Closed Band-Projected Density Algebra Must Be Girvin-MacDonald-Platzman

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The band-projected density operators in a Landau level obey the Girvin-MacDonald-Platzman (GMP) algebra, and a large amount of effort in the study of fractional Chern insulators has been directed toward approximating this algebra in a Chern band. In this Letter, we prove that the GMP algebra, up to form factors, is the *only* closed algebra that projected density operators can satisfy in two and three dimensions, highlighting the central place it occupies in the study of Chern bands in general. A number of interesting corollaries follow.

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Introduction—In the study of the fractional quantum Hall effect, the authors of Refs. [1,2] found that projected density operators in the lowest Landau level (LLL) satisfy a specific closed algebra, which would later be known as the Girvin-MacDonald-Platzman (GMP) algebra. It was also realized that higher Landau levels (LLs) satisfy a similar algebra with different so-called form factors, and it is such algebra with generic form factors that we call the GMP algebra here. [We present this algebra explicitly in Eqs. (1) and (2), below]. With kinetic energy being suppressed in Landau levels and the interaction given by density-density terms, it is understood that the GMP algebra should capture Landau level physics completely. Later on, in the search of fractional Chern insulators (FCIs) [3–5], i.e., systems that host fractional quantum Hall (FQH) effect without the application of an external magnetic field, much effort was directed toward designing bands that resemble the Landau levels. Since LL physics is captured by the GMP algebra, it is deemed desirable to reproduce the GMP algebra in a Chern band, at least in some limits. In Ref. [6], the authors demonstrated that to reproduce the long wavelength limit of the GMP algebra, the Berry curvature should be constant in the Brillouin zone. In Ref. [7], the author found the necessary and sufficient condition for a band to satisfy the GMP algebra *with LLL-like form factors* involves, besides Berry curvature, an additional condition on the quantum metric of the band, which would later be known as the ideal flatband condition. (The ideal flatband condition does not apply to GMP algebra with more general form factors).

Admittedly, there have been no experimental systems, other than Landau levels, that realize the GMP algebra exactly, and according to Ref. [8], the GMP algebra is impossible in a tight-binding model (i.e., finite number of degrees of freedom per unit cell). Nevertheless, how closely the Chern band reproduces the GMP algebra, based on quantum geometric criteria, is thought of as crucial to evaluating FCI candidates, with much theoretical efforts devoted in this direction [6,7,9–21]. One criterion that is supposed to favor stable FCI is uniform Berry curvature, as proposed in Ref. [6]. In Ref. [22], experimental evidence supported the correlation between Berry flatness and robustness of FCI, though we also note that Ref. [23] argues that this criterion is flawed, since spatial embedding affects Berry curvature distribution but does not affect FCI stability, and Ref. [8] reports numerical evidence that contradicts the correlation between Berry flatness and robustness of FCI. Another criterion that is proposed to favor stable FCI is the ideal flatband condition, as mentioned in the previous paragraph, where the quantum metric saturates certain integral bounds. Systems that satisfy ideal band condition include twisted bilayer graphene and other moiré materials (exactly so in the chiral limit, approximately so with more realistic parameters) [20,24–29]. Furthermore, in Ref. [30], a mapping was developed between systems that satisfy the ideal flatband condition to Landau levels. In Ref. [31], the authors further argued that such ideal flatbands satisfy a density commutation relation that can be built up from the GMP algebra. We note that while systems that satisfy some of the abovementioned quantum geometry criteria do not have to obey GMP algebra, these criteria are derived by mimicking systems that do satisfy GMP algebra. As such, systems with GMP algebra, as an idealized setup that can be compared with and mapped to other less ideal systems, remain a highly relevant topic in the field of FCI research and beyond.

The GMP algebra was first derived for electrons in a strong magnetic field, and this is one possible closed algebra that projected density operators can obey. Though no other closed density operator algebra is known, one may nevertheless wonder if any other closed algebra exists, which may deserve equal research interest. In this Letter, we show that *the GMP algebra in two and three dimensions, up to form factors, are the only possible closed algebra that projected density operators may satisfy*, demonstrating the unique place the GMP algebra occupies in the study of Chern bands.

In other words, if one only knows that the projected density operators for a Bloch band obey a closed algebra, then one can immediately conclude that this algebra must be of the GMP form up to form factors. This is an extremely strong statement since the only thing we require as an input is that it is a closed algebra. Previous to our Letter, a number of prior important works have been published about band projected density algebras. Besides the works already discussed in the preceding paragraphs, there is also Ref. [11], where projected density operators for composite fermions are discussed, and Ref. [13], which considers projected density operators for anisotropic Landau levels, to give a few examples. However, none of these previous results have addressed the question of whether a closed density algebra needs to be GMP, which is what we show here.

Additionally, compared to previous works such as Ref. [6] that focus on the long wavelength limit of the projected density algebra, in this Letter we study the algebra at all wavelengths instead. On top of its intrinsic theoretical interest, studying the algebra at all wavelengths is of practical significance. According to Ref. [6], the long wavelength limit of the algebra is solely dictated by the Berry curvature. However, it is clear that Berry curvature alone is not a sufficient description of Chern bands in the study of FQH effects or FCIs. For example, it is well-known that different Landau levels, despite having the same uniform Berry curvature, can host different FQH states. One needs to know the form factors of the GMP algebra as a function of wave vector as well as the interelectron interaction to fully define the Hamiltonian of interacting electrons within a Landau level [32].

Main result—For any Bloch band structure, one can define the band-projected density operator $\bar{\rho}_{\boldsymbol{q}} = \mathcal{P}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\mathcal{P}$, where $\mathcal{P}$ projects to a single band. Suppose the algebra of these operators close

$$[\bar{\rho}_{\boldsymbol{q}_1}, \bar{\rho}_{\boldsymbol{q}_2}] = f(\boldsymbol{q}_1, \boldsymbol{q}_2)\bar{\rho}_{\boldsymbol{q}_1+\boldsymbol{q}_2}, \tag{1}$$

thus defining a Lie algebra. We show that in two and three dimensions, f can only have the form

$$f(\boldsymbol{q}_1, \boldsymbol{q}_2) = 2i\frac{F(\boldsymbol{q}_1)F(\boldsymbol{q}_2)}{F(\boldsymbol{q}_1+\boldsymbol{q}_2)}\sin\left(\frac{1}{2}\boldsymbol{\Omega}\cdot(\boldsymbol{q}_1\times\boldsymbol{q}_2)\right) \tag{2}$$

for some function F that satisfies $F(-\boldsymbol{q}) = F^*(\boldsymbol{q})$ and a fixed real vector $\boldsymbol{\Omega}$. Eqs. (1) and (2) are known as the GMP algebra, F is known as the form factor, and $\boldsymbol{\Omega}$ will turn out to be the Berry curvature (a vector in three dimensions, and a vector normal to the plane in two dimensions). The commutator algebra holds true for all $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ [33], thereby fixing the properties of the Bloch band at all momentum scales.

Main proof—From the definition $\bar{\rho}_{\boldsymbol{q}} = \mathcal{P}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\mathcal{P}$, we can write the projected density operator in Bloch basis as

$$\bar{\rho}_{\boldsymbol{q}} = \int_{\mathrm{BZ}} d^2\boldsymbol{k}\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle|\boldsymbol{k}+\boldsymbol{q}\rangle\langle\boldsymbol{k}| \tag{3}$$

where $|\boldsymbol{k}\rangle$ denotes the Bloch state with momentum $\boldsymbol{k}$ and $|u_{\boldsymbol{k}}\rangle = e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}|\boldsymbol{k}\rangle$ is the periodic part of the Bloch state [34], and the integration is over the Brillouin zone (BZ). Since we will only consider a single Bloch band, the band index has been suppressed. We will choose the gauge such that the Bloch states $|\boldsymbol{k}\rangle$ vary smoothly with $\boldsymbol{k}$, but in general they do not have Brillouin zone periodicity.

Using the orthogonality of Bloch states, we have

$$\bar{\rho}_{\boldsymbol{q}_1}\bar{\rho}_{\boldsymbol{q}_2} = \int_{\mathrm{BZ}} d^2\boldsymbol{k}\langle u_{\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2}|u_{\boldsymbol{k}+\boldsymbol{q}_2}\rangle\langle u_{\boldsymbol{k}+\boldsymbol{q}_2}|u_{\boldsymbol{k}}\rangle \times|\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2\rangle\langle\boldsymbol{k}|. \tag{4}$$

We then define

$$\tilde{f}(\boldsymbol{q}_1, \boldsymbol{q}_2; \boldsymbol{k}) = \frac{\langle u_{\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2}|u_{\boldsymbol{k}+\boldsymbol{q}_2}\rangle\langle u_{\boldsymbol{k}+\boldsymbol{q}_2}|u_{\boldsymbol{k}}\rangle}{\langle u_{\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2}|u_{\boldsymbol{k}}\rangle} \tag{5}$$

and

$$f(\boldsymbol{q}_1, \boldsymbol{q}_2)_{\boldsymbol{k}} = \tilde{f}(\boldsymbol{q}_1, \boldsymbol{q}_2; \boldsymbol{k}) - \tilde{f}(\boldsymbol{q}_2, \boldsymbol{q}_1; \boldsymbol{k}). \tag{6}$$

Plugging Eqs. (4) and (3) into Eq. (1) it is easy to establish that in order for the $\bar{\rho}_{\boldsymbol{q}}$ operators to be closed under commutation it must be the case that $f(\boldsymbol{q}_1, \boldsymbol{q}_2)_{\boldsymbol{k}}$ is $\boldsymbol{k}$ independent, so that

$$f(\boldsymbol{q}_1, \boldsymbol{q}_2)_{\boldsymbol{k}} = f(\boldsymbol{q}_1, \boldsymbol{q}_2). \tag{7}$$

The aim of this Letter is to show that if the function $f(\boldsymbol{q}_1, \boldsymbol{q}_2)_{\boldsymbol{k}}$, as defined in Eqs. (5) and (6), is independent of $\boldsymbol{k}$, it must take the form of Eq. (2).

We first consider the expression $f(\boldsymbol{q}-\boldsymbol{\epsilon}, \boldsymbol{\epsilon})_{\boldsymbol{k}}$ for small $\boldsymbol{\epsilon}$. Note that it is zero for $\boldsymbol{\epsilon} = 0$. Using the definitions in Eqs. (5) and (6), we expand to first order in $\boldsymbol{\epsilon}$, and we find

$$f(\boldsymbol{q}-\boldsymbol{\epsilon}, \boldsymbol{\epsilon})_{\boldsymbol{k}} = \boldsymbol{\epsilon}\cdot\boldsymbol{G}(\boldsymbol{q})_{\boldsymbol{k}} \tag{8}$$

with

$$\boldsymbol{G}(\boldsymbol{q})_{\boldsymbol{k}} = i\boldsymbol{\mathcal{A}}(\boldsymbol{k}+\boldsymbol{q}) - i\boldsymbol{\mathcal{A}}(\boldsymbol{k}) + \boldsymbol{\nabla}_{\boldsymbol{k}}\ln\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle, \tag{9}$$

where we have defined the Berry connection as

$$\mathcal{A}(\boldsymbol{k}) = -i\langle u_{\boldsymbol{k}}|\nabla_{\boldsymbol{k}}|u_{\boldsymbol{k}}\rangle. \tag{10}$$

For the moment, we have assumed that $\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle \neq 0$ so that its logarithm is well-defined. We will revisit the issue of possibly singularities later on.

In Ref. [6], the authors found that in two dimensions (in our notations), for small $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$

$$[\bar{\rho}_{\boldsymbol{q}_1}, \bar{\rho}_{\boldsymbol{q}_2}]|\boldsymbol{k}\rangle = i\Omega(\boldsymbol{k})\hat{z}\cdot(\boldsymbol{q}_1\times\boldsymbol{q}_2)\bar{\rho}_{\boldsymbol{q}_1+\boldsymbol{q}_2}|\boldsymbol{k}\rangle + o(q^2), \tag{11}$$

where $\Omega(\boldsymbol{k})$ is the Berry curvature at $\boldsymbol{k}$, and $o(q^2)$ means that we are not writing any terms that are strictly higher powers in $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. Eq. (11) immediately implies that for the projected density operators to form a closed algebra, the Berry curvature must be constant in the Brillouin zone. The result also obviously generalizes to three dimensions,

$$[\bar{\rho}_{\boldsymbol{q}_1}, \bar{\rho}_{\boldsymbol{q}_2}]|\boldsymbol{k}\rangle = i\boldsymbol{\Omega}(\boldsymbol{k})\cdot(\boldsymbol{q}_1\times\boldsymbol{q}_2)\bar{\rho}_{\boldsymbol{q}_1+\boldsymbol{q}_2}|\boldsymbol{k}\rangle + o(q^2), \tag{12}$$

under the same limit, where $\boldsymbol{\Omega}(\boldsymbol{k}) = \nabla_{\boldsymbol{k}}\times\mathcal{A}(\boldsymbol{k})$ is the Berry curvature vector. For constant Berry curvature, we can write,

$$\mathcal{A}(\boldsymbol{k}) = \frac{1}{2}\boldsymbol{\Omega}\times\boldsymbol{k} + \nabla_{\boldsymbol{k}}\phi(\boldsymbol{k}) \tag{13}$$

for some unknown real function ϕ.

Given this form, we have

$$\boldsymbol{G}(\boldsymbol{q})_{\boldsymbol{k}} - \frac{i}{2}\boldsymbol{\Omega}\times\boldsymbol{q} = \nabla_{\boldsymbol{k}}[i\phi(\boldsymbol{k}+\boldsymbol{q}) - i\phi(\boldsymbol{k}) + \ln\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle]. \tag{14}$$

In order for our algebra to close, $f(\boldsymbol{q}_1,\boldsymbol{q}_2)_{\boldsymbol{k}}$ must be independent of $\boldsymbol{k}$, which means the left-hand side of Eq. (14) is independent of $\boldsymbol{k}$. This implies that

$$i\phi(\boldsymbol{k}+\boldsymbol{q}) - i\phi(\boldsymbol{k}) + \ln\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle = \boldsymbol{k}\cdot\boldsymbol{W}(\boldsymbol{q}) + Y(\boldsymbol{q}) \tag{15}$$

for some unknown vector function $\boldsymbol{W}$ and scalar function Y. Thus we have

$$\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle = e^{-i\phi(\boldsymbol{k}+\boldsymbol{q})+i\phi(\boldsymbol{k})+\boldsymbol{k}\cdot\boldsymbol{W}(\boldsymbol{q})+Y(\boldsymbol{q})}. \tag{16}$$

We then substitute this result back into the definition of $\tilde{f}$ [Eq. (5)] and note that all the factors of ϕ immediately cancel. We then have

$$\tilde{f}(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{k}) = e^{(\boldsymbol{k}+\boldsymbol{q}_2)\cdot\boldsymbol{W}(\boldsymbol{q}_1)+\boldsymbol{k}\cdot\boldsymbol{W}(\boldsymbol{q}_2)-\boldsymbol{k}\cdot\boldsymbol{W}(\boldsymbol{q}_1+\boldsymbol{q}_2)+Y(\boldsymbol{q}_1)+Y(\boldsymbol{q}_2)-Y(\boldsymbol{q}_1+\boldsymbol{q}_2)}. \tag{17}$$

Here we recognize e^Y as being (part of) the form factor F. Now we demand $f(\boldsymbol{q}_1,\boldsymbol{q}_2)_{\boldsymbol{k}}$ in Eq. (6) be $\boldsymbol{k}$ independent. This then requires that

$$\boldsymbol{W}(\boldsymbol{q}_1) + \boldsymbol{W}(\boldsymbol{q}_2) - \boldsymbol{W}(\boldsymbol{q}_1+\boldsymbol{q}_2) = 0, \tag{18}$$

which is Cauchy's functional equation. Given continuity of the function $\boldsymbol{W}$, the only solution is [35]

$$\boldsymbol{W}(\boldsymbol{q}) = \underline{M}\boldsymbol{q}, \tag{19}$$

where $\underline{M}$ is some fixed matrix acting on $\boldsymbol{q}$. We then obtain

$$f(\boldsymbol{q}_1,\boldsymbol{q}_2) = \frac{\tilde{F}(\boldsymbol{q}_1)\tilde{F}(\boldsymbol{q}_2)}{\tilde{F}(\boldsymbol{q}_1+\boldsymbol{q}_2)}\left[e^{\boldsymbol{q}_2\cdot\underline{M}\boldsymbol{q}_1} - e^{\boldsymbol{q}_1\cdot\underline{M}\boldsymbol{q}_2}\right] \tag{20}$$

with $\tilde{F} = e^Y$. The symmetric part of $\underline{M}$ can be absorbed into the form factors F by defining

$$F(\boldsymbol{q}) = \tilde{F}(\boldsymbol{q})e^{-\frac{1}{2}\boldsymbol{q}\cdot\underline{M}\boldsymbol{q}}, \tag{21}$$

and we then have

$$f(\boldsymbol{q}_1,\boldsymbol{q}_2) = 2\frac{F(\boldsymbol{q}_1)F(\boldsymbol{q}_2)}{F(\boldsymbol{q}_1+\boldsymbol{q}_2)}\sinh\left[\frac{1}{2}(\boldsymbol{q}_2\cdot\underline{M}\boldsymbol{q}_1 - \boldsymbol{q}_1\cdot\underline{M}\boldsymbol{q}_2)\right]. \tag{22}$$

Define vector $\Omega_i = i\epsilon_{ijk}M_{jk}$ [36], and we find that

$$f(\boldsymbol{q}_1,\boldsymbol{q}_2) = 2i\frac{F(\boldsymbol{q}_1)F(\boldsymbol{q}_2)}{F(\boldsymbol{q}_1+\boldsymbol{q}_2)}\sin\left(\frac{1}{2}\boldsymbol{\Omega}\cdot(\boldsymbol{q}_1\times\boldsymbol{q}_2)\right)$$

in the same form as Eq. (2).

Given the form of Eq. (16), as $\boldsymbol{q}\to\boldsymbol{0}$, we have $Y(\boldsymbol{0})\to 0$. Thus for small $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, the prefactor $[F(\boldsymbol{q}_1)F(\boldsymbol{q}_2)/F(\boldsymbol{q}_1+\boldsymbol{q}_2)]$ becomes unity. Now, matching the small $\boldsymbol{q}$ limit in Eq. (12), we identify the constant vector $\boldsymbol{\Omega}$ as the Berry curvature vector (in two dimensions, it simply reduces to $\boldsymbol{\Omega} = \Omega\hat{z}$, where Ω is the scalar Berry curvature). That $F(\boldsymbol{q}) = F^*(-\boldsymbol{q})$ can be easily established from its construction, as elaborated in the End Matter, and we have reached the desired result.

Singularities—Now we revisit the issue of possible singular behaviour in Eq. (9) due to vanishing $\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle$. This introduces two potential issues. First, at where $|\langle u_{\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2}|u_{\boldsymbol{k}}\rangle|$ vanishes, $f(\boldsymbol{q}_1,\boldsymbol{q}_2)$ diverges [37] and therefore is ill-defined. However, this is not an issue *per se*. This is because since overlaps are smooth, we can assume $\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle$ only vanishes on lower dimensional submanifolds. That is to say, $f(\boldsymbol{q}_1,\boldsymbol{q}_2)$ is ill-defined only on a set of measure zero. Second, it is plausible that in the space of $\boldsymbol{k}$, $\boldsymbol{q}$, the submanifold where $\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle$ vanishes could separate the space into disconnected regions. If this is the case, since vanishing $\langle u_{k+q}|u_k\rangle$ leads to logarithmic singularities in Eq. (15), we would have to check whether Eq. (15) could have different solutions in the different regions.

Consider a region free of singularity. From Eq. (16), we obtain

$$\langle u_{\boldsymbol{k}}|u_{\boldsymbol{k}+\boldsymbol{q}}\rangle = e^{-i\phi(\boldsymbol{k})+i\phi(\boldsymbol{k}+\boldsymbol{q})+(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{W}(-\boldsymbol{q})+Y(-\boldsymbol{q})}. \quad (23)$$

But $\langle u_{\boldsymbol{k}}|u_{\boldsymbol{k}+\boldsymbol{q}}\rangle = \langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle^*$, so comparing Eqs. (16) and (23) we have

$$e^{(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{W}(-\boldsymbol{q})+Y(-\boldsymbol{q})} = e^{\boldsymbol{k}\cdot\boldsymbol{W}^*(\boldsymbol{q})+Y^*(\boldsymbol{q})}. \quad (24)$$

Comparing the $\boldsymbol{k}$ dependence, we have $\boldsymbol{W}(-\boldsymbol{q}) = \boldsymbol{W}^*(\boldsymbol{q})$. Knowing that $\boldsymbol{W}(\boldsymbol{q}) = \underline{M}\boldsymbol{q}$ we find $\underline{M}^* = -\underline{M}$, i.e., $\boldsymbol{W}$ is purely imaginary. Thus

$$|\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle| = |e^{Y(\boldsymbol{q})}| \quad (25)$$

is independent of $\boldsymbol{k}$.

This means, for a fixed $\boldsymbol{q}$, in each region where $\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle$ does not vanish, $|\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle|$ must be constant. Given $\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle$ is continuous, we conclude that for a fixed $\boldsymbol{q}$, if $\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle$ is nonzero at any $\boldsymbol{k}$, it is nonzero for all $\boldsymbol{k}$. This means the only possible singular behavior in Eq. (15) is where the logarithm diverges for a submanifold of $\boldsymbol{q}$ and all $\boldsymbol{k}$. In other words, the position of singularities is only a function of $\boldsymbol{q}$, but not a function of $\boldsymbol{k}$.

As discussed, our concern is the possibility that the singularities divide the momentum subspace into different regions with independent-chosen solutions for the commutator algebra. However, this cannot be the case. Since the positions of singularities is independent of $\boldsymbol{k}$, Eq. (15) continues to hold, except on a measure zero set of $\boldsymbol{q}$. From which, following the same derivation in the main proof, we again arrive at the Cauchy's functional equation [Eq. (18)], except that we allow for discontinuities or singularities in $\boldsymbol{W}(\boldsymbol{q})$ on a set of measure zero. Allowing for such possible singularities does not change the solution to the Cauchy's equation [38]. Therefore, the main proof presented above holds even with singularities. In more intuitive terms, different regions separated by singularities are related by Eq. (19) as $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_1+\boldsymbol{q}_2$ do not have to lie in the same singularity free region. Thus we confirm that the form of Eq. (2) must always hold for a closed projected density algebra with the only possibly singularities arising where $F(\boldsymbol{q}_1+\boldsymbol{q}_2)=0$.

Further results—(1) We can also consider closed multiplication algebras

$$\bar{\rho}_{\boldsymbol{q}_1}\bar{\rho}_{\boldsymbol{q}_2} = \tilde{f}(\boldsymbol{q}_1,\boldsymbol{q}_2)\bar{\rho}_{\boldsymbol{q}_1+\boldsymbol{q}_2}. \quad (26)$$

Obviously this implies a closed commutator algebra.

Landau levels do indeed satisfy such a closed multiplication algebra as well as the closed commutation algebra. This is not a coincidence. From the form of Eq. (17) we also obtain the most general form of $\tilde{f}$, independent of $\boldsymbol{k}$. That is, projected density operators that are closed under commutation are also closed under multiplication.

(2) The quantum geometry tensor is defined as

$$Q^{ab}(\boldsymbol{k}) = \langle\partial_{k_a}u_{\boldsymbol{k}}|Q_{\boldsymbol{k}}|\partial_{k_b}u_{\boldsymbol{k}}\rangle, \quad (27)$$

where $Q_{\boldsymbol{k}} = 1-|u_{\boldsymbol{k}}\rangle\langle u_{\boldsymbol{k}}|$. Its symmetric part is the quantum metric tensor while its antisymmetric part is the Berry curvature. Assuming a GMP algebra, it is easy to use Eq. (16) to establish that the $Q_{ab}(\boldsymbol{k})$ must also be independent of $\boldsymbol{k}$, with the proof given in the End Matter. (This result was known for GMP algebra with LLL-like form factors previously [7] but, to our knowledge, not more generally.) We emphasize that the converse is not true: a band with both constant Berry curvature and constant quantum metric tensor is not necessarily GMP. An example of this is given explicitly in the End Matter.

Concluding remarks—We recall that Girvin-MacDonald-Platzman algebra is a closed commutation algebra obeyed by projected density operators that naturally arise from the study of Landau levels. In this Letter, we prove that *the GMP algebra is the only possible form of closed algebra that projected density operator of a band can take in two dimensions and three dimensions*, up to the choice of form factors. We believe this result is of mathematical interest on its own. More relevant to condensed matter research, this result demonstrates the unique place the GMP algebra occupies in the study of Chern bands, and provides some further theoretical motivation for designing Chern bands similar to Landau levels. We further remark that while in the literature the greatest focus has usually been on designing LLL-like Chern bands, for example, the formulation of ideal flatband condition, an alternative perspective would be to design Chern bands with (approximate) closed algebra, which would favor the use of somewhat different criteria, as GMP algebra with generic form factors do not obey the ideal flat band condition (see also Refs. [39,40]). We emphasize that the quantum metric tensor only reflects the long wavelength properties of the system, as further elaborated in the End Matter, where a band with flat quantum geometry tensor but without closed density algebra is explicitly constructed. We note, in addition, that quantum geometry alone is not enough to determine what type of ground state a system is likely to exhibit—the details of the interaction are clearly just as important [20,23,41].

Lastly, we note that while bands that satisfy GMP algebra can serve as benchmarks for less ideal systems; especially in the context of band design for FCIs, there is a rich amount of physics that can be explored for GMP algebra beyond the prototypical examples of simple

Landau levels. For example, graphene systems under magnetic field can host bands that combine orbital contents of different Landau levels [42,43], enriching the family of form factors that can be explored. With the development of synthetic magnetic field [44] in cold atom experiments, we expect that designing bands that host GMP algebra with an even more diverse range of orbital structures will become possible.

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[33] Except possibly on a lower-dimensional manifold, i.e., a set of measure zero, where $F(\boldsymbol{q}_1+\boldsymbol{q}_2)=0$, making $f(\boldsymbol{q}_1,\boldsymbol{q}_2)$ ill-defined.

[34] For any lattice translation vector $\boldsymbol{R}$, we have $\langle \boldsymbol{r}+\boldsymbol{R}|\boldsymbol{k}\rangle = \langle \boldsymbol{r}|\boldsymbol{k}\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{R}}$. This immediately implies $\langle \boldsymbol{r}+\boldsymbol{R}|u_{\boldsymbol{k}}\rangle = \langle \boldsymbol{r}|u_{\boldsymbol{k}}\rangle$ for $|u_{\boldsymbol{k}}\rangle = e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}|\boldsymbol{k}\rangle$.

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[37] We will quickly show that whether $|\langle u_{\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2}|u_{\boldsymbol{k}}\rangle|$ vanishes depends only on $\boldsymbol{q}_1+\boldsymbol{q}_2$ but not on $\boldsymbol{k}$, so that this statement is unambiguous.

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End Matter

Appendix A—Starting with Eq. (24) and comparing the $\boldsymbol{k}$-independent component on both sides, we find that $Y(-\boldsymbol{q}) = Y^*(\boldsymbol{q}) - \boldsymbol{q}\cdot\boldsymbol{W}(-\boldsymbol{q}) = Y^*(\boldsymbol{q}) + \boldsymbol{q}\cdot\underline{M}\boldsymbol{q}$. From Eq. (21) we have

$$F(\boldsymbol{q}) = e^{Y(\boldsymbol{q})-\frac{1}{2}\boldsymbol{q}\cdot\underline{M}\boldsymbol{q}}.$$

This means

$$\begin{aligned} F(-\boldsymbol{q}) &= e^{Y(-\boldsymbol{q})-\frac{1}{2}\boldsymbol{q}\cdot\underline{M}\boldsymbol{q}} = e^{Y^*(\boldsymbol{q})+\frac{1}{2}\boldsymbol{q}\cdot\underline{M}\boldsymbol{q}} \\ &= e^{Y^*(\boldsymbol{q})-\frac{1}{2}\boldsymbol{q}\cdot\underline{M}^*\boldsymbol{q}} = F^*(\boldsymbol{q}), \end{aligned}$$

where we have used $\underline{M}^* = -\underline{M}$. As such, we have established $F(-\boldsymbol{q}) = F^*(\boldsymbol{q})$.

Appendix B: Proof that a closed algebra implies constant quantum geometry—Following the proof in the main text, if a density operator projected into a Bloch band obeys a closed algebra, we can write, using Eqs. (16) and (19),

$$\langle u_{\boldsymbol{k}'}|u_{\boldsymbol{k}}\rangle = e^{h(\boldsymbol{k}',\boldsymbol{k})} \tag{B1}$$

with

$$h(\boldsymbol{k}',\boldsymbol{k}) = -i\phi(\boldsymbol{k}') + i\phi(\boldsymbol{k}) + \boldsymbol{k}\cdot\underline{M}(\boldsymbol{k}'-\boldsymbol{k}) + Y(\boldsymbol{k}'-\boldsymbol{k}).$$

The quantum geometry tensor, as defined by Eq. (27), is related to the above inner products by

$$Q^{ab}(\boldsymbol{k}) = [\partial_{k'_a}\partial_{k_b}\langle u_{\boldsymbol{k}'}|u_{\boldsymbol{k}}\rangle - (\partial_{k'_a}\langle u_{\boldsymbol{k}'}|u_{\boldsymbol{k}}\rangle)(\partial_{k_b}\langle u_{\boldsymbol{k}'}|u_{\boldsymbol{k}}\rangle)]|_{\boldsymbol{k}'=\boldsymbol{k}},$$

which can be simplified to

$$Q^{ab}(\boldsymbol{k}) = \partial_{k'_a}\partial_{k_b} h(\boldsymbol{k}',\boldsymbol{k})|_{\boldsymbol{k}'=\boldsymbol{k}},$$

where we have used $h(\boldsymbol{k},\boldsymbol{k})=0$. Using the explicit form of h, we find

$$Q^{ab}(\boldsymbol{k}) = -(\partial_a\partial_b Y)(\boldsymbol{0}) + M_{ba},$$

which is explicitly independent of $\boldsymbol{k}$, completing the proof.

We further note that since GMP algebra is obviously closed, this immediately means that if a band obeys GMP algebra with arbitrary form factors, its quantum geometry tensor is constant.

Appendix C: Flat quantum geometry without closed algebra—Consider Landau levels on a torus or infinite plane. We choose unit cell dimensions a_x, a_y such that $a_x a_y$ encloses one magnetic flux quantum. Define $\boldsymbol{a}_1 = (a_x, 0)$ and $\boldsymbol{a}_2 = (0, a_y)$, and the magnetic translation by $\boldsymbol{R}$ is denoted by $t_{\boldsymbol{R}}$ with $t_{-\boldsymbol{R}} = t^\dagger_{\boldsymbol{R}}$. The usual Landau level Hamiltonian is denoted $H_0 = \hbar\omega_c(a^\dagger a + \frac{1}{2})$, where $a^\dagger$ is the Landau level raising operator and ω_c is the cyclotron frequency. We use units where the magnetic length $l_B \equiv 1$ and $\hbar \equiv 1$.

We consider a system with a three-state pseudo-spin degree of freedom. Denoting $t \equiv t_{2\boldsymbol{a}_1}$ for simplicity, we consider the following Hamiltonian:

$$\begin{pmatrix} H_0 + \frac{1}{2}(t+t^\dagger)\epsilon_0 & \frac{i}{\sqrt{2}}(t-t^\dagger)a\epsilon_0 & \left(-1+\frac{1}{2}(t+t^\dagger)\right)\frac{1}{\sqrt{2}}a^2\epsilon_0 \\ \frac{i}{\sqrt{2}}(t-t^\dagger)a^\dagger\epsilon_0 & H_0 - \omega_c - (1+t+t^\dagger)\epsilon_0 & \frac{i}{2}(t-t^\dagger)a\epsilon_0 \\ \left(-1+\frac{1}{2}(t+t^\dagger)\right)\frac{1}{\sqrt{2}}(a^\dagger)^2\epsilon_0 & \frac{i}{2}(t-t^\dagger)a^\dagger\epsilon_0 & H_0 - 2\omega_c + \frac{1}{2}(t+t^\dagger)\epsilon_0 \end{pmatrix}.$$

This Hamiltonian has a flat band at $E = \frac{1}{2}\omega_c - 3\epsilon_0$, with the periodic parts of its wave functions given by (assuming ω_c and ϵ_0 are chosen to avoid accidental degeneracy)

$$u_{\boldsymbol{k}}(\boldsymbol{r}) = \begin{pmatrix} \sin(k_x a_x)u_{0\boldsymbol{k}}(\boldsymbol{r})/\sqrt{2} \\ \cos(k_x a_x)u_{1\boldsymbol{k}}(\boldsymbol{r}) \\ \sin(k_x a_x)u_{2\boldsymbol{k}}(\boldsymbol{r})/\sqrt{2} \end{pmatrix},$$

where $u_{n,\boldsymbol{k}}$ is the periodic part of the usual nth Landau level wave function. (In the context of Landau levels, we understand "periodic" as invariant under *magnetic* translation operators.) We can show that (using the same convention as Ref. [17])

$$\langle u_{\boldsymbol{k}+\boldsymbol{q}}|u_{\boldsymbol{k}}\rangle = e^{-\boldsymbol{q}^2/4}e^{iq_x(k_y+q_y/2)}\left[\cos(q_x a_x)\left(1-\frac{\boldsymbol{q}^2}{2}\right) + \frac{1}{16}\sin\left((k_x+q_x)a_x\right)\sin(k_x a_x)\boldsymbol{q}^4\right], \tag{C1}$$

where we have used the fact that $\langle u_{n,\boldsymbol{k}+\boldsymbol{q}}|u_{n,\boldsymbol{k}}\rangle = e^{-\boldsymbol{q}^2/4}e^{iq_x(k_y+q_y/2)}L_n(q^2/2)$ with L_n being the Laguerre polynomial.

It can be verified straightforwardly that the quantum geometry tensor [Eq. (27)] is $\boldsymbol{k}$ independent for the band constructed above, given by

$$\underline{Q} = \begin{pmatrix} 5/2 & i/2 \\ -i/2 & 3/2 \end{pmatrix}.$$

This means that both the Berry curvature and the quantum metric tensor are flat. However, due to the nontrivial $\boldsymbol{k}$ dependence in the $\boldsymbol{q}^4$ term in Eq. (C1), $f(\boldsymbol{q}_1,\boldsymbol{q}_2)_{\boldsymbol{k}}$ is not $\boldsymbol{k}$ independent, and the density algebra is not closed. This example shows that flat quantum geometry tensor is not a sufficient condition for closed density algebra, and it reflects that quantum geometry is a long wavelength behavior which does not fix the behavior of the system at all momentum scales. (Unless the quantum metric tensor satisfies a more restrictive condition on its determinant, which leads to GMP algebra with LLL-like form factors at all momentum scales [7]. This special case does not apply to GMP algebra with general form factors.)

Suppose we consider the following Hamiltonian instead:

$$\begin{pmatrix} H_0 + (t+t^\dagger)\epsilon_0 & i(t-t^\dagger)\epsilon_0 \\ i(t-t^\dagger)\epsilon_0 & H_0 - (t+t^\dagger)\epsilon_0 \end{pmatrix}.$$

It has a flat band at $E = \frac{3}{2}\omega_c - 2\epsilon_0$, with the periodic parts of its wave functions given by (assuming $4\epsilon \neq n\omega_c$ for any integer n to avoid accidental degeneracy)

$$\tilde{u}_{\boldsymbol{k}}(\boldsymbol{r}) = \begin{pmatrix} \sin(k_x a_x)u_{1\boldsymbol{k}}(\boldsymbol{r}) \\ \cos(k_x a_x)u_{1\boldsymbol{k}}(\boldsymbol{r}) \end{pmatrix}.$$

The overlaps are given by

$$\langle \tilde{u}_{\boldsymbol{k}+\boldsymbol{q}}|\tilde{u}_{\boldsymbol{k}}\rangle = e^{-\boldsymbol{q}^2/4}e^{iq_x(k_y+q_y/2)}\cos(q_x a_x)\left(1-\frac{\boldsymbol{q}^2}{2}\right),$$

which is the same as Eq. (C1) but without the $\boldsymbol{q}^4$ term. The quantum geometry tensor is identical with the previous example, but the generalized GMP algebra will be obeyed. This shows that knowledge about quantum geometry tensor is generally insufficient for diagnosing general whether the GMP algebra holds.