

INDEPENDENCE OF CM POINTS IN ELLIPTIC CURVES

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ABSTRACT. We prove a result which describes, for each $n \geq 1$, all linear dependencies among n images in elliptic curves of special points in modular or Shimura curves under parameterizations (or correspondences). Our result unifies and improves in certain aspects previous work of Rosen-Silverman-Kühne and Buium-Poonen.

1. INTRODUCTION AND MAIN RESULTS

Let Y be a modular (or Shimura) curve, E an elliptic curve over \mathbb{C} and $V \subset Y \times E$ an irreducible correspondence. If $(s, x) \in V$ we will call x a V -image of s . We prove a result describing, for each $n \geq 1$, all linear dependencies in E among the V -images of n special points in Y .

An example of particular interest is when V is the graph of a modular parameterization $\phi : Y \rightarrow E$ and then the V -images of special points are known as *CM points* or *Heegner points* (though the latter term is usually taken to have some further assumptions). A number of results in the literature establish linear independence of CM points under suitable hypotheses. After framing our result we compare it with previous results.

Definition. With notation as above, and $n \geq 1$, let π_{Y^n}, π_{E^n} be the projections of $Y^n \times E^n$ onto the first and second factors, respectively.

(i) A *special graph* in V^n is a component $W \subset V^n \cap (S \times B)$, where $S \subset Y^n$ and $B \subset E^n$ are special subvarieties, such that $\pi_{Y^n}(W) = S$, $\pi_{E^n}(W) \subset B$.

(ii) A special graph W in V^n is called *dependent* if B (may be taken such that it) is a proper special subvariety.

(iii) A special graph W in V^n is called *exemplary* if, setting B to be the smallest special subvariety of E^n with $\pi_{E^n}(W) \subset B$, there is no special graph W' strictly larger than W with $\pi_{E^n}(W') \subset B$.

In particular when V is the graph of a parameterization $\phi : Y \rightarrow E$, a special graph is simply the graph of the restriction of ϕ to a special subvariety $S \subset Y^n$. The special subvarieties in E^n are the cosets of abelian subvarieties by torsion points (“torsion cosets”); the special subvarieties of Y^n are described e.g. in [24].

Let $x_1, \dots, x_n \in E$ be V -images of special points $s_1, \dots, s_n \in Y$. Write $s = (s_1, \dots, s_n) \in Y^n, x = (x_1, \dots, x_n) \in E^n$. If $(s, x) \in W$ for some dependent special graph in V^n , then the points $x_1, \dots, x_n \in E$ are linearly dependent in E . Note that, for us, *linear dependence in E* is always taken

to be over $\text{End}(E)$. We have that $\text{End}(E) = \mathbb{Z}$ unless E has CM (complex multiplication), in which case $\text{End}(E)$ is an order in an imaginary quadratic field.

Conversely, if x_1, \dots, x_n are linearly dependent in E then (s, x) is contained in some exemplary dependent special graph. Note that the unique non-dependent exemplary special graph is V^n itself as a component of $V^n \cap (Y^n \times E^n)$.

The following theorem thus gives a description of every linear dependence among V -images of n special points.

Theorem 1.1. *Given $V \subset Y \times E$ as above with Y a modular curve or a Shimura curve and $n \geq 1$, there are only finitely many exemplary special graphs in V^n .*

Example. It is well known that $X_0(11)$ has the structure of an elliptic curve, so we may set $E = \text{Pic}^0(X_0(11))$. Consider the Atkin-Lehner involution w . Now on E , w is a non-trivial automorphism, so its graph must be an abelian subvariety. Set $\phi : X_0(11) \rightarrow E$ to be the identification taking $(\infty) \rightarrow 0$. Then $\phi(w(\infty)) = \phi(0)$ is a torsion point, and thus if we set $S \subset X_0(11)^2$ to be the graph of w , then S is a special curve whose ϕ -graph is exemplary.

A number of results in the literature assert linear independence properties of the V -images of CM points. The fact that only finitely many V -images of special points can be torsion was proved in [20] for modular parameterisations and Heegner points (generalized to certain Shimura curve parameterizations in [14]) and is equivalent to the assertion of Theorem 1.1 for $n = 1$. This also follows from the stronger results in [5], and was reproved as a “special point problem” within the Zilber-Pink conjecture in [24].

We deduce some consequences of Theorem 1.1 and compare with some further results in the literature. For $N \geq 1$ we let $X_N \subset Y \times Y$ be the locus of points (s_1, s_2) such that there is a cyclic isogeny of degree N between the corresponding elliptic curves (when Y is a modular curve) or abelian surfaces (when Y is a Shimura curve).

Definition. Let D be a positive integer. A set of special points $\{s_1, \dots, s_n\}$ in Y is called *D -independent* if, for each i , the discriminant $|\Delta(s_i)| > D$ and, for $i \neq j$, there is no relation $(s_i, s_j) \in X_N$ with $N \leq D$.

Corollary 1.2. *For $n \geq 1$ there exists a positive integer $D = D(Y, E, V, n)$ such that if $\{s_1, \dots, s_n\}$ is D -independent then any V -images x_1, \dots, x_n of s_1, \dots, s_n are linearly independent in E . \square*

Proof. For $s = (s_1, \dots, s_n)$ to have a V -image which is dependent requires s to lie in one of finitely many proper special subvarieties $S_1, \dots, S_k \subset Y^n$, and, for each i , $s \in S_i$ requires either that some coordinate is equal to a fixed special point, or some $(s_i, s_j) \in X_N$ for some $i \neq j$ and fixed N . These are not possible if s is D -independent for sufficiently large D . \square

Corollary 1.2 improves a result of Kühne [15] (which in turn improved a result of Rosen-Silverman [31]) by getting independence even for CM points corresponding to orders in the same CM field, if the orders are “sufficiently far apart” (i.e. if the corresponding singular moduli are modularly independent up to suitable D); the previous results required the CM fields of the s_i to be distinct. The results in [15, 31] also exclude CM elliptic curves E , though see [32], and all these results restrict to modular parameterizations of E/\mathbb{Q} . However, Kühne’s result is effective, whereas our result is not.

Let Σ denote the set of V -images in E of special points of S .

Corollary 1.3. *For $n \geq 1$ there exists a positive integer $N = N(Y, E, V, n)$ such that if $x_1, \dots, x_N \in \Sigma$ are distinct then there is a linearly independent subset of $\{x_i\}$ of size at least n .*

Proof. Given n we can find N such that any set of N distinct V -images of special points contains a subset of size n for which the corresponding special points are $D(Y, E, V, n)$ -independent. (And N is effective given D .) \square

Corollary 1.4. *Let Γ be a finitely generated subgroup of E of rank r . Then $|\Gamma \cap \Sigma| \leq N(Y, E, V, r + 1)$.* \square

This reproves a result of Buium-Poonen (and generalizes to correspondences their result for maps from Shimura curves to elliptic curves) and in a uniform way: the size of the intersection is bounded depending only on the rank of Γ . However we cannot recover their “Bogomolov”-type result.

In §2 we show that Theorem 1.1 is a consequence of the Zilber-Pink conjecture (ZP). The framing of ZP in terms of “optimal subvarieties” (as in [13]) suggests the formulation of Theorem 1.1.

Our proof of Theorem 1.1 goes via point-counting on definable sets in o-minimal structures, and utilizes a suitable Ax-Schanuel theorem, as have been employed in various earlier work to tackle special cases of ZP, and in this respect follows in particular the approach in [26] in studying “CM-points” for the multiplicative group. As there, various issues arise from the fact that we cannot prove the full Zilber-Pink statement for V^n . But unlike in [26], where we showed that no positive dimensional dependent special graphs exist, we must here deal with this possibility, which complicates the point-counting and the application of Ax-Schanuel, in view of our inability to affirm the full ZP. We must show that we are able to restrict throughout to atypical intersections of a specific form.

In effect, we must prove a very strong result of André-Oort type: each proper special subvariety of E^n has a pre-image in Y^n . This gives a countably infinite collection of subvarieties of Y^n which is not contained in any algebraic family. We must show that there are only finitely many special subvarieties of Y^n which are contained and maximal in any one of this countably infinite collection.

In the modular case we show that our results can be extended to include the Hecke orbits of a finite number of points in addition to special points. The *Hecke orbit* of $u \in Y$ is $\{v \in Y : \exists N \text{ with } (u, v) \in X_N\}$.

Definition. Let Y be a modular curve and $U \subset Y$.

- (i) A *U-special point* of Y is a point which is either special or in the Hecke orbit of some $u \in U$.
- (ii) A *U-special point* in Y^n is an n -tuple of U -special points in Y .
- (iii) A *U-special subvariety* of Y^n is a weakly special subvariety which contains a U -special point.

Now we consider again an irreducible correspondence $V \subset Y \times E$.

Definition. Let notation be as above.

- (i) A *U-special graph* in V^n is a component $W \subset V^n \cap (S \times B)$, where $S \subset Y^n$ is U -special, $B \subset E^n$ is special, $\pi_{Y^n}(W) = S$, and $\pi_{E^n}(W) \subset B$.
- (ii) A U -special graph W is *dependent* if B (may be taken such that it) is a proper special subvariety.
- (iii) A U -special graph W is *exemplary* if, setting B to be the smallest special subvariety of E^n with $\pi_{E^n}(W) \subset B$, there is no U -special graph W' strictly larger than W with $\pi_{E^n}(W') \subset B$.

Theorem 1.5. *Given $V \subset Y \times E$ as above with Y a modular curve, $U \subset Y$ finite, and $n \geq 1$, there are only finitely many exemplary U -special graphs in V^n .*

One may deduce corollaries analogous to 1.2, 1.3, and 1.4 above. The last recovers a result of Baldi ([1], obtained via equidistribution) which is also a special case of results of Dill [8, 9], affirming a conjecture of Buism-Poonen [6]; see the discussion in [1]. Baldi obtains a stronger “Bogomolov”-type result, which we do not. These results are in the circle of the “André-Pink conjecture”, see [28] and further references in [1], though Theorem 1.5 is rather an “unlikely intersection” result in such contexts. Of course it too is subsumed under the general Zilber-Pink conjecture.

With existing arithmetic estimates Theorem 1.5 and its corollaries should generalize to Shimura curves, with a suitable notion of Hecke orbit¹.

The structure of the paper is as follows. The Zilber-Pink setting is recalled in §2. The Ax-Schanuel statement and refinements we need are given in §3. Some arithmetic estimates are collected in §4. Theorems 1.1 and 1.5 are proved in §5, when everything is defined over a numberfield, and extended to \mathbb{C} in §6. In this paper, “definable” will mean “definable in the o-minimal structure $\mathbb{R}_{\text{an}, \text{exp}}$ ”; for background on o-minimality and on $\mathbb{R}_{\text{an}, \text{exp}}$ see [23].

¹There is an issue with abelian varieties that one could consider isogenies not necessarily respecting the polarization, which complicates matters.

2. THE ZILBER-PINK SETTING

We place Theorem 1.1 in the context of the Zilber-Pink conjecture (ZP) proposed independently, in slightly different formulations, by Zilber [35], Bombieri-Masser-Zannier [3], and Pink [29].

This concerns a *mixed Shimura variety* M and its collection \mathcal{S} of *special subvarieties*. One has also the larger collection of *weakly special subvarieties*. For definitions see e.g. Gao [10]. Let $Z \subset M$ be a subvariety.

For $S \in \mathcal{S}$, a component $A \subset Z \cap S$ is *atypical* if

$$\dim A > \dim Z + \dim S - \dim M.$$

(The intersection is called *unlikely* if $\dim Z + \dim S - \dim M < 0$.) ZP predicts a description in finite terms of all “atypical” intersections of Z with special subvarieties $S \in \mathcal{S}$.

For a subvariety $Z \subset M$ we let $\langle Z \rangle$ denote the smallest special subvariety of M containing Z , and by $\langle Z \rangle_{\text{ws}}$ the smallest weakly special one.

We define the *defect* $\delta(Z)$ of Z and the *weakly special defect* $\delta_{\text{ws}}(Z)$ by

$$\delta(Z) = \dim \langle Z \rangle - \dim Z, \quad \delta_{\text{ws}}(Z) = \dim \langle Z \rangle_{\text{ws}} - \dim Z.$$

Definition. Let $Z \subset M$.

- (i) A subvariety $A \subset Z$ is called *optimal* if it is maximal for its defect as a subvariety of Z . That is, if $A \subset B \subset Z$ and $\delta(B) \leq \delta(A)$ then $B = A$.
- (ii) A subvariety $A \subset Z$ is called *geodesic optimal* if it is maximal for its weakly special defect as a subvariety of Z .

The following is formally equivalent to the strongest form of ZP, namely the analogue for a mixed Shimura variety of the conjectures of Zilber and Bombieri-Masser-Zannier (for semi-abelian varieties and \mathbb{G}_m), as shown in [13]. (The notion here called “geodesic optimal” was earlier introduced as “cd-maximal” in a different context in [30] in the setting of \mathbb{G}_m .)

Conjecture 2.1 (ZP). *Let $Z \subset M$. Then Z has only finitely many optimal subvarieties.*

The ambient variety $Y^n \times E^n$ is an example of a *weakly special subvariety* of a *mixed Shimura variety* (it is *special* precisely if E has CM). Namely, let

$$\mathcal{E} \rightarrow Y$$

be the universal family over Y (of elliptic curves if Y is a modular curve, or of abelian surfaces if Y is a Shimura curve). Then \mathcal{E} is a mixed Shimura variety (see e.g. [10]), in which Y can be identified with the zero-section. If E is isomorphic to the fibre over $s \in Y$ then it may be identified with this fibre, which is weakly special. Correspondingly, $Y^n \times E^n$ may be identified with a weakly special subvariety of $\mathcal{E}^n \times \mathcal{E}^n$.

It is well-known, see e.g. Pink [29], that ZP implies a similar statement for its weakly special subvarieties, whose “special subvarieties” are simply the intersections of it with special subvarieties of the ambient mixed Shimura variety. There are corresponding notions of smallest special and weakly

special subvariety containing a given subvariety, defect and weakly special defect, and ZP can be expressed in terms of the corresponding notion of “optimal” as in 2.1; in the sequel the notation $\langle \cdot \rangle$ and defects will always be with respect to the ambient variety $Y^n \times E^n$. In particular, we have:

Definition. The *(weakly) special subvarieties* of $Y^n \times E^n$, in the above sense, are products of (weakly) special subvarieties in Y^n and E^n , where the “special subvarieties” of E^n are its torsion cosets.

It follows then that, for $Z \subset Y^n \times E^n$,

$$\langle Z \rangle = \langle \pi_{Y^n}(Z) \rangle_{Y^n} \times \langle \pi_{E^n}(Z) \rangle_{E^n}$$

and likewise for $\langle Z \rangle_{\text{ws}}$.

Given $V \subset Y \times E$, we consider ZP for $V^n \subset Y^n \times E^n$. If $x \in E^n$ is a V -image of a special point $s \in Y^n$ and x is dependent then $x \in B$ for some proper special subvariety of E^n . Then $(s, x) \in V^n \cap (\{s\} \times B)$, and since $\dim(\{s\} \times B) + \dim V^n < 2n$ this shows that any dependent image of a special point is an “unlikely” or “atypical” intersection in the sense of the Zilber-Pink conjecture.

The following shows that exemplary special graphs are optimal subvarieties of V^n , and hence that Theorem 1.1 is a consequence of ZP. However, we are not able to prove ZP for V^n (once $n \geq 3$).

Proposition 2.1. *An exemplary special graph in V^n is an optimal subvariety of V^n .*

Proof. Let $W \subset V^n \cap (S \times B)$ be an exemplary special graph with $\pi_{Y^n}(W) = S$ and $B = \langle \pi_{E^n}(W) \rangle$. Then $\dim W = \dim S$ and the smallest special subvariety of $Y^n \times E^n$ containing W is $S \times B$. Hence the defect of W is

$$\delta(W) = \dim \langle W \rangle - \dim W = \dim S + \dim B - \dim W = \dim B.$$

If W were not optimal, it would be contained in some larger subvariety $W' \subset V^n$ of the same, or lower defect. Write

$$\langle W' \rangle = S' \times B'.$$

Then $B \subset B'$ and $\dim W' \leq \dim S'$ and

$$\delta(W') = \dim \langle W' \rangle - \dim W' = \dim S' + \dim B' - \dim W'.$$

If $\delta(W') \leq \delta(W)$ we must have $B' = B$ and $\dim W' = \dim S'$, which would mean that W' is a special graph in V^n on S' , containing W , projecting into B . But by the maximality of W we have $W' = W$. \square

We will need the “weak” analogue of the above. A *weakly special graph* in V^n is a component $W \subset V^n \cap (S \times B)$ where S, B are weakly special subvarieties. It is *exemplary* if, taking $B = \langle \pi_{E^n}(W) \rangle_{\text{ws}}$, there is no weakly special graph W' strictly larger than W with $\pi_{E^n}(W') \subset B$.

Proposition 2.2. *An exemplary weakly special graph in V^n is a geodesic optimal subvariety of V^n .*

Proof. The same. \square

The Ax-Schanuel theorem only detects weakly special subvarieties, and we thus need to show (as has already been shown in several other settings, including for all pure Shimura varieties by Daw-Ren [7]) that optimal subvarieties are geodesic optimal. For this we establish the “defect condition”.

Definition. A weakly special subvariety X of a mixed Shimura variety has the *defect condition* if, for $A \subset B \subset X$, we have

$$\delta(B) - \delta_{\text{ws}}(B) \leq \delta(A) - \delta_{\text{ws}}(A),$$

the defects being with respect to the special and weakly special subvarieties of X .

Proposition 2.3. *Let S be a pure Shimura variety and T an abelian variety. Then $S \times T$ has the defect condition.*

Proof. For an abelian variety (as well as for \mathbb{G}_m^n and products of modular curves) the defect condition is established in [13], Proposition 4.3, and for a general pure Shimura variety in [7], 4.4. Since the (weakly) special subvarieties of $S \times T$ are products of (weakly) special subvarieties of the factors, we have

$$\langle A \rangle = \langle \pi_S(A) \rangle_S \times \langle \pi_T(A) \rangle_T$$

so that

$$\delta(A) - \delta_{\text{ws}}(A) = \delta(\pi_S(A)) - \delta_{\text{ws}}(\pi_S(A)) + \delta(\pi_T(A)) - \delta_{\text{ws}}(\pi_T(A)),$$

and likewise for B , and the defect condition for $S \times T$ follows from the defect conditions in S and T by addition. \square

It is conjectured in [13] that the defect condition holds in all mixed Shimura varieties. Presumably a proof can’t be too far from the above, as the weakly specials are “nearly” products, i.e. they are flat over a pure special.

Proposition 2.4. *An optimal subvariety is geodesic optimal.*

Proof. This follows formally once one has the defect condition, as in [13]. \square

3. AX-SCHANUEL

The *Ax-Schanuel property* for the uniformization map

$$u_M : D \rightarrow M$$

realizing a mixed Shimura variety M as a quotient of a suitable Hermitian symmetric domain D by a discrete arithmetic group Γ is a functional transcendence statement for u_M analogous to the classical Ax-Schanuel theorem for the exponential function $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$. For discussion and proof of such results see [19, 10]. Such a result implies a corresponding statement for each weakly special subvariety $X \subset M$, uniformized by an irreducible component of $u_M^{-1}(X)$.

The Ax-Schanuel result we need is for (all the cartesian powers of) the uniformization

$$u : \mathbb{H} \times \mathbb{C} \rightarrow Y \times E.$$

We will use the same notation $u : \mathbb{H}^n \times \mathbb{C}^n \rightarrow Y^n \times E^n$ for cartesian powers. Since this is the uniformization corresponding to a weakly special subvariety of \mathcal{E}^{2n} , the result follows from the Ax-Schanuel statement for the uniformization

$$\mathbb{H}^{2n} \times \mathbb{C}^{2n} \rightarrow \mathcal{E}^{2n}$$

and since Y^{2n} , the “pure” Shimura variety underlying \mathcal{E}^{2n} , is a special subvariety of \mathcal{A}_g , the Siegel modular variety of principally polarized abelian varieties, when $g \geq 2n$ the required Ax-Schanuel follows from the corresponding statement for the universal family \mathcal{X}_g of abelian varieties over \mathcal{A}_g , namely the Ax-Schanuel theorem for the uniformization

$$\mathbb{H}_g \times \mathbb{C}^g \rightarrow \mathcal{X}_g.$$

This theorem is due to Gao [10], Theorem 1.1, extending, for \mathcal{A}_g , the result for a general pure Shimura variety in [19], Theorem 1.1.

We will (as usual in ZP applications) use only the “two-sorted” form, which we now state for the uniformization $u : \mathbb{H}^n \times \mathbb{C}^n \rightarrow Y^n \times E^n$, after noting the following convention.

Strictly speaking \mathbb{H}^n has no “algebraic subvarieties”; by an *algebraic subvariety* of U , where $U \subset \mathbb{H}^n \times \mathbb{C}^n$ is a weakly special subvariety, we will mean an irreducible analytic component of the intersection of U with an algebraic subvariety (in the usual sense) of the ambient $\mathbb{C}^n \times \mathbb{C}^n$.

Theorem 3.1. *Let U' be a weakly special subvariety of $\mathbb{H}^n \times \mathbb{C}^n$ with image $u(U') = X'$ a weakly special subvariety of $Y^n \times E^n$. Let $Z \subset X'$, $A \subset U'$ be algebraic varieties, and Ω an irreducible analytic component of $A \cap u^{-1}(Z)$. Then*

$$\dim \Omega = \dim Z + \dim A - \dim X'$$

unless Ω is contained in a proper weakly special subvariety of U' . \square

As in [13, 7], this can be reformulated in terms of a suitable notion of “optimality”, for which we adopt the terminology used by Daw-Ren [7], §5.7-5.9, to distinguish it from “optimality” as above in §2.

Definition. Let $Z \subset Y^n \times E^n$ be a subvariety.

(i) An *intersection component* of $u^{-1}(Z)$ is an irreducible analytic component of the intersection of $u^{-1}(Z)$ with an algebraic subvariety of $\mathbb{H}^n \times \mathbb{C}^n$.

(ii) If A is an intersection component of $u^{-1}(Z)$ with Zariski closure $\langle A \rangle_{\text{Zar}}$ we define its *Zariski defect* to be

$$\delta_{\text{Zar}}(A) = \dim \langle A \rangle_{\text{Zar}} - \dim A.$$

(iii) An intersection component A of $u^{-1}(Z)$ is called *Zariski optimal* if one cannot find a larger intersection component of $u^{-1}(Z)$ which does not increase the Zariski defect.

(iv) An intersection component A of $u^{-1}(Z)$ is called *geodesic* if A is a component of $u^{-1}(Z) \cap \langle A \rangle_{\text{Zar}}$ and $\langle A \rangle_{\text{Zar}}$ is weakly special.

Proposition 3.2. *Let $Z \subset Y^n \times E^n$ be a subvariety. A Zariski optimal component of $u^{-1}(Z)$ is geodesic.*

Proof. The equivalence of 3.1 and 3.2 is purely formal and the proof is carried out in [13], below 5.12. \square

Definition. A *Möbius subvariety* of \mathbb{H}^n is an algebraic subvariety defined by setting some coordinates constant, and relating some other pairs of coordinates by elements of $\text{SL}_2(\mathbb{R})$.

We let F denote a standard fundamental domain for the uniformization of $Y \times E$. The uniformization map restricted to F is definable (in this case by results of Peterzil-Starchenko [23]), and the Möbius subvarieties of \mathbb{H}^n form a definable family.

This means that if we consider the definable family of subvarieties of $\mathbb{H}^n \times \mathbb{C}^n$ comprising all products of “Möbius subvarieties” of \mathbb{H}^n and linear subvarieties of \mathbb{C}^n , and define the set of Zariski optimal ones by the difference of their dimension and dimension of intersection with $u^{-1}(V)$, just among these which go through F , we will get the slopes (up to $\text{SL}_2(\mathbb{Z})$ and Λ) of all geodesic optimal components. This then implies the finiteness of such slopes in $Y^n \times E^n$, and any geodesic optimal component of V^n will have some pre-image component going through F .

We want the corresponding finiteness for the particular type of components we consider. Namely, if W is a dependent special graph, we consider a component U of its pre-image in $\mathbb{H}^n \times \mathbb{C}^n$. It is a component of the intersection of $u^{-1}(V^n)$ with suitable pre-image $M \times L$ of $\langle W \rangle = S \times B$, and is thus a geodesic component which projects onto M and thus has $\dim U = \dim M$.

We need to observe that, if Zariski optimal, such a component comes from a maximal dependent (weakly) special image, i.e. something of the same form. In fact we need something further along these lines in the proof of 1.1, in order to get from “something positive-dimensional algebraic” to a component of the right form.

Proposition 3.3. *Let U be of the following type: it is a component of $A \times L$ intersecting $u^{-1}(V^n)$, where A is algebraic, and L is linear which projects onto A .*

If U is maximal of this type for the given L then L (and A) are weakly special and U is Zariski optimal.

Proof. We have $\dim U = \dim A$ and so

$$\delta_{\text{Zar}}(U) \leq \dim L.$$

Suppose that $U \subset U'$, with U' Zariski optimal, and hence geodesic optimal, with U' a component of the intersection of $u^{-1}(V^n)$ with weakly special

$A' \times L'$, and $A' \times L'$ is its Zariski closure. Then

$$\delta_{\text{Zar}}(U') = \dim A' + \dim L' - \dim U'.$$

But $\dim U' \leq \dim A'$ and $L \subset L'$. If

$$\delta_{\text{Zar}}(U') \leq \delta_{\text{Zar}}(U)$$

we must have $L = L'$ and $\dim U' = \dim A'$ so that U' is a pre-image of a “dependent weakly special image”. By the maximality of U we have $U = U'$ and then $L = L'$ and $A = A'$ are weakly special. \square

Now we get the finiteness statement.

Proposition 3.4. *For each k there are only finitely many strongly special subvarieties in Y^k which have a V -image which lies in any proper weakly special in E^k*

Proof. We take the definable space of products $M \times L$ of Möbius and linear subvarieties, and take the definable subset of maximal ones in the above sense. These are Zariski optimal and hence geodesic optimal, and hence are among the finite set of slopes corresponding to the latter. \square

4. ARITHMETIC ESTIMATES

Constant $C, C', \dots, c, c', \dots$ in the following depend on E, Y, V, n and the choice of a fundamental domain F_Y for the uniformization $\mathbb{H} \rightarrow Y$. We let $\Delta = \Delta(s)$ denote the discriminant (which is negative) of a special point $s \in Y$.

Proposition 4.1. *Let $s \in Y$ be a special point and $\Delta(s)$ the discriminant of the corresponding quadratic order. Let $z \in F_Y$ be a pre-image of s . Then*

1. $h(s) \leq c(\epsilon)|\Delta|^\epsilon$ for any $\epsilon > 0$;
2. $H(z) \leq C|\Delta(s)|^c$;
3. $[\mathbb{Q}(s) : \mathbb{Q}] \leq c(\epsilon)|\Delta|^{\frac{1}{2}+\epsilon}$ for any $\epsilon > 0$;
4. $[\mathbb{Q}(s) : \mathbb{Q}] \geq c(\epsilon)|\Delta|^{\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$.

Proof. For classical singular moduli: 1. Given in [13], Lemma 4.3. 2. Elementary (with $c = 1$), given in [24]. 3. See [21] for an explicit result. 4. This is by the classical (ineffective) Landau-Siegel bound. The same bounds follow for a modular curve Y as a finite cover of $Y(1)$. For Shimura curves: 2 follows from work of the second author appearing in [25], 1 follows from [33] combined with the comparison (see e.g. [22]) of Faltings height with height of a moduli point, while for 3 and 4 see [34], in particular equation (3.10) for $O_{\text{gl}}(x) = O_{\text{cm}}(x)$, and Remark (1) on Page 3664 for the asymptotic. \square

We assume E is in Weierstrass form (but an estimate of the same form then follows if it isn't) and defined over a number field K_0 of degree $D = [K_0 : \mathbb{Q}]$. Let q denote the Néron-Tate height on E (see e.g. [4] or [16]).

We have the following Theorem E of Masser [16]. Set

$$\eta = \eta(E, K) = \inf q(x),$$

taking the infimum over non-torsion $x \in E(K)$, and let

$$\omega = \omega(E, K)$$

be the cardinality of the torsion subgroup of $E(K)$.

Theorem 4.2. *Let $x_1, \dots, x_n \in E(K)$ with Néron-Tate heights bounded by $q \geq \eta$. There is a basis for the relations*

$$m_1 x_1 + \dots + m_n x_n = 0_E, \quad m_i \in \mathbb{Z},$$

with all m_i having

$$|m_i| \leq n^{n-1} \omega \left(\frac{q}{\eta} \right)^{(n-1)/2}. \quad \square$$

To accommodate CM, we work, like Barroero [2], in E^{2n} with $x_i, \rho x_i$, where E has CM by the order $\mathbb{Z} + \mathbb{Z}\rho$. We write $\|a + b\rho\| = \max(|a|, |b|)$ for $a + b\rho \in \text{End}(E)$. Then under the previous hypotheses a set of generators for the relation group can be found with

$$\|m_i\| \leq (2n)^{2n-1} \omega \left(\frac{q}{\eta} \right)^{(2n-1)/2}.$$

Following [16] we have the following estimates for η, ω . Set $L = \log(D+2)$. We have

$$\eta \geq C^{-1} D^{-3} L^{-2}$$

by results of, respectively, Laurent (CM) and Masser (non CM) cited in [16], and

$$\omega \leq CDL$$

(see discussion in [16]).

Combining the above estimates yields the following result, where $\|m\|$ is as above in the CM case, but in the non-CM case we set $\|m\| = |m|$.

For a tuple $s = (s_1, \dots, s_n) \in Y^n$ of special points with discriminants $\Delta(s_i)$ we define the *complexity* of s by $\Delta(s) = \max(|\Delta(s_i)|)$.

Proposition 4.3. *There are constants C, C', c , depending on E, Y, V, n , with the following property. Let $(s_1, x_1), \dots, (s_n, x_n) \in Y \times E$ be V -graphs of special points with discriminants $\Delta(s_i)$ and set $\Delta = \Delta(s) = \Delta(s_1, \dots, s_n)$. Then, for $\Delta \geq C'$, there is a generating set for the linear relations satisfied by the x_i in E with*

$$\|m_i\| \leq C \Delta^c.$$

Proof. The difference $|q - h|$ is bounded on $E(\overline{K_0})$ by some constant c^* (see e.g. [4]). On the other hand, if x is a V -image of s then $H(x) \leq CH(s)^c$ and $[K_0(x) : K_0] \leq C[\mathbb{Q}(s) : \mathbb{Q}]$. Thus, $D \leq C \Delta^c$ by 4.1.3.

If the maximum h of the $h(x_i)$ is sufficiently large then we will have $h - c^* \geq \eta$ and $2h \geq q$. Then $h \leq C \Delta^c$ by 4.1.1, and now everything in 4.2 is bounded in terms of Δ . \square

Propositions 4.3 and 4.1.2 will be used in the next section to bound the height of a rational/quadratic point on a suitable definable set, while 4.1.4 will be used to show that there are “many” such points.

5. PROOF OF THEOREMS OVER $\overline{\mathbb{Q}}$

Proof of Theorem 1.1 when E, V are defined over $\overline{\mathbb{Q}}$. Let K_0 be a number-field over which E, Y, V and all elements of $\text{End}(E)$ are defined.

We consider an exemplary special graph $W \subset V^n$, a V -image of some special subvariety $S \subset Y^n$, with $\langle \pi_{E^n}(W) \rangle = B$. Then any Galois conjugate W' of W over K_0 is also an exemplary special graph (of the conjugate S' of S , with $\langle \pi_{E^n}(W') \rangle = B'$ with B' the corresponding conjugate of B), and *vice-versa*.

We can write S as a product $S = S_1 \times \{S_2\}$ of some strongly special $S_1 \subset Y^{A_1}$ on some subset $A_1 \subset \{1, \dots, n\}$ of coordinates, and a special point $S_2 \in Y^{A_2}$ where $A_2 \subset \{1, \dots, n\}$ is the complementary subset to A_1 .

By Proposition 3.4 there are only finitely many such S_1 to consider, and so we may assume they are all defined over K_0 .

We can write $W = W_1 \times W_2$ and write ξ_j, η_k for the coordinates in E^{A_1}, E^{A_2} respectively. We will show that if $\eta \in E^{A_2}$ is a V^{A_2} -image of a special point S_2 of sufficiently large complexity (depending on S_1) then W is not exemplary, and this will establish the requisite finiteness.

It may be that the projection of W_1 to E^{A_1} is contained in some proper weakly special subvariety, which means that there are some equations of the form

$$\sum_{i \in A_1} m_i \xi_i = p, \quad m_i \in \text{End}(E), \quad p \in E$$

holding on this projection. We let p_1, \dots, p_k be the points corresponding to a generating set of such relations. Note that the linear span of the p_i is $\text{Gal}(\overline{\mathbb{Q}}/K_0)$ invariant, so we can make all the p_i defined over K_0 .

If we take a generating set of all the equations over $\text{End}(E)$ satisfied by the points in $\pi_{E^n}(W)$ then this defines an algebraic subgroup B_0 of which B is a connected component. Any such equation of the form

$$\sum_{i \in A_1} m_i \xi_i + \sum_{j \in A_2} n_j \eta_j = 0, \quad m_i, n_j \in \text{End}(E),$$

entails that $\sum m_i \xi_i$ is constant on W_1 and is equivalent to some equation involving the p_i, η_j , and *vice-versa*. We consider then the system of equations

$$\sum_{i=1, \dots, k} m'_i p_i + \sum_{j \in A_2} n_j \eta_j = 0, \quad m'_i, n_j \in \text{End}(E),$$

corresponding (and equivalent) to the system defining B_0 , where η is a V^{A_2} -image of S_2 . Let d_0 be the dimension of the subvariety this cuts out in E^{A_2} .

By Proposition 4.3 there is a set of generators of all such relations with

$$\|m_i\|, \|n_j\| \leq C \Delta(S_2)^c.$$

Fix a pre-image $\nu = (\nu_1, \dots, \nu_k) \in F_E^k$ of (p_1, \dots, p_k) . Let us first suppose that E has NCM (“not CM”), and $d = \dim B$. Let G be the Grassmanian of $(d_0 + k)$ -dimensional affine linear \mathbb{C} -subspaces of \mathbb{C}^{k+n_2} where $n_2 = |A_2|$.

Take the definable set

$$X = \{(z, w, g) \in F_Y^{A_2} \times F_E^{A_2} \times G : u(z, w) \in V^{A_2}, (\nu, w) \in g\},$$

where F_E is a standard fundamental domain for the uniformization $\mathbb{C} \rightarrow E$, and, projecting, the definable set

$$Z = \{(z, g) \in F_Y^{A_2} \times G : \exists w \in F_E^{A_2} : (z, w, g) \in X\}.$$

A special point $S_2 \in Y^{A_2}$ of “large” complexity $\Delta(s)$ leads to “many” points in Z which are quadratic in the F_Y coordinates and rational (even integral) in the g coordinates. More specifically, for sufficiently large $\Delta(s)$ we get (by 4.1.4, 4.1.2, and 4.3)

$$\gg \Delta(S_2)^c \text{ such points of height at most } \ll \Delta(S_2)^{c'}.$$

Hence, by the Counting Theorem (see e.g. [27]), there is a connected, semi-algebraic set R in Z belonging to a fixed definable family, in which the z coordinates cannot be constant (since the positive-dimensional semi-algebraic sets need to account for “many” different conjugates of s). Since all of the Galois conjugates of a point have the same slopes m_i, n_j we can moreover assume that R has a fixed slope.

Lemma 5.1. *The projection of R to G is a point.*

Proof. Let β be the covering space of B_0 and $\beta' = \mathbb{C}^{A_2}/\beta$. Consider the image $R' \subset F_Y^{A_2} \times F_{\beta'}$ of the pre-image of R in X . Again by the counting theorem, R' contains a semi-algebraic set R'' belonging to a fixed definable family, with “many” rational points coming from a single Galois orbit. Now note that R'' maps into the image V' of V^n inside the product $Y^{A_2} \times E^{A_2}/B_0$. Thus by Ax-Lindemann, the image of R'' lies in a weakly special contained in V' . However, the projection of V' to Y^{A_2} is finite-to-one, and therefore the weakly special containing the image of R'' must have no abelian part, and therefore its projection to E^{A_2}/B_0 is a point, as desired. \square

By lemma 5.1 we may write $R = A \times g_0$ with $g_0 \in G$ and $A \subset \mathbb{H}^{A_2}$ semi-algebraic. Let L be the linear subspace of \mathbb{C}^{k+n_2} corresponding to g_0 . Note that L projects to some Galois conjugate of B inside E^{A_2} . Let $L_\nu \subset \mathbb{C}^k$ be the fiber of L over ν . Now, by definition of A , we have that $A \times L_\nu \cap u^{-1}(V^{A_2})$ has a component U which maps onto A . Note that the Zariski defect of U is at most d_0 .

By Proposition 3.3, there exists a weakly special A^* containing A and a component U^* of $A^* \times L_\nu \cap u^{-1}(V^{A_2})$ containing U which maps onto A^* with defect at most d_0 . Since A^* contains special points, it must in fact be special. Let S^* be the image of A^* in Y^{A_2} . It contains at least one (in fact “many”) Galois conjugates of S_2 . By definition, a suitable V -image of $S_1 \times S^*$ is contained in a coset of B_0 . We may now take a Galois conjugate of S^* which contains S_2 , thus giving a larger special graph projecting to the same torsion coset, which is a contradiction.

Now suppose that E has CM by the order $\mathbb{Z} + \mathbb{Z}\rho$. We now let G parameterize $(n_2 + 2k + d_0)$ -dimensional complex affine-linear subspaces in \mathbb{C}^{2k+2n_2} and consider the definable set

$$X = \{(z, w, g) \in F_Y^{A_2} \times F_E^{A_2} \times G : u(z, w) \in V^{A_2}, (\nu, \rho\nu, w, \rho w) \in g\}$$

and, projecting, the definable set

$$Z = \{(z, g) \in F_Y^{A_2} \times G : \exists w \in F_E^{A_2} : (z, w, g) \in X\}.$$

The rest of the proof is the same as the NCM case. \square

Proof of Theorem 1.5 when E, V, U are defined over $\overline{\mathbb{Q}}$. This is very much the same as the argument above but using different arithmetic estimates, drawn from [12], and a different definable set on which to count points.

We consider again an exemplary special graph of the form $W_1 \times W_2$, a V -image of some $S_1 \times \{S_2\}$ as above with $S_2 \in Y^{A_2}$ a special point. There are again only finitely many such decompositions to consider, by 3.4.

Let us consider U -special points $S_2 = (s_i) \in Y^{A_2}$ of a particular form, namely points in which s_i is in the Hecke orbit of a fixed $u_i \in U$ for $i \in A_2$, and all the u_i are non-special. Then there is a unique cyclic isogeny between the elliptic curves corresponding to u_i and s_i whose degree we denote N_i . For such a point S_2 we define its U -complexity by

$$\Delta(S_2) = \max\{N_1, \dots, N_n\}.$$

We observe that the height of S_2 is controlled by $\Delta(S_2)$; using the results of Faltings relating Faltings heights of isogenous elliptic curves and Silverman's comparison of Faltings height and height of the j -invariant (see the discussion in [12] on heights under isogenies in the proof of Lemma 4.2, p15) we have

$$h(S_2) \leq C \max\{1, \log N_i\}$$

(constants now depend on Y, E, V, U and n). If $(S_2, \eta) \in V^{A_2}$ the above leads (via Masser's Theorem E) to bounds of the form

$$\|m\| \leq C \Delta(S_2)^c$$

on the size of entries in a set of generators for the relation group of (p, η) .

On the other hand the degrees $[\mathbb{Q}(s_i) : \mathbb{Q}]$ are controlled by $\Delta(S_2)$ via isogeny estimates (see the discussion in [12] on degrees in §6 above proof of 1.3) which imply $[\mathbb{Q}(s_i) : \mathbb{Q}] \geq C' N_i^{1/6}$ and hence

$$[\mathbb{Q}(S_2) : \mathbb{Q}] \geq C' \Delta(S_2)^{c'}.$$

Finally, if $\nu_i \in F_Y$ is a pre-image of u_i and $z_i \in F_Y$ is a preimage of s_i then $z_i = g\nu_i$ for some $g_i \in \text{GL}_2^+(\mathbb{Q})$ with

$$H(g_i) \leq c N_i^{10}$$

(see Lemma 5.2 of [12]).

We now count points though on a different definable set as U -special points are not algebraic and the counting must be done for $\mathrm{GL}_2^+(\mathbb{Q})$ points in a definable subset of $\mathrm{GL}_2^+(\mathbb{R})$.

We fix a pre-image $\mu \in F_Y^{A_2}$ of (u_1, \dots, u_{n_2}) and consider the definable set

$$X = \{(h, \nu, w, g) \in \mathrm{GL}_2^+(\mathbb{R})^{A_2} \times F_E^k \times F_E^{A_2} \times G :$$

$$h\mu \in F_Y^{A_2}, u(h\mu, w) \in V^{A_2}, (\nu, w) \in g\}$$

and its projection

$$Z = \{(h, g) \in \mathrm{GL}_2^+(\mathbb{R})^{A_2} \times G : \exists w \in F_E^{A_2} : (h\mu, \nu, w, g) \in X\}.$$

A U -special point S_2 of the form being considered of “large” complexity leads to “many” rational points on Z . If $\Delta(S_2)$ is sufficiently large then by counting we get a real algebraic curve in Z which (since these come from “many” distinct points in $F_Y^{A_2}$ and by complexification) gives rise to a complex algebraic curve $A \subset \mathbb{H}^{A_2}$ and an intersection component of $A \times L_g$ of Zariski defect d as previously. This leads to a contradiction as in the argument above, so that $\Delta(S_2)$ is bounded for an exemplary special graph, giving finiteness for S_2 of this type.

The general case will follow by combining the treatment of special and non-special points using a suitable definable set (i.e. using F_Y for special coordinates and $\mathrm{GL}_2^+(\mathbb{R})$ for coordinates in the Hecke orbit of a non-special $u \in U$) and a combinatorial argument. \square

6. GOING FROM $\overline{\mathbb{Q}}$ TO \mathbb{C}

6.1. Setup. Let F be a finitely generated subfield of \mathbb{C} so that $V \subset Y(1) \times E$ are all defined over F . F can be thought of as the function field of an irreducible algebraic variety S over some number field $K \subset F$. Replacing S with a dense open subset, we assume that E extends to an elliptic scheme \mathcal{E} over S and V extends to a flat family \mathcal{V} over S . We pick a generic regular point $s_0 \in S(\mathbb{C})$ such that $K(s_0)$ is isomorphic to F , and pick an open ball $B \subset S(\mathbb{C})$ around s_0 , so that in B we can trivialize the homology of \mathcal{E} over S .

6.2. Ordering points in S . We will need to order points in S , so we proceed as follows. Let $f : S \rightarrow \mathbb{P}^{\dim S}$ be a quasi-finite map. Then we define the f -degree of a point s in $S(\overline{\mathbb{Q}})$ to be the degree of its image under f , and the f -height $h_f(s)$ to be the (logarithmic) height of its image under f . By Northcott’s theorem, there are finitely many points of bounded f -degree and f -height. We only consider heights for the subset S_f of S whose image lands in $\mathbb{P}^{\dim S}(K)$.

6.3. The proof. By Proposition 3.4 in a family, there are only finitely many strongly special varieties whose \mathcal{V}_u -image lies inside any proper weakly-special subvariety of \mathcal{E}_u for any $u \in B$. Thus, there are only finitely many families of special subvarieties we have to consider. By rearranging co-ordinates, we may assume they are all of the form $T \times p \times q$ where $T \subset Y(1)^m$ is a fixed strongly special subvariety, and $p \in Y(1)^k$ is a CM point, and q has co-ordinates isogenous to points in U .

Now, for the sake of contradiction let p_i, q_{i,s_0} be an infinite sequence of such points such that $T \times p_i \times q_{i,s_0}$ are projections of optimal special graphs for \mathcal{V}_{s_0} . Let A_i be the smallest torsion coset containing the \mathcal{V}_{s_0} -image of $T \times p_i \times q_{i,s_0}$. Then for each point $s \in S(\overline{\mathbb{Q}}) \cap B$ image of $T \times p_i \times q_{i,s}$ is still contained in A_i . But we've proven the statement for $\overline{\mathbb{Q}}$ -points, and thus for each s there are finitely many special varieties containing all the $T \times p_i \times q_{i,s}$ whose \mathcal{V}_s image is contained in a proper torsion coset.

Let $T_1(s_0), \dots, T_m(s_0)$ be the smallest collection of U_{s_0} -special subvarieties containing all the $T \times p_i \times q_{i,s_0}$.

Lemma 6.1. *For large enough d , for a density 1 set of points s in S_f ordered by f -height, $T_1(s), \dots, T_m(s)$ is the smallest collection of U_s -special subvarieties containing all the $T \times p_i \times q_{i,s}$.*

Proof. First, note that since the degrees of CM points tend to infinity. Thus, the set of points $s \in S_f$ such that U_s is CM is contained in a proper subvariety, and so has density 0. Next, since U -special subvarieties are defined simply by imposing isogeny relations, it is sufficient to prove that for a density 1 set of points s that u_s, v_s are not isogenous, for u, v distinct points in U .

Now, for $s \in S_f$, it follows that $h(u_s), h(v_s) \ll h_f(s)$, and thus by Masser-Wüstholz isogeny bound [18, Main Theorem] it follows that if u_s, v_s are isogenous then there is an isogeny between them of degree $O(h_f(s)^\kappa)$ for some fixed $\kappa > 0$. Now, the degree of the T_N in $X(1)^2$ is $O(N^2)$, and therefore the set of all $s \in S_f$ with $h_f(s) < X$ such that u_s, v_s are isogenous are contained in $O(h_f(s)^\kappa)$ divisors of f -degree at most $O(h_f(s)^{2\kappa})$. Now, the size of $\{s \in S_f, h(s) < X\}$ is asymptotic to $e^{X(\dim S + 1)}$ whereas the number of points in any divisor of degree d of height at most X is $O(de^{X \dim S})$. The result follows. \square

Thus we are done once we prove the following

Lemma 6.2. *Let \mathcal{E} be an elliptic scheme over S , and let $\mathcal{W} \subset \mathcal{E}^n$ be an irreducible algebraic subvariety. If \mathcal{W}_s is contained inside a proper abelian subvariety for a density 1 set of $s \in S_f$, then \mathcal{W} is contained inside an abelian subscheme.*

Proof. Replacing \mathcal{W} by its own n -fold self-sum we may assume that \mathcal{W} is a coset of an abelian subscheme. Quotienting out by the corresponding

abelian subscheme, we may further assume that \mathcal{W} is finite over S , and base changing S by a finite map we may assume that \mathcal{W} is a section over S . By the Main Theorem of [17], it follows that for a density one set of points s the n points of \mathcal{E}_s represented by \mathcal{W}_s are linearly independent. This completes the proof in the case that \mathcal{E} does not have generic CM. Otherwise, one may argue similarly, by recording an extra set of co-ordinates for the extra endomorphism of \mathcal{E} . \square

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