# THE BETHE-SALPETER EQUATION AND UNSTABLE PARTICLES

A thesis submitted in the University of Oxford

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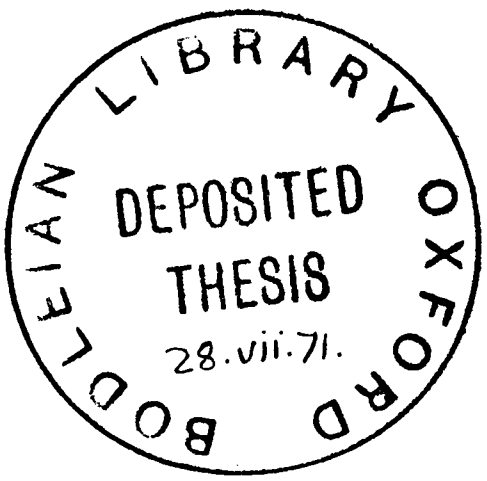


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ABSTRACT

# INTRODUCTION

This thesis contains some results about composite particles which have been obtained from field theory. The Bethe-Salpeter scattering equation is considered from various mathematical and computational points of view and a new procedure for numerically inverting the Bethe-Salpeter equation in the scattering region is described. We give the results of applying this method to the $\pi$-$\rho$ bootstrap.

The first part of Chapter I introduces the main ideas of field theory which are required, including Green's functions. The space of states of a non-interacting system is described as a direct sum of channels, and a covariant parametrization of the non-normalizable states in a channel c is given in terms of the total 4-momentum E and another parameter $s^c$ which specifies the 3-momenta of the particles constituting the channel. This is then used to define analytic continuation of the S-matrix elements with respect to the total energy. By applying the energy-analytic representation of the Green's functions and relying on certain explicitly stated assumptions about the analyticity of off-shell scattering elements, we derive a unitarity relation which is the off-shell analogue of on-shell hermitian unitarity. Our definition of a resonance-pole gives immediately the factorization of the S-matrix residues, and the off-shell unitarity relation is then used to prove that the off-shell scattering elements also have poles

with factorizable residues at a resonance-energy. This behaviour, which is easy to obtain at bound-state energies, has usually been assumed to hold at resonance-energies although this is the first proof of the fact, to our knowledge.

The next chapter is principally concerned with the extension to elastic and complex energies of the Wick-rotated scalar Bethe-Salpeter equation in the ladder approximation. Whilst the extension to real energies in the elastic region has previously been considered by a number of authors[23-27] and Greenman[31] has obtained important results for any real energy, a non-singular equation is derived here which does not require contour-distortions and gives the analytic continuation off the threshold cut directly. We prove that a unique solution exists in a correctness class of $C^1$ functions similar to that used by Lovelace[32a].

The phase shift criterion for a narrow width resonance, that $\delta_\ell$ pass sharply through $\pi/2$, causes the Bethe-Salpeter Heitler matrix (which gives $\tan\delta_\ell$ on-shell) to have an on-shell infinity. We show in Chapter III that this necessarily arises from a pole in the off-shell Heitler matrix which has a factorizable residue. Thus when $\delta_\ell = \pi/2$ the Heitler matrix has a wave-function and there is a genuine eigenvalue problem. The chapter also contains a description of the BSE in the energy-analytic representation when one subtaction is required and we give an improved approximation to the BSE. The latter

has been independently discovered by Cohen[61], with whose numerical results we have found agreement.

In Chapter IV a simple model of $\pi$- and $\rho$-mesons reacting via the Yang-Mills coupling is examined in the bootstrap limit. The wave-function renormalization-constant criterion for $\rho$-compositeness leads, formally, to an explicit expression for the $\rho$-field in terms of the $\pi$-field, and the current-field identity gives the value of a limit ($\alpha$) which appears in the analysis. The behaviour of the Green's function equations in the composite limit is also considered.

The result of substituting our expression for $\rho$ into the lagrangian produces a non-linear pion interaction which is similar in character to chiral-invariant zero-mass pion interactions[62], being of the form $\partial_\mu \pi^i g_{ij}(\underline{\pi}) \partial^\mu \pi^j$, but the isospin form $g_{ij}$ is of rank 2 and therefore singular. When the lowest order term of the lagrangian is retained, a crude estimate of the $\ell = 1$, $I = 1$ partial wave amplitude contains a resonance with the mass and width of $\rho$ provided that $\alpha$ (above) has a certain value. This compares fairly favourably with the value obtained previously.

The last chapter contains a mesh-point method for numerically inverting the energy-analytic BSE in the scattering region, and our results for the scalar equation agree with Schwarz and Zemach's figures[26] to within about one percent. Unlike configuration-space methods it is no more difficult to use

with exchange potentials in which derivative vertices appear, such as the Bethe-Salpeter $\rho$-exchange potential for $\pi\pi$-scattering. When the method was used to solve this problem it was found to be impossible, at any value of the cutoff, to choose an input mass and coupling for the exchange potential which produces a similar resonance in the direct channel.

The scattering BSE has been little used because the techniques for dealing with its singularities have only recently been developed. Now the equation contains a good deal of structure (including inelastic thresholds) and its physical predications are of considerable interest. To our knowledge the only physical scattering processes that have hitherto been calculated using the BSE are nucleon-nucleon interactions[30,63]. Our result for the $\pi$-$\rho$ bootstrap is disappointing, particularly since the corresponding N/D calculation is more successful in producing a self-consistent solution, but the relative order of magnitude of our input and output values suggest that an extended version of the model would be worth constructing. A multichannel calculation incorporating other mesons such as $\omega$, $f^0$ is likely to lead to improvements, although we are in no position, as yet, to carry it out.

Virtually no mention is made in this thesis of Regge behaviour or the Veneziano model, which may appear odd in view of the fact that latter provides the most successful description of $\pi\pi$-scattering to date. Our principal interest here is with

dynamical models, and whilst Veneziano theory may very well be correct, its possible derivation (together with Regge behaviour) from a dynamical theory cannot be discounted at the present time.

# NOTATION

Atomic units ($\hbar = c = 1$) are used throughout.

3-vectors are underlined, and when a is a 4-vector its time-space decomposition is written $(a_0, \underline{a})$. The Lorentz product of the 4-vectors a and b is $ab = a_0 b_0 - \underline{a}.\underline{b}$. When a 4-momentum E is in a centre of mass frame it is often written $(E, \underline{0})$, the symbol E representing both the 4-vector and its single energy component.

Inner products of state-vectors are linear in the left argument, and an operator sandwiched between state-vectors acts on the left-hand vector. The symbol T time-orders field operators so that the earliest is furthest left. With this convention the expressions for scattering elements are consistent with the diagram notation in which particles enter from the left and leave on the right.

# I - FIELD THEORY

## § 1. Free Fields and Fock Space

In this paragraph a covariant kinematic description of n-particle free states is given in which the states of fixed total energy are easily distinguished. This will be used when considering the analytic continuation of the scattering elements with respect to the total energy.

Consider a trivial field theory of non-interacting scalar or pseudoscalar particles. The hilbert space is a direct sum of orthogonal "channels". An n-particle channel c consists of all possible states of the n-particles and is generated by the non-normalizable vectors

$$|\underline{p}_1, \ldots, \underline{p}_n\rangle = a_1(\underline{p}_1) \ldots a_n(\underline{p}_n) |0\rangle$$

where $a_1, \ldots, a_n$ are the creation operators of the n particles defining the channel and $\underline{p}_1, \ldots, \underline{p}_n$ are their 3-momenta.

Now let E be the sum of n 4-momenta $p_1, \ldots, p_n$. Provided $\underline{E} \neq \underline{0}$ there exists a unique Lorentz transformation leaving invariant the plane defined by $(\lambda E_0, \mu \underline{E})$, which maps E into the time-vector $(\sqrt{E^2}, \underline{0})$; when $\underline{E} = \underline{0}$ the transformation is taken as the identity. $\tilde{p}_1, \ldots, \tilde{p}_n$, the images of $p_1, \ldots p_n$ are the 4-momenta measured in a centre of mass frame. This frame depends only on the direction (in 4-space) of the 4-momentum E, and for the time being we will only

consider states with total 4-momentum in a fixed direction, and assume that we are already in the centre of mass frame.

$p_i$ are on-shell and therefore satisfy $p_i^2 = m_i^2$, $p_{iO} > O$. Thus $p_{iO} = (\underline{p}_i^2 + m_i^2)^{\frac{1}{2}}$, and hence, with $E \equiv (E, \underline{O})$

$$(1.1) \qquad E = \sum_{i=1}^{n} (\underline{p}_i^2 + m_i^2)^{\frac{1}{2}}$$

$$(1.2) \qquad \underline{O} = \sum_{i=1}^{n} \underline{p}_i$$

$m_1 + \ldots + m_n$ is called the threshold energy of the channel c because it is the minimum total energy of any state.

Consider the 3n-vector $\underline{P} = (\underline{p}_1, \ldots, \underline{p}_n)$ and define $\underline{P}^2 = \underline{p}_1^2 + \ldots + \underline{p}_n^2$. Equation (1.2) restricts $\underline{P}$ to lie in a (3n - 3)-dimensional subspace W, and (1.1) confines $\underline{P}$ to a hypersurface. A half-ray from the origin intersects the hypersurface in just one point, so that the possible on-shell momenta of fixed total energy can be represented by the half-rays in W. Alternatively they can be represented by the points of the unit sphere in W, $\underline{P}^2 = 1$, $P \in W$. This sphere will be denoted by $S^c$ and a typical point on it will be written $s^c$, with 3-dimensional components $\underline{s}_1^c, \underline{s}_2^c, \ldots \underline{s}_n^c$. The physical on-shell momenta $\underline{p}_1, \ldots, \underline{p}_n$ represented by $s^c$ are given by $\underline{p}_i = \lambda \underline{s}_i^c$ where $\lambda$ satisfies

$$E = \sum_{i=1}^{n} (\lambda^2 (\underline{s}_i^c)^2 + m_i^2)^{\frac{1}{2}}$$

For fixed $s^c$ and E satisfying $E > m_1 + \ldots + m_n$, there is, as we said, precisely one positive value $\lambda(E)$ satisfying this equation. It is not difficult to show that $\lambda(E)$ is analytic in a neighbourhood of the interval $E > m_1 + \ldots + m_n$ and that as $E \longrightarrow m_1 + \ldots + m_n$, $\lambda \longrightarrow 0$. Thus, when E is near to threshold, $E - (m_1 + \ldots + m_n) \sim \lambda^2 \sum_{i=1}^{n} (\underline{s}_i^c)^2/2m_i$ and $\lambda$ has a square-root singularity at the threshold point $m_1 + \ldots + m_n$. If a cut is made along $E > m_1 + \ldots + m_n$ then $\lambda$ is analytic in a neighbourhood of the cut and satisfies $\lambda(E^*) = -\lambda^*(E)$. The on-shell 4-momenta $(p_1, p_2, \ldots, p_n)$ will be denoted by the 4n-vector P. Thus for total 4-momentum $(E, \underline{0})$ and on-shell 3-momenta represented by $s^c$, $P = P(s^c, E)$ where, for fixed $s^c$, P is analytic in a neighbourhood of the threshold cut. The result of continuation around the cut through unphysical energies E from the upper lip to the lower lip reverses the sign of all the 3-momenta and leaves the energies fixed: it is therefore equivalent to space-inversion.

Consider now all states in the channel c, and not just those with total 3-momentum equal to zero. The states $|\underline{p}_1, \ldots, \underline{p}_n\rangle$ are not proper states in the hilbert space but state-valued distributions over $R^{3n}$, with inner products

$$\langle \underline{p}_1, \ldots \underline{p}_n | \underline{q}_1, \ldots, \underline{q}_n \rangle = \prod_{i=1}^{n} 2(\underline{p}_i^2 + m_i^2)^{\frac{1}{2}} \delta^3(\underline{p}_i - \underline{q}_i)$$

The normalized states are obtained by smearing $|\underline{p}_1, \ldots \underline{p}_n\rangle$ by smooth test-functions $f(p_1, \ldots, p_n)$ of compact support, so that

the hilbert space, $H^c$ of the channel is spanned by states

$$x(f) = \int d^3\underline{p}_1 \ldots d^3\underline{p}_n \; f(\underline{p}_1 \ldots \underline{p}_n) \; |\underline{p}_1, \ldots, \underline{p}_n\rangle$$

where

$$(1.3) \quad \langle x(f) \mid x(g)\rangle = \int f \, g^* \, d\mu$$

and $$d\mu = \prod_{i=1}^{n} 2(\underline{p}_i^2+m_i^2)^{\frac{1}{2}} d\underline{p}_i$$

The completion of the space of test-functions with the inner product given by the right-hand side of (1.3) is $L^2(R^{3n},\mu)$, and since the mapping $L^2(R^{3n},\mu) \longrightarrow H^c$ given by $f \longrightarrow x(f)$ is isometric and has a dense domain and a dense range, it can be extended to a unitary equivalence.

As stated above, with respect to any standard frame a unique Lorentz transformation can be defined which sends the total 4-momentum of n on-shell 4-momenta into a time-vector. Thus, by the previous argument, there is a mapping

$$(\underline{p}_1, \ldots, \underline{p}_n) \xrightarrow{F} (E, \; s^c)$$

where E is the total 4-momentum and the directions of the 3-momenta are represented in the centre of mass frame by $s^c \in S^c$. F induces a unitary mapping

$$L^2(V^c \times S^c, \; \bar{\mu}) \longrightarrow L^2(R^{3n}, \; \mu)$$

where $d\mu/d\bar{\mu}$ is the Jacobian of F, and $V^c$ is the set of E such that $E^2 > (m_1+ \ldots +m_n)^2$, $E_0 > 0$. Thus the hilbert space $H^c$ of the channel is unitarily equivalent to $L^2(V^c \times S^c, \; \bar{\mu})$. By multiplying

the functions in this $L^2$ space by a weighting function, $\mu$ can be replaced by the product of the Lebesgue measures on $V^c$ and $S^c$, so that $d\tilde{\mu} \to d^4E\ ds^c$. The states whose total energy lies in a set A and whose 3-momenta in the centre of mass frame are represented by directions in a subset B of $S^c$ are given by functions in $L^2(V^c \times S^c)$ which have support in A x B. The non-normalizable functions $\delta^4(E-\tilde{E})\ \delta(s^c,\tilde{s}^c)$, where $\delta(s^c,\tilde{s}^c)$ is the $\delta$-function on $S^c$ with support on $\tilde{s}^c$, represents a state $|\tilde{E},\tilde{s}^c\rangle$ with total 4-momentum E and 3-momenta in the centre of mass frame pointing in the direction $\tilde{s}^c$. Clearly

$$\langle \tilde{E},\tilde{s}^c |\ E,s^c\rangle \quad = \quad \delta^4(E-\tilde{E})\ \delta(s^c,\tilde{s}^c).$$

We shall find it convenient to imagine $L^2(V^c \times S^c)$ as a hilbert bundle [1] $V^c \Delta L^2(S^c)$ of identical hilbert spaces $L^2(S^c)$, one sitting over each point of the base $V^c$. Thus the elements of $L^2(V^c \times S^c)$ are considered as cross-sections $\Phi : V^c \to L^2(S^c)$ in which $E \to \Phi_E \in L^2(S^c)$ and $\langle \Phi | \Phi \rangle < \infty$ where $\langle \Phi | \Psi \rangle = \int_{V^c} \langle \Phi_E, \Psi_E \rangle d^4E$. It is easily seen that the connexion between a function $\phi$ in $L^2(V^c \times S^c)$ and its cross-section $\Phi$ is the obvious one, namely $\phi(E,s^c) = \Phi_E(s^c)$ a.e. Hence the space $H^c$ can be identified with the set of cross sections by a unitary mapping $T^c$: $H^c \to V^c \Delta L^2(S^c)$. We refer to $L^2(S^c)$ as the space of states in c of constant 4-momentum. It can, of course, only be defined as such if E lies in $V^c$ i.e. E is above threshold. The non-normalizable states $|\tilde{s}^c\rangle \equiv \delta(s^c,\tilde{s}^c)$ in

$H^c$ satisfy $\langle \tilde{s}^c \mid s^c \rangle = \delta(s^c, \tilde{s}^c)$

The unitary correspondence

$$T : \quad H \longrightarrow \bigoplus_c \; V^c \; \Delta \; L^2(S^c)$$

will be referred to as the canonical mapping. When some of the particles comprising a channel are identical it is necessary, in the equivalence to restrict to symmetric (or in the case of fermions, antisymmetric) subspaces of $L^2(S^c)$.

## § 2. The S-Matrix

In the LSZ formulation of the theory of interacting fields[2] the behaviour of a system is such that as $t \to \pm\infty$ the elementary fields tend in weak limits to free fields, so that the momenta of the in- and out-particles can be described by free states of the asymptotic fields.

Corresponding to these asymptotic limits there are two canonical maps

$$T_\pm : \quad H \longrightarrow \bigoplus_c \; V^c \; \Delta \; L^2(S^c).$$

and the S-matrix is thus represented by the operator $S \equiv T_- \, T_+^{-1}$. The S-matrix must conserve 4-momentum, and if a final state $\phi_{out}$ has 4-momentum in a set A then its initial state $\phi_{in} \equiv S \phi_{out}$ must also have 4-momentum in A. Now if $\phi_{out} = \sum_c{}' \Phi^c$, where $\sum_c{}'$ extends over cross-sections in channels c which are open, then $\Phi^c_E = 0$ if $E \notin A$. Thus momentum is conserved provided S induces

unitary operators $S_E$ which act in the constant 4-momentum spaces of open channels and $(S \Phi^c)_E = S_E \Phi_E$. $S_E$ are called the scattering operators at constant energy. Clearly $S_E$ depends only on $E^2$, the magnitude of the total 4-momentum, and we shall consider a centre of mass frame where $E \equiv (E, \underline{0})$. At fixed E there is a unitary operator $S_E$ which maps $\bigoplus'_c L^2(S^c)$ into itself, where $\bigoplus'_c$ denotes the direct-sum over open channels. The restrictions of $S_E$ to the various open channels are written $S_E^{cd}$ where $S_E^{cd} : L^2(S^d) \rightarrow L^2(S^c)$.

A priori, $S_E^{cd}$ is only defined if E, the centre of mass energy, exceeds the thresholds of both c and d. It is, however, suggested in many models that $S_E^{cd}$ enjoys analytic continuation away from these real physical values and that it may be considered as the boundary-value of an operator $S_E^{cd}$ which is analytic in a larger domain; in such a case we shall be said to have total-energy analyticity of the S-matrix.

Clearly $S_E^{cd}$ does not have a continuous kernel, otherwise it would be a compact operator (since $S^c$ and $S^d$ are compact spaces and the $L^2$ norm would exist), but we may hypothesise that it can be represented by a kernel which is a distribution. In this case $S_E^{cd}$ is represented by a 2-variable distribution $\langle s^c | S_E | s^d \rangle$, written here in a notation which is consistent with that for the momentum-localized states $s^c$ . The terms $\langle s^c | S_E | s^d \rangle$ are called the S-matrix elements, which are more usually represented

by $_{\text{in}}\langle p_1 \cdots p_m \mid q_1 \cdots q_n \rangle_{\text{out}}$.

When there is total-energy analyticity the S-matrix elements $\langle s^c \mid S_E \mid s^d \rangle$, although distributions, may well be the boundary-values of analytic functions for fixed $s^c$ and $s^d$. The analytic S-matrix elements clearly depend on the particular parametrization of the external states of constant energy that we have chosen. In a different parametrization $\bar{s}^c$, with $s^c = s^c(\bar{s}^c, E)$, the functions at constant $s^c$ and $\bar{s}^c$ are quite different. However, it will be seen later that the one we have chosen has the merit of being consistent with hermitian symmetry, $S_E^* = S_E{}^*$ because of the space-inverting property of $\lambda(E)$, the function discussed in § 1. With another parametrization $S_E$ becomes a different unitary operator $U_E\, S_E\, U_E^*$ which does not in general satisfy hermitian symmetry.

A more ambitious form of analyticity may hold when $\langle s^c \mid S_E \mid s^d \rangle$ are analytic in all of their variables $s^c$, $s^d$, E. In this case the S-matrix elements $_{\text{in}}\langle p_1, \cdots, p_m \mid q_1, \cdots, q_n \rangle_{\text{out}}$ are boundary values of functions analytic in each of the external momenta; this is referred to as full on-shell unitarity (see § 7).

## § 3. Green's Functions

The N-point Green's function for N uncharged scalar particles described by fields $\phi_1$, $\phi_2$, ..., $\phi_N$ is defined as

$$G'(p_1, \ldots, p_N) = \int e^{-i\sum p_i x_i} \langle 0 | T\phi_1(x_1) \ldots \phi_N(x_N) | 0 \rangle \prod d^4x_i$$

where T is the time-ordering operator which puts the fields in order of time, the earliest furthest left. The off-shell scattering elements are given by

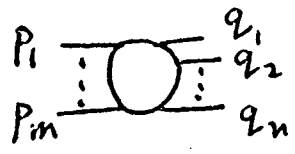
$$T'(p_1, \ldots, p_N)\,\delta^4(p_1 + \ldots + p_N) = \prod_i \left[D_i(p_i)\right]^{-1} G'(p_1, \ldots, p_N)$$

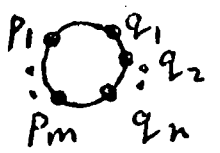
where $D_i$ is the bare propagator of the field $\phi_i$, and the term $\delta^4(p_1 + \ldots + p_N)$ arises from momentum-conservation: in future this expression will be assumed to have been removed from $G'$ as well.

By the LSZ reduction formulae[2], the physical transition matrix $T = S-I$ of a process in which these particles participate is equal to $T'$ when the incoming particles have on-shell 4-momenta $p_i$, with $p_i^2 = m_i^2$, $p_{i0} > 0$, and the outgoing particles have on-shell 4-momenta $-p_i$ so that $p_i^2 = m_i^2$, $p_{i0} < 0$. Thus for a crossed process, in which different subsets of the particles $1, 2, \ldots, N$ enter and leave, $T'$ is given by the same formula, but the two on-shell S-matrices have no overlap because of the positive energy condition.

The extension of this to charged particles with spin and internal quantum numbers is straightforward, and the particles with negative momenta are then anti-particles. When we wish to emphasize that particles with momenta $p_1, \ldots, p_m$ enter and particles with momenta $q_1, \ldots, q_n$ leave the $(m + n)$ -point functions are written as $G'(p_1, \ldots, p_m;\ q_1, \ldots, q_n)$ and $T'(p_1, p_2, \ldots, p_m;\ q_1, \ldots, q_n)$, with the outgoing momenta reversed in the above formulae. They are

represented by the bubble-diagrams

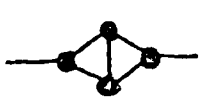
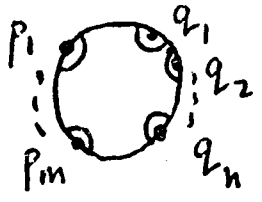
$G'(p_1, \ldots, p_m; q_1, \ldots, q_n) =$ 


$T'(p_1, \ldots, p_m; q_1, \ldots, q_n) =$ 


These diagrams are consistent with the notation $\underset{i}{\text{———}}$ for bare propagators. The 2-point function $G'(p; q)$ for the field $\phi_i$ is the clothed propagator $D_i'(p - q)$, which will be denoted $\underset{i}{\text{—+—}}$. Often it will be unnecessary to label propagator-lines.

We shall find it convenient to define the <u>truncated</u> off-shell transition matrix by

$$T(p_1, \ldots, p_N) = \prod_i \left[D_i'(p_i)\right]^{-1} G'(p_1, \ldots, p_N)$$

For physical momenta T and T' are identical, since on the mass-shell $D_i(p_i)$ and $D_i'(p_i)$ have poles with the same residue. The effect of truncation (or amputation) is to remove from G' all self-energy terms on the external legs: in perturbation-theory these look like  ,  , etc., and belong to the clothed propagators of the external particles. A truncated (m + n)-point function is denoted by a bubble 


## § 4. <u>Perturbation Theory</u>

As far as we are aware it is impossible to examine the particle-structure of Green's functions without making some reference to perturbation theory. As is well known, perturbation

expansions may be ill-defined on two accounts:

(i) The actual terms are usually divergent integrals. This may happen because of infra-red divergences which occur when zero-masses are involved, or because of ultra-violet divergences due to taking products of distributions. In renormalizable theories the expansion may be finite to any order by the use of infinite renormalization constants, but not in every theory.

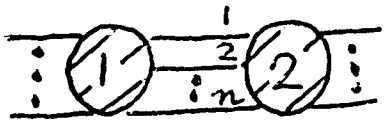
(ii) The renormalized expansion may be a divergent series, and it is conjectured that this is usually the case[3].

Taylor[7] has attempted to extend the concept of renormalizability to a larger class of interactions but the second problem remains. We shall slide over this difficulty by making the assumption that solutions of the field equations can be obtained as limits of cutoff perturbation solutions, and that arguments depending on the perturbation expansions remain valid in such limits.

Thus, in lagrangian field theories with polynomial interactions it will be considered, whenever convenient, that the Green's functions represent sums of Feynman diagrams. All statements about Feynman diagrams will be made modulo their topological rearrangement; for example a diagram is said to be <u>disconnected</u> if it is not of one piece. Henceforth (unless otherwise stated) <u>Green's functions will only include the connected graphs</u>. This makes discussions about particle-structure simpler, and in any case the full amplitudes can always be recovered as a finite sum of products of connected amplitudes.

## § 5. Exposure of Cuts

Consider a graph in the expansion of $G^1$. If it is possible to write it in the form

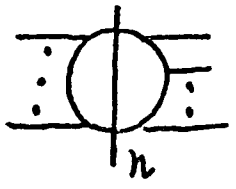



where the bubbles (1) and (2) are also connected perturbation diagrams and the propagators 1, 2, ... , n are internal then we say that the diagram has an n-particle cut in the direct channel due to the particles 1, 2, ... , n.

It is a simple consequence of perturbation theory that the diagram has a normal threshold cut (in the sense of functions of a complex variable) beginning at the branch-point $s = (m_1 + \ldots + m_n)^2$, where s is the square of the total energy in the direct channel.

An n-particle cut can therefore be represented by a line across the diagram which separates the incoming particles from the outgoing ones and intersects n internal propagators; cuts will be considered identical if they are identical when self-energy terms have been removed. A partial ordering may be put on the cuts within a diagram, with $C_1 < C_2$ if $C_1$ can be represented by a line lying to the left of a line representing $C_2$. It can easily be shown that within a diagram without any cuts of order less than m, the ordering of n-cuts is total: thus a first and a last n-cut exist and are unique (if an n-cut exists at all). The "n-particle reduced Green's function" $G^1_n$ is defined as the sum of the graphs in $G^1$ which have no m-particle cuts for $m \leqslant n$, and is denoted

by the bubble

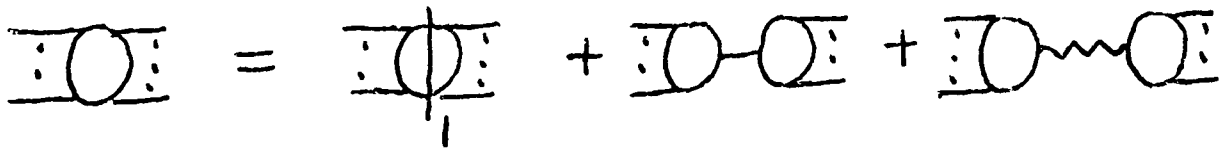



When the particles which enter the theory are not identical we include all m-cuts of different particle-types, or alternatively consider cuts with respect to <u>one</u> particular particle only.

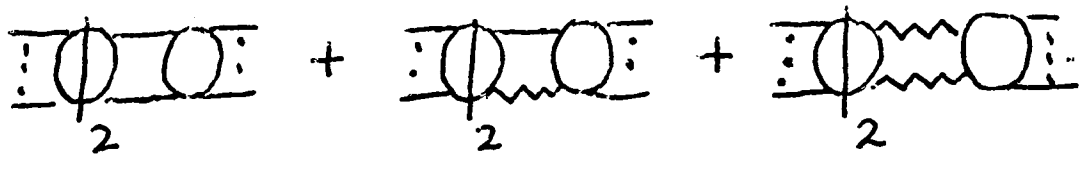
## § 6. Green's Function Equations

The Green's function equations [4,5,6] are an infinite family of equations in the complete propogators and truncated n-particle reduced amplitudes which replace the differential field equations of the field-operators. Since we shall not now be very interested in free propagators and untruncated amplitudes, the lines and bubbles which previously represented them will now represent clothed propagators and truncated amplitudes, unless state otherwise.

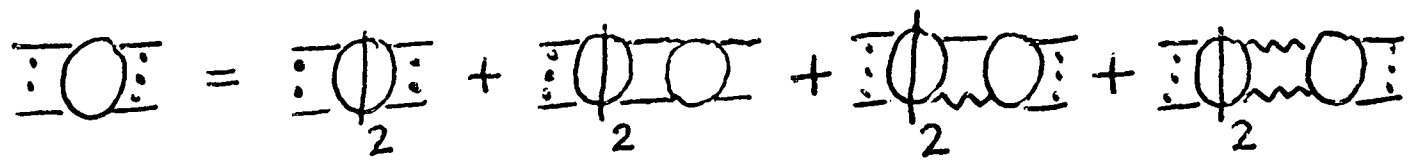
Consider a lagrangian theory of two non-identical particles A and B (here denoted by —— and ~~~ ). We exclude from the total amplitude all terms which have single (A or B) particle intermediate states; these are added in separately through the equation



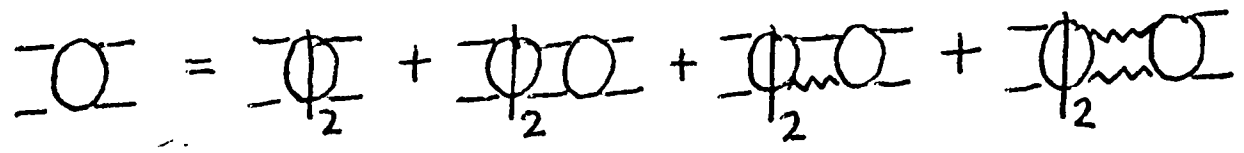

The graphs which have 2-particle intermediate states can be written



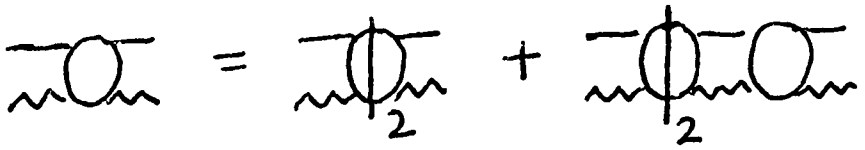
where we have "exposed" the first 2-cut and the external particles are arbitrary. Since graphs have either no 2-cut at all, or a first 2-cut,

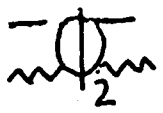



This is the first-order development of the equations relating T and $T_2$. The second-order development will relate T, $T_2$, $T_3$ by the exposure of 3-cuts, and so on. Thus a hierarchy of field equations can be generated in the reduced amplitudes of higher and higher degree. Together with the propagator and vertex function equations they form a family which should, in principle determine a solution after renormalization, but it has so far been impossible to prove that one exists[8]. In practice it is necessary to terminate these equations at some stage and make an appropriate approximation to the highest order reduced amplitudes. If the equations are terminated at the first step, we obtain for the 2-body equations

(6.1) 


together with the three similar equations in which the external particles are  and  in turn. This is the Bethe-Salpeter equation, discovered around 1951 by several authors[9,5,10,11]. The reader may be surprised to see the equation in the form we have written it, since it often appears as the single equation

(6.2) 

in which  is a reduced amplitude which has no AB cuts, but may have AA- and BB- cuts: this is because of the fact that AB-cuts are ordered. Equation (6.2) is of course the form of the equation when —— and ∿∿ are identical. In 2-body scattering, ignoring 1-particle states, T is the transition-matrix at elastic energies and the 2-particle reduced amplitudes are analogues of non-relativistic potentials; they will be denoted by V. The BSE in either of its forms (6.1) or (6.2) can be written in the operator form

$$T = V + VG_0'T \tag{6.3}$$

where T and V may be matrices (as in (6.1) and its three companions) or single terms (as in (6.2)) and $G_0'$ represents the intermediate propagators.

The BSE is the simplest possible Green's function equation and is the natural relativistic analogue of the Lippmann-Schwinger equation; in fact the two are identical in the non-relativistic limit when V is a Yukawa-type interaction[10]. Like all finitely terminated field equations it is meaningless as it stands unless approximations can be found for V and $G_0'$; if no appropriate approximations are known then it is necessary to consider more equations of greater complexity. The equation is not manifestly

crossing-symmetric and an approximate potential is usually taken which restores this symmetry to some extent.

## § 7. S-Matrix Theory

In high-energy physics the reaction-times of experiments are so short (of the order of $10^{-10}$ sec or less) that no observations can be made whilst a reaction takes place: the observations on the particles are made before and after they have reacted, at times which are great compared with the reaction-time. With this is mind, Heisenberg around 1943 conjectured that these "infinite-time" observations, which are the on-shell matrix elements, are all that is required to formulate and solve high energy problems. A great deal of effort has been applied to the S-matrix programme, and since we shall make comparisons between calculations carried out in field theory and S-matrix theory we gather together some of the main ideas.

It is probably true to say that from about 1950 until 1965, the three notions which have made S-matrix calculations possible have been

(1) Unitarity of the S-matrix

(2) The assumed analyticity (to some degree) of the S-matrix elements

(3) Crossing-symmetry.

All of these are extremely plausible in S-matrix theories derived

from field theory. As we said in § 3, the statement of crossing-symmetry in field theory is meaningless for on-shell amplitudes because on-shell S-matrices for crossed processes have no overlap. However, with sufficient analytic continuation between the domains they can be boundary-values of the same analytic function: this is the meaning of (3) above.

Analyticity is considered to be a basic axiom in Chew's book[12], where S-analyticity means maximal analyticity consistent with unitarity, and in many respects this is similar to the analyticity of Feynman diagrams in perturbation theory. When possible, it is convenient to express S-matrix elements by dispersion-relations, which are Cauchy integrals across lines or surfaces of discontinuity in the complex on-shell momentum space. In 2-body scattering with Mandelstam variables (s,t,u) the 1-dimensional dispersion relation in s with t fixed is of the form

$$t(s,t,u) = \frac{1}{\pi i}\int \frac{\rho_L(s^1)ds^1}{s-s^1} + \frac{1}{\pi i}\int \frac{\rho_R(s^1)ds^1}{s-s^1}$$

where $\rho_L$ and $\rho_R$ are the discontinuities across cuts on the real s-axis which lie to the left and right of each other. These are the dynamic and kinematic cuts respectively. Some of the 1-dimensional dispersion relations have been deduced from field theory[13]. Contrary to the above additive representation, the N/D equations[12] represent each partial-wave component of the amplitude as a quotient N(s)/D(s) in which N and D are analytic in the complex plane except for discontinuities along the left- and right-hand cuts respectively.

When an expression of unitarity is required of the amplitude, equations can be found relating N and D. Like the BSE (to which it bears some resemblance, N taking the place of V) the equations are not manifestly crossing-symmetric. If N is replaced by a dominant contribution (like an exchange graph) which has the right analytic structure, an approximate solution can be found.

The other common dispersion relation is Mandelstam's extension of the above to two variables. It takes the form

$$t(s,t,u) = \sum_{s,t,u} -\frac{1}{\pi^2} \int \frac{\rho_{st}(s^1,t^1)}{(s-s^1)(t-t^1)}\, ds^1\, dt^1$$

where $s^1$, $t^2$, $u^1$ range over the kinematic cuts in the three channels. The left-hand cuts are thus represented as crossed kinematic cuts and the representation automatically satisfies crossing-symmetry. The Mandelstam representation has been widely used but also disputed; it has not been deduced from field theory and has been shown to be inconsistent with perturbation theory in some situations.

Now although S-matrix theory can be thought to enjoy a close connexion with field theory, this in no way detracts from its right to a possible independent existence, although it cannot claim to have been given a set of minimal axioms like the Wightman axioms of axiomatic field theory, which in the context of the field equations are tantamount to boundary-conditions. Undoubtedly the greatest practical advantage of S-matrix equations is that they are

considerably simpler than field equations because they involve a smaller number of variables, although they may be accused of lacking some structure, in the sense that there is not, as yet, a definite problem to solve for a given process. Nothing significant is known about the inverse problem of a field theory given its S-matrix elements, but the inverse problem of potential scattering has been examined by Gelfand and Levitan[14]. These authors have shown that the potential from which an S-matrix may be obtained is non-unique to the extent of a discrete set of parameters. These are the analogue of arbitrary CDD poles[15] in the N/D equations, where additional resonances may appear in the amplitude without being dynamically generated. They may be inserted on the second sheet (see § 9) of D without altering its assumed analytic structure. It is thus highly unlikely that the inverse problem of field theory has a unique solution.

In S-matrix models the analytic continuation of partial wave amplitudes in the angular momentum (Regge behaviour) is becoming increasingly important. The advantage of this is that in a single Regge pole term $\frac{r(\ell, s)}{\ell - \alpha(s)}$ and infinity of particles is represented with variable angular momentum $\ell$; their masses are given by the criterion $\ell = \alpha(s)$. In addition such terms have meaningful limits as the cross-channel energies t and u tend to infinity. It is unclear how Regge behaviour is related to other conjectures about the S-matrix, e.g. the Mandelstam representation, and whether it will assume a primary or dependent role in the future. The most recent

and probably the most successful achievement of S-matrix theory has been the Veneziano model[16], which attaches great importance to Regge poles and analyticity and satisfies crossing, duality and unitarity, to some extent. We shall have little to say about Regge behaviour since its incorporation in dynamical field calculations is difficult, although the BSE predicts how Regge poles may arise in the direct channel[17], just as potential theory predicted their existence in the first place.

## § 8. Spectral Representations

As is the common practice, the term dispersion relation will be applied to on-shell S-matrix elements, and the expression spectral representation will denote any Cauchy representation of off-shell amplitudes. The simplest one is the Lehmann representation of the 2-point function:

$$D'(p) = \int du^2 \rho(u^2)\, D(p,u^2)$$

where $D(p, u^2)$ is the free propagator of a particle of mass u and the same quantum numbers, and $\rho$ is a positive distribution called the spectral function. $\rho(u^2) = \delta(u^2 - m^2) + \sigma(u^2)$ where the $\delta$-function gives the pole contribution with correct residue at the physical mass m, and the dispersive part $\sigma$ represents the intermediate many-particle states in $D'$.

The other spectral representation we shall require is the energy-analytic representation (EAR) which was proved some time

ago[18], but since it is not generally known we here reproduce the derivation. By definition of the time-ordering symbol, the n-point function is given by

$$G^1(p_1,\dots,p_n) \equiv \int \Pi d^4x_i\, e^{-i\Sigma p_i x_i} \langle 0|T\phi_1(x_1)\dots\phi_n(x_n)|0\rangle$$

$$= \sum_P \int \Pi d^4x_i\, e^{-i\Sigma p_i x_i}\, \theta(x_{P(1)} - x_{P(2)})\dots\theta(x_{P(n-1)} - x_{P(n)})$$

$$\times \langle 0|\phi_{P(1)}(x_{P(1)})\dots\phi_{P(n)}(x_{P(n)})|0\rangle .$$

$\sum_P$ extends over all permutations of (1, 2, ... , n) and $\theta(x) = 1$ if $x_0 < 0$ and $\theta(x) = 0$ otherwise. Removing the function $\delta^4(p_1 + \dots + p_n)$ and inessential constant factors,

$$G^1 = \sum_P \int \prod_{r=1}^{n-1} d^4(x_{P(r)} - x_{P(r+1)}) \cdot e^{-i\sum_{r=1}^{n-1}\left[(x_{P(r)} - x_{P(r+1)})\sum_{j=1}^{r} p_{P(j)}\right]}$$

$$\times \prod_{r=1}^{n-1} \theta(x_{P(r)} - x_{P(r+1)}) \cdot \langle 0|\prod_{r=1}^{n} \phi_{P(r)}(x_{P(r)})|0\rangle$$

If

$$\langle 0|\prod_{r=1}^{n} \phi_{P(r)}(x_{P(r)})|0\rangle \equiv F_P(x_{P(1)} - x_{P(2)},\dots,x_{P(n-1)} - x_{P(n)})$$

Then

$$G^1 = \sum_P \int \prod_{r=1}^{n-1} d^4y_r \cdot e^{-i\sum_{r=1}^{n-1} y_r Q_{P(r)}} \prod_{r=1}^{n-1} \theta(y_r) \cdot F_P(y_1,\dots,y_{n-1})$$

where $Q_{P(r)} = \sum_{j=1}^{r} p_{P(j)}$

When this Fourier transform of a product is converted into a convolution integral we have

$$G^1(p_1,\dots,p_n) = \sum_P \int \frac{dt_1 \dots dt_{n-1}\, \omega_P(t_1, \underline{Q}_{P(1)}, \dots, t_{n-1}, \underline{Q}_{P(n-1)})}{\prod_{r=1}^{n-1} (Q_{P(r),0} - t_r + i\varepsilon)}$$

where

$$\omega_P(q_1,\dots,q_{n-1}) = \int \prod_{r=1}^{n-1} d^4y_r \cdot e^{-i\sum_{r=1}^{n-1} y_r q_r} F_P(y_1,\dots,y_{n-1})$$

$G^I$ is thus expressed as a sum of spectral- (Cauchy-) integrals in which the energy-analyticity is exposed in an $(n-1)$-dimensional complex space. $G^I$ has singular surfaces which are functions of the energy-sums $Q_{P(r),O}$. We shall later make use of the observation that in the scattering channel where particles 1, 2, ... , k enter and the rest leave, the terms in the above representation which have cuts in the total energy $p_{1O} + \dots + p_{kO}$ occur in the parts of $G^I$ which result from integrals over the following domains:

(i) the direct channel region (DCR)

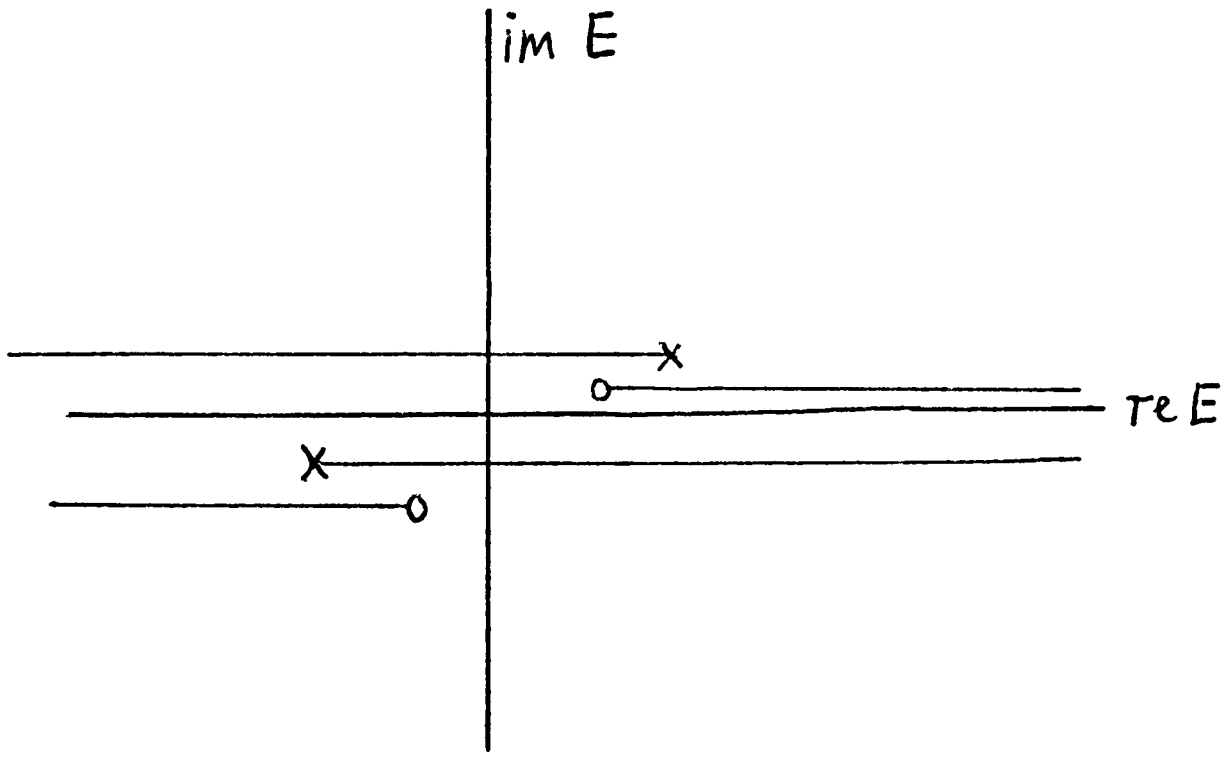
$$x_{1O}, \dots, x_{kO} < x_{k+1,O}, \dots, x_{nO}$$

(ii) the anti-direct channel region (anti-DCR)

$$x_{1O}, \dots, x_{kO} > x_{k+1,O}, \dots, x_{nO}$$

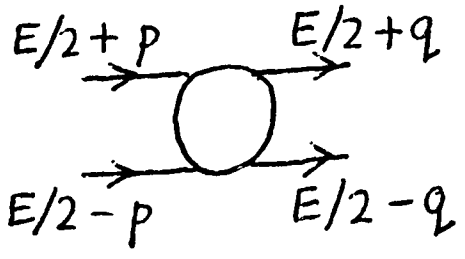
These total energy (threshold) cuts will be examined in the next paragraph.

If the total energy is written $E \equiv (E, \underline{O})$, the behaviour of $G^I$ *qua* function of E with the relative 4-momenta fixed has branch-points of the form $E = a \pm i\varepsilon$. If a does not involve the other energy variables we insert a cut $(a, \mp\infty)$; if a does involve the other variables we make a cut $(a, \pm\infty)$. The E-plane thus looks like




where (x) denotes a threshold singularity and (O) denotes a moving singularity. This cut-structure is called the physical sheet of $G^1$ in the direct channel. In the limit $\varepsilon \to 0+$ the cuts become collinear and the real axis is sandwiched between threshold and moving cuts. The physical sheet structure is defined by an analytic continuation from the real axis, before the $\varepsilon \to 0+$ limit is taken. When the particles are put on-shell many of the moving singularities disappear, but some other ones appear. These are the multi-particle analogues of the dynamical left-hand cuts, and we assume that they stay at a safe distance from the kinematic cuts, or at any rate do not prevent the continuation of scattering amplitudes away from the real axis and around the kinematic cuts.

Later on we shall require the 1-dimensional EAR for the 4-point function

$G_E^I(p;q) \equiv$ 


The restriction of the EAR to the single variable $p_o$ gives, in the truncated form of the amplitude,

$$(8.1)\quad T_E(p;q) = \int_{\alpha_+}^{\infty} \frac{\rho_E(t,\underline{p};q)}{p_0 - t + i\varepsilon}\,dt + \int_{-\infty}^{\alpha_-} \frac{\rho_E(t,\underline{p};q)}{p_0 + t - i\varepsilon}\,dt$$

Such a representation holds in many approximations related to perturbation theory (such as approximate Bethe-Salpeter equations) and will later be used as the means to obtain numerical solutions of the BSE.

## § 9. Bound-State Poles and Off-Shell Unitarity

Consider an off-shell scattering process of m + n particles which are assumed to be scalars or pseudoscalars. Then

$$G^I(p_1,\dots p_m; q_1,\dots,q_n) = \int e^{-i\sum p_i x_i} \langle 0|T \prod_i \phi_i(x_i) \prod_j \psi_j(y_j)|0\rangle e^{i\sum q_j y_j} \prod_i d^4x_i \prod_j d^4y_j$$

where i and j range from 1 to m and 1 to n respectively.

$X = \frac{x_1 + \dots + x_m}{m}$ is the centroid and $E = p_1 + \dots + p_m$ is the total momentum of the incoming particles. Relative coordinates are defined by $\bar{x}_i = x_i - X$ and relative momenta by $\bar{p}_i = p_i - E/m$. Thus

$p_i x_i = EX + \sum \bar{p}_i \bar{x}_i$. $x_i$ are not independent and neither are

$p_i$, for $\sum x_i=0$ and $\sum p_i=0$, but we preserve the symmetrical description. In a similar manner Y, F, $\bar{y}_j$ and $\bar{q}_j$ are defined for the outgoing particles and $\sum q_j y_j = FY + \sum \bar{q}_j \bar{y}_j$.

Consider the DCR $x_{1O}, \ldots, x_{mO} < y_{1O}, \ldots, y_{nO}$, which is not Lorentz invariant. If min $(\bar{x}_{iO}) = u$, max $(\bar{y}_{jO}) = v$ the DCR is described by $\theta(X_O - Y_O - u - v) = 1$ where $\theta(w) = 1$ if $w \leqslant O$ and $\theta(w) = O$ if $w > O$. The contribution of the DCR to G' is thus

$$G'_{DCR} \equiv \sum_{P,\alpha} \int e^{-i(EX+\Sigma \bar{p}_i \bar{x}_i)} \langle 0|T\Pi\phi_i(x_i)|P\alpha\rangle\langle P\alpha|T\Pi\psi_j(y_j)|0\rangle \times e^{i(FY+\Sigma q_j y_j)} \cdot \theta(X_o - Y_o - u - v)\Pi d^4x_i \; \Pi d^4y_j$$

where $|P,\alpha\rangle$ is a complete set of states of 4-momentum P and quantum numbers $\alpha$.

By using the translation operators,

$$\langle 0|T\Pi\phi_i(x_i)|P\alpha\rangle = e^{iPX}\langle 0|T\Pi\phi_i(\bar{x}_i)|P\alpha\rangle$$

The measure $\Pi d^4x_i$ is now replaced by $d^4X\, d\bar{x}$ where $d\bar{x} = \Pi d^4\bar{x}_i\, \delta^4(\bar{x}_1 + \ldots + \bar{x}_m)$. Thus

$$G'_{DCR} = \sum_{P,\alpha} \int e^{-i[(E-P)X+\Sigma \bar{p}_i \bar{x}_i]} \langle 0|T(\bar{x}_i)|P\alpha\rangle\langle P\alpha|T(\bar{y}_j)|0\rangle \times e^{i[(F-P)Y+\Sigma \bar{q}_j \bar{y}_j]} \theta(X_o - Y_o - u - v)\, d^4X\, d^4Y\, d\bar{x}\, d\bar{y}$$

Now inserting $\theta(w)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-izw}\,dz}{z-i\varepsilon}$ and performing the X- and Y-integrals, we obtain, after the momentum-preserving function $\delta^4(E-F)$ has been removed,

$$G'_{DCR} = -\frac{(2\pi)^8}{2\pi i}\sum_{P,\alpha}\int d\bar{x}\,d\bar{y}\,e^{-i\Sigma\bar{p}_i\bar{x}_i}\langle 0|T(\bar{x}_i)|P\alpha\rangle\langle P\alpha|T(\bar{y}_j)|0\rangle \tag{9.1}$$
$$\times\; e^{i\Sigma\bar{q}_j\bar{y}_j}\;\frac{e^{i(E_0-P_0)(u+v)}}{E_0-P_0+i\varepsilon}\;\delta^3(\underline{E}-\underline{P})$$

When $|P\alpha\rangle$ is a bound-state $|b\rangle$ of mass $m_b$, its contribution to this expression is obtained by replacing $\sum_{P,\alpha}$ with $\int d^4P\,\delta^+(P^2-m_b^2)$ and this gives

$$-\frac{(2\pi)^8}{4\pi i}\int d\bar{x}\,d\bar{y}\,e^{-i\Sigma\bar{p}_i\bar{x}_i}\langle 0|T(\bar{x}_i)|b\rangle\langle b|T(\bar{y}_j)|0\rangle e^{i\Sigma\bar{q}_j\bar{y}_j}$$
$$\times\;\frac{1}{E_0-\sqrt{m_b^2+\underline{E}^2}}\cdot\frac{e^{i(E_0-\sqrt{m_b^2+\underline{E}^2})(u+v)}}{\sqrt{m_b^2+\underline{E}^2}}$$

Thus a pole is generated on the upper sheet of the mass-hyperboloid $E^2=m_b^2$ and its residue is

$$-\frac{(2\pi)^8}{4\pi i}\,\frac{1}{\sqrt{m_b^2+\underline{E}^2}}\,G'(\bar{p}_1,\dots\bar{p}_m;b)\,G'(b;\bar{q}_1,\dots\bar{q}_n)$$

where half off-shell Green's functions in momentum space are defined by

$$G'(\bar{p}_1,\dots,\bar{p}_m;b)=\int d\bar{x}\;e^{-i\Sigma\bar{p}_i\bar{x}_i}\;\langle 0|T(\bar{x}_i)|b\rangle$$

$$G'(b;\bar{q}_1,\dots,\bar{q}_n)=\int d\bar{y}\;e^{i\Sigma\bar{q}_j\bar{y}_i}\;\langle 0|T(\bar{y}_j)|b\rangle$$

A similar pole is generated on the lower sheet of the hyperboloid when $|b\rangle$ makes its contribution to the anti-DCR region $x_{1O},\dots,x_{mO} > y_{1O},\dots,y_{nO}$, and the pole-structure of $G'$

near $E^2 = m_b^2$ is given by the sum of these. In terms of off-shell and half off-shell truncated amplitudes we have, in the centre of mass frame near the upper-sheet pole,

$$T(p_1, \ldots, p_m; q_1, \ldots q_n) \sim \frac{(2\pi)^8}{2\pi i} \cdot \frac{T_{m_b}(\bar{p}_1, \ldots \bar{p}_m; b)\, T_{m_b}(b; \bar{q}_1, \ldots, \bar{q}_n)}{E^2 - m_b^2}$$

where $T_E(\bar{p}_1, \ldots, \bar{p}_m; b) = \Pi\left[D_i'(p_i)\right]^{-1} G'(\bar{p}_1, \ldots, \bar{p}_m; b)$. The pole thus has the characteristic form for a stable particle in the direct channel i.e. it is a function of E and its residue factorizes into two wave-functions which are functions of the in- and out-relative momenta respectively.

We shall now assume that there are no bound-states in the theory, so that the hilbert space is spanned by the many-particle states of the elementary particles, both in the in- and out-asymptotic representations. As has long been known[19] this does not involve a loss of generality. The total 4-momentum will be put in a centre of mass frame where $E = (E, \underline{0})$.

Since little is known about the off-shell amplitudes and multiparticle on-shell amplitudes, we must make some fairly plausible assumptions.

Assumption 1 The off-shell amplitudes $T_E(\bar{p}_1, \ldots, \bar{p}_m; \bar{q}_1, \ldots, \bar{q}_n) \equiv T(p_1, \ldots, p_m; q_1, \ldots, q_n)$ are defined for physical energies as the boundary-value $E \rightarrow E+i0$ of a complex function.

When the relative 4-momenta $\bar{p}_i$, $\bar{q}_j$ are kept fixed, $T_E$ is analytic when E lies in a neighbourhood of the kinematic cut and has branch-points at the threshold energies of all channels which are not forbidden by selection-rules. $T_E$ can be analytically continued around the kinematic cut, avoiding the singularities in E which depend on the relative momenta (i.e. those which are not "fixed"). This region of analyticity, above and below the kinematic cut is defined as the physical sheet.

Assumption 2. Similar statements to the above can be made about $T_E(\bar{p}_1, \ldots, \bar{p}_m; s^c)$ and $S_E(s^c; s^d)$ in which $s^c$ and $s^d$ represent the relative momenta of on-shell particles of total energy E, as in § 1. This is a highly non-trivial extension of Assumption 1 because when some of the particle are on-shell, their relative 4-momenta are $\bar{p}_i = \bar{p}_i(s^c, E)$, etc, and when $s^c$ is kept fixed they are functions of E and vary as E is continued around the cut. Hence the path of continuation must avoid the movable real singularities in $\lambda(E)$, the function defined in § 1.

Now in the EAR it was shown that the singularities in G' which are functions of E alone (i.e. those with branch-points fixed at the threshold energies) arise from the DCR and the anti-DCR. Expression (9.1) can be decomposed into the sum of a Lorentz-invariant part

$$-\frac{(2\pi)^8}{2\pi i}\sum_{P,\alpha}\int d\bar{x}\,d\bar{y}\; e^{-i\sum \bar{p}_i\bar{x}_i}\langle 0|T(x_i)|P\alpha\rangle \times \langle P\alpha|T(y_i)|0\rangle\, e^{i\sum \bar{q}_i\bar{y}_i}\,\frac{\delta^3(\underline{P})}{E_0 - P_0 + i\varepsilon}$$

and a part which is not Lorentz-invariant but does not have any real singularities with respect to E. Since the states now belong to the continuum we can replace $\sum_{P,\alpha}$ by $\int d^4P \sum'_{s^c}$, where $\sum'_{s^c}$ denotes a sum of integrals $\int ds^c$ over channels which are open at the energy P.

The discontinuity of (9.1) across the kinematic cut is

$$\Delta G^1 \equiv (2\pi)^8 \sum'_{s^c} \sum'_{s^d} G^1_E(\bar{p}_1,\dots,\bar{p}_m; s^c) S^*_E(s^d; s^c) G^1_E(s^d; \bar{q}_1,\dots\bar{q}_n)$$

where

$$G^1_E(\bar{p}_1,\dots,\bar{p}_m; s^c) = \int d\bar{x}\; e^{-i\sum \bar{p}_i \bar{x}_i} \langle 0|T(\bar{x}_i)|E s^c\rangle_{out}$$

$$G^1_E(s^c; \bar{q}_1,\dots\bar{q}_n) = \int d\bar{y}\; e^{i\sum \bar{q}_j \bar{y}_j} {}_{in}\langle E s^c|\bar{T}(\bar{y}_j)|0\rangle$$

Truncating away the external legs, we can thus write

$$T_{E*}(\bar{p}_1,\dots,\bar{p}_m; \bar{q}_1,\dots\bar{q}_n) = T_E(\bar{p}_1,\dots,\bar{p}_m; \bar{q}_1,\dots,\bar{q}_n)$$

$$- \sum'_{s^c} \sum'_{s^d} T_E(\bar{p}_1,\dots\bar{p}_m; s^c) S^*_E(s^d; s^c) T_E(s^d; \bar{q}_1,\dots\bar{q}_n) \qquad (9.2)$$

$T_{E*}$ represents the value of $T_E$ at a point on the lower lip of the kinematic cut opposite the energy E on the upper lip, and is the continuation of $T_E$ along a path which avoids the moving singularities. The inessential factor $(2\pi)^8$ has been removed.

Consider the case when $\bar{q}_j$ are the on-shell relative momenta of an n-particle state of total energy E described by $s^a$ with $\bar{q}_j = \bar{q}_j(s^a, E)$. Then, as proved in § 1, $\bar{q}_i(s^a, E^*) = I_s\, \bar{q}_i(s^a, E)$ where $I_s$ is the space-inverting operator. Hence, for $\ell$-wave angular momentum states in the direct channel,

$$T_{E*}(\bar{p}_1, \dots, \bar{p}_m; \bar{q}_1(s^a,E), \dots) = (-)^{\ell}\, T_{E*}(\bar{p}_1, \dots, \bar{p}_m; s^a)$$

Thus $$(-)^{\ell} T_{E*}(\bar{p}_1,\cdots,\bar{p}_m;s^a) = T_E(\bar{p}_1,\cdots,\bar{p}_m;s^a) - \sum_{s^c}{}' \sum_{s^d}{}' T_E(\bar{p}_1,\cdots\bar{p}_m;s^c) S_E^*(s^d;s^c) T_E(s^d;s^a)$$

Remembering that the on-shell T- and S-matrices satisfy

$T_E = S_E - I$, and using the unitarity of $S_E$, $S_E^* S_E = I$, this gives

$$(9.3)\quad T_E(\bar{p}_1, \cdots, \bar{p}_m; s^a) = (-)^{\ell} \sum_{s^c}{}' T_{E*}(\bar{p}_1, \cdots, \bar{p}_m; s^c) \times S_E(s^a, s^c)$$

Substituting this expression for $T_E$ in (9.2),

$$(9.4)\quad T_{E*}(\bar{p}_1, \cdots, \bar{p}_m; \bar{q}_1, \cdots, \bar{q}_n) - T_E(\bar{p}_1, \cdots, \bar{p}_m; \bar{q}_1, \cdots, \bar{q}_n) = (-)^{\ell+1} \sum_{s^c}{}' T_{E*}(\bar{p}_1, \cdots, \bar{p}_m; s^c)\, T_E(s^c; \bar{q}_1, \cdots, \bar{q}_n)$$

This is the off-shell unitarity relation. A special case of it will be proved for a Bethe-Salpeter model at elastic energies in the next chapter. There the Wick-rotation will remove all singularities except the threshold cut and the details are much simpler because there are no moving singularities to bother about.

Putting $\bar{p}_1, \cdots, \bar{p}_m$ on-shell in (9.2) we obtain a similar equation to (9.3):

$$(9.5)\quad (-)^{\ell} T_E(s^a; \bar{q}_1, \cdots, \bar{q}_n) = \sum_{s^c}{}' S_E(s^c; s^a)\, T_{E*}(s^c; \bar{q}_1, \cdots, \bar{q}_n)$$

Substituting this expression for $T_E$ in (9.4),

$$(9.6)\quad T_E(\bar{p}_1,\cdots,\bar{p}_m;\bar{q}_1,\cdots,\bar{q}_n) = T_{E*}(\bar{p}_1,\cdots\bar{p}_m;\bar{q}_1\cdots\bar{q}_n) + \sum_{s^c}{}' \sum_{s^d}{}' T_{E*}(\bar{p}_1,\cdots,\bar{p}_m;s^c) S_E(s^d;s^c) T_{E*}(s^d;\bar{q}_1,\cdots,\bar{q}_n)$$

This formula will be the basis for our discussion of off-shell second-sheet structure.

The connexion between the off-shell unitarity relation (9.4) and hermitian analyticity is seen by putting all of the particles on-shell:

$$T_E{}^*(s^a;\ s^b) - T_E(s^a;\ s^b) = -\sum_{s^c}{}' T_E{}^*(s^a;\ s^c)\ T_E(s^c;\ s^b)$$

Unitarity gives

$$T_E^*\ (s^b;\ s^a) + T_E(s^a;\ s^b) = -\sum_{s^c}{}' T_E^*\ (s^c;\ s^a)\ T_E(s^c;\ s^b)$$

Thus $T_E^*\ (s^a;\ s^b) = -\ T_E{}^*(s^b;\ s^a)$, which is the formula for hermitian analyticity.

## § 10. Second Sheet Structure and Resonances

In the previous paragraph a basic minimal analyticity was assumed which is sufficient to define the continuation of the scattering elements around the kinematic cut in the physical sheet. We shall now discuss the continuation of the scattering elements downwards through the kinematic cut into the second Riemann sheet, which is called the unphysical sheet.

At real physical energy E the S-matrix operator satisfies $S_E{}^* S_E = I$. Assuming that $S_E$ possesses an analytic continuation into im $E > 0$ and remains a bounded and invertible operator, $(S_{E^*}^*)^{-1}$ is defined and analytic in a region of im $E < 0$ which is the mirror-image of the physical-sheet domain. This operator agrees with $S_E$ for real values of E, and is therefore a downward continuation of $S_E$ which is called the second- or unphysical-sheet structure of $S_E$, denoted $S_E^{II}$.

If the continuation is carried out from $E^a < E < E^b$ where $E^a$ and $E^b$ are consecutive threshold energies for channels a and b, $S_E^{II}$ is defined on the space $\bigoplus_c{}' L^2(S^c)$ and has components $(S_E^{cd})^{II}$ where c and d range over channels which are open when $E^a < E < E^b$. by choosing different consecutive thresholds $E^a$ and $E^b$ we can therefore define $(S_E^{cd})^{II}$ in several ways, and unlike the first-sheet structure, these definitions do not all agree. This is because every threshold energy is a branch-point and comparison can only be made between different values of $(S_E^{cd})^{II}$ by circling one or more of these branch-points.

To prevent such ambiguities, $(S_E^{cd})^{II}$ will, when used without qualification, denote the continuation of $S_E^{cd}$ from the interval $(E^a, E^b)$, where $E^a$ and $E^b$ are consecutive thresholds and $E^a < \text{re } E < E^b$. This is clearly the most direct path of continuation from the real axis, and physically the most significant. The physical-sheet so defined has its threshold cuts pointing vertically downwards.

The second-sheet structure of $S_E$ is important because for small $\varepsilon$, energies $E - i\varepsilon$ on the second-sheet are topologically near to the real scattering region, just as energies $E + i\varepsilon$ on the first sheet are also near. Suppose that $S_E^{II}$ has a pole $A/(E - E_r)$, where the residue A is an operator. If N is the null-space of A and R is its range, then the restriction $A: N^{\perp} \to R$ is 1-1 and onto. Now G, the product of the rotation and internal symmetry groups,

has, in the hilbert space a representation U which commutes with R and thus leaves N and $N^{\perp}$ invariant. Assuming that the pole is not accidentally degenerate, the representation induced by U in $N^{\perp}$ is irreducible and therefore finite-dimensional (since G is compact). Thus A can be written $\sum_{i=1}^{n} |\Phi_i\rangle\langle\Psi_i|$ where $\{\Phi_i\}$ is an orthonormal basis for $N^{\perp}$, $\Psi_i = A\Phi_i$ and n is the degeneracy. When these conditions are satisfied, the pole is called a resonance of complex mass $E_r$.

Since $\Phi_i$ can be obtained from the orbit (under U) of a single vector $\Phi$, and $\Psi_i$ can similarly be obtained from $\Psi \equiv A\Phi$, the pole will be denoted by $|\Phi\rangle\langle\Psi|/(E - E_r)$. (If the channels are assumed to have fixed quantum numbers, this representation is literally correct because the degeneracy is then factored away .) Putting $\Phi = \sum_c{}' \phi^c$, $\Psi = \sum_c{}' \psi^c$, we have $(S_E^{cd})^{II} \sim |\phi^c\rangle\langle\psi^d|/(E - E_r)$ near the pole, and when $\phi^c$ is non-zero the resonance is said to be coupled to channel c. The S-matrix elements satisfy $S_E^{cd}(s^c; s^d)^{II} \sim \phi^c(s^c)^* \psi^d(s^d)/(E - E_r)$ and have factorizable residues characteristic of a resonance. The factorizability is readily verifiable for a resonance provided it is possible to determine the second-sheet residues for the different processes in which the resonance appears.

Although the second-sheet is unobservable directly, a resonance pole $E = E_r \equiv m_r - i\Gamma/2$ with small width $\Gamma$, $\Gamma \ll m_r$, will cause the scattering amplitudes to show a sharp peak when the

total energy is near to $m_r$, other parameters remaining fixed; in such a case we are said to have a sharp resonance. For example, consider an $\ell$-th partial wave-amplitude.for 2-body scattering, with total energy near to a resonant energy $m_r$:

$$e^{i\delta_\ell} \sin\delta_\ell = \frac{g}{E - E_r} + f(E)$$

where f is a background term and the inelasticity factor has been absorbed into f and g. Thus

$$e^{2i\delta_\ell} = \frac{E - E_r^*}{E - E_r} \cdot \frac{g + (E - E_r)f}{g^* + (E - E_r^*)f^*}$$

$\arg(g + (E - E_r)f) \equiv \Delta_\ell$ is called the background shift and $\delta_\ell = \Delta_\ell + \tan^{-1}\left(\frac{\Gamma/2}{E-m_r}\right)$ Hence, if $\Delta_\ell$ is small (e.g. when g is nearly real and $\Gamma \ll 2|g/f|$), the phase-shift rises sharply through $\pi/2$ near to $E = m_r$ and $\Gamma \sim 2/\delta_\ell'(m_r)$. In S-matrix theory it is sometimes tacitly assumed that this is a sufficient criterion for the existence of a resonance. From our point of view we shall only apply the term "resonance" to poles in the amplitudes which are functions of the total energy, and have residues that factorize into functions of the in- and out-relative momenta; this last condition is a triviality in 2-body scattering only.

For a resonance with mass $m_r$ and width $\Gamma$, it can be shown[20] that a wave-packet with energy clustered around $m_r$ will decay at an exponential rate $e^{-t\Gamma}$, giving a mean life-time $\tau \sim \Gamma^{-1}$. It is precisely for this reason that the physical sheet of S is assumed to

have no poles; their presence, from exactly the same argument, could violate unitarity to such an extent that state-norms increase exponentially with time. Years ago the presence of such "ghost poles" in approximate theories caused some alarm until they were recognised as degenerate cuts produced in the approximation.

Let us now consider the presence of resonances in off-shell amplitudes. The equation

$$T_E(\bar{p}_1,\cdots,\bar{p}_m;\bar{q}_1,\cdots,\bar{q}_n) = T_E{}^*(\bar{p}_1,\cdots;\bar{q}_1,\cdots) + \sum_{s^c}{}' \sum_{s^d}{}' T_E(\bar{p}_1,\cdots;s^c) S_E(s^d;s^c) T_E(s^d;\bar{q}_1,\cdots) \tag{10.1}$$

can be written, when E is continued downwards into the second-sheet,

$$T_E^{II}(\bar{p}_1,\cdots,p_m;\bar{q}_1,\cdots,\bar{q}_n) = T_E^{I}(\bar{p}_1,\cdots;\bar{q}_1,\cdots) + \sum_{s^c}{}' \sum_{s^d}{}' T_E^{I}(\bar{p}_1,\cdots s^c) S_E^{II}(s^d;s^c) T_E^{I}(s^d;\bar{q}_1,\cdots) \tag{10.2}$$

where the superscript I emphasises first- (physical-) sheet values. The highly interesting thing about this equation is that at a resonance pole in $S_E^{II}$, where $S_E^{II}(s^c;\, s^d) \sim \dfrac{\phi^c(s^c)\,\psi^d(s^d)}{E - E_r}$ the off-shell unphysical-sheet amplitude $T_E^{II}(p_1, \ldots\ ;\, q_1, \ldots)$ has a pole with a factorizable residue:

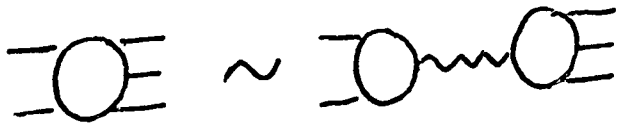
$$T_E^{II}(\bar{p}_1, \ldots ; \bar{q}_1, \ldots) \sim \frac{\chi_{in}(\bar{p}_1, \ldots)\,\chi_{out}(\bar{q}_1, \ldots)}{E - E_r}$$

where $\chi_{in}(\bar{p}_1, \ldots) = \sum_{s^c}{}' T_E^I(\bar{p}_1, \ldots ; s^c)\, \phi^c(s^c)$

$$\chi_{out}(\bar{q}_1, \ldots) = \sum_{s^d}{}' \psi^d(s^d)\; T_E^I(s^d; \bar{q}_1, \ldots)$$

This demonstrates that the second-sheet structure of the off-shell T-matrix is such that at a resonance energy the components of T have poles in E with residues which factorize into wave-functions of the incoming and outgoing relative-momenta. This is completely analagous to the first-sheet behaviour of the off-shell T-matrix in the neighbourhood of a bound-state energy and in agreement with the alternative description of resonances as unstable particles. Although it would be convenient to identify $\Phi, \Psi$ above as the state of the resonance, it must be remembered that the hilbert-space is not $\oplus' L^2(S^c)$ but an isomorphic copy of this. The isomorphism depends on the total energy E and $\Phi, \Psi$ were obtained when E assumed an unphysical complex value: thus no direct physical meaning can be attached to these vectors.

Bound-states and resonances are collectively called "composite particles" and the poles they create in amplitudes are indicated by bare propagators. Thus, for $2 \rightarrow 3$ scattering near a composite particle energy



where the bubbles in the RHS are the off-shell wave-functions.

# II - THE BETHE-SALPETER EQUATION (1)

## § 1. Summary of previous Work on its Transformation and Analysis

The first papers to consider the existence and form of solutions of the BSE appeared by Wick[21] and Cutkofsky[22] shortly after the original papers on the BSE were written. The method of the Wick rotation, by which the relative energy variables are continued analytically to the imaginary axis, converts the indefinite Lorentz metric into a posite euclidean one, and for a scalar exchange potential with total energy in the bound state region the equation becomes a non-singular equation of the fredholm type. Kemmer and Salam[23] succeeded in using the Wick rotation in the elastic scattering region; the transformed equation is non-singular except for the threshold singularity, and this does not cause any trouble in a numerical treatment. However, the limits of integration depend on the total energy variable, and we cannot discuss complex energies (such as those in the resonance region) by this method. Levine et al.[24] have used essentially the same equation as Kemmer and Salam's but removed the threshold singularity.

For quite a long time after the papers by Wick and Cutkofsky there was no progress on the rigorous analysis of the equation. Tiktopoulos[25] then showed that the Wick rotated BSE has a solution in the elastic scattering region by a method of distorting the contours of integration, thus producing a hilbert-schmidt kernel.

It is the configuration-space equivalent of this which is used in Schwartz and Zemach's numerical method[26]. Ciafaloni[27] has also shown that the eigenvalues of the kernel can be analytically continued away from the bound-state region into the whole complex plane of the total energy, with the physical threshold cut excluded.

Broido and Taylor[28] (see also Pagnamenta and Taylor[29] and Ito et al.[30]) have given a procedure for transforming the equation into an equation involving the spectral function of the energy-analytic representation: this will be discussed in the next chapter. The method is used at real values of the total energy, when the branch-points in the relative energy plane lie on the real axis; at complex total energies the branch-points move to non-collinear points, and an analysis based on an infinite family of cuts seems unlikely to be tractable. In ref.[28] a very complicated method is described to remove every singularity in the energy-analytic equation, but Greenman has shown more recently in two papers[31] that solutions exist using a method which treats the equation as it stands. In each of Greenman's papers the analysis is carried out in a banach space where the kernel is shown to be compact.

After a brief justification for the use of exchange potentials we shall quickly summarize the Wick rotation at bound-state energies. The rotated partial wave BSE with the exchange potential is then transformed so that its singularities are removed by

Taylor's subtraction procedure[32]. This leaves an equation which is non-singular (except for a threshold cut) in a complex domain which is the natural extension of the elastic scattering region. It is then shown that the kernel of the equation is compact in a certain banach space, using a method which Lovelace used on the Lippmann-Schwinger equation[32a]. The second-sheet structure is also examined.

The equation could be used as a numerical method for finding on-shell residues at elastic resonances, for example, and the only additional complication is that the equation is, as we should expect, complex.

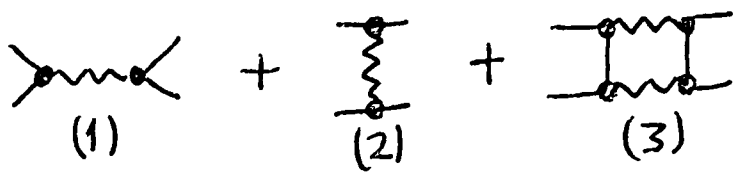
Although the solution is obtained for Wick rotated relative energies, it is obtained from a non-singular equation which produces the on-shell solutions and contains a significant amount of total energy analyticity..

## § 2. The Choice of an approximation to $T_2$

As we have said, the 2-particle reduced amplitude $T_2$ is, like T itself, unknown since it consists of an infinite number of graphs. Thus in order to obtain solutions, approximate potentials are always taken.

In a theory of scalar particles with fields $\phi, \psi$ reacting via an interaction $g\phi^2\psi$, the perturbation development of

the 2 $\phi$ -cut reduced amplitude in 2$\phi$ - scattering is given by

$T_2$ =  + exchange graphs for (2), (3), etc.



The omission of graphs such as (1), which have a pole in the direct channel, will lead to their exclusion in T. If is an elementary particle field, the pole in T requires mass renormalisation, but in a theory where $\psi$ is composite the pole arises directly from the BSE when $T_2$ excludes the graph (1). In a later chapter this difference will be considered, but for the moment it will be assumed that no elementary particles are coupled directly with the scattering channel.

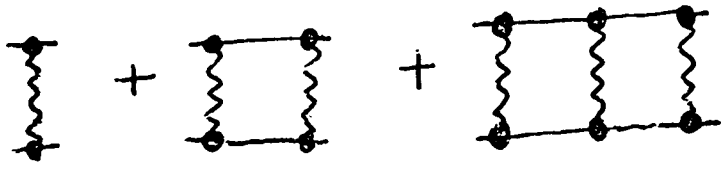
Now the first order contribution (in powers of g) to $T_2$ is $\frac{1}{2}(V + V')$ where V =  , $V'$ =  , and these exchange-diagrams are usually taken as a satisfactory approximation to $T_2$. This may be heuristically justified in several ways:-

(1) When g is small the rest of the series is of the second order.

(2) The cross-channel poles are expected to dominate the kernel's behaviour for small energies near the mass-shell, which is usually taken to be the principal region of the interaction.

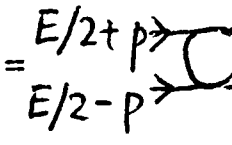
(3) The resulting approximate solution satisfies elastic unitarity, and this is important for the following reason. Using the approximate potential the following "ladder series" is

obtained:

$T \sim T_L =$ 

+ exchange diagrams

Now this approximation excludes such diagrams as  which are of the fourth order and above in g, and there is no reason, a priori, to suppose that they are any less important than the fourth order approximations included in $T_L$. The fact that the S-matrix satisfies elastic unitarity implies, however, that cancellations occur in those excluded terms to improve the approximation in this energy range. For exactly the same reason the ladder series should be viewed with some suspicion above the inelastic threshold; as has been shown by Levine, Tjon and Wright[24], the resulting S-matrix can violate unitarity to such an extent that scattering probabilities exceed unity.

In dealing with the ladder approximation, the particle propagators are approximated consistently by bare propagators. The notation $T(p; q) =$  where $E \equiv (E, \underline{0})$ will be used throughout this chapter, and the approximate BSE then assumes the form

$$T(p; r) = V(p; r) + i \int V(p; q)\, G_0(q)\, T(q; r)\, d^4q$$

where, for scalar exchange, $V(p; q) = \frac{g^2}{(2\pi)^4} \cdot \frac{1}{(p-q)^2 - \mu^2 + i\varepsilon}$ and

$$G_0(q) = \frac{1}{(E/2+q)^2 - m^2 + i\varepsilon} \cdot \frac{1}{(E/2-q)^2 - m^2 + i\varepsilon} .$$

The c.m. energy E is usually fixed and so its explicit dependence is omitted in T and V.

The highly singular BSE is required to possess a causal boundary-value in the limit $\varepsilon \to 0+$.

## § 3. The Wick Rotation

Consider a change of scale, $p_0 \to z\, p_0$, etc. in equation (2.1), where z is real and positive. The solution of

$$(3.1)\quad T^z(p;\, r) = V(p^z;\, r^z) + iz\int V(p^z;\, q^z)\, G_0(q^z)\, T^z(q;\, r)\, d^4q$$

where $p^z = (z\, p_0^0,\, \underline{p})$, etc., trivially satisfies

$$T^z(p;\, r) = T(p^z;\, r^z)$$

assuming that a unique solution exists. Thus

$$T^z(0,\underline{p};\, 0,\underline{r}) = T(0,\underline{p};\, 0,\underline{r})$$

for all real z. Now provided $0 \leqslant \arg z \leqslant \pi$ and $\varepsilon > 0$, the equation remains non-singular and formally analytic when E is a bound-state energy, $|E| < 2m$.

If $0 < \arg z < \pi$ it is easy to show that, after replacing $T^z$ by $\bar{T}^z(p;\, r) \equiv G_0^{\frac{1}{2}}(p^z)\, T^z(p;\, r)\, G_0^{\frac{1}{2}}(r^z)$, $\bar{T}^z$ satisfies

$$(3.2)\quad \bar{T}^z(p;\, r) = \bar{K}^z(p;\, r) + iz\int \bar{K}^z(p;\, q)\, \bar{T}^z(q;\, r)\, d^4q$$

where the kernel $\bar{K}^z = G_0^{\frac{1}{2}}\, V\, G_0^{\frac{1}{2}}$ is a hilbert-schmidt operator over $\mathbb{R}^4$. The kernel and hence the solution are operator-analytic with respect to z and $\bar{T}^z(p;\, r)$ is continuous with respect to p and r. Thus $\bar{T}^z(0,\underline{p};\, 0,\underline{r})$ is independent of z and defines $T(0,\underline{p};\, 0,\underline{r})$ at bound-state energies E.

Let arg z = $\theta$: then equation (3.2) is operator-analytic for z and E varying independently in a region which, as $\varepsilon \to 0$, becomes $-m + im\,E.\tan\theta < re\,E < m + im\,E.\tan\theta$. Thus equation (3.2) and hence its solution $\bar{T}^z$ are operator-analytic in a neighbourhood of any point (z, E) where (3.2) is defined.

The region of definition is connected, and so by continuation $\bar{T}_z(0,\underline{p};\ 0,\underline{r};)$ is independent of z for complex E. This gives a suitable definition of $T(0,\underline{p};\ 0,\underline{r})$ for any E lying in the complex plane outside small neighbourhoods of the cuts $(\pm 2m,\ \pm\infty)$. For E not lying on the cuts the limit $\varepsilon \to 0+$ exists in the uniform topology, and hence the Feynman boundary value can be found for any E off the physical cuts.

There remains the question of whether a full off-shell solution $T^z(p;\ r)$ can be continued backwards to the real z-axis. Such a continuation is not provided by this method because for a given E there does not exist a curve connecting z to the real axis along which the equation is well-defined, except when E lies in the bound-state region. This is the difficulty caused by the complicated sheet-structure of $T_E$ in the complex 3-dimensional space of E, $p_0$, $r_0$.

When z = i and E lies in the bound-state region, the kernel of the symmetrised equation is real, symmetric and hilbert-schmidt. This is the Wick-rotated form of the BSE which was originally obtained by a slightly different method. At bound-state energies,

when the kernel has an eigenvalue, it follows from the form of the kernel that the residue of the pole factorizes into wave-functions, in agreement with the behaviour of the exact "unrotated" amplitude.

## § 4. The Partial Wave Projections

The off-shell partial wave decomposition of T will be written as follows:

$$T(p_0, \underline{p}; q_0, \underline{q}) = -\frac{1}{4\pi |\underline{p}||\underline{q}|} \sum_{\ell=0}^{\infty} (2\ell+1) T_\ell(p_0, |\underline{p}|; q_0, |\underline{q}|) P_\ell(z).$$

where $z = \underline{p}.\underline{q} \,/\, |\underline{p}|\,|\underline{q}|$. The decomposition of the potential V is given by an exactly similar formula. When partial wave equations are considered p will denote $|\underline{p}|$ (rather than 4-momentum) and the 2-vector $(p_0, p)$ will be written $\underset{\sim}{p}$; $\hat{\underset{\sim}{p}}$ denotes the on-shell momentum $(0, \hat{p})$. The on-shell scattering amplitude is given by

$$f_\ell \equiv \frac{e^{i\delta_\ell} \sin\delta_\ell}{\hat{p}} = \frac{\pi^2}{2E\hat{p}^2} T_\ell(\hat{\underset{\sim}{p}}; \hat{\underset{\sim}{p}})$$

The partial wave BSE can now be written

$$(4.1) \quad T_\ell(\underset{\sim}{p}; \underset{\sim}{r}) = V_\ell(\underset{\sim}{p}; \underset{\sim}{r}) - i \int V_\ell(\underset{\sim}{p}; \underset{\sim}{q})\, G_0(\underset{\sim}{q})\, T_\ell(\underset{\sim}{q}; \underset{\sim}{r}) d^2\underset{\sim}{q}$$

or $\quad T_\ell = V_\ell - i K_\ell T_\ell$

After the Wick rotation, $T_\ell = V_\ell + K_\ell T_\ell$, with $p_0$ replaced by $ip_0$, etc. in $V_\ell$ and $K_\ell$. The following notation for the propagators will also be used: $G_0(\underset{\sim}{q}) = G_+(\underset{\sim}{q})\; G_-(\underset{\sim}{q})$, where $G_\pm(\underset{\sim}{q}) = [q_0^2 + \omega_\pm^2(q)]^{-1}$, $\omega_\pm(q) = \pm E/2 + \omega(q)$, $\omega(q) = (m^2+q^2)^{\frac{1}{2}}$.

In the case of scalar exchange,

$$V_\ell(\underset{\sim}{p};\underset{\sim}{q}) = \frac{f}{2}\int \frac{pq\,P_\ell(z)\,dz}{(p_0-q_0)^2+p^2+q^2-2pqz+\mu^2} \equiv \frac{f}{2}\,Q_\ell\left(\frac{(p_0-q_0)^2+p^2+q^2+\mu^2}{2pq}\right)$$

where $f \equiv g^2/4\pi^3$ and g is the coupling constant.

## § 5. Analytic Continuation of the bound-state solution into the Elastic Scattering and Resonance regions

To obtain the analytic continuation of $T_\ell$ away from the bound-state region we shall begin with the rotated unsymmetrical partial wave equation

$$(5.1)\quad T(\underset{\sim}{p}) = V(\underset{\sim}{p};\underset{\sim}{r}) + \int d^2\underset{\sim}{q}\, K(\underset{\sim}{p};\underset{\sim}{q})\, T(\underset{\sim}{q})$$

where, for convenience, the explicit dependence of $T_\ell(\underset{\sim}{p};\underset{\sim}{r})$ on $\underset{\sim}{r}$ and $\ell$ is suppressed for the time being. The solution will be obtained in a banach space, but in the bound-state region it will be clear that the equation is satisfied by the solution obtained using the previous method.

The kernel $K(\underset{\sim}{p};\underset{\sim}{q}) = V(\underset{\sim}{p};\underset{\sim}{q})\,G_0(\underset{\sim}{q})$ of equation (5.1) is non-singular so long as $|\mathrm{re}\ E| < 2m$; for $|\mathrm{re}\ E| > 2m$ there is trouble when $G_-(q)$ becomes infinite. To deal with this two singularities are removed using Taylor's method[31], by making subtractions in $T(\underset{\sim}{p})$ with respect to $p_0$ at the points $\pm i\omega_-(p)$, and writing

$$T(\underset{\sim}{p}) = \frac{p_0+i\omega_-(p)}{2i\omega_-(p)}\,a_+(p) - \frac{p_0-i\omega_-(p)}{2i\omega_-(p)}\,a_-(p) + [p_0^2+\omega_-^2(p)]\,b(\underset{\sim}{p})$$

where $a_\pm(p) = T(\pm i\omega_-(p),\ p)$. This decomposition will be required

to hold for $p \leq \alpha$, where $\alpha$ is a number to be determined later. If $p \geq \alpha$ we put $T(\underset{\sim}{p}) = c(\underset{\sim}{p})$. Substituting these expressions into equation (5.1), 4 simultaneous integral equations are obtained:

$$(5.2^{\pm})\quad a_{\pm}(p) = V(\pm i\omega_{-}(p), p; \underset{\sim}{T}) + \int_{S_1} \frac{dq}{2i\omega_{-}(q)} \left[ W_{+}(\pm i\omega_{-}(p), p; q) a_{+}(q) \right.$$

$$\left. - W_{-}(\pm i\omega_{-}(p), p; q) a_{-}(q) \right] + \int_{S_2} d^2\underset{\sim}{q}\, V(\pm i\omega_{-}(p), p; \underset{\sim}{q})\, G_{+}(\underset{\sim}{q})\, b(\underset{\sim}{q})$$

$$+ \int_{S_3} d^2\underset{\sim}{q}\, V(\pm i\omega_{-}(p), p; \underset{\sim}{q})\, G_{0}(\underset{\sim}{q})\, c(\underset{\sim}{q}) \qquad (0 \leq p \leq \alpha)$$

$$(5.3)\quad b(\underset{\sim}{p}) = \hat{V}(\underset{\sim}{p}; \underset{\sim}{T}) + \int_{S_1} \frac{dq}{2i\omega_{-}(q)} \left[ \hat{W}_{+}(\underset{\sim}{p}; q)\, a_{+}(q) \right.$$

$$\left. - \hat{W}_{-}(\underset{\sim}{p}; q)\, a_{-}(q) \right] + \int_{S_2} d^2\underset{\sim}{q}\, \hat{V}(\underset{\sim}{p}; \underset{\sim}{q})\, G_{+}(\underset{\sim}{q})\, b(\underset{\sim}{q})$$

$$+ \int_{S_3} d^2\underset{\sim}{q}\, \hat{V}(\underset{\sim}{p}; \underset{\sim}{q})\, G_{0}(\underset{\sim}{q})\, c(\underset{\sim}{q}) \qquad (0 \leq p \leq \alpha)$$

$$(5.4)\quad c(\underset{\sim}{p}) = V(\underset{\sim}{p}; \underset{\sim}{T}) + \int_{S_1} \frac{dq}{2i\omega_{-}(q)} \left[ W_{+}(\underset{\sim}{p}; q)\, a_{+}(q) \right.$$

$$\left. - W_{-}(\underset{\sim}{p}; q)\, a_{-}(q) \right] + \int_{S_2} d^2\underset{\sim}{q}\, V(\underset{\sim}{p}; \underset{\sim}{q})\, G_{+}(\underset{\sim}{q})\, b(\underset{\sim}{q})$$

$$+ \int_{S_3} d^2\underset{\sim}{q}\, V(\underset{\sim}{p}; \underset{\sim}{q})\, G_{0}(\underset{\sim}{q})\, c(\underset{\sim}{q}) \qquad (\alpha \leq p)$$

$S_1$, $S_2$, $S_3$ denote the regions

$$S_1 = \{ p: \quad 0 \leq p \leq \alpha \}$$

$$S_2 = \{(p_0, p) : 0 \leq p \leq \alpha \}$$

$$S_3 = \{(p_0, p) : p \geqslant \alpha \}$$

The following functions are used above:

$$W_{\pm}(\underset{\sim}{p};q) = \int_{-\infty}^{\infty} dq_0 \; \frac{V(\underset{\sim}{p};\underset{\sim}{q})\, G_+(\underset{\sim}{q})}{q_0 \mp i\omega_-(q)}$$

$$[p_0^2+\omega_-^2(p)]\, \hat{V}(\underset{\sim}{p};\underset{\sim}{q}) =$$

$$V(\underset{\sim}{p};\underset{\sim}{q}) - \frac{p_0+i\omega_-(p)}{2i\omega_-(p)} V(i\omega_-(p),p;\underset{\sim}{q}) + \frac{p_0-i\omega_-(p)}{2i\omega_-(p)} V(-i\omega_-(p),p;\underset{\sim}{q})$$

$$[p_0^2+\omega_-^2(p)]\, \hat{W}_{\pm}(\underset{\sim}{p};q) =$$

$$W_{\pm}(\underset{\sim}{p},q) - \frac{p_0+i\omega_-(p)}{2i\omega_-(p)} W_{\pm}(i\omega_-(p),p;q) + \frac{p_0-i\omega_-(p)}{2i\omega_-(p)} W_{\pm}(-i\omega_-(p),p;q)$$

Equations (5.2)-(5.4) will be written

$$\mathcal{J} = \mathcal{J}^0 + \mathcal{K}\mathcal{J} \tag{5.5}$$

where $\mathcal{J} = (a_+, a_-, b, c)^T$, $\mathcal{J}^0 = (a_+^0, a_-^0, b^0, c^0)$. $\mathcal{K} \equiv [K_{ij}]$ is is a 4 x 4 matrix of integral operators.

It is our purpose to show that equation (5.5) is non-singular and has a solution in the elastic scattering and resonance regions. Before proving this some definitions and a lemma are required:-

Definitions

(1) $B_1$ is the banach space of continuous functions mapping $S_1 \equiv [0, \alpha]$ into $\mathbb{C}$ with a continuous derivative and norm

$$\|f\| = \sup_{p \in S_1} |f(p)| + \sup_{p \in S_1} |f'(p)|$$

(2) $B_2$ is the banach space of continuous functions mapping $S_2 \equiv \mathbb{R} \times [0, \alpha]$ into $\mathbb{C}$ such that $p_0^2 f(p_0, p)$ is bounded, with norm

$$\|f\| = \sup_{(p_0, p) \in S_2} |p_0^2 f(p_0, p)|$$

Let X be the banach space of bounded continuous functions mapping $S \subseteq \mathbb{R}^2$ into $\mathbb{C}$ with norm

$$\|f\| = \sup_{(p_0, p) \in S} |f(p_0, p)|$$

Then (3) $B_3 = X$ when $S = S_3 = \mathbb{R} \times [\alpha, \infty)$

(4) $B_0 = X$ when $S = S_0 = \mathbb{R} \times [0, \infty)$

A vector in a banach space, or a continuous linear map F between banach spaces, which depends on a variable z will be called *continuous, *analytic, etc., if it is continuous, analytic, etc., as a function of z with respect to the norm (uniform) topology. The ordinary operations of calculus carry over without difficulty to vector-valued functions in normed spaces (see Taylor[33]).

Lemma

Let $K_E(x; y)$ be the kernel of an integral linear transformation $F(E)$: $B_i \to B_j$ $(i, j = 1, 2, 3)$ which depends on the complex variable E.

Define $f_{y,E}(x) = K_E(x;\ y)$. Then

(1) F is a compact operator analytic for E in an open domain $D \subset \mathbb{C}$ if

(a) $f_{y,E} \in B_j$ for $(y,\ E) \in S_i \times D$, and the map $(y,\ E) \longrightarrow f_{y,E}$ is *continuous in y, E and *analytic in E

(b) $\int_{S_i} \| f_{y,E} \|\, dy < \infty$

(c) $\int_{S_i} \| \partial f_{y,E}/\partial E \|\, dy$ converges uniformly.

(2) Let i = 1 (so that $y \in [0,\alpha]$), and $K_E(x;\ y) = \dfrac{L_E(x;\ y)}{y-\hat{p}}$

where $\hat{p} = \sqrt{E^2/4 - m^2}$. Define $g_{y,E}(x) = L_E(x;\ y)$. Then F(E) satisfies (1) above with $D \to D - [2m, \infty)$ and the limits $F_{\pm}(E) \equiv \lim_{\varepsilon \to 0+} F(E \pm i\varepsilon)$ exist and are compact when $E \in [2m, \infty) \cap D$ provided

(a) $g_{y,E} \in B_j$ for $(y,\ E) \in S_1 \times D$ and the map $(y,\ E) \to g_{y,E}$ is

(i) *continuous in y, E

(ii) *analytic with respect to E

(iii) *continuously differentiable with respect to y

(b) $g_{0,\ 2m} = 0$

Proof of the Lemma

(1) Let S(y, E) be the linear operator of rank 1 mapping $B_i$ into $B_j$ by

$$\left[S(y,\ E)\ h\right](x) = f_{y,\ E}(x)\ h(y) \qquad (h \in B_i)$$

By 1(a) this operator is *continuous, and so by 1(b) the integral $F(E) = \int_{S_i} S(y,E)dy$ exists as a bounded linear operator.

$F(E)$ is also compact, since it is the limit of a sum of compact (in fact rank 1) operators (cf. definition of the integral).

Now $S(y, E)$ is *analytic, and by complete analagy with the theorems of complex analysis we may differentiate under the integral sign with respect to E to prove that $F(E)$ is *analytic, provided condition (c) is also satisfied. It can be seen that when $S_i$ is compact 1(b) and 1(c) are redundant.

(2) In view of the last remark and conditions 2a(i) - (ii), it is clear that $T(E)$ satisfies (1) above when $D \rightarrow D - [2m, \infty)$

Let $[T(y, E)\ h](x) = g_{y,E}(x)\ h(y) \qquad (h \in B_1)$

Now $G(E \pm i\varepsilon) = \int_0^\alpha \frac{T(y, E \pm i\varepsilon)\, dy}{y - (\hat{p} \pm i\zeta)}$

$$= \int_0^\alpha \frac{T(\hat{p} \pm i\zeta, E \pm i\varepsilon)}{y - (\hat{p} \pm i\zeta)}\, dy + \int_0^\alpha \frac{T(y, E \pm i\varepsilon) - T(\hat{p} \pm i\zeta, E \pm i\varepsilon)}{y - (\hat{p} \pm i\zeta)}\, dy$$

where $E \in [2m, \infty) \cap D$, $\zeta = \zeta(\varepsilon)$ and $\zeta \rightarrow 0$ as $\varepsilon \rightarrow 0$

The first integral can be done analytically and clearly has a limit. (When $\hat{p} = 0$ there could be trouble, but this is is provided for in condition 2(b), since $T(\hat{p} \pm i\zeta, E \pm i\varepsilon)$ will tend to zero faster than the logarithmic infinity $\log(i\zeta)$; as will be shown later we can assume that $\hat{p}$ is never equal to $\alpha$ .) The second integral is non-singular in the limit when $T(y, E)$ is *continuously differentiable with respect to y at $\hat{p}$. The *derivative, $\partial T(y,E)/\partial y$, when it exists, is given by

$$\left[\frac{\partial T}{\partial y}(y,E)\,h\right](x) = \frac{\partial g_{y,E}(x)}{\partial y}\,h(y) + g_{y,E}(x)\,\frac{\partial h(y)}{\partial y}$$

and the conditions for its existence and *continuity are given in 2(a). Finally, $T_{\pm}(E)$ are the limits of compact operators, and are hence compact.

We mention in passing that the conditions (2) could be relaxed by replacing 2a(iii) with a Lifschitz condition, but for our purposes this is an unnecessary refinement.

Proof of the Theorem

We are now able to prove that equation (5.5) possesses solutions both on and off the physical cut, and has continuations into the second sheet.

Let $D = \{E: -2m < \text{re } E < 2m + \mu\}$, $C = [2m, \infty)$. If V is the partial wave potential due to the exchange of a particle of mass $\mu$ then

I For every $E_0 \in D$ there exists a neighbourhood $N \subset D$ of $E_0$ such that at least one of the following is true:

(1) $N \subset \{E: |\text{re } E| < 2m\}$. The kernel K of equation (5.1) is non-singular for $E \in N$ and defines a compact operator over $B_0$ which is analytic in N. The inhomogeneous term belongs to $B_0$ and is also analytic. Thus a solution exists in $B_0$ and is analytic in the neighbourhood N.

(2) There exists $\alpha > 0$ (which is arbitrary within an open

interval) such that the kernel $\mathcal{K}$ of equation (5.5) is non-singular for $E \in N - C$, and defines a compact operator over $B \equiv B_1 \times B_1 \times B_2 \times B_3$. The uniform limits $\mathcal{K}_{\pm}$ exist and are compact when E approaches C from above and below. $\mathcal{J}^0$ belongs to B and is analytic in N, so that by inversion a unique analytic solution of equation (5.5) exists which has limits on the cut C.

II There is an analytic continuation between the solutions in (1) and (2) above, hence T is defined and analytic in D-C, and has boundary values $T_{\pm}$ on C. (It can be seen that T is defined rather like a manifold, with various local solutions fitting together globally.)

III The wave-functions $\chi(\underset{\sim}{p}) = T(\underset{\sim}{p}; \hat{\underset{\sim}{p}})$ and the on-shell scattering matrix $t(s) = T(\hat{\underset{\sim}{p}}; \hat{\underset{\sim}{p}})$ can be continued from the physical region C into the domain $\{ E: |\text{re } E| < 2m + \mu ; \quad |\text{im } \hat{p}| < \mu \}$

Proof

I It will be sufficient for us to restrict our attention to the second case (2). The proof of the first follows by considering c and $K_{44}$ with $\alpha = 0$. For neighbourhoods which allow both (1) and (2) to apply (and we shall show that they exist) the solutions must be identical, since by the addition of equations (5.2) - (5.4), T satisfies (5.1). This therefore disposes of I(1) and II provided the theorem can be proved for the second case when D is replaced by some open cover $\tilde{D}$ for $\{ E: \ 2m < \text{re } E < 2m + \mu \}$. We shall take, in fact, $\tilde{D} = \{ E: \ \varepsilon < \text{re } E < 2m + \mu \}$, where $|\varepsilon| < 2m$ and $\varepsilon$ will be further restricted later.

Let $E \in \tilde{D}$ and assume

(5.6) $\alpha^2 > (\mathrm{re}\ E)^2/4 - m^2$.

We first make the observation that only $V(\pm i\omega_+(p),\ p;\ \underset{\sim}{q})$ can cause $K_{ij}(j = 3,4)$ to be singular, since $G_+$ is regular and $G_-$ is non-singular over the domain of $K_{i4}(i = 1, \ldots, 4)$.

$$V(i\omega_-(p), p; \underset{\sim}{q}) = \frac{f}{2}\int_{-1}^{1} \frac{pq P_\ell(z)\,dz}{[q_0 - i(\omega_-(p) - A)][q_0 - i(\omega_-(p) + A)]}$$

where $A = (p^2 + q^2 + \mu^2 - 2pqz)^{\frac{1}{2}}$. Since $p \leq \alpha$, the imaginary part of the first denominator in the integrand is never zero if

(5.7) $-\mathrm{re}\ E/2 + (m^2 + \alpha^2)^{\frac{1}{2}} - \mu < 0$

whilst the second denominator is non-singular so long as

$\mathrm{re}\ E/2 < m + \mu$.

It is clear that (5.6) and (5.7) can be satisfied with $\alpha$ varying over a suitable interval over the positive reals, and E varying (independently of $\alpha$) over a neighbourhood of $\tilde{D}$. Similarly $V(-i\omega_-(p),\ p;\ \underset{\sim}{q})$ is non-singular with the same conditions on $\alpha$. This argument also proves that the inhomogeneous term $\mathcal{J}^0$ is regular. It is precisely because $\alpha$ has an upper limit in (5.7) that we cannot put $\alpha = \infty$ and forget about c(p).

The kernels $K_{ij}$ $(j = 1, 2)$ can have singularities if $W_+$ are singular. The evaluation of $W_+$ can be done analytically, and when V is written as $V_+ + V_-$ where

$$V_\pm = \frac{f}{4}\int_{-1}^{1} \frac{pq\,P_\ell(z)\,dz}{A(A \mp (q_0 - p_0))} \qquad \text{(A as above)}$$

we have

$$W_+(\underset{\sim}{p};q) = -\frac{\pi}{\omega(q)}\left[\frac{V_+(\underset{\sim}{p};i\omega_+(q),q)}{2\omega_+(q)} + \frac{V_-(\underset{\sim}{p};i\omega_-(q),q)}{E}\right]$$

$$W_-(\underset{\sim}{p};q) = -\frac{\pi}{\omega(q)}\left[\frac{V_+(\underset{\sim}{p};-i\omega_-(q),q)}{2\omega_-(q)} + \frac{V_-(\underset{\sim}{p};-i\omega_+(q),q)}{E}\right]$$

It is straightforward to verify that all the $W_+$ appearing in the equations are non-singular, and hence that $\omega_-(q)^{-1}$ is the only singularity, which occurs in $K_{ij}$ when j is 1 or 2.

We have thus shown that for any $E \in \tilde{D}$ there exists an $\alpha$ and a neighbourhood of E for which equation (5.5) is non-singular, except for the threshold singularity. For values on $C_\pm$, the upper and lower lips of the physical cut, the $i\varepsilon$ - prescription $\frac{1}{q-\hat{p}\mp i\varepsilon}$ must be used, but when the principal value is given to the pole the solution gives $\tan\delta_\ell$ on the shell, instead of $e^{i\delta_\ell}\sin\delta_\ell$ (see next chapter).

Now in order to prove part I of the theorem it is sufficient to verify that:

(1) $\mathcal{J}^0 \in$ B and is *analytic w.r.t. E

(2) Kernels $K_{ij}$ $(j = 3,4)$ satisfy the conditions of the Lemma, Part 1.

(3) The remaining kernels $K_{ij}$ $(j = 1,2)$ satisfy the conditions of the Lemma, Part 2.

This is a simple, but tedious exercise of estimation, and the details are omitted. All that is necessary to verify is that

when a certain function $f_z(x)$ in a banach space B is, for example, required to be *continuous with respect to the parameter z, then it is continuous with respect to z subject to uniform conditions in x being applied. These are the restrictions that make $f_z$ belong to B.

That T is independent of $\alpha$ can be seen by observing that the components of $\mathcal{J}$ as given by equations (5.2) - (5.4) can be extended (or restricted) outside (or within) their respective ranges of definition ($p \gtrless \alpha$), and they continue to belong to the same banach spaces for different values of $\alpha$. If $\mathcal{J}'$ is defined by this extension as an element of B for a different $\alpha$ (say $\alpha'$), then $\mathcal{J}'$ satisfies the equations with $\alpha'$ replacing $\alpha$, and so gives the same (unique) solution of the equations. The off-shell amplitude $T(\underset{\sim}{p})$ can be found at all $\underset{\sim}{p}$ by extending $c(\underset{\sim}{p})$, given in equation (5.4), to arbitrary values of p.

For real $\underset{\sim}{r}$, $\mathcal{J}^0(\underset{\sim}{p}; \underset{\sim}{r})$ is *continuous w.r.t. $\underset{\sim}{r}$ and hence $T(\underset{\sim}{p}; \underset{\sim}{r})$ is continuous as a function of both $\underset{\sim}{p}$ and $\underset{\sim}{r}$. It is symmetric in $\underset{\sim}{p}$ and $\underset{\sim}{r}$ because this is trivially verifiable in the bound-state region, and T is analytic in the whole domain D.

Finally, to prove Part III it is sufficient only to verify that the inhomogeneous term $\mathcal{J}^0$ is non-singular and belongs to B when $\underset{\sim}{r} = \hat{\underset{\sim}{p}}$, provided we impose $|\text{im } p| < \mu$ and have re $E > 2m - \mu^2/m$; thus at the outset we should have stated that $\tilde{D} = \{ E: \varepsilon < \text{re } E < 2m + \mu \}$

where $2m > \varepsilon > 2m - \mu^2/m$. Because $\mathcal{T}^0(\underset{\sim}{p}; \hat{\underset{\sim}{p}})$ belongs to B and is *analytic w.r.t. E, the same analysis goes through, and the wave-function $\chi(\underset{\sim}{p}) = T(\underset{\sim}{p}; \hat{\underset{\sim}{p}})$ is defined and analytic within

$$D_1 = \left\{ -2m < \text{re } E < 2m + \mu, \quad |\text{im } \hat{p}| < \mu \right\} .$$

The on-shell solution t(s) is obtained from equation (5.4):

$$t(s) = V(\hat{\underset{\sim}{p}}; \hat{\underset{\sim}{p}}) + \int_{S_1} \frac{dq}{2i\omega(q)} \left[ W_+(\hat{\underset{\sim}{p}}; q) a_+(q) + W_-(\hat{\underset{\sim}{p}}; q) a_-(q) \right]$$
$$+ \int_{S_2} d^2\underset{\sim}{q}\, V(\hat{\underset{\sim}{p}}; \underset{\sim}{q}) G_+(\underset{\sim}{q})\, b(\underset{\sim}{q}) + \int_{S_3} d^2\underset{\sim}{q}\, V(\hat{\underset{\sim}{p}}; \underset{\sim}{q}) G_0(\underset{\sim}{q})\, c(\underset{\sim}{q})$$

The right hand side is a continuous linear function for E in the above domain, with dynamical ("left-hand") cuts ($\pm i\mu/2$, $\pm i\infty$) arising from the born term $V(\hat{\underset{\sim}{p}}; \hat{\underset{\sim}{p}})$.

We have thus provided a continuation of the solution into $\{ E: \ -2m < \text{re } E < 2m + \mu \}$. We could do a similar analysis for $\{ -(2m + \mu) < \text{re } E < 2m \}$ but there is no need: the solutions are even in E, as can be seen from equation (5.1).

Second-Sheet Structure

The second-sheet continuations of the rotated amplitudes may now be obtained very easily.

All the awkward moving cuts that were met in Chapter I have been rotated away, hence it is unnecessary to make any assumptions about the analytic structure. Second-sheet amplitudes are denoted by a superscript II, and defined in $\{ E: \ |\text{re } E| < 2m + \mu, \ |\text{im } \hat{p}| < \mu \}$. Subscripts $\pm$ denote, as usual, values on the upper and lower lip of the physical cut. We must remember that $\hat{p}$ is

double-valued, and $\hat{p}_- = -\hat{p}_+$.

On C, $\mathcal{J}_\pm = \mathcal{J}^0 + \mathcal{K}_\pm \mathcal{J}_\pm$ and so $\Delta\mathcal{J} = \Delta\mathcal{K}\mathcal{J}_- + \mathcal{K}_+ \Delta\mathcal{J}$

where $\Delta\mathcal{J} = \mathcal{J}_+ - \mathcal{J}_-$, $\Delta\mathcal{K} = \mathcal{K}_+ - \mathcal{K}_-$

Now $\Delta\mathcal{K}$ is found from $\Delta\left(\frac{1}{\omega_-(q)}\right) = \frac{\pi E}{\hat{p}}\,\delta(q-\hat{p})$

Thus $\Delta\mathcal{K}\mathcal{J}_-(\underset{\sim}{p};\underset{\sim}{r}) = i\lambda(-)^{\ell+1}\mathcal{J}^0_+(\underset{\sim}{p};\underset{\sim}{\hat{p}})\,T_-(\underset{\sim}{\hat{p}};\underset{\sim}{r})$

where $\lambda = \pi^2/E\hat{p}$, and so

$$(1-\mathcal{K}_+)\Delta\mathcal{J} = i\lambda(-)^{\ell+1}\mathcal{J}^0_+(\underset{\sim}{p};\underset{\sim}{\hat{p}})\,T_-(\underset{\sim}{\hat{p}};\underset{\sim}{r})$$

$$\Delta\mathcal{J} = i\lambda(-)^{\ell+1}\mathcal{J}_+(\underset{\sim}{p};\underset{\sim}{\hat{p}})\,T_-(\underset{\sim}{\hat{p}};\underset{\sim}{r})$$

Hence $\Delta T(\underset{\sim}{p};\underset{\sim}{r}) = i\lambda(-)^{\ell+1} T_+(\underset{\sim}{p};\underset{\sim}{\hat{p}})\,T_-(\underset{\sim}{\hat{p}};\underset{\sim}{r})$, which is the rotated form of the off-shell unitarity relation. On putting $\underset{\sim}{r} = \underset{\sim}{\hat{p}}$,

$$\chi_+(\underset{\sim}{p}) \equiv T_+(\underset{\sim}{p};\underset{\sim}{\hat{p}}) = \frac{(-)^{\ell+1}T_-(\underset{\sim}{p};\underset{\sim}{\hat{p}})}{1-i\lambda t_-(s)} = \frac{(-)^{\ell+1}\chi_-(\underset{\sim}{p})}{1-i\lambda t_-(s)}$$

Thus $\chi$ and t have continuations through the cut given by

$$\chi^{II}(\underset{\sim}{p}) = \frac{(-)^{\ell+1}\chi(\underset{\sim}{p})}{1+i\lambda\, t(s)}\,,\quad t^{II}(s) = \frac{t(s)}{1+i\lambda\, t(s)}$$

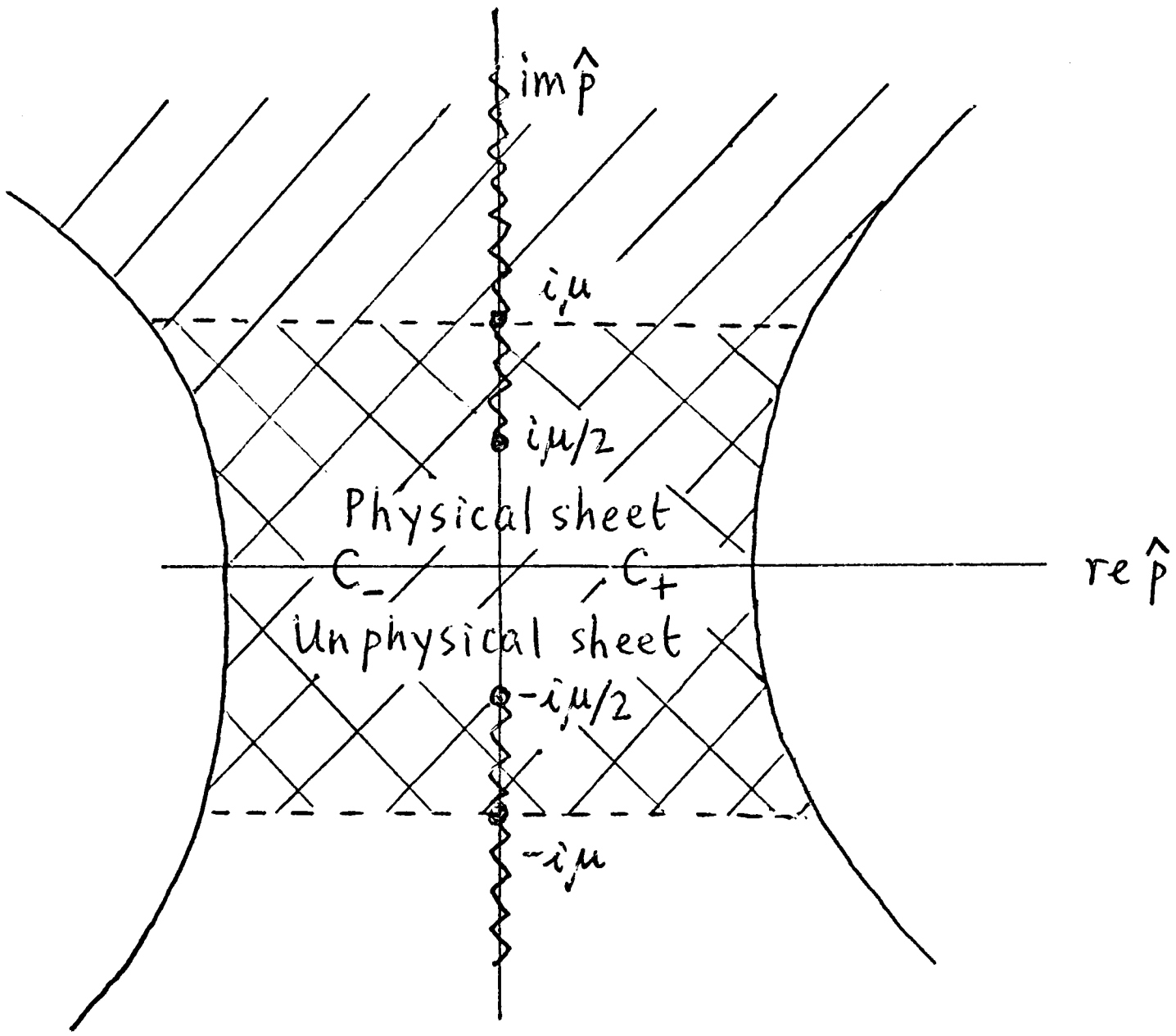
As a consequence, the second-sheet structure of T is given by

$$T^{II}(\underset{\sim}{p};\underset{\sim}{r}) = T(\underset{\sim}{p};\underset{\sim}{r}) + i\lambda\frac{T(\underset{\sim}{p};\underset{\sim}{\hat{p}})\,T(\underset{\sim}{\hat{p}};\underset{\sim}{r})}{1+i\lambda\, t(s)}$$

The singularity at E = 2m is of the square-root type because the transformations $T \longrightarrow T^{II}$, etc., are involutions and T is not infinite at the branch-point.

Because T, $\chi$, t are even functions of E, they are analytic functions of $s \equiv E^2$ with the threshold cut $s \geqslant 4m^2$. In terms of the variable $\hat{p}$ instead, the physical sheet is the upper $\hat{p}$ - plane and the unphysical sheet is the lower half. In the diagram below

the region of analyticity of T is shaded, and the region of analyticity of $\chi$ and t is double shaded. The curve is the inelastic boundary $|\text{re } E| = 2m + \mu$, and the wavy lines are the dynamical cuts $|\text{im } \hat{p}| > \mu/2$.



III - THE BETHE-SALPETER EQUATION (2)

## § 1. The Heitler Matrix

Consider the scattering of two scalar or pseudoscalar particles of masses $m_1$ and $m_2$ in the angular momentum state $\ell$. Then, in the unrotated form of the BSE, $T_\ell = V_\ell - iV_\ell G_0' T_\ell$, where $V_\ell$ is an exact or approximate potential, and $G_0' = D_1' D_2'$.

Now let $\tilde{G}_0' = G_0' + 2\pi^2 \delta^+(D_1^{-1}) \delta^+(D_2^{-1})$ represent the stationary 2-particle propagator, where the elastic-cut discontinuity $(2\pi i)^2 \delta^+(D_1^{-1}) \delta^+(D_2^{-1})$ has been half-removed. If $R_\ell$ satisfies

$$(1.1) \qquad R_\ell = V_\ell - iV_\ell G_0 R_\ell$$

then $R_\ell(\underset{\sim}{p}; \underset{\sim}{r}) = V_\ell(\underset{\sim}{p}; \underset{\sim}{r}) - iK V_\ell(\underset{\sim}{p}; \hat{\underset{\sim}{p}}) R_\ell(\underset{\sim}{p}; \underset{\sim}{r}) + V_\ell \tilde{G}_0' R_\ell$

where $K = \pi^2/2E\hat{p}$. Thus

$$R_\ell(\underset{\sim}{p}; \underset{\sim}{r}) = T_\ell(\underset{\sim}{p}; \underset{\sim}{r}) - iK T_\ell(\underset{\sim}{p}; \hat{\underset{\sim}{p}}) R_\ell(\underset{\sim}{p}; \underset{\sim}{r})$$

Puting $\underset{\sim}{p} = \hat{\underset{\sim}{p}}$, $R_\ell(\hat{\underset{\sim}{p}}; \underset{\sim}{r}) = T_\ell(\hat{\underset{\sim}{p}}; \underset{\sim}{r}) - iK T_\ell(\hat{\underset{\sim}{p}}; \hat{\underset{\sim}{p}}) R_\ell(\underset{\sim}{p}; \underset{\sim}{r})$

and hence

$$(1.2) \qquad R_\ell(\underset{\sim}{p}; \underset{\sim}{r}) = T_\ell(\underset{\sim}{p}; \underset{\sim}{r}) - iK \frac{T_\ell(\underset{\sim}{p}; \hat{\underset{\sim}{p}}) T_\ell(\hat{\underset{\sim}{p}}; \underset{\sim}{r})}{1 + iK T_\ell(\hat{\underset{\sim}{p}}; \hat{\underset{\sim}{p}})}$$

The conjugate equation is similarly obtained;

$$T_\ell(\underset{\sim}{p}; \underset{\sim}{r}) = R_\ell(\underset{\sim}{p}; \underset{\sim}{r}) + iK \frac{R_\ell(\underset{\sim}{p}; \hat{\underset{\sim}{p}}) R_\ell(\hat{\underset{\sim}{p}}; \underset{\sim}{r})}{1 - iK R_\ell(p; p)}$$

Now on-shell, in the elastic region,

$$T_\ell(\hat{\underset{\sim}{p}}; \hat{\underset{\sim}{p}}) = e^{i\delta_\ell} \sin\delta_\ell / K$$

so that $R_\ell(\hat{\underset{\sim}{p}}; \hat{\underset{\sim}{p}}) = \tan\delta_\ell / K$. $R_\ell$ is the relativistic analogue of the Heitler reaction matrix. Even if $V_\ell$ is approximate,

$T_\ell$ will satisfy elastic unitarity so long as $R_\ell$ is real on-shell.

When looking for sharp resonances by examination of the phase-shifts, we shall expect a second-sheet pole in the elastic region when the phase-shift rises sharply through $\pi/2$ and $R_\ell$ is therefore infinite on-shell. There are various ways that a function of several variables may become infinite at a point, but equation (1.2) tells us precisely how this comes about. Provided $T_\ell\ (\hat{\underset{\sim}{p}};\ \hat{\underset{\sim}{p}})$ is sufficiently smooth, $R_\ell\ (\underset{\sim}{p};\ \underset{\sim}{q})$ has poles at the zeros of $1 + ikT_\ell\ (\hat{\underset{\sim}{p}};\ \hat{\underset{\sim}{p}})$, and at a pole the residue factorizes into two "wave-functions" for the incoming and outgoing particles:

$$R_\ell\ (\underset{\sim}{p};\ \underset{\sim}{r}) \sim -\frac{k}{\left(\frac{d\delta_\ell}{dE}\right)_{E=E_r}} \cdot \frac{T_\ell\ (\underset{\sim}{p};\ \hat{\underset{\sim}{p}})\, T_\ell\ (\hat{\underset{\sim}{p}};\ \underset{\sim}{r})}{E - m_r} \quad \text{near } E = m_r,$$

where $m_r$ is the mass of the resonance.

This factorization, it should be observed, is a characteristic property of the equation, and holds whether $V_\ell$ is exact or not. There thus arises a homogeneous equation for the wave-function:

$\chi_\ell = -iV_\ell\, \tilde{G}_0'\, \chi_\ell$ at $E = m_r$, where $\chi_\ell(\underset{\sim}{p}) = T_\ell\ (\underset{\sim}{p};\ \hat{\underset{\sim}{p}})$.

Interestingly, this is just the equation for bound-states in $|m_1 - m_2| < E < m_1 + m_2$, since here $\tilde{G}_0' = G_0'$ (the $\delta$-functions vanish) and so $R_\ell = T_\ell$. The bound-state eigenvalue equation is, however, derived under the assumption that at a bound-state energy $E$, $E^2$ is an eigenvalue of the mass-operator $P^\mu P_\mu$, whereas we used the phase-shift criterion for resonances.

When the analytic distribution $\delta^+(D_1^{-1})\ \delta^+(D_2^{-1})$ is analytically continued to the imaginary axis by the Wick rotation, it is responsible for replacing the $i\xi$ - prescription on the 2-particle threshold by a principal value, and it is this boundary-value which gives the Heitler matrix. In both the method described in the previous chapter, and the method to follow, the equations for $R_\ell$ in the elastic scattering region are real, and hence elastic unitarity is automatically satisfied. In addition we have a well-defined condition, the existence of non-trivial solutions of a homogeneous equation, at real energies that are expected to indicate resonance-poles on the second-sheet.

## § 2. Energy-Analytic Representation of the Partial Wave Equation

As was stated in § 8 of Chapter I, the 2-body scattering amplitude satisfies a spectral representation in the relative energy of the incoming particles. It is most probable that this representation holds for all approximate amplitudes which are derived from perturbation theory (such as the ladder series), and this can be proved for the BSE when the approximate potential consists of sums of exchange graphs, or continuous sums of exchange graphs, such as the complete exchange-propagator  or the diagram  occuring in a $\phi^4$ - theory.

Again we consider scalar-or pseudoscalar-scattering, and take the scattered particles identical, putting the outgoing momenta on-shell. The relaxation of these conditions is trivial,

but the equations are about twice as long. Although the direct propagators will be replaced by bare propagators, the complete kernel can be recovered by representing the clothes direct propagators as integrals of bare ones having variable mass, using weight-functions given by the Lehmann representation.

Since the method will be used in Chapter V to calculate phase-shifts it is presented in some detail. It will be required that the approximate potential can be expresessed in the spectral form $V_\ell(\underset{\sim}{p}, \underset{\sim}{q}) = \int \frac{\sigma_\ell(\underset{\sim}{p}, \underset{\sim}{q}, u)\, du^2}{-(p_o-q_o)^2 + u^2 - i\varepsilon}$, where $\sigma_\ell$ is taken as real. The dependence of $\sigma_\ell$ on $\underset{\sim}{p}$ and $\underset{\sim}{q}$ accounts for spin, tensor forces, etc., and it will be assumed that $\sigma_\ell$ is a polynomial in $p_o$ and $q_o$, vanishing when $u^2 < (p - q)^2 + \mu^2$ . The polynomial condition is simply inserted to make the method work, but is covers a large number of possibilities. The support condition defines $\mu$ to be the lowest mass of the exchanged particles.

With $V_\ell$ written

$$(2.1)\quad V_\ell = V_+ + V_- \equiv -\int \frac{\sigma_\ell\, du}{p_o-q_o-u+i\varepsilon} + \int \frac{\sigma_\ell\, du}{p_o-q_o+u-i\varepsilon},$$

it follows immediately from the BSE that $T_\ell$ has an EAR in $p_o$ like that in § 8 of Chapter I. Bearing in mind the restrictions we imposed, $T_\ell$ is an even function of $p_o$ and the EAR can be written

$$T_\ell(\underset{\sim}{p}) = \int \frac{\rho_\ell(p, s)}{p_o^2 - s^2}\, ds^2$$

where the outgoing momenta are on-shell and therefore ignored, and it has been assumed that no subtractions are necessary. If n subtractions are required the EAR can be written

$$T_{\ell\ \underset{\sim}{(p)}} = T_{\ell}^{(0)}(p) + p_0^2\, T_{\ell}^{(1)}(p) + \ldots + p_0^{2n} \int \frac{\rho_\ell\ (p,s)ds^2}{p_0^2 - s^2}$$

where $p_0 = 0$ has been taken as the subtraction-point. It must be observed that unlike in S-matrix theory where divergent integrals are often renormalized by subtractions requiring arbitrary constants, here no such procedure is possible. This is because the subtraction "constants" are functions of p, and more general than the on-shell solution itself. Thus the subtraction constants must be determined by the BSE, which is another way of saying that the integral in the inhomogeneous term of the BSE must be convergent as it stands.

Now the purpose of the exercise is to rewrite the BSE as an equation in $\rho_\ell$ , and we shall consider the case when one subtraction is necessary, so that

$$T_{\ell\ \underset{\sim}{(p)}} = a(p) + p_0^2 \int \frac{\rho\,(p,\ s)ds^2}{p_0^2 - s^2}$$

The angular momentum index will now be suppressed, and spacial momenta usually omitted from the potential: thus $\sigma_\ell(\underset{\sim}{p},\ \underset{\sim}{q},\ u)$ will be written as $\sigma(p_0,\ q_0,\ u)$.

In the partial-wave BSE the inhomogeneous term $-iVG_0T$ is

$$-i\int \frac{\sigma(P_0,q_0,u)}{-(P_0-q_0)^2+u^2}\cdot\frac{1}{[q_0^2+\omega_+^2(q)][q_0^2+\omega_-^2(q)]}\cdot\left[a(q)+q_0^2\int\frac{\rho(q,t)}{q_0^2-t^2}\right]du^2\,d^2\underset{\sim}{q}$$

Using equation (2.1) the $q_0$-integral can be done by two contour-integrals, around the lower and upper planes, for $V_+$ and $V_-$ respectively to give

$$2\pi\int\frac{\sigma(P_0,\omega_+(q),u)}{P_0-\omega_+(q)-u}\cdot\frac{1}{4E\omega_+(q)\omega(q)}\cdot\left[a(q)+\omega_+^2(q)\int\frac{\rho(q,t)dt^2}{\omega_+^2(q)-t^2}\right]dq\,du$$

$$-2\pi\int\frac{\sigma(P_0,\omega_-(q),u)}{P_0-\omega_-(q)-u}\cdot\frac{1}{4E\omega_-(q)\omega(q)}\cdot\left[a(q)+\omega_-^2(q)\int\frac{\rho(q,t)dt^2}{\omega_-^2(q)-t^2}\right]dq\,du$$

$$+2\pi\int\frac{\sigma(P_0,t,u)}{P_0-t-u}\cdot\frac{1}{[t^2-\omega_+^2(q)][t^2-\omega_-^2(q)]}\cdot t^2\rho(q,t)\,dt\,dq\,du$$

plus three similar terms in which $p_0$ is replaced by $-p_0$.

The value at $p_0 = 0$ is

$$\text{(2.2)}\quad \frac{\pi}{E}\int\frac{dq}{\omega(q)}\left[\frac{v(\omega_+(q))}{\omega_+(q)}-\frac{v(\omega_-(q))}{\omega_-(q)}\right]a(q)$$

$$-2\pi\int dq\,dt\left[\frac{t\omega_+(q)v(\omega_+(q))}{E\omega(q)[\omega_+^2(q)-t^2]}-\frac{t\omega_-(q)v(\omega_-(q))}{E\omega(q)[\omega_-^2(q)-t^2]}+\frac{2t^2v(t)}{[\omega_+^2(q)-t^2][\omega_-^2(q)-t^2]}\right]\rho(q,t)$$

where $$v(t)\equiv v(p,q,t)=-\int\frac{\sigma(0,p,t,q,u)\,du}{t+u}$$

Expression (2.2) is denoted

$$\int dq\, K_{00}(p;\, q)\, a\,(q) + \int dqdt\, K_{01}(p;\, q,t)\, \rho\,(q,\, t)$$

The discontinuity of the inhomogeneous term across the cut $p_0 = s - i\varepsilon$ is, at $p_0 = s$, given by the expression

$$(2.3)\ \frac{\pi}{2Es^2}\int\frac{dq}{\omega(q)}\left[\frac{\sigma(s,\omega_+(q),s-\omega_+(q))}{\omega_+(q)} - \frac{\sigma(s,\omega_-(q),s-\omega_-(q))}{\omega_-(q)}\right]a(q)$$

$$+\frac{\pi}{s^2}\int dq\,dt\left[\frac{t\,\omega_+(q)\,\sigma(s,\omega_+(q),s-\omega_+(q))}{E\,\omega(q)[\omega_+^2(q)-t^2]} - \frac{t\omega_-(q)\,\sigma(s,\omega_-(q),s-\omega_-(q))}{E\,\omega(q)[\omega_-^2(q)-t^2]}\right.$$

$$\left. + \frac{2t^2\sigma(s,t,s-t)}{[\omega_+^2(q)-t^2][\omega_-^2(q)-t^2]}\right]\rho(q,t)$$

Expression (2.3) is denoted

$$\int dq\, K_{10}(p,s;\, q)\, a\,(q) + \int dqdt\, K_{11}(p,s;\, q,t)\, \rho\,(q,t)$$

We are now in a position to write down the energy-analytic form of the BSE. If the solution is represented by $X \equiv \begin{pmatrix} a \\ \rho \end{pmatrix}$ the inhomogeneous term is $X^0 = \begin{pmatrix} a^0 \\ \rho^0 \end{pmatrix}$ where $a^0(p) = \nu(p,\hat{p},0)$ and $\rho^0(p,s) = \sigma(s,p,0,\hat{p},s)$. Thus

$$X = X^0 + \mathcal{K}\,X$$

where $\mathcal{K}$ is the 2 x 2 matrix of integral operators $K_{ij}$ $(i,\ j = 0,\ 1)$.

Let $\mathcal{S}$ be the set of pairs $(a,\rho)$ with $\rho(p,s)$ having support in $S \equiv \left\{ s \geqslant -E/2 + \left[(m+\mu)^2 + p^2\right]^{\frac{1}{2}} \right\}$. Then $\mathcal{K}$ maps $\mathcal{S}$ into itself. In fact, because of the support condition on $\sigma$, for <u>any</u> pair X, the image $\tilde{X}$ under the first two terms of $K_{10}$ and $K_{11}$ has $\tilde{\rho} = 0$ if $s < -E/2 + \left[(m+\mu)^2 + p^2\right]^{\frac{1}{2}}$. Also, provides $X \in \mathcal{S}$ its

image $\tilde{X}$ under the third term of $K_{11}$ has $\tilde{\rho}(p,s) = 0$ if $s < -E/2 + \left[(m + 2\mu)^2 + p^2\right]^{\frac{1}{2}}$.

The inhomogeneous term $X^0$ has $\rho^0 = 0$ if $s < \left[\mu^2 + (p - \hat{p})^2\right]^{\frac{1}{2}}$, so that $X^0 \in \mathcal{S}$. Hence the energy-analytic BSE leaves the set of functions $\mathcal{S}$ invariant, and S is the smallest set for which this happens. $\mathcal{S}$ will be taken as the correctness class in which to seek solutions for the half off-shell amplitude.

Singularities of the Kernel

For the moment we shall ignore the singularities which may be associated with $\sigma$ and $v$, i.e. the particular potential in question; these will be examined in Chapter V when exchange-potentials are considered.

There are poles in the kernel $\mathcal{K}$ when $(E > 0)$, $\omega_-(q) = 0$ (i.e. the threshold cut), $t \pm \omega_-(q) = 0$, and singularities where $v$ becomes singular because of threshold conditions.

(i) $t = \omega_-(q)$, $t = -\omega_+(q)$ are non-singular when $\mathcal{K}$ is operating on elements of $\mathcal{S}$.

(ii) At $t = \omega_+(q)$ the residue of the pole is zero and the pole is only apparent (although it must be considered in the numerical treatment, Chapter V).

(iii) $t = -\omega_-(q)$ will lie in the support of the kernels if $E > 2m + \mu$ i.e. when E lies above the inelastic threshold. When this happens it is necessary to decompose it according to (iv) below, into real and imaginary components.

(iv) The threshold singularity is, as usual written

$$\frac{1}{\omega_-(q)} = \frac{P}{\omega_-(q)} + \frac{i\pi E}{2\hat{p}}\,\delta(q-\hat{p})$$

The contribution of the $\delta$-function in $K_{oo}$ and $K_{o1}$ corresponds exactly to $V(G_o - \tilde{G}_o)T$ in § 1. Thus the EAR gives T when $\mathcal{K}$ has the $i\varepsilon$-prescription, and R when $\mathcal{K}$ is given the principal value.

(v) $V$ will develop end-point singularities in $K_{oo}$ and $K_{o1}$ when $\omega_-(q) < -\left[\mu^2 + (p-q)^2\right]^{\frac{1}{2}}$ for some q. This happens when $E > 2m + 2\mu$ i.e. E is above the second inelastic threshold. For E in this range a logarithmic-type singularity develops and a continuation must be made around it. These kernels are then complex.

Thus, in the elastic region $E < 2m + \mu$ the only singularity is $\omega_-(q) = 0$, and with the principal value the energy-analytic equation is real, and unitarity is exactly satisfied. Furthermore, as was shown in § 1, elastic resonances will cause the equation $X = \bar{\mathcal{K}}\,X$ to have a non-trivial solution, where $\bar{\mathcal{K}}$ denotes $\mathcal{K}$ with a principal-value threshold.

It is thus seen that by taking the energy-analytic form of the equation (what Broido and Taylor call the "generalized Wick rotation") many of the singularities have been removed. Although the moving singularities in the potential will reappear in $\sigma$ (see Chapter V) they do so in a less vicious form which can be handled quite easily in the numerical solution.

The equation we have derived is essentially the same as that obtained in[28]; except that we have admitted a more general potential and considered the case where a subtraction may be required.

## § 3. Approximate Solutions

Since the accurate solution of the Bethe-Salpeter equation involves the solution of 2-dimensional equations which are usually of a rather irregular type, there have been attempts to obtain approximate solutions which may give qualitatively, if not quantitatively, significant approximations to the accurate solutions. Whilst the techniques for solving the BSE are rapidly improving, and such approximations may not be considered vitally important in themselves, their possible use in 3-body scattering, where relativistic calculations are much more complex, could be valuable.

After a brief review of two of these methods another one is presented which combines the advantages of each of these, and is no more complicated than the second.

### (i) The Method of Cohen, Pagnamenta and Taylor[34]

This is a method which, it is claimed, rather than considering the singularities of the BSE as an inconvenience, uses them to good effect.

In the BSE the curves of singularity $D_1^{-1} = 0$, $D_2^{-1} = 0$ of the direct propagators intersect at the on-shell momentum. Let us

assume that the principal contribution to $V_\ell\, G_0\, T_\ell$ comes from the region around this highly singular point; then if we approximate $T_\ell$ by its half on-shell value, $T_\ell(\underset{\sim}{p};\underset{\sim}{r}) \sim T_\ell(\hat{\underset{\sim}{p}};\underset{\sim}{r})$, the resulting equation is

$$T_\ell(\underset{\sim}{p};\underset{\sim}{r}) \sim V_\ell(\underset{\sim}{p};\underset{\sim}{r}) + \Delta_\ell(\underset{\sim}{p})\, T_\ell(\hat{\underset{\sim}{p}};\underset{\sim}{r})$$

where $\Delta_\ell(p) = -i\int V_\ell(\underset{\sim}{p};\underset{\sim}{q}) G_0(\underset{\sim}{q})\, d^2\underset{\sim}{q}$ and we have been able to carry out the $d^2\underset{\sim}{q}$ integral directly because of the approximation. Putting $\underset{\sim}{p} = \underset{\sim}{r} = \hat{\underset{\sim}{p}}$,

$$T_\ell(\hat{\underset{\sim}{p}};\hat{\underset{\sim}{p}}) = \frac{V_\ell(\hat{\underset{\sim}{p}},\hat{\underset{\sim}{p}})}{1 - \Delta_\ell(\hat{\underset{\sim}{p}})}$$

The off-shell approximate solution can also be obtained:

$$T_\ell(\underset{\sim}{p};\underset{\sim}{r}) = V_\ell(\underset{\sim}{p};\underset{\sim}{r}) + \frac{\Delta_\ell(\underset{\sim}{p})\, V_\ell(\hat{\underset{\sim}{p}};\underset{\sim}{r})}{1 - \Delta_\ell(\hat{\underset{\sim}{p}})}$$

This approximation is explicit, and requires only the born-term and $\Delta_\ell$ to be evaluated. Apart from its extreme simplicity (it requires just one integral to be evaluated) it can be said in its favour that it satisfies elastic unitarity, but on the other hand it does not contain inelastic contributions even in the inelastic region.

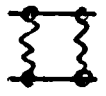
The off-shell form of the approximation is clearly unsymmetrical and therefore violates time reversal invariance. Although it is accurate in perturbation theory to the first order only, the approximation has bound-state and resonance poles in the off-shell amplitude when $\Delta_\ell(\hat{\underset{\sim}{p}}) = 1$, and furthermore the

residue factorizes into incoming and outgoing wave-functions, even though they are different.

It is worth comparing the CPT approximation with the (1, 1) Pade approximant (see Appendix).

$$T_\ell(\underset{\sim}{p};\underset{\sim}{r}) \sim V_\ell(\underset{\sim}{p};\underset{\sim}{r}) + \frac{\left[V_\ell(\underset{\sim}{p};\underset{\sim}{r})\right]^2}{V_\ell(\underset{\sim}{p};\underset{\sim}{r}) - V_\ell^{(2)}(\underset{\sim}{p};\underset{\sim}{r})}$$

where $V_\ell^{(2)}$ is the second born term. This approximation requires about as much labour as the CPT method and satisfies both unitarity and time-reversal. In addition it is accurate to the second order in perturbation theory; but although it is capable of producing bound-states and resonances at poles of the on-shell amplitude, when $V_\ell(\hat{\underset{\sim}{p}}, \hat{\underset{\sim}{p}}) = V_\ell^{(2)}(\hat{\underset{\sim}{p}}; \hat{\underset{\sim}{p}})$, these poles do not continue off the shell as functions of the total energy alone.

When V is an exchange graph  , $\Delta_\ell$ is the partial wave projection of the triangle graph  , with a scalar vertex at the end, and $V_\ell^{(2)}$ is the box-diagram  . On the mass-shell, each of these can be evaluated by a dispersion-relation.

(ii) <u>The Blankenbeclar-Sugar Approximation</u>[35]

The Blankenbeclar-Sugar kernel is an approximation to the kernel of the BSE which is 1-dimensional and possesses the same discontinuity across the elastic cut as the exact kernel; in fact the approximation is only slightly different from the non-relativistic limit of the Bethe-Salpeter equation.

Below the inelastic threshold the discontinuity of $K_\ell \equiv -i\, V_\ell\, G_0$ across $E > m_1 + m_2$ is given by the Cutkofsky rule:

$$[K_\ell] = -i(2\pi i)^2\, V_\ell\, \delta^+(D_1^{-1})\ \delta^+(D_2^{-1})$$

$$= -i(2\pi i)^2\, V_\ell\, (\underset{\sim}{p};\ \hat{\underset{\sim}{p}})\, \frac{\delta^2(\underset{\sim}{q} - \hat{\underset{\sim}{p}})}{E\ \hat{p}}$$

The discontinuity is the same as that for the operator

$$\frac{\pi}{2}\, V_\ell(\underset{\sim}{p};\ \hat{\underset{\sim}{p}}) \int \frac{\delta(q_0 - \hat{p}_0)\,\delta(q - \hat{p})}{E'\ \hat{p}'\,(E'^2 - E^2)}\, dE'^2$$

$$\frac{\pi}{2}\, V_\ell(\underset{\sim}{p};\ p_0, p)\, \frac{\delta(q - \hat{p}_0)}{\omega(q)(q^2 - \hat{p}^2)} \equiv J_\ell(\underset{\sim}{p};\ q)\,\delta(q_0 - \hat{p}_0)$$

In the B-S approximation $K_\ell$ is approximated by $J_\ell(\underset{\sim}{p};\ q) \cdot (q_0 - \hat{p}_0)$ and hence

$$T_\ell(\underset{\sim}{p};\ \underset{\sim}{r}) \sim V_\ell(\underset{\sim}{p};\ \underset{\sim}{r}) + \int J_\ell(\underset{\sim}{p};\ q)\, T_\ell(p_0,\ q;\ \underset{\sim}{r})\, dq$$

Putting $p_0$ on-shell and defining $\hat{T}_\ell(p;\ r) = T_\ell(\hat{p}_0,\ p;\ \hat{p}_0,\ r)$, $\hat{J}_\ell(p;\ q) = J_\ell(\hat{p}_0,\ p;\ q)$ and $\hat{V}_\ell(p;\ r) = V_\ell(\hat{p}_0,\ p;\ \hat{p}_0,\ r)$

$$\hat{T}_\ell \sim \hat{V}_\ell + \hat{J}_\ell\, \hat{T}_\ell$$

Thus in this approximation $\hat{T}_\ell$ satisfies a one-dimensional integral equation. In equal mass scattering

$$\hat{J}_\ell(p;\ q) = \frac{\pi}{2} \cdot \frac{\hat{V}_\ell(p;\ q)}{(m^2 + q^2)^{\frac{1}{2}}\,(q^2 - \hat{p}^2)},$$

which only differs from the non-relativistic Lippmann-Schwinger kernel by the factor $(m^2 + q^2)^{-\frac{1}{2}}$. When V is an exchange graph the non-relativistic limit $\hat{V}_\ell(p,\ q)$ is the Yukawa potential.

Whilst the argument from elastic unitarity will recommend any other propagator differing from $G_0$ by a function which is regular on the elastic cut, the Blankenbeclar-Sugar choice has a satisfactory behaviour for large E in its favour.

(iii) Extension of (i) and (ii)

As we said in (i) above, the greatest contribution to $-i\, K_\ell T_\ell$ is expected to come from the mass-shell region, where the lines of discontinuity intersect. Higher approximations to the CPT approximation will involve the derivatives of $T_\ell$ at this point. Now in a half off-shell calculation of equal-mass scattering, $T_\ell$ is even in $p_0$, so that its $p_0$-derivative vanishes at the mass-shell momentum. We should therefore expect quite a reasonable approximation by keeping $p_0$ fixed at 0 and allowing p to vary, thus reducing the equation to one-dimensional form. The kernel is just $K_{oo}$ obtained in § 2:

$$K_{oo}(p;\, q) = \frac{\pi}{E\omega(q)} \left[ \frac{\nu(p,\, q,\, \omega_+(q))}{\omega_+(q)} - \frac{\nu(p,\, q,\, \omega_-(q))}{\omega_-(q)} \right]$$

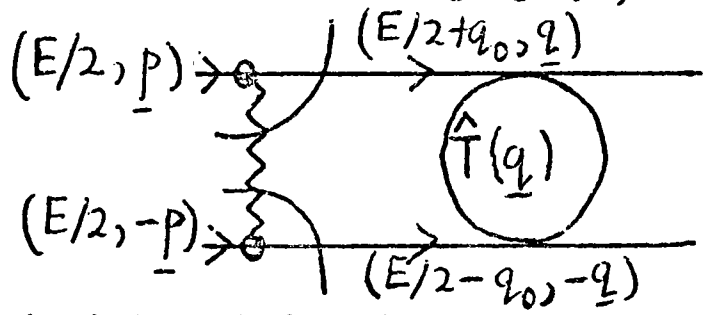
where $\nu$ is the positive frequency component (with respect to t) of $V_\ell$ (0,p; t,q), and the inhomogeneous term is $\hat{V}_\ell$ (p; r). The solution $\hat{T}_\ell$ (p; r), which is an approximation to $T_\ell$ (0,p; 0,r), is not symmetrical in p and r: thus time-reversal is certainly violated.

The discontinuity of $K_{oo}$ over the elastic cut is correct, so that it is a Blankenbeclar-Sugar type of approximation, but the

advantage of this approximation (which is born out by the numerical results in Chapter V) is that $p_0$- analyticity is much better represented, and the off-shell extension of $T_\ell$ contains the leading cuts in the $p_0$-plane.

This approximation contains no inelastic effects below the second inelastic threshold, and for higher energies unitarity is broken in a rather anomalous way. As explained in § 2, the term $\mathcal{V}(p,q,\omega_-(q))$ becomes singular when $E/2 > \left[(m+\mu)^2 + p^2\right]^{\frac{1}{2}}$ for some q, but this is not due to a normal direct-channel threshold, as can be seen by looking at $K_{00}\,\hat{T}(\underline{p})$ (not in partial waves). When the potential is an exchange-graph,

$K_{00}\,\hat{T}(\underline{p})$ =




The variable $(q_0,q)$ is integrated. There are singularities associated with normal cuts on the incoming external lines, and these occur when $(E/2 \pm p)^2 > (m+\mu)^2$. They appear in the exact amplitude, but are absent when the external particles are on-shell. However, because we have insisted in the approximation that $p_0 \equiv 0$, the sheet-structure is upset, and singularities $|E/2| > \left[(m+\mu)^2 + \underline{p}^2\right]^{\frac{1}{2}}$ persist in the iteration of $\hat{T}$. Thus, even on the mass-shell there are singularities $|E/2| > \left[(m+\mu)^2 + \underline{p}^2\right]^{\frac{1}{2}}$ on the real E-axis, although they do not result from a direct-channel threshold.

A subsequent comparison of the numerical results in Chapter V will show that this approximate Blankenbeclar-Sugar type kernel is capable of giving good accuracy for little effort, and is a significant improvement on (i) and (ii) above. Although we have no exact figures for unequal particle scattering, we expect that the comparison would be less good for the reasons given above.

Each method (i) - (iii) involves the approximation of the behaviour of $T_\ell(p_o, p)$ under the action of the kernel. Now by far the most singular part of $T_\ell$ comes from the born term, which has a cross-channel pole for particle-exchange. It might therefore be expected that the results would be considerably improved if this term were removed, and the approximate method applied to equations where the inhomogeneous term is the <u>second</u> born term (calculated exactly). By the same token, the critical values of the coupling strength where resonances occur should be more accurate than the phase-shifts, since the former result from equations which are homogeneous. These conjectures are indeed confirmed numerically.

# IV - COMPOSITE PARTICLES AND BOOTSTRAPS

## § 1. Introduction

The problem of whether the structure of the particle-spectrum can be derived from a theory involving a small number of so-called elementary stable particles which react together to create composite particles (bound-states and unstable resonances) remains, of course, unsolved. In recent years, however, the question has arisen of whether this approach is the best one, or whether it may be more expedient (or more correct) to involve all of the particles, elementary and composite, directly in the theory at the outset. The physical constants of the composite particles are then established by requiring that, as input parameters in the theory, they are consistent with the same physical constants when obtained as output phenomena from the dynamical equations.

The mechanism by which this might be concretely formulated will be explained later, but we mention here the fundamental difference in the strategy. For whereas in the first approach (which we shall call the synthetic method) the equations involve the elementary particles alone, in the second approach the dynamical equations involve all of the particles, so that composite particles may be created as composites of both elementary particles and themselves. This is the so-called bootstrap principle of particle-physics which was originally

introduced via S-matrix theory. The two approaches are not necessarily inconsistent, for it may be possible to reduces a bootstrap theory to a synthetic theory (we shall give a formal example of this) and in either case the aim is to determine the structure of the particle-spectrum from the physical constants of a few elementary particles.

In the present state of knowledge the complete bootstrap hypothesis is wild speculation. However, the magnitude of the forces involved suggest that when discussing interactions of the hadrons the other particles may be neglected, and the more stable hadrons (such as the nucleons and pions) considered as stable and possibly 'elementary'. Even within this limited sphere it appears that some processes are dominated by a small number of exchange forces, and that certain subsystems can be spproximately decoupled in partially bootstrapped schemes.

Now the original investigation of $\pi\pi$ -scattering was done by Chew, Mandelstam and Noyes[36] at a time before the existence of the vector $\rho$-meson had been firmly established. Their method was to assume a double dispersion relation (which implies crossing-symmetry) and use the N/D equations and unitarity to define a non-linear integral equation. It was found that an S-wave dominated solution with little scattering in the higher spin channels was consistent with the equation. This solution therefore fails to include the well-known effects in $\pi\pi$ -scattering

such as the $\rho$- and $f^0$-resonances. There is a multitude of possible reasons for why this happens; these include the questions of whether a double dispersion relation holds, to what extent inelastic contributions are important, and the non-uniqueness of the solution. Interesting work on the Mandelstam[37] equations has been carried out more recently by Atkinson using fixed-point methods.

After the $\rho$-resonance had been discovered and shown to be a highly dominant feature of P-wave $\pi\pi$-scattering as well as of many other processes (this had been predicted by Sakurai), it seemed more promising tactics[38] to attempt a reconstruction of the $\pi\pi$-binding force from the resonance data, by assuming the presence of second-sheet $\rho$-poles in the cross-channels. As it is most natural to consider $\rho$ as a $\pi\pi$-resonant state, we have the peculiar situation in which the force which creates the resonance is "produced by" the resonance itself. This approximately bootstrapped process is solved, in principle, by setting-up a dynamical equation for P-wave $\pi\pi$-scattering with a $\rho$-dominated potential, and adjusting the mass and width of $\rho$ so that a resonance is produced in the direct-channel with the same mass, width and quantum-numbers.

Such a calculation was carried out by Zachariason and Zemach[39] using N/D equations in which the force term N(s) was approximated by a narrow P-wave resonance. The method gives a self-consistent

resonance at about the correct physical coupling, but too small a $\rho$-mass. When a large contribution is assumed to come from the $\pi\omega$-channel self-consistency can be achieved at about the physical $\rho$-parameters, and this was followed by a more ambitious bootstrap model[40].

We can now make a brief comparison between the methodology of these two approaches. It is quite possible, for all we know, that the $\pi\pi$ cross-section can be explained by assuming an elementary pion interaction (perhaps a $\pi^4$ type of interaction would work) but a complete solution of this kind of problem is not possible at the present time. Even approximate solutions are not likely to be greatly successful, mainly because of the absence of any features in the input forces, like exchange poles, which are likely, a priori, to be dominant. Now with the hindsight of the physical facts that we have about $\pi\pi$-scattering, it would appear to be a more reliable procedure to assume that such resonances as we know about appear in the amplitude, and use them as exchange-forces in the input potential. An undetermined pion interaction may be considered to have "sparked-off" the reaction, but its precise form ceases to be important; and whichever way you look at it (whether from the synthetic or bootstrap points of view) the end-result is the same. As a practical matter, however, it is important that the multitude of possible elementary reactions is restricted only by selection-rules, whereas a force-field which is known to produce bound-states and resonances with certain

masses and quantum-numbers is distinctly less arbitrary.

A later $\rho$-bootstrap calculation used the equivalent potential method[41]. Here the input potential for a non-relativistic Schrodinger calculation is obtained from a relativistic crossing relation. Self-consistency could be achieved at about the correct $\rho$-mass but with too large a width.

In each of the $\rho$-bootstraps described above, the input width of the resonance-pole is approximated to zero. The more accurate representation by a true Breit-Wigner form has been used in both N/D[42] and equivalent potential[43] calculations; in the first case self-consistency is destroyed, but the finite width correction has little effect on the equivalent potential calculation. There have also been successful attempts[44, 45] to incorporate $f^0$ in a double bootstrap.

This variety of techniques and predictions reveals our incomplete knowledge about both the source of the $\pi\pi$-interaction, and how it may be accurately approximated in dynamical calculations. The best model of $\pi\pi$-scattering is the Veneziano model[46], which is not dynamical (at any rate within the usual meaning of the term). We can, however, be encouraged that all of the $\rho$-bootstrap calculations bear some resemblance to reality (even when self-consistency is impossible), which justifies the assumption that the $\pi\pi$-force is "dynamically generated", principally by $\rho$-exchange.

According to Coulter[47], high energy effects make significant contributions to the N/D bootstrap calculations. Thus a Bethe-Salpeter calculation in which high energy (inelastic) effects are automatically included has an advantage in this respect. Such a calculation has been performed and is described in the next chapter. Self-consistency was found to be impossible at any $\rho$-mass, and although a zero-width input pole was used (the method seems intractable otherwise) it appears unlikely that a finite-width correction would alter this. We are thus forced to the conclusion that either the Bethe-Salpeter potential is less accurate than the equivalent potential (which we doubt - for the BSE is a type of equivalent potential model which is <u>fully</u> convariant) or else the effects of the $\omega$, $f^0$ mesons are not negligible. The latter conclusion is born out by the improvement in other calculations when these effects are included.

## § 2. <u>Field Theory and Composite Particles</u>

The concept of a composite particle will now be considered from the stand-point of field theory, and we shall examine in some detail an exact $\pi$-$\rho$ bootstrap. The approach of Broido and Taylor (ref. [48] and authors quoted therein) will be adopted.

It will be assumed, without proper justification, that both stable and unstable particles may be represented by fields. Retaining the notion that particle masses belong to the (point-)spectrum of the hamiltonian, complex masses may be

accomodated and a spectral theory preserved provided that the hamiltonian is taken to be a <u>normal</u> operator. This requires that mass ($\equiv (H + H^{*})/2$) and lifetime ($\equiv (H - H^{*})/2i$) are simultaneously measureable. To ensure that state norms diminish (rather than increase) with time the spectrum of H must be contained within the lower-half plane.

A particle with field $\psi$ is defined to be a composite of particles with fields $\phi_1, \phi_2, \ldots, \phi_n$ if there exists a relation of the form $\psi = f(\phi_1, \ldots, \phi_n)$, where f is a local function of $\phi_1, \ldots, \phi_n$ which does not involve their derivatives to a higher order than the first. The possibility is not excluded that $\phi_1, \ldots, \phi_n$ include the composite field itself, but in such a case $\partial\psi$ may not appear; this is a slight relaxation of the usual condition that an explicit expression for $\psi$ is required. The derivative condition ensures that some awkward situations[48] cannot occur.

Now the way that such equations are made to arise from the field equations is as follows. All fields are assumed, at the beginning, to represent elementary particles, and the renormalized field equations (or their Green's function equivalents) are generated in the usual way. One of the fields $\psi$, say, is then said to become composite if it is possible to take a limit of the field equations with respect to the renormalized constants (masses and coupling strengths) in such a way that the wave-function

renormalization constant $Z_\psi$ tends to zero consistently with the pole-structure of the propagators being preserved.

For example, consider the simplest case of two scalar fields $\phi, \psi$ reacting via a renormalized interaction $g\,(\psi - f(\phi,\psi))^2$. If the $\psi$-field is wave-function renormalized the interaction lagrangian is

$$L_I = \tfrac{1}{2}(Z_\psi - 1)\left((\partial\psi)^2 + m_\psi^2\,\psi^2\right) + g\left(\psi - f(\phi,\psi)\right)^2$$

The $\psi$-field equation

$$Z_\psi(\Box + m_\psi^2)\psi = g\left(\psi - f(\phi,\psi)\right)\left(1 - \frac{\partial f(\phi,\psi)}{\partial\psi}\right)$$

becomes, in a completely formal manner, in the limit $Z_\psi \to 0$,

$$\left(\psi - f(\phi,\psi)\right)\left(1 - \frac{\partial\phi}{\partial\psi}(\phi,\psi)\right) = 0$$

and provided the second factor can be cancelled, the composite field criterion $\psi = f(\phi,\psi)$ is obtained. The self-energy term $g\,\psi^2$ in $L_I$ may be interpreted as the mass-renormalization term $\frac{1}{2}\delta m_\psi^2 Z_\psi \psi^2$ which we did not include, provided $g = \frac{1}{2}\delta m_\psi^2 Z_\psi$. Thus a composite field condition can be produced in a theory of <u>a priori</u> elementary particles. This can be extended to more complete cases of bootstrapping, e.g. when the field $\phi$ is absent but $\psi$ is bootstrapped to itself via an equation $\psi = f(\psi)$, or when $\phi$ and $\psi$ are reciprocally bootstrapped to each other via equations

$$\psi = f(\phi,\psi)\ ;\ \phi = g(\phi,\psi)$$

We shall now apply this procedure to a simple theory of pseudoscalar particles (pions) and vector particles (rho-mesons)

each with isospin 1, which are decoupled from other particles.

Bearing in mind the quantum-numbers of $\pi$ and $\rho$, the simplest $\pi$-$\rho$ interaction is $g(\partial_\mu \underline{\pi} \wedge \underline{\pi}) \cdot \underline{\rho}^\mu$ where the 3-vector notation refers to the isospin index. The field equations arising from this interaction are not consistent with the gauge-invariance condition $\partial_\mu \underline{\rho}^\mu = 0$ but this can be satisfied if the interaction is extended to the more complicated Yang-Mills interaction

$$\begin{aligned} L_{YM} &= g(\partial_\mu \underline{\pi} \wedge \underline{\pi}) \cdot \underline{\rho}^\mu - \tfrac{1}{2} g^2 (\underline{\pi} \wedge \underline{\rho}^\mu) \cdot (\underline{\pi} \wedge \underline{\rho}_\mu) \\ &\quad - g\, \partial_\mu \underline{\rho}_\nu \cdot (\underline{\rho}^\mu \wedge \underline{\rho}^\nu) - \tfrac{1}{4} g^2 (\underline{\rho}^\mu \wedge \underline{\rho}^\nu) \cdot (\underline{\rho}_\mu \wedge \underline{\rho}_\nu) \\ &\equiv g L_1 + g^2 L_2 + g L_3 + g^2 L_4 \end{aligned}$$

The gauge-invariance condition arises as a consequence of the conserved I-spin current. It is not possible (even with mass-renormalization terms) to express this in the form $(\rho - f(\rho, \pi))^2$ but it is still possible, formally at least, to write down the renormalized field equations and carry out the limiting procedure.

The free lagrangian is

$$L_0 = -\tfrac{1}{2}\left((\partial \underline{\pi})^2 + m_\pi^2 \underline{\pi}^2\right) - \tfrac{1}{2}\left((\partial \underline{\rho})^2 + m_\rho^2 \underline{\rho}^2\right)$$

whilst the renormalized interaction is

$$\begin{aligned} L_I &= -\tfrac{1}{2}(Z_\rho - 1)\left((\partial \underline{\rho})^2 + m_\rho^2 \underline{\rho}^2\right) + \tfrac{1}{2} \delta m_\rho^2 Z_\rho \underline{\rho}^2 \\ &\quad + g Z_\rho^{\frac{1}{2}} L_1 + g^2 Z_\rho L_2 + g Z_\rho^{3/2} L_3 + g^2 Z_\rho^2 L_4 . \end{aligned}$$

Defining a wave-function renormalized coupling-constant by $g_r = Z_\rho^{\frac{1}{2}} g$, the second line of this expression can be written

$$g_r L_1 + g_r^2 L_2 + g_r Z_\rho L_3 + g_r^2 Z_\rho L_4$$

which becomes $g_r L_1 + g_r^2 L_2$ in the limit $Z_\rho \to 0$, whilst the field equation $\delta L/\delta \rho = 0$ has the limit

$$Z_\rho \delta m_\rho^2 \underline{\rho}^\mu + g_r \partial^\mu \underline{\pi} \wedge \underline{\pi} - g_r^2 (\underline{\pi} \wedge \underline{\rho}^\mu) \wedge \underline{\pi} = 0$$

From this it follows that:

(2.1) $$\underline{\rho}^\mu . \underline{\pi} = 0$$

giving $$Z_\rho \delta m_\rho^2 \underline{\rho}^\mu + g_r \partial^\mu \underline{\pi} \wedge \underline{\pi} - g_r^2 \underline{\pi}^2 \underline{\rho}^\mu = 0$$

and so

(2.2) $$\underline{\rho}^\mu = g_r \frac{\partial^\mu \underline{\pi} \wedge \underline{\pi}}{g_r^2 \underline{\pi}^2 - \alpha}$$

where

(2.3) $$\alpha = \lim_{Z_\rho \to 0} Z_\rho \delta m_\rho^2 .$$

In order for this to make sense at all, it is essential that some meaning can be attached to the limit in (2.3). Provided that this can be taken care of, the functional dependence of $\underline{\rho}$ upon $\underline{\pi}$ is of the required type and exists as an explicit relationship. If the expression (2.2) for $\rho$ is substituted into the lagrangian, we obtain in the limit $Z_\rho \to 0$,

$$\begin{aligned} L &= L_0(\underline{\pi}) + \tfrac{1}{2} Z_\rho \delta m_\rho^2 \underline{\rho}^2 + g_r L_1 + g_r^2 L_2 \\ &= L_0(\underline{\pi}) + \tfrac{1}{2}\alpha \underline{\rho}^2 + (g_r^2 \underline{\pi}^2 - \alpha)\underline{\rho}^2 - \tfrac{1}{2} g_r^2 \underline{\pi}^2 \underline{\rho}^2 \\ &= L_0(\underline{\pi}) + \frac{g_r^2}{2} . \frac{(\partial^\mu \underline{\pi} \wedge \underline{\pi}) . (\partial \mu \underline{\pi} \wedge \underline{\pi})}{g_r^2 \underline{\pi}^2 - \alpha} \\ &\equiv L_0(\underline{\pi}) + \tilde{L}(\underline{\pi}) \end{aligned}$$

where $L_0(\underline{\pi})$ is the free lagrangian of the $\underline{\pi}$ -field.

Since $\tilde{L} = \frac{1}{2} g_r(\partial_\mu \underline{\pi} \wedge \underline{\pi}).\underline{\rho}^\mu$ , $f = \frac{1}{2} g_r$ where f is the renormalized $\pi\pi\rho$ coupling constant ($f^2/4\pi \sim 2.5$).

The current-field identity of the $\rho$ -meson derived from current-algebra is[50]

$$\underline{J}^\mu = \frac{m_\rho^2}{f} \underline{\rho}^\mu$$

where $\underline{J}^\mu$ is the conserved isospin current. Now in our model,

$$\underline{J}^\mu = \frac{\alpha\, \partial^\mu \underline{\pi} \wedge \underline{\pi}}{g_r^2 \underline{\pi}^2 - \alpha} = \frac{\alpha}{g_r} \underline{\rho}^\mu$$

so that the expression for $\underline{\rho}^\mu$ is consistent with the CFI if the expression

$$\alpha = 2 m_\rho \qquad (2.4)$$

is taken for the limit (2.3).

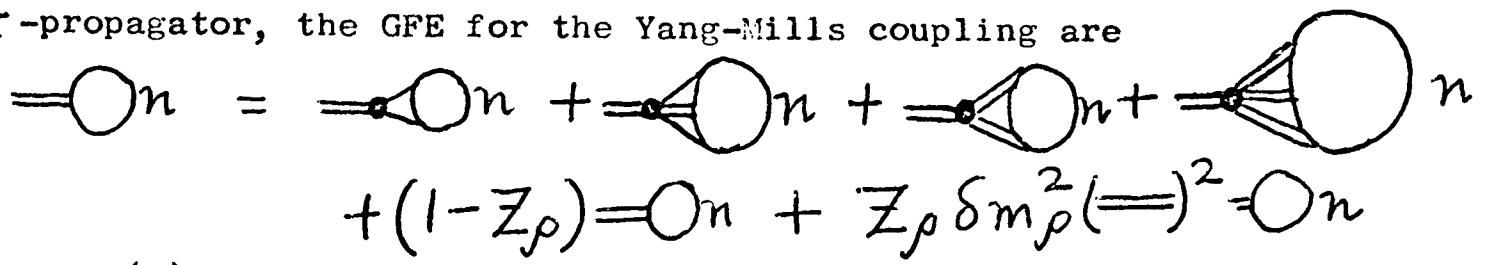
Thus, at any rate in a formal sense, the theory is reduced to the investigation of a derivative pion interaction $\tilde{L}(\underline{\pi})$. There has been some recent work[51] on the renormalization of this type of non-linear lagrangian. The procedure is, roughly speaking, to expand $\tilde{L}$ in a formal Taylor series with respect to a scale parameter on $\underline{\pi}$ and so obtain an interaction in the form of an infinite series, which it is the object of the theory to regularize, e.g. by Borel summation. In our example all of these terms involve derivatives of $\underline{\pi}$ and the lowest order term is $-\bar{g}(\partial \underline{\pi} \wedge \underline{\pi})^2$ where $\bar{g} = g_r^2/2\alpha$. According to equation (2.4) $\bar{g} \sim 1\ m_\pi^{-2}$.

This interaction should be compared with the hypothetical non-derivative interaction $(\underline{\pi}\cdot\underline{\pi})^2$ which is often considered in models of self-interacting pions. It is clear that the two are vastly different.

## § 3. The Green's Function Equations in the $\pi$-$\rho$ Bootstrap

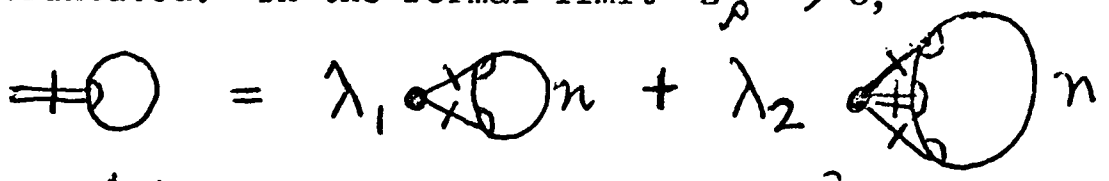
It will now be shown how the Green's function equations for § 2 behave in the bootstrap limit.

If $=\!=$ denotes the bare $\rho$-propagator and $—$ the bare $\pi$-propagator, the GFE for the Yang-Mills coupling are




where (n) denotes an n-particle state ($n \geq 2$) and the amplitudes are not truncated. In the formal limit $Z_\rho \to 0$,

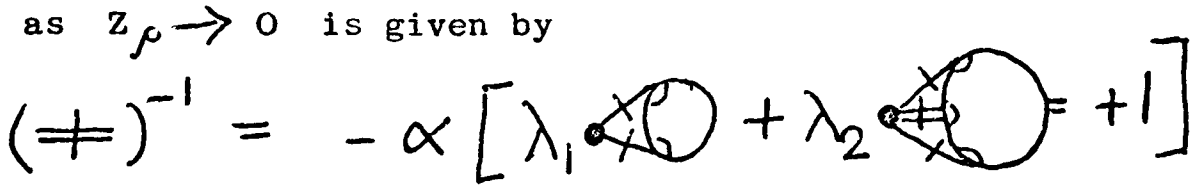
(3.1)




where $\lambda_i = -g_r^i/\alpha$ $(i = 1,2)$ ; $\alpha = \lim_{Z_\rho \to 0} Z_\rho \delta m_\rho^2$

In an exactly similar manner, the limit of the $\rho$-propagator equation as $Z_\rho \to 0$ is given by

(3.2)




Single lines and bubbles will again now represent complete propagators and truncated amplitudes, but in equation (3.1) the amplitudes contain their disconnected components.

Now the expressions for the bare and clothed propagators of a vector particle are

$$D_{\mu\nu}(p,\ m^2) \;=\; \frac{g_{\mu\nu} - p_\mu p_\nu / m^2}{p^2 - m^2}$$

$$D'_{\mu\nu}(p) \;=\; \int dm^2\, \rho(m^2)\, D_{\mu\nu}(p,\ m^2)$$

where $\rho(m^2)$ is the Lehmann spectral function.

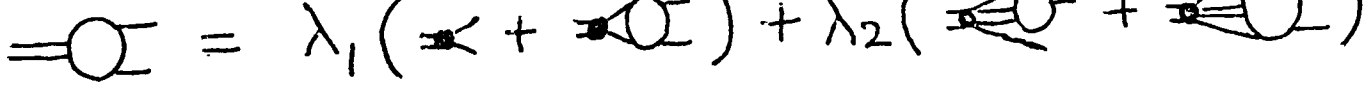
When $p = (s^{\frac{1}{2}},\ \underline{0})$,

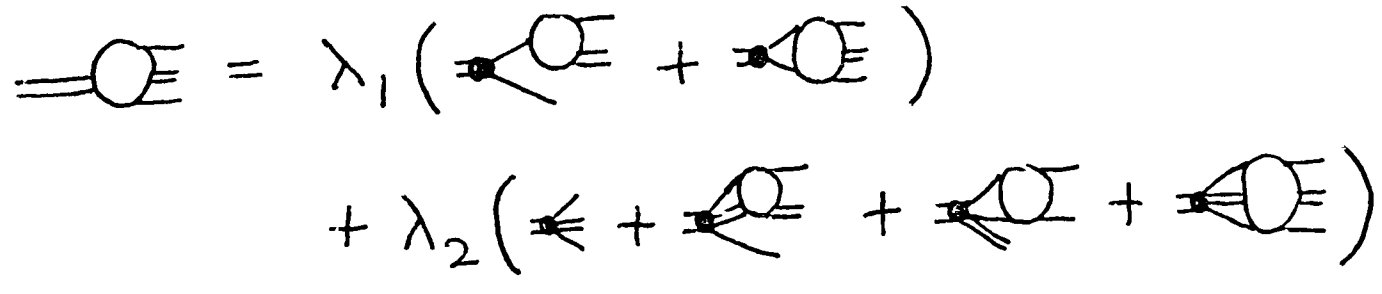
$$D_{\mu\nu}(p,\ m^2) \;=\; \begin{bmatrix} -1 & & & \\ & -\frac{1}{s-m^2} & & \\ & & -\frac{1}{s-m^2} & \\ & & & -\frac{1}{s-m^2} \end{bmatrix}$$

It follows that near the energy-shell $s = m_\rho^2$, $[D']^{-1}_{\mu\nu} \sim -(s - m_\rho^2)$ for $\mu, \nu = 1,2,3$ and this provides conditions on the value and first two derivatives of the RHS of (3.2), which must be satisfied at $s = m_\rho^2$.

Letting the bubbles now represent connected amplitudes,

(3.1) gives for the 3- and 4-point functions

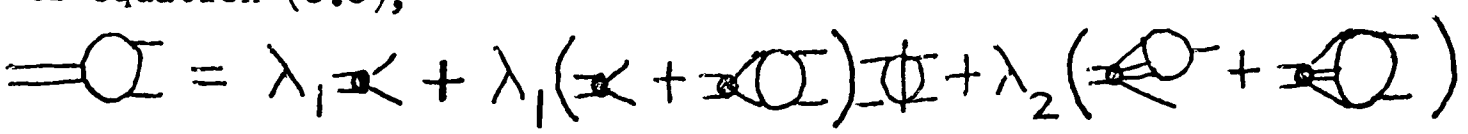
(3.3) 

(3.4) 

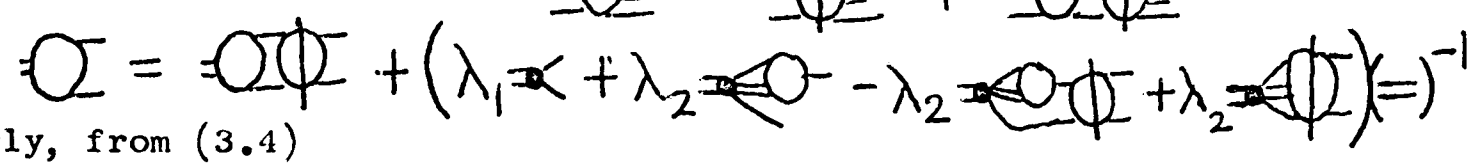
Assuming the dependence of $\rho$ upon $\pi$, the intermediate $2\pi$ states may be exposed in the BSE

(3.5) 

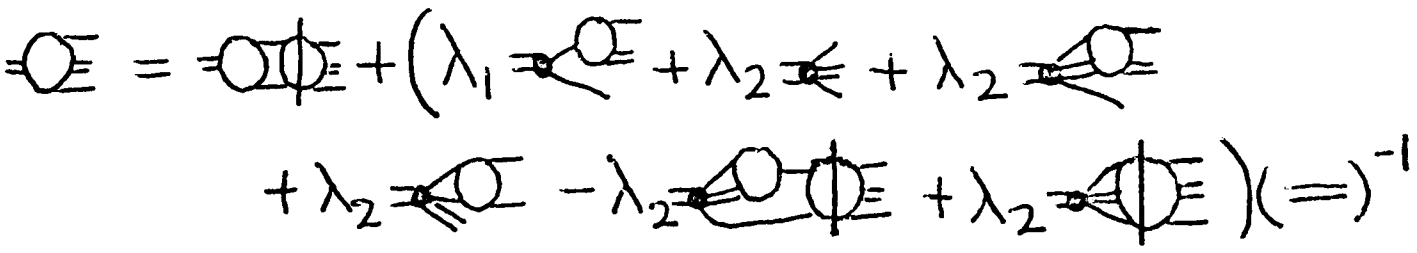
where the pole contribution of $\rho$ is contained in the last term (see also § 4). Substituting this into the second term of the RHS of equation (3.3),

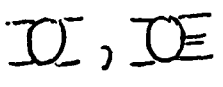


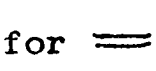
Using (3.3) again, as well as 

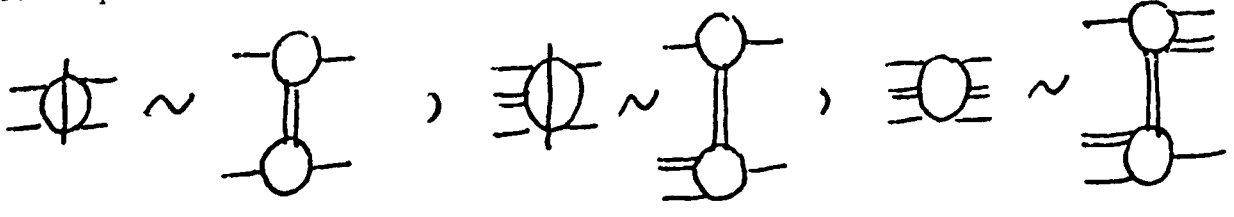
(3.6) 

Similarly, from (3.4)

(3.7) 

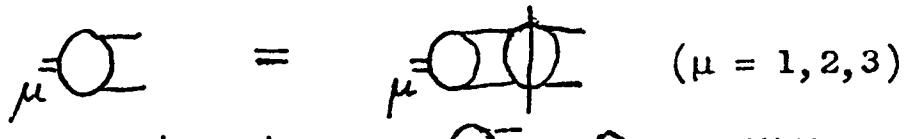
It is clear that the 3-, 4-, ... point Green s function equations (3.3), (3.4), ... in the bootstrap limit explicitly display the $\rho$-particle pole in the amplitudes  etc.

The bootstrap problem is equivalent to the solution of equations (3.2), (3.3), (3.6), (3.7), which are coupled integral equations for  ,  and the "vertex functions"  and  , in terms of  and the potentials  ,  and  . Bearing in mind their cut-structure and the assumed dominance of pole contributions, appropriate approximations to these potentials are

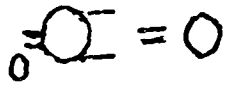


With a suitable ansatz for ——— , the problem is thus reduced to four coupled integral equations which are highly non-linear. Their solution is beyond present-day techniques, although some progress was made by Taylor[8] on similar equations in euclidean theories.

When the total energy is put on the mass-shell $s = m_\rho^2$, equation (3.6) becomes



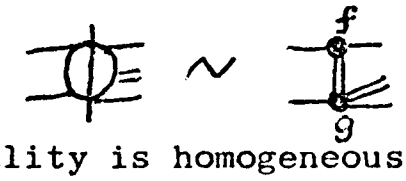
$(\mu = 1,2,3)$

whilst from gauge-invariance,  . With the "ladder" substitution



in which the vertex interaction is given by $f(\partial_\mu \underline{\pi} \wedge \underline{\pi}) \cdot \underline{\rho}^\mu$ and f is the physical coupling constant, the usual homogeneous linear equation is obtained.

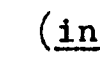
If the same procedure is used on (3.7), putting



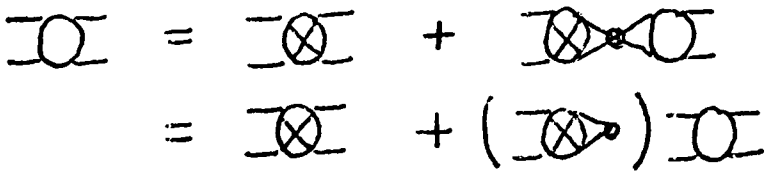

the resulting equality is homogeneous in the 4-point physical coupling strength g, so that it does not seem possible to obtain g in this way.

## § 4. Approximate Models of Self-Interacting Pions.

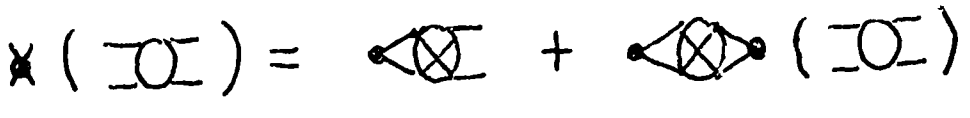
Consider a pion interaction which has a lowest order term of the 4th order, such as $(\underline{\pi} \cdot \underline{\pi})^2$, $(\partial \underline{\pi} \wedge \underline{\pi})^2$, $(\partial_\mu \underline{\pi} \wedge \partial_\nu \underline{\pi}) \cdot (\partial^\mu \underline{\pi} \wedge \partial^\nu \underline{\pi})$.

Let  denote the part of the off-shell scattering amplitude  (<u>including</u>  ) which, when expanded in perturbation theory does not include "nodal" graphs of the form  .

Then, by exposing the "first node" we have



where  represents the 4-point bare vertex, here used as an integral operator. Hence



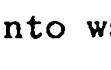
and so, from the last two equations,



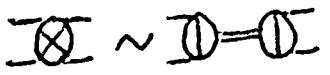
Composite states will be produced at poles of  and these occur when the symmetric integral operator

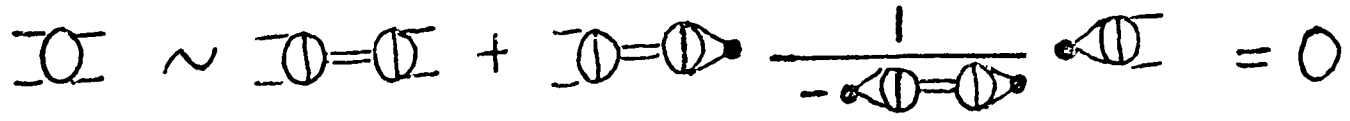
$$K_E(p,q) \equiv$$ 

satisfies a homogeneous equation $K_E f = 0$. At such a pole the residue will factorize into wave-functions if $1/K_E$ has the factorizable form.

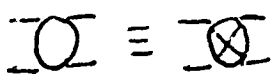
Thus, for a scalar interaction composite states can occur at zeros of $f(s) \equiv ig(2\pi)^4 -$  $= 0$, and only S-wave states can be produced in this way.

It might be imagined that in addition to this, composite states are also produced at the poles of  . If 

factorizes near a pole into  , then near the pole,



and so the residue of  is zero.

Thus if the bare 4-vertex  allows transmission of states in a channel with certain quantum-numbers, composite states are produced in that channel if and only if $K_E$ satisfies the above criterion. Otherwise the nodal graphs vanish and  in that channel.

We shall now examine in more detail the $-\bar{g}\left(\partial \underline{\pi} \wedge \underline{\pi}\right)^2$ vector interaction obtained in § 2 as the leading term in $\tilde{L}(\underline{\pi})$. The bare vertex $V(1,2,3,4) =$  has the value

$$\bar{g} \sum_{(2,3,4)} (p_1 - p_2)(p_3 - p_4)\left[\delta_{13}\delta_{24} - \delta_{14}\delta_{23}\right]$$

where $\sum$ extends over the cyclic permutations of (2,3,4). Thus

$$V(1,2,3,4) = \bar{g}\left[(C-B)\delta_{12}\delta_{34} + (A-C)\delta_{13}\delta_{24} + (B-A)\delta_{14}\delta_{23}\right]$$

where $A = (p_1-p_2)\ (p_3-p_4)$ , $B = (p_1-p_3)\ (p_4-p_2)$, $C = (p_1-p_4)\ (p_2-p_3)$ and $A + B + C = 0$. The Kronecker deltas are to be interpreted as follows. When each particle is in some I-spin state corresponding to a basis-element of an orthonormal I-spin basis then $\delta_{ij}$ is 1 if particles i and j are in the same I-spin state and zero otherwise. The formula for V can then be extended by linearity to arbitrary I-spin states.

In the $(12) \longrightarrow (34)$ channel V has the I-spin resolution

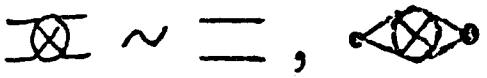
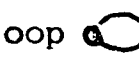
| I | V |
|---|---|
| 0 | $-2(B-C)g$ |
| 1 | $3Ag$ |
| 2 | $(B-C)g$ |

Putting $$p_1 = E/2 + p \ , \ p_2 = E/2 - p$$

$$-p_3 = E/2 + q \ , -p_4 = E/2 - q$$

$$B - C = 2(p^2 + q^2 - E^2) \ , \ A = -4p\ q$$

Thus the nodal graphs have only S-wave contributions when I = 0 or 2, and only P-wave contributions when I = 1 (there is actually an S-wave contribution when I = 1 which arises from the term $p_0 q_0$ in A, but this is ignored because it vanishes on-shell, when $p_0 = 0$).

If —— is approximated by a bare propagator and  is given its zeroth order approximation,  is the "self-energy" loop  which can be evaluated by a dispersion-integral, and its I = 1 component has the value

$$\left[-\lambda \int_4^\infty \frac{\rho(s')\,ds'}{s - s' + i\varepsilon}\right] \times$$

where $\lambda = \bar{g}/16\pi^2$ and $\rho(s) = (s-4)^{3/2}/s^{\frac{1}{2}}$, in units where $m_\pi = 1$.

Now on-shell, in the (1,1) channel

$$e^{i\delta} \sin\delta = \frac{\pi\lambda\rho(s)}{1 + \lambda \int_4^\infty \frac{\rho(s')\,ds'}{s - s' + i\varepsilon}}$$

The integral is linearly divergent and therefore requires two subtractions. Writing

$$\int_4^\infty \frac{\rho(s')\,ds'}{s - s'} = A + Bs + s^2 \int_4^\infty \frac{\rho(s')\,ds'}{s'^2 (s - s')}$$

where A and B are subtraction-constants, the integral can be done numerically and a resonance found at the position of the complex $\rho$ -mass on the unphysical sheet of s.

With (*) $m_\rho$ = 765 MeV the following figures are obtained for various values of the $\rho$ -width $\Gamma_\rho$ :

| $\Gamma_\rho$ (MeV) | $(\lambda^{-1} + A)/m_\pi^2$ | B |
|---|---|---|
| 100 | 527 | -17.5 |
| (*)125 | 508 | -16.9 |
| 150 | 490 | -16.3 |

(*) indicates approximate physical value.

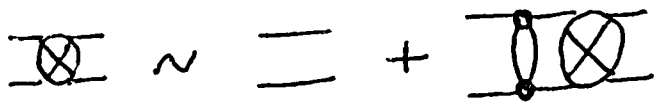
The numbers $\lambda$ , A and B are not independent, and if g is the physical $4\pi$ -coupling constant calculated at the threshold energy $s = 4m^2$,

$$\int \frac{\rho(s')\,ds'}{s - s'} = 0 , \quad \text{i.e.} \quad A + 4B - 2.7 = 0$$

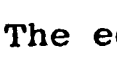
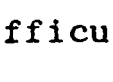
A value 0.4 $m_\pi^{-2}$ is then obtained for $\bar{g} \equiv 16\pi^2\lambda$, and this is not very sensitive to the choice of the subtraction-point. It compares to an order of magnitude with the value 1 $m_\pi^{-2}$ obtained in § 2.

This simple little model of the (1, 1) channel for $\pi\pi$ scattering thus satisfies elastic unitarity, and is capable of producing a resonance at the $\rho$ -mass with a factorizable residue proportional to the $4\pi$ -vertex. The coupling constant bears some relation to a value obtained from current-algebra. The solution is, in fact, identical with the one iterated N/D equations in which the force term N(s) is approximated by a separable potential (the vector vertex).

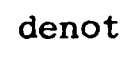
In spite of the disparaging remarks made in § 1 about the possible accuracy of pion-interaction calculations, the only assumption made here concerns the validity of the approximation  . To obtain higher degrees of approximation a closer approximation to  is required, for example through a Bethe-Salpeter estimate



It is important to remember that, as shown above, it is not necessarily true that composite states are created when the kernel has a eigenfunction, and this is certainly not the case in the (1,1) channel with the $-\bar{g}\left(\partial\underline{\pi}\wedge\underline{\pi}\right)^2$ interaction.

The equations in this type of approach are difficult to renormalize, chiefly because of the importance we attached to the "nodeless" graphs. The trouble is that the clothed vertex  is what we are looking for, namely  .

We now turn to the other line of attack, which is the exposure of two-particle cuts in the scattering amplitude, and in particular in cases where a composite $2\pi$ -state is known to exist.

For the moment, unless otherwise stated,  denotes the <u>set</u> of graphs in the amplitude (<u>excluding</u>  ) and  is the <u>set</u> of graphs which have no $2\pi$-cuts in the s-channel. The operator $\theta_\alpha$ ($\alpha$ = s,t,u) crosses the s- and $\alpha$-channels of a bubble to which it is applied. Thus $\theta_t$  =  etc., and $\theta_s$ = I.

Since the pions are pseudoscalar,

(4.1)  $\subset$  ($\alpha$ = t, u)

By virtue of the BSE  and  are complements of each other with respect to the global set  , and it follows from (4.1) that

 ($\alpha$ = s, t, u)

are mutually disjoint. The amplitude may therefore be expressed in the set-theoretic form



where  and the elements of the RHS are mutually disjoint.

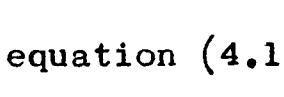
The 2-cut approximation[51] consists in making the approximation

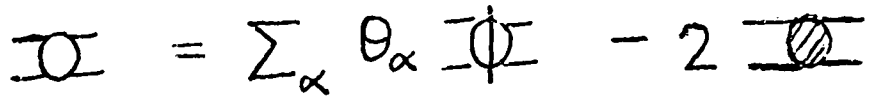


(here bubbles have their <u>numerical value</u>) where  is a renormalized

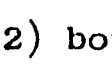
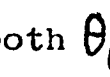
vertex, and the equation is assumed to have been renormalized by differentiation and integration on the external momenta. Thus, in the 2CA the only parts of the amplitude that are included are those which have a $2\pi$ -cut in one channel. The BSE and the 2CA together form a pair of integral equations which should, in principle, be capable of a solution for both the amplitude and the potential.

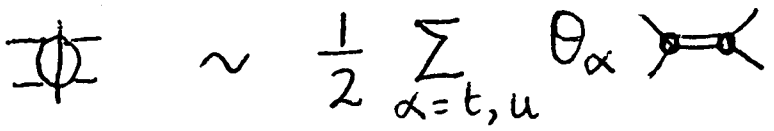
The 2CA has been discussed by Broido[53] where the amplitudes were imbedded in a commutative $*$-algebra. To our knowledge no one has yet pointed out that this discussion is incorrect because the crossing transformations are not isomorphisms. For example it is very easy to prove that $\theta_t (T. T_2) \neq \theta_t T. \theta_t T_2$. A different analysis of the 2CA is due to Taylor[8]. Here solutions in n-dimensional euclidean theories are show to exist by using the Banach fixed point theorem; for n > 2 a cutoff is required.

Now from equation (4.1)  , which shows that the 2-particle potentials "cover" the amplitude. The numerical equivalent of this is

(4.2) 

If  has a composite state  of mass $m_b$,  and  do not have the composite particle pole at $s = m_b^2$ because they have no $2\pi$ -intermediate states. Thus, from

equation (4.2) both $\theta_t$  and $\theta_u$  must have the direct-channel pole at $s = m_b^2$, and we are justified in making the ansatz



which is the 1-particle exchange potential for a composite particle.

# V - CALCULATION OF BETHE-SALPETER PHASE SHIFTS

## § 1. Introduction

The most accurate calculations of phase shifts for the scalar Bethe-Salpeter equation are generally believed to be those of Schwarz and Zemach[26], who applied the Schwinger variational principal to the configuration-space form of Tiktopoulos's equation[25]. It is necessary at one stage to take the Fourier transform and so return to momentum space, where unpleasant singularities occur. These were accommodated by using analytic trial functions which permit the continuation of the relative energy variable around the cuts that appear when the total energy exceeds threshold. Judging by the rapid convergence of the figures with the increase in the number of trial functions, it would appear that this method is extremely accurate, and it has been extended to the case of spin-scattering by McInnis[54]. The fact that it uses configuration space is a considerable disadvantage when derivative interactions are used, and another method was devised by Saenger[55] which retains the simple momentum equation (in the Wick rotated form). It is possible to cope with the singularities by a transformation of the $p_0$-variable which permits the necessary continuation, but the figures obtained were not very accurate.

Another attempt to solve the scalar equation was made by Levine et al.[24] who used essentially the same equation that Kemmer and Salam derived, but they removed the threshold cut.

Their results are accurate to within about 5 per cent and the method can also be extended to inelastic energies. Haymaker[56] obtained the only figures which hitherto bear accurate comparison with Schwartz and Zemach's, and these were obtained by continuing the bound-state solutions of the Bethe-Salpeter equation into the elastic scattering region using a numerical analytic continuation of the total energy (see Appendix).

In this chapter it is shown how the energy-analytic form of the BSE can be used to calculate phase-shifts. This form of the equation was used by Ito et al.[30] to evaluate a few ladder diagrams for NN-scattering, from which approximate phase shifts were obtained by using variational principles. The authors imagined that the singularities of the equation are so awkward that a mesh-point method would be impractical. We show how these singularities can be dealt with and numerical solutions found by inversion of the kernel which agree to within a percent with those of Schwarz and Zemach.

The off-shell amplitude may be calculated at any point in the $p_0$-plane using the energy-analytic representation and there is no reason why the method should not be used for the inelastic region, when the equations become only slightly more complicated. There is no difficulty when the vertices have derivative couplings, except that cutoffs are then necessary.

The method has been used to calculate elastic phase shifts for $2\pi$ -scattering when the potential represents the exchange of a vector particle. The results were examined for the existence of a self-consistent particle in the direct channel. An accurate calculation of this has not been performed before.

## § 2. Numerical Treatment of the Singularities in the Energy-Analytic Equation

It was shown in Chapter III that in the elastic region the only singularity of the energy-analytic equation is apparently the 2-particle threshold. This was assuming, however, that the potential's spectral function is free from singularity, and in the cases we shall be interested in, of single particle exchange, this is far from true.

Consider an exchange potential $V(p, q) \equiv$  where the exchanged particle has mass $\mu$. Now

$$V(p; q) \sim \frac{1}{(p-q)^2 - \mu^2} = \int \frac{\delta(u^2-(\underline{p}-\underline{q})^2-\mu^2)}{(p_0-q_0)^2-u^2+i\varepsilon}\, du^2$$

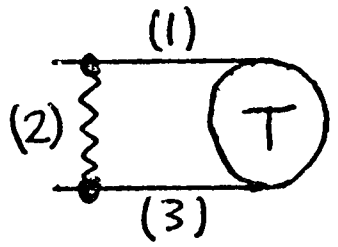
with unimportant modifications for particles with spin. The full spectral function thus involves a $\delta$-function. This singularity is rendered less drastic by taking the partial wave components, and we have

$$\sigma_\ell = \frac{|\underline{p}|\,|\underline{q}|}{2}\int \delta\left(u^2-(\underline{p}-\underline{q})^2-\mu^2\right) P_\ell(z)\,dz$$

$$= \begin{cases} \frac{1}{4}\, P_\ell\left(\dfrac{|\underline{p}|^2+|\underline{q}|^2+\mu^2-u^2}{2|\underline{p}|\,|\underline{q}|}\right) & \text{if } (|\underline{p}|-|\underline{q}|)^2+\mu^2 < u^2 < (|\underline{p}|+|\underline{q}|)^2 \\ 0 & \text{otherwise} \end{cases}$$

At the boundary of its support, $|\sigma_\ell| = 1/4$ so that $\sigma_\ell$ has a step-singularity instead of the $\delta$ -function.

The kernels $K_{ij}$ $(i, j = 0,1)$ will now refer to those in § 2 of Chapter III and p will denote $|\underline{p}|$, etc. In $K_{10}$ there appears a singular part which behaves like $\dfrac{\sigma(s, \omega_-(q), s-\omega_-(q))}{q-\hat{p}}$ (the explicit dependence of $\sigma$ on $\ell$, p,q is again omitted). The lines of singularity in $\sigma$ are parallel to the t-axis and these coincide with the singularity $q = \hat{p}$ when $s = s_+ = \left[(p \pm \hat{p})^2 + \mu^2\right]^{\frac{1}{2}}$. The conjunction of the step-singularities and the pole results in logarithmic singularities in $\rho$ along the lines $s = s_\pm(p)$. These are due to the anomalous triangle singularity[57] which occurs by pinching of the integration-contour by the three lines (1), (2), (3) in the homogeneous term




In the diagram below these singularities are shown together with three threshold lines A, B, C.

(i) The first parts of $K_{10}$, $K_{11}$ make contributions to $\rho$ above

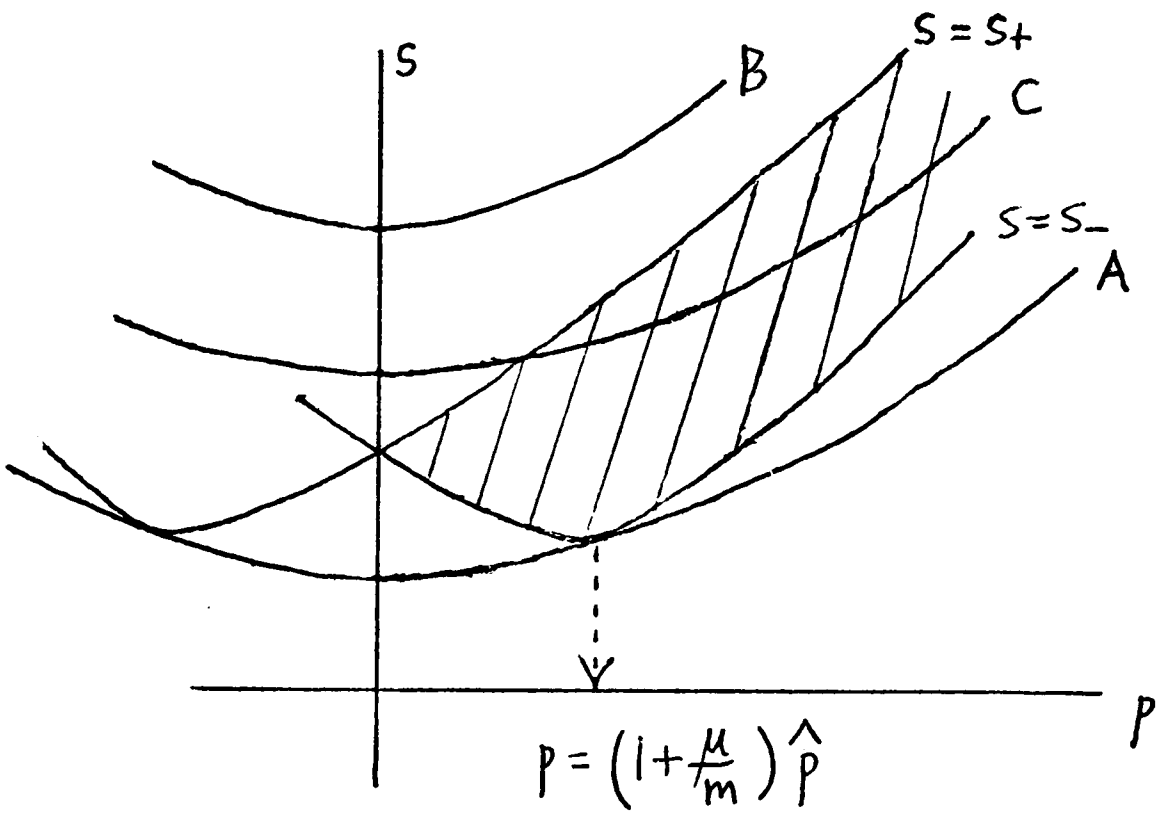
$A \equiv s = E/2 + \left[p^2+(m+\mu)^2\right]^{\frac{1}{2}}$

(ii) The second parts of $K_{10}$, $K_{11}$ make contributions to $\rho$ above

$$B \equiv s = E/2 + \left[ p^2+(m + \mu)^2 \right]^{\frac{1}{2}}$$

(iii) The third part of $K_{11}$ makes contributions to $\rho$ above

$$C \equiv s = -E/2 + \left[ p^2+(m + 2\mu)^2 \right]^{\frac{1}{2}}$$




These are not the only singularitires possessed by $\rho$ ; others corresponding to normal and anomalous thresholds occur by pinching of more lines in the ladder diagrams. From the point of view of computation, however, these are the more important singularities and may require special attention. For a satisfactory numerical treatment the domain of $\rho$ must be divided into regions where $\rho$ is well behaved, except at the boundaries of these regions. Now A and B are square-root singularities and C can be ignored because it is only as singular as $x^2 \log x$. Thus an appropriate subdivision of the p-s plane above A is made by B and $s = s_{\pm}$. The inhomogeneous born term fits in the shaded region between $s_{\pm}$.

The step-singularities in $K_{10}$ and $K_{11}$ which arise from $\sigma$ must now be considered. The supports of various parts of the kernels lie in the following regions:

For $K_{10}$, (i) $\omega_-(q) + \left[(p-q)^2 + \mu^2\right]^{\frac{1}{2}} \leqslant s \leqslant \omega_-(q) + \left[(p+q)^2 + \mu^2\right]^{\frac{1}{2}}$

(ii) $\omega_+(q) + \left[(p-q)^2 + \mu^2\right]^{\frac{1}{2}} \leqslant s \leqslant \omega_+(q) + \left[(p+q)^2 + \mu^2\right]^{\frac{1}{2}}$

For $K_{11}$, (iii) as in (i) above

(iv) as in (ii) above

(v) $t + \left[(p-q)^2 + \mu^2\right]^{\frac{1}{2}} \leqslant s \leqslant t + \left[(p+q)^2 + \mu^2\right]^{\frac{1}{2}}$

Since these regions actually vary with p and s no further subdivision of the (q, t) range can create regions in which the kernel is non-singular for all values of p, s. The way that this problem is met is best illustrated by a simpler example.

Consider a 1-dimensional integral equation on the reals given by a kernel $K(x, y)$ whose support lies in $\alpha(x) \leqslant y \leqslant \beta(x)$. If K is smooth in this region but may have step singularities at the boundaries the solution f of $f = f^{(0)} + K f$ will be regular provided $\alpha$, $\beta$ and the inhomogeneous term $f^{(0)}$ are smooth functions.

Now let $x_1, x_2, \ldots, x_n$ be mesh-points for a solution and $f_1, f_2, \ldots, f_n$ the values of f at these points. We approximate the value of f at a general point x by a linear interpolation formula

$$f(x) \sim \sum_{j=1}^{n} c_j(x)\, f_j$$

so that the value of Kf at x is approximately

$$\sum_{j=1}^{n} f_j \int_{\alpha(x)}^{\beta(x)} K(x,y)\, c_j(y)\, dy$$

Hence, if $y_1(x), y_2(x), \ldots, y_I(x)$ are a set of points for numerical integration over $[\alpha(x), \beta(x)]$ and $w_1(x), w_2(x), \ldots, w_I(x)$ are their weights, $Kf(x)$ can be approximated by a finite sum

$$\sum_{j=1}^{n} f_j \sum_{r=1}^{I} K(x, y_r(x))\, c_j(y_r(x))\, w_r(y_j(x))$$

The integral equation is thus approximated numerically by an n x n system of linear equations

$$f_i = f_i^{(0)} + \sum_{j=1}^{n} k_{ij} f_j$$

where $k_{ij} = \sum_{r=1}^{I} K(x_i, y_r(x_i))\, c_j(y_r(x_i))\, w_r(y_j(x_i))$. These may be solved when $k_{ij}$ have been evaluated.

In the energy-analytic equation the kernels $K_{10}$ and $K_{11}$ consist of a sum of kernels with moving singularities in the variable q like the discontinuities in this example. If the values of a and $\rho$ are linearly interpolated from a set of mesh-points for q (with the transformed value of t fixed in the case of $K_{11}$) then the integrals over the supports of various bits of $K_{10}$ and $K_{11}$ can be performed numerically, exactly as in the example above. Thus, by this linear interpolation procedure a satisfactory numerical approximation to the problem is possible, which is only a little more complicated than if the discontinuities were absent altogether. It is clear that, as pointed out in ref.(30), a straightforward mesh-point integration over the q-t plane would

be hopelessly inaccurate.

There also remains the threshold singularity appearing in $K_{00}$ and $K_{10}$. This principal value is dealt with by subtracting out the principal part and putting

$$P \int \frac{f(q)\, dq}{q-\hat{p}} = \int \frac{f(q)- u(q)\, f(\hat{p})}{q-\hat{p}}\, dq + f(\hat{p})\; P \int \frac{u(q)\, dq}{q-\hat{p}}$$

Provided $u(\hat{p}) = 1$ the first integral is non-singular, and for suitable choices of u the second integral can be done analytically. For example, if the q-range of integration is finite we may put $u(q) \equiv 1$, and if the range is infinite we may put $u(q) = \hat{p}/q$. Finally, the pole $t = \omega_+(q)$, which was shown to have zero residue in § 3 of Chapter III cannot here be ignored. This is because the different bits of $K_{11}$ where it appears are calculated (i.e. interpolated) separately and the necessary cancellation of poles need not occur. The principal value is therefore taken in each part, using the above procedure.

In extending the method to the first inelastic region $2m + \mu < E < 2m + 2\mu$, the only new singularity is the pole $t = \omega_-(q)$ in $K_{01}$ and $K_{11}$. This can be divided into its real (i.e. principal) part and its imaginary part, and the principal value evaluated as above. Also, when E exceeds $2m + 2\mu$, the function $\nu(p, q, \omega_-(q)$ has complex logarithmic singularities. These present no essential complication, but complex arithmetic must be used.

## § 3. Results for Elastic Scattering with the Scalar Equation

Some of the more technical details of the computation are gathered together in the Appendix. The equation of Chapter III with one subtraction in $p_0$ was used for the scalar equation, although this is not essential. The advantage is that the threshold singularity is transferred to kernels $K_{00}$ and $K_{11}$ of lower degree.

Our S-wave values of $E \tan \delta / 2\hat{p}$ for scalar exchange with $m = \mu = 1$ and various couplings $\lambda \equiv g^2/(4\pi)^2$ are shown in Table 3.1. They were obtained using a 17-entry array for $a(p)$ and a 160-entry array for $\rho(p, s)$. These figures are compared with the revised figures of Schwarz and Zemach[26] and the following approximations:

(i) The (1, 1) Pade approximant

(ii) The Cohen-Pagnamenta-Taylor (CPT) approximation[34]

(iii) The Blankenbeclar-Sugar (B-S) approximation[35]

(iv) Use of the approximate kernel $K_{00}$.

It is clear that our results are in very good agreement with Schwarz and Zemach's except at $p^2 = 0.2$ when $\lambda = 5$. This disagreement is due to the fact that the phase shift is very nearly $\pi$, and the calculated values of the phase shift agree, in fact, to within about $\frac{1}{2}$ per cent.

| | $g^2/(4\pi)^2$ | EAR | S-Z | (1,1)PADE | CPT | B-S | $K_{00}$ |
|---|---|---|---|---|---|---|---|
| $\hat{p}^2 = 0.2$ | .1 | .24607 | .24607 | .24574 | .26083 | .25024 | .24762 |
| | .7 | 3.1970 | 3.1962 | 2.8097 | 8.3010 | 4.2798 | 3.5063 |
| | 1 | 9.5540 | 9.5447 | 5.8742 | -15.337 | 58.826 | 13.238 |
| | 3 | -2.0539 | -2.0541 | -8.4327 | -2.8247 | -1.1013 | -1.8147 |
| | 5 | .05372 | .06298 | -5.6705 | -2.4284 | 1.3737 | .17719 |
| $\hat{p}^2 = 0.8$ | .1 | .14138 | .14138 | .14117 | .14771 | .14419 | .14246 |
| | .7 | 1.0827 | 1.0826 | 1.0002 | 1.4569 | 1.2689 | 1.1415 |
| | 1 | 1.7106 | 1.7102 | 1.4376 | 2.6165 | 2.2307 | 1.8524 |
| | 3 | -5.4865 | -5.503 | 4.4961 | -10.989 | -2.5483 | -4.6924 |
| | 5 | -1.1367 | -1.1413 | 7.8261 | -5.3869 | -.47012 | -1.1033 |

Table 3.1 S-wave values of $E \tan \delta/2\hat{p}$ obtained from the energy-analytic Bethe-Salpeter equation for the elastic scattering of scalar particles of mass 1 with the exchange-potential of a scalar particle of mass 1. $\delta$ is the phase shift, $E$ is the centre of mass energy, $\hat{p}$ is the on-shell momentum and $g$ is the coupling constant. Our figures (EAR) are compared with Schwarz and Zemach's calculations and with certain approximations.

All of the approximations except (i) reproduce the first "resonant-state" ($\tan\delta = \infty$) reasonably well, but (iv) gives by far the best overall approximation. As is to be expected, (i) is the best approximation for small $\lambda$ because it alone is accurate to the second order.

Table 3.2 shows the values of $\lambda$ which produce a first bound state at the threshold energy E = 2m in the S- and P-channels.

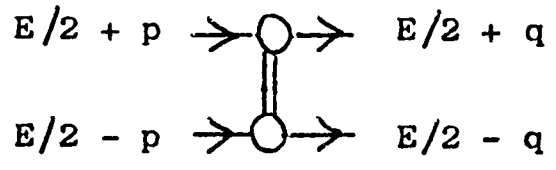
| | EAR | S-Z | $K_{OO}$ |
|---|---|---|---|
| S-wave | .7611 | .7622 | .7325 |
| P-wave | 7.364 | 7.377 | 6.685 |

Table 3.2 S- and P-wave values of $\lambda \equiv g^2/(4\pi)^2$ which produce a threshold (E = 2m) bound-state in the scalar equation when m = $\mu$. Our figures (EAR) are compared with Schwarz and Zemach's and with those obtained from the approximate kernel $K_{OO}$.

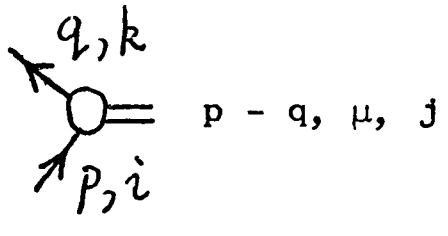
From these tables it is clear that our calculations confirm the accuracy of the Schwarz-Zemach figures. Table 3.1 also verifies what was suggested in § 3 of Chapter III, that bound- and resonant-state values of the coupling which are obtained from $K_{OO}$ should be more accurate than the corresponding approximate phase shifts.

## § 4. Bethe-Salpeter Calculation of the $\pi$-$\rho$ bootstrap

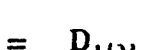
Taking $\rho$-exchange as the dominant term in the $2\pi$-potential,

$V(p;\ q) \sim$ 


If a further simplification is made as in § 3 of Chapter IV and the clothed vertex




is approximated by $-f\ (2\pi)^4\ (p + q)_\mu\, \varepsilon_{ijk}$, the potential is determined to within a coupling constant. In this derivative bare-vertex interaction i, j, k indicate the I-spin indices, $\mu$ is the polarization index of $\rho$ and f is the <u>physical</u> $\pi\pi\rho$ coupling constant ($f^2/4\pi \sim 2.5$). It is now necessary to decide on an appropriate approximation for the clothed propagator. From the Lehmann representation,

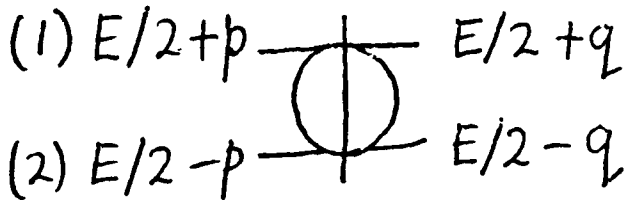
$$D'_{\mu\nu}(p) \equiv \text{  } = D_{\mu\nu}(p,\ m_\rho^2) + \int dm^2\ \sigma(m^2)\ D_{\mu\nu}(p;\ m^2)$$

where $D_{\mu\nu}(p,\ m^2) = \dfrac{g_{\mu\nu} - p_\mu p_\nu / m^2}{p^2 - m^2}$ is the bare vector propagator. If the contribution of the dispersive part were small we should be justified in putting $D'_{\mu\nu}(p) \sim D_{\mu\nu}(p,\ m_\rho^2)$, but there is no reason to suppose that this is the case. In the absence of this information it seems most sensible to retain the poles in the propagator; these occur on the mass-shell, but not in every component. Putting $p = (p_0,\ \underline{0})$,

$$D_{\mu\nu}(p,\ m_\rho^2) = \begin{bmatrix} -\frac{1}{m_\rho^2} & & & \\ & -\frac{1}{p^2-m_\rho^2} & & \\ & & -\frac{1}{p^2-m_\rho^2} & \\ & & & -\frac{1}{p^2-m_\rho^2} \end{bmatrix}$$

By taking $D'_{\mu\nu} \sim D_{\mu\nu}$ we should include all of the pole contributions, but at the expense of having $D_{00}$ constant; this leads to bad ultraviolet divergences. In view of our ignorance about the remaining contribution to $D'_{\mu\nu}$, the renormalization of this would appear rather futile, and in any case these contributions at infinity are quite likely cancelled by a more realistic vertex factor. We therefore choose to make the approximation $D'_{\mu\nu}(p) \sim \frac{g_{\mu\nu}}{p^2-m_\rho^2}$, so that although the extra pole is introduced, the ultraviolet divergences are considerably diminished. This is further justified by the fact that when the pions are on the mass-shells the residue of this additional pole in the potential is zero anyway.

With these approximations we have




$$\sim -\frac{if^2}{(2\pi)^4}\,\frac{E^2-(p+q)^2}{(p-q)^2-m_\rho^2}\,(\delta_{12}\delta_{34}-\delta_{13}\delta_{24})$$

where the Kronecker deltas refer to isospin, and have the same meaning as in Chapter IV. In the direct channel the isospin factor has the resolution (2, 1, -1), corresponding to $I = 0, 1, 2$. It is readily seen by power counting that the kernel arising from this potential is logarithmically divergent, but with a cutoff on $|\underline{p}|$ a solution may be shown to exist, e.g. by slightly adapting the proof in Chapter II.

For partial-wave projections with $\ell > 0$ we can replace

the term $\dfrac{E^2 - (p+q)^2}{(p-q)^2 - m_\rho^2}$ in the potential by $\dfrac{E^2 + m_\rho^2 - 2(p^2+q^2)}{(p-q)^2 - m_\rho^2}$

and use the energy-analytic form of the BSE described in Chapter I, assuming that the narrow width of p can be neglected; a single subtraction is needed. It is clear that the cutoff on $|\underline{p}|$ is conveniently chosen to preserve the energy-analytic structure of the equation, and it causes only trivial modifications to the method (see Appendix).

It was thus possible to calculate phase shifts in the $\ell = 1$, $I = 1$ channel of $2\pi$ scattering with a $\rho$-exchange potential, and carry out the following procedure:

(i) Fix the cutoff and the $\rho$-mass $m_\rho$ ;

(ii) Calculate the input value of the $\pi\pi\rho$ coupling constant necessary to make $\tan\delta$ infinite when the direct energy is $m_\rho$ .

(iii) Assuming that this represents a narrow width resonance, evaluate its width by the formula of § 10 in Chapter I. The

corresponding output coupling is then determined by the unitarity condition.

By carrying out such consistent mass calculations, it was possible to discover whether, by varying the cutoff and $m_\rho$ a resonance may exist in both the direct and cross channels which has a consistent mass and a consistent coupling-constant (i.e. width).

It was found that this condition could not be satisfied. Although the input and output couplings each diminish slowly as the cutoff increases (with $m_\rho$ fixed), the output value is consistently too large.

Input and output values of $f^2/4\pi$ in consistent mass calculations with (i) $m_\rho = 3m_\pi$ and (ii) $m_\rho = 765$ MeV, $m_\pi = 140$ MeV (approximate physical values), are shown in Table 4.1. The first value of the cutoff for comparison with reference[39].

| | cutoff | $(f^2/4\pi)_{in}$ | $(f^2/4\pi)_{out}$ |
|---|---|---|---|
| $m_\rho = 3m_\pi$ | $\sqrt{34}$ m | 8.40 | 14.8 |
| | 40 m | 3.92 | 6.6 |
| $m_\rho = 765$ MeV | $\sqrt{34}$ m | 14.2 | 73. |
| | 40 m | 4.17 | 11.0 |

Table 4.1 Input and output couplings in a Bethe-Salpeter consistent mass calculation.

The input values appear to be stable against variation of the mesh size, etc. However, the cutoff causes the spectral function to be rougher (see Appendix) and our accuracy is rather less than in the scalar case. Thus, since the output values depend on numerical differentiation, they are not expected to have an accuracy of less than a few per cent.

Figure 4.2 shows the typical variation in the input and output values of $f^2/4\pi$ with the increase in the cutoff for two self consistent masses. It is clear that this Bethe-Salpeter model with vector-exchange is incapable of producing the same vector resonance in the direct channel within a very large range for the cutoff. For the most part these couplings are too large to indicate narrow width resonances in either channel, and the nomenclature "coupling constant" is purely formal over most of the range.

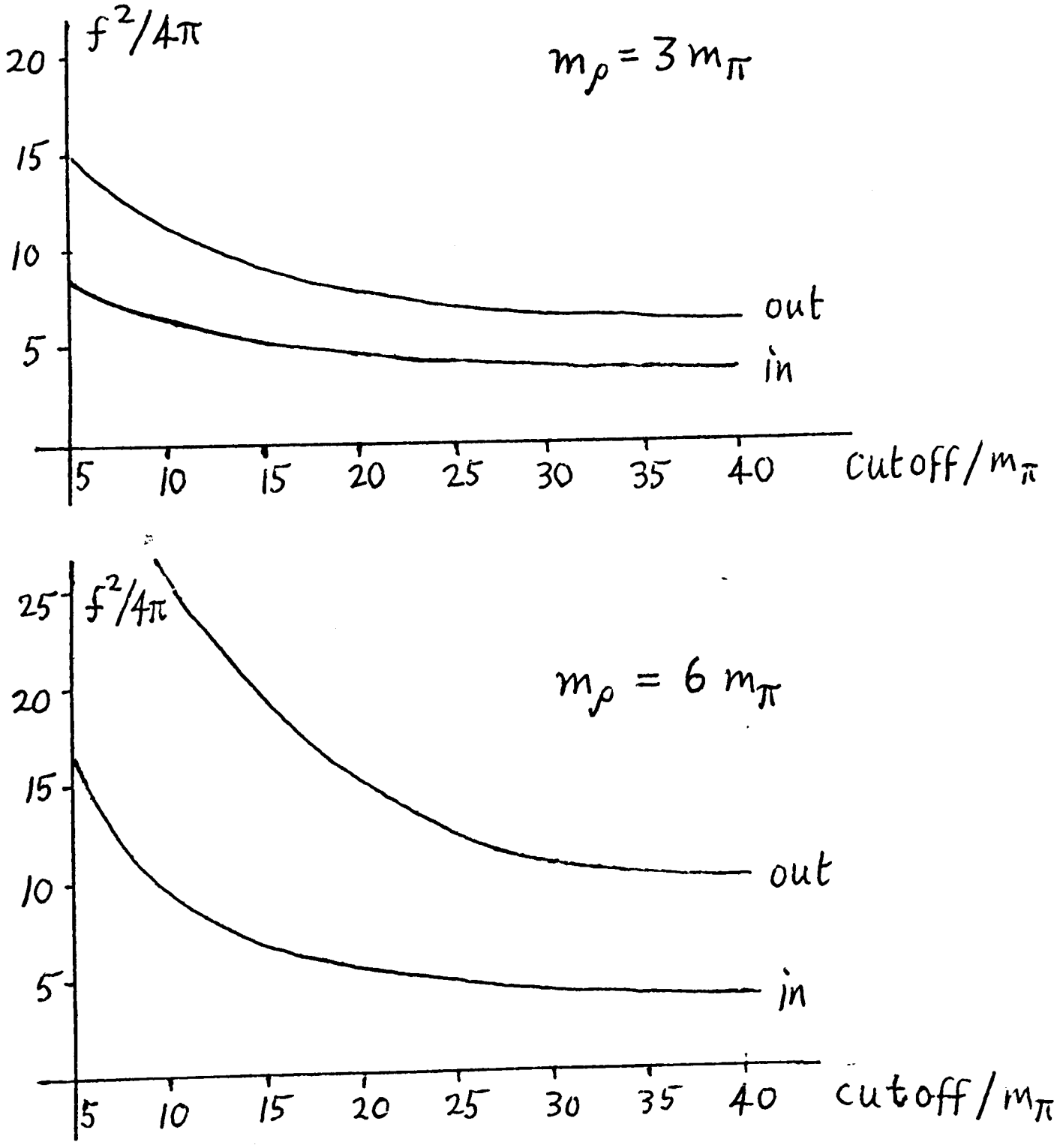



Figure 4.2 Variation of the input and output coupling constants with change of cutoff in a consistent mass Bethe-Salpeter calculation. The cutoff is measured in units of $m_\pi$.

# APPENDIX

## A1. Transformation of the $\rho$ -Domain

The regions (both finite and infinite) into which the $\rho$ -domain was divided in § 2 of Chapter VI were each transformed into the unit rectangle $[0, 1] \times [0, 1]$ . The transformations were of the form $q = q(u)$, $t = t(q, v)$ with $u, v \in [0, 1]$ . At boundaries $v = 0, 1$ where $\rho$ is singular, care was taken to reduce the singularity by making (in the case of $u = 0$) $t \sim O(v^2)$ as $v \to 0$.

The mesh-points chosen were those corresponding to a Gaussian integration. This is convenient not only for integration using Gaussian weights (the $\rho$ -domain is integrated over with the kernel $K_{01}$) but also as regards interpolation, because a polynomial interpolation with these mesh-points is of the Tchebyshev type.

It was also possible to ensure that the supports of the $\sigma$ -functions were calculable analytically in terms of the transformed variables by judicious choice of the transformations.

The method of Gaussian quadrature was used throughout.

For the vector equations, the sharp cutoff in the kernel reduces the smoothness of $\rho$ along lines where the boundary of a $\sigma$ -function in the kernel coincides with the line $q$ = cutoff. Making transformations near these lines similar to those

described above led to an improvement in the accuracy, and these lines did not, fortunately, intersect the other singular lines.

A2. Inversion of the Numerical Kernels and the Pade Approximant

Let $f(\lambda)$ be a complex function, analytic in a neighbourhood of the origin. The (N, D) Pade approximant to f near the origin is a rational function $n(\lambda)/d(\lambda) \equiv f_{N,D}(\lambda)$ where n and d are a pair of polynomials of degrees N and D respectively such that

$$(1) \qquad f(\lambda) = f_{N,D}(\lambda) + O(\lambda^{N+D+1})$$

i.e. $f_{N,D}$ has maximum contact with f at the origin.

If $f(\lambda) = \sum_{i=0}^{\infty} f_i \lambda^i$, $n(\lambda) = \sum_{i=0}^{N} n_i \lambda^i$ and $d(\lambda) = \sum_{i=0}^{D} d_i \lambda^i$, then without loss of generality we may take $d_0 = 1$ and it follows that $n_0 = f_0$. Equating powers of $\lambda$ in equation (1),

$$(2) \qquad f_i + \sum_{j=1}^{D} f_{i-j}\, d_j = 0 \qquad N+1 \leqslant i \leqslant N+D$$

$$(3) \qquad f_i + \sum_{j=1}^{D} f_{i-j}\, d_j = n_i \qquad 1 \leqslant i \leqslant N$$

Equations (2) give $d_1, d_2, \ldots, d_D$ by inversion and if this solution is unique the approximant is called "normal". $n_1, n_2, \ldots, n_N$ can then be obtained from (3) by back-substitution.

Theorem (de Balloire[58])

If all of the Pade approximants to f are normal and f is

meromorphic with k poles in a disc $\Delta$ then as $N \to \infty$ the sequence of Pade approximants $f_{N,k}$ tends uniformly to f in any compact subset of $\Delta$ where f is non-singular, and the poles of $f_{N,k}$ tend to the poles of f.

There are some conjectured extensions of this result for arbitrary (N, D) approximants which can be found in Baker[59].

The theorem is applicable to the solution of a large number of linear equations

$$\underline{x} = \underline{x}^{(0)} + \lambda K \underline{x}$$

whose solution $\underline{x}(\lambda) = (1 - \lambda K)^{-1} \underline{x}^{(0)}$ is meromorphic as a function of $\lambda$. To calculate the (N, D) Pade approximant of $\underline{x}$ it is sufficient to evaluate the first N+D+1 terms of the Neumann series, viz $\underline{x}^{(i)} = K^i \underline{x}^{(0)}$, (i = 0, ... , N+D), and apply the method of the theorem to each coordinate. When $\underline{x}$ has to be calculated for a large number of values of $\lambda$ this method is particularly fast and efficient because it is virtually independent of the number of solutions required.

When applied to the equations of Chapter V where K is the numerical kernel of the Bethe-Salpeter equation and $\lambda$ is in the ranges quoted, 5-figure accuracy could be achieved for 150 equations with the (6, 6) approximant in about 20 seconds of KDF 9 computer-time. This is considerably less than the time required for a direct inversion of the equations with only one

value of $\lambda$ . Moreover, a matrix of this size practically exhausted the storage of our machine, whereas the rows of K could conveniently be held on the discs and taken off one-by-one when the Born series is generated.

The Pade method has considerable interest in off-shell scattering problems independently of numerical treatments like our own. The essential question to be settled is the analytic structure of the scattering matrix as a function of the coupling constant; this may often be proved by showing that the kernel is compact in some banach space. For example in Chapter II we showed that the Wick-rotated kernel was compact so that the solution is meromorphic everywhere in the $\lambda$ -plane (and a much stronger result than this asserts that the (N, N) approximants converge to the solution of the BSE[60]).

The benefit of using Pade approximants in $\lambda$ is that from only a few terms of the perturbation series the solution may be continued beyond its radius of convergence. Now it is often said that a great advantage of the Bethe-Salpeter equation over perturbation theory is that it can give solutions when the perturbation series is "useless" i.e. the expansion is not convergent. The point is that although we require only the perturbation expansion, it is necessary to construct a sufficient number of its terms for them to have "settled down" . A systematic method of doing this is therefore usually necessary and ad hoc

methods are likely to be tediously long; moreover there may be no guarantee that the solution has the necessary analytic properties. Thus we are skeptical about the accuracy of methods that have been used which purport to approximate the S-matrix using a few Feynman diagrams obtained from renormalized perturbation theory.

We mention in passing that the Pade method is also used in scattering theory in quite a different way, when the perturbation parameter is the total energy variable. For example, in the N/D equations the unknown quantities can be represented by a Stieltjes integral (i.e. a Cauchy integral with positive "weight" ). A different theorem is available and, unlike in the above theorem, upper and lower bounds can be put on the solution from a knowledge of certain approximants. Haymaker[56] has also continued bound-state solutions of the Bethe-Salpeter equation into the elastic scattering region by using Pade approximants in the total energy variable, but estimates like those obtainable in the latter method cannot be found here.

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Abstract of a Thesis Entitled

# THE BETHE-SALPETER EQUATION AND UNSTABLE PARTICLES

by

JOHN DENMEAD SMITH, M.A.

St John's College

March 1971

Using certain assumptions about off-shell elements, we show that Green's functions have poles with factorizable residues at those values of the direct channel energy which produce resonance poles in the second-sheet of the S-matrix.

An extension of the Wick-rotated Bethe-Salpeter equation (BSE) in the ladder approximation is obtained for elastic and complex energies, and an existence theorem is proved for an equation which is non-singular except on the threshold cut. This method does not require contour-distortions, and the continuation to complex energies is direct. We show also that the Bethe-Salpeter Heitler matrix has a factorization property, and derive the energy-analytic form of the BSE when a subtraction is required. An approximate 1-dimensional reduction of the BSE is described which gives more accurate results than previous methods.

A theory of pi- and rho-mesons coupled via the Yang-Mills interaction is considered in the bootstrap limit, and this leads to an expression for the rho-field. An equivalent pion interaction can then be obtained which is similar to the chiral-invariant lagrangians for zero-mass pions, and it is used in a rough calculation. The bootstrap limit of the Green's function equations is also given.

We give a new exact method of numerically inverting the BSE

using the energy-analytic representation, and for the scalar equation our figures agree with Schwarz and Zemach's to within about 1 per cent. There is no additional difficulty with derivative couplings, and the procedure was used to calculate P-wave pi-pi scattering with a Bethe-Salpeter rho-exchange potential. A self-consistent resonance in the direct channel was not predicted at any mass and cutoff.