram contains at least one (1)-handle, and depicts a transitive representation of $\Delta^*(2, 3, 7)$ in which t acts as an odd permutation. We have omitted from the specifications the nature of any remaining handles.

<u>Diagram</u>	<u>Specification</u>
J_2	: 42 : a (b9) 3^1 <u>11^1</u> <u>17^1</u>
M_1	: 57 : a (b16) 6^1 9^1 <u>11^1</u> 13^1
$E(3)K$: 72 : a (<u>b27</u>) 8^1 <u>13^1</u> 22^1
$G_1(2)M_2(2)H$: 115 : a (b8) 4^1 10^1 <u>17^1</u> 22^2 30^1
$P_1(1)P_3$: 144 : a (b10) 2^1 5^1 10^1 16^1 <u>17^1</u> 22^1 60^1
$V_2(3)E(1)G_1(2)H$: 187 : a (b20) 2^1 8^1 18^1 20^1 24^2 26^1 <u>43^1</u>
$W(1)U_1$: 216 : a (b29) 2^1 12^2 14^1 16^1 28^1 <u>31^1</u> 34^1 36^1
$E(3)L$: 77 : a (b20) 8^1 12^1 16^1 <u>19^1</u>
$E(3)D$: 36 : a (b10) <u>5^1</u> 8^1 <u>11^1</u>
$J_2(3)K$: 93 : a (<u>b9</u>) 3^1 <u>13^1</u> 22^1 44^1
$D(3)E(1)J_1(1)G_1(2)H$: 136 : a (b12) 2^2 <u>5^1</u> 8^1 11^1 22^1 24^3
$G_1(2)S(2)H$: 151 : a (b8) 4^1 <u>17^1</u> 24^2 26^1 46^1
$V_2(3)P_4$: 180 : a (b20) 2^2 18^1 <u>19^1</u> 20^1 <u>29^1</u> 68^1
$V_1(2)M_2(2)H$: 195 : a (b50) 4^1 10^1 12^1 14^1 <u>23^1</u> 26^2 28^1

Notice that each contains at least one useful cycle of prime length, which has (as before) been underlined.

Now suppose n is any positive integer. There is a unique integer d such that $n \equiv d \pmod{14}$ and an appendage with d vertices is specified in the list on the previous page.

Let $n - d = 14s$, say.

We first suppose that $s \geq 0$.

Take the appendage with d vertices, and join it by (1)-composition to the stock X_s .

Now let us consider the resulting (transitive) permutation representation of $\Delta^*(2,3,7)$. The degree is $d + 14s$, which is n , and t acts as an odd permutation. Next there is a useful cycle of xyt whose length is a prime p which divides the length of no other cycle of xyt . Notice that this useful p -cycle may result from the composition of the appendage to the stock. (For example, when $d = 72$ we take $p = 13, 37$ or 41 , depending on whether $s = 0$ or $s = 2$ or otherwise; and when $d = 93$ we take $p = 13, 19$ or 23 in the same way.) Also it is quite clear that $p < n - 2$ (in all cases).

By Corollary 1.12 (and the definition of a useful cycle), the group generated by xyt and t is alternating or symmetric on the vertices of the diagram. But t is odd, so this group must act as the symmetric group S_n . On the other hand, by Lemma 1.4 the elements xyt and t generate the group $\Delta^*(2,3,7)$.

Thus S_n is a homomorphic image of $\Delta^*(2,3,7)$.

Furthermore, the even permutations x and y generate a subgroup of index 2 in $\Delta^*(2,3,7)$, so this subgroup must act as the alternating group A_n on the vertices of the diagram. In particular, A_n is a Hurwitz group!

In each case, t acts as

We turn now to the cases where $n - d < 0$. The latter part of the above argument may be applied to the following diagrams:

F	: 22	: $5^1 6^1 \underline{11}^1$
E(1)G ₁	: 49	: $2^1 4^1 8^1 9^2 \underline{17}^1$
E(3)J ₃	: 63	: $1^1 2^1 7^1 8^1 \underline{13}^1 32^1$
E(1)G ₁ (2)H	: 79	: $2^1 4^1 8^1 \underline{17}^1 24^2$
H(2)M ₂	: 87	: $4^1 7^1 10^1 15^1 \underline{23}^1 28^1$
F(2)P ₂	: 94	: $6^1 \underline{11}^1 18^1 21^1 38^1$
F(2)M ₂ (2)F	: 101	: $4^1 6^2 10^1 12^1 18^1 \underline{19}^1 26^1$
U ₂	: 102	: $21^1 \underline{23}^1 58^1$
D(3)E(1)P ₃	: 108	: $2^1 5^2 8^1 \underline{11}^1 \underline{17}^1 60^1$
K(3)N	: 109	: $\underline{13}^1 22^1 24^1 50^1$
H(2)M ₂ (2)H	: 117	: $4^1 10^1 22^1 \underline{23}^1 28^1 30^1$
F(2)G ₁ (1)P ₃	: 122	: $2^1 5^1 6^1 14^1 \underline{17}^1 20^1 58^1$
E(1)U ₁	: 123	: $2^1 4^1 8^1 12^1 14^1 \underline{19}^1 28^1 36^1$
F(2)V ₁	: 130	: $1^1 6^1 12^1 14^1 24^1 26^1 \underline{47}^1$
I(2)U ₂	: 137	: $\underline{7}^1 10^1 58^1 62^1$
D(3)E(1)U ₁	: 138	: $2^1 \underline{5}^1 8^1 11^1 12^1 14^1 22^1 28^1 36^1$
F(2)P ₂ (3)K	: 145	: $6^1 \underline{13}^1 18^1 22^1 38^1 48^1$
U ₁ (1)M ₁	: 159	: $2^1 6^1 9^1 \underline{11}^1 12^1 13^1 14^1 28^2 36^1$
V ₃ (3)N	: 166	: $11^1 \underline{13}^1 14^1 16^1 24^1 30^1 58^1$
H(2)G ₁ (1)G ₁ (2)M ₂ (2)H	: 173	: $2^1 4^1 10^1 16^1 18^1 \underline{23}^1 24^3 28^1$
H(2)P ₂ (3)P ₄	: 174	: $2^2 18^1 \underline{19}^1 \underline{29}^1 52^2$
F(2)V ₃ (3)K	: 181	: $6^1 \underline{13}^1 14^1 16^1 22^1 24^1 30^1 56^1$
F(2)V ₁ (1)P ₃	: 202	: $2^1 5^1 6^1 12^1 14^1 \underline{17}^1 24^1 26^1 96^1$

In each case, t acts as an odd permutation, so $\Delta^*(2,3,7)$ acts

as the symmetric group on the vertices of the diagram, and $\Delta(2,3,7)$ as the alternating group.

It is clear that if the symmetric group S_n is a homomorphic image of $\Delta^*(2,3,7)$, then there must be a coset diagram for the latter group with n vertices and such that the element t acts as an odd permutation on these vertices. Also if $n \geq 3$ then the subgroup $\Delta(2,3,7)$ must be transitive on the vertices. (Actually this subgroup will act as the alternating group, therefore will be $(n-2)$ -fold transitive.) In particular, n must be an H^* -number (cf. Chapter 2). Furthermore, by Proposition 2.2 we know that n cannot be one of $8, 9, 14, 15, 21, 24, 30, 44, 51, 60, 66$ or 80 .

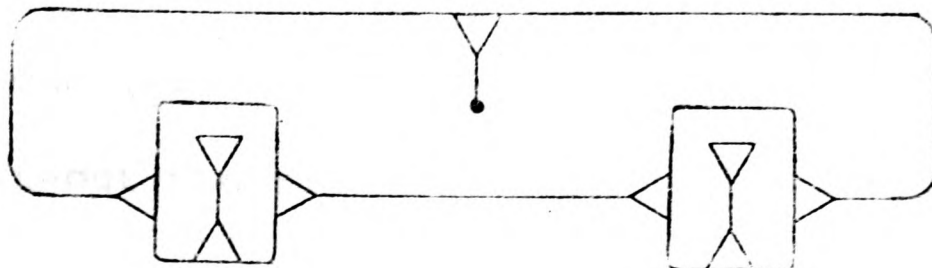
This leaves us with just 16 values of n , namely $1, 2, 28, 35, 37, 52, 58, 65, 73, 116, 124, 132, 152, 153, 160$ and 188 .

We deal with these one-by-one below.

$n=1$ This case is trivial.

$n=2$ The subgroup $\Delta(2,3,7)$ has index 2 in $\Delta^*(2,3,7)$, hence S_2 is a homomorphic image of the latter group.

$n=28$ The following coset diagram for $\Delta^*(2,3,7)$ has specification $28 : \text{odd} : 2^1 9^1 17^1$.

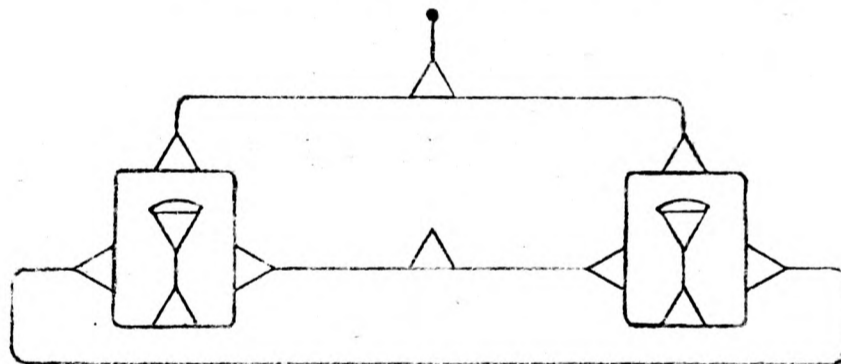


Since $(xyt)^{18}$ acts as a 17-cycle, we know by Corollary 1.11 that $\Delta^*(2,3,7)$ acts as S_{28} on the vertices of the diagram.

$n = 35$ The composite diagram $C(1)E$ has specification $35 : \text{odd} : 2^1 4^1 8^1 21^1$. By Corollary 2.3 the corresponding representation of $\Delta^*(2,3,7)$ must be primitive. Hence we can apply Theorem 1.13, noticing that $(xyt)^{24}$ has order 7 and degree 21. The result is that S_{35} is a homomorphic image of $\Delta^*(2,3,7)$.

$n = 37$ The coset diagram below has specification

$$37 : \text{odd} : 14^1 23^1 .$$



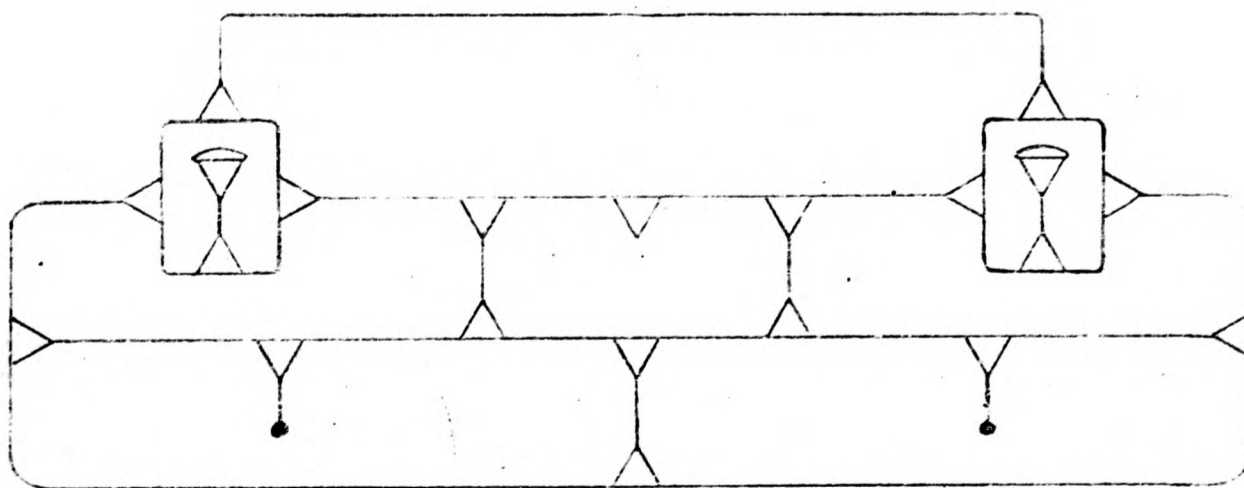
Since $(xyt)^{14}$ acts as a 23-cycle, we know by Corollary 1.11 that $\Delta^*(2,3,7)$ acts as S_{37} .

If $n = 52, 116, 124, 132, 152, 160$ or 188 , then from the genus formula (1.3) it follows that the involution x fixes no points in any transitive representation of $\Delta(2,3,7)$ of degree n . Consequently Lemma 1.4 (c) tells us that the cycles of xyt of odd length (if any) must occur in pairs. For this reason the theorem of Jordan (1.10) is not applicable. We must also deduce primitivity without the aid of useful cycles of xyt (of odd prime length).

$n = 52$ The composite diagram $F(2)H$ has specification
 $52 : \text{odd} : 6^1 20^1 26^1$. By Corollary 2.3 the corresponding
 representation is primitive, hence we can apply Theorem 1.13.
 The element $(xyt)^{60}$ acts as a permutation of order 13 and
 degree 26, so $\Delta^*(2,3,7)$ acts as S_{52} .

$n = 58$ The basic diagram N has specification
 $58 : \text{odd} : 7^1 24^1 27^1$. The corresponding representation of
 $\Delta^*(2,3,7)$ is primitive (by Corollary 2.3), the element $(xyt)^{216}$
 acts as a 7-cycle, so by Jordan's theorem $\Delta^*(2,3,7)$ acts as
 S_{58} .

$n = 65$ The following coset diagram for $\Delta^*(2,3,7)$ has
 specification $65 : \text{odd} : 5^2 8^1 47^1$.



The element $(xyt)^{40}$ acts as a 47-cycle, so by Corollary 1.11
 the group $\Delta^*(2,3,7)$ acts as S_{65} .

$n = 73$ The composite diagram $D(3)N$ has specification
 $73 : \text{odd} : 5^1 10^1 24^1 34^1$. Again by Corollary 2.3 the
 corresponding representation is primitive; and since the element

$(xyt)^{120}$ acts as a permutation of order 17 and degree 34,

Theorem 1.13 tells us that $\Delta^*(2,3,7)$ acts as S_{73} .

$n = 116$ The composite diagram $H(2)R$ has specification

$116 : \text{odd} : 12^2 26^1 28^1 33^1$.

Assume the corresponding representation is imprimitive. Then there must be 58 blocks of size 2, for 58 is the only H^* -number which is a non-trivial divisor of 116. From the diagram it is clear that only one of the blocks can be fixed by y , and this block must contain the points marked η and θ in the diagram R . Now if z denotes the element $xyxyxy^2xy^2xyxyx$, then $\eta z = \theta$, so z fixes the block $\{\eta, \theta\}$. But that implies $\theta z = \eta$, which is visibly not so.

Hence the representation must be primitive.

Finally the element $(xyt)^{1092}$ acts as a permutation of order 19 and degree 38, so by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts as S_{116} .

$n = 124$ The composite diagram $H(2)T_2$ has specification

$124 : \text{odd} : 8^1 10^1 12^1 28^1 66^1$. The corresponding

representation is primitive (by Corollary 2.3), the element

$(xyt)^{1848}$ acts as a permutation of order 5 and degree 10, so

by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts as S_{124} .

$n = 132$ The composite diagram $H(2)U_2$ has specification

$132 : \text{odd} : 36^1 38^1 58^1$.

Assume the corresponding representation is imprimitive. Then

there must be either 22 blocks of size 6, or 44 blocks of size 3, or 66 blocks of size 2. In any case there will be at least one block which is fixed by xy , and any such block must be fixed pointwise by xy . The points which are fixed by xy have been labelled as ε and ζ (in diagram H) and κ, λ, μ and ν (in diagram U_2). Now if z denotes the element $yxyx^2yxy^2$, then we see from the diagram that $\varepsilon z = \zeta$, hence either ε and ζ lie in the same block, or z moves the block containing ε to that which contains ζ . But none of $\zeta z, \kappa z, \lambda z, \mu z$ or νz is fixed by xy , so this is impossible!

Hence the representation must be primitive.

Finally the element $(xyt)^{684}$ acts as a permutation of order 29 and degree 58, so by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts as S_{132} .

$n=152$ The composite diagram $H(2)G_1(1)T_1$ has specification $152 : \text{odd} : 2^1 14^1 22^1 24^2 28^1 38^1$.

Assume the corresponding representation is imprimitive. Then there must be 8 blocks of size 19, for 8 is the only H^* -number which is a non-trivial divisor of 152. Now these 8 blocks must be permuted by $\Delta^*(2,3,7)$ as in the following coset diagram:



In particular, the five points which are fixed by xy lie in the same block, and the elements $yxyx$ and yxy^2x map this block to two different blocks, both of which are fixed by y . But let

us look at the points marked ξ and ζ in the diagram H . These two points are fixed by xy , hence $\xi yxyx$ and ζyxy^2x lie in different blocks, both of which are fixed by y . From the diagram we see that $\zeta yxy^2xy = \xi yxyx$, so this is impossible!

Hence the representation must be primitive.

Finally the element $(xyt)^{1848}$ acts as a permutation of order 19 and degree 38, so by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts as S_{152} .

$n = 153$ The composite diagram $H(2)S(2)H$ has specification
 $153 : \text{odd} : 4^1 23^1 24^1 26^1 30^1 46^1$

Assume the corresponding representation is imprimitive. Then there must be either 9 blocks of size 17, or 51 blocks of size 3, for 9 and 51 are the only H^* -numbers which are non-trivial divisors of 153. Now the element $(xyt)^{1560}$ acts as a permutation of order 23 and degree 69; if there were 9 blocks of size 17 then this element would fix each block and therefore fix every point; hence there must be 51 blocks of size 3. But on the other hand the element $(xyt)^{2760}$ acts as a permutation of order 13 and degree 26; it will fix at least 42 blocks, therefore all 51 blocks, and therefore all 153 points. We have a contradiction.

Hence the representation must be primitive.

Finally Theorem 1.13 tells us that the group $\Delta^*(2,3,7)$ acts as S_{153} .

$n = 160$ The composite diagram $H(2)P_2(3)N$ has specification
 $160 : \text{odd} : 18^1 24^1 28^1 38^1 52^1$.

Assume the corresponding representation is imprimitive. Then there must be either 8 blocks of size 20, or 80 blocks of size 2, for 8 and 80 are the only H^* -numbers which are non-trivial divisors of 160. In both cases at least two blocks must be fixed by y , and therefore at least four points are fixed by y . But we can see from the diagrams that this is not so.

Hence the representation must be primitive. (In fact any transitive permutation representation of $\Delta^*(2,3,7)$ of degree 160 must be primitive.)

Finally the element $(xyt)^{6552}$ acts as a permutation of order 19 and degree 38, so by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts as S_{160} .

$n = 188$ The composite diagram $F(2)G_1(1)V_1(2)H$ has specification $188 : \text{odd} : 2^1 6^1 12^1 14^2 20^1 26^1 34^1 60^1$.

Assume the corresponding representation is imprimitive. Then there must be 94 blocks of size 2, for 94 is the only H^* -number which is a non-trivial divisor of 188. Now from the diagrams it is clear that only one of the blocks can be fixed by y , and this block must contain the points marked γ and δ in the diagrams F and G_1 respectively. Next if z denotes the element xy^2xyxy^2xyxy , then we see that $\delta z = \gamma$, so z fixes the block $\{\gamma, \delta\}$. But this implies $\gamma z = \delta$, which is not so.

Hence the representation must be primitive.

Finally the element $(xyt)^{5460}$ acts as a permutation of order 17 and degree 34, so by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts

<u>Appendage</u>	<u>Specification</u>
$G_1(2)G_3$: 56 : a (b8) $4^1 7^1 \underline{17^1} 18^1$
$J_2(3)M_1$: 99 : a (b9) $1^1 3^1 6^1 13^1 \underline{17^1} 22^1 26^1$
P_3	: 72 : a (b48) $5^1 \underline{17^1}$
$M_1(3)P_4$: 129 : a (b16) $2^2 6^1 13^1 \underline{19^1} \underline{29^1} 40^1$
$V_1(2)O$: 172 : a (b48) $1^1 8^1 \underline{11^1} 12^2 14^1 26^1 38^1$
$V_1(2)M_2(2)F$: 187 : a (b50) $4^1 6^1 10^1 12^1 14^1 18^1 \underline{19^1} 26^2$
$W(2)U_2$: 216 : a (b22) $1^1 12^1 16^1 52^1 \underline{53^1} 58^1$
$G_1(2)I$: 63 : a (b8) $\underline{7^1} 10^1 36^1$
O	: 64 : a (b10) $8^1 12^1 13^1 \underline{19^1}$
$E(3)N$: 79 : a (b14) $8^1 24^1 \underline{31^1}$
V_1	: 108 : a (b35) $12^1 14^1 \underline{19^1} 26^1$
$V_2(3)E(1)E(3)D$: 165 : a (b20) $2^1 5^1 8^2 \underline{11^1} 18^1 20^1 28^1 \underline{43^1}$
$D(3)E(1)W(2)H$: 180 : a (b44) $2^1 5^1 8^1 \underline{11^1} 12^1 16^1 34^1 46^1$
$P_1(1)P_1(1)E(3)N$: 223 : a (b10) $2^2 8^1 10^2 16^2 22^3 24^1 26^1 \underline{31^1}$

Again we can (1)-compose each appendage to each of the stocks X_s in turn, and this time we find that $\Delta^*(2,3,7)$ has a representation as the alternating group A_n for all but finitely many values of n .

The usual argument can be applied also to the following diagrams:

D	: 15 : $3^1 \underline{5^1} 7^1$
G_2	: 28 : $4^1 \underline{7^1} 8^1 9^1$

I	: 35	: $\underline{7}^1$ 10^1 18^1
J_3	: 42	: 2^1 7^1 $\underline{13}^1$ 20^1
K	: 51	: $\underline{13}^1$ 16^1 22^1
D(3)E(1)E	: 57	: 2^1 4^1 $\underline{5}^1$ 8^2 $\underline{11}^1$ $\underline{19}^1$
$G_2(2)H$: 58	: 4^1 $\underline{7}^1$ $\underline{23}^1$ 24^1
H(2)I	: 65	: $\underline{7}^1$ 10^1 48^1
D(3)K	: 66	: $\underline{5}^1$ 13^1 22^1 26^1
D(3)L	: 71	: $\underline{5}^1$ 12^1 16^1 19^2
Q	: 85	: 6^1 8^1 9^1 $\underline{13}^1$ 15^1 16^1 18^1
D(3) P_2	: 87	: $\underline{5}^1$ 18^2 22^1 24^1
H(2)O	: 94	: 1^1 8^1 $\underline{11}^1$ 12^1 28^1 34^1
D(3)L(2)H	: 101	: $\underline{5}^1$ 12^1 16^1 34^2
U_1	: 102	: 1^1 $\underline{11}^1$ 12^1 14^1 28^1 36^1
F(2) $M_2(2)H$: 109	: 4^1 6^1 10^1 12^1 $\underline{23}^1$ 26^1 28^1
H(2)Q	: 115	: 6^1 8^1 9^1 $\underline{13}^1$ 16^1 30^1 33^1
D(3) $P_2(2)H$: 117	: $\underline{5}^1$ 18^2 24^1 52^1
K(3) P_2	: 123	: $\underline{13}^1$ 18^1 22^2 48^1
H(2) $G_1(1)P_3$: 130	: 2^1 5^1 $\underline{17}^1$ 24^2 58^1
H(2) $G_1(1)E(3)N$: 137	: 2^1 8^1 24^4 $\underline{31}^1$
H(2) V_3	: 138	: 9^1 14^1 26^1 28^1 30^1 $\underline{31}^1$
D(3)E(1) $M_1(3)K$: 144	: 2^1 $\underline{5}^1$ 6^1 8^1 11^1 13^2 22^1 28^1 36^1
F(2)S(2)H	: 145	: 4^1 6^1 14^1 $\underline{23}^1$ 26^1 30^1 42^1
$M_2(2)T_2$: 151	: 4^1 8^2 10^2 12^1 $\underline{13}^1$ 28^1 58^1
$G_2(2)G_1(1)U_1$: 158	: 2^1 4^1 7^1 12^1 14^1 $\underline{17}^1$ 18^1 20^1 28^1 36^1
K(3) V_2	: 159	: 1^1 $\underline{13}^1$ 18^1 20^1 21^1 22^1 64^1
O(1) U_1	: 166	: 2^1 8^1 12^2 $\underline{13}^1$ 14^1 $\underline{19}^1$ 22^1 28^1 36^1
C(1) $V_2(3)K$: 173	: 2^1 $\underline{13}^1$ 18^1 20^1 22^1 34^1 64^1

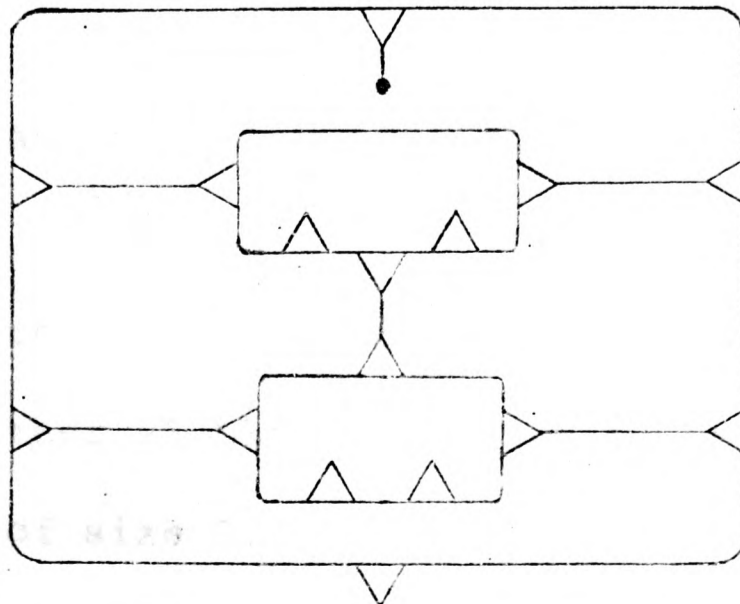
$P_3(1)U_1$: 174	: 2^1	5^1	12^1	14^1	<u>17^1</u>	28^1	36^1	60^1		
$N(3)E(1)U_1$: 181	: 2^1	8^1	12^1	14^1	24^1	26^1	28^1	<u>31^1</u>	36^1	
$D(3)V_2(1)P_3$: 195	: 2^1	5^2	12^1	<u>17^1</u>	18^1	20^1	46^1	70^1		
$P_2(2)G_1(1)U_1$: 202	: 2^1	<u>11^1</u>	12^1	14^1	18^1	20^1	21^1	28^1	36^1	40^1
$H(2)M_2(2)G_1(1)T_1$: 209	: 2^1	4^1	10^1	14^1	16^1	22^1	<u>23^1</u>	24^1	28^2	38^1

If the alternating group A_n is to be a homomorphic image of $\Delta^*(2,3,7)$, then we require that n is an H^* -number (or, of course, $n = 2$), and further, by Proposition 2.2, that n is not one of $8, 9, 14, 21, 22, 24, 30, 36, 37, 44, 50, 52, 60, 73$ or 80 . We are left with just ten values of n , which we deal with below.

$n = 1$ This case is trivial

$n = 2$ This case is trivial

$n = 49$ The following coset diagram for $\Delta^*(2,3,7)$ has specification $49 : \text{even} : 7^2$ 11^1 12^2 .



We see that the 11 -cycle of xyt is a useful one, so the group

$\Delta^*(2,3,7)$ acts as A_{49} on the vertices of the diagram.

$n = 116$ The composite diagram $F(2)T_2$ has specification

$116 : \text{even} : 6^1 \cdot 8^1 \cdot 10^1 \cdot 12^1 \cdot 28^1 \cdot 52^1$.

Assume the corresponding representation is imprimitive. Then there must be 58 blocks of size 2, and one of these blocks will contain the points marked γ and λ in the diagrams F and T_2 respectively. Now if z denotes the element $xyxy^2xy^2xyxy^2xyxyx$, then we see that $\lambda z = \gamma$, hence z fixes the block $\{\gamma, \lambda\}$, but that implies $\gamma z = \lambda$, which is not so.

Hence the representation must be primitive.

Finally the element $(xyt)^{2184}$ acts as a permutation of order 5 and degree 10, so by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts as A_{116} .

$n = 124$ The composite diagram $F(2)U_2$ has specification

$124 : \text{even} : 6^1 \cdot 28^1 \cdot 32^1 \cdot 58^1$. The corresponding representation

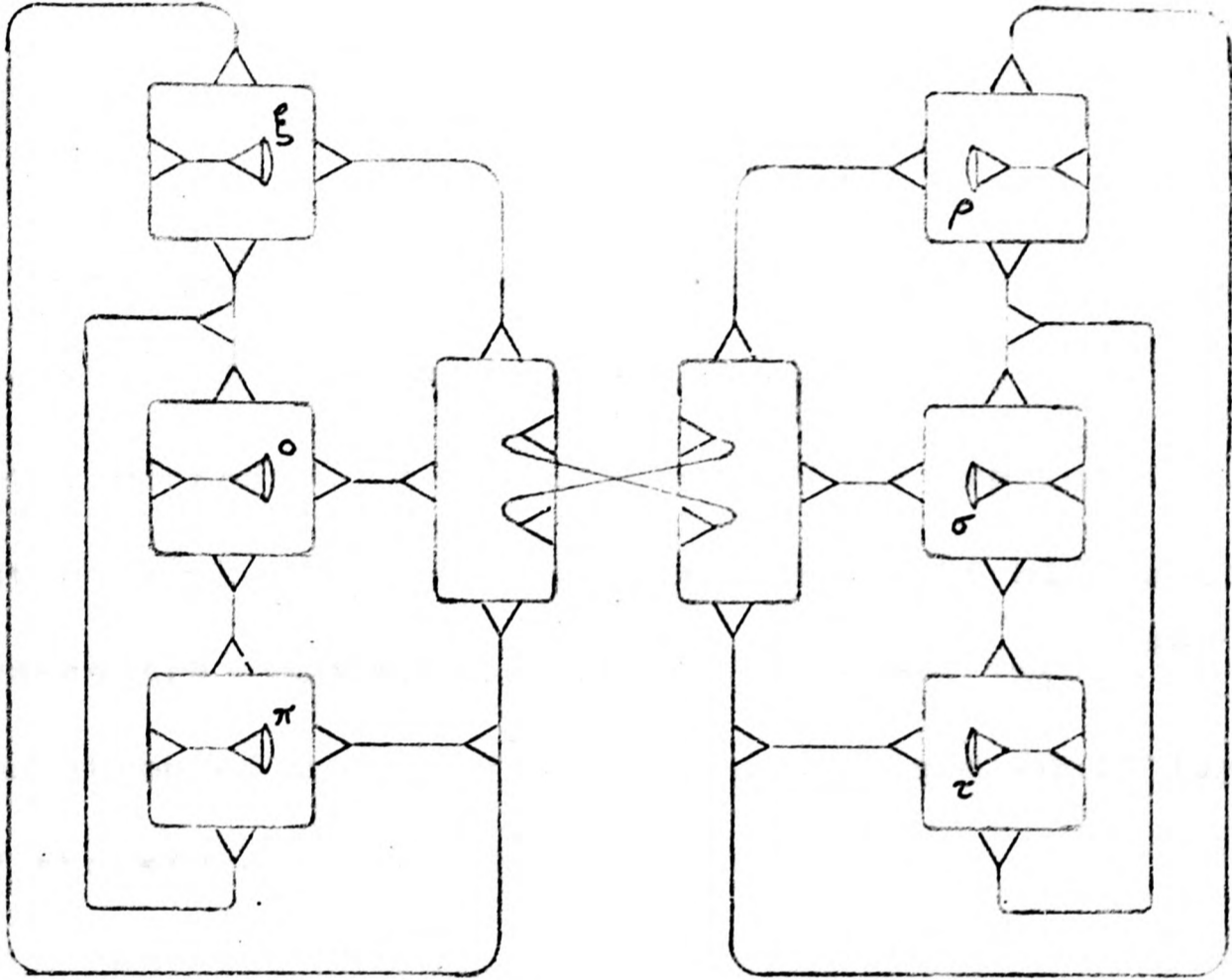
is primitive (by Corollary 2.3), the element $(xyt)^{672}$ acts as a permutation of order 29 and degree 58, so by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts as A_{124} .

$n = 132$ The diagram on the following page has specification

$132 : \text{even} : 13^2 \cdot 53^2$.

Assume the corresponding representation is imprimitive. Then there must be either 22 blocks of size 6, or 44 blocks of size 3, or 66 blocks of size 2. In any case the six fixed points of xy , namely those labelled $\xi, \theta, \pi, \rho, \sigma$ and τ in the

diagram, lie in one or more blocks which are fixed pointwise by xy . Now if z denotes the element $xyx^2yxy^2xyx^2y^2$, then we see that $\xi z = o$ and $\sigma z = \rho$, but none of $o z$, πz , ρz or τz is fixed by xy , hence $\{\xi, \sigma\}$ must be a block.



But on the other hand, if w denotes the element $xyx^2yxyx^2yxy^2$ then $\xi w = \pi$ while σw is not fixed by xy , hence $\{\xi, \sigma\}$ cannot be a block.

Hence the representation must be primitive.

Finally the element $(xyt)^{13}$ acts as a permutation of order 53 and degree 106, so by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts as A_{132} .

$n = 152$ The composite diagram $C(1)V_1(2)H$ has specification

152 : even : $2^1 12^1 14^1 26^1 34^1 64^1$. The corresponding representation is primitive, for the same reason that we gave when showing that the symmetric group S_{152} is a homomorphic image of $\Delta^*(2,3,7)$. This time t acts as an even permutation, the element $(xyt)^{17472}$ acts as a permutation of order 17 and degree 34, and so by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts as the alternating group A_{152} .

n = 153 The composite diagram $D(3)V_3(2)H$ has specification
153 : even : $5^1 12^1 14^1 26^1 28^1 30^1 38^1$.

Assume the corresponding representation is imprimitive. Then there must be either 9 blocks of size 17, or 51 blocks of size 3. Both cases are impossible, for the element $(xyt)^{5460}$ acts as a permutation of order 19 and degree 38 (and would fix each block and therefore fix every point.) .

Hence the representation must be primitive.

Finally by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts as A_{153} .

n = 160 The composite diagram $H(2)G_1(1)U_1$ has specification
160 : even : $2^1 12^1 14^1 20^1 24^2 28^1 36^1$. The corresponding representation must be primitive, as we noticed when we showed that the symmetric group S_{160} is a homomorphic image of $\Delta^*(2,3,7)$. Next the element $(xyt)^{504}$ acts as a permutation of order 5 and degree 20, so by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts as A_{160} .

n = 188 The composite diagram $R(2)U_2$ has specification

188 : even : $12^2 26^1 34^1 46^1 58^1$.

Assume the corresponding representation is imprimitive. Then there must be 94 blocks of size 2, and one of these blocks will contain the points marked η and θ in the diagram R. But this leads to a contradiction, in exactly the same way that it did when we showed that the symmetric group S_{116} is a homomorphic image of $\Delta^*(2,3,7)$.

Hence the representation must be primitive.

Finally the element $(xyt)^{60996}$ acts as a permutation of order 29 and degree 58, so by Theorem 1.13 the group $\Delta^*(2,3,7)$ acts as A_{188} .

4.2 THEOREM

The alternating group A_n is a homomorphic image of $\Delta^*(2,3,7)$ precisely when n is one of

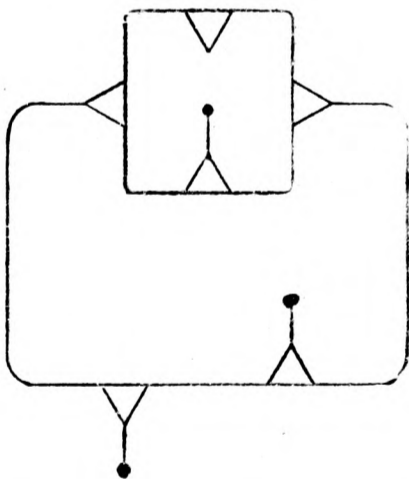
1	2																		
15																			
28								35											
42								49		51									
56	57	58						63	64	65	66								
70	71	72						77	78	79									
84	85	86	87					91	92	93	94								
98	99	100	101	102				105	106	107	108	109							
112	113	114	115	116	117			119	120	121	122	123	124						
126	127	128	129	130		132	133	134	135	136	137	138							
140	141	142	143	144	145			147	148	149	150	151	152	153					
154	155	156	157	158	159	160	161	162	163	164	165	166							

or n is any integer greater than 167 .

As a corollary of either of Theorems 4.1 and 4.2, we obtain that all but finitely many of the alternating groups A_n are Hurwitz groups. (For the subgroup $\Delta(2,3,7)$ has index 2 in $\Delta^*(2,3,7)$, and acts always as a group of even permutations.)

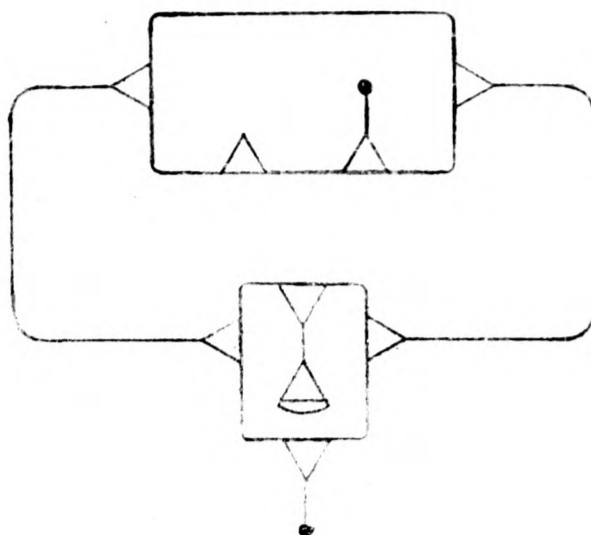
There are eight integers n for which neither S_n nor A_n is a factor group of $\Delta^*(2,3,7)$ but nevertheless A_n is a Hurwitz group. We know that any such n must be an H-number (cf. Chapter 2), other than 7, 8, 9, 14, 24, 30, 44 or 60; this leaves us with the following.

$n=21$ On the vertices of the coset diagram for $\Delta(2,3,7)$ below, the commutator $xy^{-1}xy$ acts as a permutation with two cycles of length 2, and one of length 17.

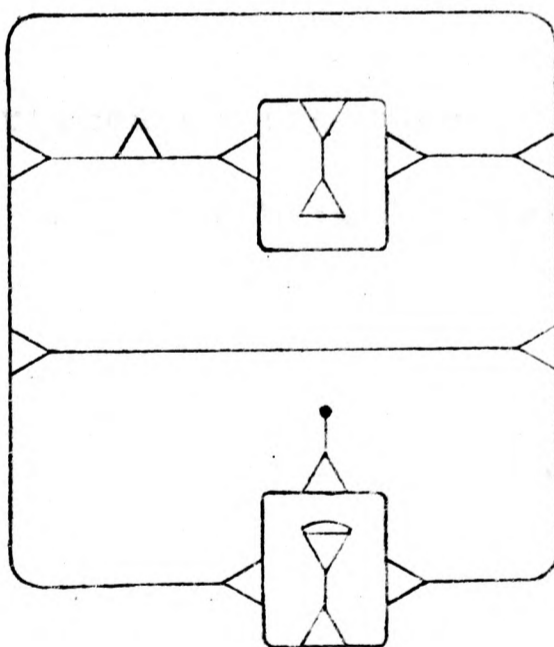


In particular, the element $(xy^{-1}xy)^2$ acts as a 17-cycle, so by Corollary 1.11 the group $\Delta(2,3,7)$ acts as A_{21} .

$n=29$ On the vertices of the following coset diagram for $\Delta(2,3,7)$, the commutator $xy^{-1}xy$ acts as a permutation with two cycles of length 3, and one of length 23. In particular, the element $(xy^{-1}xy)^3$ acts as a 23-cycle, so by Corollary 1.11 the group $\Delta(2,3,7)$ acts as A_{29} .



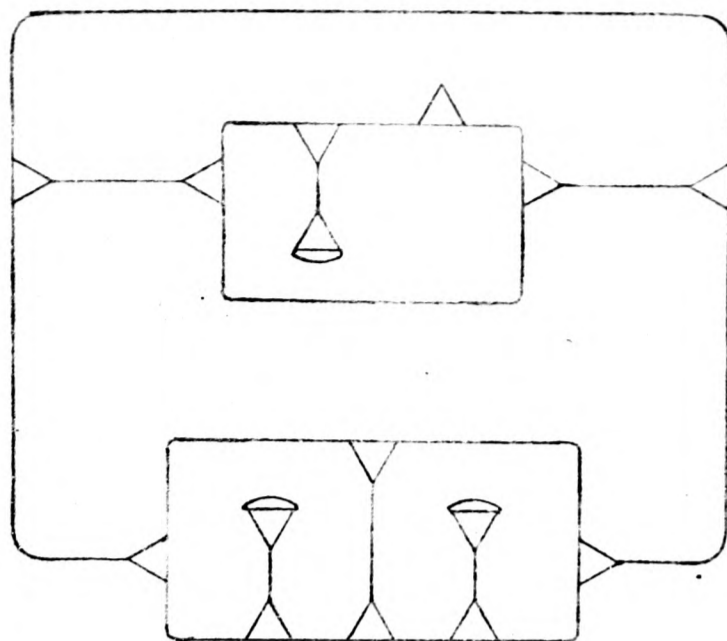
$n = 43$ On the vertices of the following diagram, the commutator $xy^{-1}xy$ acts as a permutation with one fixed point, two cycles of length 3, one of length 11, and one of length 25.



Since 43 is prime, the corresponding representation of $\Delta(2,3,7)$ must be primitive. The element $(xy^{-1}xy)^{75}$ acts as an 11-cycle, so by Jordan's theorem the group $\Delta(2,3,7)$ acts as A_{43} .

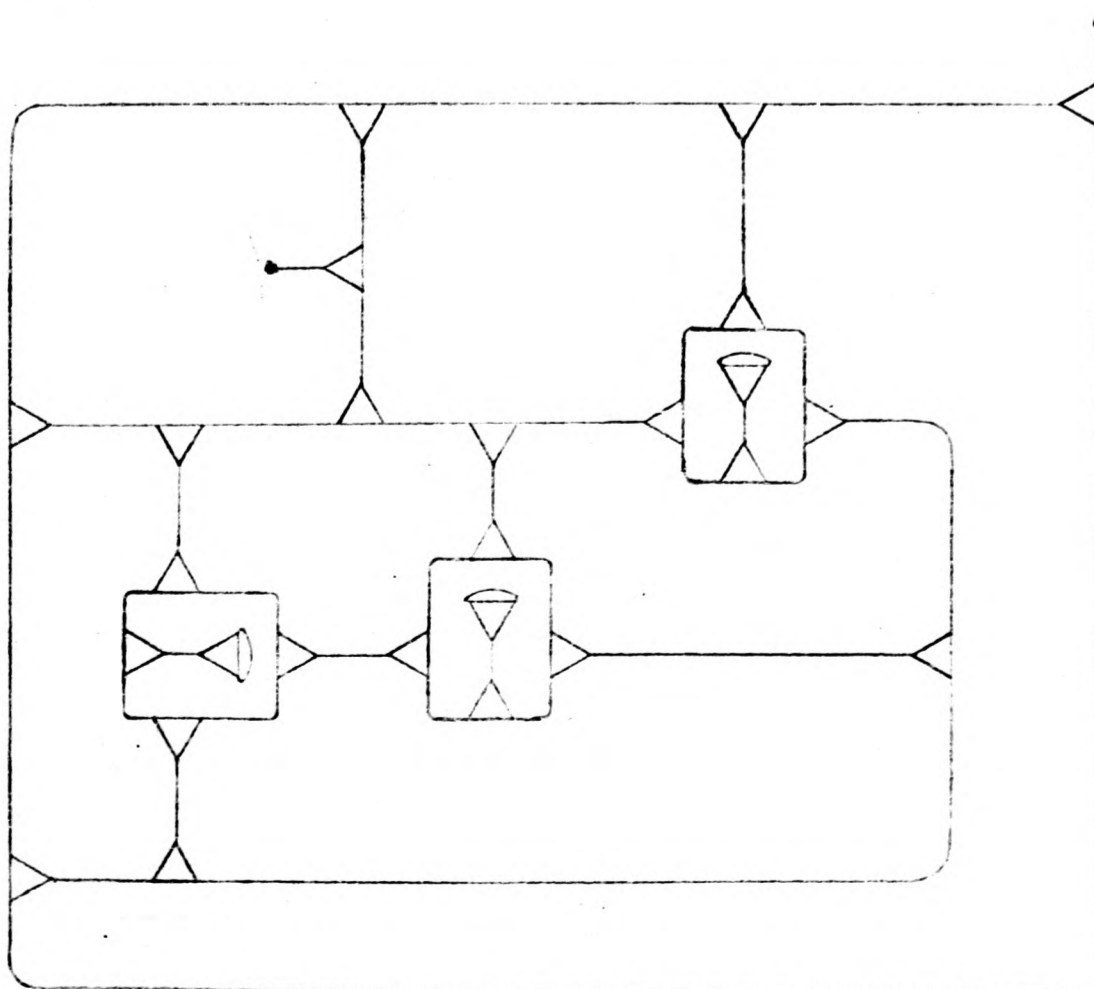
$n = 45$ On the vertices of the following coset diagram for $\Delta(2,3,7)$, the element $xy^2(xy)^2xy^2$ acts as a permutation with

two fixed points, two cycles of length 3, and one of length 37.



In particular, the element $(xy^2(xy)^2xy^2)^3$ acts as a 37-cycle, so by Corollary 1.11 the group $\Delta(2,3,7)$ acts as A_{45} .

$n = 80$ On the vertices of the coset diagram below, the commutator $xy^{-1}xy$ acts with two cycles of length 9, and two of length 31.

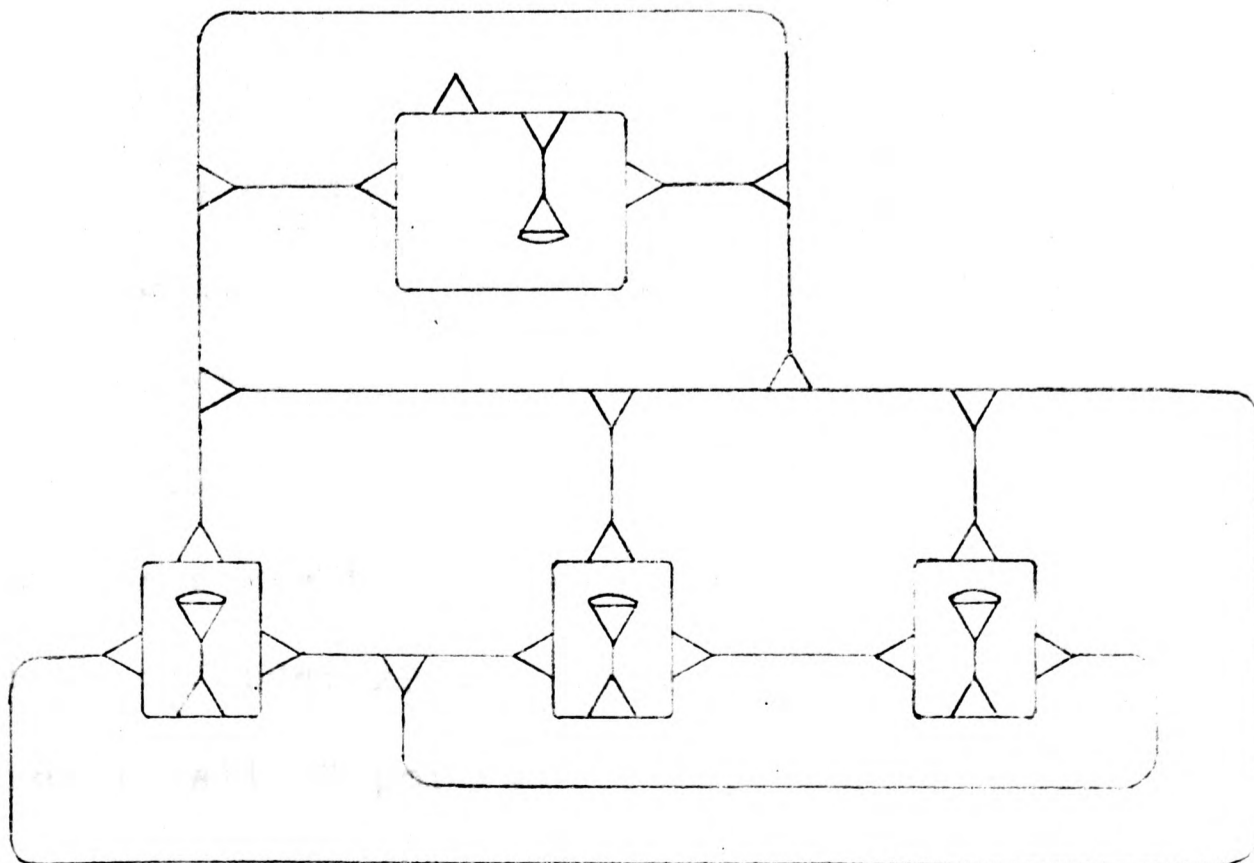


Assume the corresponding representation of $\Delta(2,3,7)$ is imprimitive. Then there must be 8 blocks of size 10, for the group $\Delta(2,3,7)$ permutes the blocks transitively and 8 is the only H-number which is a non-trivial divisor of 80. Now the element $(xy^{-1}xy)^9$, which acts as a permutation of order 31, will fix each block and therefore fix every point, but this is impossible!

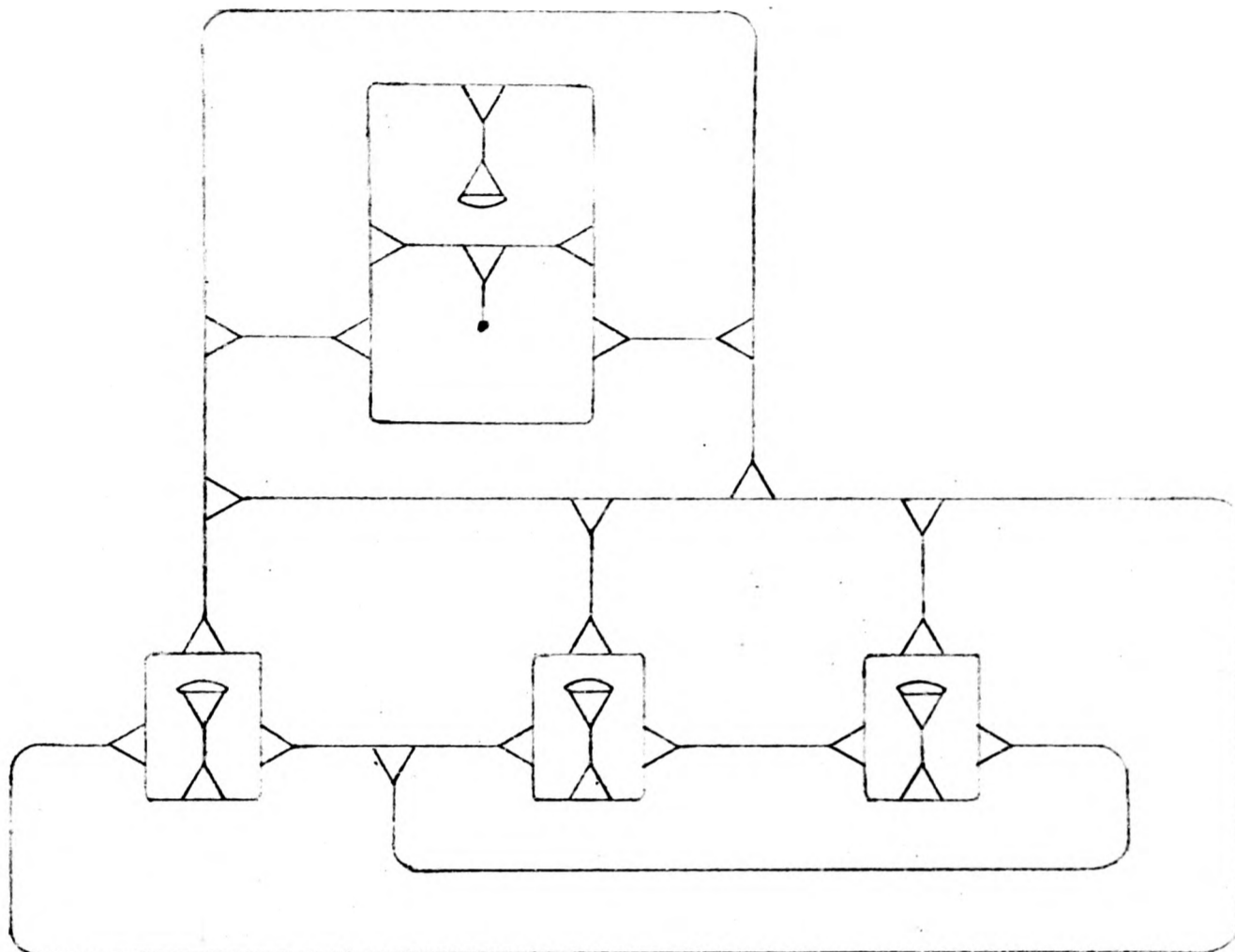
Hence the representation must be primitive.

Finally by Theorem 1.13 the group $\Delta(2,3,7)$ acts as A_{80} .

$n=81$ On the vertices of the coset diagram below, the element $(xy^2)^2(xy)^2$ acts as a permutation with two fixed points, two cycles of length 3, and one of length 73. In particular, the cube of this element acts as a 73-cycle, so by Corollary 1.11 the group $\Delta(2,3,7)$ acts as A_{81} .



$n = 88$ On the vertices of the following coset diagram for $\Delta(2,3,7)$, the commutator $xy^{-1}xy$ acts with two cycles of length 3, two of length 12, and two of length 29.

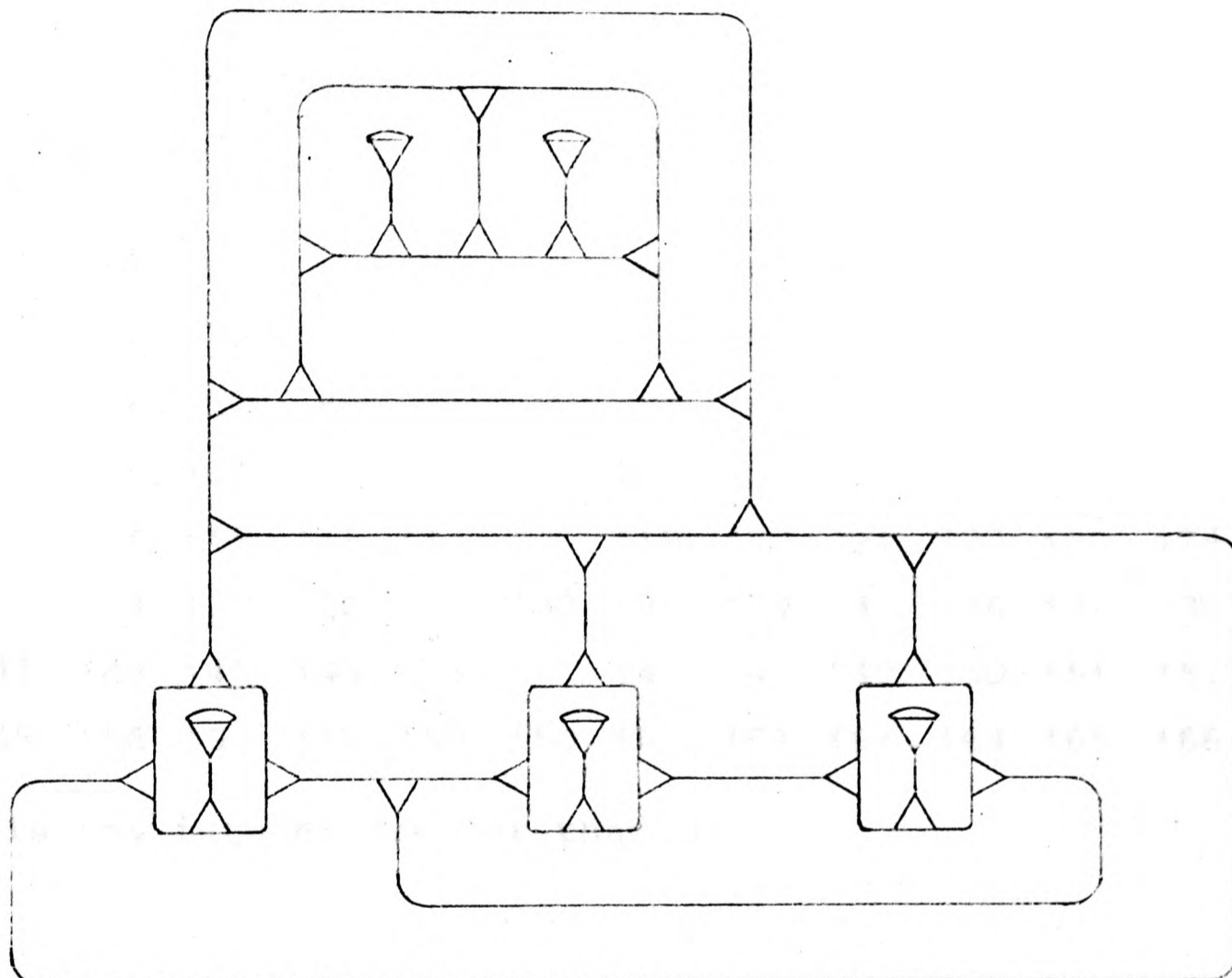


Assume the corresponding representation is imprimitive. Then there must be either 8 blocks of size 11, or 22 blocks of size 4, or 44 blocks of size 2. In the first and third cases, the 3-element y will fix at least two blocks and therefore fix at least four points; but from the diagram it is clear that there is only one fixed point of y . On the other hand, the second case is impossible since the element $(xy^{-1}xy)^{12}$, acting as a permutation of order 29 and degree 58, would fix each block and therefore fix all 88 points!

Hence the representation must be primitive.

Finally by Theorem 1.13 the group $\Delta(2,3,7)$ acts as A_{88} .

$n = 96$ On the vertices of the following coset diagram for $\Delta(2,3,7)$, the commutator $xy^{-1}xy$ acts as a permutation with two cycles of length 14, and two of length 34. In particular, the element $(xy^{-1}xy)^{14}$ acts with order 17 and degree 68.



Assume the corresponding representation is imprimitive. Then there must be either 8 blocks of size 12, or 24 blocks of size 4. In the first case, the element $(xy^{-1}xy)^{14}$ will fix each block, but that is impossible. In the second case, the 7-element xy will fix at least three blocks and therefore fix at least twelve points, but visibly that does not happen.

Hence the representation must be primitive.

Finally by Theorem 1.13 the group $\Delta(2,3,7)$ acts as A_{96} .

It is possible to obtain variants of the preceding result. For example, we can prove (using a modified list of basic diagrams) that for all integers n greater than 167 the alternating group A_n is a homomorphic image of the group $\Delta(2,3,7;360360)$. Furthermore the author (acting upon a suggestion of Graham Higman) has found that for all integers n greater than 617 the alternating group A_n is a homomorphic image of $\Delta(2,3,7;1260)$. This result can almost certainly be improved.

Chapter 5 : MORE GENERATORS FOR ALTERNATING AND SYMMETRIC GROUPS

In this, the final chapter, we prove the following theorem.

5.1 THEOREM

Let k be any integer greater than six. Then there exists $N_k > 0$ such that for all integers $n \geq N_k$:

(a) both A_n and S_n can be generated by elements x, y, t which satisfy $x^2 = y^3 = (xy)^k = t^2 = (xt)^2 = (yt)^2 = (xyt)^{2m} = 1$;

(b) A_n , and if k is even then also S_n , can be generated by elements x, y which satisfy $x^2 = y^3 = (xy)^k = (xy^{-1}xy)^m = 1$,

where $m = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 360360$.

The proof uses for each value of k a list of four or five coset diagrams for the group $\Delta^*(2,3,k)$. Because our resources are not infinite, we cannot draw each diagram separately; however we have a technique which allows us to present them here, and which also makes the proof considerably easier.

Specifically, each of the diagrams which follow (on pages 92 to 116) represents for some fixed $h \in \{7,8,9,10,11,12\}$ a sequence of coset diagrams, one for $\Delta^*(2,3,k)$ for every k such that $k-h$ is a non-negative multiple of six. At various places in the given diagram, configurations shaped like




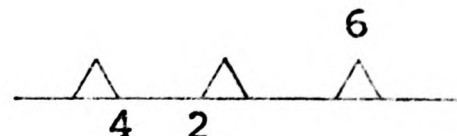
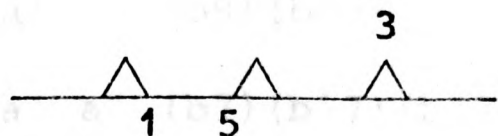
have been drawn, such that their occurrences are symmetric about the vertical axis. To obtain the coset diagram for $\Delta^*(2,3,h+6d)$, one just replaces each such configuration by a string of d triangles, thus:

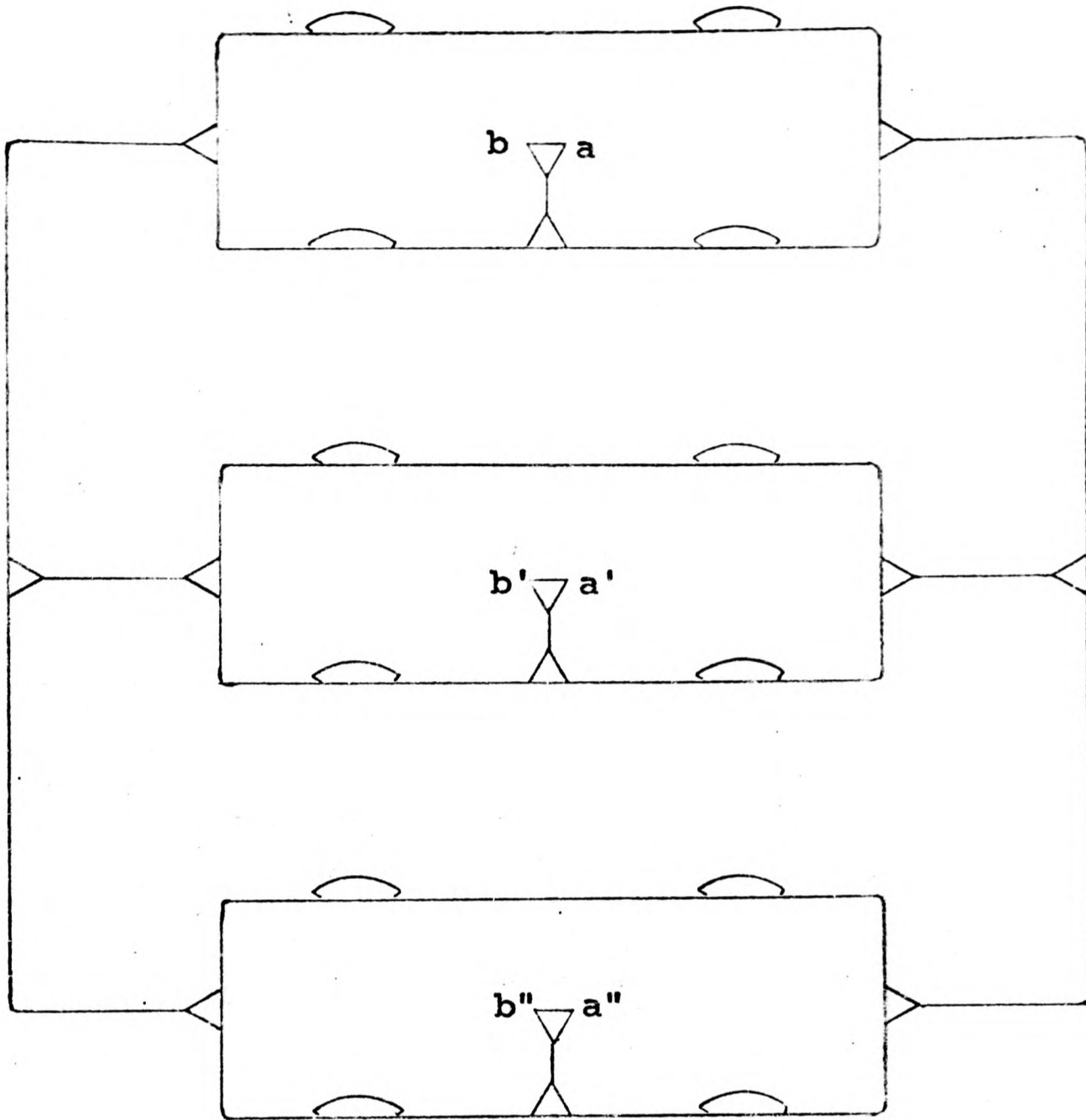


The reader should observe that the configurations have been placed in such a way that the length of each cycle of the element xy does become $h+6d$ (or at least a divisor of $h+6d$).

We have given each coset diagram a label, either $S(h,d)$ or $T(h,d)$ or $U(h,d)$ or $V(h,d)$ or $W(h,d)$, and also we have stated the number of its vertices, the parity of the actions of the elements x and t , and have described the cycle structure of the element xyt (just as we did for the diagrams in Chapter 4).

It is fortunate (for our purposes) that as d increases above 2, the only change that occurs to the cycle structure of xyt is the addition of further 6-cycles. This phenomenon is due to the fact that we add a string of d triangles at each place marked , and that the placings are symmetric about the vertical axis. To illustrate this, we have drawn below part of a coset diagram for some $\Delta^*(2,3,k)$; the vertices labelled 1 to 6 form a cycle of xyt .



Diagram $S(7,d)$ $42 + 36d$ vertices x even t even

xyt has cycle structure:

 $a \ a' \ a'' \ (b12) \ (b'12) \ (b''12)$ when $d = 0$
 $a \ a' \ a'' \ (b9) \ (b'9) \ (b''9) \ 3^3 \ 12^3$ when $d = 1$
 $a \ a' \ a'' \ (b7) \ (b'7) \ (b''7) \ 3^3 \ 6^{3+6(d-2)} \ 8^3 \ 12^3$ when $d \geq 2$

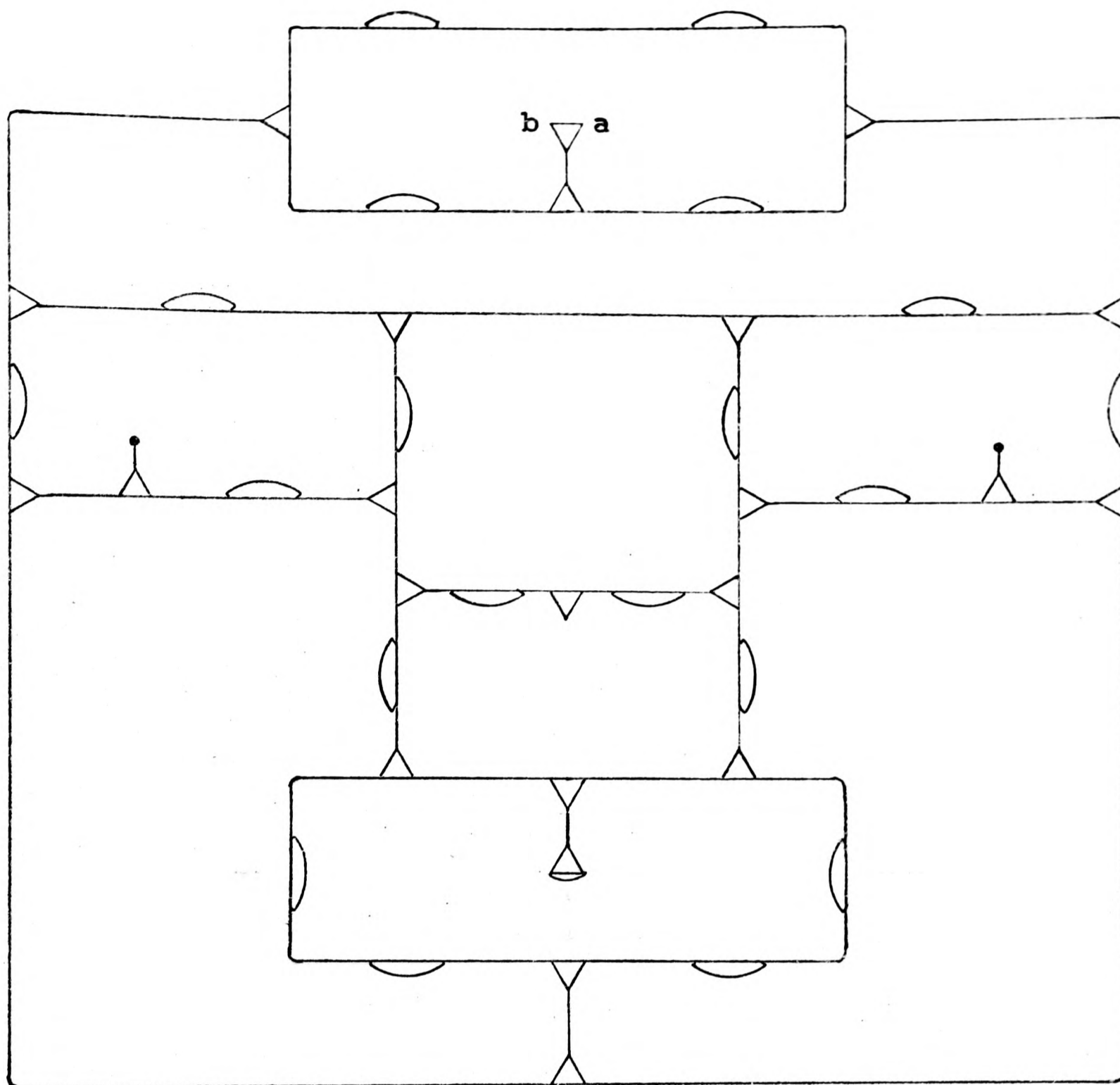


Diagram $T(7,d)$

$71 + 60d$ vertices

x even

t even

xyt has cycle structure:

a (b28) $6^1 15^1 20^1$ when $d = 0$

a (b9) $3^3 4^1 6^1 8^3 9^1 12^1 15^2 26^1$ when $d = 1$

a (b7) $3^3 4^1 6^{3+10(d-2)} 7^1 8^8 9^1 10^1 12^4 13^1$ when $d \geq 2$

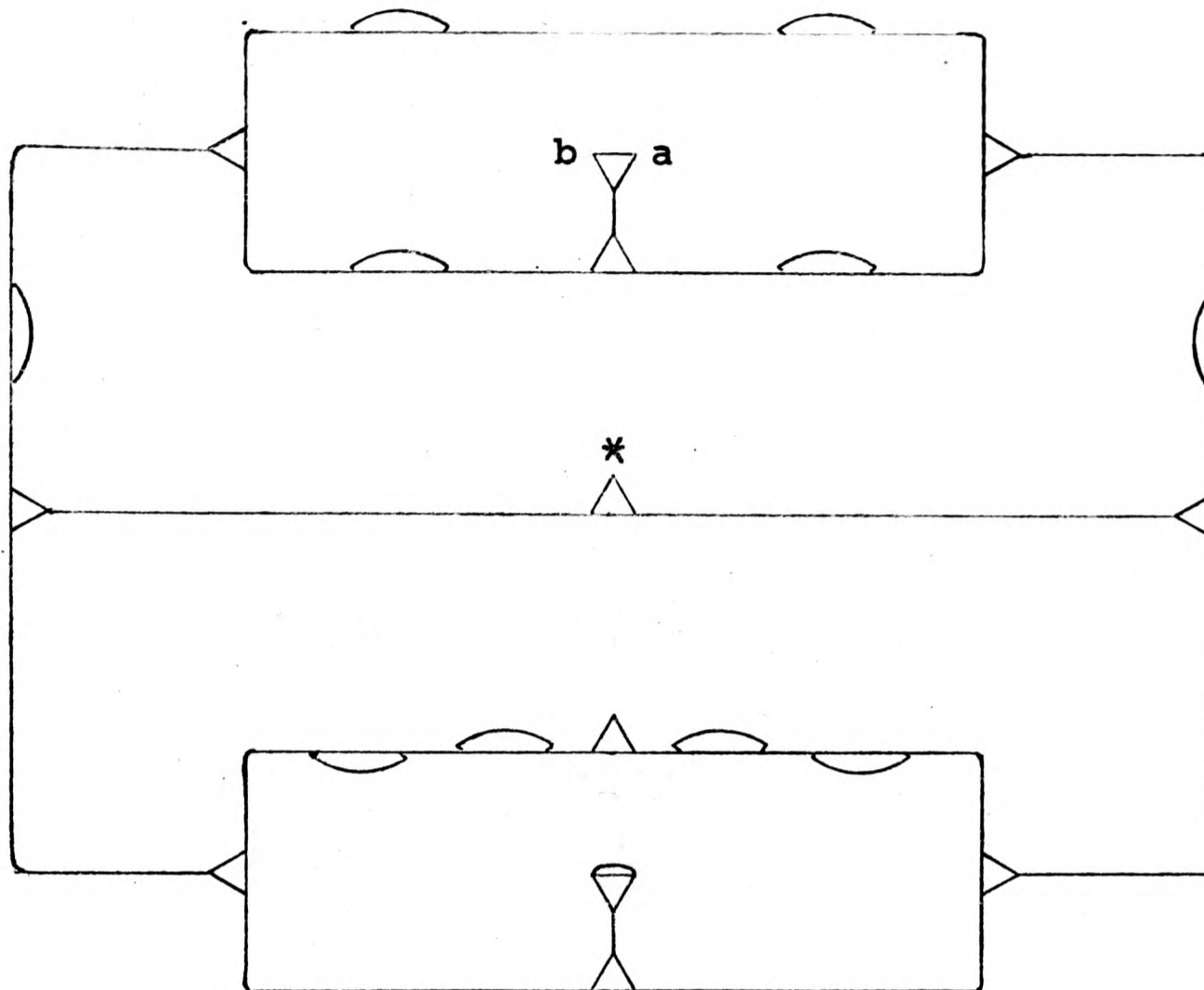


Diagram $U(7,d)$

$36 + 30d$ vertices

x even

t is odd when $d \equiv 0 \pmod{2}$, even when $d \equiv 1 \pmod{2}$

xyt has cycle structure:

a (b10) $5^1 8^1 11^1$ when $d = 0$

a (b9) $3^2 4^1 8^2 9^2 11^1$ when $d = 1$

a (b7) $3^2 4^1 6^{2+5(d-2)} 7^2 8^5 11^1$ when $d \geq 2$

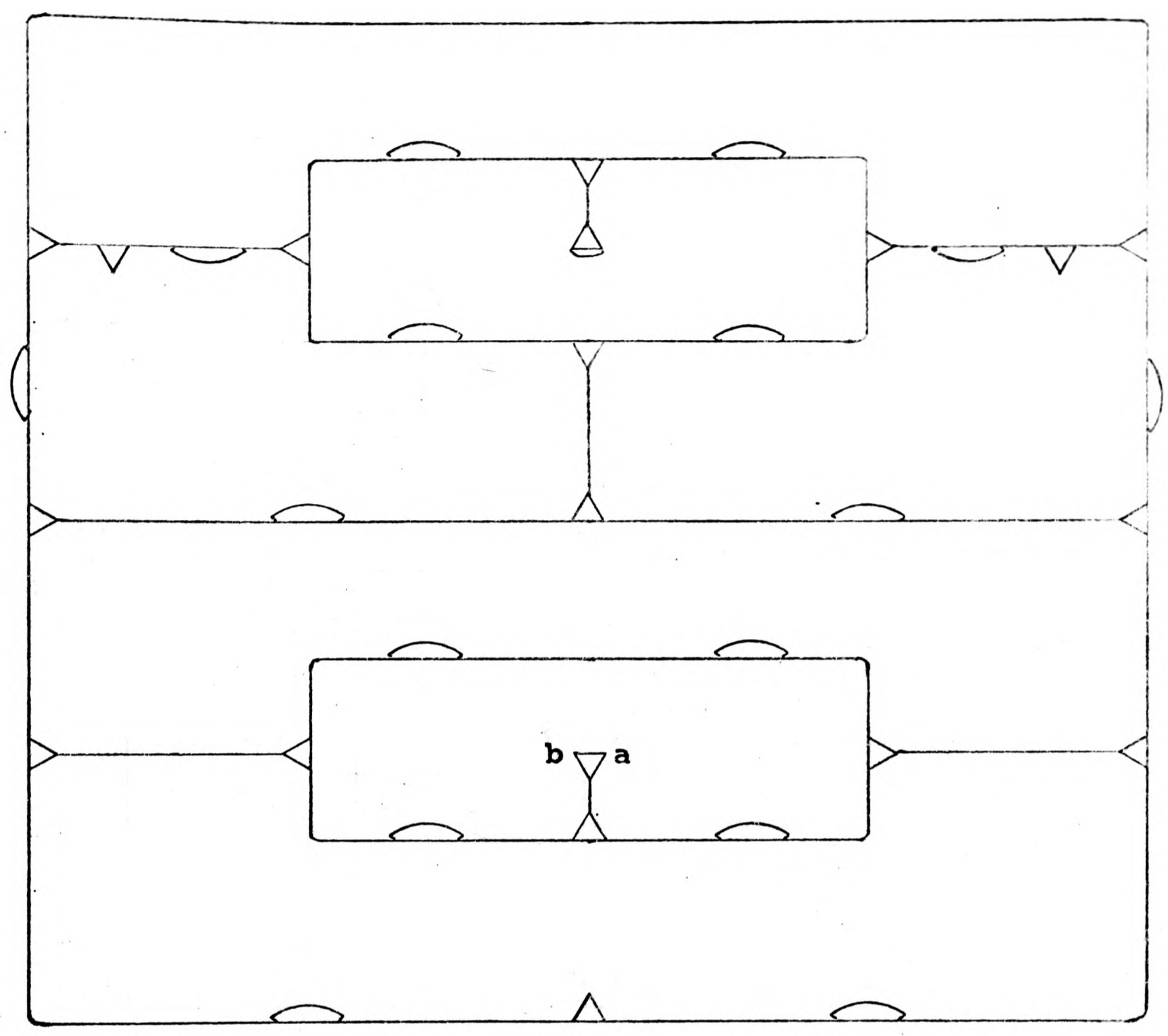


Diagram $V(7,d)$

$57 + 48d$ vertices

x even

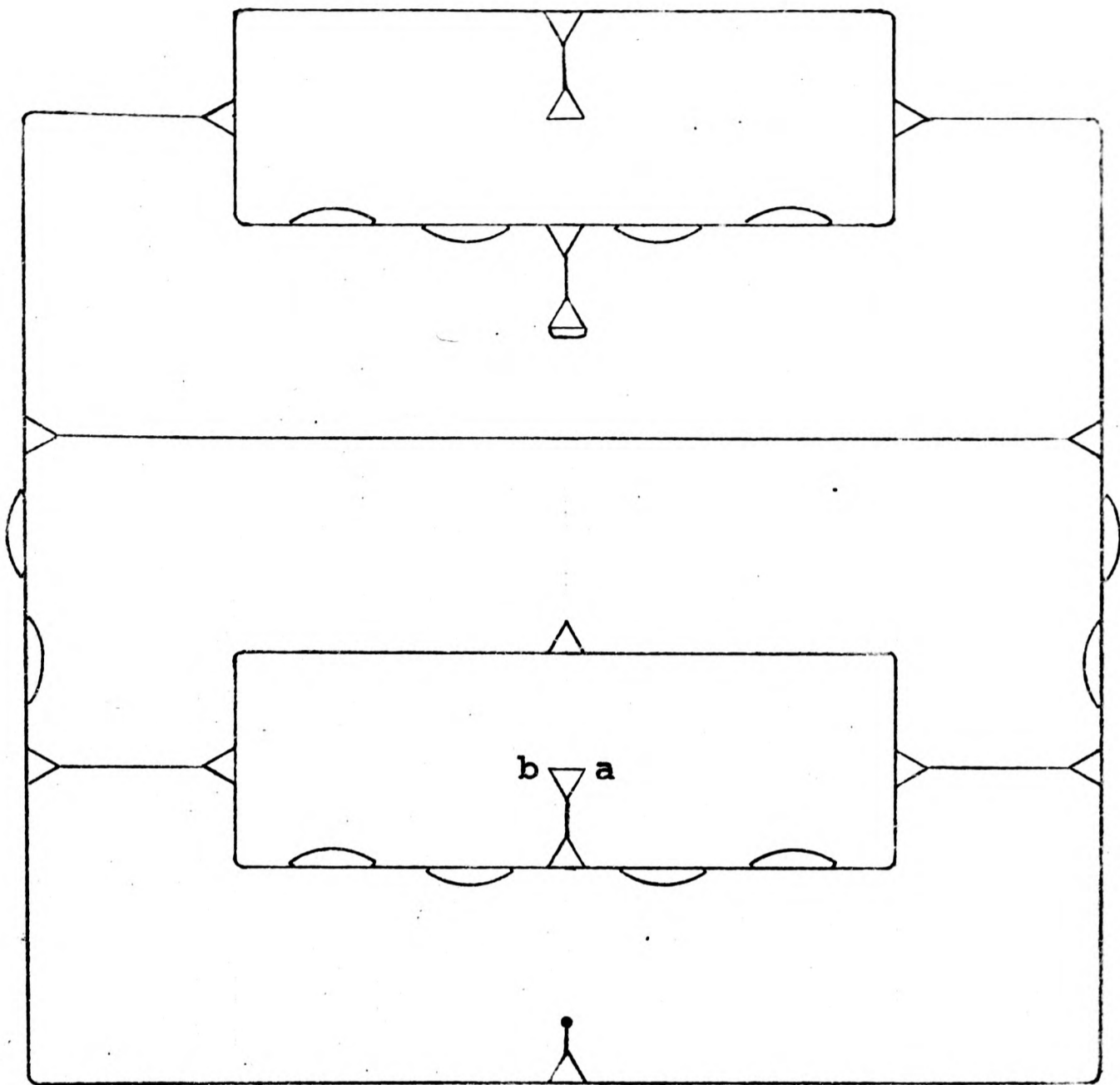
t odd

xyt has cycle structure:

$a (b16) 8^1 9^2 13^1$ when $d = 0$

$a (b9) 3^3 7^3 8^3 12^2 16^1$ when $d = 1$

$a (b7) 3^3 5^2 6^{3+8(d-2)} 7^1 8^7 10^2 12^2$ when $d \geq 2$

Diagram $T(8,d)$ $49 + 36d$ vertices x even t even xyt has cycle structure:
 $a (b12) 1^1 5^1 6^1 8^1 15^1$ when $d = 0$
 $a (b5) 1^1 3^1 4^3 6^1 7^1 9^1 12^2 16^1$ when $d = 1$
 $a (b5) 1^1 3^1 4^3 5^1 6^{1+6(d-2)} 7^2 8^4 9^1 10^2 12^1$ when $d \geq 2$

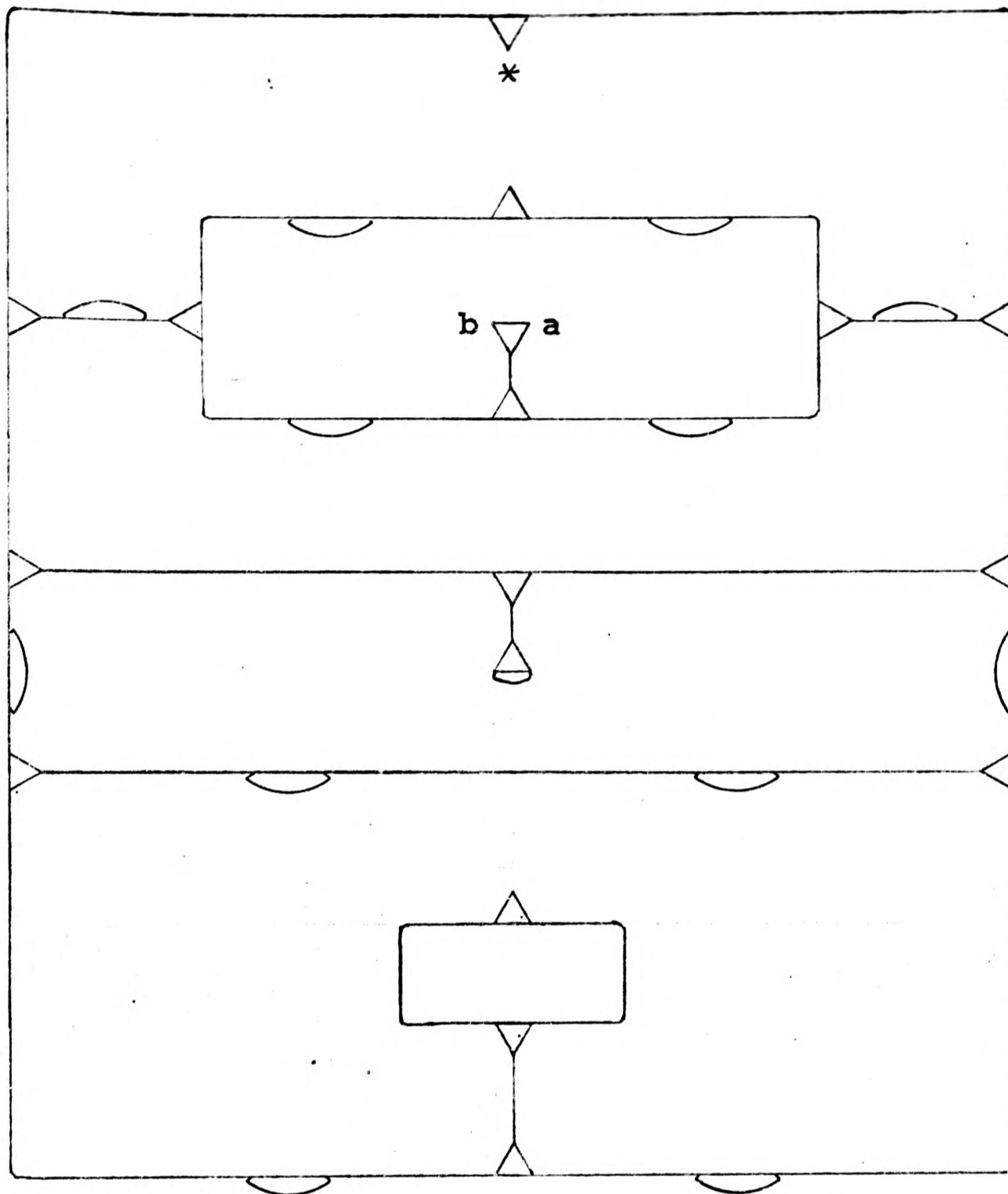


Diagram $U(8,d)$

$51 + 36d$ vertices

x odd

t odd

xyt has cycle structure:

a (b8) $3^1 5^1 10^1 11^1 12^1$ when $d = 0$

a (b5) $3^1 4^1 7^4 8^2 9^2 11^1$ when $d = 1$

a (b5) $3^1 4^1 5^3 6^{1+6(d-2)} 7^3 8^7 11^1$ when $d \geq 2$

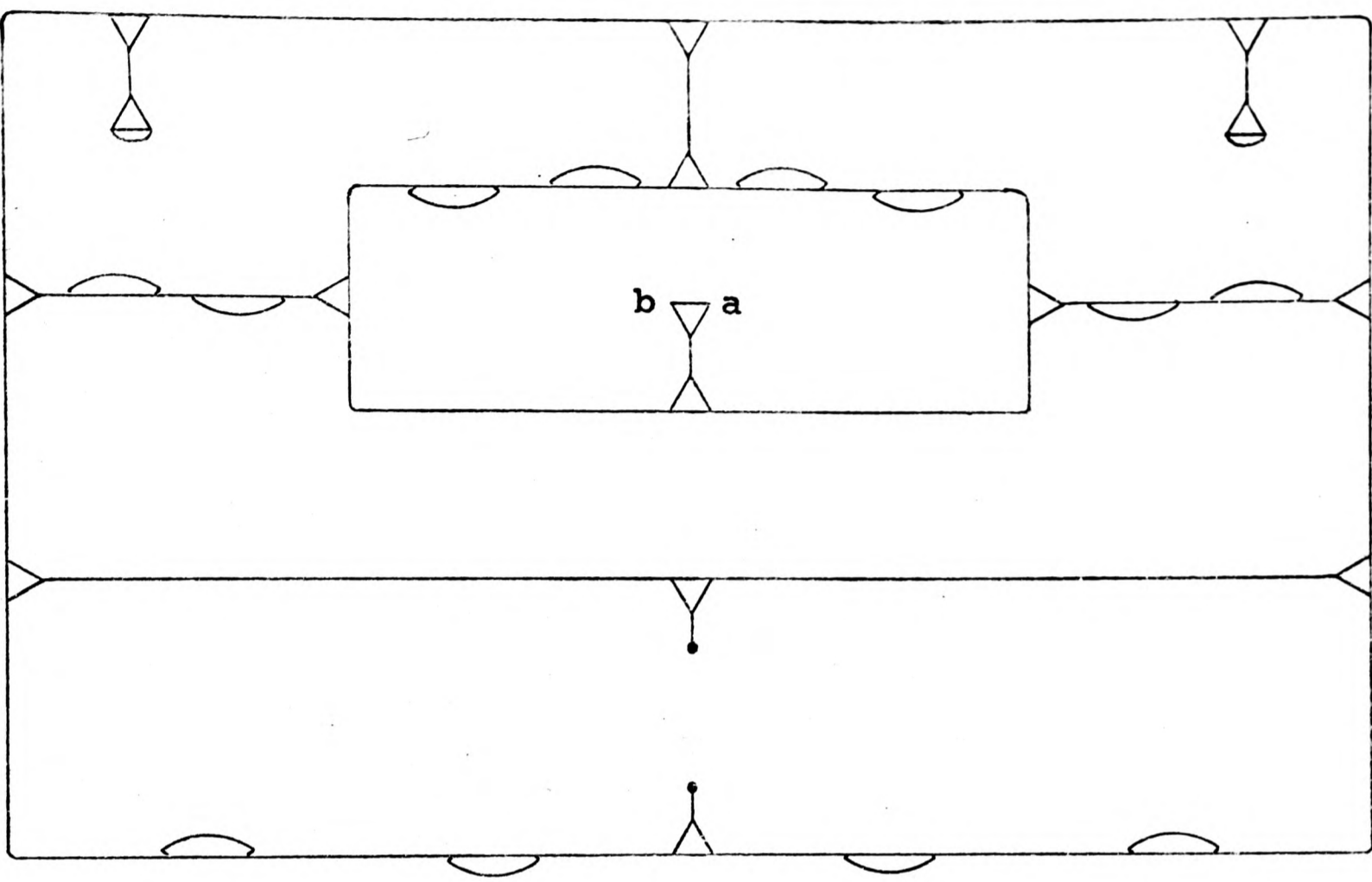


Diagram $V(8,d)$

$50 + 36d$ vertices

x even

t odd

xyt has cycle structure:

- a (b16) $4^1 10^1 18^1$ when $d = 0$
- a (b24) $3^1 4^3 5^1 6^1 8^2 18^1$ when $d = 1$
- a (b9) $3^1 4^3 5^2 6^{1+6(d-2)} 7^2 8^4 16^1 18^1$ when $d \geq 2$

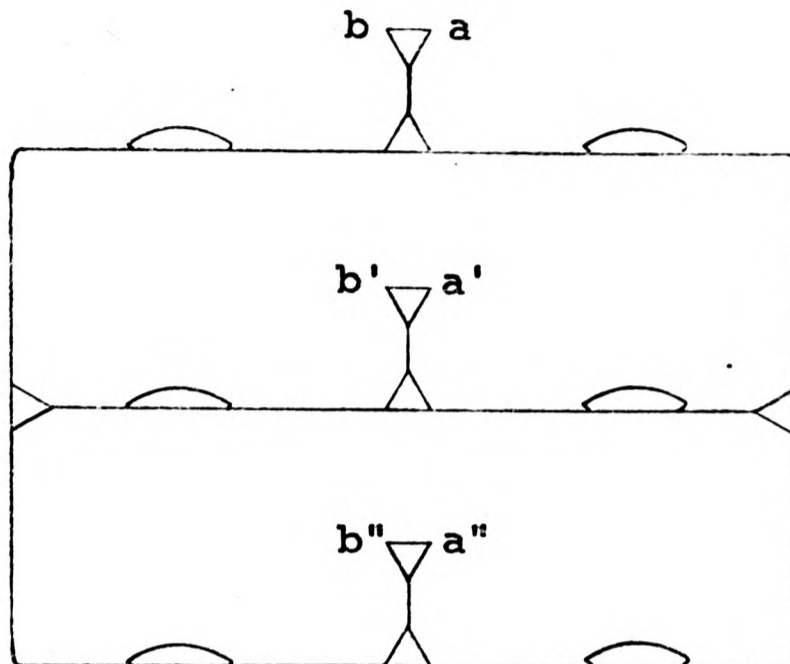


Diagram $W(8,d)$

$24 + 18d$ vertices

x odd

t is odd when $d \equiv 0 \pmod{2}$, even when $d \equiv 1 \pmod{2}$

xyt has cycle structure:

$a \ a' \ a'' \ (b_6) \ (b'_6) \ (b''_6)$ when $d = 0$

$a \ a' \ a'' \ (b_9) \ (b'_9) \ (b''_9) \ 3^3$ when $d = 1$

$a \ a' \ a'' \ (b_7) \ (b'_7) \ (b''_7) \ 3^3 \ 6^{3(d-2)} \ 8^3$ when $d \geq 2$

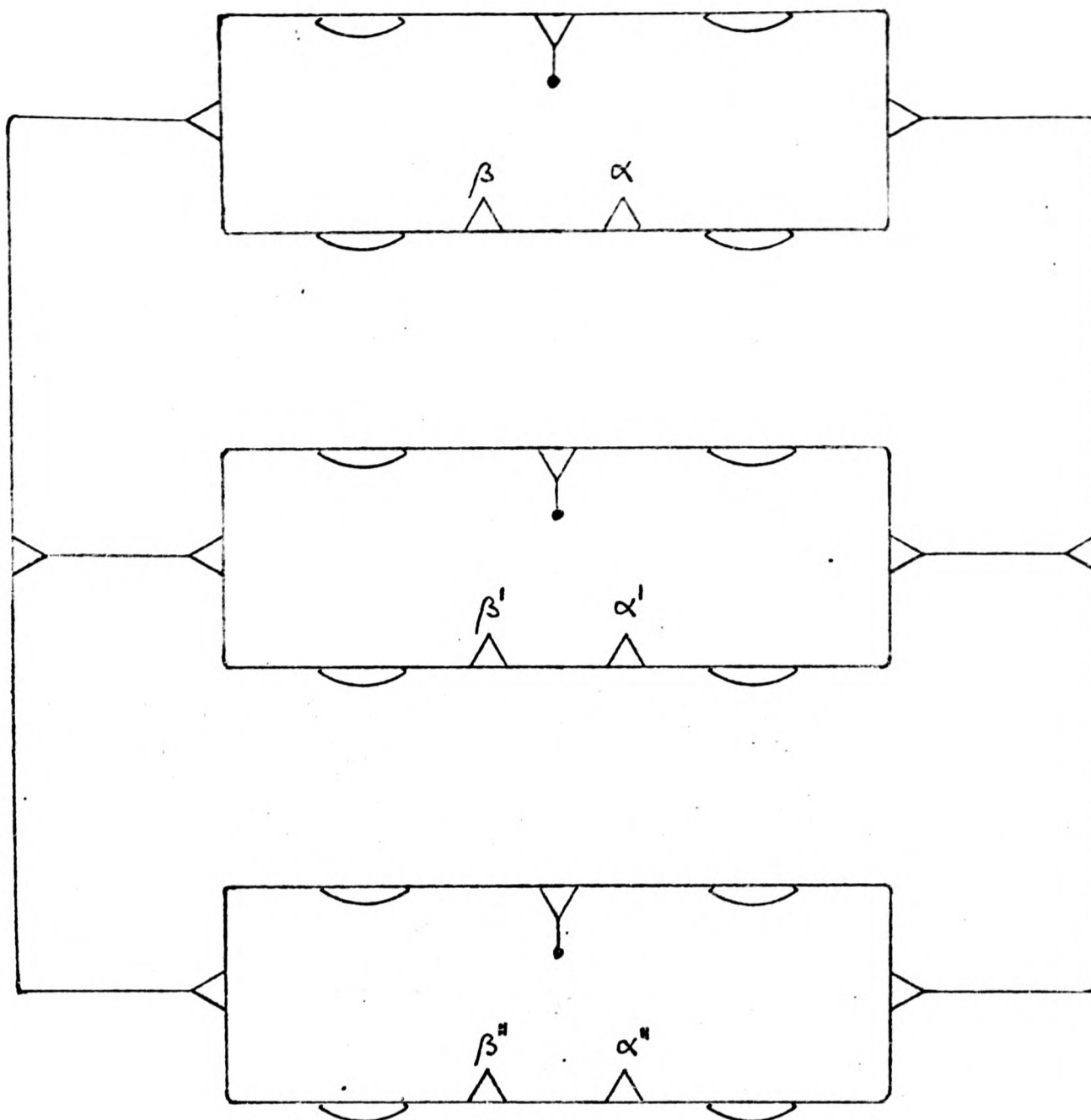


Diagram $S(9, d)$

$54 + 36d$ vertices

x even

t even

xyt has cycle structure:

$(\alpha 6) (\beta 10) (\alpha' 6) (\beta' 10) (\alpha'' 6) (\beta'' 10)$ when $d = 0$

$(\alpha 13) (\beta 3) (\alpha' 13) (\beta' 3) (\alpha'' 13) (\beta'' 3)$ $3^3 9^3$ when $d = 1$

$(\alpha 7) (\beta 3) (\alpha' 7) (\beta' 3) (\alpha'' 7) (\beta'' 3)$ $3^3 6^{6(d-2)} 7^3 8^3 12^3$ when $d \geq 2$

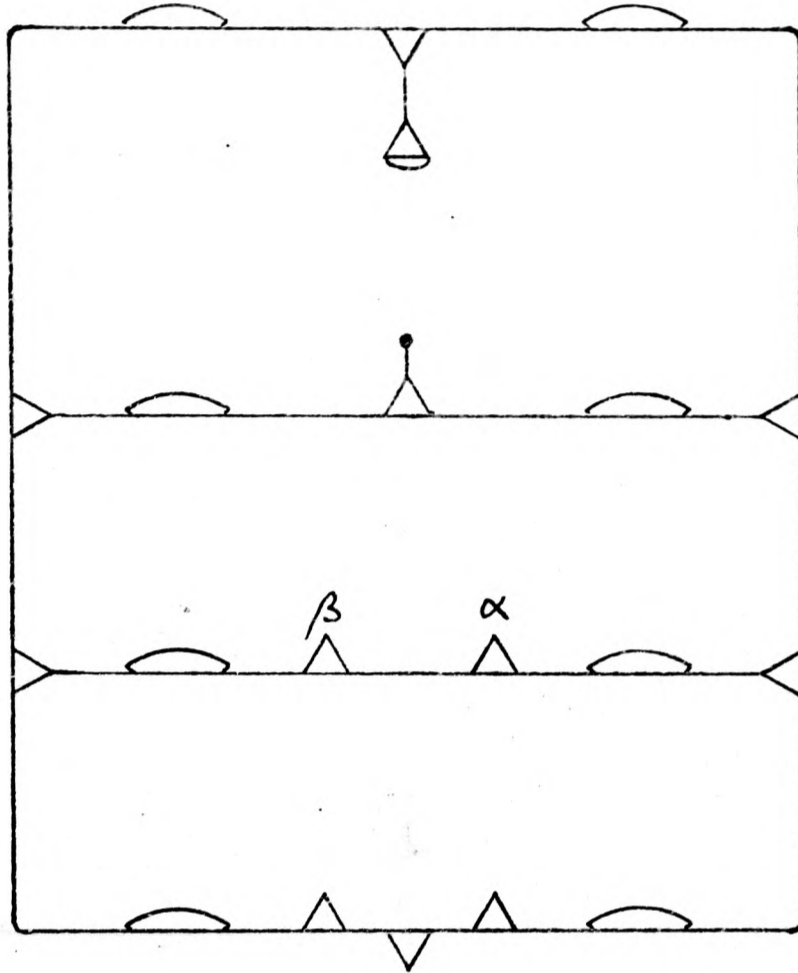


Diagram $T(9,d)$

$37 + 24d$ vertices

x even

t even

xyt has cycle structure:

$$(\alpha 12) (\beta 6) 4^2 9^1 \quad \text{when } d = 0$$

$$(\alpha 5) (\beta 10) 3^1 4^1 5^1 7^1 8^2 9^1 \quad \text{when } d = 1$$

$$(\alpha 5) (\beta 5) 3^1 4^1 5^2 6^{1+4(d-2)} 7^2 8^3 12^1 \quad \text{when } d \geq 2$$

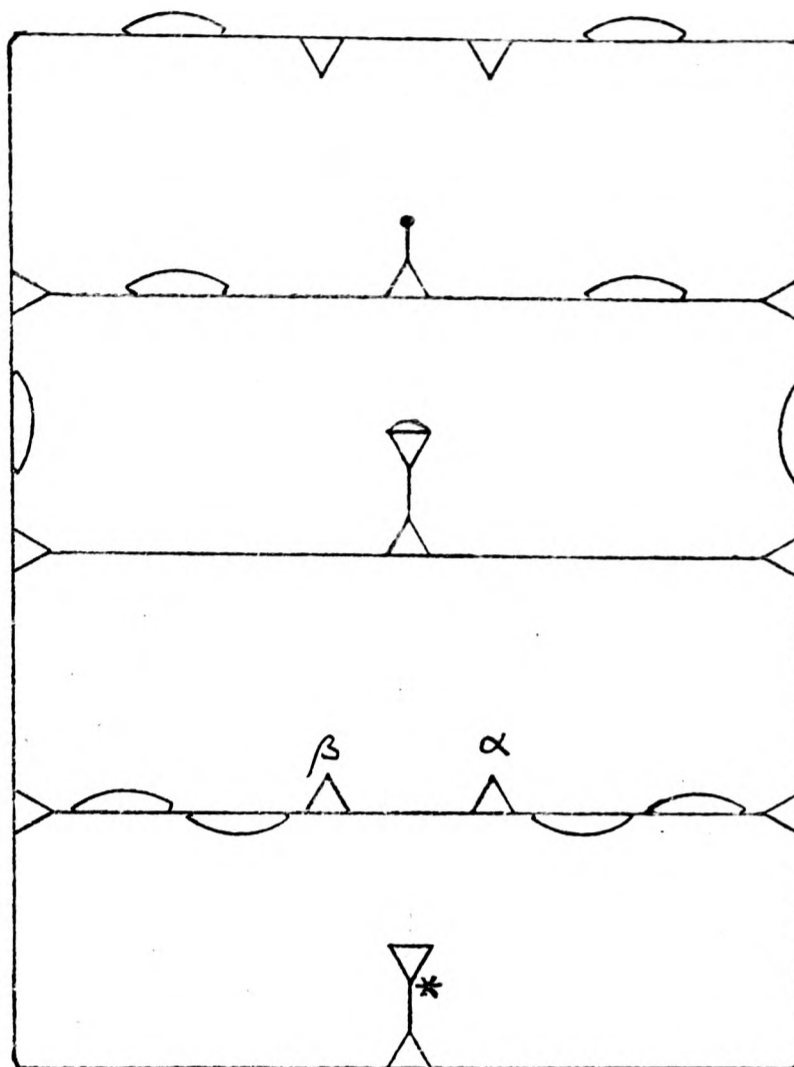


Diagram $U(9,d)$

$46 + 30d$ vertices

x even

t is even when $d \equiv 0 \pmod{2}$, odd when $d \equiv 1 \pmod{2}$

xyt has cycle structure:

$$(\alpha 12) (\beta 4) 1^1 7^2 13^1 \quad \text{when } d = 0$$

$$(\alpha 13) (\beta 3) 1^1 3^1 4^2 5^1 9^2 10^1 13^1 \quad \text{when } d = 1$$

$$(\alpha 7) (\beta 3) 1^1 3^1 4^2 5^1 6^{5(d-2)} 7^2 8^5 10^1 13^1 \quad \text{when } d \geq 2$$

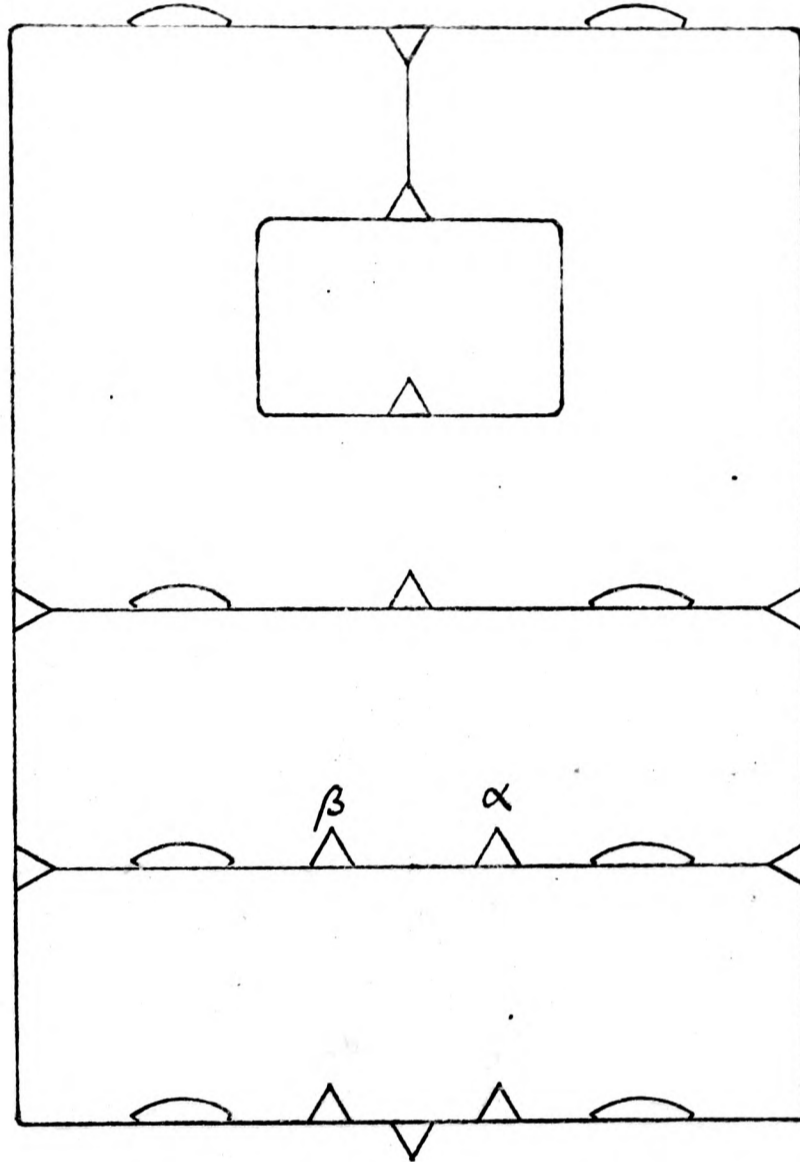


Diagram $V(9, d)$

$39 + 24d$ vertices

x even

t odd

xyt has cycle structure:

$(\alpha 13) (\beta 6) 2^1 4^2 8^1$ when $d = 0$

$(\alpha 5) (\beta 10) 2^1 3^1 4^1 5^1 8^4$ when $d = 1$

$(\alpha 5) (\beta 5) 2^1 3^1 4^1 5^2 6^{2+4(d-2)} 8^4 12^1$ when $d \geq 2$

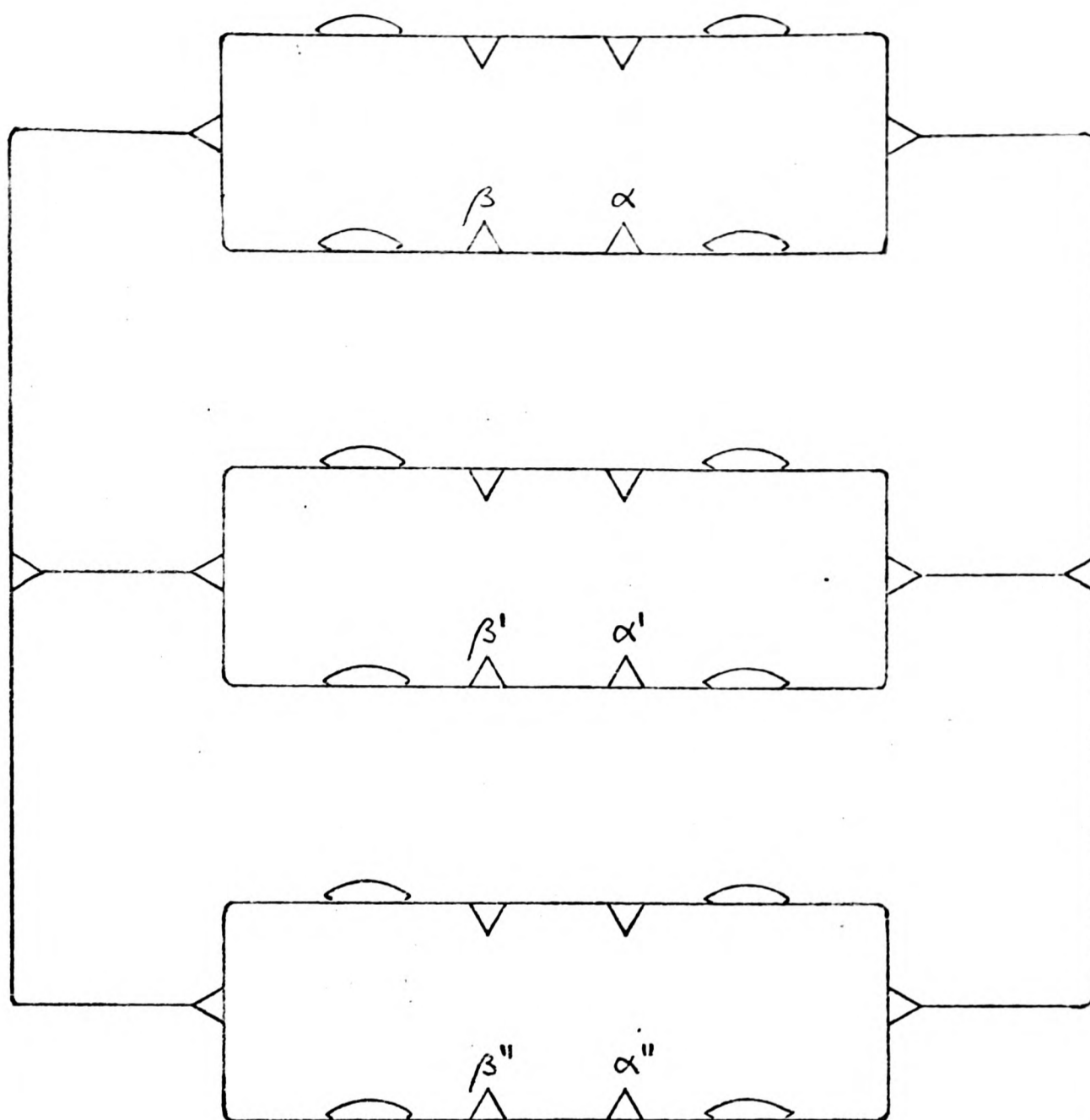


Diagram $S(10, d)$

$60 + 36d$ vertices

x even

t even

xyt has cycle structure:

$(\alpha 9) (\beta 9) (\alpha' 9) (\beta' 9) (\alpha'' 9) (\beta'' 9)$ when $d = 0$

$(\alpha 5) (\beta 7) (\alpha' 5) (\beta' 7) (\alpha'' 5) (\beta'' 7)$ $4^3 14^3$ when $d = 1$

$(\alpha 5) (\beta 5) (\alpha' 5) (\beta' 5) (\alpha'' 5) (\beta'' 5)$ $4^3 6^{6(d-2)} 8^6 12^3$ when $d \geq 2$

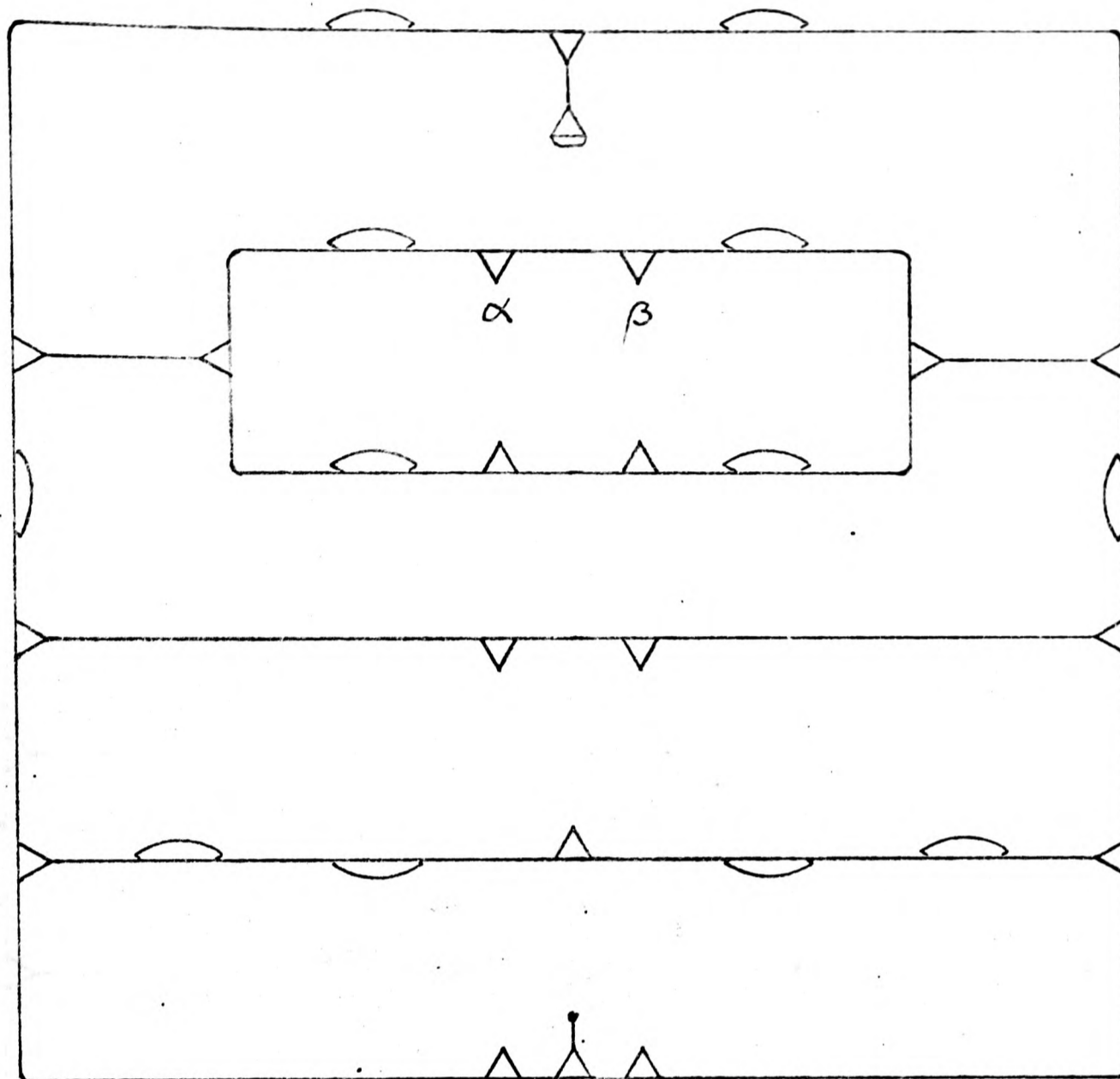


Diagram $T(10, d)$

$61 + 36d$ vertices

x even

t even

xyt has cycle structure:

$$(\alpha 9) (\beta 10) 3^1 7^1 9^1 10^1 11^1 \quad \text{when } d = 0$$

$$(\alpha 15) (\beta 3) 3^1 4^2 6^1 7^2 8^2 9^1 10^1 11^1 \quad \text{when } d = 1$$

$$(\alpha 7) (\beta 3) 3^1 4^2 5^2 6^{2+6(d-2)} 7^1 8^5 9^1 10^2 12^1 \quad \text{when } d \geq 2$$

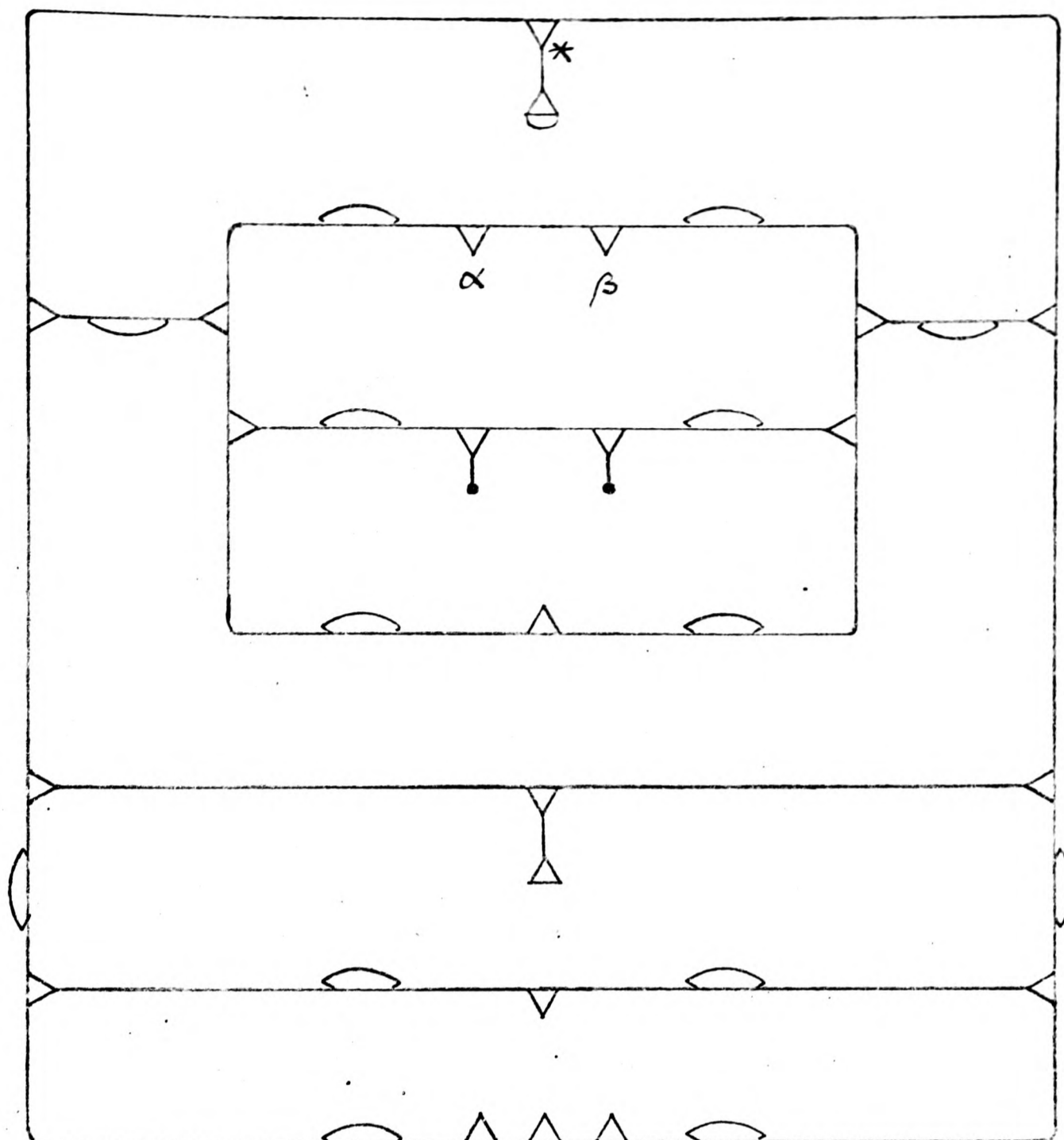


Diagram $U(10, d)$

$71 + 42d$ vertices

x odd

t is even when $d \equiv 0 \pmod{2}$, odd when $d \equiv 1 \pmod{2}$

xyt has cycle structure:

$$(\alpha 9) (\beta 11) 1^1 3^1 7^2 13^1 18^1 \quad \text{when } d = 0$$

$$(\alpha 9) (\beta 3) 1^1 3^2 4^1 5^2 6^1 7^1 8^3 13^1 28^1 \quad \text{when } d = 1$$

$$(\alpha 7) (\beta 3) 1^1 3^2 4^1 5^3 6^{3+7(d-2)} 8^6 12^2 13^1 14^1 \quad \text{when } d \geq 2$$

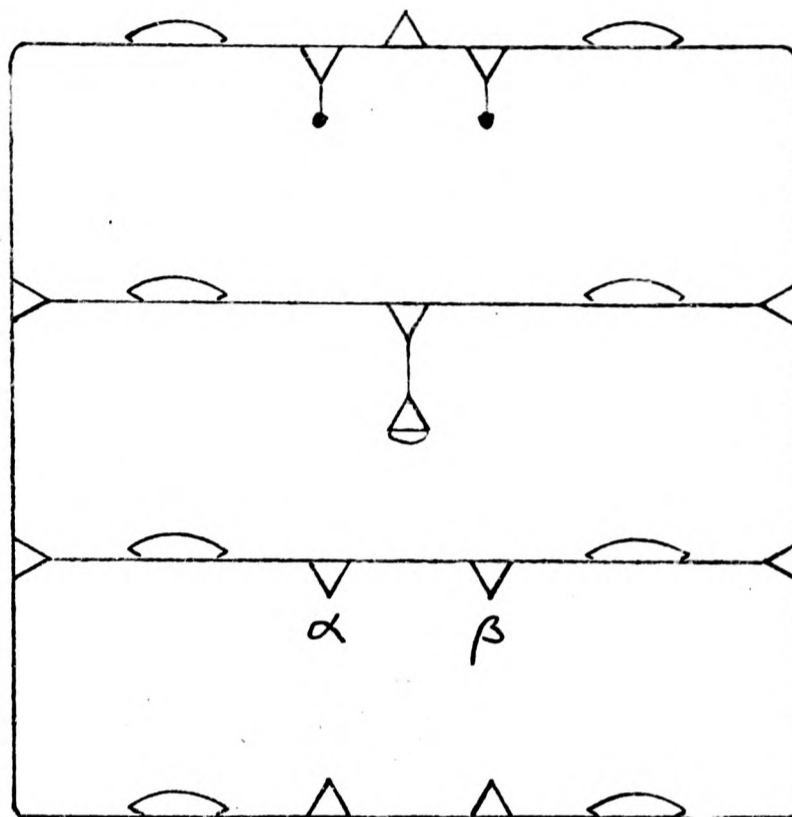


Diagram $V(10, d)$

$41 + 24d$ vertices

x even

t odd

xyt has cycle structure:

$$(\alpha 9) (\beta 13) 7^1 10^1$$

when $d = 0$

$$(\alpha 14) (\beta 3) 6^1 7^2 8^2 10^1$$

when $d = 1$

$$(\alpha 7) (\beta 3) 5^1 6^{2+4(d-2)} 7^2 8^3 10^1 12^1$$

when $d \geq 2$

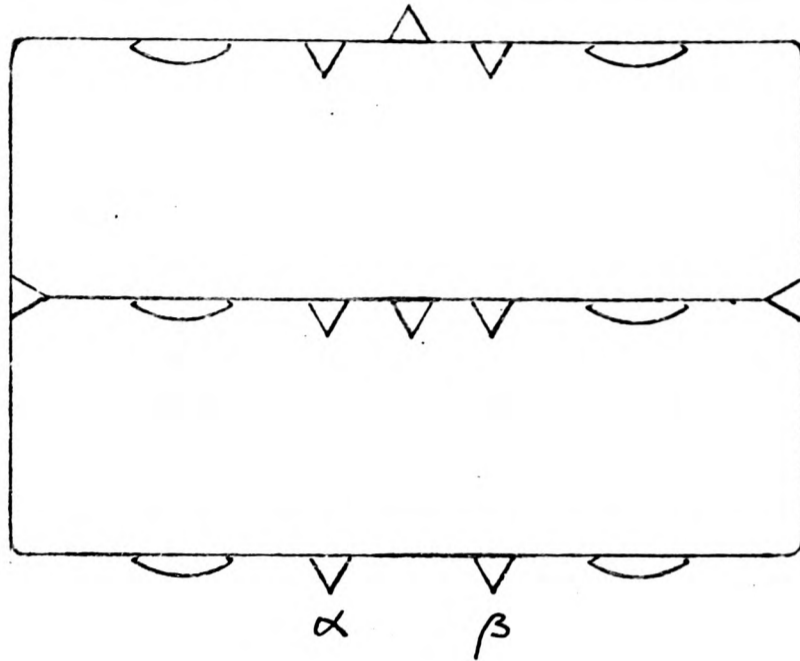


Diagram $W(10, d)$

$30 + 18d$ vertices

x odd

t is even when $d \equiv 0 \pmod{2}$, odd when $d \equiv 1 \pmod{2}$

xyt has cycle structure:

$$(\alpha 7) (\beta 7) 3^1 4^1 7^1 \quad \text{when } d = 0$$

$$(\alpha 5) (\beta 7) 3^1 4^1 5^1 6^1 8^2 \quad \text{when } d = 1$$

$$(\alpha 5) (\beta 5) 3^1 4^1 5^1 6^{3+3(d-2)} 8^3 \quad \text{when } d \geq 2$$

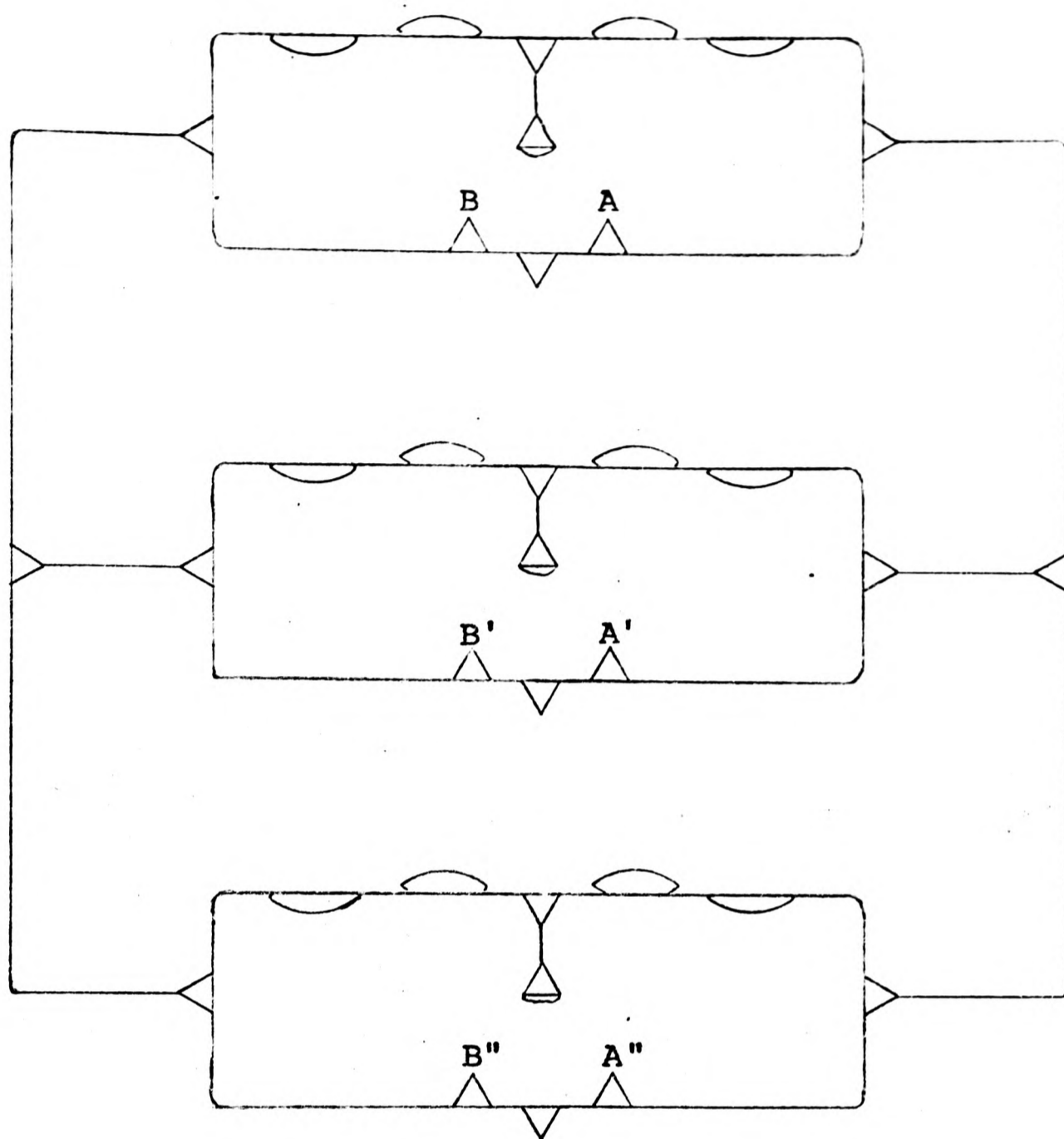


Diagram S(11, d)

69 + 36d vertices

x even

t even

xyt has cycle structure:

$$(A3) (B12) (A'3) (B'12) (A''3) (B''12) 6^3 \quad \text{when } d = 0$$

$$(A3) (B9) (A'3) (B'9) (A''3) (B''9) 4^3 7^3 10^3 \quad \text{when } d = 1$$

$$(A3) (B9) (A'3) (B'9) (A''3) (B''9) 4^3 5^3 6^{6(d-2)} 7^3 8^3 9^3$$

when $d \geq 2$

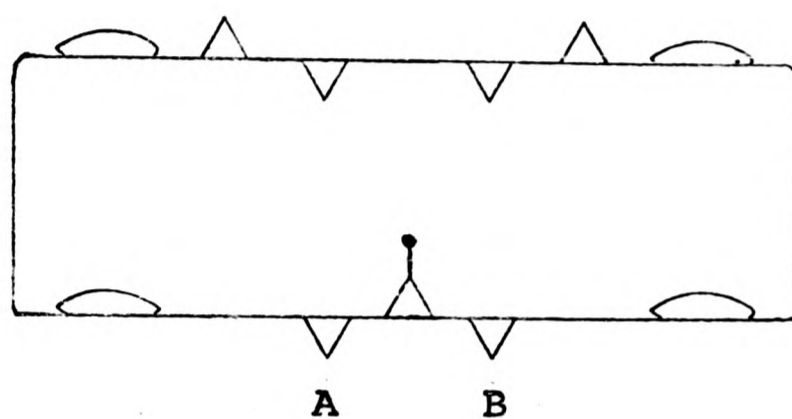


Diagram $T(11,d)$

$22 + 12d$ vertices

x even

t even

xyt has cycle structure:

$(A4)(B7) 4^1 5^1$ when $d = 0$

$(A4)(B3) 4^2 8^1 9^1$ when $d = 1$

$(A4)(B3) 4^2 6^{1+2(d-2)} 7^1 8^2$ when $d \geq 2$

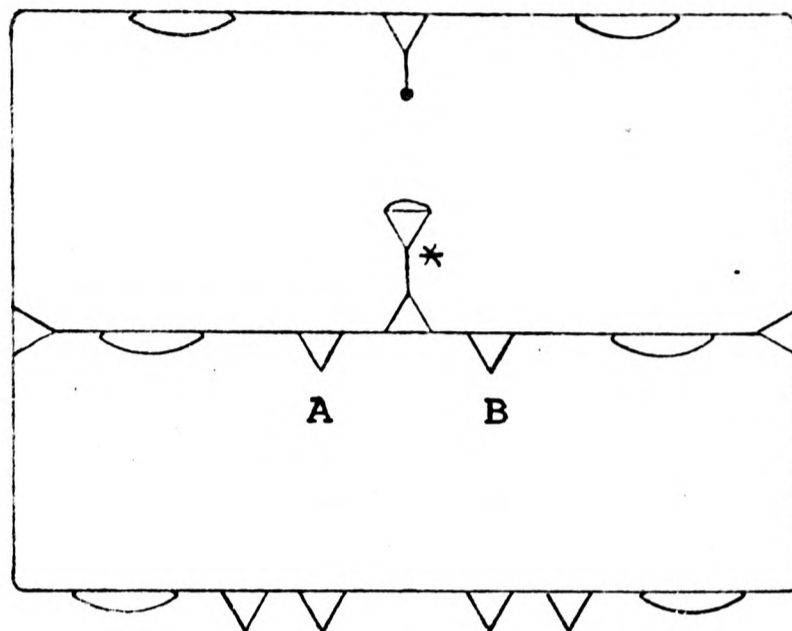


Diagram $U(11, d)$

$34 + 18d$ vertices

x even

t is odd when $d \equiv 0 \pmod{2}$, even when $d \equiv 1 \pmod{2}$

xyt has cycle structure:

(A6) (B4) $6^1 7^1 9^1$ when $d = 0$

(A6) (B7) $3^1 5^1 6^2 8^1 9^1$ when $d = 1$

(A6) (B5) $3^1 5^1 6^{3+3(d-2)} 7^1 8^3$ when $d \geq 2$

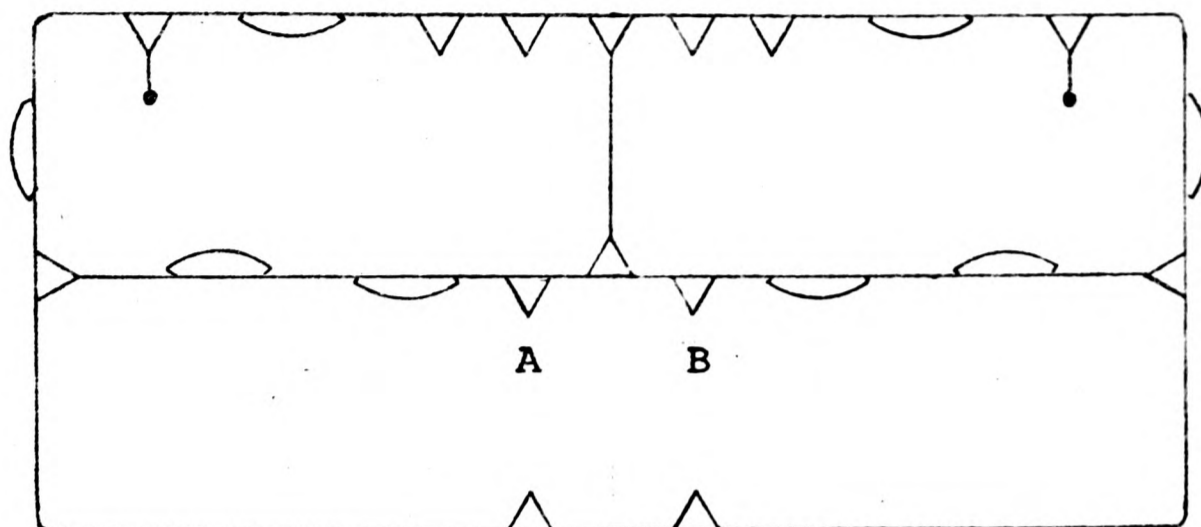


Diagram $V(11, d)$

$44 + 24d$ vertices

x even

t odd

xyt has cycle structure:

(A9) (B7) $3^1 9^1 14^1$ when $d = 0$

(A9) (B9) $3^1 4^1 5^1 6^1 8^1 10^1 12^1$ when $d = 1$

(A9) (B5) $3^1 4^1 5^1 6^{2+4(d-2)} 8^5 12^1$ when $d \geq 2$

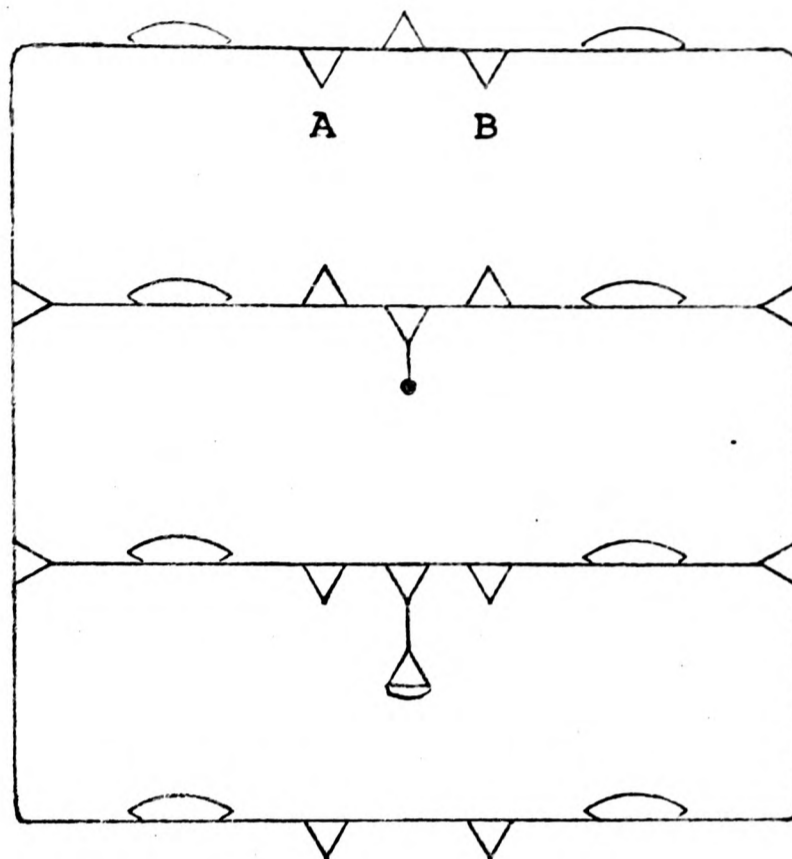


Diagram $T(12, d)$

$49 + 24d$ vertices

x even

t even

xyt has cycle structure:

(A3) (B7) $3^1 5^1 8^1 10^1 11^1$ when $d = 0$

(A3) (B3) $3^1 4^2 5^2 8^2 10^1 18^1$ when $d = 1$

(A3) (B3) $3^1 4^2 5^2 6^{1+4(d-2)} 7^1 8^4 11^1 12^1$ when $d \geq 2$

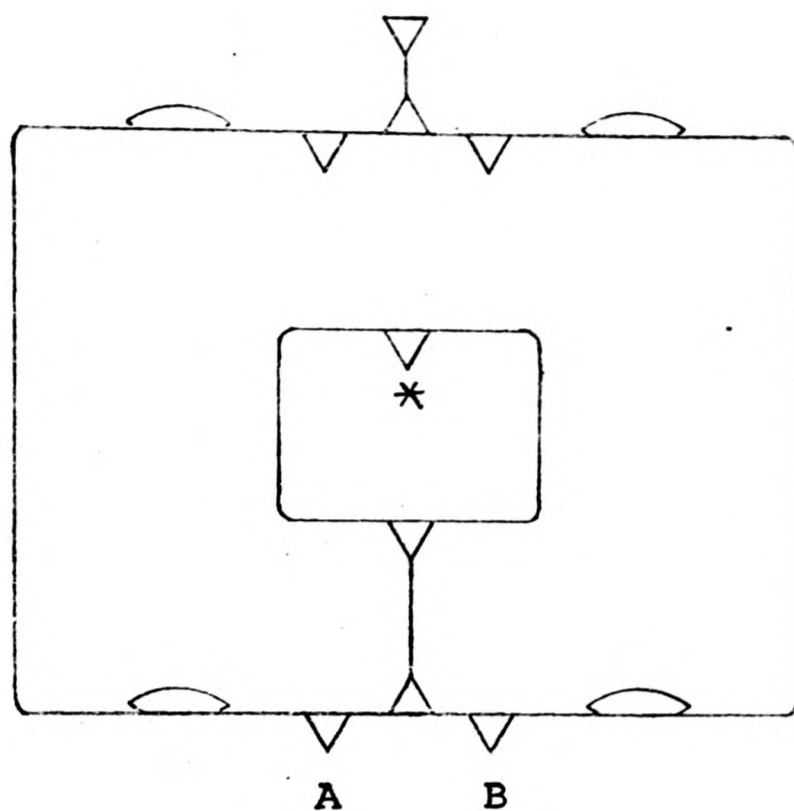


Diagram $U(12, d)$

$27 + 12d$ vertices

x even

t odd

xyt has cycle structure:

$$(A7) (B3) 1^1 2^1 6^2$$

when $d = 0$

$$(A7) (B3) 1^1 2^1 4^2 6^1 10^1$$

when $d = 1$

$$(A7) (B3) 1^1 2^1 4^2 6^{1+2(d-2)} 7^2 8^1$$

when $d \geq 2$

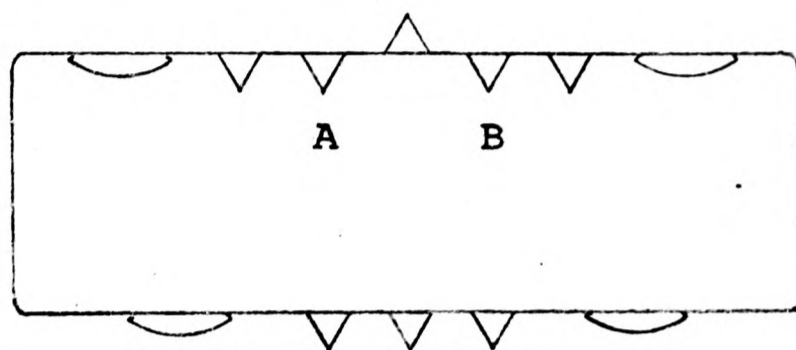


Diagram $V(12, d)$

$24 + 12d$ vertices

x even

t odd

xyt has cycle structure:

$(A3)(B7) 3^1 4^1 5^1$ when $d = 0$

$(A3)(B5) 3^1 4^1 5^1 6^1 8^1$ when $d = 1$

$(A3)(B5) 3^1 4^1 5^1 6^{3+2(d-2)} 8^1$ when $d \geq 2$

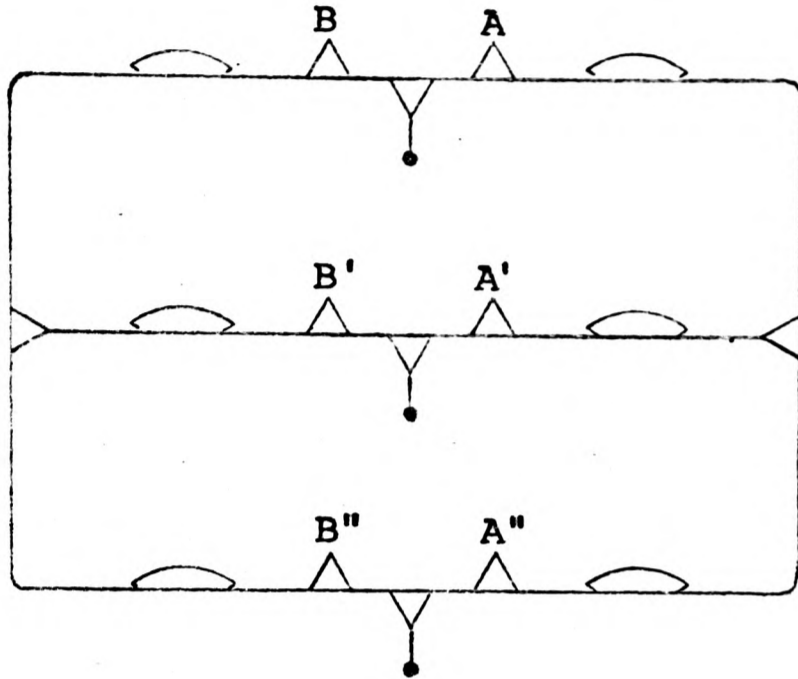


Diagram $W(12, d)$

$36 + 18d$ vertices

x odd

t is odd when $d \equiv 0 \pmod{2}$, even when $d \equiv 1 \pmod{2}$

xyt has cycle structure:

$(A4) (B6) (A'4) (B'6) (A''4) (B''6)$ when $d = 0$

$(A4) (B7) (A'4) (B'7) (A''4) (B''7) 5^3$ when $d = 1$

$(A4) (B5) (A'4) (B'5) (A''4) (B''5) 5^3 6^{3(d-2)} 8^3$ when $d \geq 2$

Before we continue, let $S(8,d)$ denote a (1)-composite of two copies of the diagram $W(8,d)$, and let $S(12,d)$ denote a (3)-composite of two copies of $W(12,d)$, for each non-negative integer d .

It may now be noticed that for every pair (h,d) the diagram $S(h,d)$ contains at least three (j) -handles, where j is the integer $[\frac{1}{2}(h-5)]$. Also each of the other three or four coset diagrams for the group $\Delta^*(2,3,h+6d)$ contains at least one (j) -handle. Next the element t acts as an odd permutation on the vertices of the diagram $V(h,d)$. Finally if h is even then the element x acts as an odd permutation on the vertices of $W(h,d)$. (When h is odd there is no such diagram $W(h,d)$, for in that case the subgroup $\langle x,y \rangle$ acts always as a group of even permutations.)

We proceed to a proof of the theorem.

Suppose that k is any integer greater than six.

Then there ^{are} unique integers h, d with $h \in \{7,8,9,10,11,12\}$ and $d \geq 0$ such that $k = h + 6d$. Let j be the integer $[\frac{1}{2}(h-5)]$. Next denote by S, T, U and V the numbers of vertices of the diagrams $S(h,d), T(h,d), U(h,d)$ and $V(h,d)$ respectively. If h is even, then take W to be the number of vertices of the diagram $W(h,d)$. If h is odd, then let $W = 0$.

5.2 LEMMA T is coprime to S .

Proof We deal with the case $h=7$, the other cases being very similar. Any common divisor of S (which is $42+36d$) and T (which is $71+60d$) must divide also the number

$$3T - 5S = 213 + 180d - (210 + 180d) = 3.$$

But 3 does not divide $71+60d$. Hence the greatest common divisor of S and T is 1.

It follows from the lemma that every sufficiently large integer is expressible in the form $fS + gT$, where f and g are integers which satisfy $f > g \geq 0$. In fact, since the residue of T modulo S generates the ^{additive group} $\langle \quad \rangle$ of integers modulo S , every integer exceeding $S^2 + (S-1)T$ is expressible in this way.

Now take N_k to be the integer $S^2 + (S-1)T + U + V + W$. It is clear that N_k is of the form $\lambda k^2 + \mu k + \nu$, where λ, μ and ν are integers which depend only on the value of h , since each of S, T, U, V and W depends linearly on k . In particular, N_k is of the order of k^2 .

Suppose n is any integer greater than N_k .

Then by the above remarks, each of the numbers $n - U$, $n - (U+V)$, $n - (U+W)$ and $n - (U+V+W)$ can be expressed in the form $fS + gT$ with $f > g \geq 0$.

So let f and g be integers which satisfy the latter inequality. Take f copies of the diagram $S(h,d)$, and join them by (j) -composition into a chain. (There are, of course, many ways of doing this, but for our purposes any one will suffice.) Now

use $g+1$ of the remaining (j) -handles to adjoin g copies of the diagram $T(h,d)$, and a single copy of the diagram $U(h,d)$. There are still at least two free (j) -handles, which may be used to adjoin a copy of $V(h,d)$, or (if h is even) a copy of $W(h,d)$, or copies of both.

In this way we obtain either two or four coset diagrams for $\Delta^*(2,3,k)$ on n vertices. If k is even we have four diagrams, one on which x and t both act as even permutations, one on which x is even while t is odd, one on which x is odd while t is even, and one on which x and t are both odd.

The reader should understand that a copy of $V(h,d)$ is added in order to change the parity of t , while addition of a copy of $W(h,d)$ changes the parity of x . If k is odd we have just two diagrams with n vertices, one on which t acts as an even permutation, and the other on which t is odd. (Recall that in the latter case x is the product of xy and y^{-1} , two elements of odd order, and so x acts always as an even permutation.)

We look now at the cycle structure of the element xyt in any of the corresponding permutation representations of the group $\Delta^*(2,3,k)$. First, the length of every cycle (even every cycle which results from composition of the basic diagrams) divides the number 720720. Next the point marked with an asterisk in the diagram $U(h,d)$ is fixed by t and lies in a cycle of prime length p (where $p = 11$ if $h = 7, 8$ or 11 , while $p = 13$ if $h = 9, 10$ or 12). This p -cycle is a useful cycle of xyt , and results from (j) -composition of the single

copy of the diagram $U(h,d)$. Finally p divides the length of no other cycle of xyt .

By Corollary 1.12 and Lemma 1.4 (a), the group $\Delta^*(2,3,k)$ acts as A_n or S_n on the vertices of the diagram.

Furthermore, the subgroup $\Delta(2,3,k)$ is alternating if x is even, or symmetric if x is odd. In particular, if k is an odd integer, then $\Delta(2,3,k)$ must act as A_n on the vertices.

Finally by Lemma 1.4 (b) the order of the commutator $xy^{-1}xy$ divides $\frac{1}{2}(720720)$, that is, 360360.

The proof is complete.

We have taken N_k to be of the order of k^2 , but there is plenty of evidence to suspect that the theorem is true with say $N_k = 24k$. Certainly this is the case when $k=7$; and the work done in Chapter 4 could easily be adapted to other cases, however it is probably not worth the trouble to do this.

Also it is likely that the theorem is true for much smaller values of m .

Since each of the groups $\Delta^*(2,3,k)$ is generated by its elements xy and t , the above proof yields also the result overleaf.

5.3 COROLLARY For every integer k greater than six, all but finitely many of the alternating and symmetric groups can be generated by elements u, v which satisfy

$$u^2 = v^k = (uv)^{2k+2} = 1$$

We complete this chapter by making the following conjecture. It seems very likely that methods similar to those presented here could be used to verify this, certainly in cases where $r=2$.

5.4 CONJECTURE

If r, s, t are integers satisfying $2 \leq r \leq s \leq t$ and $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} < 1$, then all but finitely many of the alternating groups A_n possess (r, s, t) -generating pairs, and if any two of r, s, t are even, then the same is true of (all but finitely many of) the symmetric groups S_n as well.

Appendix 1 : TWO-ELEMENT GENERATION OF THE GROUP OF THE
'MAGIC CUBE'

We assume here that the reader is familiar with the notation used by David Singmaster in his booklet Notes on the 'Magic Cube' (cf. [22]).

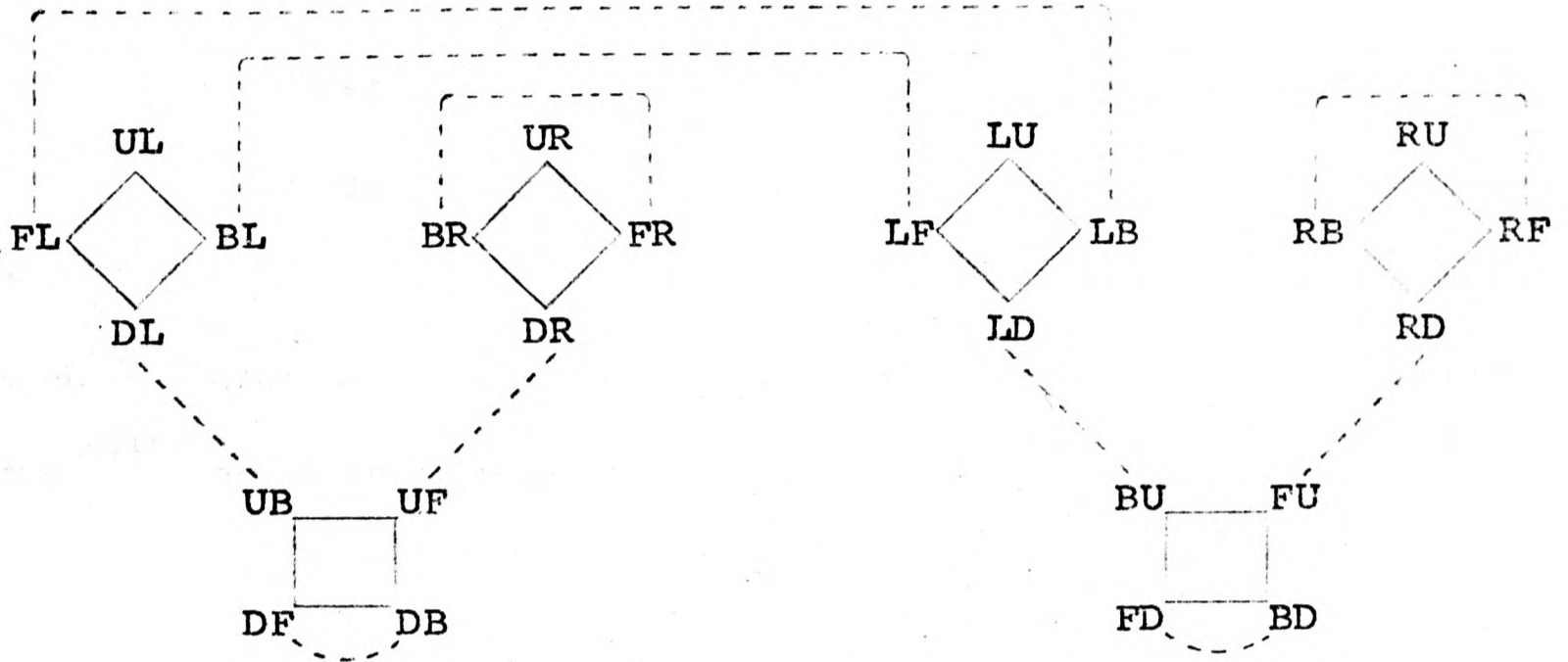
The group G in question is the group generated by the basic moves R, F, U, L, B, D , and has order $\frac{8! \cdot 12!}{2} \cdot \frac{3^8}{3} \cdot \frac{2^{12}}{2}$ (which is $2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11$). On the twelve edge-pieces of the cube, G acts as an extension of an elementary Abelian group of order 2^{11} by the symmetric group S_{12} ; while on the eight corner-pieces G acts as an extension of an elementary Abelian group of order 3^7 by the symmetric group S_8 .

Since S_8 is not a factor group of the modular group (cf. [3] or [20]), neither is G . In particular, for no integer k does G possess a $(2,3,k)$ -generating pair.

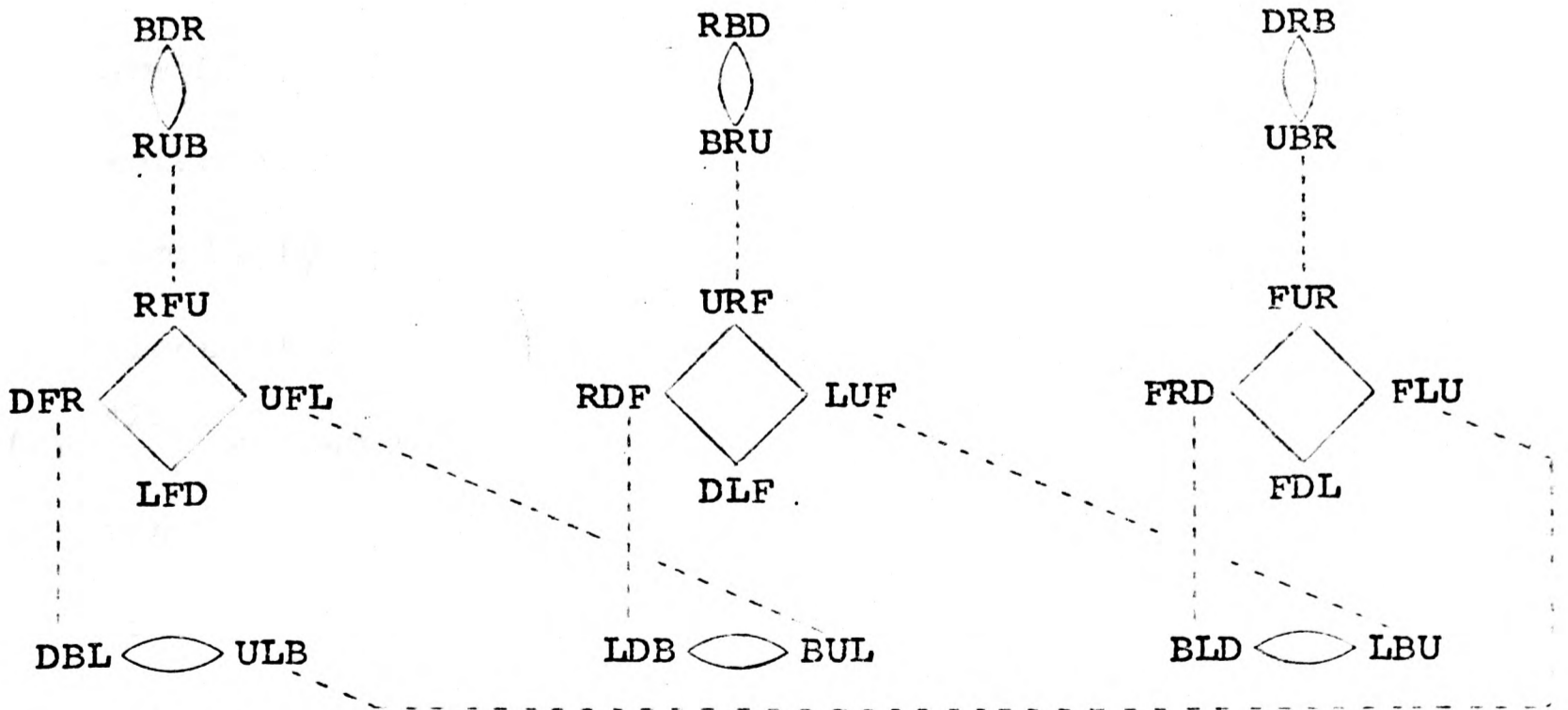
We can, however, generate G with an element of order two and an element of order four. (Such elements will then form a minimal generating pair for G .) We can even choose the generators in such a way that their product has maximal order (viz. 1260) in the group G .

For suppose there exist elements x and y in G such that x acts as $(FL, LB) (BR, FR) (DL, UB) (DR, UF) (DF, DB)$
 $\times (RFU, RUB) (DFR, DBL) (UFL, BUL)$
 and y acts as $(UR, BR, DR, FR) (UF, UB, DF, DB) (UL, FL, DL, BL)$
 $\times (RFU, DFR, LFD, UFL) (RUB, BDR) (DBL, ULB)$

Well, x has order two and y has order four, and the subgroup they generate is transitive both on edge-pieces and on corner-pieces. This is perhaps best illustrated by the diagrams below.



Action on edges



Action on corners

We see that the product xy acts as

$$(FL, LU)_+ (BL, LD, FD, FU, RF, RD, BU)_+ (BR, UR) \\ (RFU, BDR, RUB, DFR, ULB)_+ (DBL, LFD, UFL)_-$$

and in particular has order $2^2 \cdot 3^2 \cdot 5 \cdot 7$, that is, 1260, the largest possible order for any element of G (cf. Section 10-C of [22]). Notice that

$(xy)^{180}$ acts as $(BL, DR, UF, DL, BU, RF, FD)$, a 7-cycle on edges,
 $(xy)^{630}$ acts as $(FL)_+ (LU)_+$, flipping two edge-pieces,
 $(xy)^{252}$ acts as $(RFU, BRU, LBU, DRB, DFR)$, a 5-cycle on corners,
 $(xy)^{420}$ acts as $(DBL)_+ (LFD)_+ (UFL)_+$, twisting three corners clockwise.

Let H be the subgroup generated by x and y .

Now on the positions of edge-pieces, H is transitive, contains an odd permutation (x , for example), and also contains a 7-cycle. Therefore by Corollary 1.11 this group acts as S_{12} . Moreover, H contains an element which flips two edge-pieces, and now by conjugation within H we can obtain a flip of any two edge-pieces.

Similarly the 5-cycle above shows that H acts as S_8 on the positions of corner-pieces, and by conjugation of the element $(xy)^{420}$ we can obtain an element of H which twists any three corners.

Consequently H has order at least equal to that of G , and so x and y actually generate G .

It remains for us to produce such elements x and y using the basic moves R, F, U, L, B, D . But this can be done easily

- for example, by dismantling the cube, reassembling it into that state prescribed by the action of the appropriate element, and then using any algorithm for restoring the cube to the START position.

Appendix 2 : THE NON-ABELIAN SIMPLE GROUPS OF ORDER LESS THAN 10^5

Order	Group
60	$A_5 \cong \text{PSL}(2,4) \cong \text{PSL}(2,5)$
168	$\text{PSL}(3,2) \cong \text{PSL}(2,7)$
360	$A_6 \cong \text{PSL}(2,9)$
2520	A_7
5616	$\text{PSL}(3,3)$
6048	$\text{PSU}(3,3)$
7920	M_{11}
20160	$A_8 \cong \text{PSL}(4,2) \cong O_6^+(2)$
20160	$\text{PSL}(3,4)$
25920	$\text{PSU}(4,2) \cong \text{Sp}(4,3) \cong O_5(3) \cong O_6^-(2)$
29120	$\text{Sz}(8)$
62400	$\text{PSU}(3,4)$
95040	M_{12}
126000	$\text{PSU}(3,5)$
175560	J_1 (Janko)
181440	A_9
372000	$\text{PSL}(3,5)$
443520	M_{22}
604800	J_2 (Hall-Janko)
979200	$\text{Sp}(4,4) \cong O_5(4)$
$(q-1)q(q+1)$	$\text{PSL}(2,q)$, where $q = 4, 8, 16, 32, 64$
$\frac{1}{2}(q-1)q(q+1)$	$\text{PSL}(2,q)$, where q is an odd prime power with $5 \leq q \leq 125$

Simple groups other than those named in the above list can be accounted for by the isomorphisms

$$\text{PSL}(2,q) \cong \text{PSU}(2,q) \cong \text{Sp}(2,q) \cong O_3(q)$$

$$\text{PSL}(2,q^2) \cong O_4^-(q)$$

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