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INEFFICIENCIES ON LINKING DECISIONS

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Inefficiencies on Linking Decisions*

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Abstract

Jackson and Sonnenschein (2006) show that by linking collective decisions the incentive costs can become negligible and, at the limit, ex-ante efficiency can be achieved. In a voting situation this implies that the agents' intensity of preferences can be taken into account even in the absence of monetary transfers. Rather than considering a limiting result we want to analyse what can be achieved while we consider a *finite* number of linked decisions. We first characterise the set of implementable mechanisms and show that ex-ante efficiency can never be achieved. We then proceed to relax the efficiency requirement and prove that, even when we just require unanimity, the mechanism cannot be sensitive to the agents' intensity of preference.

JEL Classification: C72, D70, D80

Keywords: Linking Decisions, Mechanism Design, Multidimensional Screening, Strategy-Proofness, Intensity Problem, Separable Preferences

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1 Introduction

Jackson and Sonnenschein (JS 2006) show that the linking of collective decisions leads to large efficiency gains: in the limit, incentive costs can be totally overcome and the ex-ante efficient allocation can be achieved. Their paper is inspired by the *storable votes* mechanism proposed by Casella (2005) where voters, who interact repeatedly over time, are allowed to trade-off their interests across time by *storing* unused votes for future decisions.

Both these papers show that if a number of public decisions are linked the resulting allocation is superior to that achieved when the decisions are made separately. Using a mechanism that links decisions improves the overall allocation by allowing the intensities of agents' preferences to affect the outcomes of individual decisions. However, the design (JS 2006) or welfare properties (Casella 2005) of these mechanisms rely heavily on the prior distribution of preferences. In this note we extend their line of reasoning to examine whether mechanisms exist that not only take into account the intensity of preferences when transfers are forbidden, but are also robust to any specification on the priors.[1] Section 3 provides an affirmative answer to this enquiry. However, any incentive compatible mechanism can only depend on the agents' relative valuation across decisions (i.e. it will bunch all proportional types). This implies that the ex-ante efficient outcome can never be reached as long as the number of linked decisions is finite. It also implies that a *taxation principle* can be stated for this environment: any mechanism can be replicated by a *point-voting mechanism* where agents are endowed with a perfectly divisible point that can be allocated freely between the linked public decisions.[2]

Since the limiting ex ante efficiency result of JS (2006) does not hold when the number of decisions is finite, in Section 4 we consider whether a weaker form of efficiency is achievable. We show that no mechanism that is sensitive to the intensity of preferences can satisfy even the (much weaker) *unanimity* property for a finite number of decisions.[3] We finish this note by relating our work to the existing literature in Section 5.

Our analysis has a wide variety of applications. Examples of public decisions where transfers are simply not viable include the allocation of resources within a government department, and the regulation of mergers. Perhaps more significant, however, are situations such as policy referenda, or the allocation of pupils to schools, where transfers may be feasible but are ruled out on moral grounds.

[1] We consider *robustness* in the sense of Bergemann and Morris (2005). In our setting this is equivalent to strategy-proofness (i.e. implementation in dominant strategies).

[2] In a previous paper (Hortala-Vallve 2006), we have shown the properties of a mechanism that endows agents with a fixed number of votes that can be distributed among a predetermined number of issues. When there is a majority of votes towards the approval (dismissal) of any issue, the issue is approved (dismissed).

[3] Unanimity requires an alternative to be implemented with certainty if all agents wish so.

2 The model

In this section we focus on an example of voting over two alternatives based on Example 1 in JS (2006). There are n agents facing a decision problem $\mathcal{D}$ over the set of alternatives $D = \{a, b\}$; u_i is is the difference in agent i's utility between the two outcomes $u_i = u_i(b) - u_i(a) \in O \subset \mathbb{R}$; the joint distribution from which these differences are drawn is common knowledge. A *social choice function* f is a mapping from utility differences $(u_1, ..., u_n)$ to the probability of implementing alternative b (alternative a is implemented with the complementary probability).

Definition 1 *(JS 2006) A social choice function f on a decision problem $\mathcal{D}$ is* ***ex ante Pareto efficient*** *if there does not exist any social choice function f' on $\mathcal{D}$ such that $E_{(u_1,...,u_n)}\{u_i \cdot f'(u_1, ..., u_n)\} \geq E_{(u_1,...,u_n)}\{u_i \cdot f(u_1, ..., u_n)\}$ for all i with strict inequality for some i.*

A much weaker efficiency requirement than ex-ante Pareto efficiency is the *unanimity* property. Unanimity is a necessary condition for ex-post efficiency which is, in turn, a necessary condition for ex-ante efficiency.

Definition 2 *A social choice function f on a decision problem $\mathcal{D}$ satisfies* ***Unanimity*** *whenever it implements the alternative preferred by all agents (whenever such alternative exists). That is $f(u_1, ..., u_n) = 1$ if $u_i > 0$ for all i and $f(u_1, ..., u_n) = 0$ if $u_i < 0$ for all i.*

We consider a series of K linked decision problems $\mathcal{D}$. Given the Revelation principle, we restrict our attention to direct revelation mechanisms g that map the revealed preferences over the K decision problems into the set of probability distributions on D^K (i.e. $\Delta\left(D^K\right) = [0,1]^K$). Hereafter, $u^k = \left(u_1^k, ..., u_n^k\right)$ denotes the agents' preferences on decision problem k, $u_i = (u_i^1, ..., u_i^K)$ denotes agent i's preferences over the K decision problems, and u denotes the profile of all agents preferences over all decision problems. $g^k(u)$ denotes the marginal distribution under g on the k-th decision problem when agents declare u. The utility over the linked decisions is simply the sum of utilities over each decision. Finally, a strategy for agent i is a mapping from his own preferences to his revealed ones.

3 Robust linking mechanisms

The mechanism proposed by JS (2006) relies heavily on knowing the prior distribution from which the utility profiles are drawn. The same occurs with the equilibria and optimality properties of the alternative mechanisms to Majority Rule that Casella (2005) and Hortala-Vallve (2006) propose.

Majority Rule is a mechanism that, in the present setting with K linked decision problems, is *robust* to any specification of the priors (i.e. it is always optimal to truthfully reveal the type on each decision problem), but fails to take into account the intensity of agents' preferences.[4] It remains to be shown whether there are mechanisms that satisfy both properties; this is our aim in this Section.

Bergemann and Morris (2005) show that requiring a mechanism to be *robust* to any specification of the priors (interim implementation for all possible type spaces) in private value environments is equivalent to *ex-post implementation* and is also equivalent to *dominant strategy implementation* or *strategy-proofness.* A mechanism is strategy-proof whenever it is a dominant strategy for any agent to reveal his preferences truthfully. That is,

$$u_i \in \arg\max_{\hat{u}_i \in O^K} \sum\nolimits_{k=1}^{K} u_i^k \cdot g^k\left(\hat{u}_i, u_{-i}\right) \text{ for all } i = 1, .., n \text{ and all } u_i \in \mathbb{R}^K.$$

We define the indirect utility of agent i in equilibrium as $U\left(u_i, u_{-i}\right) = \sum\nolimits_{k=1}^{K} u_i^k \cdot g^k\left(u_i, u_{-i}\right).$ Rochet (1985) shows that we can extend the usual one-dimensional screening techniques, due to Mirrlees (1971), to our environment and characterise all implementable mechanisms in the following way.

Lemma 1 *(Rochet 1985) The linking mechanism $g : O^{Kn} \to [0,1]^K$ is strategy-proof if and only if*

1. $\nabla_{u^i} U\left(u_i, u_{-i}\right) = g\left(u_i, u_{-i}\right)$ *for almost all* $\left(u_i, u_{-i}\right) \in O^{Kn}$
2. U *is convex on* $u_i \in O^K$ *for almost all* $\left(u_i, u_{-i}\right) \in O^{Kn}$

where $\nabla_{u^i} U\left(u_i, u_{-i}\right) := \left(\frac{\partial U\left(u_i, u_{-i}\right)}{\partial u_i^1}, ..., \frac{\partial U\left(u_i, u_{-i}\right)}{\partial u_i^K}\right).$

The necessity of both conditions stems from applying the envelope theorem and using the first and second order conditions that arise from the truthtelling condition. The sufficiency is immediate given the multilinear specification of the agents' utilities.

[4] See Dasgupta and Maskin (2003) for a further defense of the robustness of MR in a standard Social Choice framework.

Condition (1) together with the definition of U imply that any strategy-proof mechanism should satisfy the following linear first-order partial differential equation for almost all preference profiles

$$u^i \cdot \nabla_{u^i} U^i(\theta) = U^i(\theta).$$

An extended version of Euler's Theorem (see Osborne 2006), implies that the former equality is satisfied if and only if U is homogeneous of degree one on $u_i \in O^K$ (HD1, i.e. $U(\lambda u_i, u_{-i}) = \lambda U(u_i, u_{-i}), \lambda \in \mathbb{R}_+$).[5] We can thus characterise all strategy-proof mechanisms in a very simple way.

Proposition 1 *The linking mechanism $g : O^{Kn} \to [0,1]^K$ is strategy-proof if and only if the agents' indirect utilities are HD1 and convex on their own preferences. That is, U is HD1 and convex on u_i for almost all $(u_i, u_{-i}) \in O^{Kn}$.*

This result implies that g_k, the marginal distribution on decision problem k, is homogeneous of degree zero. In other words, the mechanism bunches all proportional types. This fact is not surprising but characterising all implementable allocation under the convexity and homogeneity conditions is, to our knowledge, new to the literature.[6]

Our result is stronger than the standard result that agents' incentives in any game are not changed if the von Neumann Morgenstern utilities are multiplied by a constant. Strategy-proofness implies that the expected utility of declarations u_i and λu_i ($\lambda > 0$) coincide. However, this is weaker than the requirement that the mechanism remain unchanged on *every* dimension when the agent's declaration is multiplied by a positive scalar.[7] Moreover, our result is not solely a necessary condition for strategy-proofness but, together with the convexity condition, it is a *full characterisation of all strategy-proof mechanisms.*

JS (2006) states that by linking more decision problems "the utility costs associated with incentive constraints become negligible". This result can be reviewed at the light of Proposition 1. The homogeneity result on g_k implies that the message space of any strategy-proof mechanism has one dimension less than the preference space. Hence, as we link more decision problems together the homogeneity condition tends to bite less and we can get arbitrarily close to the first best.. Proposition 1 also implies that, as long as we consider a finite number of linked problems, the

[5] Euler's Theorem can only be applied where the function is differentiable. Nevertheless, given that U is convex , we know that it is continuous and differentiable almost everywhere thus, by continuity we can extend the homogeneity result to those points where the function may not be differentiable (I am indebted to Steven Matthews for bringing this fact to my attention).

[6] Proposition 1 can be applied to any multidimensional situation without transfers. For instance, in a Bayesian Nash setting, the convexity and homogeneity conditions need simply be imposed on the interim utilities.

[7] It could be the case that the allocation (i.e. the K-dimensional vector of probabilities) changes when we multiply the declaration of player i by a positive scalar while keeping his expected payoff constant. This may ease the truthtelling constraints of a different player and could affect the welfare properties of the mechanism.

ex-ante Pareto efficient allocation can never be achieved. Such an allocation would require us to compare the valuations of all the agents in each decision problem, which is unfeasible given the homogeneity result.

Finally, Proposition 1 implies that any agent is indifferent between declaring his own preferences or declaring his own preferences normalised by, say, the L_1 norm (i.e. such that the absolute value of its components sum to 1). As a consequence, a Taxation Principle can be stated in our environment allowing us to replicate any direct mechanism with a very simple indirect one.

Corollary 1 ***(Taxation Principle)*** *Any strategy-proof mechanism of K linked decisions problems can be replicated by a mechanism which endows agents with a perfectly divisible point that can be distributed among the decision problems.*

We could rephrase the Corollary as follows: any multidimensional screening problem without transferable utility can be replicated by a mechanism with *money.* The role of the numeraire is simply to smooth transactions and allow the mechanism to compare the preferences of individual agents across decision problems.[8]

Finally note that there are plenty of mechanisms that satisfy the conditions in Proposition 1 and still take into account the agents' intensity of preferences. Consider for instance the mechanism that normalises the declared preferences of every agent by its euclidean norm and adds these values for all agents:[9]

$$g^k(u_1,...,u_n) = \frac{1}{2n} \cdot \left(\frac{u_1^k}{\sqrt{\left(u_1^1\right)^2 + ... + \left(u_1^k\right)^2}} + ... + \frac{u_n^k}{\sqrt{\left(u_n^1\right)^2 + ... + \left(u_n^k\right)^2}} + n \right), k = 1...K.$$

4 An impossibility result on linking decisions

We have seen that it is not possible to implement the ex-ante Pareto efficient social choice function when K is finite. It remains to be shown whether a weaker form of efficiency can be achieved in the finite setting.

In this Section we will show that even using the considerably weaker condition of Unanimity, the linking mechanism cannot do any better than a mechanism that deals with each decision problem

[8] Similar reasoning is developped in Abdulkadiroglu (2004).

[9] We can extend the set of strategy-proof mechanisms that are sensitive to the agents' intensity of preferences as follows:

$$g^k(u_1,...,u_n) = \frac{1}{2n} \cdot \left(sign(u_1^k) \cdot \left(\frac{\left|u_1^k\right|}{\left\|u_1\right\|_p} \right)^{p-1} + ... + sign(u_n^k) \cdot \left(\frac{\left|u_n^k\right|}{\left\|u_n\right\|_p} \right)^{p-1} + n \right), p > 1.$$

separately. That is, any strategy-proof and unanimous mechanism cannot be *sensitive* to the agents' intensity of preferences.

Definition 3 *The linking mechanism $g : O^{Kn} \rightarrow [0,1]^K$ is **sensitive** whenever there exists two preference profiles with identical sign in all decision problems that implement different allocations. That is, $\exists (u_1, ..., u_n), (v_1, ..., v_n) \in O^{Kn}$ such that $sign(u_1, ..., u_n) = sign(v_1, ..., v_n)$ and $g(u_1, ..., u_n) \neq g(v_1, ..., v_n)$.*[10]

The impossibility of implementing sensitive and unanimous mechanisms in the case with two agents is proved by induction on the number of linked decision problems. The first step in the inductive proof is given by Proposition 1: intensity of preferences can play no role when we consider a single decision problem. We then show that the unanimity property implies that any mechanism inherits the properties of mechanisms dealing with fewer decisions problems. This is proven by constructing an iterative process during which the preferences of each agent are modified so that by the end there is a unanimous will in a decision problem. Under this scenario, the unanimous will is implemented –and the inductive hypothesis implies that the mechanism cannot be strategy-proof, unanimous and sensitive. Finally, we show that in order to preserve strategy-proofness in every step of our iterative process, the mechanism can never be sensitive.

The Theorem itself is then proved by induction on the number of agents. We define a linking mechanism for $n-1$ agents from the one with n agents as follows: $\tilde{g}(u_2, u_3, ..., u_n) = g(u_2, u_2, u_3, ..., u_n)$. The proof relies on showing that the mechanism with n agents inherits the unanimous, sensitive and strategy-proof properties of the linking mechanism with $n-1$ agents. Therefore, the inductive hypothesis together with the impossibility of implementing sensitive and unanimous mechanisms in the case with two agents proves the Theorem that is stated below.

Theorem 1 *(Impossibility) There exists no mechanism for K linked decisions problems that is strategy-proof, unanimous and sensitive.*

Needless to say, there are various ways of avoiding our impossibility result. On the one hand, we can consider situations where unanimous wills do not exist. On the other hand, we can relax the implementability concept to Bayesian Nash (see Casella 2005 and Hortala-Vallve 2006). Both paths lead to the existence of sensitive mechanisms.

[10] The operator *sign* should be interpreted as a vector of minus ones, zeros and ones according to the sign of each coordinate.

5 Related literature

Most of the literature on mechanism design assumes the availability of transfers. In the multidimensional case, Rochet and Chone (1998), using *calculus of variations with an inequality constraint* techniques, establish that bunching is robust even when the distribution of types is *very* regular. Proposition 1 extends this bunching result to the case where monetary transfers are not allowed.

Within the literature on mechanism design without transfers, the Gibbard-Satterthwhaite (G-S) Theorem states that in an election with three or more outcomes and where we assume agents' preferences have universal domain, the only strategy-proof and onto mechanisms are dictatorial. Gibbard (1977) shows further that the G-S Theorem still holds when we allow lotteries across outcomes: any strategy-proof mechanism is dictatorial or is restricted to implement a fixed pair of alternatives. We depart from this work by allowing the mechanism to use cardinal information on the agents' preferences, and by not requiring the allocation to belong to a $(K-1)$-dimensional simplex. In this way we avoid the impossibility result and provide a set of mechanisms that take into account the agents' intensity of preferences.[11]

We have seen that requiring the mechanism to satisfy the unanimity property implies that the agents' intensity of preferences can no longer play a role. A parallel result in Hylland (1980) shows the G-S Theorem holds whenever the unanimity property is also imposed.[12] Once again, our work differs in not requiring the final allocation to belong to a $(K-1)$-dimensional simplex. In other words, our mechanism implements the probabilities of approving K independent decisions (i.e. an element in $[0,1]^K$) rather than the probability of electing 1 representative out of K candidates (i.e. an element in Δ^{K-1})

Finally, it is worth noting that interest in the intensity problem when transfers are forbidden has been growing in various fields. Eliaz, Ray and Razin (2006) analyse a situation where, depending on their relative aversion towards disagreement, , voters may choose to abstain; Borgers and Postl (2006) show that no efficient mechanism exists for a setting where two agents have to elect a representative out of three; and Abdulkadiroglu (2006) provides an improved mechanism for the allocation of indivisible goods where intensity of preferences can be elicited and the achieved allocation is at least as good as the *random serial dictatorship* one.

[11] Our work also shows that whenever we allow the use of *cardinal* preferences and randomisation across outcomes the results on *voting by committees* no longer hold. Barbera *et al* (1991) shows that in a multidimensional setting where agents have separable preferences, only mechanisms where each agent announces a subset of 'chosen' alternatives can be implemented. LeBreton and Sen (1999) extends this result and show that such mechanisms need to be decomposable in the sense that the allocation on each dimension depends only on the message sent on that dimension.

[12] Recent papers have provided alternative and simpler proofs of the Hylland Theorem (see Dutta *et al* (2005) and references therein). A related impossibility result in situations without transfers is the impossibility result on allocating indivisible goods by Zhou (1990).

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6 Appendix (Impossibility Theorem proof)

Definition 4 *The linking mechanism* $g : O^{Kn} \to [0,1]^K$ *is* ***anonymous*** *if it is insensitive to the labelling of agents. That is,* $g(u_1, ..., u_n) = g\left(u_{\sigma(1)}, ..., u_{\sigma(n)}\right)$*, where* $\sigma(\cdot)$ *denotes any permutation of* n *elements.*

Definition 5 *The linking mechanism* $g : O^{Kn} \to [0,1]^K$ *is* ***neutral with respect to decision problem k*** *(*$k = 1, ..., K$*) if it does not depend on the positive or negative labelling of decision problem* k*. That is,*

$$\begin{cases} g^k(u_1, ..., u_n) = 1 - g^k(\tilde{u}_1, ..., \tilde{u}_n) \\ g^l(u_1, ..., u_n) = g^l(\tilde{u}_1, ..., \tilde{u}_n) \text{ for } l \neq k \end{cases}$$

where $\tilde{u}_i$ *is equal to* u_i *except that all preferences in problem* k *are multiplied by minus one.*

Lemma 2 *We can assume, without loss of generality, that any strategy-proof, unanimous and sensitive mechanism is anonymous and neutral with respect to decision problem* k *for any* $k = 1, ..., K$*.*

Proof. *Consider first a strategy-proof, unanimous and sensitive linking mechanism that is not anonymous with respect to agents* 1 *and* 2 *and define the new linking mechanism* $\tilde{g}(u_1, ..., u_n) = (g(u_1, u_2, ..., u_n) + g(u_2, u_1, ..., u_n))\frac{1}{2}$*. It is immediate to prove that, given the properties of* g*, the new linking mechanism* $\tilde{g}$ *is strategy-proof, unanimous, sensitive and anonymous with respect to agents* 1 *and* 2*.*

Second, consider a strategy-proof, unanimous and sensitive linking mechanism that is not neutral with respect to decision problem 1 *and define a new linking mechanism as follows:*

$$\begin{cases} \tilde{g}^1(u_1, ..., u_n) = \frac{1}{2}\left(1 + g^1(u_1, ..., u_n) - g^1(\tilde{u}_1, ..., \tilde{u}_n)\right) \\ \tilde{g}^k(u_1, ..., u_n) = \frac{1}{2}\left(g^k(u_1, ..., u_n) + g^k(\tilde{u}_1, ..., \tilde{u}_n)\right) \text{ for } k \neq 1 \end{cases}$$

It is immediate to prove that, given the properties of g*, the new linking mechanism* $\tilde{g}$ *is strategy-proof, unanimous, sensitive and neutral with respect to decision problem* 1*.* ■

Lemma 3 *If the linking mechanism* $g : O^{Kn} \to [0,1]^K$ *is sensitive, there exists* $i \in \{1, ..., n\}$ *and* $k \in \{1, ..., K\}$ *such that the mechanism is sensitive to agent* i*'s intensity over decision problem* k*.*

Proof. Since g is sensitive there exists two profiles $(u_1, ..., u_n)$ and $(v_1, ..., v_n)$ with the same sign that reach different allocations. Now consider the iterative process in which we switch the preference of one agent on a decision problem from u_i^k to v_i^k. At the beginning of the process the mechanism implements $g(u_1, ..., u_n)$ and at the end it implements a different allocation, $g(v_1, ..., v_n)$. Therefore, in at least one step of the process the mechanism must vary its allocation. That is, whenever one agent varies the intensity of his preference in one decision problem, the mechanism implements a different allocation. ■

Lemma 4 *Whenever* $n = 2$*, there exists no mechanism of* K *linked decisions problems that is strategy-proof, unanimous and sensitive.*

Proof. We prove this result by induction on the number of linked decision problems. Strategy-proofness (Proposition 1) implies that the mechanism cannot be sensitive when $K = 1$. Now, assume the claim is true for $K-1$ decisions problems and that it is *not* true for K decisions problems. That is, there exists a strategy-proof, unanimous and sensitive K-linking mechanism. Relabelling the decision problems if necessary and taking into account Lemmas 2 and 3 we have that there exists, $u_1, u_2 >> 0$ and $a > b > 0$ such that:[13]

$$g^1\left(\left(a, u_1^2, ..., u_1^K\right), u_2\right) > g^1\left(\left(b, u_1^2, ..., u_1^K\right), u_2\right) \tag{1}$$

Whenever agent 1 declares a negative preference in problem 2 $\left(-u_1^2\right)$ the mechanism can no longer be sensitive. This is because there are unanimous wills in decision problem 2 hence the mechanism deals with $K-1$ decision problems –to which the inductive hypothesis applies. Therefore, $g^1\left(\left(x, -u_1^2, ..., u_1^K\right), u_2\right) = \bar{g}_1$ for all $x > 0$. Strategy-proofness implies that when x is large enough, agent 1 should not have incentives to declare $\left(a, u_1^2, ..., u_1^K\right)$, therefore $\bar{g}^1 \geq g^1\left(\left(a, u_1^2, ..., u_1^K\right), u_2\right)$.

Neutrality with respect to decision problem 2 implies $g^1\left(\left(b, u_1^2, ..., u_1^K\right), \left(u_2^1, -u_2^2, ..., u_2^K\right)\right) = \bar{g}^1$ for any $u_2^1 < 0$. In particular, for large enough $\left|u_2^1\right|$ we have that the above reasoning applies again and $\bar{g}^1 \leq g^1\left(\left(b, u_1^2, ..., u_1^K\right), u_2\right)$ needs to hold. Both weak inequalities are incompatible with (1). ■

Lemma 5 *If the linking mechanism* $g : O^{Kn} \to [0,1]^K$ *is not sensitive, the allocation in each decision problem can only depend on the agents' valuations on that decision problem.*

Proof. Non-sensitiveness implies that the same allocation is implemented given the profiles u_i or $u_i(\lambda) := \left(\lambda u_i^1, \frac{u_i^2}{\lambda}, ..., \frac{u_i^K}{\lambda}\right)$, for all $\lambda > 0$ and $u_i \in O^K$. If the allocation in a decision problem depends on the valuation over various decision problems, we have that there exists a large enough $\lambda > 0$ for which agent i has incentives to manipulate his declaration. ■

Note that Lemma 5 implies that whenever the mechanism is anonymous and not sensitive, the allocation in each decision problem can only depend on the number of agents that value that decision problem positively and the number of agents that value it negatively. In other words, anonymity and non-sensitivity together imply that each decision problem is evaluated independently.

Theorem *There exists no mechanism of* K *linked decisions problems that is strategy-proof, unanimous and sensitive.*

Proof. We prove the result by induction on the number of agents. Lemma 4 implies that the mechanism cannot be sensitive when $n = 2$. Assume the claim is true for $n-1$ agents and that it is *not* true for n agents. That is, there exists a strategy-proof, unanimous and sensitive n-agent mechanism.

We now define a linking mechanism for $n-1$ agents from the one with n agents:[14]

$$\tilde{g}\left(u_2, u_3, ..., u_n\right) = g\left(u_2, u_2, u_3, ..., u_n\right).$$

[13] $u_i >> 0$ should be read as $u_i^k \geq 0$ for all $k = 1, ..., K$.

[14] We borrow this *trick* from the Hylland Theorem' proof in Dutta *et al* (2005).

STEP 1: *$\tilde{g}$ is strategy-proof and unanimous*

It is immediate that $\tilde{g}$ inherits the unanimity property from g. Strategy-proofness for agents 3 to n also follows immediately from the strategy-proofness of g. We just need to prove that agent 2 has no incentives to deviate. That is, the following inequality should hold

$$\sum_{k=1}^{K} \tilde{g}^k (u_2, u_3, ..., u_n) \cdot u_2^k \geq \sum_{k=1}^{K} \tilde{g}^k (u_1, u_3, ..., u_n) \cdot u_2^k, \text{ for all } (u_1, ..., u_n) \in O^{Kn}.$$

Strategy-proofness and anonymity of g implies that the following chain of inequalities holds.

$$\sum_{k=1}^{K} g^k (u_2, u_2, u_3, ..., u_n) \cdot u_2^k \geq \sum_{k=1}^{K} g^k (u_2, u_1, u_3, ..., u_n) \cdot u_2^k =$$
$$= \sum_{k=1}^{K} g^k (u_1, u_2, u_3, ..., u_n) \cdot u_2^k \geq \sum_{k=1}^{K} g^k (u_1, u_1, u_3, ..., u_n) \cdot u_2^k \ \blacksquare$$

STEP 2: *g cannot be sensitive when $u_i = u_j$, for $i \neq j$.*

Step 1 implies that $\tilde{g}$ is a strategy-proof and unanimous mechanism for $n-1$ agents. Hence, the inductive hypothesis applies and $\tilde{g}$ cannot be sensitive. Now consider a preference profile where two agents value all decision problems equally. Lemma 2 allows us to assume, without loss of generality, that these agents are the first two. First note that g cannot be sensitive to the preferences of any agents other than 1 or 2 (otherwise $\tilde{g}$ would be sensitive). Second, assume that the mechanism is sensitive with respect to agent 1's preferences on decision problem 1:

$$g^1 \left(\left(u_1^1, u_1^2 ..., u_1^K \right), u_1, u_3, ..., u_n \right) > g^1 \left(\left(\tilde{u}_1^1, u_1^2 ..., u_1^K \right), u_1, u_3, ..., u_n \right) \text{ where } u_1^1 > \tilde{u}_1^1 > 0. \quad (2)$$

Strategy-proofness implies that g^1 cannot increase when agent 2's valuation in decision problem 1 decreases:

$$g^1 \left(\left(\tilde{u}_1^1, u_1^2 ..., u_1^K \right), u_1, u_3, ..., u_n \right) \geq g^1 \left(\left(\tilde{u}_1^1, u_1^2 ..., u_1^K \right), \left(\tilde{u}_1^1, u_1^2 ..., u_1^K \right), u_3, ..., u_n \right). \quad (3)$$

Finally, note that the LHS of (2) and the RHS of (3) need to coincide given that $\tilde{g}$ is not sensitive. Therefore, g cannot be sensitive whenever two agents value all decision-problems equally. $\blacksquare$

STEP 3: *g cannot be sensitive with respect to any decision-problem where there are at least two agents with positive preferences and two agents with negative preferences.*

Assume agents 1 and 2 value decision problem 1 positively, and agents 3 and 4 value it negatively. Moreover, assume that the mechanism is sensitive to agent 1's preferences on decision-problem 1.

$$g^1(\underbrace{(a, u_1^2 ..., u_1^K)}_{u_a}, ...) > g^1(\underbrace{(b, u_1^2 ..., u_1^K)}_{u_b}, ...) \text{ where } a > b > 0. \quad (4)$$

Step 2 implies that g cannot be sensitive under profiles $(\mathbf{u}_a, \mathbf{u}_a, u_3, u_4, ...)$ and $(u_b, u_2, \mathbf{u}_3, \mathbf{u}_3, ...)$. Under both preference profiles there are the same number of agents with positive and negative valuations on decision problem 1. Thus, by Lemma 5, $g^1 (u_a, u_a, u_3, u_4, ...) = g^1 (u_b, u_2, u_3, u_3, ...) =$

$\bar{g}$.

Consider now the profile $(u_a, u_a, u_3, u_4, ...)$ and assume that agent 2's preference on decision problem 1 increases by an arbitrary amount. Strategy-proofness implies that $\bar{g} \geq g^1(u_a, u_2, u_3, u_4, ...)$. Under profile $(u_b, u_2, u_3, u_3, ...)$, strategy-proofness implies that when agent 4's preference on decision problem 1 decreases arbitrarily, $\bar{g} \leq g^1(u_b, u_2, u_3, u_4, ...)$. Both weak inequalities are incompatible with (4). ■

STEP 4: *g cannot be sensitive with respect to any decision problem*

Strategy-proofness implies that a sensitive linking mechanism varies its allocation in at least two decision problems in response to a variation in the intensity of an agent's preferences. Given Step 3, we know that in these two decision problems the minority views are only represented by one agent. There are three possible scenarios: (1) there is an agent that holds minority views on both decision problems; (2) the minority views are always represented by a different agent and there are more than three agents; and (3) the minority views are always represented by a different agent and there are only three agents.

Scenario 1: Agent 1 holds minority (and positive) views on decision problems 1 and 2. u and $\tilde{u}$ are two preference profiles with equal sign such that $g^1(u) \neq g^1(\tilde{u})$. An analogous argument to the one used in the proof of Step 3 implies that $g^1(u)$ and $g^1(\tilde{u})$ are bounded below by $g^1(u_1, u_2, u_2..., u_n)$. Whenever agent 1 has a negative valuation on decision problem 2, g^2 needs to be equal to zero (unanimity property) and an inductive argument analagous to the one used in the proof of Lemma 4 implies that the mechanism can no longer be sensitive. Therefore, $g^1(u)$ and $g^1(\tilde{u})$ are bounded above by $g^1(\left(u_1^1, -u_1^2, u_1^3, ..., u_1^K\right), u_2, ..., u_n)$. Finally note that Lemma 5 implies that the two bounds coincide, so $g^1(u) = g^1(\tilde{u})$.

Scenario 2: Whenever agent 1 ($u_1 >> 0$) only holds minority views on decision problem 1 we can replicate the proof from scenario 1. We just need to highlight that when agent 1 has a negative valuation on decision problem 2, there are at least two agents with positive preferences and two agents with negative ones in decision problem 2 (this is true provided there are strictly more than three agents). Therefore, Step 3 and an inductive argument similar to that used in Lemma 4 imply that the mechanism is not sensitive.

Scenario 3: We finally need to prove that the mechanism cannot be sensitive when there are three agents and none holds minority views in more than one decision problem. Given the existence of only three agents, the mechanism cannot be sensitive with respect to more than three decision problems at the same time. Otherwise, there would exist an agent that holds minority views on more than one decision problem (scenario 1).

Whenever the mechanism is sensitive with respect to three decision problems the proof is analogous to those given above. We just need to realise that when an agent switches the sign of his preference on a decision problem where he is in majority, the mechanism can no longer be sensitive because there is an agent that holds minority views in more than one decision problem (scenario 1).

We now prove the case where the mechanism is sensitive with respect to two decisions problems. Assume agents 1 and 2 hold minority (and positive) views on decision problems 1 and 2 respectively. When the the linking mechanism is sensitive we can assume, without loss of generality, $g^k(a, b) := g^k((a, -1), (-1, b), (-1, -1)) > g^k((1, -1), (-1, 1), (-1, -1)) = \bar{g}^k$ for some $a, b > 1$ $(k = 1, 2)$.

Note that $g^k(a,b)$ is weakly increasing in both arguments (Proposition 1) and takes values on the compact set $[0,1]$ therefore $\lim_{a\to\infty} g^1(a,a) = g^\infty$ is well defined.

For large enough a, we have that agent 1's positive valuation on decision problem 2 should have only a small effect on the allocation in the first decision problem (otherwise he would have incentives to misreport his true valuation). That is,

$$\forall\delta > 0, \exists\bar{a} > 0 : \forall a > \bar{a}, \left|g^1\left((a, +\mathbf{1}), (-1, a), (-1, -1)\right) - g^\infty\right| \leq \delta.$$

At the same time, strategy-proofness for agent 2 implies that the allocation in decision problem 2 should be arbitrarily close to the allocation that is insensitive to the intensity of agents' preferences when minority views are negative (which we denote $1 - \hat{g}^2$). If this were not true he would have incentives to misrepresent his true sign on decision problem 1. That is,

$$\forall\delta > 0, \exists\tilde{a} > 0 : \forall a > \tilde{a}, \left|g^2\left((a, +1), (-1, a), (-1, -1)\right) - \left(1 - \hat{g}^2\right)\right| \leq \delta.$$

Note that the allocation in decision problem 2 is arbitrarily close to the highest possible allocation in the absence of unanimous wills. Therefore, strategy-proofness implies that this allocation is almost insensitive to the preference intensities of agent 1. That is,

$$\forall\delta > 0, \exists\hat{a} > 0 : \forall a > \hat{a} \text{ and } \forall\lambda < a, \left\|g\left((\lambda, 1), (-1, a), (-1, -1)\right) - \left(g^\infty, \left(1 - \hat{g}^2\right)\right)\right\| \leq \delta.$$

Whenever $\lambda = 1$ and agent 2 holds preferences $(-1, 1)$, neutrality with respect to decision problem 2 (Lemma 2) and anonymity imply that

$$g^1\left((1, 1), (-1, 1), (-1, -1)\right) = g^1\left((1, -1), (-1, -1), (-1, 1)\right) = \bar{g}^1 (< g^\infty). \tag{5}$$

Now consider the LHS in (5) and reduce agent 1's preference in decision problem 1. Strategy-proofness implies that $g^1\left((\lambda, 1), (-1, 1), (-1, -1)\right) \leq \bar{g}^1$, for any $\lambda < 1$. Moreover, with λ arbitrarily small, strategy-proofness for agent 1 implies that the allocation on decision problem 2 should be arbitrarily close to $\left(1 - \hat{g}^2\right)$.

The final step of the proof is to note that under the preference profile $\left((\lambda, 1), (-1, a), (-1, -1)\right)$, agent 2 has incentives to lie and declare $(-1, 1)$ so that he improves the allocation on decision problem 1 at *negligible* cost for decision problem 2. ■