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**Constant Rank Operators:
Lower Semi-continuity
and
 L^1 -estimates**

By

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Declaration of Authorship

I hereby declare that this thesis is entirely the work of my own except where otherwise indicated. All direct or indirect sources are acknowledged as references. It has not been submitted, either wholly or substantially, for another course of this Department or University, or for a course at any other institution.

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28th of September 2018

To Ruxi

Abstract

We contribute to the study of variational integrals

$$\mathcal{E}[w] := \int_{\Omega} f(x, w(x)) \, dx \quad \text{constrained by} \quad \mathcal{A}w = 0.$$

Here $\mathcal{A}$ is a linear partial differential operator accounting for constraints on the physical quantity w , which is measured by the energy functional $\mathcal{E}$ (e.g., if $w = \nabla u$ is the strain of a deformation u with elastic energy $\mathcal{E}[w]$, one takes $\mathcal{A} = \text{curl}$). Under the mild *constant rank* assumption on (the symbol map of) $\mathcal{A}$, i.e.,

$$\text{rank } \mathcal{A}(\xi) \text{ is independent of } \xi \neq 0,$$

the lower semi-continuous relaxation of $\mathcal{E}$ on L^p for energy densities f of p -growth, i.e., $|f| \leq c(1 + |\cdot|^p)$, was extensively studied, particularly for $1 < p < \infty$.

We propose shifting focus to a simpler model (which has the advantage that one can easily incorporate boundary conditions), namely

$$\tilde{\mathcal{E}}[u] := \int_{\Omega} f(x, \mathbb{B}u(x)) \, dx,$$

by constructing a linear differential operator $\mathbb{B}$ such that

$$\ker \mathcal{A}(\xi) = \text{im } \mathbb{B}(\xi) \quad \text{for all } \xi \neq 0.$$

In this case, we show that the $\mathcal{A}$ -quasiconvex envelope of f is obtained by testing with compactly supported vector fields $\mathbb{B}u$ and that $\mathcal{A}$ -free Young measures are generated by sequences $(\mathbb{B}u_j)$ modulo a shift by the barycentre. In the linear growth setting, $p = 1$, we use these ideas to prove that $\mathcal{A}$ -free generalized Young measures with absolutely continuous barycentre can be characterized by duality with $\mathcal{A}$ -quasiconvex functions.

In this case, it is of interest to study the function space $\{u : \mathbb{B}u \in L^1\}$. Equivalently, one studies regularity properties of linear equations

$$\mathbb{B}u = w, \tag{0.1}$$

for $\mathbb{B}$ of constant rank and $w \in L^1$. Such linear equations are well understood only if $w \in L^p$ for $1 < p < \infty$, i.e., if $\mathbb{B}$ has order l , then (0.1) does not, in general, have a solution u such that $D^l u \in L^p$, unless $\mathcal{A}w = 0$ (here and it what follows, we restricted considerations to functions defined on $\mathbb{R}^n$). If $p = 1$, by Ornstein's Non-inequality, (0.1) may have no solutions such that $D^l u \in L^1$. Sharp results

on the Sobolev scale were given by VAN SCHAFTINGEN, who showed that if $\mathbb{B}$ is (overdetermined) *elliptic* and $j = 1, 2, \dots, \min\{l, n-1\}$, then

$$\|D^{l-j}u\|_{L^{\frac{n}{n-j}}} \leq c\|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty \quad (0.2)$$

if and only if $\mathbb{B}$ is *canceling* (C) (see Section 1.1.4 for definitions).

We will show that the endpoint embedding for $j = n \leq l$:

$$\|D^{l-n}u\|_{L^\infty} \leq c\|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty \quad (0.3)$$

is equivalent, in the class of elliptic operators, with a *weaker canceling* condition (WC).

We will also extend the inequalities (0.2) and (0.3) to the case of constant rank operators: There exists a projection map π onto $\ker \mathbb{B}$ such that

$$\|D^{l-j}(u - \pi u)\|_{L^{\frac{n}{n-j}}} \leq c\|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty \quad (0.4)$$

if and only if $\mathbb{B}$ satisfies (C) (for $1 \leq j \leq \min\{l, n-1\}$) or $\mathbb{B}$ satisfies (WC) (for $j = n \leq l$ and the convention $n/0 = \infty$). Inequalities on different scales (fractional, Besov, Triebel–Lizorkin, Lorentz, Hardy), due to VAN SCHAFTINGEN for elliptic operators, are extended to constant rank operators analogously to (0.4).

We will also investigate pointwise regularity for maps in the Sobolev-type space $\{u: \mathbb{B}u \in L^1\}$. We will show that the existence of critical L^q -Taylor expansions for such maps is equivalent with the embeddings (0.2) and (0.3).

We will also investigate embeddings on lower dimensional manifolds (trace theorems) in the spirit of ADAMS, i.e.,

$$\|D^{l-1}u\|_{L^{\frac{n}{n-s}}(\Sigma; \mathcal{H}^{n-s})} \leq c\|\mathbb{B}u\|_{L^1(\mathbb{R}^n; \mathcal{L}^n)}, \quad (0.5)$$

where $\Sigma \subset \mathbb{R}^n$ is $(n-s)$ -regular, with $s \in (0, 1]$. This generalizes (0.2), which corresponds to the case $s = 0$. We show that, if $s \in (0, 1)$, then (0.5) holds if and only if $\mathbb{B}$ is elliptic and satisfies (C). If $s = 1$, we show that (0.5) implies ellipticity and a *stronger canceling* condition (SC). We show that for first order operators ($l = 1$), the trace inequality (0.5) is equivalent to (SC).

Lastly, we investigate boundary estimates, by which we mean

$$\|D^{l-1}u\|_{L^{\frac{n}{n-1}}(\Omega)} \leq c\left(\|\mathbb{B}u\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)}\right) \quad \text{for } \mathbb{L}u = 0 \text{ on } \partial\Omega, \quad (0.6)$$

where $\mathbb{L}$ is a differential operator on $\partial\Omega$. Inspired by the comparison between GÅRDING's Inequality (replaced here by (0.2) with $l = 1$) and ARONSZAJN's Inequality (replaced here by (0.6) with $\mathbb{L} \equiv 0$), we show that the Gagliardo–Nirenberg–Sobolev inequality on domains

$$\|D^{l-1}u\|_{L^{\frac{n}{n-1}}(\Omega)} \leq c\left(\|\mathbb{B}u\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)}\right) \quad \text{for all } u \in C^\infty$$

is equivalent with the complex ellipticity (Lopatinskiĭ–Shapiro complementing) condition $\ker_{\mathbb{C}} \mathbb{B}(\xi + i\eta) = 0$ for all $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$. This should only be the first step of a considerably larger program.

Some of these results are the author's own work, while others are obtained in collaboration with F. Gmeineder, J. Kristensen, and J. Van Schaftingen (see the precise list in Section 1.9).

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¹allegedly

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CHAPTER 1

Prologue

1.1. Context and Overview

The main motivation for the present thesis comes from the study of minimization of variational integrals

$$\mathcal{E}[w] := \int_{\Omega} f(x, w(x)) \, dx \quad (1.1)$$

defined on vector fields w that satisfy a linear partial differential constraint

$$\mathcal{A}w = 0, \quad (1.2)$$

for continuous integrands $f: \Omega \times W \rightarrow \mathbb{R}$ that satisfy (standard) growth conditions

$$0 \leq |f(x, \eta)| \leq c(1 + |\eta|^p) \quad x \in \Omega, \eta \in W,$$

for some $1 \leq p < \infty$ (no growth condition is required if $p = \infty$). Here $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and W is a finite dimensional Euclidean space. For better exposition, the first part of this introductory chapter is confined to the case $1 < p < \infty$, although in general we will focus on the limiting case $p = 1$. The case $p = \infty$ is beyond the scope of this thesis.

The motivation for studying such functionals or the Young measures generated by sequences satisfying differential constraints stems from physics (e.g., electrostatics, conductivity, elasticity, plasticity, elasto-plasticity, fluid mechanics, conservation laws [85, 184, 137, 168, 19, 88]), specific aspects of which will not be pursued here. Instead, we will focus on the interplay between algebraic assumptions on the compensating operator $\mathcal{A}$ and (semi-)continuity properties of the energy functional $\mathcal{E}$ or on the function space properties of the competitors w . We aim to complement the existing unified framework for such problems described below with new results pertaining mostly to the linear growth case $p = 1$.

We now outline the organization of the introductory chapter. Section 1.1 consists essentially of a survey of existing results pertaining to lower semi-continuity of functionals (1.1) under (1.2), on L^p -estimates for linear equations, and on the connections between them. The author's results are only briefly highlighted in Section 1.1.

Most of the results presented in the lower semi-continuity Subsections 1.1.1 and 1.1.2 will be examined in the following examples of constraints (1.2):

- (a) unconstrained case $\mathcal{A} \equiv 0$;
- (b) the prototypical case of the Calculus of Variations, $w = Du$, i.e., $\mathcal{A} = \text{curl}$;
- (c) general *constant rank* operators $\mathcal{A}$.

Subsection 1.1.3 is devoted to L^p coercive estimates for $1 < p < \infty$, where the author brought no new contribution. In Subsection 1.1.4 we present quite recent L^1 -estimates for linear equations.

Sections 1.2–1.8 contain more technical descriptions of each of the author's contributions, particularly ideas of proof and some sketches.

1.1.1. Lower semi-continuity Motivated by describing the interaction between non-linear functionals and weak-convergence, MURAT and TARTAR introduced in [142, 143, 183, 184] the Theory of Compensated Compactness, which we briefly describe now.

Even in the unconstrained case $\mathcal{A} \equiv 0$, it is easy to see that if $w_j \xrightarrow{*} w$ in L^∞ , then it need not follow that $f(\cdot, w_j) \xrightarrow{*} f(\cdot, w)$ in L^∞ (in fact, this is the case only if f is affine [103, Cor. 5.2.4]). The more appropriate notion of continuity in this case is (weakly- $*$ sequential) lower semi-continuity of $\mathcal{E}$, i.e.,

$$\liminf_{j \rightarrow \infty} \mathcal{E}[w_j] \geq \mathcal{E}[w], \quad (1.3)$$

which is consistent with minimization of $\mathcal{E}$. In this case, lower semi-continuity restricts the class of admissible integrands f , in the sense that (1.3) holds whenever $w_j \xrightarrow{*} w$ in L^∞ if and only if f is convex [103, Thm. 5.2.3]. If f is not convex, it is empirically clear that one can construct a highly oscillating sequence such that (1.3) is violated.

The typical example of a compensating operator is $\mathcal{A} = \text{curl}$, in which case $\mathcal{A}w = 0$ implies $w = Du$. In this case, the class of integrands f for which $\mathcal{E}$ is lower semi-continuous, strictly smaller than the class convex functions, was defined by MORREY in [140]. We do not dwell on this point now, as we will come back to it later in detail.

Instead, following [189, Ch. 17], we discuss the case in which the integrand f is a quadratic form on W and the operator $\mathcal{A}$ is an arbitrary first order, homogeneous linear differential operator with constant coefficients. More precisely,

$$\mathcal{A}w := \sum_{j=1}^n \partial_j \mathcal{A}_j w, \quad (1.4)$$

where $\mathcal{A}_j \in \text{Lin}(W, X)$ are linear maps between (finite dimensional) Euclidean spaces W, X . In this case, one can show by use of Plancherel's Theorem that (1.3) holds whenever $w_j \rightharpoonup w$ in L^2 satisfy $\mathcal{A}w_j = 0$ if and only if $f \geq 0$ in the wave cone

$$\Lambda_{\mathcal{A}} := \bigcup_{\xi \in \mathbb{R}^n \setminus \{0\}} \ker \mathcal{A}(\xi), \quad (1.5)$$

where the (Fourier) symbol map $\mathcal{A}(\cdot): \mathbb{R}^n \rightarrow \text{Lin}(W, X)$ is defined by

$$\mathcal{A}(\xi)\hat{w}(\xi) = i\widehat{\mathcal{A}w}(\xi), \quad \text{i.e.,} \quad \mathcal{A}(\xi) := \sum_{j=1}^n \xi_j \mathcal{A}_j.$$

Our convention for the Fourier Transform is

$$\hat{u}(\xi) := \int_{\mathbb{R}^n} u(x) e^{-ix \cdot \xi} dx \quad \text{for } u \in \mathcal{S}(\mathbb{R}^n).$$

More generally, it is sufficient to assume that w_j is strongly compact in H_{loc}^{-1} :

Theorem 1.1.1 ([189, Thm. 17.3]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\mathcal{A}$ be as in (1.4). Let f be a quadratic form on W such that*

$$f(\eta) \geq 0 \quad \text{for all} \quad \eta \in \Lambda_{\mathcal{A}}.$$

Then for all $w_j, w \in L^2(\Omega, W)$ such that

- (a) $w_j \rightharpoonup w$ in $L^2(\Omega, W)$,
- (b) $(\mathcal{A}w_j)_j$ is compact in $H_{\text{loc}}^{-1}(\Omega, X)$,

we have that

$$\liminf_{j \rightarrow \infty} \int_{\Omega} f(w_j(x)) dx \geq \int_{\Omega} f(w(x)) dx.$$

Theorem 1.1.1 can be viewed as a generalization of the celebrated div-curl Lemma [189, Lem. 7.2]. The emergence of the wave cone in the statement of Theorem 1.1.1 is by no means accidental. One can show, by analogy with the way in which MORREY showed that quasiconvexity implies rank-one convexity [140], that weakly-* sequential lower semi-continuity of $\mathcal{E}$ on L^∞ implies convexity of f along lines in the wave cone:

Proposition 1.1.2 ([189, Lem. 17.1]). *Let $f \in C(W)$ be such that*

$$\liminf_{j \rightarrow \infty} \int_{\Omega} f(w_j(x)) dx \geq \int_{\Omega} f(w(x)) dx$$

whenever $w_j \xrightarrow{} w$ in $L^\infty(\Omega, W)$ are such that $\mathcal{A}w = 0$. Then*

$$f(tw_1 + (1-t)w_2) \leq tf(w_1) + (1-t)f(w_2)$$

for $t \in [0, 1]$ and $w_i \in W$ with $w_1 - w_2 \in \Lambda_{\mathcal{A}}$.

As a side-note, this fact implies that weakly-* sequentially continuous functionals $\mathcal{E}$ have integrands f that are affine in directions of the wave cone (compare with quasilinear functions [61, Sec. 5.3.1], as introduced by BALL in [26]; see also [30]).

On a different note, the wave cone generally encodes condensed oscillation and concentration effects in weakly convergent sequences, constrained by $\mathcal{A}w_j = 0$. To

briefly illustrate this phenomenon, we present an example: Let H be a hyperplane in $\mathbb{R}^n$ with unit normal given by ν . Let

$$w = (\mathcal{H}^{n-1} \llcorner H) w_0,$$

where $w_0 \in W$ is a constant vector, so that w is a Radon measure. This measure is, in the author's opinion, the simplest example that is not absolutely continuous with respect to $\mathcal{L}^n$ when checking $\mathcal{A}$ -freeness. One can then show that

$$\mathcal{A}w = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n, X) \iff \mathcal{A}(\nu)w_0 = 0.$$

Do note that this example is consistent with what is observed if $\mathcal{A} = \text{curl}$, in which case $\Lambda_{\mathcal{A}}$ is the cone of matrices of rank at most equal to one: let $u \in \text{BV}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ be given by $u = \chi_{H^+} u_0$, where H^+ denotes a half-space bounded by H and $u_0 \in \mathbb{R}^N$. Then, up to a choice of sign, $Du = (\mathcal{H}^{n-1} \llcorner H) u_0 \otimes \nu$.

Many of the considerations above pertain to TARTAR's work. In turn, MURAT's approach relied on the assumption that the symbol map $\mathcal{A}(\cdot)$ is a matrix field of constant rank away from zero [143]. Although this restriction is not necessary to prove Theorem 1.1.1, it turns out to be flexible enough to lower semi-continuity results for much more general integrands than quadratic forms.

Definition 1.1.3. *A homogeneous, linear differential operator $\mathcal{A}$ on $\mathbb{R}^n$ with constant coefficients is said to be of constant rank if*

$$\text{rank } \mathcal{A}(\xi) = r \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\} \quad (1.6)$$

for some integer r .

The constant rank condition was introduced in the context of linear differential operators by SCHULENBERGER and WILCOX in [166] to obtain coerciveness estimates in the absence of ellipticity, but it may have been used in linear algebra before.

The technical advantage brought by this condition is that the orthogonal projection onto $\ker \mathcal{A}(\xi)$ in frequency space defines a Fourier multiplier

$$\mathbb{P}(\xi) := \text{Proj}_{\ker \mathcal{A}(\xi)}, \quad (1.7)$$

i.e., $\mathbb{P} \in C^\infty(\mathbb{R}^n \setminus \{0\}, \text{Lin}(W, W))$ is zero-homogeneous. To see that $\mathbb{P}$ has the appropriate homogeneity, one writes $\mathbb{P}(\xi) = \text{Id} - \mathcal{A}^\dagger(\xi) \mathcal{A}(\xi)$, which is possible by the geometric properties of the Moore–Penrose generalized inverse [52]. We wrote $M^\dagger$ for the pseudo-inverse of a matrix M , as defined in [139, 34, 151]. Roughly speaking, this gives a formal “Helmoltz” decomposition in frequency space

$$\hat{w}(\xi) = \mathbb{P}(\xi) \hat{w}(\xi) + (\text{Id}_W - \mathbb{P}(\xi)) \hat{w}(\xi), \quad (1.8)$$

which, under suitable boundary conditions, can be turned into a decomposition of w that can deal well with constraints such as Thm. 1.1.1(b). These ideas were used by FONSECA and MÜLLER to prove appropriate generalizations of Theorem 1.1.1 in [85]. Their result is fundamental for the present work, so that we will describe it in a broader context.

Firstly, let us recall the cornerstone result of MORREY, which, when embedded in the MURAT–TARTAR framework described above, corresponds to $\mathcal{A} = \text{curl}$:

Theorem 1.1.4 ([140, Thm. 2.1-2]). *Let $f \in C(\mathbb{R}^{N \times n})$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then*

$$\liminf_{j \rightarrow \infty} \int_{\Omega} f(Du_j(x)) \, dx \geq \int_{\Omega} f(Du(x)) \, dx$$

whenever $Du_j \xrightarrow{} Du$ in $L^\infty(\Omega, \mathbb{R}^{N \times n})$ if and only if f is quasiconvex, i.e.,*

$$\int_{[0,1]^n} f(\eta + D\phi(x)) \, dx \geq f(\eta)$$

for all $\eta \in \mathbb{R}^{N \times n}$ and $\phi \in W_0^{1,\infty}([0,1]^n, \mathbb{R}^N)$.

Roughly speaking, lower semi-continuity along sequences $Du_j \xrightarrow{*} Du$ in L^∞ is equivalent with affine functions $u(x) = \eta x$ being minimizers with respect to their own boundary conditions. In contrast with Theorem 1.1.1, quasiconvexity is not a directional convexity condition in general. This can be seen from ŠVERAK's counterexample [181] and Proposition 1.1.2 for $\mathcal{A} = \text{curl}$. Furthermore, quasiconvexity is not a local condition, as shown by KRISTENSEN in [118] (cp. [76]). Numerous refinements and generalizations of Theorem 1.1.4 can be found in [1, 132, 58, 136, 17, 84, 82, 119], to mention a few; also see DACOROGNA's monograph [61] and the references therein for more details.

A first attempt to generalize Compensated Compactness Theory (Theorem 1.1.1) in the spirit of Theorem 1.1.4 is due to DACOROGNA [59, Ch. I.2], where it is shown that if $f \in C(W)$ is such that

$$\int_{[0,1]^n} f(\eta + w(x)) \, dx \geq f(\eta)$$

for all $\eta \in W$ and all $w \in L^\infty((0,1)^n, W)$ such that $\int w = 0$ and $\mathcal{A}w = 0$ in the sense of distributions, then

$$\liminf_{j \rightarrow \infty} \int_{\Omega} f(w_j(x)) \, dx \geq \int_{\Omega} f(w(x)) \, dx$$

whenever $w_j \xrightarrow{*} w$ in $L^\infty(\Omega, W)$ are such that $\mathcal{A}(w_j - w) = 0$. In the terminology of [59, Eq. (2.2)], such f is said to be $\mathcal{A}$ -quasiconvex, notion which is not clearly necessary for lower semi-continuity. We will not use this terminology here, but that of FONSECA and MÜLLER, who showed that:

Theorem 1.1.5 ([85, Thm. 3.6-7]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $1 < p < \infty$, and $f \in C(W)$ be such that $0 \leq f < c(1 + |\cdot|^p)$. Let $\mathcal{A}$ be an operator as in (1.4) that satisfies the constant rank condition (1.6). Then*

$$\liminf_{j \rightarrow \infty} \int_{\Omega} f(w_j(x)) \, dx \geq \int_{\Omega} f(w(x)) \, dx$$

whenever $w_j \rightarrow w$ in $L^p(\Omega, W)$ and $\mathcal{A}w_j \rightarrow 0$ in $W^{-1,p}(\Omega, X)$, if and only if f is $\mathcal{A}$ -quasiconvex.

The more appropriate notion of $\mathcal{A}$ -quasiconvexity used here, i.e., under periodic boundary conditions, was introduced in the same paper:

Definition 1.1.6 ([85, pp. 1357]). *A function $f \in C(W)$ is said to be $\mathcal{A}$ -quasiconvex if and only if*

$$\int_{[0,1]^n} f(\eta + w(x)) \, dx \geq f(\eta)$$

for all $\eta \in W$ and $w \in C_{\text{per}}^\infty([0,1]^n, W)$ such that $\int w = 0$ and $\mathcal{A}w = 0$.

This is consistent with MORREY's quasiconvexity by [61, Prop. 5.13]. There are many developments following Theorem 1.1.5, but these are beyond the scope of this thesis. For the reader's convenience, we give a few references [44, 81, 24, 21, 116, 62].

When reading the proof of Theorem 1.1.5 in [85], one notices that significant technical difficulty is incurred when proving sufficiency of $\mathcal{A}$ -quasiconvexity for lower semi-continuity. This is due to the lack of potential functions $w = \mathbb{B}u$, where u can be taken to be compactly supported. This fact was already observed in [60], where a conditional result is given. We will not state the result of [60, Thm. 4] precisely, but rather describe it roughly: Assume that there exists a first order, homogeneous linear differential operator $\mathbb{B}$ on $\mathbb{R}^n$ from V to W with constant coefficients such that

$$\mathcal{A} \circ \mathbb{B} \equiv 0. \tag{1.9}$$

We will say that $w_j \rightarrow_{\mathcal{A}, \mathbb{B}} w$ for functions $w_j, w \in L^p(\Omega, W)$ if there exist functions $u_j \in W_0^{1,p}(\Omega, V)$ and $g_j \in L^p(\Omega, W)$ (for some $1 < p < \infty$) such that

- (a) $w_j - w = \mathbb{B}u_j + g_j$ for all j ,
- (b) $\mathbb{B}u_j \rightharpoonup 0$ in $L^p(\Omega, W)$,
- (c) $g_j \rightarrow 0$ in $L^p(\Omega, W)$.

If $f \in C(W)$ is such that $0 \leq f \leq c(| \cdot |^p + 1)$, then $w_j \rightarrow_{\mathcal{A}, \mathbb{B}} w$ implies

$$\liminf_{j \rightarrow \infty} \int_{\Omega} f(w_j(x)) \, dx \geq \int_{\Omega} f(w(x)) \, dx,$$

if and only if f is $\mathcal{A}$ - $\mathbb{B}$ -quasiconvex, i.e.,

$$\int_{[0,1]^n} f(\eta + \mathbb{B}u(x)) \, dx \geq f(\eta) \tag{1.10}$$

for all $\eta \in W$ and $u \in W_0^{1,\infty}([0,1]^n, V)$.

We remark that there is little hope to obtain a Helmholtz-type decomposition as in (a) that satisfies (b) and (c) unless $\mathcal{A}$ has constant rank. This is so since, in the absence of the constant rank condition, the multipliers in (1.7) and (1.8) are discontinuous [52].

One of the main results of this thesis is that, under the constant rank assumption on $\mathcal{A}$, DACOROGNA's result [60, Thm. 4] presented above is unconditional. We explain this below and postpone the details to Section 1.2.

Using a powerful algebraic result of DECELL in [63], we show in Theorem 2.1.1 ([156, Thm. 1]) that $\mathcal{A}$ has constant rank if and only if there exists a homogeneous linear differential operator $\mathbb{B}$ such that

$$\ker \mathcal{A}(\xi) = \operatorname{im} \mathbb{B}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \quad (1.11)$$

Of course, (1.11) is a strengthening of (1.9)¹. As a consequence of this fact, we have that the notions of $\mathcal{A}$ -quasiconvexity (Definition 1.1.6) and $\mathcal{A}$ - $\mathbb{B}$ -quasiconvexity (1.10) coincide; this is proved in Corollary 2.3.2.

On the other hand, the convergence $w_j \rightarrow_{\mathcal{A}, \mathbb{B}} w$ for $w_j, w \in L^p(\Omega, W)$ can be weakened to the topology of compensated compactness, i.e.,

- (a) $w_j \rightharpoonup w$ in $L^p(\Omega, W)$,
- (b) $\mathcal{A}w_j \rightarrow \mathcal{A}w$ in $W_{\text{loc}}^{-1,p}(\Omega, X)$,

by showing that there exist maps $u_j \in C_c^\infty(\mathbb{R}^n, V)$ such that $\mathbb{B}u_j \rightharpoonup 0$ in $L^p(\Omega, W)$ and

$$\liminf_{j \rightarrow \infty} \int_{\Omega} f(w_j(x)) \, dx = \liminf_{j \rightarrow \infty} \int_{\Omega} f(w(x) + \mathbb{B}u_j(x)) \, dx. \quad (1.12)$$

This reduction will be discussed in Section 1.2 as a consequence of Proposition 2.4.3.

In view of these considerations, the proof of Theorem 1.1.5 reduces to the following statement:

Proposition 1.1.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $1 < p < \infty$, and $f \in C(W)$ be such that $0 \leq f < c(1 + |\cdot|^p)$. Let $\mathbb{B}$ be a homogeneous linear differential operator on $\mathbb{R}^n$ from V to W with constant coefficients that satisfies the constant rank condition (1.6). Then*

$$\liminf_{j \rightarrow \infty} \int_{\Omega} f(w(x) + \mathbb{B}u_j(x)) \, dx \geq \int_{\Omega} f(w(x)) \, dx$$

for $u_j \in C_c^\infty(\Omega, V)$, $w \in L^p(\Omega, W)$, and $\mathbb{B}u_j \rightharpoonup 0$ in $L^p(\Omega, W)$, if and only if

$$\int_{[0,1]^n} f(\eta + \mathbb{B}u(x)) \, dx \geq f(\eta)$$

for all $\eta \in W$ and $u \in C_c^\infty([0,1]^n, V)$.

This statement is much easier to prove using quite classical ideas from [140, 136, 60]. A proof will be given in [154], a paper which is not included in this document.

An important point of the present thesis is to suggest a paradigm shift from functionals

$$\mathcal{E}[w] = \int_{\Omega} f(x, w(x)) \, dx \text{ under } \mathcal{A}w = 0 \quad \text{to} \quad \tilde{\mathcal{E}}[u] = \int_{\Omega} f(x, \mathbb{B}u(x)) \, dx,$$

¹In general, $\mathbb{B}$ need not be of first order, as in the setup of [60]. This is absolutely no restriction, particularly if $1 < p < \infty$; in fact, all the results presented in this Section are easily adapted to operators $\mathcal{A}$ and $\mathbb{B}$ of arbitrary (different) orders. None of the proofs in Part I of this thesis are restricted to first order operators.

under the constant rank assumption on $\mathcal{A}, \mathbb{B}$ satisfying (1.11). Roughly speaking, the considerations above show that the lower semi-continuity of $\mathcal{E}$ under the topology of compensated compactness can be inferred from the lower semi-continuity of $\tilde{\mathcal{E}}$, which is much easier to analyze with classical methods. Another significant advantage of $\mathcal{E}$ is that one can easily incorporate boundary conditions, so that variational boundary value problems can be tackled. This is part of the objective of PROSINSKI and the author in [154]. More detail will be given in Section 1.2.

It is important to mention that differential operators that satisfy the exact relation (1.11) must be of constant rank by Theorem 2.1.1. In particular, the properties of functionals such as $\mathcal{E}$ cannot be transferred to or inferred from functionals such as $\tilde{\mathcal{E}}$ if either $\mathcal{A}$ or $\mathbb{B}$ fail the constant rank condition. There are very few studied examples where this is the case, see, e.g., [186, 145, 127], which are discussed for academic purposes. Concerning more practical applications, the author is not aware of any examples where either $\mathcal{A}$ or $\mathbb{B}$ do not have constant rank.

1.1.2. Young measures Most of the results in Section 1.1.1 can be described efficiently in terms of Young measures, which are a functional analytical and measure theoretical tool introduced by YOUNG in [206, 207, 208] in order to describe interactions between non-linear functionals and highly oscillating sequences (see also [128, 129]). Young measures were and are used extensively in the Calculus of Variations to describe lower semi-continuity properties of energy functionals and their relaxation phenomena. In fact, many variational principles that do not have classical solutions, have instead generalized solutions that are Young measures. For details on Young measures and their properties the reader is referred to the monographs [146, 149, 80] (see also [29, 194]).

Young measures were also used in other areas of the study of partial differential equations. For instance, TARTAR used them in conjunction with the method of compensated compactness to prove existence of entropy solutions of a hyperbolic equation, namely a conservation law. His result was later extended by DIPERNA to cover systems of conservation laws [69] (see also [70, 71, 185]). On a different note, Young measures were later generalized to account for concentration effects [72, 13], which are particularly relevant for variational problems of linear growth.

To formally introduce (classical or oscillation) Young measures, we appeal to a variant of the Fundamental Theorem 2.4.1, which lacks a lot of generality but is adequate for our purposes: Let $w_j \rightharpoonup w$ in $L^p(\Omega, W)$ for some $1 \leq p < \infty$ ($\overset{*}{\rightharpoonup}$ for $p = \infty$), where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Then there exists a non-relabelled subsequence of w_j and a weakly- $*$ measurable map $x \mapsto \nu_x \in \mathcal{P}(W)$ (a *parametrized measure* $\boldsymbol{\nu} = (\nu_x)_{x \in \Omega}$, where ν_x is a probability measure for $\mathcal{L}^n$ -a.e. $x \in \Omega$) such that

$$\liminf_{j \rightarrow \infty} \int_{\Omega} f(w_j(x)) \, dx \geq \int_{\Omega} \langle f, \nu_x \rangle \, dx,$$

for all $f \in C(W)$ that is bounded from below. Moreover, $\{f(w_j)\}_j$ is uniformly integrable if and only if

$$\lim_{j \rightarrow \infty} \int_{\Omega} f(w_j(x)) \, dx = \int_{\Omega} \langle f, \nu_x \rangle \, dx.$$

From a functional analytic perspective, the Young measure ν is a weakly- $*$ limit (understood in $L^1(\Omega, C_0(W))^* \simeq L_{w^*}^\infty(\Omega, \mathcal{M}(W))$) of the sequence of parametrized measures $(\delta_{w(x)})_{x \in \Omega}$. Such limit points exist by the Banach–Alaoglu Theorem. We say that the Young measure ν is *generated by the sequence* $(w_j)_j$.

From the point of view of the Fundamental Theorem of Young Measures (see, e.g., Theorem 2.4.1), one can assert that Young measures are, to some extent, dual objects to lower semi-continuity of energy functionals. On the other hand, in Section 1.1.1, we argued extensively that lower semi-continuity of functionals defined on vector fields constrained by a compensating operator $\mathcal{A}$ is essentially dual to $\mathcal{A}$ -quasiconvexity of integrands that satisfy suitable growth conditions. We will now describe which Young measures are dual objects to $\mathcal{A}$ -quasiconvex integrands of p -growth. Firstly, we note that the link between the two is made through Jensen-type inequalities

$$\langle f, \nu_x \rangle \geq f(\bar{\nu}_x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega, \quad (1.13)$$

where $\bar{\mu} := \langle \text{Id}, \mu \rangle$ denotes the *centre of mass* (barycentre) of a measure μ . Such inequalities are easily observed in the unconstrained case $\mathcal{A} \equiv 0$, where lower semi-continuity of $\mathcal{E}$ is equivalent to convexity of f (recall the notation from (1.1)). On the other hand, assuming that we are given a homogeneous Young measure ν (effectively a probability measure), then Jensen’s Inequality implies that for convex f , we have that

$$\langle f, \nu \rangle \geq f(\bar{\nu}). \quad (1.14)$$

Conversely, it is easy to see that if (1.14) holds for all convex f , then ν must be a probability measure.

A first significant step towards describing Young measures generated by constrained sequences was made by KINDERLEHRER and PEDREGAL in [111], where it is shown, roughly speaking, that a Young measure ν such that $\text{spt } \nu_x \subset K$ for some compact set $K \subset W$ is generated by a sequence $Du_j \xrightarrow{*} Du$ in $L^\infty(\Omega, W)$ for Lipschitz maps $u_j \in W^{1,\infty}(\Omega, V)$ (in this case, say $V \simeq \mathbb{R}^N$ and $W \simeq \mathbb{R}^{N \times n}$) if and only if the Jensen-type inequality (1.13) holds for all quasiconvex $f \in C(W)$. In turn, we will give a precise statement in the case of $W^{1,p}$ gradient Young measures, which are those Young measures generated by weakly convergent sequences in L^p :

Theorem 1.1.8 ([112, 117, 182]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. Let $\nu = (\nu_x)_{x \in \Omega}$ be a Young measure. Then ν is a $W^{1,p}$ gradient Young measure if and only if*

- (a) *there exists $u \in W^{1,p}(\Omega, \mathbb{R}^{N \times n})$ such that $Du(x) = \langle \text{Id}, \nu_x \rangle$ for $\mathcal{L}^n$ -a.e. $x \in \Omega$;*
- (b) *ν has finite p -th moment, i.e.,*

$$\int_{\Omega} \int_{\mathbb{R}^{N \times n}} |w|^p d\nu_x(w) dx < \infty;$$

- (c) *The Jensen-type inequality (1.13) holds for all quasiconvex $f \in C(\mathbb{R}^{N \times n})$ of p -growth, $|f| \leq c(1 + |\cdot|^p)$.*

Theorem 1.1.8 was extended by FONSECA and MÜLLER to constraints given by compensating operators of constant rank as follows:

Theorem 1.1.9 ([85, Thm. 4.1]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $1 < p < \infty$, and $\mathcal{A}$ be a differential operator as in (1.4) that satisfies the constant rank condition (1.6). Let $\nu = (\nu_x)_{x \in \Omega}$ be a Young measure. Then ν is generated by a p -uniformly integrable sequence $w_j \in L^p(\Omega, W)$ such that $\mathcal{A}w_j = 0$ if and only if*

- (a) *there exists $w \in L^p(\Omega, W)$ such that $\mathcal{A}w = 0$ and $w(x) = \langle \text{Id}, \nu_x \rangle$ for $\mathcal{L}^n$ -a.e. $x \in \Omega$;*
- (b) *ν has finite p -th moment, i.e.,*

$$\int_{\Omega} \int_W |w|^p d\nu_x(w) dx < \infty;$$

- (c) *The Jensen-type inequality (1.13) holds for all $\mathcal{A}$ -quasiconvex $f \in C(W)$ of p -growth, $|f| \leq c(1 + |\cdot|^p)$.*

The statements of both Theorems 1.1.8 and 1.1.9 also hold for $p = 1$. We chose to exclude this possibility since the weak- L^1 topology is not compatible with the standard coercivity of energy functionals of linear growth. Indeed, assuming that f has linear growth

$$|\eta| \leq f(\eta) \leq c(|\eta| + 1) \quad \text{for } \eta \in W,$$

minimizing sequences $(w_j)_j$ of $\mathcal{E}$ as in (1.1) are bounded in L^1 , but this does not guarantee weak compactness, unless $(w_j)_j$ is uniformly integrable. On the other hand, $(w_j)_j$ is weakly- $*$ compact in $\mathcal{M}(\Omega, W) \simeq C_0(\Omega, W)^*$, which is consistent with practical examples. For instance, both image segmentation applications [141, 18] ($w_j = Du_j$, for $u_j \in \text{BV}(\Omega, \mathbb{R}^n)$) and perfect plasticity applications [190] ($w_j = \mathcal{E}u_j$, where $\mathcal{E}u := \frac{1}{2}(Du + Du^t)$ and $u \in \text{BD}(\Omega)$) must allow for jump discontinuities, which are of course inconsistent with uniform integrability of the minimizing sequence $(w_j)_j$.

In this case, a reasonable extension (see, e.g., [21]) of the topology of compensated compactness theory is the following:

- (a) $w_j \xrightarrow{*} w$ in $\mathcal{M}(\Omega, W)$,
- (b) $\mathcal{A}w_j \rightarrow 0$ in $W_{\text{loc}}^{-1,q}(\Omega, X)$ for some $1 < q < n/(n-1)$,

with respect to which lower semi-continuity theorems are significantly harder to prove for reasons that we now explain.

We begin with a quite important point that was deliberately omitted so far, in order not to over-complicate exposition. We will slightly postpone a precise statement to Theorem 1.1.10 below. If $f \in C(W)$ is $\mathcal{A}$ -quasiconvex and has p -growth, $1 < p < \infty$, we have that

$$\mathcal{E}[w] = \int_{\Omega} f(w(x)) dx$$

$$= \bar{\mathcal{E}}[w] := \inf \left\{ \liminf_{j \rightarrow \infty} \mathcal{E}[w_j] : w_j \rightharpoonup w \text{ in } L^p, \mathcal{A}w_j \rightarrow \mathcal{A}w \text{ in } W^{-1,p} \right\}.$$

In this case $\bar{\mathcal{E}}$ denotes the lower semi-continuous relaxation (Lebesgue–Serrin extension) of $\mathcal{E}$ with respect to the stated topology. If f is not $\mathcal{A}$ -quasiconvex, then $\bar{\mathcal{E}}$ can be strictly smaller than $\mathcal{E}$ and the physical meaning of relaxation is to link the microscopic description of the minimized energy $\mathcal{E}$ to the macroscopic one given by $\bar{\mathcal{E}}$.

This phenomenon fails for weakly- $*$ star convergence of measures, as is already seen for $\mathcal{A} \equiv 0$ and $f = |\cdot|$ in the one dimensional case, essentially by the typical example of failure of Fatou’s Lemma for a simple concentrating sequence. By taking $\Omega = (0, 1)$, $w = \delta_0$ and $w_j = n\chi_{(0, 1/n)}$, we see that $w_j \xrightarrow{*} w$ in $\mathcal{M}(0, 1)$ and $w_j \rightarrow 0$ pointwisely, but

$$\bar{\mathcal{E}}[w_j] = 1 > \mathcal{E}[0].$$

This example shows that, since f is convex, the definition of $\mathcal{E}$ should be extended to accomodate for $\mathcal{E}[\delta_0] = 1$. To achieve this, we follow [18, Ch. 2.6]. Let $\mu = \mu^a + \mu^s$ be a Radon–Nikodým decomposition of $\mu \in \mathcal{M}(\bar{\Omega}, W)$, and define, for convex $f \in C(W)$ of linear growth

$$\mathcal{E}[\mu] := \int_{\Omega} f \left(\frac{d\mu}{d\mathcal{L}^n}(x) \right) dx + \int_{\bar{\Omega}} f^{\infty} \left(\frac{d\mu^s}{d|\mu|}(x) \right) d|\mu|(x), \quad (1.15)$$

where $f^{\infty} \in C(W)$ is a positively 1-homogeneous *recession function* given by

$$f^{\infty}(\eta) := \lim_{t \rightarrow \infty} \frac{f(t\eta)}{t}. \quad (1.16)$$

In more generality, much care has to be taken in order to properly define recession functions. In the present situation this is not the case and, moreover, this technical issue is not in the scope of the present thesis.

Indeed, defining $\mathcal{E}$ as in (1.15) and (1.16), we obtain that (for convex f):

$$\mathcal{E}[\mu] = \bar{\mathcal{E}}[\mu], \quad (1.17)$$

where the relaxation is taken with respect to $\mu_j \xrightarrow{*} \mu$ in $\mathcal{M}(\bar{\Omega}, W)$ [18, Thm. 2.38], a result that is attributable to RESHETNYAK.

A lot of work has been done in the case when $w_j = Du_j$ for $u_j \xrightarrow{*} u$ in $BV(\Omega, \mathbb{R}^N)$, i.e., for $\mathcal{A} = \text{curl}$. Indeed, even if the integrand f is convex, the concentration effects due to weakly- $*$ convergence of Du_j make lower-semicontinuous relaxation results quite intricate, as can be seen from [97]. Of course, one would expect that $\mathcal{E}[Du] = \bar{\mathcal{E}}[Du]$ holds with respect to weakly- $*$ convergence in BV for quasiconvex integrands, but this was established much later in [17, 84]:

Theorem 1.1.10 ([17, Thm. 4.1]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $f \in C(\mathbb{R}^{N \times n})$ be a quasiconvex function of linear growth $0 \leq f \leq c(|\cdot| + 1)$. Then*

$$\int_{\Omega} f(\nabla u(x)) dx + \int_{\Omega} f^{\infty} \left(\frac{dDu^s}{d|Du|}(x) \right) d|Du|(x)$$

$$= \inf \left\{ \liminf_{j \rightarrow \infty} \int_{\Omega} f(\nabla u_j(x)) \, dx : C^1(\Omega, \mathbb{R}^N) \ni u_j \rightarrow u \text{ in } L^1(\Omega, \mathbb{R}^N) \right\}$$

for all $u \in \text{BV}(\Omega, \mathbb{R}^N)$.

Here the recession function is defined as in (1.16) with $\lim$ replaced by $\limsup$ (see [144]) and we write ∇u for the approximate derivative of $u \in \text{BV}$ (which agrees with the Radon–Nikodým derivative of Du with respect to Lebesgue measure almost everywhere [75]). The result stated is slightly stronger than what we announced, i.e., the relaxation being taken with respect to $u_j \xrightarrow{*} u$ in $\text{BV}(\Omega, \mathbb{R}^N)$, but the difference is insignificant if one takes into account the Rellich–Kondrachov Compactness Theorem.

We refer the reader to [18, Ch. 5] and the references therein for the ample history of Theorem 1.1.10, which is much more difficult to prove than if we considered the relaxation with respect to weak convergence in $W^{1,1}$, which can be tackled using MORREY’s ideas [140]. Here, we only recall the blow–up argument in [83], where it was shown that the functional

$$\tilde{\mathcal{E}}[u] := \int_{\Omega} f(\nabla u(x)) \, dx \quad \text{defined for } u \in W^{1,1}(\Omega, \mathbb{R}^N)$$

is lower semi–continuous along $W^{1,1}$ sequences $u_j \rightarrow u \in L^1(\Omega, \mathbb{R}^N)$ and Alberti’s Rank–one Theorem [11] (see also [133]), which enables one to analyze the singular part with respect to Lebesgue measure of measures

$$S \mapsto \int_S f \left(\frac{dDu}{d|Du|}(x) \right) d|Du|(x) \quad \text{for Borel sets } S \subset \mathbb{R}^n.$$

The Rank–one Theorem (or rather its generalization in [65]) will play a particularly important role in this thesis, so we take a moment to describe its statement.

Recall from Section 1.1.1 that if $u = \chi_{H^+} u_0$ for H^+ a half–space bounded by a hyper–plane H with unit normal ν , then up to a choice of sign, we have that $Du = (\mathcal{H}^{n-1} \llcorner H) u_0 \otimes \nu$ (up to a choice of sign). In view of the Structure Theorem for BV–maps, this observation generalizes to a decomposition

$$\frac{dDu}{d|Du|}(x) = (u^+(x) - u^-(x)) \otimes \nu$$

for $\mathcal{H}^{n-1}$ –a.e. jump points $x \in J_u$ of u (see, e.g., [18, Thm. 3.77]; see [18, Ch. 3.7] for details on fine properties of BV–functions). The main result in [11] states that the same property holds for the Cantor part of Du :

$$\text{rank} \left(\frac{dDu}{d|Du|}(x) \right) \leq 1 \quad \text{for } |Du^s| \text{--a.e. } x. \quad (1.18)$$

This result is crucially used in [17, 84] to prove variants of Theorem 1.1.10.

The absence of a generalization of (1.18) to other compensating operators has prevented the development of satisfactory generalizations of Theorem 1.1.5 in the spirit of Theorem 1.1.10. From this point of view, a lot of effort was required to obtain proofs of lower semi–continuity for problems of linear growth under other differential constraints $\mathcal{A}w = 0$ [160, 161, 24].

Quite recently, DE PHILIPPIS and RINDLER proved the following long awaited generalization of (1.18):

Theorem 1.1.11 ([65]). *Let $\mathcal{A}$ be a homogeneous linear differential operator on $\mathbb{R}^n$ with constant coefficients and let μ be a (vectorial) Radon measure on $\mathbb{R}^n$ such that $\mathcal{A}\mu$ is a Radon measure on $\mathbb{R}^n$. Then*

$$\frac{d\mu}{d|\mu|}(x) \in \Lambda_{\mathcal{A}} \quad \text{for } |\mu^s| \text{-a.e. } x \in \mathbb{R}^n.$$

Here $\mathcal{A}$ can be taken to be of arbitrary order, in which case the definition (1.5) extends as is. The statement of Theorem 1.1.11 extends to inhomogeneous compensating operators $\mathcal{A}$ (i.e., having lower order terms, in which case the wave cone is defined only by the principal part of $\mathcal{A}$) and to operators with suitably regular variable coefficients.

Interestingly, Theorem 1.1.11 is true in the absence of the constant rank assumption for $\mathcal{A}$, as is the case for VAN SCHAFTINGEN's co-canceling estimate [201, Thm. 1.4], which will be discussed in Section 1.1.4.

The relevance of Theorem 1.1.11 for the study of Calculus of Variations under differential constraints, measure solutions of linear partial differential equations, and examples in geometric measure theory cannot be over-stated. We refer the reader to the survey [67] for a detailed exposition of numerous consequences. Here, we only make a few remarks: Firstly, Theorem 1.1.11 substantiates the empirical remark that the wave cone $\Lambda_{\mathcal{A}}$ encodes the possible values of condensed singularities. Secondly, by [10, Thm. 3], in general, there exist measures μ such that

$$\mathcal{A}\mu = 0, \quad \text{but} \quad \mathcal{A}(\mu^s) \neq 0 \text{ in any open set.} \quad (1.19)$$

In particular, the compensating constraint does not transfer to the singular part of measures converging in the topology of compensated compactness in an obvious way. In turn, Theorem 1.1.11 shows to what extent the differential constraint is transferred to their singular parts by weakly-* convergence of tangent measures. Of course, in view of the proof of Theorem 1.1.10, which relies heavily on (1.18), proofs of its quite general extensions to constant rank compensating conditions are now readily available [21, Thm. 1.2, 1.7, 1.8]. The proofs of these results make use of generalized Young measures, i.e., parametrized measures that account for both oscillation and concentration effects, as introduced by DIPERNA and MAJDA in [72] and later developed in [13, 124, 122], among others.

We recall a streamlined definition of generalized Young measures (termed in what follows simply *Young measures*) for the convenience of the reader. We will define Young measures by duality with the set of integrands

$$\mathbf{E}(\Omega, W) := \{f \in C(\Omega \times B_W) : Tf \text{ satisfies (1.20)}\},$$

where B_W denotes the unit ball in W ,

$$Tf(x, w) := (1 - |w|)f\left(x, \frac{w}{1 - |w|}\right) \quad \text{for } x \in \Omega, w \in B_W,$$

and

$$Tf \text{ extends to a uniformly continuous function on } \overline{\Omega \times B_W}. \quad (1.20)$$

Such integrands f admit suitable recession functions f^∞ that we do not define carefully until Chapter 3 (see [122, 21] for detailed discussions). Then $\mathbf{E}(\Omega, W)$ is a separable Banach space when endowed with the norm $f \mapsto \|Tf\|_{L^\infty(\Omega \times B_W)}$. Then for each sequence of uniformly bounded measures $(\mu_j)_j \subset \mathcal{M}(\Omega, W)$ (in the total variation norm), there exists a non-relabelled sub-sequence and a parametrized measure $\nu = (\nu_x, \lambda_\nu, \nu_x^\infty)_{x \in \bar{\Omega}}$ such that, for all $f \in \mathbf{E}(\Omega, W)$,

$$\begin{aligned} \int_{\Omega} f\left(x, \frac{d\mu_j}{d\mathcal{L}^n}(x)\right) dx + \int_{\bar{\Omega}} f\left(x, \frac{d\mu_j^s}{d|\mu_j|}(x)\right) d|\mu_j|(x) \\ \rightarrow \int_{\Omega} \langle f(x, \cdot), \nu_x \rangle dx + \int_{\bar{\Omega}} \langle f^\infty(x, \cdot), \nu_x^\infty \rangle d\lambda_\nu(x) =: \langle f, \nu \rangle. \end{aligned}$$

Here $(\nu_x)_{x \in \Omega} \subset \mathcal{P}(W)$ is a weakly-* measurable family with respect to $\mathcal{L}^n$ of probability measures (the oscillation measure), $\lambda_\nu \in \mathcal{M}(\bar{\Omega})^+$ is a bounded, non-negative measure (the concentration measure), and $(\nu_x^\infty)_{x \in \bar{\Omega}} \subset \mathcal{P}(\mathbb{S}_W)$ is weakly-* measurable, with respect to λ_ν , family of probability measures on the unit sphere $\mathbb{S}_W$ of W (the concentration-angle measure). We say that $(\mu_j)_j$ *generates the Young measure* ν and write $\mathbf{Y}(\Omega, W) \subset \mathbf{E}(\Omega, W)^*$ for the space of Young measures.

In this setting, it is natural to ask what is the generalization of the Kinderlehrer–Pedregal Theorem 1.1.8, namely, what is the characterization by duality with linear growth quasiconvex functions of Young measures generated by gradients of BV-maps? This would amount to an effective description of the interaction between non-linear integrands and weakly-* convergence of gradients as in the lower semi-continuity Theorem 1.1.10.

A positive answer to this question was given only recently by KRISTENSEN and RINDLER in [121, 122] (see also [162]):

Theorem 1.1.12 ([122, Thm. 9]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\nu \in \mathbf{Y}(\Omega, \mathbb{R}^{N \times n})$ be a Young measure such that $\lambda_\nu(\partial\Omega) = 0$. Then there exists a sequence $u_j \in \text{BV}(\Omega, \mathbb{R}^N)$ such that Du_j generates ν if and only if*

- (a) *there exists $u \in \text{BV}(\Omega, \mathbb{R}^N)$ such that the barycentre $\langle \text{Id}, \nu \rangle$ equals Du ;*
- (b) *$\langle f, \nu_x \rangle + \langle f^\infty, \nu_x^\infty \rangle \lambda_\nu^a(x) \geq f(\langle \text{Id}, \nu_x \rangle + \langle \text{Id}, \nu_x^\infty \rangle \lambda_\nu^a(x))$ for $\mathcal{L}^n$ -a.e. $x \in \Omega$;*
- (c) *$\langle f^\infty, \nu_x^\infty \rangle \geq f^\infty(\langle \text{Id}, \nu_x^\infty \rangle)$ for λ_ν^s -a.e. $x \in \bar{\Omega}$;*

for all quasiconvex $f \in C(\mathbb{R}^{N \times n})$ of linear growth $|f| \leq c(|\cdot| + 1)$.

In the statement of the Theorem, both Radon–Nikodým decompositions $\mu = \mu^a + \mu^s$ are taken with respect to the $\mathcal{L}^n$ -measure on Ω . It is remarkable that the Jensen-type inequality at singular points in (c) is not necessary to characterize BV–Young measures, as was shown by KIRCHHEIM and KRISTENSEN in [113, Sec. 6]. Moreover, their argument relies on a powerful abstract result of convex analysis [113, Thm. 1.1] which can be applied to constraints given by other compensation operators, provided that the wave cone $\Lambda_{\mathcal{A}}$ spans W ².

²This assumption is far from restrictive, as can be seen from [21, Sec. 2.5].

To explain this point, recall that the method of proof used in [21] is to establish Jensen-type inequalities at singular and regular points. Let $\mathcal{A}$ be a k -homogeneous operator of constant rank on $\mathbb{R}^n$ from W to X and let $w_j, w \in \mathcal{M}(\Omega, W)$ be such that $w_j \xrightarrow{*} w$ in $\mathcal{M}$ and $\mathcal{A}w_j \rightarrow 0$ in $W^{-k,q}$ for some $1 < q < n/(n-1)$. Let w_j generate a Young measure $\nu = (\nu_x, \lambda_\nu, \nu_x^\infty)_{x \in \bar{\Omega}}$ (termed $\mathcal{A}$ -free Young measure [21, Def. 2.23]). It is shown in [21, Sec. 3] by a localization procedure that, for all $\mathcal{A}$ -quasiconvex $f \in C(W)$ with linear growth $0 \leq f \leq c(|\cdot| + 1)$, the following Jensen-type inequalities hold:

- (a) $\langle f, \nu_x \rangle + \langle f^\infty, \nu_x^\infty \rangle \lambda_\nu^a(x) \geq f(\langle \text{Id}, \nu_x \rangle + \langle \text{Id}, \nu_x^\infty \rangle \lambda_\nu^a(x))$ for $\mathcal{L}^n$ -a.e. $x \in \Omega$;
- (b) $\langle f^\infty, \nu_x^\infty \rangle \geq f^\infty(\langle \text{Id}, \nu_x^\infty \rangle)$ for λ_ν^s -a.e. $x \in \bar{\Omega}$.

A suitable extension to signed integrands seems quite feasible. In particular, the inequality in (a) is easily established for signed integrands by using Lemma 3.3.4 and the standard localization result in Lemma 3.2.3.

As for attempting to establish a converse, we make two remarks: Firstly, if one would like to show that the Jensen-type inequalities above characterize $\mathcal{A}$ -free Young measures, the observations in [113, Sec. 6] extend to show that only (a) would suffice for a characterization. More plainly, since f^∞ is an $\mathcal{A}$ -quasiconvex and positively 1-homogeneous function, by Proposition 1.1.2 and the main result [113, Thm. 1.1], we have that f^∞ is convex at points in $\Lambda_{\mathcal{A}}$. Using this fact and Theorem 1.1.11, the inequality in (b) follows, meaning that a characterization of $\mathcal{A}$ -free Young measures by duality with $\mathcal{A}$ -quasiconvex functions would not depend by an inequality at singular points.

Secondly, only few examples, except for Theorem 1.1.12, of such characterizations are known. One such example is the characterization of BD-Young measures in [66], where it was shown using [113, Thm. 1.1] that the Jensen-type inequality at regular points (a) with respect to symmetric-quasiconvex functions of linear growth does indeed characterize Young measures generated by sequences $(\mathcal{E}u_j)_j$ for u_j converging weakly- $*$ in $\text{BD}(\Omega)$ (this case corresponds to $\mathcal{A} = \text{curl} \circ \text{curl}$ on $\mathbb{R}^n$, i.e., the Saint-Venant compatibility conditions). An earlier result was established in [25], which has the drawbacks that the method used is confined to first order compensating operators and requires a Morrey condition [25, Eq. (1.4)] which cannot be checked a priori for all $\mathcal{A}$ -free measures except for some examples. On a different note, effective descriptions of concentration and oscillation effects of weak- L^p convergence under differential constraints were given in [86, 110, 79] for $p > 1$. In this case, oscillations and concentration carry differential structure separately, i.e., phenomena such as (1.19) are not possible (at the level of the parametrized measures).

Pertaining to the characterization of $\mathcal{A}$ -free Young measures by duality with $\mathcal{A}$ -quasiconvex functions of linear growth, one of the results of this thesis, obtained jointly with KRISTENSEN, can be stated vaguely as follows: Under the constant rank assumption on $\mathcal{A}$, the Jensen-type inequality at regular points (a) is sufficient to characterize $\mathcal{A}$ -free Young measures with absolutely continuous barycentre. The technical assumption on the barycentre means that the Young measure is generated by a sequence $w_j \xrightarrow{*} w$ with $w \in L^1$ and is, unfortunately, quite restrictive; one would expect the result to be true for w being simply a Radon measure. On the other hand, the result is an improvement of the case when

$w_j \rightharpoonup w$ in L^1 [85, Thm. 4.1], when $\lambda_\nu \equiv 0$. In our case, the concentration measure can be an arbitrary positive measure (see [122, Prop. 5]). We believe that this first step can be extended to the general case $w \in \mathcal{M}$.

In particular, we show that if a Young measure ν with $\mathcal{A}$ -free barycentre $w \in L^1$ satisfies (a), then ν is generated by a sequence $w + \mathbb{B}u_j$, where $\mathbb{B}$ is the constant rank operator constructed in [156] that satisfies (1.11). The result is stated precisely in Section 1.3 and is proved in Chapter 3.

1.1.3. L^p -coerciveness estimates As outlined in Sections 1.1.1 and 1.1.2, we are interested in variational problems to minimize functionals of the form

$$\tilde{\mathcal{E}}[u] = \int_{\Omega} f(x, \mathbb{B}u(x)) \, dx \quad (1.21)$$

subject to various boundary conditions. Here $\Omega \subset \mathbb{R}^n$ is a bounded open set, $u: \mathbb{R}^n \rightarrow V$ is a weakly differentiable function, and $\mathbb{B}$ is an l -homogeneous linear partial differential operator on $\mathbb{R}^n$ from V to W (finite dimensional normed vector spaces over $\mathbb{R}$), i.e.,

$$\mathbb{B}u := \sum_{|\alpha|=l} \partial^\alpha \mathbb{B}_\alpha u, \quad \mathbb{B}(\xi) := \sum_{|\alpha|=l} \xi^\alpha \mathbb{B}_\alpha \in \text{Lin}(V, W) \text{ for } \xi \in \mathbb{R}^n \quad (1.22)$$

for linear maps $\mathbb{B}_\alpha \in \text{Lin}(V, W)$ and multi-indices $\alpha \in \mathbb{N}_0^n$. The integrand f is assumed, as in the previous sections, to have p -growth $1 \leq p < \infty$ from above and below, i.e.,

$$|\eta|^p \leq f(x, \eta) \leq c(|\eta|^p + 1) \quad \text{for all } \eta \in W, x \in \Omega. \quad (1.23)$$

In this context, the Direct Method of the Calculus of Variations can be applied to $\tilde{\mathcal{E}}$ defined on (a Dirichlet class) in the Sobolev space $W^{l,p}(\Omega, V)$. However, in the absence of a coerciveness inequality, which we write formally³ as

$$\|D^l u\|_{L^p} + ? \leq \|\mathbb{B}u\|_{L^p} + ? \quad \text{for } u \in ? \subset W^{l,p}, \quad (1.24)$$

one cannot conclude that minimizing sequences of $\tilde{\mathcal{E}}$ are bounded in $W^{l,p}$. In this case, no (weak) compactness of the minimizing sequences is available, and it is not at all clear how one can conclude that minimizers exist. Assuming availability of a suitable coerciveness inequality, in view of the lower semi-continuity results in Section 1.1.1, it is reasonable to suspect that weak limit points of minimizing sequences are minimizers of $\tilde{\mathcal{E}}$, provided that the integrand f satisfies an appropriate quasiconvexity assumption. For constant rank operators $\mathbb{B}$, there exists an operator $\mathcal{A}$ such that the exact relation (1.11) holds, so that the correct quasiconvexity assumption in this case is $\mathcal{A}$ -quasiconvexity (see Theorems 1.1.5). It is likely that one can address the lower semi-continuity problem for $\tilde{\mathcal{E}}$ also for operators that are not of constant rank using classical methods (e.g., [140, 58, 61]), but we do not pursue this here. An important reason is that the author is not aware of any theory of coerciveness inequalities (1.24) for operators $\mathbb{B}$ that may fail the constant rank

³Many variants of inequalities such as (1.24) are used in different contexts, so we chose to give a rather general, if not vague, presentation.

condition. On the other hand, the class of constant rank operators is broad enough to include numerous examples in Physics [31, 168, 169, 88, 85, 99, 19, 177, 137].

Consequently, to round-off the overview of the application of the Direct Method to functionals such as $\tilde{\mathcal{E}}$, we will next discuss coerciveness inequalities (1.24) for operators $\mathbb{B}$ of constant rank in more detail.

A first example that we recall is Gårding's inequality [92], which states that

$$\|D^{2m}u\|_{L^2(\mathbb{R}^n)} \leq c \|\langle D^m u, Q D^m u \rangle\|_{L^2(\mathbb{R}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n), \quad (1.25)$$

provided that the symmetric tensor Q is positive definite, in which case we say that the operator $\mathbb{B}u := \langle D^m u, Q D^m u \rangle$ is elliptic. Another old example is given by Korn's L^2 -inequality,

$$\|Du\|_{L^2(\mathbb{R}^n)} \leq c \|\mathcal{E}u\|_{L^2(\mathbb{R}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n),$$

where $\mathcal{E}u := \frac{1}{2}(Du + Du^t)$, which is relevant in elasticity theory [99]. KORN's example shows that neither the algebraic structure (quadratic forms), nor the fact that the maps considered are scalar are necessary for an inequality

$$\|D^l u\|_{L^p(\mathbb{R}^n)} \leq c \|\mathbb{B}u\|_{L^p(\mathbb{R}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V). \quad (1.26)$$

Indeed, it was essentially proved by CALDERÓN and ZYGMUND in [49] that (1.26) holds for $1 < p < \infty$ if and only if $\mathbb{B}$ is overdetermined *elliptic* [6, Def. 6.3], i.e.,

$$\ker \mathbb{B}(\xi) = \{0\} \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \quad (1.27)$$

We will refer to such operators plainly as “elliptic”. Do note that elliptic operators have constant rank by the Rank-Nullity Theorem.

For the remainder of this section, the exponent p is assumed to satisfy $1 < p < \infty$. Note that the inequality (1.26) essentially settles the issue of coercivity for the functional $\tilde{\mathcal{E}}$ under Dirichlet boundary conditions. An overview of the existence and relaxation theory for the Dirichlet problem for $\tilde{\mathcal{E}}$ will appear in [154] (which is not included in this thesis).

It may, however, be of interest to consider more general boundary conditions. For the issues of coercivity, this would amount to proving inequalities of the form

$$\|D^l u\|_{L^p(\Omega)} \leq c (\|\mathbb{B}u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}) \quad \text{for } \mathbb{L}u = 0 \text{ on } \partial\Omega. \quad (1.28)$$

Here it is more natural to allow for $\mathbb{B}$ and the boundary differential operator $\mathbb{L}$ to have space-dependent coefficients, for reasons that we now briefly explain. It was observed by ARONSZAJN in [20] in the case $\mathbb{L} \equiv 0$ and $p = 2$ (although the first to make similar observations was probably LOPATINSKIĬ in [126]) that, for the estimate (1.28) to hold, one must impose, apart from ellipticity at each $x \in \Omega$, a condition on $\mathbb{B}$ at each $x \in \partial\Omega$. We state a generalization of ARONSZAJN's results, which can be proved using the ideas in [102, Ch. XX]:

Theorem 1.1.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $\mathbb{B}$ be an operator as in (1.22), with coefficients $\mathbb{B}_\alpha \in C^\infty(\bar{\Omega}, \text{Lin}(V, W))$, and $1 < p < \infty$. Then the inequality*

$$\|D^l u\|_{L^p(\Omega)} \leq c (\|\mathbb{B}u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}) \quad \text{for all } u \in C^\infty(\mathbb{R}^n, V)$$

holds if and only if both of the following hold:

- (a) $\mathbb{B}$ is elliptic at each $x \in \Omega$, i.e., $\ker \mathbb{B}(x; \xi) = \{0\}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$;
- (b) for each $x \in \partial\Omega$, we have that $\ker_{\mathbb{C}} \mathbb{B}(x; \mathbf{t} + i\mathbf{n}_x) = \{0\}$ for all $\mathbf{t}$ in the tangent plane of $\partial\Omega$ at x ; $\mathbf{n}_x$ denotes a unit normal to this plane.

Here we denoted the symbol map of $\mathbb{B}$ by

$$\mathbb{B}(x; \xi) := \sum_{|\alpha|=l} \xi^\alpha \mathbb{B}_\alpha(x) \quad \text{for } x \in \bar{\Omega}, \xi \in \mathbb{R}^n.$$

We also specify, for the sake of clarity, that $\ker_{\mathbb{C}} L := \{v \in V + iV : Lv = 0\}$ for $L \in \text{Lin}(V, W)$.

In more generality, an the estimate (1.28) holds if and only if $\mathbb{B}$ is elliptic in Ω and the pair $(\mathbb{B}, \mathbb{L})$ satisfies the Lopatinskiĭ–Shapiro complementing condition on $\partial\Omega$. This was proved in [7, 8] for square (determined) operators $\mathbb{B}$ (see [100, Sec. 2.3] for results for overdetermined elliptic operators). Of course, there is a vast literature on the subject and we refer the reader to the monograph [102].

Motivated by the problem of extending the result in [20] to Lipschitz domains, SMITH investigated the statement of Theorem 1.1.13 under the additional assumption that $\mathbb{B}$ has constant coefficients [171, 172], assumption which will be used henceforth throughout the entire thesis. We will use the terminology of [45]:

Definition 1.1.14. *A homogeneous linear differential operator with constant coefficients $\mathbb{B}$ on $\mathbb{R}^n$ as in (1.22) is said to be $\mathbb{C}$ -elliptic if and only if*

$$\ker_{\mathbb{C}} \mathbb{B}(\xi) = \{0\} \quad \text{for all } \xi \in \mathbb{C}^n \setminus \{0\}.$$

Examples of operators $\mathbb{B}$ that are $\mathbb{C}$ -elliptic are D^l , $\mathcal{E}$ and $\mathcal{E}^D := \mathcal{E} - \text{Id}_n \text{div}$ on $\mathbb{R}^n$ for $n \geq 3$. If $n = 2$, $\mathcal{E}^D$ is elliptic, but not $\mathbb{C}$ -elliptic.

Of course, even for operators with constant coefficients, geometric information is lost when assuming $\mathbb{C}$ -ellipticity. This can be seen for instance from the estimate for $1 < p < \infty$

$$\|D^l u\|_{L^p(\{x \cdot \mathbf{n} > 0\})} \leq c \|\mathbb{B}u\|_{L^p(\{x \cdot \mathbf{n} > 0\})} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V)$$

which holds provided that $\mathbb{B}$ is elliptic (1.27) and $\ker_{\mathbb{C}} \mathbb{B}(\xi + i\mathbf{n}) = \{0\}$ for all $\xi \in \mathbf{n}^\perp$ (such an estimate can be obtained by using methods from [102]).

On the other hand, assuming a strong algebraic condition enabled SMITH to use powerful algebraic tools to obtain strong analytical results. We record a Proposition that can be extracted from [172, 108] that proved to be crucial for the more recent developments in [45, 94]:

Proposition 1.1.15. *Let $\mathbb{B}$ be as in (1.22). The following are equivalent:*

- (a) $\mathbb{B}$ is $\mathbb{C}$ -elliptic;
- (b) There exists an integer d and a homogeneous linear differential operator $\mathbb{A}$ on $\mathbb{R}^n$ with constant coefficients such that $D^d = \mathbb{A} \circ \mathbb{B}$;
- (c) $\dim\{u \in \mathcal{D}'(\Omega, V) : \mathbb{B}u = 0\} < \infty$ for an (all) open set(s) $\Omega \subset \mathbb{R}^n$.

Moreover, if the statements above hold, the kernel in (c) consists of V -valued polynomials and is independent of Ω .

We remark that in the case of the symmetric gradient, one can compute explicitly D^2u in terms of $\mathcal{E}u$ as in (b), i.e.,

$$\partial_{ij}^2 u_k = \partial_i \left(\frac{\partial_j u_k + \partial_k u_j}{2} \right) - \partial_k \left(\frac{\partial_i u_j + \partial_j u_i}{2} \right) + \partial_j \left(\frac{\partial_k u_i + \partial_i u_k}{2} \right).$$

This idea was used by NEČAS in conjunction with the Negative Norm Theorem to obtain the Korn inequality on bounded domains [99]:

$$\|Du\|_{L^2(\Omega)} \leq c \left(\|\mathcal{E}u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right) \quad \text{for all } u \in C^\infty(\mathbb{R}^n, \mathbb{R}^n).$$

We do not know if this idea was known to SMITH, who showed, in more generality, that the following result holds:

Theorem 1.1.16. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $\mathbb{B}$ be as in (1.22) have constant coefficients, and $1 < p < \infty$. Then $\mathbb{B}$ is $\mathbb{C}$ -elliptic if and only if*

$$\|D^l u\|_{L^p(\Omega)} \leq c \left(\|\mathbb{B}u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right) \quad \text{for all } u \in C^\infty(\mathbb{R}^n, V).$$

Despite the fact that Theorem 1.1.16 is not the most general, we highlight it for its rather elegant proof (see also [108, 109]), as well as for the relevance of the tools used to Chapter 7 (see also [45]).

Summarizing a bit, the viewpoint taken so far in the discussion of inequalities (1.24) was to impose boundary conditions. In particular, for any of the above estimates ellipticity of $\mathbb{B}$ is necessary (if $1 < p < \infty$). In fact, one can show that ellipticity of $\mathbb{B}$ is equivalent with

$$\{u \in \mathcal{D}' : \mathbb{B}u \in L_{\text{loc}}^p\} = W_{\text{loc}}^{l,p}.$$

In particular, all local properties of the competitor maps of $\tilde{\mathcal{E}}$ (recall (1.21)) are inherited from Sobolev spaces. On the other hand, this is not the case for estimates near the boundary.

More can be said about the coerciveness problem (1.24) for elliptic operators $\mathbb{B}$ (see [5, 165], the lecture notes [147], and the recent papers [203, 204]), but we will refrain from going into more details. Instead, we will record one result for constant rank, but possibly non-elliptic operators.

Firstly, we sketch a proof of the fact that (1.26) holds for $\mathbb{B} = \Delta$. For $u \in C_c^\infty(\mathbb{R}^n)$, we have that $\widehat{\Delta u}(\xi) = -|\xi|^2 \hat{u}(\xi)$, which can be algebraically inverted to $\hat{u}(\xi) = -|\xi|^{-2} \widehat{\Delta u}(\xi)$, so that

$$\widehat{D^2 u} = \frac{\xi \otimes \xi}{|\xi|^2} \widehat{\Delta u}(\xi).$$

By the Hörmander–Mihlin Multiplier Theorem, the estimate (1.26) follows for $1 < p < \infty$.

Note that the same argument can be applied for any overdetermined elliptic operator $\mathbb{B}$, by using a left inverse $\mathbb{B}^\dagger(\xi) = [\mathbb{B}^*(\xi)\mathbb{B}(\xi)]^{-1}\mathbb{B}^*(\xi)$ (and $\mathbb{B}^\dagger(\xi) = 0$ for $\xi = 0$) of the symbol map $\mathbb{B}(\cdot)$:

$$\widehat{D^l u}(\xi) = \mathbb{B}^\dagger(\xi) \widehat{\mathbb{B}u}(\xi) \otimes \xi^{\otimes l}, \quad (1.29)$$

and the Multiplier Theorem yields (1.26) for elliptic operators.

In the absence of ellipticity, one can still retrieve an algebraic relation that generalizes (1.29). For $u \in C_c^\infty(\mathbb{R}^n, V)$ we can write

$$\hat{u}(\xi) - \text{Proj}_{\ker \mathbb{B}(\xi)}[\hat{u}(\xi)] = \mathbb{B}^\dagger(\xi) \widehat{\mathbb{B}u}(\xi),$$

where $\mathbb{B}^\dagger(\cdot)$ denotes the Moore–Penrose generalized inverse (pseudo-inverse) of $\mathbb{B}(\cdot)$ [52] (cp. (1.7), (1.8)). We briefly recall that a matrix $M \in \mathbb{R}^{N \times n}$ has a pseudo-inverse $M^\dagger \in \mathbb{R}^{n \times N}$ uniquely defined by the geometric properties that $MM^\dagger = \text{Proj}_{\text{im } M}$ and $M^\dagger M = \text{Proj}_{\text{im } M^*}$. It is then easy to speculate that

$$\mathcal{F}^{-1} \left[\xi \mapsto \text{Proj}_{\ker \mathbb{B}(\xi)}[\hat{u}(\xi)] \right] = \text{Proj}_{\ker \mathbb{B}(\xi)} u,$$

where the latter projection is the $L^2(\mathbb{R}^n, V)$ -orthogonal projection and $\mathcal{F}$ denotes the Fourier Transform. However, this can be justified only if the 0-homogeneous multiplier

$$\mathbb{P}(\xi) := \text{Proj}_{\ker \mathbb{B}(\xi)} = \text{Id}_V - \mathbb{B}^\dagger(\xi) \mathbb{B}(\xi)$$

is smooth away from zero. In turn, this is the case only if $\mathbb{B}$ has constant rank. Writing

$$\pi u = \mathcal{F}^{-1}[\mathbb{P}(\xi) \hat{u}(\xi)], \quad (1.30)$$

the above considerations above sum up to the result of SCHULENBERGER and WILCOX in [166, 167]:

$$\|D^l u\|_{L^2(\mathbb{R}^n)} \leq c \|\mathbb{B}u\|_{L^2(\mathbb{R}^n)} \quad \text{for all } u \in W^{l,2}(\mathbb{R}^n, V) \cap \ker \mathbb{B}.$$

Of course, in view of the Multiplier Theorem, we can state more general coerciveness estimates that address the coerciveness problem (1.24) in this spirit.

Theorem 1.1.17. *Let $\mathbb{B}$ be a constant rank operator as in (1.22) and $1 < p < \infty$. Then*

$$\|D^l(u - \pi u)\|_{L^p(\mathbb{R}^n)} \leq c \|\mathbb{B}u\|_{L^p(\mathbb{R}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V).$$

Closely related inequalities are given in [85, Prop. 2.13(iii)] and [21, Lem. 2.13]. We stress that in order to use the Multiplier Theorem, smoothness of $\mathbb{B}^\dagger(\cdot)$ away from zero is crucial and that this property fails if $\mathbb{B}$ does not satisfy the constant rank condition.

We conclude the present Subsection with the remark that, if $\mathbb{B}$ is elliptic, we have that $\mathbb{P}$ and, hence π , are identically zero, so Theorem 1.1.17 reduces to (1.26). In particular, we can view estimates for constant rank operators as “quotient” elliptic estimates. More precisely

$$\frac{\{u \in \mathcal{D}' : \mathbb{B}u \in L_{\text{loc}}^p\}}{\ker \mathbb{B}} = W_{\text{loc}}^{l,p},$$

i.e., each $\mathbb{B}u \in L_{\text{loc}}^p$ has a representative $\mathbb{B}v$ such that $v \in W_{\text{loc}}^{l,p}$. Equivalently, equations

$$\mathbb{B}u = f \in L^p(\mathbb{R}^n, W)$$

have all (resp. one) solutions in $W^{l,p}$ if $\mathbb{B}$ is elliptic (resp. of constant rank). To the author’s knowledge, no theory of coerciveness inequalities (1.24) for operators that may fail the constant rank condition has been developed.

1.1.4. Linear L^1 -estimates In Subsection 1.1.3 we discussed L^p -coerciveness inequalities (1.24) for $1 < p < \infty$ in the context of applying the Direct Method to functionals $\mathcal{E}$ as in (1.21) with integrands f of p -growth (1.23). Roughly speaking, the coerciveness inequalities enable one to relax the functional $\mathcal{E}$ in Sobolev spaces and prove existence of generalized minimizers in $W^{l,p}$.

We henceforth focus on the case $p = 1$, which is strikingly different since the coerciveness inequalities (1.24) hold, roughly speaking, only for $\mathbb{B} = D^l$. This fact follows from the so-called Orntein's Non-inequality:

Theorem 1.1.18 ([148, 113]). *Let $\mathbb{B}$ be an operator as in (1.22). Then the following are equivalent:*

- (a) *There exists a constant $c > 0$ such that $\|D^l u\|_{L^1} \leq c \|\mathbb{B}u\|_{L^1}$ for all $u \in C_c^\infty(\mathbb{R}^n, V)$.*
- (b) *There exists a linear map $L \in \text{Lin}(W, V \odot^l \mathbb{R}^n)$ such that $D^l u = L(\mathbb{B}u)$ holds for all $u \in C^\infty(\mathbb{R}^n, V)$.*
- (c) *There exists a constant $c > 0$ such that $|D^l u(x)| \leq c |\mathbb{B}u(x)|$ for all $x \in \mathbb{R}^n$ and $u \in C^\infty(\mathbb{R}^n, V)$.*

Here $V \odot^l \mathbb{R}^n$ denotes the vector space of l -linear, symmetric, V -valued maps on $\mathbb{R}^n$. In particular, if $\mathcal{E}$ is initially defined on $W^{l,1}$, we cannot in general conclude that minimizing sequences $(u_j)_j$ are bounded in $W^{l,1}$. Moreover, from $\|\mathbb{B}u_j\|_{L^1} \leq c$, we cannot infer the existence of a weak limit point in L^1 due to lack of reflexivity. We can only hope for

$$\mathbb{B}u_j \xrightarrow{*} \mu \quad \text{in } \mathcal{M}(\Omega, W),$$

where $\mathcal{M}(\Omega, W)$ denotes the space of W -valued Radon measures on Ω of finite total variation. We are thus lead naturally to the spaces

$$BV^{\mathbb{B}}(\Omega) := \{u \in L^1(\Omega, V) : \mathbb{B}u \in \mathcal{M}(\Omega, W)\}, \quad (1.31)$$

which is a Banach space under

$$\|u\|_{BV^{\mathbb{B}}(\Omega)} := \|u\|_{L^1(\Omega)} + |\mathbb{B}u|(\Omega).$$

Probably the most studied examples of such spaces are BV , the space of maps of bounded variation, when $\mathbb{B} = D$ [18] and BD , the space of maps of bounded deformation [114, 178, 16], when $\mathbb{B} = \mathcal{E}$.

In particular, Theorem 1.1.18 implies that the local structure of $BV^{\mathbb{B}}$ -maps is in general less regular than that of BV^l -maps (i.e., $BV^{\mathbb{B}}$ for $\mathbb{B} = D^l$, where many properties can be inferred from the BV structure of the $(l-1)$ -th derivatives). It is also the case that the trace operator in $BV^{\mathbb{B}}(\Omega)$ (whenever one can be defined) is not weakly- $*$ continuous, which significantly complicates the analysis of the Dirichlet problem [160, 45].

These considerations reveal one possible motivation to study properties of $BV^{\mathbb{B}}$ -maps and linear equations with L^1 -data, which is the scope of Part II of this thesis.

Indeed, although an inequality

$$\|D^l u\|_{L^1} \leq c \|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V)$$

holds only for essentially trivial $\mathbb{B}$, it may be the case that strong-type estimates for lower order derivatives are available. This discrepancy between Ornstein's Non-inequality and weaker estimates can be seen from STRAUSS's example

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq c \|\mathcal{E}u\|_{L^1(\mathbb{R}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n). \quad (1.32)$$

By Theorem 1.1.18, this inequality is a strengthening of the Gagliardo–Nirenberg–Sobolev inequality, which we now discuss in some detail. For $1 < p < n$, we sketch a proof of the Sobolev inequality which is not far from the original proof in [173].

Until otherwise specified, we will assume that $n \geq 2$ for the remainder of this subsection. For $u \in C_c^\infty(\mathbb{R}^n)$, we have by [93, Lem. 7.14] that

$$u(x) = c \int_{\mathbb{R}^n} \frac{\langle Du(y), x - y \rangle}{|x - y|^n} dy \quad \text{for all } x \in \mathbb{R}^n, \quad (1.33)$$

so that

$$|u(x)| \leq c \int_{\mathbb{R}^n} \frac{|Du(y)|}{|x - y|^{n-1}} dy = c |(I_1 |Du|)(x)| \quad \text{for all } x \in \mathbb{R}^n,$$

where I_α denotes the Riesz potential of order $\alpha \in (0, n)$ [175, Ch. 5.1]. It is well-known that I_1 is bounded between $L^p(\mathbb{R}^n)$ and $L^{np/(n-p)}(\mathbb{R}^n)$, which is not the case for $p = 1$, where we have

$$\|I_1 f\|_{L^{\frac{n}{n-1}}_{\text{weak}}} \leq c \|f\|_{L^1} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n). \quad (1.34)$$

Recall the weak- L^q space, defined for $1 \leq q < \infty$ by

$$\|f\|_{L^q_{\text{weak}}} := \sup \left\{ \lambda > 0 : \lambda \mathcal{L}(\{|f| > \lambda\})^{1/q} \right\} < \infty.$$

It is easy to see that the estimate (1.34) is sharp by taking $f = \delta_0$ (justified by taking a sequence of standard mollifiers). Despite this fact, the Sobolev inequality for $p = 1$, $n \geq 2$, i.e.,

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq c \|Du\|_{L^1(\mathbb{R}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n), \quad (1.35)$$

was proved independently by GAGLIARDO [91] and NIRENBERG [147]. There is no inconsistency between (1.34) and (1.35) since no Dirac mass is a gradient for $n \geq 2$ (one can compute explicitly that $\text{curl } \delta_0 w \neq 0$ for all $w \in \mathbb{R}^n \setminus \{0\}$). In particular, the vectorial structure of the gradient allows for a cancellation phenomenon to occur and yield a strong-type estimate.

In view of (1.32), one can ask what is the class of operators $\mathbb{B}$ as in (1.22) for which the estimate

$$\|D^{l-1} u\|_{L^{\frac{n}{n-1}}} \leq c \|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V) \quad (1.36)$$

holds. Such estimates give interior (lower order) Sobolev regularity for $BV^{\mathbb{B}}$ -maps.

We have an insight as to what some necessary assumptions for (1.36) should be. One can reasonably easily show that ellipticity of $\mathbb{B}$ (1.27) is necessary [201, Prop. 5.1]. In this case, by analogy with (1.29), we have that

$$\widehat{D^{l-1}u}(\xi) = \mathbb{B}^\dagger(\xi)\widehat{\mathbb{B}u}(\xi) \otimes \xi^{\otimes(l-1)} =: m(\xi)\widehat{\mathbb{B}u}(\xi),$$

where $m \in C^\infty(\mathbb{R}^n \setminus \{0\}, \text{Lin}(W, V \odot^{l-1} \mathbb{R}^n))$ is a (-1) -homogeneous function. We can formally invert the Fourier Transform to obtain

$$D^{l-1}u = \check{m} \star \mathbb{B}u =: K_{\mathbb{A}} \star \mathbb{B}u \quad \text{for } u \in C_c^\infty(\mathbb{R}^n, V), \quad (1.37)$$

where $K_{\mathbb{A}} \in C^\infty(\mathbb{R}^n \setminus \{0\}, \text{Lin}(W, V \odot^{l-1} \mathbb{R}^n))$ is $(1-n)$ -homogeneous (this fact is formalized in [43, Lem. 2.1]). In particular, we obtained a generalization of (1.33) to elliptic operators which implies the pointwise bound $|D^{l-1}u| \leq c|I_1|\mathbb{B}u|$. Collecting, we expect:

Proposition 1.1.19 (Weak-type estimate). *Let $\mathbb{B}$ be an operator as in (1.22). Then the estimate*

$$\|D^{l-1}u\|_{L_{\text{weak}}^{\frac{n}{n-1}}} \leq c\|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V)$$

holds if and only if $\mathbb{B}$ is elliptic.

Assume now that $\mathbb{B}$ is elliptic. In view of the standard counterexample to strong boundedness of Riesz potentials on L^1 described above, we plug in $\mathbb{B}u = \delta_0 w \neq 0$ for $w \in W$ in (1.37) (assuming that this is admissible). We obtain $D^{l-1}u = K_{\mathbb{A}}w \neq 0$, which is $(1-n)$ -homogeneous. In particular, we have that $D^{l-1}u \notin L_{\text{loc}}^{n/(n-1)}$ and we essentially proved that the estimate (1.36) implies

$$\mathbb{B}u = \delta_0 w \implies w = 0. \quad (1.38)$$

Fourier transforming, we obtain the condition

$$\bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} \text{im } \mathbb{B}(\xi) = \{0\}, \quad (\text{C})$$

which was introduced by VAN SCHAFTINGEN in [201]. We say that an operator $\mathbb{B}$ satisfying (C) is *canceling*.

So far we argued that (1.36) implies ellipticity and cancellation (EC). The striking result obtained by BOURGAIN and BREZIS in [38, Thm. 25] (for first order operators) and VAN SCHAFTINGEN in [201] is that EC is also sufficient for (1.36):

Theorem 1.1.20 (Strong-type estimate). *Let $\mathbb{B}$ be an operator as in (1.22) and $n > 1$. Then the estimate*

$$\|D^{l-1}u\|_{L^{\frac{n}{n-1}}} \leq c\|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V)$$

holds if and only if $\mathbb{B}$ is elliptic and canceling.

Since Theorem 1.1.20 is particularly relevant to all new L^1 -estimates presented in this thesis, we will discuss its proof in detail. We also refer the reader to the overarching survey [202].

We begin by mentioning that, to the author's best knowledge, no direct proof of Theorem 1.1.20 is available. We will present VAN SCHAFTINGEN's argument by duality, which consists of work done in [195, 196, 199, 201], building on earlier results by BOURGAIN and BREZIS in [35, 36, 37, 38] and BOURGAIN, BREZIS, and MIRONESCU in [41].

A first surprising geometric estimate was proved in [41, Prop. 4]: For a closed, rectifiable, oriented curve $\Gamma \subset \mathbb{R}^n$ with unit tangent $\vec{t}_\Gamma$, we have the estimate

$$\int_{\Gamma} \langle \varphi, \vec{t}_\Gamma \rangle d\mathcal{H}^{n-1} \leq c|\Gamma| \|D\varphi\|_{L^n} \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n), \quad (1.39)$$

where $|\Gamma|$ denotes the length of Γ . The surprising feature of (1.39) in connection with Sobolev embeddings can be observed by a sloppy application of Hölder's inequality to the LHS, which would then require the embedding $\dot{W}^{1,n} \hookrightarrow L^\infty$. The latter is well-known to fail (see also the more recent paper [198] on related topics) and a much more refined argument is needed. An elementary proof of (1.39) was given in [195].

The connection between the estimate (1.39) and critical estimates for differential operators was pointed out in [38, Thm. 1] and can be seen from the fact that the convex hull of $\{\vec{t}_\Gamma \mathcal{H}^{n-1} \llcorner \Gamma\}_\Gamma$ is weakly- $*$ dense in the space

$$\mathcal{M}_{\text{div}} := \{\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^n) : \text{div } \mu = 0 \text{ in } \mathcal{D}'\}$$

of solenoidal measures [170]. We have that, for all $f, \varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \langle f, \varphi \rangle dx \leq c \|f\|_{L^1} \|D\varphi\|_{L^n} \quad \text{if } \text{div } f = 0. \quad (1.40)$$

In other words, $\mathcal{M}_{\text{div}} \subset (\dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n))^* = \dot{W}^{-1,n/(n-1)}$. See [196] for an elementary proof. This fact can be used to give an alternative proof of the Gagliardo–Nirenberg–Sobolev Inequality. We present an argument in dimensions $n = 2$, where it is particularly easy to see. Let $u \in C_c^\infty(\mathbb{R}^2)$. Then

$$\|u\|_{L^2} \leq c \|Du\|_{\dot{W}^{-1,2}} = c \|(\partial_2 u, -\partial_1 u)\|_{(\dot{W}^{1,2})^*} \leq c \|Du\|_{L^1},$$

where the last inequality follows from (1.40) since $\text{div}(\partial_2 u, -\partial_1 u) = 0$.

In order for one to prove Theorem 1.1.20 along these lines, one should try to obtain a characterization of homogeneous “compensating” operators $\mathcal{A}$ such that

$$\int_{\mathbb{R}^n} \langle f, \varphi \rangle dx \leq c \|f\|_{L^1} \|D\varphi\|_{L^n} \quad \text{if } \mathcal{A}f = 0 \quad (1.41)$$

for all $f, \varphi \in C_c^\infty(\mathbb{R}^n, W)$. By plugging in $f = \delta_{x_0} w \neq 0$ for some $w \in W$ (if admissible) in (1.41), we obtain that $\|\varphi\|_{L^\infty} \leq c \|D\varphi\|_{L^n}$, a contradiction. It is thus necessary for (1.41) to have

$$\mathcal{A}(\delta_0 w) = 0 \implies w = 0.$$

By Fourier transforming, we obtain the *co-canceling* condition for $\mathcal{A}$, introduced by VAN SCHAFTINGEN in [201] to address (1.41):

$$\bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} \ker \mathcal{A}(\xi) = \{0\}.$$

Remarkably, the co-canceling condition is also sufficient for (1.41). This follows immediately for first order operators $\mathcal{A}$ from (1.40) by a change of variables [201, Prop. 2.5]. The case of higher order differential operators $\mathcal{A}$, however, cannot be inferred easily from the methods of BOURGAIN and BREZIS. One issue in this case is that it is not clear what is an appropriate higher order generalization of (1.40). A second order condition was introduced in [197] to generalize the inequality (1.32) (recall that the symmetric gradient $\mathbb{B} = \mathcal{E}$ does not satisfy a first order condition, but the Saint–Venant compatibility conditions). Another instance of an estimate (1.41) can be found in [38, Cor. 14]. The drawback of both of these examples is that neither is invariant under all isometries of $\mathbb{R}^n$, so that one cannot use them to obtain the class of co-canceling operators by a change of variable. An appropriate condition was suggested by VAN SCHAFTINGEN in [199], where it is shown that (1.41) holds for

$$\mathcal{A}f = \sum_{|\alpha|=k} \partial^\alpha f_\alpha =: \operatorname{div}^k f,$$

and this condition is invariant. We slightly abused notation to stress that div^k is essentially the adjoint of D^k . In this case, one can use [201, Prop. 2.5] to prove the following:

Theorem 1.1.21 ([201, Thm. 1.4]). *Let $\mathcal{A}$ be a homogeneous linear differential operator on $\mathbb{R}^n$ from W to X with constant coefficients, where $n > 1$. Then the estimate*

$$\|f\|_{\dot{W}^{-1, \frac{n}{n-1}}} \leq c \|f\|_{L^1} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n, W) \text{ with } \mathcal{A}f = 0$$

holds if and only if $\mathcal{A}$ is co-canceling.

It is now quite easy to outline a proof of sufficiency of EC in Theorem 1.1.20. Let $u \in C_c^\infty(\mathbb{R}^n, V)$. Using ellipticity of $\mathbb{B}$, (1.37), and boundedness of singular integrals on $L^{n/(n-1)}$, we have that

$$\|D^{l-1}u\|_{L^{\frac{n}{n-1}}} \leq c \|\mathbb{B}u\|_{\dot{W}^{-1, \frac{n}{n-1}}} = c \|\mathbb{B}u\|_{(\dot{W}^{1,n})^*} \leq c \|\mathbb{B}u\|_{L^1},$$

where in the last inequality we applied Theorem 1.1.21. This is possible due to the fact that for an elliptic operators $\mathbb{B}$ there exists an operator $\mathcal{A}$ such that $\operatorname{im} \mathbb{B}(\xi) = \ker \mathcal{A}(\xi)$ for all $0 \neq \xi \in \mathbb{R}^n$ [201, Sec. 4.2].

Here we make a hiatus in the discussion of elliptic L^1 -estimates and assume that $\mathbb{B}$ has constant rank. A modification of Theorem 1.1.17 gives

$$\|D^{l-1}(u - \pi u)\|_{L^{\frac{n}{n-1}}} \leq c \|\mathbb{B}u\|_{(\dot{W}^{1,n})^*},$$

and the only thing needed to obtain a suitable generalization of Theorem 1.1.20 is to construct an exact annihilator $\mathcal{A}$ of $\mathbb{B}$. This is granted by the author's result, Theorem 1.2.1. In particular, the following result will be discussed in Section 1.5 and Chapter 4:

Theorem 1.1.22 ([157]). *Let $\mathbb{B}$ be a constant rank operator as in (1.22) and $n > 1$. Then the estimate*

$$\|D^{l-1}(u - \pi u)\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq c\|\mathbb{B}u\|_{L^1(\mathbb{R}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V)$$

holds if and only if $\mathbb{B}$ is canceling. Here $\pi: C_c^\infty(\mathbb{R}^n, V) \rightarrow \ker \mathbb{B}$ denotes the projection operator in (1.30).

We close the hiatus, assume that $\mathbb{B}$ is elliptic. Then an iteration of the Sobolev inequality and [201, Prop. 5.5] show that the estimate

$$\|D^{l-j}u\|_{L^{n/(n-j)}} \leq c\|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V), \quad j = 1, \dots, \min\{n-1, l\} \quad (1.42)$$

holds if and only if $\mathbb{B}$ is canceling. The case $j = 0$ is ruled out by Ornstein's Non-inequality (Theorem 1.1.18). Assume for simplicity that $\mathbb{B}$ has order $l = n$, the general case being obtained by replacing u by $D^{l-n}u$. Then the canceling assumption guarantees that

$$\|Du\|_{L^n} \leq c\|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V),$$

so that another iteration of the Sobolev inequality yields

$$\|u\|_{\text{BMO}} \leq c\|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V),$$

which is not critical, since it is well-known that $\dot{W}^{n,1}(\mathbb{R}^n) \hookrightarrow L^\infty$ (see [152] for a quite recent and surprising result related to this embedding).

In particular, the inequality

$$\|u\|_{L^\infty} \leq c\|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V) \quad (1.43)$$

cannot be derived for EC operators from Theorem 1.1.20. The fact that EC is sufficient for (1.43) was proved by a different method by BOUSQUET and VAN SCHAFTINGEN in [43, Thm. 1.3]. However, they also pointed out that the simple example $\mathbb{B}u := u'$ in dimension $n = 1$ is elliptic, non-canceling, but satisfies (1.43). Examples of elliptic, non-canceling operators that satisfy (1.43) in any odd dimension were given by the author in [158, p.18-19].

These examples show that, somewhat surprisingly the inequality (1.43) is equivalent to a condition that is strictly weaker than (C) in the class of elliptic operators. This condition was identified by the author in [155] as a *weaker canceling condition*⁴

$$\mathcal{L} \left[\bigcap_{\xi \in \mathbb{S}^{n-1}} \text{im } \mathbb{B}(\xi) \right] = \{0\}, \quad \text{where } \mathcal{L} := \int_{\mathbb{S}^{n-1}} \mathbb{B}^\dagger(\xi) d\mathcal{H}^{n-1}(\xi), \quad (\text{WC})$$

⁴In the definition of (WC), we assume that the spaces $\mathbb{R}^n$, V , W on which $\mathbb{B}$ is defined are not just normed spaces, but also have Euclidean structure, so that condition (WC) may depend on the choice of Euclidean structure, whereas the embedding (1.43) does not. This is not the case, as is shown in Proposition 5.4.3.

which is written here under the assumption that $\mathbb{B}$ has order n . Recall also that the left inverse of the symbol map of $\mathbb{B}$ is given by $\mathbb{B}^\dagger(\xi) = [\mathbb{B}^*(\xi)\mathbb{B}(\xi)]^{-1}\mathbb{B}^*(\xi)$. In particular, the embedding (1.43) holds for all elliptic operators in odd dimensions. On the other hand, in even dimensions, we will give examples of elliptic, non-canceling operators that satisfy (WC).

Equivalence of condition (WC) and the inequality (1.43) will be discussed in Section 1.4 and proved in Chapter 5 (Sections 5.3 and 5.4).

The inequalities of Theorem 1.36 and (1.43) can be translated into embedding properties of the function spaces $BV^\mathbb{B}$, defined in (1.31); for instance by Theorem 1.1.20, we have by a standard density argument that $BV^\mathbb{B}(\mathbb{R}^n) \subset \dot{W}^{k-1,n/(n-1)}$. On the other hand, since the inequalities at our disposal can be viewed as interior Sobolev regularity statements, we can speculate that the Sobolev-type inequalities discussed so far may be used to deduce pointwise regularity results for $BV^\mathbb{B}$ -maps.

The pointwise properties we have in mind are L^p -differentiability, which generalizes the Lebesgue Differentiation Theorem and can be used to deduce lower semi-continuity [17] or approximation results [123]. More generally, L^p -Taylor expansions were introduced by CALDERÓN and ZYGMUND in [50], where their existence for $W^{l,p}$ -maps ($1 < p < \infty$) and their stability under mapping by singular integrals is extensively studied (see also [23, 78]). Recall that a map $u \in L^1_{\text{loc}}$ has an l -th order L^p -Taylor expansion at $x \in \mathbb{R}^n$ if and only if there exists a polynomial $P = P_x^l u$ such that

$$\|u - P\|_{L^p(B_r(x))} = o\left(r^{l+n/p}\right) \quad \text{as } r \downarrow 0. \quad (1.44)$$

With the convention $n/\infty = 0$, note that L^∞ -Taylor expansions are classical Taylor expansions. We do not recall the results in [50] here, but rather mention the subsequent paper [51], where it is shown that BV -maps are $L^{n/(n-1)}$ -differentiable (i.e., has a first order expansion) $\mathcal{L}^n$ -almost everywhere. We also mention the result of DORRONSORO in [73], where L^p -Taylor expansions of BV^l maps are analyzed, particularly dimensional estimates on the sets of non-differentiability are given.

In the context of $BV^\mathbb{B}$ -maps, it seems quite challenging to prove such dimensional estimates, particularly since not even the L^p -differentiability $\mathcal{L}^n$ -a.e. for critical p is well-understood, as we will now explain. Of course, we should also mention that L^p -differentiability of $BV^\mathbb{B}$ -maps cannot be inferred from [73] since, by Ornstein's Non-inequality, $BV^\mathbb{B}_{\text{loc}} \not\subset BV_{\text{loc}}$. To not encumber notation, we will assume that $\mathbb{B}$ has first order. Also, "almost everywhere" refers strictly to the n -dimensional Lebesgue measure.

We first discuss the case $\mathbb{B} = \mathcal{E}$, i.e., when $BV^\mathbb{B} = BD$. The L^1 -differentiability of BD -maps was established by AMBROSIO, COSCIA, and DAL MASO in [16, Sec. 7] (see also [98], in which apparently the question of approximate differentiability a.e. of BD -maps was proved first). In view of Ornstein's Non-inequality, this result is quite surprising since it implies that a map $u \in BD$ has an L^1 -derivative $(DP_x^1 u)(x) = \nabla u(x)$ for which it may be that the measurable map $x \mapsto \nabla u(x)$ is not locally integrable. The method of HAJLÁSZ in [98] was improved by ALBERTI, BIANCHINI, and CRIPPA in [12], where it is shown that BD -maps are L^p -differentiable a.e. for $1 \leq p < n/(n-1)$. In fact, it is shown in [12, Thm. 3.4]

that for a $(1-n)$ -homogeneous convolution kernel $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and measures $\mu \in \mathcal{M}(\mathbb{R}^n)$, the convolution $K \star \mu$ is L^p -differentiable a.e. for $p < n/(n-1)$. In analogy with the discussion of the (sharp) weak-type estimate (1.34), their method cannot cover the limit case $p = n/(n-1)$. Finally, by using the critical Sobolev-type inequality [178, Prop. 2.4], the fact that BD-maps are $L^{n/(n-1)}$ -differentiable a.e. was proved very recently in [106, Cor. 2.6].

The first contribution of the author was given in [95], which is joint work with GMEINER and is not included in this thesis. It was shown that maps in $BV^\mathbb{B}$ are $L^{n/(n-1)}$ -differentiable a.e., provided that the first order operator $\mathbb{B}$ is $\mathbb{C}$ -elliptic (recall Definition 1.1.14). The main contribution of the author in this direction is to characterize operators $\mathbb{B}$ in terms of their $L^{p,q}$ -differentiability properties. Here $L^{p,q}$ denotes the Lorentz space defined for $1 \leq p < \infty$ by the quasi-norm

$$\|u\|_{L^{p,q}} := p^{1/q} \left(\int_0^\infty \left[\lambda \mathcal{L}^n(\{|u| > \lambda\})^{1/p} \right]^q \frac{d\lambda}{\lambda} \right)^{1/q}$$

for $q < \infty$, and $L^{p,\infty} := L_{\text{weak}}^p$. In particular, $L^{p,p} = L^p$. We define $L^{p,q}$ -Taylor expansions as in (1.44) with the L^p -norm replaced by the $L^{p,q}$ -quasi-norm.

The following results can be inferred from [155] (presented in Chapter 5):

Proposition 1.1.23 (“Weak-type”). *Let $\mathbb{B}$ be a first order operator as in (1.22), $n > 1$. Then all maps in $BV_{\text{loc}}^\mathbb{B}$ are $L^{n/(n-1),\infty}$ -differentiable a.e. if and only if $\mathbb{B}$ is elliptic.*

The Proposition above should be compared with Proposition 1.1.19. The following Theorem should be compared with Theorem 1.1.20 (more precisely, with [201, Thm. 8.5]):

Theorem 1.1.24 (“Strong-type”). *Let $\mathbb{B}$ be a first order operator as in (1.22), $n > 1$, and $1 < q < \infty$. Then all maps in $BV_{\text{loc}}^\mathbb{B}$ are $L^{n/(n-1),q}$ -differentiable a.e. if and only if $\mathbb{B}$ is elliptic and is canceling (i.e., satisfies (C)).*

The higher order variants are presented in Chapter 5. Of course, when the order of $\mathbb{B}$ exceeds the space dimension n , one should replace (C) by the weakly canceling condition (WC).

It is not known whether the parameter $q = 1$ is admissible in Theorem 1.1.24. This is so since the corresponding Lorentz-Sobolev embedding has not been proved yet [201, Open Prob. 8.3].

Another statement of interior regularity in $BV^\mathbb{B}$ would be to investigate quantitative properties of their restrictions to lower dimensional subsets (interior traces).

As a starting point for our investigation, we take ADAMS’ extension of the Sobolev inequality in [2]. The original idea was to prove trace theorems on rough sets Σ that only satisfy a mild regularity condition by proving boundedness of Riesz potentials between $L^p(\mathbb{R}^n; \mathcal{L}^n)$ and $L^q(\mathbb{R}^n; \mu)$, where μ satisfies a Morrey condition that reflects the regularity of Σ . For instance, say $\mu = \mathcal{H}^{n-s} \llcorner \Sigma$ if $\mu(B_r(x) \cap \Sigma) \leq cr^{n-s}$ for $x \in \Sigma$, where Σ is a Borel set of co-dimension s . We make this precise, without going into too much generality: We are interested in the boundedness of

$$(I_1 f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}} dy \quad \text{for } x \in \mathbb{R}^n \text{ (and measurable } f: \mathbb{R}^n \rightarrow \mathbb{R}),$$

between $L^p(\mathbb{R}^n; \mathcal{L}^n)$ and $L^q(\mathbb{R}^n; \mu)$, where $\mu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^n)$ is such that

$$\|\mu\|_{M^{1,n-s}} := \sup \left\{ \frac{|\mu|(B_r(x))}{r^{n-s}} : x \in \mathbb{R}^n, r > 0 \right\} < \infty.$$

Here we chose $M^{p,\lambda}$ as a notation for the Morrey space in order to avoid possible confusion, though it is not standard.

The result in [2] implies that, for $1 < p < n$, $n - s > n - p$, and $q = \frac{(n-s)p}{n-p}$, we have that, for non-negative μ ,

$$\|I_1 f\|_{L^q(\mathbb{R}^n; \mu)} \leq c \|\mu\|_{M^{1,n-s}} \|f\|_{L^p(\mathbb{R}^n; \mathcal{L}^n)} \quad \text{for } f \in L^p(\mathbb{R}^n; \mathcal{L}^n) \quad (1.45)$$

Since maps in $W^{1,p}$ can be redefined on sets of Hausdorff dimension less than $n - p$, for such a trace inequality it is safe to assume that $n - s \geq n - p$. However, the case $s = p$ is quite delicate and we omit its presentation. Instead, we refer the reader to the recent paper [115].

In analogy with the Sobolev inequality, no such result is possible for $p = 1$, unless the set of competitors f is reduced. In particular, one can show that if we take $f = Du$, we have that, for non-negative μ ,

$$\|u\|_{L^{\frac{n-s}{n-1}}(\mathbb{R}^n; \mu)} \leq c \|\mu\|_{M^{1,n-s}} \|Du\|_{L^1(\mathbb{R}^n; \mathcal{L}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n) \quad (1.46)$$

for $s \in [0, 1)$. In fact, in this case one can also cover the capacitary co-dimension $s = 1$, see [210, Thm. 5.13.1]. As a side-remark, note that this is consistent with taking $p = 1$ in [115, Thm. 1.2] as $L^{1,1} = L^1$.

Jointly with GMEINER and VAN SCHAFTINGEN, the author has studied trace inequalities such as (1.46) in [96], by replacing the right hand side with the weaker quantity $\|\mathbb{B}u\|_{L^1(\mathbb{R}^n, \mathcal{L}^n)}$. One of the somewhat surprising conclusions was that, in general, the estimate (1.46) has to be analyzed depending on whether $s \in [0, 1)$ or $s = 1$; in particular, the trace estimate on sets of co-dimension one is strictly stronger than the estimate on sets of co-dimension strictly less than one.

By a careful modification of the proof of Theorem 1.1.20, which employs (1.45), one can prove the following:

Theorem 1.1.25 ([96, Thm. 1.1], Thm. 6.1.1). *Let $\mathbb{B}$ be an operator as in (1.22), $n > 1$, and $s \in [0, 1)$. The following are equivalent:*

- (a) $\mathbb{B}$ is elliptic and canceling (i.e., satisfies (C));
- (b) The estimate

$$\|D^{l-1}u\|_{L^{\frac{n-s}{n-1}}(\mathbb{R}^n; \mu)} \leq c \|\mu\|_{M^{1,n-s}} \|\mathbb{B}u\|_{L^1(\mathbb{R}^n; \mathcal{L}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V)$$

holds for all non-negative measures $\mu \in M^{1,n-s}(\mathbb{R}^n)$;

- (c) For all Borel sets $\Sigma \subset \mathbb{R}^n$ such that $\mathcal{H}^{n-s} \llcorner \Sigma \in M^{1,n-s}$, there exists a linear trace operator Tr_Σ such that

$$\|\text{Tr}_\Sigma(D^{l-1}u)\|_{L^{\frac{n-s}{n-1}}(\Sigma; \mathcal{H}^{n-s})} \leq c |\mathbb{B}u|(\mathbb{R}^n) \quad \text{for all } u \in \text{BV}^\mathbb{B}(\mathbb{R}^n).$$

We mention that this is not exactly the statement we give in Chapter 6, but we will close this gap in Section 1.7. To the author's knowledge, this result is new for the space BD of maps of bounded deformation.

Unfortunately, the limiting case $s = 1$ of (b) seems to be particularly difficult. This is why we restrict considerations to particular measures $\mu = \mathcal{H}^{n-1} \llcorner \Sigma$, where $\Sigma \subset \mathbb{R}^n$ is a hyperplane of co-dimension one (this case would correspond to classical trace theorems).

In this case, one can show that (b) implies ellipticity and a *stronger canceling* condition

$$\bigcap_{\xi \in H \setminus \{0\}} \operatorname{im} \mathbb{B}(\xi) = \{0\} \quad \text{for all } H \leq \mathbb{R}^n, \dim H = 2. \quad (\text{SC})$$

Condition (SC) was mentioned, presumably first, in the author's survey [158, p.25]. Simple examples show that condition (SC) is strictly stronger than condition (C). On the other hand, it is not clear whether (SC) implies a trace inequality

$$\|D^{l-1}u\|_{L^1(\Sigma; \mathcal{H}^{n-1})} \leq c \|\mathbb{B}u\|_{L^1(\mathbb{R}^n; \mathcal{L}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V), \quad (1.47)$$

unless $\mathbb{B}$ is a first order operator, i.e., $l = 1$. In this case, an ad hoc argument shows that if $\mathbb{B}$ is elliptic and satisfies (SC), then $\mathbb{B}$ is $\mathbb{C}$ -elliptic (see Definition 1.1.14) and one can use the main result in [45] to prove that (1.47) holds. However, it is entirely unclear whether one can use this inequality to prove that the analogue for $s = 1$ of (b) holds for first order operators.

Finally, we discuss some L^1 -estimates near the boundary. Very few examples have been investigated so far, out of which only [48, 152] are known to the author. We begin by mentioning that the counterpart of Theorem 1.1.13 is not known even in the case when Ω is a half-space and $\mathbb{B}$ has constant coefficients (see also [202, Open. Prob. 3]).

The contribution of the author in this direction is to generalize SMITH's Theorem 1.1.16, jointly with GMEINER in [94]. The precise statement, together with its consequence for $BV^{\mathbb{B}}$ is as follows:

Theorem 1.1.26. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $\mathbb{B}$ be an operator as in (1.22), and $n > 1$. The following are equivalent:*

- (a) $\mathbb{B}$ is $\mathbb{C}$ -elliptic;
- (b) $\|D^{l-1}u\|_{L^{n/(n-1)}(\Omega)} \leq c \left(\|\mathbb{B}u\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)} \right)$ for all $u \in C^\infty(\bar{\Omega}, V)$;
- (c) $BV^{\mathbb{B}}(\Omega) \subset W^{l-1, n/(n-1)}(\Omega, V)$.

In particular, we obtain a generalization of the Gagliardo–Nirenberg–Sobolev inequality on domains and of the Korn–Sobolev inequality in [178, Prop. 1.2].

The proof of the embedding follows, as is the case for Theorem 1.1.16, by extending maps in $C^\infty(\bar{\Omega}, V)$ to full-space boundedly and by employing a suitable estimate in full-space. The latter is ensured by the new fact we prove, that $\mathbb{C}$ -ellipticity implies (C), which enables us to use Theorem 1.1.20. In Section 1.8 we will sketch an analytic proof that is slightly different from the one given originally

in [94, Sec. 3]. An algebraic proof of this fact is given in Lemma 6.3.2, which will appear in [96].

The matter of extending to full-space is quite delicate. We give a technically challenging approach similar to the one in [107] (see also [45]). We see no other way around it, as simple methods inspired by [172, 109] fail for L^1 -estimates. We will sketch such a method in Section 1.8. On the other hand, for first order operators $\mathbb{B}$, the trace theorem in [45] enables us to use trivial extension by zero to $\mathbb{R}^n$. For higher order operators, this idea does not have a generalization. However, one can deduce using the methods from [45] the trace inequality

$$\|D^{l-1}u\|_{L^1(\partial\Omega)} \leq c \left(\|\mathbb{B}u\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)} \right) \quad \text{for all } u \in C^\infty(\bar{\Omega}, V)$$

(the relevant modifications are presented in the proof of Theorem 6.5.1). Provided that a bounded lifting operator $\mathbf{L}: BV^{l-1}(\partial\Omega, V) \rightarrow BV^l(\mathbb{R}^n \setminus \bar{\Omega}, V)$ would exist, then we could extend $u \in BV^\mathbb{B}(\Omega)$ to $\mathbb{R}^n$ by $\mathbf{L}(\text{Tr}_{\partial\Omega} u)$. This is, however, not the case, as shown by MIRONESCU and RUSS in [138] (see also [193]), where it is shown, roughly speaking, that the space of traces of maps in $BV^l(\Omega)$ is $B_1^{l-1,1}(\partial\Omega)$, which is strictly smaller than $BV^{l-1}(\partial\Omega)$ in general for $l \geq 2$ [47, Rk. A.1].

We also mention that the implication of EC by $\mathbb{C}$ -ellipticity is strict, as can be seen from the simple example

$$\mathbb{B}u = (\partial_1 u_1 - \partial_2 u_2, \partial_2 u_1 + \partial_1 u_2, \partial_3 u_1, \partial_3 u_2), \quad (1.48)$$

defined on $\mathbb{R}^3$ from $\mathbb{R}^2$ to $\mathbb{R}^4$. In particular, there exist operators for which $BV^\mathbb{B}(\mathbb{R}^n) \hookrightarrow \dot{W}^{l-1,n/(n-1)}(\mathbb{R}^3, \mathbb{R}^2)$, but $BV^\mathbb{B}(\Omega) \not\hookrightarrow W^{l-1,n/(n-1)}(\mathbb{R}^3, \mathbb{R}^2)$ for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$.

The remainder of this introductory chapter consists mostly of highlights of the author's contributions, particularly ideas of proofs and some sketches.

1.2. $\mathcal{A}$ -free framework vs. $\mathbb{B}$ -framework

We now discuss the author's paper [156], presented in Chapter 2.

It was explained in Section 1.1.1 that the theory of $\mathcal{A}$ -quasiconvexity can be simplified by the following:

Theorem 1.2.1 ([156, Thm. 1.1]). *Let $\mathcal{A}$ (resp. $\mathbb{B}$) be a homogeneous linear differential operator on $\mathbb{R}^n$ from W to X (resp. from V to W) with constant coefficients. Then there exists a homogeneous differential operator $\mathbb{B}$ (resp. $\mathcal{A}$) on $\mathbb{R}^n$ from V to W (resp. from W to X) with constant coefficients such that*

$$\ker \mathcal{A}(\xi) = \text{im } \mathbb{B}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\} \quad (1.49)$$

if and only if $\mathcal{A}$ (resp. $\mathbb{B}$) satisfies the constant rank condition (1.6).

The construction of $\mathcal{A}$ given $\mathbb{B}$ will be used in Chapter 4 to obtain L^1 -estimates for constant rank operators. The construction of $\mathbb{B}$ given $\mathcal{A}$ will be used in the study of lower semi-continuity of $\mathcal{A}$ -quasiconvex integrals and of $\mathcal{A}$ -free Young measures [120, 156]. In this section we focus on the latter.

We first explain why one would expect that such a result holds. Using the geometric properties of Moore–Penrose generalized inverses [52], it is easy to see that

$$\mathbb{P}(\xi) := \text{Proj}_{\ker \mathcal{A}(\xi)} = \text{Id}_W - \mathcal{A}^\dagger(\xi)\mathcal{A}(\xi),$$

so the choice $\mathbb{P}(\cdot) = \mathbb{B}(\cdot)$ obviously satisfies (1.49), but has the drawback that $\mathbb{P}$ is not a differential operator in general (it is zero-homogeneous). On the other hand, a reasonably widely acknowledged fact is that $\mathbb{P} \in C^\infty(\mathbb{R}^n \setminus \{0\}, \text{Lin}(W, W))$ (although, the only proof known to the author is the elementary one in the Appendix of [153]). In fact, by [153, Thm. 4.6], it is the case that $\mathcal{A}^\dagger(\cdot)$ is smooth away from zero. If $\mathcal{A}$ is invertible, we have that $\mathcal{A}^\dagger(\xi) = [\det \mathcal{A}(\xi)]^{-1} \text{adj} \mathcal{A}(\xi)$, where $\text{adj} M$ denotes the adjugate of a square matrix M . In this case, it is elementary to check that $\mathcal{A}^\dagger(\cdot)$ is a homogeneous rational function. Also, considering $\mathbb{B}(\xi) := \det \mathcal{A}(\xi) \mathbb{P}(\xi)$ solves the problem, as $\text{adj} \mathcal{A}(\cdot)$ and $\det \mathcal{A}(\xi)$ are clearly homogeneous polynomials⁵. Similar considerations can be applied if $\mathcal{A}$ has a one-sided inverse (as is the case for $\mathcal{A} \in \{\text{div}, \Delta, D, \mathcal{E}\}$). Assume that $\mathcal{A}(\xi)$ has a right-inverse, so that

$$\mathcal{A}^\dagger(\xi) = \mathcal{A}^*(\xi)[\mathcal{A}(\xi)\mathcal{A}^*(\xi)]^{-1},$$

in which case one takes $\mathbb{B}(\xi) := \det[\mathcal{A}(\xi)\mathcal{A}^*(\xi)]\mathbb{P}(\xi)$.

It is then natural to speculate that the smoothness of $\mathcal{A}^\dagger(\cdot)$ away from zero can be boosted to show that $\mathcal{A}^\dagger(\cdot)$ is a rational function. This is indeed the case by an application of the main result in [63]. The constant rank condition plays a crucial role to prove that

$$\mathcal{A}^\dagger(\xi) = a^{-1}(\xi)\mathcal{A}_0(\xi),$$

where a^{-1} and $\mathcal{A}_0$ are coprime homogeneous polynomials with $a(\xi) \neq 0$ for $\xi \neq 0$ (a is scalar valued and $\mathcal{A}_0$ is $\text{Lin}(W, W)$ -valued). One then sets

$$\mathbb{B}(\xi) := a(\xi)\mathbb{P}(\xi) = a(\xi)\text{Id}_W - \mathcal{A}_0(\xi)\mathcal{A}(\xi)$$

to conclude the sufficiency of the constant rank condition for (1.49).

The proof of sufficiency of the constant rank condition of $\mathbb{B}$ to extrapolate existence of $\mathcal{A}$ is proved analogously, starting from

$$\mathcal{Q}(\xi) := \text{Proj}_{\text{im } \mathbb{B}(\xi)} = \text{Id}_W - \mathbb{B}(\xi)\mathbb{B}^\dagger(\xi),$$

obtaining $\mathbb{B}^\dagger(\xi) = b^{-1}(\xi)\mathbb{B}_0(\xi)$, and setting $\mathcal{A}(\xi) := b(\xi)\mathcal{Q}(\xi)$.

Necessity of the constant rank condition for (1.49) is proved in Lemma 2.2.2. The argument is elementary; the claim follows from the fact that the topological dimension of $\{\xi : \text{rank } P(\xi) = r\}$ decreases with r (here P is a matrix-valued polynomial).

As it stands, the algebraic identity (1.49) does *not* extend immediately to functions. Although we always have $\mathcal{A}(\mathbb{B}u) = 0$, if $\mathcal{A}w = 0$ in a ball B , then it

⁵Here the reader will immediately realize that the method we propose has no control over the order of the differential operator $\mathbb{B}$.

does not follow that $w = \mathbb{B}u$ in B for some distribution u without making some assumptions on the boundary conditions of w . This is easily seen from the trivial example $\mathcal{A} = D$, when $\mathbb{B} \equiv 0$, but $\mathcal{A}w = 0$ clearly does not imply $w = 0$. On the other hand, it is also quite clear that $\mathcal{A}w = 0$ implies $w = \mathbb{B}u$ whenever the Fourier Transform can be used. This is the case if w is periodic of zero average or if w is a Schwartz function (see Lemma 2.3.1).

As a consequence, one can use standard arguments in the Calculus of Variations to prove that a map $f \in C(W)$ is $\mathcal{A}$ -quasiconvex if and only if

$$\int_{(0,1)^n} f(\eta + \mathbb{B}u(x)) \, dx \geq f(\eta) \quad \text{for all } \eta \in W, u \in C_c^\infty((0,1)^n, V).$$

In particular, under suitable growth conditions, the (semi-)convexity notion equivalent to lower semi-continuity of energy functionals as in (1.1) in the topology of compensated compactness can be defined in terms of the (potential) operator $\mathbb{B}$ only.

Moreover, for $1 < p < \infty$, we will show that Young measures ν generated by sequences $w_j \rightharpoonup w \in L^p(\Omega, W)$ such that $\mathcal{A}w_j \rightarrow \mathcal{A}w$ in $W_{\text{loc}}^{-k,p}(\Omega, X)$ have p -uniformly integrable generating sequences of the form $w + \mathbb{B}u_j$, where $u_j \in C_c^\infty(\Omega, V)$ and $\mathbb{B}u_j \rightarrow 0$ in $L^p(\Omega, W)$. We mention that the proof of this fact in Proposition 2.4.3 uses ideas from [85, Lem. 2.15]. A very similar argument can be given to prove (1.12).

1.3. Characterization of $\mathcal{A}$ -free generalized Young measures

We next discuss the work [120] (in progress), presented in Chapter 3, which is joint work of KRISTENSEN and the author.

Let $\mathcal{A}$ be a homogeneous constant rank operator of order k on $\mathbb{R}^n$. Then by Theorem 1.2.1 there exists an operator $\mathbb{B}$ such that (1.49) holds. In this case one can prove the following:

Theorem 1.3.1. *Let $\nu = ((\nu_x), \lambda, (\nu_x^\infty))$ be a generalized Young measure with $\lambda(\partial\Omega) = 0$ and denote its barycentre by $w = \bar{\nu}_x \mathcal{L}^n + \bar{\nu}_x^\infty \lambda$. We assume that $w \in L^1(\Omega, W)$.*

Then there exist sequences (w_j) in $C^\infty(\bar{\Omega}, W)$ and (u_j) in $C_c^\infty(\Omega, V)$ satisfying

$$\begin{cases} \mathcal{A}w_j = 0 \text{ in the sense of distributions in } \Omega_j \uparrow \Omega \\ w_j \rightarrow w \text{ in } L^1(\Omega, V) \\ \|u_j\|_{W^{l-1,1}} \rightarrow 0 \\ w_j + \mathbb{B}u_j \xrightarrow{Y} \nu \end{cases}$$

if and only if $\mathcal{A}w = 0$ in the sense of distributions in Ω and

$$\langle f, \nu_x \rangle + \langle f^\infty, \nu_x^\infty \rangle \frac{d\lambda}{d\mathcal{L}^n}(x) \geq f\left(\bar{\nu}_x + \bar{\nu}_x^\infty \frac{d\lambda}{d\mathcal{L}^n}(x)\right)$$

holds for $\mathcal{L}^n$ -a.e. $x \in \Omega$ and all $\mathcal{A}$ -quasiconvex f of linear growth.

A few words on the proof: We first characterize homogeneous $\mathcal{A}$ -free Young measures, i.e., independent of x , by a Hahn–Banach argument reminiscent of

[111, 112] which is quite elementary. We then give an approximation argument on dyadic cubes, where the use of $\mathbb{B}$ ensures that the approximation does not have bad jumps across the boundaries of the cubes. The convergence of the approximation is then concluded by the martingale convergence theorem.

To remove the technical assumption $u \in L^1$, one would have to substantially refine the approximation procedure. Should one try to use the ideas of the proof presented in Chapter 3 to tackle the general case $w \in \mathcal{M}(\Omega, W)$, the interaction between the singular parts of w and λ (with respect to $\mathcal{L}^n$) cannot be controlled appropriately.

1.4. The L^∞ -embedding

We now briefly discuss Theorem 5.1.3, which is one of the main results in the author's paper [155], presented in Chapter 5.

We proceed with a sketch of the proof of Theorem 5.1.3. Let $\mathbb{B}$ as in (1.22) be elliptic of order $l \geq n$. A computation in Fourier space gives:

$$\widehat{D^{l-n}u}(\xi) = \mathbb{B}^\dagger(\xi) \widehat{\mathbb{B}u}(\xi) \otimes \xi^{\otimes(l-n)} =: T(\xi) \widehat{\mathbb{B}u}(\xi),$$

where T is smooth away from zero and $(-n)$ -homogeneous. Essentially by [101, Thm. 7.1.20], we have that T has an extension T^* to $\mathbb{R}^n$ that is a tempered distribution and

$$\mathcal{F}^{-1}[T^*] = H_0 + \log|\cdot| \int_{\mathbb{S}^{n-1}} T(\xi) d\mathcal{H}^{n-1}(\xi) =: H_0 + \log|\cdot| \mathcal{L},$$

where H_0 is smooth away from zero and zero-homogeneous. In particular, we have that

$$D^{l-n}u = H_0 \star (\mathbb{B}u) + \log|\cdot| \star (\mathcal{L}\mathbb{B}u) \quad \text{for } u \in C_c^\infty(\mathbb{R}^n, V). \quad (1.50)$$

We aim to identify the condition on $\mathbb{B}$ for which $D^{l-n}u$ is bounded, the first term above is irrelevant by Young's convolution inequality. In turn, if $\mathbb{B}u = \delta_0 w$ for a vector w such that $\mathcal{L}w \neq 0$, then $D^{l-n}u$ is unbounded near zero. This yields necessity of the condition

$$\mathcal{L}w = \int_{\mathbb{S}^{n-1}} \mathbb{B}^\dagger(\xi) w \otimes^{l-n} \xi d\mathcal{H}^{n-1}(\xi) = 0 \quad \text{for all } w \in \bigcap_{\xi \in \mathbb{S}^{n-1}} \text{im } \mathbb{B}(\xi). \quad (\text{WC})$$

In turn, to prove that (WC) is sufficient for

$$\|D^{l-n}u\|_{L^\infty} \leq c \|\mathbb{B}u\|_{L^1},$$

one can use a modification of [43, Lem. 2.2] (see Lemma 5.3.1).

In particular, one can append VAN SCHAFTINGEN's inequalities (1.42) with the case $n = 1$. We adopt the convention $n/0 = \infty$.

Theorem 1.4.1. *Let $\mathbb{B}$ as in (1.22) be elliptic, $n \geq 1$, and $j = 1, 2, \dots, \min\{l, n\}$. Then the estimate*

$$\|D^{l-j}u\|_{L^{\frac{n}{n-j}}} \leq c \|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V)$$

holds if and only if

- (a) $\mathbb{B}$ satisfies (C) for $j < n$;
- (b) $\mathbb{B}$ satisfies (WC) for $j = n \leq l$.

As a function space consequence of Theorem 1.4.1(b), we record the following, assuming for simplicity that $\mathbb{B}$ has order $l = n$. By strict density of smooth functions in the space of bounded measures (see, e.g., [45, Thm. 2.8]), we have that $BV^{\mathbb{B}}(\mathbb{R}^n) \subset L^\infty$. On the other hand, recall the surprising fact due to TARTAR (see [152]), that $BV^n(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ if $n \geq 2$. This fact contradicts the intuition gained from the one-dimensional case, where BV-maps may well have jump discontinuities.

In our set up, it is easy to see from (1.50) that if $\mathbb{B}$ has order n , is elliptic, non-canceling, and satisfies (WC) (such operators exist since in odd dimensions all elliptic operators satisfy (WC) by homogeneity of T), then for $\mathbb{B}u = \delta_0 w \neq 0$, we obtain from (1.50) that $u = H_0 w$. If u were continuous, since H_0 is zero-homogeneous, we would have that u is constant and $w = 0$, a contradiction. In particular, there are operators defined on $\mathbb{R}^n$ for which $BV^{\mathbb{B}} \subset L^\infty$, but $BV^{\mathbb{B}} \not\subset C$.

On the other hand, it is far from being the case that $\mathbb{B} = D^n$ is an isolated example in which continuity of maps in $BV^{\mathbb{B}}$ holds, as can be seen from [152]. In particular, it is easy to see that for the elliptic and canceling operator on $\mathbb{R}^2$

$$\mathbb{B}u := (\Delta u, \partial_1 \partial_2 u),$$

we have from [152, Thm. 1.3] and Lemma 5.2.2 that $BV^{\mathbb{B}} \subset C$.

Open Problem 1.4.2. *Let $n \geq 2$ and $\mathbb{B}$ be an n -th order elliptic and canceling operator. Is it the case that $BV^{\mathbb{B}}(\mathbb{R}^n)$ -maps are continuous?*

1.5. L^1 -estimates for constant rank operators

We now discuss the author's paper [157], presented in Chapter 4.

We begin with a few remarks pertaining to extensions of Theorem 1.1.20 to other space scales which were omitted in Subsection 1.1.4. Firstly, the result extends on the fractional Sobolev scale [201, Thm. 8.1]: Let $\mathbb{B}$ be elliptic of order l on $\mathbb{R}^n$, $n > 1$, and $s \in (0, n)$. Then the estimate

$$\|u\|_{\dot{W}^{k-s, \frac{n}{n-s}}} \leq c \|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V)$$

holds if and only if $\mathbb{B}$ is canceling. This result shows that VAN SCHAFTINGEN's approach gives critical Sobolev estimates arbitrarily close to endpoints $s = 0$, which is ruled out by Ornstein's Non-inequality (Theorem 1.1.18), and $s = n$, which corresponds to Theorem 1.4.1(b), due to the author.

Similar results hold true for other refinements of the Sobolev scale [201, Sec. 8] (see also [200]).

Theorem 1.5.1. *Let $\mathbb{B}$ as in (1.22) be elliptic and $n > 1$. Then the estimate*

$$\|u\|_X \leq c \|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V)$$

holds if and only if $\mathbb{B}$ is canceling. Here X is one of the following:

- (a) $X = \dot{W}^{k-j, n/(n-j)}$ for some $1 \leq j \leq \min\{k, n-1\}$.
- (b) $X = \dot{W}^{k-s, n/(n-s)}$ for some $s \in (0, n)$.
- (c) $X = \dot{B}_q^{k-s, n/(n-s)}$ for some $s \in (0, n)$, $1 < q < \infty$.
- (d) $X = \dot{F}_q^{k-s, n/(n-s)}$ for some $s \in (0, n)$, $1 \leq q \leq \infty$.
- (e) $X = \dot{W}^{k-j} L^{n/(n-j), q}$ for some $1 \leq j \leq \min\{k, n-1\}$, $1 < q < \infty$.

By the same reasoning as for Theorem 1.1.22, which relies on the construction of an exact annihilator of constant rank operators $\mathbb{B}$ in Theorem 1.2.1, we can easily use the estimates in [201, Sec. 8] for co-canceling operators to obtain:

Theorem 1.5.2. *Let $\mathbb{B}$ as in (1.22) have constant rank and $n > 1$. Then the estimate*

$$\|u - \pi u\|_X \leq c \|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V)$$

holds if and only if $\mathbb{B}$ is canceling. Here X is as in Theorem 1.5.1 and the map $\pi: C_c^\infty(\mathbb{R}^n, V) \rightarrow \ker \mathbb{B}$ is given by (1.30).

Few examples of estimates that fail the constant rank condition are known, for instance [201, Prop. 5.4] (see also [152] and [43, Sec. 5]).

1.6. L^p -differentiability

We now discuss the L^p -differentiability results in the author's paper [155], presented in Chapter 5.

We refrain from discussing higher order operators and the extension of L^p -differentiability on the Lorentz scale, as we will do in Chapter 5. Instead, we present the simplest variant of the result:

Theorem 1.6.1. *Let $\mathbb{B}$ be as in (1.22) with $l = 1$ (first order), $n > 1$. Then $\mathbb{B}$ is elliptic and cancelling if and only if all $BV_{\text{loc}}^{\mathbb{B}}$ -maps are $L^{n/(n-1)}$ -differentiable $\mathcal{L}^n$ -almost everywhere.*

We outline a very streamlined sketch.

Necessity of ellipticity follows from the fact that, if $\mathbb{B}$ is not elliptic, there exist non-zero ξ and v such that $\mathbb{B}(\xi)v = 0$. Therefore the plane waves $u(x) = f(x \cdot \xi)v$ are in $BV_{\text{loc}}^{\mathbb{B}}$ provided that $f \in L_{\text{loc}}^1$. It is then easy to construct an irregular f such that u is nowhere $L^{n/(n-1)}$ -differentiable. To prove necessity of cancellation, one considers elliptic non-canceling $\mathbb{B}$, i.e., there exists $u \in BV_{\text{loc}}^{\mathbb{B}}$ such that $\mathbb{B}u = \delta_0 w \neq 0$, so that, in particular, u is not $L^{n/(n-1)}$ -integrable in any neighbourhood of zero. One can then formulate a Baire Category argument to show that there exist maps in $BV^{\mathbb{B}}$ that are not $L^{n/(n-1)}$ -integrable in any open set.

As for sufficiency of EC, one can infer from Theorem 1.1.20 that

$$\|v\|_{L^{\frac{n}{n-1}}(B_r(x))} \leq c \left(|\mathbb{B}v| \left(\overline{B_{2r}(x)} \right) + r^{-1} \|v\|_{L^1(B_{2r}(x))} \right) \quad (1.51)$$

for all $v \in \text{BV}_{\text{loc}}^{\mathbb{B}}$. Let now $u \in \text{BV}_{\text{loc}}^{\mathbb{B}}(\mathbb{R}^n)$. It is essentially known from [12, Thm. 3.4], via the convolution formula (1.37) that u has a first order L^1 -Taylor expansion $u = P_x^1 u + R_x^1 u$ at almost every $x \in \mathbb{R}^n$ (recall the definition (1.44)). We can write $P_x^1 u(y) = \nabla u(x)(y - x) + u(x)$. We apply the estimate (1.51) to $R_x^1 u$ to get

$$\|R_x^1 u\|_{L^{\frac{n}{n-1}}(B_r(x))} \leq c \left(|\mathbb{B}u - \mathbb{B}(P_x^1 u)| \left(\overline{B_{2r}(x)} \right) + r^{-1} \|R_x^1 u\|_{L^1(B_{2r}(x))} \right).$$

As $r \downarrow 0$, the first term equals $o(r^n)$ by Lebesgue's differentiation theorem for measures, whereas the second term equals $o(r^n)$ since u is L^1 -differentiable at x . By the same definition (1.44), $L^{n/(n-1)}$ -differentiability of u at x follows.

1.7. Embeddings on lower dimensional sets

We now discuss aspects of the paper [96], presented in Chapter 6, which is joint work with GMEINER and VAN SCHAFTINGEN.

The statement of Theorem 1.1.25(b), as presented in Section 1.1.4, does not coincide with the statement given in Chapter 6, Theorem 6.1.1. We now explain how this can be fixed.

We are only concerned with the equivalence of (a) and (b), since the equivalence of (a) and (c) is established explicitly in Section 6.4. The fact that (b) implies (a) is established in Section 6.2 by constructing measures $\mu = \mathcal{H}^{n-s} \llcorner \Sigma$ and maps u such that $D^{l-1}u \notin L_{\text{loc}}^{(n-s)/(n-1)}(\mathbb{R}^n; \mu)$. To see that (a) implies (b) holds, we note that the proof of Proposition 6.2.1 in fact achieves the following:

Proposition 1.7.1. *Let $\mathbb{B}$ be an elliptic operator as in (1.22) and $s \in [0, 1)$. Then for any $\gamma \in (s(n-1)/(n-s), 1)$, there exists a constant $C = C(n, s, \gamma) > 0$ such that for all $u \in C_c^\infty(\mathbb{R}^n, V)$ there holds*

$$\left(\int_{\Sigma} |D^{l-1}u|^{\frac{n-s}{n-1}} d\mu \right)^{\frac{n-1}{n-s}} \leq C \|\mu\|_{M^{1,n-s}} \|\mathbb{B}u\|_{\dot{F}_2^{\gamma-1, \frac{n}{n+\gamma-1}}(\mathbb{R}^n, W)}$$

whenever $\mu \in M^{1,n-s}(\mathbb{R}^n)^+$.

Although perhaps obvious for experts, the elliptic trace estimate in Proposition 1.7.1 is probably new.

One can then use [201, Prop. 8.7], which gives a Triebel–Lizorkin variant of Theorem 1.1.21 and states that

$$\|\mathbb{B}u\|_{\dot{F}_2^{\gamma-1, \frac{n}{n+\gamma-1}}(\mathbb{R}^n, W)} \leq c \|\mathbb{B}u\|_{L^1(\mathbb{R}^n, W)}, \quad (1.52)$$

provided that $\mathbb{B}$ is canceling, to conclude that (b) holds. In a nutshell, the coupling of the elliptic estimate of Proposition 1.7.1 with the critical estimate (1.52) is one of the main novelties in [96].

This approach is ruled out if $s = 1$ (only) due to the use of ADAMS' result (1.45). However, this failure cannot be circumvented, as we now explain.

It was observed in [89] (see also [45, Sec. 4.3]) that the function $u(z) = z^{-1}$ for $z \in \mathbb{C}$, which satisfies $\partial_{\bar{z}} u = 2\pi\delta_0$, does not admit traces in L_{loc}^1 along lines

through the origin. In the same fashion, one can compute explicitly that

$$\mathbb{B}u = 2\pi \left(\mathcal{H}^1 \llcorner \{x \in \mathbb{R}^3 : x_1 = 0 = x_2\} \right) (1, 0, 0) \quad \text{for} \quad u(x_1, x_2, x_3) := \frac{(x_1, -x_2)}{x_1^2 + x_2^2}$$

where $\mathbb{B}$ is the example in (1.48). In this case, it is clear that u does not admit L^1_{loc} -traces along planes that contain the x_3 -axis.

In more generality, assume that for a first order elliptic operator $\mathbb{B}$ on $\mathbb{R}^n$ there exists a hyperplane $H \subset \mathbb{R}^n$ with $\dim H = n - 2$, a vector $w \neq 0$, and $u \in L^1_{\text{loc}}$ such that

$$\mathbb{B}u = (\mathcal{H}^{n-2} \llcorner H) w. \quad (1.53)$$

Then an exercise in Fourier analysis (see Lemma 6.3.1) shows that, up to addition of maps in $\ker \mathbb{B}$ and a choice of coordinates (say $H = \{x_1 = x_2 = 0\}$),

$$u(x_1, x_2, \dots, x_n) = F(x_1, x_2),$$

where F is smooth away from zero and (-1) -homogeneous. In particular, as F does not admit locally integrable traces on lines through zero, u does not admit locally integrable traces on hyperplanes of co-dimension one that contain H . This argument is formalized in Section 6.3, where it is shown that condition (SC) (obtained from (1.53) by Fourier transforming) is necessary for the trace inequality

$$\|u\|_{L^1(\Sigma; \mathcal{H}^{n-1})} \leq c \|\mathbb{B}u\|_{L^1(\mathbb{R}^n; \mathcal{L}^n)} \quad \text{for } u \in C_c^\infty(\mathbb{R}^n, V)$$

for all hyper-planes Σ of co-dimension one. The converse, i.e., ellipticity of $\mathbb{B}$ and (SC) imply the trace inequality, is true for first order operators (see Theorem 6.1.4) and we speculate that the same is true for higher order operators.

1.8. Boundary L^1 -estimates

We now discuss some aspects of the main result in [94], presented in Chapter 7, which is joint work of GMEINER and the author.

We recall the steps used to prove sufficiency of $\mathbb{C}$ -ellipticity for Theorem 1.1.26:

- (a) Prove that $\mathbb{C}$ -ellipticity implies condition (C).
- (b) Construct a bounded linear extension operator $E: \text{BV}^{\mathbb{B}}(\Omega) \rightarrow \text{BV}^{\mathbb{B}}(\mathbb{R}^n)$.
- (c) Use Theorem 1.1.20 and (a) to estimate

$$\|D^{l-1}u\|_{L^{\frac{n}{n-1}}(\Omega)} \leq c \|D^{l-1}(Eu)\|_{L^{\frac{n}{n-1}}(\Omega)} \leq \|\mathbb{B}(Eu)\|_{L^1(\mathbb{R}^n)} \leq c \|u\|_{\text{BV}^{\mathbb{B}}(\Omega)}.$$

Sketch of (a). If $\mathbb{B}$ is $\mathbb{C}$ -elliptic, it is clearly elliptic. Assume that $\mathbb{B}$ is not canceling, so that (1.38) fails, i.e., there exists a vector $w \neq 0$ and an L^1_{loc} -map u such that $\mathbb{B}u = \delta_0 w$ in $\mathcal{D}'$. Then by (1.37), we have that $D^{l-1}u$ is non-zero and $(1-n)$ -homogeneous, so that $|D^{l-1}u(x)| \rightarrow \infty$ as $|x| \rightarrow 0$. In particular, $\mathbb{B}u = 0$ in $B := B_1(e)$, where $|e| = 1$. By Proposition 1.1.15, we have that $D^{l-1}u$ is a polynomial in B . This contradicts the fact that $D^{l-1}u$ blows up at 0. $\square$

As to (b), we perform a Jones-type extension, by replacing u by polynomials in $\ker \mathbb{B}$ in $\mathbb{R}^n \setminus \bar{\Omega}$ on small cubes near $\partial\Omega$. Similar ideas in the context of general L^1 boundary estimates appeared in [45, Sec. 3-4]. We could not find a simpler argument, mostly due to the fact that singular integrals are not bounded on L^1 . Below, we sketch an argument which implies Theorem 1.1.16 for $1 < p < \infty$.

A slightly different (but essentially equivalent) approach to SMITH's Theorem 1.1.16 was given in [108, 109], where averaged Taylor polynomials are used to write down SMITH's representation formulas in the form

$$u(x) = \mathcal{P}^d u(x) + \int_{\Omega} K(x, y) \mathbb{B}u(y) \, dy, \quad \text{for } x \in \Omega,$$

where $\mathcal{P}^d u$ denotes the d -th order averaged Taylor polynomial and K is smooth and has appropriate decay in $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$ (see Theorem 7.5.3 for the precise statement). Here we need to require that Ω is star-shaped with respect to a ball, but this restriction is easily removed. In this case, one can use an extension operator

$$(Eu)(x) = \rho(x) \left(\mathcal{P}^d u(x) + \int_{\Omega} K(x, y) \mathbb{B}u(y) \, dy \right), \quad \text{for } x \in \mathbb{R}^n,$$

where $\rho \in C_c^\infty(\mathbb{R}^n)$ equals one in a neighbourhood of Ω . It is not difficult then to check that $\|D^l(Eu)\|_{L^p(\mathbb{R}^n)} \leq c(\|\mathbb{B}u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)})$, but the argument uses boundedness of singular integrals on L^p .

This argument proves the estimate in Theorem 1.1.16 directly, modulo the restriction that Ω is assumed to be star-shaped with respect to a ball. To prove the statement for Lipschitz domains, one decomposes Ω into finitely many star-shaped domains (this is possible by [134, Sec. 1.1]) and employs the estimate in each subdomain.

1.9. A few notes on the text

The remainder of the document consists of several preprints, each corresponding to a chapter, following the theme of the thesis as presented in the quite lengthy Introduction. The notation is nearly consistent throughout, with some small exceptions that we outline now. In any case, the correct notation is recalled in each chapter to prevent confusion.

- (a) The symbol maps for differential operators are denoted by “ $\mathcal{A}(\xi)$ ” or “ $\mathbb{B}(\xi)$ ” in Chapters 1–4 and by “ $\mathcal{A}[\xi]$ ” or “ $\mathbb{B}[\xi]$ ” in Chapters 5–7. The reason for the distinction is that it seemed to the author that the notation with “[$\cdot$]” makes certain computations easier to follow in the later chapters, although the notation with “ $(\cdot)$ ” is more conceptually pleasing.
- (b) The order of the operators: In Chapters 1–3, the order of $\mathcal{A}$ is k and the order of $\mathbb{B}$ is l ; in Chapters 4–7, the order of $\mathcal{A}$ is irrelevant, whereas the order of $\mathbb{B}$ is k .
- (c) The convention for the Fourier transform, though not particularly relevant, is $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \, dx$, except in Chapter 2, where $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} u(x) \, dx$ is used for sheer computational convenience.

A note of warning: Despite the author's best efforts to eradicate this recurring mistake, constants arising from Fourier transforms of derivatives may be missing in several places. Fortunately, there is a good mathematical reason why such errors can be ignored, namely that non-zero, possibly complex, multiplicative constants do not change the estimates proved, nor the algebraic assumptions on the operators, particularly since we only consider homogeneous differential operators.

Having warned the reader of the shortcomings of the exposition, we conclude the introductory chapter with the list of preprints included in the document:

- B. Raită: Potentials of $\mathcal{A}$ -quasiconvexity;
- J. Kristensen and B. Raită: Characterization of Young measures generated by $\mathcal{A}$ -free measures (in progress);
- B. Raită: L^1 -estimates for constant rank operators;
- B. Raită: Critical L^p -differentiability of $BV^{\mathbb{A}}$ -maps and canceling operators;
- F. Gmeineder, B. Raită, and J. Van Schaftingen: On limiting trace inequalities for differential operators;
- F. Gmeineder and B. Raită: Embeddings for $\mathbb{A}$ -weakly differentiable functions on domains.

Also, we will mention various aspects from the following papers, which are not included in the thesis:

- A. Prosinski and B. Raită: On the well-posedness of some variational problems;
- F. Gmeineder and B. Raită: On critical L^p -differentiability for BD-maps (accepted);
- B. Raită: L^1 -estimates and $\mathbb{A}$ -weakly differentiable functions (survey).

Part I

Calculus of Variations under differential constraints

CHAPTER 2

Potentials for $\mathcal{A}$ -quasiconvexity

This chapter consists of the author's work.

2.1. Introduction

A challenging question in the study of non-linear partial differential equations is to find which non-linear functionals are well-behaved with respect to weak convergence, which represents the typical topology consistent with physical measurements and has satisfactory compactness properties. In the context of the Calculus of Variations, answering this question amounts, roughly speaking, to describing semi-continuity properties of functionals

$$\mathcal{E}[w] = \int_{\Omega} f(w(x)) \, dx \quad (2.1)$$

with respect to weak convergence in certain weakly closed, convex subsets $\mathfrak{C}$, say, of L^p -spaces, $1 < p < \infty$), under growth conditions

$$0 \leq f \leq c(|\cdot|^p + 1) \quad (2.2)$$

on the integrands f . Such subsets $\mathfrak{C}$ can account for differential constraints and boundary conditions. Modulo terms removed for simplicity of exposition, such functionals could model, for instance, the energy arising from the deformation of a solid body Ω , viewed as a sufficiently regular open subset of $\mathbb{R}^n$, where f is a continuous energy density map characterized by the constitutive properties of the material. In accordance with the Direct Method in the Calculus of Variations, imposing a suitable bound from below on f ensures existence and weak compactness of minimizing sequences w_j . The appropriate continuity property of $\mathcal{E}$ in this case is that of lower semi-continuity with respect to weak convergence in L^p

$$w_j \rightharpoonup w \implies \liminf_{j \rightarrow \infty} \mathcal{E}[w_j] \geq \mathcal{E}[w],$$

which, if satisfied, implies existence of a minimizer $w \in \mathfrak{C}$.

It is well-known that if $\mathfrak{C}$ consists of the whole of L^p , then $\mathcal{E}$ is weakly sequentially lower semi-continuous if and only if f satisfying (2.2) is convex. Of course, convexity of f is sufficient for lower semi-continuity (always understood as weakly sequential throughout this note) in any reasonable class $\mathfrak{C}$, but it is hardly necessary in general. For instance, if $\mathfrak{C}$ is the space of weak gradients in L^2 and f

is a quadratic form, then one can easily show that f being positive on rank-one matrices implies lower semi-continuity. This example, that we will later come back to in more generality, is of particular relevance, as it provides the insight for a second convexity condition, which is necessary for lower semi-continuity with the constraint $w = \nabla u$: if $\mathcal{E}$ is lower semi-continuous, then f is convex along rank-one lines. In particular, for integrands f of class C^2 , this is equivalent to the so-called Legendre–Hadamard ellipticity condition

$$\frac{\partial^2 F(X)}{\partial X_{ij} \partial X_{\alpha\beta}} a_i a_\alpha b_j b_\beta \geq 0 \quad \text{for all } X, a, b,$$

where summation over repeated indices is adopted. From this point of view, lower semi-continuity of $\mathcal{E}$ acting on gradients reflects a semi-convexity condition on f . Indeed, it was shown by MORREY in [140] that lower semi-continuity of $\mathcal{E}$ is equivalent with *quasiconvexity* of f , i.e., the Jensen-type inequality

$$f(\eta) \leq \int_Q f(\eta + \nabla u(x)) \, dx$$

holds for all η and all smooth maps u with compact support in the open cube Q . On one hand, the quasiconvexity assumption is a plausible constitutive relation for energy functionals arising in solid mechanics [27]; on the other hand, it is but a minor improvement of the lower semi-continuity concept, which makes it particularly difficult to check in applications. The counterexample of ŠVERÁK [181] rules out the possibility of quasiconvexity being a type of directional convexity (see also [30, Ex. 3.5] for the case of higher order gradients). A tractable sufficient condition for quasiconvexity is *polyconvexity*, i.e., f is a convex functions of the minors, also introduced by MORREY in [140] in connection with lower semi-continuity and used by BALL to obtain existence theorems under very mild growth conditions, giving very satisfactory existence results in non-linear elasticity [26]. The fact that quasiconvexity does not imply polyconvexity is much easier to see, at least in higher dimensions, and follows from an old observation of TERPSTRA concerning quadratic forms [191] (see also [28, 14] and the references therein).

The above considerations show that a considerable amount of work was devoted to the treatment of lower semi-continuity in the case when $\mathfrak{C}$ consists of gradients (see [1, 132] and the monographs [61, 149]). However, for instance in continuum mechanics, it is often the case that $\mathfrak{C}$ consists of those L^p -fields w that satisfy a linear, typically under-determined, partial differential constraint, say $\mathcal{A}w = 0$, assumption that we make henceforth. Examples arise in elasticity, plasticity, elasto-plasticity, electromagnetism, and others. The $\mathcal{A}$ -free framework originates in the pioneering work of MURAT and TARTAR in compensated compactness [142, 183, 184] and can be correlated with the question of finding energy functionals that are continuous with respect to weak convergence in $\mathfrak{C}$ [143]. The latter question was also studied in generality by BALL, CURRIE, and OLVER in [30], leading to the generalization of polyconvexity to the case where energy functionals depend on higher order derivatives. In this case, the definition of quasiconvexity extends mutatis mutandis [136]. As to the question of lower semi-continuity, the analysis of the case when f is a quadratic form (see, e.g., [189, Ch. 17] or [185, Thm. 2]) reveals a different necessary condition of directional convexity, namely

with respect to the so-called wave cone of $\mathcal{A}$. It was shown by DACOROGNA in [59, Thm. I.2.3] that, in order to have lower semi-continuity, it is sufficient to assume the following generalization of quasiconvexity, namely that

$$f(\eta) \leq \int_Q f(\eta + w(x)) \, dx$$

for all η and all bounded w such that $\int_Q w = 0$ and $\mathcal{A}w = 0$. However, it is not clear whether this condition is necessary. More recently, FONSECA and MÜLLER showed in [85] that if one assumes in addition that the fields w are periodic, in which case f is called $\mathcal{A}$ -*quasiconvex*, then one indeed obtains a necessary and sufficient condition¹ (under suitable growth assumptions on f). Their result holds under the assumption that the symbol map $\mathcal{A}(\cdot)$ of $\mathcal{A}$ is a constant rank matrix-valued field away from 0. This condition, introduced in [166, Def. 1.5] to prove coerciveness inequalities for non-elliptic systems, was first used in the context of compensated compactness by MURAT and ensures, as noted on [142, p.502], the continuity of the map

$$0 \neq \xi \mapsto \text{Proj}_{\ker \mathcal{A}(\xi)}, \quad (2.3)$$

making tools from pseudo-differential calculus available. In the absence of the constant rank assumption, little is known about the lower semi-continuity problem. One of the few results in this direction was proved by MÜLLER in [145], answering a long standing question of TARTAR (see also [127] for a generalization).

In the proof of the main result of [85], considerable difficulty is encountered when proving sufficiency of $\mathcal{A}$ -quasiconvexity. One reason for this is the absence of potential functions for $\mathcal{A}$, which, if available, should allow one to test with compactly supported functions in the definition of $\mathcal{A}$ -quasiconvexity and, perhaps, use more standard methods.

The main result of the present work is to show that the existence of such a potential in Fourier space is equivalent with the constant rank condition.

Theorem 2.1.1. *Let $\mathcal{A}$ be a linear, homogeneous differential operator with constant coefficients on $\mathbb{R}^n$. Then $\mathcal{A}$ has constant rank if and only if there exists a linear, homogeneous differential operator $\mathbb{B}$ with constant coefficients on $\mathbb{R}^n$ such that*

$$\ker \mathcal{A}(\xi) = \text{im } \mathbb{B}(\xi) \quad (2.4)$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Here $\mathcal{A}(\cdot)$, $\mathbb{B}(\cdot)$ denote the (tensor-valued) symbol maps of, respectively, $\mathcal{A}$, $\mathbb{B}$. We say that $\mathcal{A}$ has *constant rank* if the map $0 \neq \xi \mapsto \text{rank } \mathcal{A}(\xi)$ is constant (see Section 2.2 for detailed notation). We will regard $\mathbb{B}$ as the *potential* and $\mathcal{A}$ as the *annihilator*, although this terminology is not standard.

¹For comparison, see also SEREGIN's work [169] in incompressible linearized elasticity, where the methods used to project on solenoidal fields do not require Fourier analysis.

It is important to mention that the algebraic relation (2.4) does *not*, in general, imply for vector fields w that

$$\mathcal{A}w = 0 \implies w = \mathbb{B}u \quad \text{for some } u. \quad (2.5)$$

To see this, simply take $\mathcal{A} = \nabla^k$. In turn, if we impose restrictions on w that allow for usage of the Fourier transform, (2.5) can be shown to hold (Lemma 2.3.1). As a consequence, standard arguments in the Calculus of Variations lead to the fact that a map f is $\mathcal{A}$ -quasiconvex if and only if

$$f(\eta) \leq \int_Q f(\eta + \mathbb{B}u(x)) \, dx$$

for all η and all smooth vector fields u supported in an open cube Q (Corollary 2.3.2). It is also the case that under the constant rank condition, the notions of $\mathcal{A}$ -quasiconvexity [85, p.1357] and DACOROGNA's $\mathcal{A}$ - $\mathbb{B}$ -quasiconvexity [59, Eq. (A.12)] coincide. In particular, one can define $\mathcal{A}$ -quasiconvexity via integration over arbitrary domains (Lemma 2.3.4). As a consequence, the lower semi-continuity properties of functionals (2.1) in the topologies considered in [85, 21], which are natural from the point of view of compensated compactness theory, rely only on the structure of $\mathbb{B}$.

In fact, we will show that the $\mathcal{A}$ -quasiconvex relaxation of a continuous integrand can be described in terms of $\mathbb{B}$ only. From this point of view, it is natural to investigate the Young measures generated by sequences satisfying differential constraints [85, Sec. 4], as they efficiently describe the minimization of energies that are not lower semi-continuous. We recall that the role of parametrized measures for non-convex problems in the Calculus of Variations was first recognized by YOUNG in the pioneering works [206, 207, 208]. See the monographs [146, 149] for a modern, detailed exposition.

Roughly speaking, for $1 < p < \infty$, we consider a sequence w_j converging weakly in L^p which is asymptotically $\mathcal{A}$ -free and generates a Young measure ν . Technically speaking, it suffices to take $\mathcal{A}w_j$ to be strongly compact in $W_{\text{loc}}^{-k,p}$, where k is the order of $\mathcal{A}$. This is (slightly more general than) the topology considered in [85, Rk. 4.2(i)] and is consistent with the topology considered in compensated compactness (see, e.g., [189, Thm. 17.3], which essentially deals with the case of linear equations). In this setting, we will show that the Young measure ν is generated by a sequence of smooth maps $\mathbb{B}u_j$, modulo a shift by the barycentre.

To sum up, under the constant rank condition on the annihilator $\mathcal{A}$, the objects characterizing the lower semi-continuous relaxation of functionals defined on $\mathcal{A}$ -free vector fields (i.e., $\mathcal{A}$ -quasiconvex envelopes and $\mathcal{A}$ -free Young measures) can be described only in terms of the potential $\mathbb{B}$ constructed in Theorem 2.1.1. From this point of view, it is the author's opinion that the study of functionals

$$\mathcal{E}[w] = \int_{\Omega} f(x, w(x)) \, dx \text{ for } \mathcal{A}w = 0 \quad \text{and} \quad \mathcal{F}[u] = \int_{\Omega} f(x, \mathbb{B}u(x)) \, dx$$

is essentially dual (*strictly* under the constant rank condition). See also [60] and the Appendix of [59].

Since testing with the appropriate quantity is fundamental in the study of partial differential equations, we hope that the observations made in this work will increase the flexibility of analyzing functionals in either class described above. On the other hand, the functional $\mathcal{F}$ seems better suited for incorporating boundary conditions. This will be pursued elsewhere.

This chapter is organized as follows: In Section 2.2 we prove the main Theorem 2.1.1, in Section 2.3 we prove that $\mathcal{A}$ -quasiconvexity can be tested with compactly supported fields $w = \mathbb{B}u$ (Corollary 2.3.2), and in Section 2.4 we prove that $\mathcal{A}$ -free Young measures are shifts of Young measures generated by sequences $\mathbb{B}u_j$.

2.2. Proof of Theorem 2.1.1

We take a moment to clarify notation. By a k -homogeneous, linear differential operator $\mathcal{A}$ on $\mathbb{R}^n$ from W to X we mean

$$\mathcal{A}w := \sum_{|\alpha|=k} \partial^\alpha \mathcal{A}_\alpha w \quad \text{for } w: \mathbb{R}^n \rightarrow W, \quad (2.6)$$

where $\mathcal{A}_\alpha \in \text{Lin}(W, X)$ for all multi-indices α such that $|\alpha| = k$, for finite dimensional inner product spaces W, X . We also define the (Fourier) symbol map

$$\mathcal{A}(\xi) := \sum_{|\alpha|=k} \xi^\alpha \mathcal{A}_\alpha \in \text{Lin}(W, X) \quad \text{for } \xi \in \mathbb{R}^n.$$

We also recall the condition mentioned above that $\mathcal{A}$ is of *constant rank* if there exists a natural number r such that

$$\text{rank} \mathcal{A}(\xi) = r \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

As to the resolution of Theorem 2.1.1, we recall the notion of (*Moore–Penrose*) *generalized inverse*, introduced independently in [139, 34, 151], to which we refer plainly as the *pseudo-inverse*, although the terminology is not standard. For a matrix $M \in \mathbb{R}^{N \times m}$, its pseudo-inverse $M^\dagger$ is the unique $m \times N$ matrix defined by the relations

$$MM^\dagger M = M, \quad M^\dagger MM^\dagger = M^\dagger, \quad (MM^\dagger)^* = MM^\dagger, \quad (M^\dagger M)^* = M^\dagger M,$$

where M^* denotes the adjoint (transpose) of M . Equivalently, the pseudo-inverse is determined by the geometric property that $MM^\dagger$ and $M^\dagger M$ are orthogonal projections onto $\text{im } M$ and $(\ker M)^\perp$ respectively. We refer the reader to the monograph [52] for more detail on generalized inverses.

With these considerations in mind, it is easy to see that the projection map $\mathbb{P} \in C^\infty(\mathbb{R}^n \setminus \{0\}, \text{Lin}(W, W))$ defined in (2.3) can be represented as

$$\mathbb{P}(\xi) = \text{Id}_W - \mathcal{A}^\dagger(\xi) \mathcal{A}(\xi) \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}. \quad (2.7)$$

The smoothness of $\mathbb{P}$ is well-known [85, Prop. 2.7]; for a proof using pseudo-inverses see [153, Sec. 4]. By the basic properties of pseudo-inverses, it is easy to see that, with the choice $\mathbb{B} = \mathbb{P}$, we have that (2.4) holds; however, the tensor-valued map $\mathbb{P}$ is 0-homogeneous, hence not polynomial in general. In particular, $\mathbb{P}$ cannot define a differential operator.

On the other hand, motivated by a similar construction in [201, Rk. 4.1], one can speculate that $\mathbb{P}$ and, in fact, $\mathcal{A}^\dagger(\cdot)$ are rational functions. This is indeed the case, as a consequence of the main result of DECELL in [63], building on the fundamental result of PENROSE [151, Thm. 2] and the Cayley–Hamilton Theorem.

Theorem 2.2.1 (DECELL [63, Thm. 3]). *Let $M \in \mathbb{R}^{N \times m}$ and denote by*

$$p(\lambda) := (-1)^N (a_0 \lambda^N + a_1 \lambda^{N-1} + \dots + a_N) \quad \text{for } \lambda \in \mathbb{R}$$

the characteristic polynomial of MM^ , where $a_0 = 1$. Define*

$$r := \max\{j \in \mathbb{N} : a_j \neq 0\}. \quad (2.8)$$

Then, if $r = 0$, we have that $M^\dagger = 0$; else

$$M^\dagger = -a_r^{-1} M^* [a_0 (MM^*)^{r-1} + a_1 (MM^*)^{r-2} + \dots + a_{r-1} \text{Id}_{N \times N}].$$

Proof of Theorem 2.1.1 (sufficiency). Suppose that $\mathcal{A}$ has constant rank. We put $M := \mathcal{A}(\xi)$ in the above Theorem for $\xi \in \mathbb{R}^n \setminus \{0\}$, and abbreviate $\mathcal{H}(\xi) := \mathcal{A}(\xi)\mathcal{A}^*(\xi)$. The first, perhaps most crucial, observation is that $r(\xi)$, as defined by (2.8), equals the number of non-zero eigen-values of MM^* , which equals the number of singular values of M . This is, in turn, equal to $\text{rank } M$, which is independent of ξ by the constant rank assumption on $\mathcal{A}$.

Therefore, if $r(\xi) = r = 0$, we have that $\mathcal{A}(\xi) = 0_{N \times m}$, $\mathcal{A}^\dagger(\xi) = 0_{m \times N}$, so we can simply choose $\mathbb{B}(\xi) = \text{Id}_W$, which satisfies (2.4) and gives rise to a linear, 0-homogeneous differential operator. Otherwise, if $r(\xi) = r > 0$, we obtain

$$\mathcal{A}^\dagger(\xi) = -a_r(\xi)^{-1} \mathcal{A}^*(\xi) [a_0(\xi) \mathcal{H}(\xi)^{r-1} + a_1(\xi) \mathcal{H}(\xi)^{r-2} + \dots + a_{r-1}(\xi) \text{Id}_X].$$

It is easy to see that $\mathcal{H}(\cdot)$ is a tensor-valued polynomial in ξ . The scalar fields a_j , $j = 1 \dots r$, are such that $a_j(\xi)$ is a coefficient of the characteristic polynomial of $\mathcal{H}(\xi)$, hence a linear combination of minors. In particular, a_j are scalar-valued polynomials in ξ .

It then follows that, with $\mathbb{P}$ as in (2.3),

$$\mathbb{B}(\xi) := a_r(\xi) \mathbb{P}(\xi) = a_r(\xi) \text{Id}_W - a_r(\xi) \mathcal{A}^\dagger(\xi) \mathcal{A}(\xi) \quad \text{for } \xi \in \mathbb{R}^n \quad (2.9)$$

defines a tensor-valued polynomial that satisfies (2.4). In particular, (2.9) gives rise to a linear differential operator. To check that it is homogeneous, it suffices to see that $a_r(\cdot)$ is a linear combination of minors of the same order of $\mathcal{H}(\cdot)$, which is homogeneous since $\mathcal{A}(\cdot)$ is. $\square$

The necessity of the constant rank condition in Theorem 2.1.1 follows from the following Lemma (with $P = \mathcal{A}(\cdot)$ and $Q = \mathbb{B}(\cdot)$) and the Rank–Nullity Theorem.

Lemma 2.2.2. *Let $S \subset \mathbb{R}^n$ be a set of positive Lebesgue measure and P, Q be two matrix-valued polynomials on $\mathbb{R}^n$. Suppose that there exists s such that*

$$\text{rank } P(\xi) + \text{rank } Q(\xi) = s \quad \text{for } \xi \in S.$$

Then both P and Q have constant rank in S .

Proof. We abbreviate $R_P := \text{rank } P$, $R_Q := \text{rank } Q$ and assume for contradiction that R_P is not constant in S . Say $R_P(S) = \{r_1, r_1 + 1, \dots, r_2\}$ for natural numbers $r_1 < r_2$. We also write M_d for the map that has input a matrix and returns (a vector of) all its minors of order d . In particular, $M_d P$, $M_d Q$ are vector-valued polynomials on $\mathbb{R}^n$. We then have that

$$R_P^{-1}(\{r_1, r_1 + 1, \dots, r_2 - 1\}) \subset \{\xi \in \mathbb{R}^n : M_{r_2} P(\xi) = 0\},$$

so that either $M_{r_2} P \equiv 0$ (which is not the case by definition of r_2) or $R_P^{-1}(\{r_1, r_1 + 1, \dots, r_2 - 1\})$ is Lebesgue-null². On the other hand,

$$\begin{aligned} R_P^{-1}(\{r_2\}) \cap S &= R_Q^{-1}(\{s - r_2\}) \cap S \\ &\subset R_Q^{-1}(\{s - r_2, s - r_2 + 1, \dots, s - r_1 - 1\}) \\ &\subset \{\xi \in \mathbb{R}^n : M_{s-r_1} Q(\xi) = 0\}, \end{aligned}$$

which is Lebesgue-null by the same argument. Since

$$S = [R_P^{-1}(\{r_1, r_1 + 1, \dots, r_2 - 1\}) \cap S] \cup [R_P^{-1}(\{r_2\}) \cap S],$$

it follows that S is Lebesgue-null and we arrive at a contradiction. $\square$

It is natural to ask the reversed question, whether a constant rank operator $\mathbb{B}$ admits an exact annihilator $\mathcal{A}$. This is indeed the case, as can be shown by a simple modification of the argument above:

Remark 2.2.3. *Let $\mathbb{B}$ be a linear, homogeneous, differential operator of constant rank on $\mathbb{R}^n$ from V to W . Then, we can choose $M := \mathbb{B}(\xi)$ for $\xi \in \mathbb{R}^n \setminus \{0\}$ in Theorem 2.2.1, so that*

$$\mathcal{A}(\xi) := a_r(\xi) \left[\text{Id}_W - \mathbb{B}(\xi) \mathbb{B}^\dagger(\xi) \right] \quad \text{for } \xi \in \mathbb{R}^n$$

satisfies (2.4) and gives rise to a differential operator. In particular, the formula is consistent with [201, Eq. (4.3)], which pertains to (overdetermined) elliptic operators $\mathbb{B}$, i.e., $\text{rank } \mathbb{B}(\xi) = \dim V$ for $\xi \neq 0$. In that case, one can compute both $\mathbb{B}^\dagger(\cdot)$ and $a_r(\cdot)$ explicitly from first principles. The new construction can be used to extend the L^1 -estimates in [201, 43] to constant rank operators.

We conclude the discussion of algebraic properties with two remarks: Firstly, it is quite convenient that the two constructions presented are explicitly computable. On the other hand, performing the computations on simple examples, e.g., involving only div, grad, curl, one easily notices that the operators constructed via our formulas are often overcomplicated. Perhaps more computationally efficient methods, e.g., in the spirit of [201, Sec. 4.2] can be developed.

2.3. $\mathcal{A}$ -quasiconvexity

The relevance of Theorem 2.1.1 for analysis can be seen, for instance, from the fact that periodic $\mathcal{A}$ -free fields have differential structure:

²For an elementary proof of this fact, see [53].

Lemma 2.3.1. *Let $\mathcal{A}, \mathbb{B}$ be linear, homogeneous, differential operators of constant rank with constant coefficients on $\mathbb{R}^n$ from W to X , and from V to W , respectively. Assume that (2.4) holds. Then for all $w \in C^\infty(\mathbb{T}_n, W)$ such that $\mathcal{A}w = 0$ and $\int_{\mathbb{T}_n} w(x) dx = 0$, there exists $u \in C^\infty(\mathbb{T}_n, V)$ such that $w = \mathbb{B}u$. Similarly, for all $w \in \mathcal{S}(\mathbb{R}^n, W)$ such that $\mathcal{A}w = 0$, there exists $u \in \mathcal{S}(\mathbb{R}^n, V)$ such that $w = \mathbb{B}u$.*

Here $\mathbb{T}_n$ denotes the n -dimensional torus, identified in an obvious way with (a quotient of) $[0, 1]^n$. The Fourier transform is defined as

$$\hat{u}(\xi) := \int_{\mathbb{T}_n} u(x) e^{-2\pi i x \cdot \xi} dx,$$

for $\xi \in \mathbb{Z}^n$.

Proof. Let $w \in C^\infty(\mathbb{T}_n, W)$ have zero average and satisfy $\mathcal{A}w = 0$, so that

$$w(x) = \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} \hat{w}(\xi) e^{2\pi i x \cdot \xi},$$

for $x \in \mathbb{T}_n$, where the coefficients $\hat{w}(\xi) \in \ker \mathcal{A}(\xi)$ decay faster than any polynomial as $|\xi| \rightarrow \infty$. We define

$$u(x) := \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} \mathbb{B}^\dagger(\xi) \hat{w}(\xi) e^{2\pi i x \cdot \xi},$$

for $x \in \mathbb{T}_n$, which is smooth by homogeneity of $\mathbb{B}^\dagger(\cdot)$: say $\mathbb{B}$ has order l , then $\mathbb{B}^\dagger(\cdot)$ is $(-l)$ -homogeneous. We can thus differentiate the sum term by term to obtain

$$\begin{aligned} \mathbb{B}u(x) &= (2\pi i)^l \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} \mathbb{B}(\xi) \mathbb{B}^\dagger(\xi) \hat{w}(\xi) e^{2\pi i x \cdot \xi} \\ &= (2\pi i)^l \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} \hat{w}(\xi) e^{2\pi i x \cdot \xi} \\ &= (2\pi i)^l w(x), \end{aligned}$$

where the exactness relation (2.4) is used in the second equality, along with the geometric properties of the pseudo-inverse. The proof of the first case is complete.

We give an analogous argument for the case when $w \in \mathcal{S}(\mathbb{R}^n, W)$ is $\mathcal{A}$ -free. We have the pointwise relation $\mathcal{A}(\xi) \hat{w}(\xi) = 0$, so that (2.4) implies that $w \in \text{im } \mathbb{B}(\xi)$ and we can define

$$\hat{u}(\xi) := \mathbb{B}^\dagger(\xi) \hat{w}(\xi),$$

which satisfies the required properties. $\square$

We conclude this Section by showing that one can test with compactly supported smooth maps in the definition of $\mathcal{A}$ -quasiconvexity.

Corollary 2.3.2. *Let $\mathcal{A}, \mathbb{B}$ be as in Lemma 2.3.1 and $f: W \rightarrow \mathbb{R}$ be Borel measurable and locally bounded. Then*

$$Q_{\mathcal{A}}f(\eta) := \inf \left\{ \int_{\mathbb{T}_n} f(\eta + w(x)) dx : w \in C^\infty(\mathbb{T}_n, W), \mathcal{A}w = 0, \int_{\mathbb{T}_n} w(x) dx = 0 \right\},$$

$$Q^{\mathbb{B}}f(\eta) := \inf \left\{ \int_{[0,1]^n} f(\eta + \mathbb{B}u(x)) \, dx : u \in C_c^\infty((0,1)^n, V) \right\}$$

are equal for all $\eta \in W$. Moreover, if $\mathbb{B}$ has order l and $\alpha \in [0, 1)$, we have

$$Q_{\mathcal{A}}f(\eta) = \inf \left\{ \int_{[0,1]^n} f(\eta + \mathbb{B}u(x)) \, dx : u \in C_c^\infty((0,1)^n, V), \|u\|_{C^{l-1,\alpha}} < \varepsilon \right\} \quad (2.10)$$

for any $\eta \in W$ and $\varepsilon > 0$.

The proof follows standard arguments; in particular we follow [61, Prop. 5.13] and [113, Thm. 4.2] and include the proof for completeness of the present work.

Proof. It is obvious that $Q_{\mathcal{A}}f \leq Q^{\mathbb{B}}f$. To prove the opposite inequality, let $\varepsilon > 0$, $\eta \in W$, and w be a periodic field as in the definition of $Q_{\mathcal{A}}f(\eta)$. We will construct $v \in C_c^\infty((0,1)^n, V)$ such that

$$\int_{[0,1]^n} f(\eta + \mathbb{B}v(x)) \, dx \leq \int_{[0,1]^n} f(\eta + w(x)) \, dx + \varepsilon. \quad (2.11)$$

By Lemma 2.3.1, we have that $w = \mathbb{B}u$ for a periodic field $u \in C^\infty(\mathbb{T}_n, V)$. Say, as before, that $\mathbb{B}$ has order l and define $u_N(x) := N^{-l}u(Nx)$ for N sufficiently large. This does not change the value of the integral over the cube. Next, let $\delta > 0$ be sufficiently small and truncate to obtain $u_N^\delta := \rho^\delta u_N$, where $\rho^\delta \in C_c^\infty([0,1]^n)$ is such that $\rho^\delta(x) = 1$ if $\text{dist}(x, \partial[0,1]^n) > \delta$ and $|\nabla^j \rho^\delta| \leq C\delta^{-j}$ for $j = 0 \dots l$ and some constant $C > 0$. We impose $\delta N \geq 1$ and leave δ to be determined. It follows, for $c_1 \geq 1$ depending on $\mathbb{B}$ only, that

$$\begin{aligned} |\mathbb{B}u_N^\delta| &\leq |\rho^\delta \mathbb{B}u_N| + c_1 \sum_{j=1}^l |\nabla^j \rho^\delta| |\nabla^{l-j} u_N| \\ &\leq c_1 C \left(\|\mathbb{B}u\|_{L^\infty} + \sum_{j=1}^l (\delta N)^{-j} \|\nabla^{l-j} u\|_{L^\infty} \right) \\ &\leq c_1 C \left(\|\mathbb{B}u\|_{L^\infty} + \sum_{j=0}^{l-1} \|\nabla^j u\|_{L^\infty} \right) =: c_1 C \|u\|_{W^{\mathbb{B}, \infty}}. \end{aligned}$$

Say f is bounded by $M > 0$ on $B(0, |\eta| + c_1 C \|u\|_{W^{\mathbb{B}, \infty}})$. Hence, if we choose δ such that $\mathcal{L}^n(\{x \in [0,1]^n : \text{dist}(x, \partial[0,1]^n) \leq \delta\}) \leq M^{-1}\varepsilon$, we obtain

$$\begin{aligned} \int_{[0,1]^n} f(\eta + \mathbb{B}u_N^\delta(x)) \, dx &\leq \int_{\text{dist}(x, \partial[0,1]^n) < \delta} M \, dx + \int_{[0,1]^n} f(\eta + \mathbb{B}u_N(x)) \, dx \\ &\leq M \times M^{-1}\varepsilon + \int_{[0,1]^n} f(\eta + w(x)) \, dx, \end{aligned}$$

which implies (2.11) with $v := u_N^\delta$. To prove the equality of the two envelopes, we distinguish two cases: If $Q_{\mathcal{A}}f(\eta) > -\infty$, we can choose w such that

$$\int_{[0,1]^n} f(\eta + w(x)) \, dx \leq Q_{\mathcal{A}}f(\eta) + \varepsilon,$$

and we conclude that $Q_{\mathcal{A}}f(\eta) = Q^{\mathbb{B}}f(\eta)$ by (2.11) since $\varepsilon > 0$ is arbitrary. If $Q_{\mathcal{A}}f(\eta) = -\infty$, we choose w such that

$$\int_{[0,1]^n} f(\eta + w(x)) \, dx \leq -\varepsilon^{-1},$$

so that we can conclude by (2.11) that $Q^{\mathbb{B}}f(\eta) = -\infty$.

To prove (2.10), we need only show that the infimum is smaller than the envelope. Firstly, note as above that by replacing u with $u_N(x) = N^{-l}u(Nx)$, where u is extended by periodicity to $\mathbb{R}^n$, the value of the integral does not change. It suffices to choose N large enough so that u_N has small $C^{l-1,\alpha}$ -norm. Note that for $j = 0 \dots l-1$ we have

$$\|\nabla^j u_N\|_{\infty} = N^{j-l} \|\nabla^j u\|_{\infty},$$

which can clearly be made arbitrarily small.

Finally, to check the Hölder bound, say that $\{z_i + [0, N^{-1}]^n\}_{i=1}^{N^n}$ is a covering of $[0, 1]^n$ by cubes of side-length N^{-1} that can only touch at their boundaries and let $x, y \in [0, 1]^n$. If x, y lie in the same cube $z_i + [0, N^{-1}]^n$, we have that

$$\begin{aligned} |\nabla^{l-1} u_N(x) - \nabla^{l-1} u_N(y)| &= N^{-1} |\nabla^{l-1} u(Nx - z_i) - \nabla^{l-1} u(Ny - z_i)| \\ &\leq \|\nabla^l u\|_{\infty} |x - y| \\ &\leq (\sqrt{n}N^{-1})^{1-\alpha} \|\nabla^l u\|_{\infty} |x - y|^{\alpha}, \end{aligned}$$

which can be made small since $1 - \alpha > 0$. If x, y lie in different cubes, which we label Q_x, Q_y . Let $\bar{x} \in \partial Q_x \cap (x, y)$, $\bar{y} \in \partial Q_y \cap (x, y)$, so that $|x - y| \geq |x - \bar{x}| + |y - \bar{y}|$, $|x - \bar{x}|, |y - \bar{y}| \leq \sqrt{n}N^{-1}$, and all derivatives of u_N vanish near $\bar{x}, \bar{y}$. Using these facts and the previous step we get

$$\begin{aligned} |\nabla^{l-1} u_N(x) - \nabla^{l-1} u_N(y)| &\leq |\nabla^{l-1} u_N(x) - \nabla^{l-1} u_N(\bar{x})| \\ &\quad + |\nabla^{l-1} u_N(y) - \nabla^{l-1} u_N(\bar{y})| \\ &\leq (\sqrt{n}N^{-1})^{1-\alpha} \|\nabla^l u\|_{\infty} (|x - \bar{x}|^{\alpha} + |y - \bar{y}|^{\alpha}) \\ &\leq (\sqrt{n}N^{-1})^{1-\alpha} \|\nabla^l u\|_{\infty} 2^{-\alpha} |x - y|^{\alpha}, \end{aligned}$$

where the last inequality follows by concavity and monotonicity of $0 \leq t \mapsto t^{\alpha}$. The proof is complete. $\square$

Remark 2.3.3. Using the argument in Corollary 2.3.2, one can show for constant rank operators $\mathcal{A}$ that $\mathcal{A}$ -quasiconvexity, as defined by FONSECA and MÜLLER in [85, p.1357], coincides with $\mathcal{A}$ - $\mathbb{B}$ -quasiconvexity, as introduced by DACOROGNA in [59, 60] (to be precise, in the original definition of $\mathcal{A}$ - $\mathbb{B}$ -quasiconvexity, the operator $\mathbb{B}$ is assumed to be of first order, but this is only a minor technical restriction). In this case, it is not difficult to prove that [60, Thm. 4] is unconditional. A proof of this fact will be given elsewhere.

We also have that $\mathcal{A}$ -quasiconvexity can be defined by integrals over arbitrary domains, instead of cubes.

Lemma 2.3.4. *Let $\mathcal{A}, \mathbb{B}$ be as in Lemma 2.3.1 and $f: W \rightarrow \mathbb{R}$ be Borel measurable, locally bounded, and $\mathcal{A}$ -quasiconvex, and Ω be a bounded open set. Then*

$$f(\eta) \leq \int_{\Omega} f(\eta + \mathbb{B}v(y)) \, dy$$

for all $\eta \in W$ and $v \in C_c^\infty(\Omega, V)$.

The proof follows from a simple argument in the Calculus of Variations [61, Prop. 5.11].

Proof. Fix $\eta \in W$, $v \in C_c^\infty(\Omega, V)$, extended by zero to $\mathbb{R}^n$. By the argument in the proof of Corollary 2.3.2, we write $C := (0, 1)^n$ and have that

$$f(\eta) \leq \int_C f(\eta + \mathbb{B}u(x)) \, dx$$

for all $u \in C_c^\infty(C, V)$. For sufficiently small $\varepsilon > 0$, we can find $x_0 \in \mathbb{R}^n$ such that $x_0 + \varepsilon\Omega \subset C$. We define

$$u(x) := \varepsilon^l v\left(\frac{x - x_0}{\varepsilon}\right),$$

so that

$$\begin{aligned} f(\eta) &\leq \int_C f(\eta + \mathbb{B}u(x)) \, dx = |C \setminus (x_0 + \varepsilon\Omega)| f(\eta) + \int_{x_0 + \varepsilon\Omega} f(\eta + \mathbb{B}u(x)) \, dx \\ &= (1 - \varepsilon^n |\Omega|) f(\eta) + \int_{\Omega} f(\eta + \mathbb{B}v(y)) \varepsilon^n \, dy. \end{aligned}$$

Rearranging the terms we obtain the conclusion. $\square$

2.4. $\mathcal{A}$ -free Young measures

We recall the definition of (oscillation) Young measures, while also giving a simplified variant of the Fundamental Theorem of Young measures.

Theorem 2.4.1 (FTYM, [146, 149]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set and $z_j \in L^1(\Omega, \mathbb{R}^d)$ be a bounded sequence in L^1 . Then there exists a subsequence (not relabelled) and a weakly- $*$ measurable map $\nu: \Omega \rightarrow \mathcal{P}(\mathbb{R}^d)$ (or parametrized measure $\nu = (\nu_x)_{x \in \Omega}$) such that for all $f \in C(\Omega \times \mathbb{R}^d)$ bounded from below we have that*

$$\liminf_{j \rightarrow \infty} \int_{\Omega} f(x, z_j(x)) \, dx \geq \int_{\Omega} \langle f(x, \cdot), \nu_x \rangle \, dx.$$

Moreover,

$$\lim_{j \rightarrow \infty} \int_{\Omega} f(x, z_j(x)) \, dx = \int_{\Omega} \langle f(x, \cdot), \nu_x \rangle \, dx$$

if and only if the sequence $f(\cdot, z_j)$ is uniformly integrable.

Above, $\mathcal{P}(\mathbb{R}^d)$ denotes the space of probability measures on $\mathbb{R}^d$. In the notation of Theorem 2.4.1, we say that z_j *generates the Young measure* ν (in symbols, $z_j \xrightarrow{\mathbf{Y}} \nu$). We also recall that a sequence z_j is said to be uniformly integrable if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all borel sets $E \subset \Omega$, we have that

$$\mathcal{L}^n(E) < \delta \implies \sup_j \int_E |z_j| dx < \varepsilon,$$

or, equivalently, if

$$\lim_{\alpha \rightarrow \infty} \sup_j \int_{\{|z_j| > \alpha\}} |z_j| dx = 0.$$

If $|z_j|^p$ is uniformly integrable, we say that z_j is *p-uniformly integrable*.

Lemma 2.4.2 ([85, Prop. 2.4]). *Let z_j generate a Young measure ν and $\tilde{z}_j \rightarrow \tilde{z}$ in measure. Then $z_j + \tilde{z}_j$ generates the Young measure μ given by $\mu_x = \nu_x \star \delta_{\tilde{z}(x)}$ for $\mathcal{L}^n$ a.e. x , i.e.,*

$$\langle \varphi, \mu_x \rangle = \langle \varphi(\cdot + \tilde{z}(x), \nu_x) \rangle$$

for any $\varphi \in C_0$.

The following is an extension of [85, Lem. 2.15]. The first two steps of the present proof are almost a repetition of their arguments, which we include since the original proof only covers first order annihilators $\mathcal{A}$.

Proposition 2.4.3. *Let $\mathcal{A}, \mathbb{B}$ be as in Lemma 2.3.1 and have orders k, l , respectively, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and $1 < p < \infty$. Let $w_j, w \in L^p(\Omega, W)$ be such that*

$$\begin{aligned} w_j &\rightharpoonup w \text{ in } L^p(\Omega, W), \\ \mathcal{A}w_j &\rightarrow \mathcal{A}w \text{ in } W_{\text{loc}}^{-k,p}(\Omega, X), \\ w_j &\xrightarrow{\mathbf{Y}} \nu. \end{aligned}$$

Then there exists a sequence $u_j \in C_c^\infty(\Omega, V)$ such that

$$\begin{aligned} \mathbb{B}u_j &\rightharpoonup 0 \text{ in } L^p(\Omega, W), \\ \mathbb{B}u_j + w &\xrightarrow{\mathbf{Y}} \nu. \end{aligned}$$

Moreover, u_j can be chosen such that $(\mathbb{B}u_j)_j$ is *p-uniformly integrable*.

A Young measure ν satisfying the assumptions of Proposition 2.4.3 is said to be an $\mathcal{A}$ -free Young measure.

Proof. By Lemma 2.4.2 and linearity we can assume that $w = 0$. We will identify maps defined on Ω with their extensions by zero to full-space without mention. Uniform integrability considerations strictly refer to sequences defined on Ω .

Step I. We construct *p*-uniformly integrable $\tilde{w}_j \in C_c^\infty(\Omega, W)$ such that $\tilde{w}_j \rightharpoonup 0$ in $L^p(\Omega, W)$, $\mathcal{A}\tilde{w}_j \rightarrow 0$ in $W^{-k,q}(\mathbb{R}^n, X)$ for some $1 < q < p$, and $\tilde{w}_j$ generates ν .

Recall the truncation operators, defined for $\alpha > 0$ by

$$\tau_\alpha A := \begin{cases} A & \text{if } |A| \leq \alpha \\ \alpha A/|A| & \text{if } |A| > \alpha, \end{cases}$$

which are clearly Carathéodory integrands. By Theorem 2.4.1, we have that

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \lim_{j \rightarrow \infty} \int_{\Omega} |\tau_\alpha w_j|^p dx &= \lim_{\alpha \rightarrow \infty} \int_{\Omega} \int_W |\tau_\alpha A|^p d\nu_x(A) dx \\ &= \int_{\Omega} \int_W |A|^p d\nu_x(A) dx < \infty, \end{aligned}$$

so that we can choose a diagonal subsequence $\alpha_j \uparrow \infty$ such that $\int_{\Omega} |\tau_{\alpha_j} w_j|^p dx$ equals the p -th moment of ν . It also follows from Theorem 2.4.1 that $(\tau_{\alpha_j} w_j)_j$ is p -uniformly integrable.

We now show that $\tau_{\alpha_j} w_j$ generates ν . Since w_j converges weakly in $L^p(\Omega, W)$, it converges weakly in L^1 , hence is uniformly integrable, so that $\tau_{\alpha_j} w_j - w_j \rightarrow 0$ in measure. It also follows by elementary manipulations that $\tau_{\alpha_j} w_j - w_j \rightarrow 0$ in L^p , so that, indeed, $\tau_{\alpha_j} w_j$ generates ν by Lemma 2.4.2.

Let $1 < q < p$. We have that

$$\|\tau_{\alpha_j} w_j - w_j\|_{L^q(\Omega, W)} \leq \int_{\{|w_j| > \alpha_j\}} 2^q |w_j|^q dx \leq 2^q \alpha_j^{q-p} \int_{\{|w_j| > \alpha_j\}} |w_j|^p dx \rightarrow 0,$$

so that $\mathcal{A}\tau_{\alpha_j} w_j \rightarrow 0$ in $W_{\text{loc}}^{-k, q}(\Omega, X)$. We also record that $\tau_{\alpha_j} w_j$ is precompact in $W^{-1, q}(\Omega, W)$, so that $D^\beta \tau_{\alpha_j} w_j \rightarrow 0$ in $W^{-k, q}(\Omega, X)$ for $|\beta| < k$.

We can therefore choose a sequence of cut-off functions $\rho_j \in C_c^\infty(\Omega, [0, 1])$ such that $\rho_j \uparrow 1$ in Ω and $\|\rho_j \mathcal{A}\tau_{\alpha_j} w_j\|_{W^{-k, q}(\mathbb{R}^n, X)} \rightarrow 0$ and

$$\mathcal{A}(\rho_j \tau_{\alpha_j} w_j) = \rho_j \mathcal{A}\tau_{\alpha_j} w_j + \sum_{m=1}^k B_m[D^m \rho_j, D^{k-m} \tau_{\alpha_j} w_j] \rightarrow 0 \quad \text{in } W^{-k, q}(\mathbb{R}^n, X),$$

where B_m are fixed bi-linear pairings given by the Leibniz rule. To see that this is possible, consider $\Omega_j := \{x \in \Omega : \text{dist}(x, \partial\Omega) < j\}$, where $s_j \downarrow 0$ will be determined. We require that $\rho_j = 1$ in $\Omega \setminus \Omega_{s_j}$, $\rho_j = 0$ in Ω_{2s_j} and $|D^m \rho_j| \leq c s_j^{-m}$, $m = 1, \dots, k$. It is easy to see that the sum above is controlled in $W^{-k, q}$ by

$$\sum_{m=1}^k \|D^m \rho_j\|_{L^\infty} \|D^{k-m} \tau_{\alpha_j} w_j\|_{W^{-k, q}} \leq c \sum_{m=1}^k s_j^{-m} \|D^{k-m} \tau_{\alpha_j} w_j\|_{W^{-k, q}},$$

so that it suffices to choose any $s_j \geq \max_{m=1, \dots, k} \|D^{k-m} \tau_{\alpha_j} w_j\|_{W^{-k, q}}^{1/(2m)} \downarrow 0$ as $j \rightarrow \infty$. Alternatively, one can consider a different cut-off sequence $\rho_i \uparrow 1$ and employ a diagonalization argument.

We define

$$\tilde{w}_j := (\rho_j \tau_{\alpha_j} w_j) \star \eta_{\varepsilon(j)},$$

where $\eta_{\varepsilon(j)}$ denotes a standard sequence of (radial, positive) mollifiers and $\varepsilon(j) \downarrow 0$ is such that $\tilde{w}_j \in C_c^\infty(\Omega, W)$ and, therefore, $\mathcal{A}\tilde{w}_j \rightarrow 0$ in $W^{-k,q}(\mathbb{R}^n, X)$. The latter inequality follows since, for all $\varphi \in C_c^\infty(\mathbb{R}^n, W)$ with $\|\varphi\|_{W^{k,q}} \leq 1$,

$$\begin{aligned} \langle \mathcal{A}\tilde{w}_j, \varphi \rangle &= \langle \mathcal{A}(\rho_j \tau_{\alpha_j} w_j), \varphi \star \eta_{\varepsilon(j)} \rangle \leq \|\mathcal{A}(\rho_j \tau_{\alpha_j} w_j)\|_{W^{-k,q}} \|\varphi \star \eta_{\varepsilon(j)}\|_{W^{k,q}} \\ &\leq \|\mathcal{A}(\rho_j \tau_{\alpha_j} w_j)\|_{W^{-k,q}} \rightarrow 0. \end{aligned}$$

It is also clear that $\|\tilde{w}_j - \tau_{\alpha_j} w_j\|_{L^p} \rightarrow 0$, so that $\tilde{w}_j$ is p -uniformly integrable, converges weakly to 0 in L^p , and generates ν .

Step II. We project $\tilde{w}_j$ on the kernel of $\mathcal{A}$ in $\mathbb{R}^n$ and show that $\mathbb{P}\tilde{w}_j$ are p -uniformly integrable in Ω , converge weakly to zero in L^p , and generate ν . Here the L^2 -orthogonal projection operator $\mathbb{P}$ is given by the multiplier in (2.7),

$$\widehat{\mathbb{P}w}(\xi) := \mathbb{P}(\xi)\hat{w}(\xi) = [\text{Id}_W - \mathcal{A}^\dagger(\xi)\mathcal{A}(\xi)]\hat{w}(\xi) \quad \text{for } w \in \mathcal{S}(\mathbb{R}^n, W).$$

Since the symbol $\mathbb{P}(\cdot)$ is homogeneous of degree zero, $\mathbb{P}$ is a singular integral operator of convolution type; in particular $\mathbb{P}$ maps Schwartz functions to Schwartz functions. Moreover, we have that

$$\mathcal{F}(\tilde{w}_j - \mathbb{P}\tilde{w}_j)(\xi) = \mathbb{B}^\dagger(\xi)\mathbb{B}(\xi)\mathcal{F}\tilde{w}_j(\xi) = \mathcal{A}^\dagger\left(\frac{\xi}{|\xi|}\right)\frac{\widehat{\mathcal{A}\tilde{w}_j}(\xi)}{|\xi|^k},$$

so that, by boundedness of singular integrals on L^q

$$\|\tilde{w}_j - \mathbb{P}\tilde{w}_j\|_{L^q(\mathbb{R}^n, W)} \leq c \left\| \mathcal{F}^{-1}\left(\frac{\mathcal{A}\tilde{w}_j}{|\cdot|^k}\right) \right\|_{L^q(\mathbb{R}^n, X)} = c\|\mathcal{A}\tilde{w}_j\|_{W^{-k,q}(\mathbb{R}^n, X)} \rightarrow 0.$$

It immediately follows by Lemma 2.4.2 that $\mathbb{P}\tilde{w}_j$ generates ν . To see that $\mathbb{P}\tilde{w}_j \rightarrow 0$ in $L^p(\Omega, W)$, we note that, since $\mathbb{P}$ is (pointwisely) self-adjoint, we have, for any $g \in L^{p/(p-1)}(\Omega, W)$,

$$\int_\Omega \langle g, \mathbb{P}\tilde{w}_j \rangle dx = \int_\Omega \langle \mathbb{P}g, \tilde{w}_j \rangle dx \rightarrow 0,$$

since $\mathbb{P}g \in L^{p/(p-1)}(\Omega, W)$ by boundedness of singular integrals.

To see that $\mathbb{P}\tilde{w}_j$ is p -uniformly integrable, we use the idea in [85, Lem. 2.14.(iv)]. We first note, by boundedness of $\mathbb{P}$ on L^p , that

$$\sup_j \|\mathbb{P}\tilde{w}_j - \mathbb{P}\tau_{\alpha_j}\tilde{w}_j\|_{L^p(\mathbb{R}^n, W)} \leq c \sup_j \|\tilde{w}_j - \tau_{\alpha_j}\tilde{w}_j\|_{L^p(\mathbb{R}^n, W)} \rightarrow 0 \quad \text{as } \alpha \rightarrow \infty$$

by p -uniform integrability of $\tilde{w}_j$. Note that for each fixed α , $\mathbb{P}\tau_{\alpha}\tilde{w}_j$ is bounded in L^r for any $p < r < \infty$, hence is p -uniformly integrable. Let $\varepsilon > 0$. We choose $\alpha > 0$ such that

$$\sup_j \|\mathbb{P}\tilde{w}_j - \mathbb{P}\tau_{\alpha}\tilde{w}_j\|_{L^p(\mathbb{R}^n, W)} < \varepsilon$$

and also choose $\delta > 0$ such that for each Borel set $E \subset \Omega$ with $\mathcal{L}^n(\Omega) < \delta$, we have that $\int_E |\mathbb{P}\tau_{\alpha}\tilde{w}_j|^p dx < \varepsilon$ for all j . It follows that for all such E ,

$$\int_E |\mathbb{P}\tilde{w}_j|^p dx \leq 2^{p-1} \left(\sup_j \int_E |\mathbb{P}\tilde{w}_j - \mathbb{P}\tau_{\alpha}\tilde{w}_j|^p dx + \sup_j \int_E |\mathbb{P}\tau_{\alpha}\tilde{w}_j|^p dx \right) < (2\varepsilon)^p,$$

where the right hand side is independent of j . The second step is concluded.

Step III. Using Lemma 2.3.1, we can write $\mathbb{P}\tilde{w}_j = \mathbb{B}u_j$, where $\hat{u}_j(\xi) := \mathbb{B}^\dagger(\xi)\widehat{\mathbb{P}\tilde{w}_j}(\xi)$, so that $u_j \in \mathcal{S}(\mathbb{R}^n, V)$. It remains to cut-off u_j suitably.

Since $\mathbb{B}$ has order l , we first note that

$$\widehat{D^l u}(\xi) = \mathbb{B}^\dagger(\xi)\widehat{\mathbb{B}u}(\xi) \otimes \xi^{\otimes l},$$

so that $\mathbb{B}u \mapsto D^l u$ is a singular integral operator of convolution type. It follows that $D^l u_j$ is bounded in $L^p(\mathbb{R}^n)$ (recall here that $\mathbb{B}u_j = \mathbb{P}\tilde{w}_j$ is bounded in L^p as $\tilde{w}_j \in C_c^\infty(\Omega, W)$ is a weakly convergent sequence), so u_j is bounded in $W^{l,p}(\Omega, V)$.

By compactness of the embedding $W^{l,p}(\Omega) \hookrightarrow W^{l-1,p}(\Omega)$, we have that $u_j \rightarrow u$ in $W^{l-1,p}(\Omega, V)$. Since $\mathbb{B}u_j \rightharpoonup 0$, we have that $\mathbb{B}u = 0$. On the other hand, $u = \mathcal{F}^{-1}[\mathbb{B}^\dagger(\cdot)] \star (\mathbb{B}u) = 0$, so that $D^{l-m}u_j \rightarrow 0$ in $L^p(\Omega)$ for $m = 1, \dots, l$.

We now proceed similarly to Step I. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be such that $\rho_j = 1$ in $\Omega \setminus \Omega_{s_j}$ and $|D^m \rho_j| \leq c s_j^{-m}$, $m = 1, \dots, l$, where

$$s_j := \max_{m=1, \dots, l} \|D^{l-m} u_j\|_{L^p(\Omega)}^{1/(2m)} \rightarrow 0.$$

We can then estimate

$$\begin{aligned} \|\mathbb{B}u_j - \mathbb{B}(\rho_j u_j)\|_{L^p(\Omega)} &\leq \|(1 - \rho_j)\mathbb{B}u_j\|_{L^p(\Omega)} + \sum_{m=1}^l \|B_m[D^m \rho_j, D^{l-m} u_j]\|_{L^p(\Omega)} \\ &\leq \|\mathbb{B}u_j\|_{L^p(\Omega_{s_j})} + c \sum_{m=1}^l s_j^{-m} \|D^{l-m} u_j\|_{L^p(\Omega)}, \end{aligned}$$

which tends to zero by p -uniform integrability of $\mathbb{B}u_j$ and the choice of s_j . Here B_m is another collection of bi-linear pairings given by the product rule. It then follows that $\mathbb{B}(\rho_j u_j)$ converges weakly to zero in $L^p(\Omega, W)$, is p -uniformly integrable, and generates ν . The proof is complete. $\square$

The limit points $p = 1$ and $p = \infty$ of Proposition 2.4.3 are significantly more involved and deserve special attention. As is the case for the entire thesis, the case $p = \infty$ is not in our scope (yet), but the limit case $p = 1$ very much is. The natural set-up in that case, as explained in the Introduction, is that of generalized Young measures and is significantly more difficult to analyze since one cannot perform shifts in a desirable way. However, a partial result in this direction is very much in the scope of the next Chapter (Theorem 3.1.1).

CHAPTER 3

Characterization of Young Measures generated by $\mathcal{A}$ -free measures

This chapter consists of ongoing joint work with J. Kristensen.

3.1. Introduction

Since their Introduction in [206, 207, 208], Young measures are widely acknowledged as particularly effective tools to describe minimization of non-convex energy functionals in the Calculus of Variations. Young measures have also been successfully used in several other branches of partial differential equations, as tools to describe the interaction between weak convergence and non-linear functionals, see for instance the works of TARTAR [183, 184] and DiPERNA [69, 70, 71].

In the study of lower semi-continuity of variational integrals

$$\mathcal{E}[u] = \int_{\Omega} f(Du) \, dx$$

defined over spaces of weakly differentiable functions, Young measures can be viewed as one-point statistics of weakly converging gradients that describe the relaxation of the energy $\mathcal{E}$. It was shown by KINDERLEHRER and PEDREGAL in [111, 112] that Young measures generated by sequences of gradients weakly convergent in L^p can be characterized by duality with quasiconvex functions through Jensen-type inequalities, thereby showing that the classical lower semi-continuity theorem of MORREY [140] can be described completely by Young measures and quasiconvex functions. More technically, the inequality

$$f(\langle \text{Id}, \nu_x \rangle) \leq \langle f, \nu_x \rangle \quad \text{for a.e. } x \in \Omega \quad (3.1)$$

holds for all quasiconvex f of p -growth if and only if the Young measure ν (which is a parametrized measure $(\nu_x)_{x \in \Omega}$) is generated by a sequence Du_j that converges weakly in L^p .

As a generalization to the MURAT-TARTAR framework of compensated compactness [142, 143, 183, 184, 185], FONSECA and MÜLLER showed in [85] that the Kinderlehrer-Pedregal Theorem can be generalized to characterize Young measures generated by sequences satisfying general differential constraints. To briefly outline their result, we introduce a compensating differential operator $\mathcal{A}$ on $\mathbb{R}^n$

from W to X (finite dimensional Euclidean spaces) and its symbol map $\mathcal{A}(\cdot)$:

$$\mathcal{A} = \sum_{|\alpha|=k} A_\alpha \partial^\alpha \quad \text{and} \quad \mathcal{A}(\xi) = \sum_{|\alpha|=k} \xi^\alpha A_\alpha \quad \text{for } \xi \in \mathbb{R}^n. \quad (3.2)$$

We will assume that $\mathcal{A}$ satisfies the so-called *constant rank* condition,

$$\text{rank } \mathcal{A}(\xi) = r \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \quad (3.3)$$

The constant rank condition is a fairly mild algebraic condition introduced by MURAT in the context of compensated compactness and used earlier in [166, 167] in connection with some degenerate coerciveness inequalities. It may be the case that matrix-valued polynomial fields of constant rank were studied even earlier in algebra.

In this case, the notion of quasiconvexity that respects the differential constraint was introduced in [85] and is termed $\mathcal{A}$ -quasiconvexity (see Section 3.2.3 for a definition). It can then be shown that $\mathcal{A}$ -quasiconvex functions of p -growth can be characterized by duality with sequences v_j that satisfy $\mathcal{A}v_j = 0$ and are weakly convergent in L^p , formally, by the same inequality (3.1).

In the case of linear growth functionals, $p = 1$, weak convergence of minimizing sequences cannot, in general, be inferred from standard coercivity assumptions. Rather, one should consider lower semi-continuity with respect to weakly* convergence of measures, which cannot be described by (classical/oscillation) Young measures. In this case, a generalization of the Kinderlehrer-Pedregal Theorem characterizing weakly* convergence of BV-maps by duality with quasiconvex functions was proved by KRISTENSEN and RINDLER in [122], using generalized Young measures, as introduced in [72] and developed in [13]. The result was later simplified in [113, Sec. 6] to a form that we present now.

Generalized Young measures are defined as (triples of) parametrized measures $\nu = ((\nu_x)_x, \lambda, (\nu_x^\infty))$, where ν_x are probability measures describing oscillations, λ is a positive measure describing location of concentrations, and ν_x^∞ are probability measures describing the directions of concentration. In this case, the inequality (3.1) takes the form

$$f(\langle \text{Id}, \nu_x \rangle + \langle \text{Id}, \nu_x^\infty \rangle \lambda^a(x)) \leq \langle f, \nu_x \rangle + \langle f^\infty, \nu_x^\infty \rangle \lambda^a(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega, \quad (3.4)$$

where $\lambda = \lambda^a \mathcal{L}^n \llcorner \Omega + \lambda^s$ is the Radon-Nikodým decomposition with respect to Lebesgue measure and f^∞ denotes the *recession function*

$$f^\infty(\xi) = \limsup_{t \rightarrow \infty} \frac{f(t\xi)}{t}.$$

It was shown in [122, 113] that the inequality (3.4) holds for all quasiconvex f of linear growth if and only if ν is generated by a sequence Du_j , where $u_j \in \text{BV}$.

Few generalizations to other compensating operators are known [66, 25]. The aim of this work is to provide a generalization to all constant rank operators $\mathcal{A}$, modulo the stringent technical assumption that we only characterize those Young

measures with barycentre in L^1 . We also require the mild algebraic assumption that

$$\text{span } \Lambda_{\mathcal{A}} = W, \quad \text{where} \quad \Lambda_{\mathcal{A}} = \bigcup_{\xi \in \mathbb{R}^n \setminus \{0\}} \ker \mathcal{A}(\xi). \quad (3.5)$$

This assumption is not too restrictive since $\mathcal{A}$ -free, periodic maps are valued in $\text{span } \Lambda_{\mathcal{A}}$ [21, Sec. 2.5]. To simplify the argument, we will use the recent construction in [156] of a differential operator

$$\mathbb{B} = \sum_{|\alpha|=l} B_{\alpha} \partial^{\alpha} \quad (3.6)$$

such that

$$\ker \mathcal{A}(\xi) = \text{im } \mathbb{B}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \quad (3.7)$$

We will assume, without mention, throughout this chapter that the operator $\mathcal{A}$ satisfies assumptions (3.3) and (3.5). The former is equivalent with the existence of an operator $\mathbb{B}$ as in (3.6) such that the exact relation (3.7) holds. Although the preliminary results in Section 3.2 hold in more generality, the proof of the main result uses both assumptions.

The main result of the present work is the following:

Theorem 3.1.1. *Let $\nu = ((\nu_x), \lambda, (\nu_x^{\infty})) \in Y(\Omega, W)$ be a Young measure with $\lambda(\partial\Omega) = 0$ and denote its barycentre by $w = \bar{\nu}_x \mathcal{L}^n + \bar{\nu}_x^{\infty} \lambda$. We assume that $w \in L^1(\Omega, W)$.*

Then there exist sequences (w_j) in $C^{\infty}(\bar{\Omega}, W)$ and (u_j) in $C_c^{\infty}(\Omega, V)$ satisfying

$$\begin{cases} \mathcal{A}w_j = 0 \text{ in } \mathcal{D}'(\Omega_j, W), \text{ where } \Omega_j \uparrow \Omega \\ w_j \rightarrow w \text{ in } L^1(\Omega, V) \\ \|u_j\|_{W^{l-1,1}} \rightarrow 0 \\ w_j + \mathbb{B}u_j \xrightarrow{Y} \nu \end{cases} \quad (3.8)$$

if and only if $\mathcal{A}w = 0$ in the sense of distributions on Ω and

$$\int_W f \, d\nu_x + \int_{\mathbb{S}_W} f^{\infty} \, d\nu_x^{\infty} \frac{d\lambda}{dx} \geq f \left(\bar{\nu}_x + \bar{\nu}_x^{\infty} \frac{d\lambda}{dx} \right) \quad (3.9)$$

holds for $\mathcal{L}^n$ almost all $x \in \Omega$ and all $\mathcal{A}$ -quasiconvex f of linear growth.

The proof in Section 3.3.3 is not written in an optimal way, but in a way which shows clearly at which step the technical assumption $w \in L^1$ enters. We hope that the method we present can be strengthened to cover the case $w \in \mathcal{M}(\Omega, W)$, which would settle the question of characterizing $\mathcal{A}$ -free generalized Young measures by duality with $\mathcal{A}$ -quasiconvex functions of linear growth.

3.2. Preliminaries

Let W be a finite dimensional real vector space. We fix an inner product on W and consider W as an Euclidean space: for $v, w \in W$ their inner product is $v \cdot w$ and the norm is $|v| = \sqrt{v \cdot v}$. The closed unit ball and the unit sphere in W are denoted by B_W and $\mathbb{S}_W$, respectively. We write $B_W(y, r)$ for the open ball with centre $y \in W$ and radius $r > 0$.

3.2.1. Young measures Let Ω be a bounded open subset of $\mathbb{R}^n$. A real-valued Borel function $f: \Omega \times W \rightarrow \mathbb{R}$ is called an integrand. We say it has linear growth if there exists a constant c such that

$$|f(x, z)| \leq c(|z| + 1)$$

holds for all $(x, z) \in \Omega \times W$.

We use the sphere-compactification of W , leading to the class of integrands $\mathbb{E} = \mathbb{E}(\Omega, W)$ and corresponding Young measures $\nu = ((\nu_x), \lambda, (\nu_x^\infty))$ that were introduced in [13, 72]. We use the notation and set-up given in [122, Sections 2.4 and 3]. We write $\mathbf{Y}(\Omega, W)$ for the set of Young measures. The reader can also find more background information on these Young measures in the recent book [163]. We also collect a few basic facts and definitions below.

We say that a sequence of measures $\mu_j \in \mathcal{M}(\Omega, W)$ generates $\nu \in \mathbf{Y}(\Omega, W)$ and write $\mu_j \xrightarrow{\mathbf{Y}} \nu$ if and only if

$$\int_{\Omega} \Phi(\mu_j^a) dx + \int_{\Omega} \Phi^\infty \left(\frac{d\mu_j^s}{d|\mu_j|} \right) d|\mu_j|^s \rightarrow \int_{\Omega} \langle \nu_x, \Phi \rangle dx + \int_{\Omega} \langle \nu_x^\infty, \Phi^\infty \rangle d\lambda =: \langle \nu, \Phi \rangle$$

for all $\Phi \in \mathbb{E}(\Omega, W)$. In this case, the *barycentre* of ν is

$$\bar{\nu} = \langle \text{Id}, \nu_x \rangle \mathcal{L}^n \llcorner \Omega + \langle \text{Id}, \nu_x^\infty \rangle = \bar{\nu}_x \mathcal{L}^n \llcorner \Omega + \bar{\nu}_x^\infty \lambda$$

and we have that $\mu_j \xrightarrow{*} \bar{\nu} \llcorner \Omega$ in $\mathcal{M}(\Omega, W)$.

We also recall that if μ_j are uniformly bounded in the total variation norm, then the sequence μ_j generates a Young measure [121, Thm. 7].

Finally, following [21, Def. 2.23], we let $q \in (1, n/(n-1))$ and say that a Young measure is $\mathcal{A}$ -free if it is generated by a sequence μ_j such that $\mathcal{A}\mu_j \rightarrow 0$ in $W^{-k,q}(\Omega, X)$.

3.2.2. Partial differential operators We use this subsection to fix the notation for the two homogeneous differential operators used throughout. Let V, W, X be finite dimensional normed vector spaces over $\mathbb{R}$ and let $k \in \mathbb{N}$. For multi-indices $\alpha \in \mathbb{N}_0^n$ of length $|\alpha| = k$ we are given linear maps $A_\alpha \in \text{Lin}(W, X)$ that act as coefficients of partial differential operators

$$\mathcal{A} = \sum_{|\alpha|=k} A_\alpha \partial^\alpha \tag{3.10}$$

Thus for a W -valued distribution w defined on an open subset Ω of $\mathbb{R}^n$ the expressions $\mathcal{A}w$ defines in the obvious way a X -valued distribution on Ω .

Throughout this chapter, the homogeneous partial differential operator $\mathbb{B}$ introduced in (3.6) as

$$\mathbb{B} = \sum_{|\alpha|=l} B_\alpha \partial^\alpha$$

has order $l \in \mathbb{N}$, acts on V -valued distributions, and has coefficients $B_\alpha \in \text{Lin}(V, W)$. Both $\mathcal{A}$ and $\mathbb{B}$ will be assumed to have constant rank and satisfy the exact relation $\ker \mathcal{A}(\xi) = \text{im } \mathbb{B}(\xi)$ for all non-zero $\xi \in \mathbb{R}^n$ [156].

3.2.3. $\mathcal{A}$ -quasiconvex integrands We recall from [85, Def. 3.1] that a map $f \in C(W)$ is said to be $\mathcal{A}$ -quasiconvex if and only if

$$f(\xi) \leq \int_{\mathbb{X}} f(\xi + w(x)) \, dx$$

holds for all $\xi \in W$ and all $w \in C_{\text{per}}^\infty(\mathbb{X}, W)$ such that $\int_{\mathbb{X}} w \, dx = 0$ and $\mathcal{A}w = 0$. We wrote $\mathbb{X}$ for the unit cube $(0, 1)^n$. If $\Phi \in C(W)$ may fail to be $\mathcal{A}$ -quasiconvex, then one defines the $\mathcal{A}$ -quasiconvex envelope Φ^{qc} of Φ as

$$\Phi^{\text{qc}}(\xi) = \inf \left\{ \int_{\mathbb{X}} \Phi(\xi + w(x)) \, dx : w \in C_{\text{per}}^\infty(\mathbb{X}, W), \int_{\mathbb{X}} w = 0, \mathcal{A}w = 0 \right\}.$$

We have the following elementary property for integrands of linear growth:

Lemma 3.2.1. *Suppose that $\mathcal{A}$ is an operator of the form (3.10) for which the wave cone is spanning, i.e., $\text{span } \Lambda_{\mathcal{A}} = W$. Let $\Phi \in C(W)$ have linear growth $|\Phi| \leq c(|\cdot| + 1)$ and be such that its $\mathcal{A}$ -quasiconvex envelope satisfies*

$$\Phi^{\text{qc}}(\xi_0) > -\infty \quad \text{for some } \xi_0 \in W.$$

Then Φ^{qc} is real valued, $\mathcal{A}$ -quasiconvex and has linear growth, $|\Phi^{\text{qc}}| \leq c(|\cdot| + 1)$.

Proof. By [85, Proposition 3.4], we have that Φ^{qc} is convex in the directions of $\Lambda_{\mathcal{A}}$. In particular, whenever $\Phi^{\text{qc}}(\eta) > -\infty$, it is also the case that $\Phi^{\text{qc}} > -\infty$ on $\eta + \Lambda_{\mathcal{A}}$. Now let $\xi \in W$. Since $\Lambda_{\mathcal{A}}$ spans W , we can find vectors $v_1, \dots, v_s \in \Lambda_{\mathcal{A}}$ such that $\xi - \xi_0 = \sum_{i=1}^s v_i$. By the previous consideration, it follows that $\Phi^{\text{qc}}(\xi) > -\infty$, hence Φ^{qc} is real valued.

We choose a basis $\{e_i\}_{i=1}^d \subset \Lambda_{\mathcal{A}}$ of W , where $d = \dim W$. Let M be a change of basis matrix such that $\{M^{-1}e_i\}_{i=1}^d$ is an orthonormal basis of W . Then $\Phi^{\text{qc}} \circ M$ is separately convex. Using the facts that norms are equivalent on finite dimensional spaces and $\Phi^{\text{qc}} \leq \Phi$, we have that

$$\Phi^{\text{qc}} \circ M \leq c(M)(|\cdot|_M + 1),$$

where $|\cdot|_M$ denotes the Euclidean norm in the new coordinate system. By [119, Lemma 2.5], we have that $\Phi^{\text{qc}} \circ M$ has linear growth from below. Again by norm equivalence, we have that

$$\Phi^{\text{qc}} \geq -c(M)(|\cdot| + 1).$$

The fact that Φ^{qc} is $\mathcal{A}$ -quasiconvex follows by an adaptation of the arguments in Case 1 of the proof of [85, Proposition 3.4], where first order operators are considered. $\square$

We also record the following equivalent definition of $\mathcal{A}$ -quasiconvex envelopes in terms of $\mathbb{B}$, which will be used to prove the characterization of homogeneous $\mathcal{A}$ -free Young measures:

Lemma 3.2.2 ([156, Corollary 5]). *Let $\Phi \in C(W)$, $\mathbb{X} = (0, 1)^n$, and $\varepsilon > 0$. Then*

$$\Phi^{\text{qc}}(\xi) = \inf \left\{ \int_{\mathbb{X}} \Phi(\xi + \mathbb{B}u) : u \in C_c^\infty(\mathbb{X}, V), \|D^{l-1}u\|_{L^\infty} < \varepsilon \right\} \quad (3.11)$$

for all $\xi \in W$.

3.2.4. Localization It is well-known that localization principles make Young measures particularly useful tools to deduce lower semi-continuity of integral functionals. In the linear growth set up under differential constraints we have the following variant, which is proved in the appendix of [21].

Proposition 3.2.3. *Assume w_j, w are W -valued Radon measures on Ω , $Q \subset \mathbb{R}^n$ is a cube, and $\mathcal{A}$ has constant rank. Let $w_j \xrightarrow{*} w$ and $\mathcal{A}w_j \rightarrow 0$ in $W^{-k,q}(\Omega, X)$ for some $q \in (1, \frac{n}{n-1})$. If $w_j \xrightarrow{Y} \nu$, then there exist an $\mathcal{L}^n$ -null set $N^a \subset \Omega$ and a λ^s -null set $N^s \subset \Omega$ with the following properties:*

(a) *For each $x \in \Omega \setminus N^a$ the Young measure*

$$\nu(x) = \left((\nu_x)_{y \in Q}, \frac{d\lambda}{d\mathcal{L}^n}(x) \mathcal{L}^n \llcorner Q, (\nu_x^\infty)_{y \in Q} \right)$$

is generated by a sequence $(v_j) \subset C^\infty(\bar{Q}, W)$ satisfying

$$v_j \mathcal{L}^n \llcorner Q \xrightarrow{*} \left(\bar{\nu}_x + \bar{\nu}_x^\infty \frac{d\lambda}{d\mathcal{L}^n}(x) \right) \mathcal{L}^n \llcorner Q \quad \text{and} \quad \mathcal{A}v_j = 0.$$

(b) *For each $x \in \Omega \setminus N^s$ and each $\tau \in \text{Tan}(\lambda^s, x)$ the Young measure*

$$\nu(x) = ((\delta_0)_{y \in Q}, \tau, (\nu_x^\infty)_{y \in Q})$$

is generated by a sequence $(\tilde{w}_j) \subset C^\infty(\bar{Q}, W)$ satisfying

$$\tilde{w}_j \mathcal{L}^n \llcorner Q \xrightarrow{*} \bar{\nu}_x^\infty \tau \quad \text{and} \quad \mathcal{A}\tilde{w}_j \rightarrow 0 \text{ in } W^{-k,q}(\Omega, X).$$

3.3. Proof of characterization Theorem 3.1.1

3.3.1. Functional set-up Let $\mathbb{H}$ be the space of continuous integrands $\Phi: W \rightarrow \mathbb{R}$ of linear growth that admit a regular recession integrand:

$$\Phi^\infty(z) = \lim_{t \rightarrow \infty} \frac{\Phi(tz)}{t} \quad \text{exists locally uniformly in } z \in W.$$

This is exactly the class of integrands $\Phi: W \rightarrow \mathbb{R}$ for which the transformed integrand

$$\hat{z} \mapsto (1 - |\hat{z}|) \Phi \left(\frac{\hat{z}}{1 - \hat{z}} \right)$$

is bounded and uniformly continuous on the open unit ball $B_W(0, 1)$. We endow $\mathbb{H}$ with the norm

$$\|\Phi\| = \sup_{z \in W} \frac{|\Phi(z)|}{1 + |z|}.$$

The mapping $T: \mathbb{H} \rightarrow C(B_W)$ defined by

$$T\Phi(\hat{z}) = \begin{cases} (1 - |\hat{z}|) \Phi \left(\frac{\hat{z}}{1 - \hat{z}} \right) & \text{if } |\hat{z}| < 1 \\ \Phi^\infty(\hat{z}) & \text{if } |\hat{z}| = 1, \end{cases}$$

is easily seen to be an isometric isomorphism. Hence so is the dual mapping $T^*: C(B_W)^* \rightarrow \mathbb{H}^*$ and thus in view of the Riesz representation theorem we find

that each $\ell \in \mathbb{H}^*$ can be uniquely identified with a pair (μ^0, μ^∞) of (signed) Radon measures $\mu^0 \in \mathcal{M}(W)$, $\mu^\infty \in \mathcal{M}(\mathbb{S}_W)$, where the total variation of μ^0 has a finite first moment, $\langle |\mu^0|, |\cdot| \rangle < \infty$, and

$$\ell(\Phi) = \int_W \Phi \, d\mu^0 + \int_{\mathbb{S}_W} \Phi^\infty \, d\mu^\infty.$$

If w is a bounded W -valued Radon measure on $\mathbb{X} = (0, 1)^n$ with Lebesgue-Radon-Nikodým decomposition $w = w^a \, dx + w^s$, then we define $\varepsilon_w \in \mathbb{H}^*$ by

$$\varepsilon_w(\Phi) = \int_{\mathbb{X}} \Phi(w^a(x)) \, dx + \int_{\mathbb{X}} \Phi^\infty(w^s) \quad (\Phi \in \mathbb{H})$$

Fix $\xi \in W$ and denote by Y the set of all pairs $(\nu^0, \nu^\infty) \in \mathcal{M}_1^+(W) \times \mathcal{M}^+(\mathbb{S}_W)$ for which we can find a sequence (u_j) in $C_c^\infty(\mathbb{X}, V)$ such that, as $j \rightarrow \infty$, $\|u_j\|_{W^{l-1,1}} \rightarrow 0$ and

$$\int_{\mathbb{X}} \Phi(\xi + \mathbb{B}u_j(x)) \, dx \rightarrow \int_W \Phi \, d\nu^0 + \int_{\mathbb{S}_W} \Phi^\infty \, d\nu^\infty \quad \forall \Phi \in \mathbb{H}.$$

Obviously, we can consider Y as a subset of $\mathbb{H}^*$ and we clarify that $\mathcal{M}^+(\mathbb{S}_W)$ denotes the set of positive Radon measures on $\mathbb{S}_W$, whereas $\mathcal{M}_1^+(W)$ denotes the set of probability measures on W .

Lemma 3.3.1. *The family $\{\varepsilon_{\xi + \mathbb{B}u} : u \in C_c^\infty(\mathbb{X}, V)\}$ is a weakly- * dense subset of Y .*

Proof. It suffices to show that $\varepsilon_{\xi + \mathbb{B}u} \in Y$ for each $u \in C_c^\infty(\mathbb{X}, V)$. But this is a consequence of a standard exhaustion argument that runs as follows: We extend u to $\mathbb{R}^n$ by zero without changing notation and subdivide each side of $\mathbb{X}$ into $j \in \mathbb{N}$ disjoint congruent open intervals, and consider the resulting mesh of j^n disjoint congruent cubes. We fix an arrangement of the cubes, say

$$\mathbb{X}_i = x_i + j^{-1}\mathbb{X}, \quad i \in \{1, \dots, j^n\},$$

and define

$$\phi_j(x) = \sum_{i=1}^{j^n} j^{-l} u(j(x - x_i)).$$

It is then simple to compute that

$$\begin{aligned} \int_{\mathbb{X}} \Phi(\xi + \mathbb{B}u) \, dx &= \int_{\mathbb{X}} \Phi(\xi + \mathbb{B}\phi) \, dx \\ \|D^{l-1}\phi\|_{L^1(\mathbb{X})} &= j^{-1} \|D^{l-1}\phi\|_{L^1(\mathbb{X})} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

The conclusion follows by applying Poincaré's inequality with zero boundary values iteratively. $\square$

Lemma 3.3.2. *The set Y is weakly * closed in $\mathbb{H}^*$.*

Proof. Let $\mu = (\mu^0, \mu^\infty) \in \overline{Y}^{w^*}$. Let $\{\Phi_j\}_{j \in \mathbb{N}}$ be a countable and dense subset of $\mathbb{H}$ and put $\Phi_0 = |\cdot|$. Now for each $i \in \mathbb{N}$ we may select $u_i \in C_c^\infty(\mathbb{X}, V)$ such that

$$\|u_i\|_{W^{l-1,1}} < \frac{1}{i} \quad , \quad \max_{0 \leq j \leq i} \left| \int_{\mathbb{X}} \Phi_j(\xi + \mathbb{B}u_i) \, dx - \mu(\Phi_j) \right| < \frac{1}{i}. \quad (3.12)$$

Since $\Phi_0 = |\cdot|$ is included above we infer that the sequence $(\xi + \mathbb{B}u_i)$ is bounded in $L^1(\mathbb{X}, W)$ and so, by Banach-Alaoglu's compactness theorem and separability of $\mathbb{H}$, each of its subsequences admit a further subsequence (not relabelled) so

$$\int_{\mathbb{X}} \Phi(\xi + \mathbb{B}u_i) \, dx \rightarrow \ell(\Phi) \quad \forall \Phi \in \mathbb{H} \quad (3.13)$$

for some $\ell \in \mathbb{H}^*$ that at this stage of course can depend on the particular subsequence. But the density of $\{\Phi_j\}_{j \in \mathbb{N}_0}$ and the identification of $\mathbb{H}^*$ with pairs of measures allow us by virtue of (3.12) to conclude that $\ell = \mu$ and so that (3.13) in fact holds true for the full sequence. Taken together with (3.12) we have shown that $\mu \in Y$, and the proof is complete. $\square$

Lemma 3.3.3. *The set Y is convex.*

Proof. Let $\nu_0, \nu_1 \in Y$, $t \in (0, 1)$ and $\nu_t = (1 - t)\nu_0 + t\nu_1$. In view of Lemmas 3.3.1 and 3.3.2 it suffices to show that $\nu_t \in Y$ when $\nu_0 = \varepsilon_{\xi + \mathbb{B}u_0}$, $\nu_1 = \varepsilon_{\xi + \mathbb{B}u_1}$ and $t = \left(\frac{p}{q}\right)^n$, where $u_0, u_1 \in C_c^\infty(\mathbb{X}, V)$ and $p, q \in \mathbb{N}$ are coprime. In fact, we shall show that $\nu_t = \varepsilon_{\xi + \mathbb{B}\phi}$ for some $\phi \in C_c^\infty(\mathbb{X}, V)$.

In order to see this we subdivide each side of $\mathbb{X}$ into q disjoint congruent half-open intervals, and consider the resulting mesh of q^n disjoint congruent cubes. We fix an arrangement of the cubes, say

$$\mathbb{X}_i = x_i + q^{-1}\mathbb{X}, \quad i \in \{1, \dots, q^n\},$$

and define

$$\phi(x) = \sum_{i=1}^{p^n} q^{-l} u_1(q(x - x_i)) + \sum_{i=p^n+1}^{q^n} q^{-l} u_0(q(x - x_i)) \quad (x \in \mathbb{X})$$

where we extend u_0, u_1 to $\mathbb{R}^n \setminus \mathbb{X}$ by 0. Clearly, $\phi \in C_c^\infty(\mathbb{X}, V)$ and for $\Phi \in \mathbb{H}$ we check that $\varepsilon_{\xi + \mathbb{B}\phi}(\Phi) = \nu_t(\Phi)$, and therefore $\nu_t \in Y$ by Lemma 3.3.1. $\square$

3.3.2. The Hahn-Banach argument For a continuous integrand $f: W \rightarrow \mathbb{R}$ of linear growth we define the upper recession integrand by

$$f^\infty(z) = \limsup_{\substack{t \rightarrow \infty \\ z' \rightarrow z}} \frac{f(tz')}{t} \quad (z \in W)$$

Hereby $f^\infty: W \rightarrow \mathbb{R}$ is a positively 1-homogeneous integrand of linear growth. When the wave cone for $\mathcal{A}$ spans the space W , then $\mathcal{A}$ -quasiconvex integrands f of linear growth are Lipschitz continuous. In that case, the upper recession integrand f^∞ is also positively 1-homogeneous, Lipschitz and $\mathcal{A}$ -quasiconvex. Furthermore we have the simpler formula

$$f^\infty(z) = \limsup_{t \rightarrow \infty} \frac{f(tz)}{t} \quad (z \in W).$$

Lemma 3.3.4. *Let $\nu = (\nu^0, \nu^\infty) \in \mathcal{M}_1^+(W) \times \mathcal{M}^+(\mathbb{S}_W)$. Then $\nu \in Y$ if, and only if, $\bar{\nu}^0 + \bar{\nu}^\infty = \xi$ and*

$$\int_W f \, d\nu^0 + \int_{\mathbb{S}_W} f^\infty \, d\nu^\infty \geq f(\xi)$$

holds for all $\mathcal{A}$ -quasiconvex integrands $f: W \rightarrow \mathbb{R}$ of linear growth.

Proof. To prove the *only if* part we fix $\nu \in Y$ and an $\mathcal{A}$ -quasiconvex integrand $f: W \rightarrow \mathbb{R}$ of linear growth. Select a sequence (u_j) in $C_c^\infty(\mathbb{X}, V)$ such that $\|u_j\|_{W^{l-1,1}} \rightarrow 0$ and

$$\int_{\mathbb{X}} \Phi(\xi + \mathbb{B}u_j) \, dx \rightarrow \nu(\Phi)$$

for each $\Phi \in \mathbb{H}$. Thus for $\Phi \in \mathbb{H}$ with $\Phi \geq f$ we get by $\mathcal{A}$ -quasiconvexity of f at ξ that

$$\nu(\Phi) \geq \limsup_{j \rightarrow \infty} \int_{\mathbb{X}} f(\xi + \mathbb{B}u_j) \, dx \geq f(\xi).$$

To obtain the required bound we use the approximation result [13, Lemma 2.3]. Hereby one finds a descending sequence (Φ_k) in $\mathbb{H}$ with $\Phi_k \searrow f$, $\Phi_k^\infty \searrow f^\infty$ pointwise. The Lebesgue monotone convergence theorem completes the argument.

We turn to the *if* part of the statement. According to Lemmas 3.3.2 and 3.3.3, Y is a weakly* closed and convex subset of $\mathbb{H}^*$, hence by the Hahn-Banach separation theorem we can write $Y = \bigcap H$, where we take intersection over all weakly* closed half-spaces H in $\mathbb{H}^*$ that contain Y . Fix such a half-space H ; then for some $\Phi \in \mathbb{H}$, $t \in \mathbb{R}$ we have

$$H = \{\ell \in \mathbb{H}^* : \ell(\Phi) \geq t\}.$$

Because $Y \subset H$ it follows from Lemma 3.3.1 that in particular $\varepsilon_{\xi + \mathbb{B}u} \in H$ for all $u \in C_c^\infty(\mathbb{X}, V)$. Consequently,

$$t \leq \varepsilon_{\xi + \mathbb{B}u}(\Phi) = \int_{\mathbb{X}} \Phi(\xi + \mathbb{B}u) \, dx$$

holds for all $u \in C_c^\infty(\mathbb{X}, V)$. But then the relaxation formula (3.11) yields $t \leq \Phi^{\text{qc}}(\xi)$ and it follows that the envelope Φ^{qc} is real-valued, $\mathcal{A}$ -quasiconvex and of linear growth by Lemma 3.2.1. Returning to our assumptions on ν with this information we deduce that

$$\nu(\Phi) \geq \nu(\Phi^{\text{qc}}) \geq \Phi^{\text{qc}}(\xi) \geq t,$$

that is, $\nu \in H$. □

3.3.3. Proof of the main result

Proof of Theorem 3.1.1. The proof of necessity of (3.9) follows at once by localization from the Proposition 3.2.3 and the Jensen-type inequality in Lemma 3.3.4.

Conversely, suppose that (3.9) holds for all $\mathcal{A}$ -quasiconvex f of linear growth. We write $\lambda = \lambda^a \mathcal{L}^n \llcorner \Omega + \lambda^s$ for the Radon-Nykodým decomposition of λ . Let $\{\phi_j \otimes \Phi_j\}_{j \in \mathbb{N}}$ be a countable and dense subset of $\mathbb{E}(\Omega, W)$ such that each ϕ_j ,

Φ_j is Lipschitz (such a set exists [21, Lem. 2.4]). Put $\phi_0 = \chi_{\Omega}$, $\Phi_0 = |\cdot|$ and $L_m = \max\{\text{lip}(\phi_j) + \text{lip}(\Phi_j) + \|\phi_j\|_{\infty} : 0 \leq j \leq m\}$. Let

$$\begin{aligned} g_j(x) &:= \phi_j(x) (\langle \nu_x, \Phi_j \rangle + \langle \nu_x^{\infty}, \Phi_j^{\infty} \rangle \lambda^a(x)) \\ h_j(x) &:= \phi_j(x) \langle \nu_x^{\infty}, \Phi_j^{\infty} \rangle, \end{aligned}$$

so that $g_j \in L^1(\Omega; \mathcal{L}^n)$ and $h_j \in L^1(\Omega; \lambda^s)$. Let $\varepsilon > 0$ to be chosen depending on m . By Lusin's Theorem, we can find a compact subset C of $\text{spt} \lambda^s$ and continuous functions $\tilde{h}_j \in C(\bar{\Omega})$ such that $h_j = \tilde{h}_j$ on C and $\lambda^s(\Omega \setminus C) < \varepsilon$. By uniform continuity of $\tilde{h}_j$, we can choose a regular covering $\mathcal{F}_{\delta}$ of $\mathbb{R}^n$ by cubes of side length equal to $\delta > 0$ such that if x, y lie in the same cube, we have that $|\tilde{h}_j(x) - \tilde{h}_j(y)| < \varepsilon$. Write

$$\mathcal{F}_{\delta}^s := \{Q \in \mathcal{F}_{\delta} : \lambda^s(Q \cap C) > 0\}.$$

We record that, since $|C| = 0$, we can choose $\delta > 0$ small enough so that $|\bigcup \mathcal{F}_{\delta}^s| < \varepsilon$. We now employ Lusin's Theorem for g_j to find $\tilde{g}_j \in C(\bar{\Omega})$ and a large compact set $K \subset \Omega$ such that $|\Omega \setminus K| < \varepsilon$. We write

$$\mathcal{F}_{\delta}^a := \{Q \in \mathcal{F}_{\delta} : |Q \cap K| > 0\} \setminus \mathcal{F}_{\delta}^s.$$

By taking δ small enough, we can ensure that $|\tilde{g}_j(x) - \tilde{g}_j(y)| < \varepsilon$ whenever $x, y \in Q \in \mathcal{F}_{\delta}^a$. We also have that $\lambda^s(\bigcup \mathcal{F}_{\delta}^a) < \lambda^s(\Omega \setminus C) < \varepsilon$ and $|\Omega \setminus \bigcup \mathcal{F}_{\delta}^a| < |\Omega \setminus K| + |\bigcup \mathcal{F}_{\delta}^s| < 2\varepsilon$. There is no loss of generality in also requiring

$$\left| \int_{\Omega} g_j \, dx - \int_{\bigcup \mathcal{F}_{\delta}^a} \tilde{g}_j \, dx \right| < \varepsilon \quad \text{and} \quad \left| \int_{\Omega} h_j \, d\lambda^s - \int_{\bigcup \mathcal{F}_{\delta}^s} \tilde{h}_j \, d\lambda^s \right| < \varepsilon,$$

so that we can proceed to estimate

$$\left| \sum_{Q \in \mathcal{F}_{\delta}^a} \int_Q \tilde{g}_j \, dx - \sum_{Q \in \mathcal{F}_{\delta}^a} |Q| g_j(x_Q) \right| \leq \sum_{Q \in \mathcal{F}_{\delta}^a} |Q| \int_Q |\tilde{g}_j - g_j(x_Q)| \, dx < \varepsilon |\Omega|,$$

where x_Q is an arbitrary point in $Q \cap K$. We will define a map $u_m \in C_c^{\infty}(\Omega, V)$ iteratively on cubes in $\mathcal{F}_{\delta}$. Firstly, $u_m = 0$ on cubes that are not in $\bigcup \mathcal{F}_{\delta}^a$ or $\bigcup \mathcal{F}_{\delta}^s$. Next, we use Lemma 3.3.4 to define u_m on $Q \in \mathcal{F}_{\delta}^a$ such that

$$\left| |Q| g_j(x_Q) - \phi_j(x_Q) \int_Q \Phi_j(w^a(x_Q) + \mathbb{B}u_m) \, dx \right| < \varepsilon |Q|$$

(so far we only assume that $w \in \mathcal{M}(\Omega, W)$). We can assume without loss of generality that there also exists a function $\tilde{w}^a$ that satisfies with respect to w^a all the properties $\tilde{g}_j$ satisfies with respect to g_j . We have that

$$\left| \phi_j(x_Q) \int_Q \Phi_j(w^a(x_Q) + \mathbb{B}u_m(x)) - \Phi_j(\tilde{w}^a(x) + \mathbb{B}u_m(x)) \, dx \right| \leq \varepsilon L_m^2 |Q|.$$

Using the fact that $\tilde{w}^a$ can be chosen arbitrarily L^1 -close to w^a , we can collect the estimates and show that

$$\left| \int_{\Omega} g_j \, dx - \sum_{Q \in \mathcal{F}_{\delta}^a} \phi_j(x_Q) \int_Q \Phi_j(w^a + \mathbb{B}u_m) \, dx \right| \leq \varepsilon c(m) |\Omega|. \quad (3.14)$$

We will next obtain a similar estimate near singular points. In the argument above we constructed the part of the generating sequence that approximates the the Young measure ν near regular points of λ . For singular points of λ , the analysis will be similar.

We first show that $(\delta_0, t\nu_x^\infty) \in Y$ for λ^s -a.e. $x \in \Omega$ where $Q \subset \Omega$ is a cube containing x . We will use Lemma 3.3.4. To this end, let f be $\mathcal{A}$ -quasiconvex of linear growth and let $x \in \Omega$ be such that $\bar{\nu}_x^\infty \in \Lambda_{\mathcal{A}}$. Such points exist λ^s -a.e. by [65]. The inequality we want to check is

$$f(0) + \langle f^\infty, t\nu_x^\infty \rangle \geq f(t\bar{\nu}_x^\infty),$$

which is easily re-written as

$$\langle f^\infty, \nu_x^\infty \rangle \geq \frac{f(t\bar{\nu}_x^\infty) - f(0)}{t}.$$

Since f is $\mathcal{A}$ -quasiconvex, so is f^∞ , thus convex in the directions in $\Lambda_{\mathcal{A}}$, so we can apply [113, Thm. 1.1] to obtain

$$\langle f^\infty, \nu_x^\infty \rangle \geq f^\infty(\bar{\nu}_x^\infty) = \limsup_{s \rightarrow \infty} \frac{f(s\bar{\nu}_x^\infty) - f(0)}{s} \geq \frac{f(t\bar{\nu}_x^\infty) - f(0)}{t}.$$

This completes the proof of the fact that $(\delta_0, t\nu_x^\infty) \in Y$ for λ^s -a.e. $x \in \Omega$.

By using this fact for $x = x_Q$ arbitrary points in $C \cap Q$ for $Q \in \mathcal{F}_\delta^s$ and $t = \lambda^s(Q)/|Q|$, we can then show by analogy with the argument given for regular points that

$$\left| \int_{\Omega} h_j \, d\lambda^s - \sum_{\bigcup \mathcal{F}_\delta^s} \phi_j(x_Q) \int_Q \Phi_j(\mathbb{B}u_m) \, dx \right| \leq \varepsilon c(m) \lambda^s(\Omega), \quad (3.15)$$

where we *crucially* use the fact that $w \in L^1(\Omega)$ (so that $\bar{\nu}_{x_Q}^\infty = 0$ as $w^s = \bar{\nu}_x^\infty \lambda^s$).

From (3.14), (3.15), the fact that $|\bigcup \mathcal{F}_\delta^s| < \varepsilon$ we can deduce that

$$\left| \int_{\Omega} g_j \, dx + \int_{\Omega} h_j \, d\lambda^s - \sum_{\bigcup \mathcal{F}_\delta^a \cup \bigcup \mathcal{F}_\delta^s} \phi_j(x_Q) \int_Q \Phi_j(w^a + \mathbb{B}u_m) \, dx \right| < \varepsilon c(m, \Omega). \quad (3.16)$$

By choosing $\varepsilon = m^{-1}c(m, Q)$ in (3.16), we can quite easily establish that $(w + \mathbb{B}u_m)_m$ is a generating sequence for ν (under our technical assumption, $w = w^a$). First, take $j = 0$ in (3.16) to get that

$$\left| \langle \nu, |\cdot| \rangle - \int_{\Omega} |w + \mathbb{B}u_m| \, dx \right| < \frac{1}{m},$$

so that $(w + \mathbb{B}u_m)_m$ is bounded in L^1 independently of m . We then note that

$$\left| \phi_j(x_Q) \int_Q \Phi_j(w + \mathbb{B}u_m) \, dx - \int_Q \phi_j \Phi_j(w + \mathbb{B}u_m) \, dx \right| \leq \delta c(m) \int_Q |w + \mathbb{B}u_m| \, dx.$$

Therefore, assuming that $0 < \delta < \varepsilon$ and making the correct a priori choice of ε above, we have that (3.16) implies

$$\left| \langle \nu, \phi_j \otimes \Phi_j \rangle - \int_{\Omega} \phi_j \Phi_j (w + \mathbb{B}u_m) dx \right| \leq \frac{1}{m} \quad \text{for all } j = 0 \dots m. \quad (3.17)$$

By L^1 -boundedness of $(w + \mathbb{B}u_m)_m$, we have that there exists a subsequence $(w + \mathbb{B}u_{m_i})_i$ that generates a Young measure $\tilde{\nu}$. By (3.17), we have that ν and $\tilde{\nu}$ agree on $\{\phi_j \otimes \Phi_j\}_j$, hence $\tilde{\nu} = \nu$.

Since smooth mollifications $(w \star \eta_i)_i$ of w can be chosen L^1 -arbitrarily close to w , we finally conclude that the sequence $(w \star \eta_i + \mathbb{B}u_{m_i})_i$ satisfies the assumptions of the Theorem. The proof is complete. $\square$

Should one try to perform a similar proof in the general case $w \in \mathcal{M}(\Omega, W)$, the approximating sequence that arises naturally for w^s is given by $\sum \lambda^s(Q)/|Q| \tilde{\nu}_{x_Q}^\infty \chi_Q$, where the sum is taken over singular cubes. It is not difficult to check by direct computation in examples (e.g., line measure in dimension 2) that this approximation does not match the expected smooth approximation $w^s \star \eta$ in a desirable way. In particular, the general problem requires more involved ideas. However, the authors are quite confident that a proof can be obtained in the near future.

Part II

Linear L^1 -estimates

CHAPTER 4

L^1 -estimates for constant rank operators

This chapter consists of the author's work.

4.1. Introduction

Recently, VAN SCHAFTINGEN proved in [201] that the surprising estimate

$$\|D^{k-1}u\|_{L^{n/(n-1)}} \leq c\|\mathbb{B}u\|_{L^1} \quad (4.1)$$

for vector fields $u \in C_c^\infty$ holds if and only if the k -homogeneous differential operator $\mathbb{B}$ with constant coefficients is elliptic and canceling. Here (overdetermined) ellipticity is defined by injectivity of the symbol map (Fourier multiplier) of $\mathbb{B}$, whereas cancellation, for elliptic operators, can be defined as the property that the equation $\mathbb{B}u = \delta_0 w$ has no distributional solutions if $w \neq 0$.

The result of VAN SCHAFTINGEN can be viewed as a critical a priori estimate for linear elliptic equations on the Sobolev scale. This is in sharp contrast with the case $1 < p < \infty$ of L^p -estimates,

$$\|D^k u\|_{L^p} \leq c\|\mathbb{B}u\|_{L^p} \quad (4.2)$$

for $u \in C_c^\infty$, which is attributable to CALDERÓN and ZYGMUND [49]. On the other hand, the estimate (4.2) holds for $p = 1$ only if $D^k u = L[\mathbb{B}u]$ for all u , where L is a linear map between finite dimensional spaces. In other words, only trivial estimates such as (4.2) hold for $p = 1$ [148, 113].

The purpose of the present chapter is to relax the ellipticity assumption to the constant rank condition, which gives the largest class in which existent methods apply (see Remark 4.2.7). Only relatively few examples outside this class are known and it is safe to say that developing a theory in this regime is a challenging problem (see Remark 4.2.7). Of course, ellipticity is necessary for (4.1), as is for (4.2), but as noted in [166] the inequality (4.2) holds for $p = 2$ and operators of constant rank if fewer competitor maps $u \in C_c^\infty$ are allowed (to be precise, if $u \in [\ker \mathbb{B}]^\perp$). We will show that

$$\|D^{k-1}(u - \pi u)\|_{L^{n/(n-1)}} \leq c\|\mathbb{B}u\|_{L^1} \quad (4.3)$$

holds for vector fields $u \in C_c^\infty$ if and only if $\mathbb{B}$ is canceling. Here π denotes the L^2 -projection onto $\ker \mathbb{B}$ (the complete statement as well as explicit definitions are given in Section 4.2).

The proof follows from the fact that $u - \pi u$ can be written as a Fourier multiplier of $\mathbb{B}u$ and of the dual estimate [201, Thm. 1.4]. To apply the latter, we employ the recent construction of exact annihilators of constant rank operators [156]. We do not claim that the ideas used are original, but we make a case of the fact that a considerable ammount of work has led to, among other developments, the characterization of operators that satisfy (4.1) [39, 40, 41, 35, 37, 36, 38, 195, 196, 199, 200] and to round off the theory with the case of constant rank operators seems welcome.

With similar methods, we obtain critical inequalities on the full Sobolev scale, fractional scales, Lorentz scale, Hardy and Hardy–Sobolev inequalities.

4.2. The inequalities

4.2.1. Definitions and notation Throughout this chapter we will work with a homogeneous linear differential operator of order k with constant coefficients on $\mathbb{R}^n$ from V to W , i.e.,

$$\mathbb{B}u := \sum_{|\alpha|=k} \mathbb{B}_\alpha \partial^\alpha u \quad \text{for } u \in C^\infty(\mathbb{R}^n, V), \quad (4.4)$$

where $\mathbb{B}_\alpha \in \text{Lin}(V, W)$ are linear maps for all multi-indices α with $|\alpha| = k$ and V, W are finite dimensional inner product spaces. The symbol map of $\mathbb{B}$ is a $\text{Lin}(V, W)$ -valued homogeneous polynomial of degree k given by

$$\mathbb{B}(\xi) = \sum_{|\alpha|=k} \xi^\alpha \mathbb{B}_\alpha \in \text{Lin}(V, W) \quad \text{for } \xi \in \mathbb{R}^n.$$

The symbol map is defined such that $\widehat{\mathbb{B}u} = (-i)^k \mathbb{B}(\cdot) \hat{u}$ for u in the Schwartz space $\mathcal{S}(\mathbb{R}^n, V)$. Our convention for the Fourier transform is

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \, dx.$$

An operator $\mathbb{B}$ is said to be of *constant rank* if for some integer r ,

$$\text{rank } \mathbb{B}(\xi) = r \quad \text{for all } 0 \neq \xi \in \mathbb{R}^n.$$

The constant rank condition was introduced in [166] to prove coerciveness estimates for non-elliptic systems and used in the theories of compensated compactness [143] and lower semi-continuity [85]. It is well-known that the constant rank condition is equivalent with smoothness of the $(-k)$ -homogenous map $\xi \mapsto \mathbb{B}^\dagger(\xi)$ away from zero (here $M^\dagger$ denotes the *Moore–Penrose generalized inverse* of a matrix M ; see [52] for a general presentation). This fact was recently improved in [156], where it is shown, as an immediate consequence of the result in [63], that $\mathbb{B}^\dagger(\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\}, \text{Lin}(W, V))$ is a rational function.

An operator $\mathbb{B}$ is said to be (overdetermined) *elliptic* if $\mathbb{B}$ is of full constant rank ($r = \dim V$); equivalently, the symbol map $\mathbb{B}(\xi)$ is injective for all $0 \neq \xi \in \mathbb{R}^n$. An operator $\mathbb{B}$ is said to be *canceling* if

$$\bigcap_{\xi \in \mathbb{S}^{n-1}} \text{im } \mathbb{B}(\xi) = \{0\}.$$

It follows from various considerations in [201] that, if $\mathbb{B}$ is elliptic, then it is canceling if and only if $\mathbb{B}u = \delta_0 w$ implies $w = 0$. The same holds true for constant rank operators:

Lemma 4.2.1. *Let $\mathbb{B}$ as in (4.4) have constant rank. Then $\mathbb{B}$ is canceling if and only if for $u \in L^1_{\text{loc}}$ and $w \in W$,*

$$\mathbb{B}u = \delta_0 w \implies w = 0. \quad (4.5)$$

Moreover, if $\mathbb{B}$ is non-canceling, (4.5) has a solution $u_h \in C^\infty(\mathbb{R}^n \setminus \{0\}, V)$ such that $D^l u_h$ is $(k - n - l)$ -homogeneous for all $l \geq \max\{0, k - n + 1\}$.

This fact follows just as in [155, Lem. 2.5], with the modification that $\mathcal{A}$ in the respective proof should be replaced with $\mathcal{A}$ as constructed in [156, Rk. 4]. We will return to this point in detail later.

Schematically, the strategy in [201] for elliptic and canceling $\mathbb{B}$ can be summarized as follows:

- (a) By ellipticity, write $u = K \star (\mathbb{B}u)$ where $\hat{K} = \mathbb{B}^\dagger(\cdot)$ [43, Lem. 2.1];
- (b) Use an elliptic estimate (in principle, boundedness of singular integral operators) to obtain

$$\|u\|_X \leq c \|\mathbb{B}u\|_Y \quad \text{for } u \in C_c^\infty(\mathbb{R}^n, V).$$

For example, in the case of (4.1), $X = \dot{W}^{k-1, n/(n-1)}$ and $Y = \dot{W}^{-1, n/(n-1)}$.

- (c) Use a dual, critical L^1 -estimate

$$\|w\|_Y \leq c \|w\|_{L^1}$$

for vector fields $w \in C_c^\infty(\mathbb{R}^n, W)$ in the kernel of an underdetermined differential operator, $\mathcal{A}w = 0$ (see [201, Thm. 1.4]).

The operators that satisfy (c) are said to be *cocanceling* and are defined by

$$\bigcap_{\xi \in \mathbb{S}^{n-1}} \ker \mathcal{A}(\xi) = \{0\}.$$

For applicability of (c) to improve the estimate in (b) with $w = \mathbb{B}u$, we want

$$\text{im } \mathbb{B}(\xi) = \ker \mathcal{A}(\xi) \quad \text{for all } 0 \neq \xi \in \mathbb{R}^n, \quad (4.6)$$

in which case we say that the differential operator $\mathcal{A}$ is an exact annihilator of $\mathbb{B}$. Given a differential operator $\mathbb{B}$, a naive way to construct $\mathcal{A}$ would be to set

$$\mathcal{A}(\xi) = \text{Proj}_{[\text{im } \mathbb{B}(\xi)]^\perp} = \text{Id}_W - \mathbb{B}(\xi) \mathbb{B}^\dagger(\xi) =: \mathcal{P}(\xi),$$

but this expression does not define a differential operator in general. However, if $\mathbb{B}$ is elliptic, one can explicitly compute that $\mathbb{B}^\dagger(\xi) = [\mathbb{B}^*(\xi) \mathbb{B}(\xi)]^{-1} \mathbb{B}^*(\xi)$, so that setting

$$\mathcal{A}(\xi) := \det[\mathbb{B}^*(\xi) \mathbb{B}(\xi)] \mathcal{P}(\xi)$$

defines a polynomial, and hence $\mathcal{A}$ thus defined is a homogeneous linear differential operator [201, Sec. 4.2].

If we only assume that $\mathbb{B}$ has constant rank, one still has that $\mathbb{B}^\dagger(\xi) = p(\xi)^{-1}\mathcal{Q}(\xi)$, where p is a scalar-valued homogeneous polynomial with no non-zero roots in $\mathbb{R}^n$ and $\mathcal{Q}$ is a $\text{Lin}(W, V)$ -valued polynomial. In particular,

$$\mathcal{A}(\xi) := p(\xi)\mathcal{P}(\xi)$$

defines a linear differential operator that satisfies (4.6). Collecting, we proved that:

Lemma 4.2.2 ([156, Rk. 4]). *Let $\mathbb{B}$ be a constant rank operator as in (4.4). Then there exists a constant rank, homogeneous, linear differential operator $\mathcal{A}$ with constant coefficients such that the exact relation (4.6) holds.*

Lemma 4.2.2 deals with the duality of estimates (b) and (c) (see also Remark 4.2.9). On the other hand, when the ellipticity assumption in (a) is relaxed to the constant rank condition, the convolution $K \star (\mathbb{B}u)$ is no longer a representation of a Schwartz function u . Instead, we can compute in frequency space:

$$\begin{aligned} \mathbb{B}^\dagger(\xi)\widehat{\mathbb{B}u}(\xi) &= \mathbb{B}^\dagger(\xi)\mathbb{B}(\xi)\hat{u}(\xi) = \hat{u}(\xi) - \left[\text{Id}_V - \mathbb{B}^\dagger(\xi)\mathbb{B}(\xi) \right] \hat{u}(\xi) \\ &= \hat{u}(\xi) - \text{Proj}_{\ker \mathbb{B}(\xi)} \hat{u}(\xi), \end{aligned}$$

so that we actually have

$$u - \pi u = K \star (\mathbb{B}u) \quad \text{where} \quad \hat{K} = \mathbb{B}^\dagger(\cdot) \quad \text{and} \quad \hat{\pi} = \text{Id}_V - \mathbb{B}^\dagger(\cdot)\mathbb{B}(\cdot). \quad (4.7)$$

In fact, it is easy to see that, since $\xi \mapsto \text{Id}_V - \mathbb{B}^\dagger(\xi)\mathbb{B}(\xi)$ is zero-homogeneous, π maps $\mathcal{S}(\mathbb{R}^n, V)$ into $\mathcal{S}(\mathbb{R}^n, V)$. It can also be checked by direct computation that π is the L^2 -orthogonal projection onto $\ker \mathbb{B} \cap \mathcal{S}$.

Of course, if $\mathbb{B}$ is elliptic, then $\pi \equiv 0$, so all our inequalities for constant rank operators exactly reduce to the known existing inequalities for elliptic operators.

Finally, we refer the reader to [192] for background on Besov and Triebel–Lizorkin spaces and to [210] for background on Lorentz spaces.

4.2.2. The statements The considerations in Section 4.2.1 are sufficient so that we need not write complete proofs. All necessary details can be found in [201, 43, 155] and the references therein. We will indicate which result should be quoted for each statement. We use the notation in (4.7).

Theorem 4.2.3 (L^p -scales). *Let $\mathbb{B}$ be a constant rank operator as in (4.4) and $n > 1$. Then the estimate*

$$\|u - \pi u\|_X \leq c \|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V)$$

holds if and only if $\mathbb{B}$ is canceling. Here X is one of the following:

- (a) $X = \dot{W}^{k-j, n/(n-j)}$ for some $1 \leq j \leq \min\{k, n-1\}$.
- (b) $X = \dot{W}^{k-s, n/(n-s)}$ for some $s \in (0, n)$.

- (c) $X = \dot{B}_q^{k-s, n/(n-s)}$ for some $s \in (0, n)$, $1 < q < \infty$.
- (d) $X = \dot{B}_q^{k-s, n/(n-s)}$ for some $s \in (0, n)$, $1 \leq q \leq \infty$.
- (e) $X = \dot{W}^{k-j} L^{n/(n-j), q}$ for some $1 \leq j \leq \min\{k, n-1\}$, $1 < q < \infty$.

The limiting case $s = 0$ is ruled out by Ornstein's Non-inequality. Theorem 4.2.3(a) can be refined on a Hardy–Sobolev scale (cp. [43, Prop. 2.3]):

Theorem 4.2.4 (Hardy–type inequalities). *Let $\mathbb{B}$ be a constant rank operator as in (4.4), $n > 1$, $1 \leq j \leq \min\{n-1, k\}$, and $q \in [1, n/(n-j)]$. Then the estimate*

$$\left(\int_{\mathbb{R}^n} \left(\frac{|D^{k-j}(u - \pi u)(x)|}{|x|^{n/q - (n-j)}} \right)^q dx \right)^{1/q} \leq c \int_{\mathbb{R}^n} |\mathbb{B}u(x)| dx \quad \text{for } u \in C_c^\infty(\mathbb{R}^n, V)$$

holds if and only if $\mathbb{B}$ is canceling.

It was already noticed in [155] for elliptic operators that, in the limiting cases $s = n$ of Theorem 4.2.3(b), (c), $j = n$ of Theorem 4.2.3(a) and Theorem 4.2.4 (which only makes sense if $q = \infty$), the canceling condition should be replaced by a strictly weaker condition [155, Eq. (1.7)]. In the present context, we extend [155, Thm. 1.3]:

Theorem 4.2.5 (L^∞ -estimate). *Let $\mathbb{B}$ be a constant rank operator as in (4.4) and $k \geq n \geq 1$. Then the estimate*

$$\|D^{k-n}(u - \pi u)\|_{L^\infty} \leq c \|\mathbb{B}u\|_{L^1} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V)$$

holds if and only if $\mathbb{B}$ satisfies

$$\int_{\mathbb{S}^{n-1}} \mathbb{B}^\dagger(\xi) w \otimes^{k-n} \xi d\mathcal{H}^{n-1}(\xi) = 0 \quad \text{for all } w \in \bigcap_{\xi \in \mathbb{S}^{n-1}} \text{im } \mathbb{B}(\xi). \quad (4.8)$$

We can also interpret Theorem 4.2.5 as the limiting case $j = n$ of Theorem 4.2.3(e) with $q = 1$ (see [155, Sec. 7]).

The proof of necessity for all three results stems from the following Lemma:

Lemma 4.2.6. *Let $\mathbb{B}$ be a constant rank operator as in (4.4). Let $w \in W$ be such that $w \in \text{im } \mathbb{B}(\xi)$ for all $\xi \neq 0$ and $\rho \in C_c^\infty(B_2)$ be such that $\rho = 1$ in B_1 . Then the locally integrable function $\rho K w$ can be approximated by C_c^∞ -functions in the semi-norm $u \mapsto |\mathbb{B}u|(\mathbb{R}^n)$. Here $|\mu|$ denotes the total variation measure of a measure μ , K is such that $\hat{K} = \mathbb{B}^\dagger(\cdot)$, and $B_r := B(0, r)$. Moreover,*

$$\rho K w - \pi(\rho K w) = K w + \Psi, \quad (4.9)$$

where $\Psi \in C^\infty(\mathbb{R}^n, V)$.

Proof. We first recall that $\mathbb{B}^\dagger(\cdot)$ is a $(-k)$ -homogeneous distribution in $\mathbb{R}^n \setminus \{0\}$, so it extends to a tempered distribution (see the proof of [43, Lem. 2.1]). Moreover, $K \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L_{\text{loc}}^1(\mathbb{R}^n)$ valued in $\text{Lin}(W, V)$ and $\mathbb{B}(K w) = \delta_0 w^1$, where w is as in the statement of the Lemma. This is due to the fact that

$$\mathcal{F}[\mathbb{B}(K w)](\xi) = \mathbb{B}(\xi) \mathbb{B}^\dagger(\xi) w = w,$$

¹We can take $u_h = K w$ in Lemma 4.2.1.

where the last equality follows from the fact that $MM^\dagger$ is the orthogonal projection onto $\text{im } M$. By the chain rule and $\rho(0) = 1$, we have that

$$\mathbb{B}(\rho K w) = \delta_0 w + \psi,$$

where $\psi \in C_c^\infty(B_{5/2} \setminus B_{1/2}, W)$, so that $\mathbb{B}(\rho K w)$ is a compactly supported measure.

Let now η_ε be a sequence of radially symmetric, positive (standard) mollifiers. Then $(\rho K w) \star \eta_\varepsilon \in C_c^\infty(\mathbb{R}^n, V)$ and

$$\mathbb{B}[(\rho K w) \star \eta_\varepsilon] = [\mathbb{B}(\rho K w)] \star \eta_\varepsilon = \eta_\varepsilon w + \psi \star \eta_\varepsilon.$$

For small $\varepsilon > 0$, the two smooth terms above are disjointly supported, so

$$\begin{aligned} \int_{\mathbb{R}^n} |\mathbb{B}[(\rho K w) \star \eta_\varepsilon]| dx &= \int_{B_{1/2}} |\eta_\varepsilon w| dx + \int_{B_{5/2} \setminus B_{1/2}} |\psi \star \eta_\varepsilon| dx \\ &\rightarrow |w| + \|\psi\|_{L^1} = |\mathbb{B}(\rho K w)|(\mathbb{R}^n) \quad \text{as } \varepsilon \downarrow 0. \end{aligned}$$

Furthermore, by (4.7), we have that

$$\rho K w - \pi(\rho K w) = K \star [\mathbb{B}(\rho K w)] = K \star (\delta_0 w + \psi),$$

and (4.9) follows with $\Psi := K \star \psi$. $\square$

As a minor remark, it is well-known that mollifications of bounded measures converge strictly, but we presented an elementary proof in this case, as it was readily available.

Using Lemma 4.2.6, which essentially shows that $u = \rho K w$ is admissible in all estimates investigated, we can give a slightly different proof of [201, Prop. 5.5, 8.23], and [43, Prop. 3.1], which is contained in the following:

Proof of necessity of algebraic assumptions for Theorems 4.2.3, 4.2.4, 4.2.5. We begin by assuming that $\mathbb{B}$ is not canceling and show that the embedding in (e) fails.

Let $u := \rho K w$ be as in Lemma 4.2.6, $1 \leq j \leq \min\{n-1, k\}$, $1 < q < \infty$. By (4.9), it suffices to show that $D^{k-j} K w$ is not in $L^{n/(n-j), q}$ in a neighbourhood of zero.

Essentially by Lemma 4.2.1 (or [43, Lem. 2.1]), we have that $D^{k-j} K$ is $(j-n)$ -homogeneous, and so is $D^{k-j} K w \not\equiv 0$. Since $D^{k-j} K w$ is continuous on $\mathbb{S}^{n-1}$, it follows that there exists a constant $c_0 > 0$ such that

$$|D^{k-j} K(x) w| \geq c_0 |x|^{j-n} \quad \text{for all } x \in \mathcal{C},$$

where the set $\mathcal{C} = \{x \in \mathbb{R}^n : x_1^2 + x_2^2 + \dots + x_{n-1}^2 < a x_n^2\}$ for some $a > 0$ is a cone (represented in a new coordinate system). We can then have, with $B := B_1$,

$$\begin{aligned} \|D^{k-j} K w\|_{L^{n/(n-j), q}(B)}^q &= \int_0^\infty \left[\lambda^{\frac{n}{n-j}} \mathcal{L}^n(\{x \in B : |D^{k-j} K(x) w| > \lambda\}) \right]^{\frac{q(n-j)}{n}} \frac{d\lambda}{\lambda} \\ &\geq \int_0^{c_0} \left[\lambda^{\frac{n}{n-j}} \mathcal{L}^n(\{x \in B \cap \mathcal{C} : |x| < (\lambda/c_0)^{\frac{1}{j-n}}\}) \right]^{\frac{q(n-j)}{n}} \frac{d\lambda}{\lambda} \\ &= \infty, \end{aligned}$$

since $\mathcal{L}^n(B_r \cap \mathcal{C}) \sim r^n$. This proves necessity of cancellation for Theorem 4.2.3(e) and (a).

The necessity of cancellation for Theorem 4.2.4 follows almost along the same lines, by computing

$$\int_B \frac{|D^{k-j}K(x)w|^q}{|x|^{n-q(n-j)}} dx \geq c_0 \int_{B \cap \mathcal{C}} \frac{|x|^{q(j-n)}}{|x|^{n-q(n-j)}} dx = \infty.$$

Necessity of cancellation for Theorem 4.2.3(b), (c), (d) follows from the arguments in, respectively, [201, Prop. 8.19-21].

Finally, we prove necessity of Condition (4.8) for Theorem 4.2.5. Let $k \geq n$. Recall from [155, Sec. 3] that $\mathbb{B}^\dagger(\cdot)$ extends to a tempered distribution that we do not relabel for which

$$\begin{aligned} D^{k-n} \mathcal{F}^{-1}[\mathbb{B}^\dagger(\cdot)]\mathbf{w} &= H_0 \mathbf{w} + \log |\cdot| \int_{\mathbb{S}^{n-1}} \mathbb{B}^\dagger(\xi) \mathbf{w} \otimes \xi^{\otimes(k-n)} d\mathcal{H}^{n-1}(\xi) \\ &=: H_0 \mathbf{w} + \log |\cdot| \mathcal{L} \mathbf{w} \quad \text{for } \mathbf{w} \in W, \end{aligned}$$

where $H_0 \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is a tensor-valued zero-homogeneous function and $\mathcal{L}$ is a linear map between finite dimensional spaces. Therefore, we can write $D^{k-n}K = H_0 + \log |\cdot| \mathcal{L}$.

We now assume that Condition (4.8) fails, so that there exists $w \in W$ such that $w \in \text{im } \mathbb{B}(\xi)$ for all $\xi \neq 0$ and, in addition, $\mathcal{L}w \neq 0$. For small $|x|$ we have

$$\begin{aligned} |D^{k-n}K(x)w| &\geq |H_0(x)w| + \log |x| |\mathcal{L}w| \\ &\geq -\log |x| |\mathcal{L}w| - \|H_0\|_\infty |w| \rightarrow \infty \quad \text{as } x \rightarrow 0. \end{aligned}$$

By Lemma 4.2.6, it follows that the embedding fails for $u = \rho K w$. $\square$

Proof of sufficiency of the algebraic assumptions for Theorems 4.2.3, 4.2.4, 4.2.5. We begin by proving that the embedding in (c) holds. This includes sufficiency of cancellation for (b) (by choosing $q = n/(n-s)$) and for (a) (by further choosing $s = j$).

The proof follows as in [201, Sec. 8]. We write $f := \mathbb{B}u$. Then, by homogeneity of $\mathbb{B}$, we have that

$$[\mathcal{F}(u - \pi u)](\xi) = \mathbb{B}^\dagger(\xi) \hat{f}(\xi) = \mathbb{B}^\dagger\left(\frac{\xi}{|\xi|}\right) |\xi|^{-k} \hat{f}(\xi) = \mathbb{B}^\dagger\left(\frac{\xi}{|\xi|}\right) \widehat{I_k f}(\xi),$$

where I_k denotes the k -th homogeneous Riesz potential. By [192, Thm. 5.2.2-3],

$$\|u - \pi u\|_{\dot{B}_q^{k-s, n/(n-s)}} \leq c \|I_k f\|_{\dot{B}_q^{k-s, n/(n-s)}} \leq c \|f\|_{\dot{B}_q^{-s, n/(n-s)}} = c \|\mathbb{B}u\|_{(\dot{B}_q^{s, n/s})^*}.$$

One then employs Lemma 4.2.2 and [201, Prop. 8.8] with $L(D) = \mathcal{A}$ and $f = \mathbb{B}u$ to conclude. The proof is similar in the case (d) of Triebel–Lizorkin spaces, using [201, Prop. 8.7] instead of [201, Prop. 8.8].

As for Theorem 4.2.3(e), we start with the case $j = 1$ and use boundedness of singular integrals between Lorentz spaces and [201, Prop. 8.10] (as above, for $f = \mathbb{B}u$ and $L(D) = \mathcal{A}$ with $\mathcal{A}$ given by Lemma 4.2.2) to get

$$\|D^{k-1}(u - \pi u)\|_{L^{n/(n-1), q}} = \|D^{k-1}K \star \mathbb{B}u\|_{L^{n/(n-1), q}} \leq c \|\mathbb{B}u\|_{\dot{W}^{-1} L^{n/(n-1), q}}$$

$$= c\|\mathbb{B}u\|_{(\dot{W}^1 L^{n,q/(q-1)})^*} \leq c\|\mathbb{B}u\|_{L^1}.$$

The case when $j = 2, 3, \dots, \min\{n-1, k\}$ follows by an iteration of [187, Prop. 8].

The proof of Theorem 4.2.4 is completed by inserting $G = K$ in the proof of [43, Prop. 2.3], which then requires that [43, Prop. 2.2] holds for constant rank and canceling operators. This is true and can be seen by replacing $L(D)$ with $\mathcal{A}$ as given by Lemma 4.2.2 in the proof of [43, Prop. 2.2]. Similarly, the proof of Theorem 4.2.5 follows from the proof of [155, Thm. 1.3], noting that [155, Lem. 3.1] (which is a minor modification of [43, Lem. 2.2]) extends to constant rank operators by Lemma 4.2.2. $\square$

4.2.3. Concluding remarks

Remark 4.2.7 (Non-constant rank operators). *We recall from [156, Thm. 1] that if (4.6), holds, then both differential operators $\mathbb{B}$ and $\mathcal{A}$ are of constant rank. In particular, if $\mathbb{B}$ does not have constant-rank, then one cannot argue by duality using estimates for cocanceling operators. Even if one relaxes (4.6) to*

$$\ker \mathcal{A}(\xi) \subset \operatorname{im} \mathbb{B}(\xi) \quad \text{for } \xi \neq 0,$$

in which case cancellation of $\mathbb{B}$ implies cocancellation of $\mathcal{A}$, one still cannot use the Fourier multiplier techniques to prove the inequalities of this chapter. This is due to the fact that $0 \neq \xi \mapsto \mathbb{B}^\dagger(\xi)$ is necessarily discontinuous.

In particular, if one dispenses of the assumption that $\mathbb{B}$ is canceling, inequalities for non-elliptic operators may become available, but the methods have to be new. This is the case for [201, Prop. 5.4] and the family of pure Hardy-type inequalities in [43, Sec. 5], which are derived by elementary methods. On the other hand, by Remark 4.2.8 below, whenever such inequalities hold for non-elliptic operators, these do not have constant rank.

Remark 4.2.8 (Indispensability of the projection). *If $\mathbb{B}$ is not elliptic but of constant rank, one cannot dispense of the projection π defined in (4.7) in any inequality from Section 4.2.2. In other words, inequalities as in (4.1) cannot hold for constant rank, non-elliptic operators. This so, since by [156, Thm. 1], there exists a differential operator $\mathfrak{B} \neq 0$ such that $\operatorname{im} \mathfrak{B}(\xi) = \ker \mathbb{B}(\xi) \neq 0$ for all $\xi \neq 0$. Plugging $u := \mathfrak{B}\psi$ for $0 \neq \psi \in C_c^\infty$ in (4.1) we obtain*

$$\|D^{k-1}\mathfrak{B}\psi\|_{L^{n/(n-1)}} \leq c\|\mathbb{B} \circ \mathfrak{B}\psi\|_{L^1} = 0,$$

which is an obvious contradiction. In particular, if [201, Open Prob. 8.1] would be concluded in the positive, a counter-example should fail the constant rank condition. The same is true if [201, Open Prob. 5.1] would be concluded in the negative.

Thus, the inequalities in Section 4.2.2 can also be interpreted as follows: If $\mathbb{B}$ has constant rank, among all solutions u of $\mathbb{B}u = f \in \mathcal{M}$, there is one that has the appropriate regularity; moreover the statement is optimal if $\mathbb{B}$ is non-elliptic. Interestingly, for the operators in [201, Prop. 5.4], [43, Sec. 5], which necessarily have non-constant rank whenever non-elliptic, one obtains critical regularity for all solutions u .

Remark 4.2.9 (Cocanceling estimates from canceling estimates). *Our proof of sufficiency of cancellation for Theorem 4.2.3 uses the method from [201] to derive “primal” estimates for canceling operators from “dual” estimates for cocanceling operators, by which we mean*

$$\|K \star \mathbb{B}u\|_X \leq c\|\mathbb{B}u\|_Y \leq c\|\mathbb{B}u\|_{L^1} \quad \text{for } u \in C_c^\infty(\mathbb{R}^n, V),$$

where K is as in (4.7), X is as in the statement of Theorem 4.2.3, and Y is given by the elliptic estimates used in the proof. In the present context, one can also derive the “dual” estimates from “primal” estimates. We give an example to illustrate this, namely when $X = \dot{W}^{k-1, n/(n-1)}$, so that $Y = (W^{1, n})^*$.

Suppose that we are given operators $\mathbb{B}$ and $\mathcal{A}$ such that the exact relation (4.6) holds (hence both have constant rank by Remark 4.2.7) such that $\mathcal{A}$ is cocanceling, so $\mathbb{B}$ is canceling. Let $w \in C_c^\infty$ be such that $\mathcal{A}w = 0$. By (4.6), we have that $u := K \star w \in \mathcal{S}$ is such that $\mathbb{B}u = w$ (see also [156, Lem. 5]). Assuming that Theorem 4.2.3(a) holds, we have that

$$\begin{aligned} \|w\|_Y &= \|\mathbb{B}u\|_{(\dot{W}^{1, n})^*} \leq c\|D^{k-1}(K \star \mathbb{B}u)\|_{L^{n/(n-1)}} \\ &\leq c\|\mathbb{B}u\|_{L^1} = c\|w\|_{L^1}, \end{aligned}$$

where the second inequality follows from (4.7) and the assumed Theorem.

We conclude this chapter with two typical examples of constant rank, non-elliptic operators: $\mathbb{B} = \text{curl}$ and $\mathbb{B} = \text{div}$:

Example 4.2.10. Let $\mathbb{B} = \text{curl}$ on $\mathbb{R}^n$, $n \geq 2$. Then $\mathbb{B}$ is canceling and the estimates in Theorems 4.2.3 and 4.2.4 hold. Moreover, one can use the inequality

$$\|u - \pi u\|_{L^{n/(n-1)}} \leq c\|\text{curl } u\|_{L^1} \quad \text{for } u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \quad (4.10)$$

to prove that

$$\|u\|_{L^{n/(n-1)}} \leq c\|\text{curl } u\|_{L^1} \quad \text{for } u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \text{ with } \text{div } u = 0. \quad (4.11)$$

In this case, we recover [36, Thm. 2], which states that (4.11) holds for $n = 3$, but the approach we give is not new, as the proof below is essentially given in [196].

Proof of (4.11) using (4.10). By definition of π (see (4.7) and the lines above), we have

$$u = \pi u + (u - \pi u) =: \nabla \phi + \psi$$

is an L^2 -Helmholtz decomposition of $u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ for unique $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\psi \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^n)$ with $\text{div } \psi = 0$. If we assume that $\text{div } u = 0$, it follows that $\Delta \phi = 0$ (e.g., by [43, Lem. 2.1]), so that $u - \pi u = \psi$. The proof is complete. $\square$

Example 4.2.11. Let $\mathbb{B} = \text{div}$ on $\mathbb{R}^n$, $n \geq 1$. Then $\mathbb{B}$ is not canceling and the estimates in Theorems 4.2.3 and 4.2.4 fail (cp. [38, Rk. 5]). If $n = 1$, Condition (4.8) trivially holds, so the estimate of Theorem 4.2.5 holds, but this is not very interesting as the inequality follows by the Fundamental Theorem of Calculus anyway.

4.3. Appendix: Weak-type estimates

We briefly discuss the remaining limiting cases $q \in \{0, \infty\}$ of Theorem 4.2.3(c), (e). If $q = 1$, the problem is open (cp. [201, Open Prob. 8.2-3]). If $q = \infty$, one obtains, by analogy with [201, Prop. 8.22, 8.24] and [155, Prop. 7.5]:

Proposition 4.3.1. *Let $\mathbb{B}$ be a constant rank operator as in (4.4), $n \geq 1$, $s \in (0, n)$, and $1 \leq j \leq \min\{n, k\}$. We have that*

$$\begin{aligned} \|D^{k-1}(u - \pi u)\|_{L^{n/(n-j), \infty}} &\leq c \|\mathbb{B}u\|_{L^1} \\ \|u - \pi u\|_{\dot{B}_{\infty}^{k-s, n/(n-s)}} &\leq c \|\mathbb{B}u\|_{L^1} \end{aligned}$$

for all $u \in C_c^\infty(\mathbb{R}^n, V)$.

If $j = n$, we write $L^{\infty, \infty}$ for the space weak- L^∞ , as defined in [33].

CHAPTER 5

Critical L^p -differentiability of $BV^{\mathbb{B}}$ -maps and canceling operators

This chapter consists of the author's work.

5.1. Introduction

5.1.1. First order operators Approximate differentiability properties of weakly differentiable functions are well understood [75, Ch. 6]. In particular, CALDERÓN and ZYGMUND proved in [50, 51] that a map $u \in BV_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ is $L^{n/(n-1)}$ -differentiable at $\mathcal{L}^n$ -a.e. $x \in \mathbb{R}^n$, i.e., there exists a matrix $\nabla u(x)$ (called *approximate gradient*) such that

$$\|R_x^1 u\|_{L^{n/(n-1)}(B_r(x))} = o(r^n) \quad \text{as } r \downarrow 0.$$

Here $R_x^1 u$ denotes the first order Taylor remainder

$$(R_x^1 u)(y) := u(y) - u(x) - \nabla u(x)(y - x)$$

for all $y \in \mathbb{R}^n$. It was recently proved in [95] that the same property holds true of the space BD of maps of bounded deformation, building on the results in [12, 16, 98]. The space BD, arising naturally in plasticity problems [19, 88, 178], is defined as the space of integrable maps $u: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that the distributional symmetrized gradient $\mathcal{E}u := (Du + Du^t)/2$ is a bounded measure. The $L^{n/(n-1)}$ -differentiability of BD_{loc} -maps cannot be extrapolated from the BV result since by Ornstein's Non-inequality [57, 113, 148], there are BD-maps for which the full distributional gradient is not a bounded measure. Instead, the view point taken in [12, 98] is that test functions can be retrieved from their symmetrized gradient via convolution. In analogy with the fact that the Riesz potential I_1 is bounded from L^1 into $L^{n/(n-1)}_{\text{weak}}$ (and *not* into $L^{n/(n-1)}$), the harmonic analysis techniques used in [12, 98] yield L^p -differentiability of BD_{loc} -maps for all $1 \leq p < n/(n-1)$, but cannot easily cover the critical case. The idea used in [95] was to use the embedding $BD \hookrightarrow L^{n/(n-1)}$ [178, Prop. 1.2] in the form of a Poincaré–Sobolev-type inequality. In fact, the arguments in [95] cover a class of differential operators that contains $\mathcal{E}$. To be precise, we define the space $BV^{\mathbb{B}}(\Omega)$ as the space of $u \in L^1(\Omega, V)$ such that $\mathbb{B}u \in \mathcal{M}(\Omega, W)$ is a bounded measure, for open sets $\Omega \subset \mathbb{R}^n$, where the differential operator $\mathbb{B}$ is defined, independently of coordinate

choice, by

$$\mathbb{B}u := A(Du) \quad (5.1)$$

for smooth maps $u: \mathbb{R}^n \rightarrow V$, a linear map $A \in \text{Lin}(V \times \mathbb{R}^n, W)$, and finite dimensional normed vector spaces V, W . For simplicity of exposition, we assume in addition that $\mathbb{R}^n, V, W$ are all equipped with fixed inner products. This is no restriction, as we will explain later. Although in this section we assume that $n > 1$, the embedding presented in Theorem 5.1.3 will enable us to also cover the case $n = 1$, which is ruled out for similar results, e.g., [12, Thm. 3.4] or [95, Thm. 1.1].

The main result in [95] states that if the solution space of $\mathbb{B}u = 0$ in $\mathcal{D}'(\mathbb{R}^n, V)$ is finite dimensional (when we say that $\mathbb{B}$ has *FDN*), then any map in $BV_{\text{loc}}^{\mathbb{B}}$ is $L^{n/(n-1)}$ -differentiable $\mathcal{L}^n$ -a.e.. Of course, this covers the physically relevant case $BV^{\mathbb{B}} = \text{BD}$, since $\mathcal{E}u = 0$ is satisfied only by rigid deformations [99, Thm. 3.2]. It is however natural to ask the mathematically relevant question to determine the class of operators $\mathbb{B}$ such that all maps in $BV_{\text{loc}}^{\mathbb{B}}$ are $L^{n/(n-1)}$ -differentiable $\mathcal{L}^n$ -a.e.. What we know is that such a class contains the FDN operators and it is not difficult to see that it is contained in the class of elliptic operators. Indeed, we will establish this in Lemma 5.5.2 (recall that an operator $\mathbb{B}$ as in (5.1) is (overdetermined) *elliptic* if the symbol map $V \ni v \mapsto A(v \otimes \xi) =: \mathbb{B}[\xi]v$ is injective). It was already shown in [95, Rk. 3.2] that ellipticity is not sufficient for differentiability. On the other hand, as explained in [45, 94], the FDN condition is equivalent with boundary regularity, respectively, of traces of maps in $BV^{\mathbb{B}}(\Omega)$ and of solutions of $\mathbb{B}u = 0$ in $\mathcal{D}'(\Omega, V)$. It does not a priori seem likely that FDN is equivalent to a pointwise regularity property, as is $L^{n/(n-1)}$ -differentiability. In light of the proof of [95, Thm. 1.1], it is however feasible that $L^{n/(n-1)}$ -differentiability of $BV_{\text{loc}}^{\mathbb{B}}$ -maps is equivalent with a homogeneous Sobolev-type embedding. Indeed, we will prove in Lemma 5.5.1 that the embedding

$$\|u\|_{L^{n/(n-1)}} \lesssim \|\mathbb{B}u\|_{L^1} \quad (5.2)$$

for all $u \in C_c^\infty(\mathbb{R}^n, V)$ is sufficient to prove the required $L^{n/(n-1)}$ -differentiability. The precise conditions on $\mathbb{B}$ for (5.2) to hold, namely elliptic and *canceling* (EC), were established in [38, 201]. The canceling condition, i.e.,

$$\bigcap_{\xi \in \mathbb{S}^{n-1}} \text{im } \mathbb{B}[\xi] = \{0\},$$

was introduced by VAN SCHAFTINGEN in [201] and can indeed be interpreted as an interior regularity (integrability) condition, as we will discuss in Lemma 5.2.5. In fact, in view of the Lorentz space embedding [201, Thm. 8.5] (see also [150, 15, 104, 187, 188] for previous results on Lorentz-Sobolev embeddings), we say that a map u is $L^{n/(n-1),q}$ -differentiable at x if

$$\|R_x u\|_{L^{n/(n-1),q}(B_r(x))} = o(r^n) \quad \text{as } r \downarrow 0,$$

where $1 \leq q \leq \infty$, then we can obtain the following refined statement:

Theorem 5.1.1. *Let $\mathbb{B}$ be as in (5.1), $1 < q < \infty$, $n > 1$. Then $\mathbb{B}$ is EC if and only if all maps in $BV_{\text{loc}}^{\mathbb{B}}$ are $L^{n/(n-1),q}$ -differentiable $\mathcal{L}^n$ -a.e.*

Our proof of sufficiency of EC is elementary and somewhat novel, in the sense that all other proofs of L^p -differentiability of BV- or BD-maps using Poincaré(-type) inequalities or Sobolev(-type) embeddings that we traced in the literature cannot easily be adjusted to rely on (5.2) only (cp. [16, 18, 75, 95]). We also see no alternative way to link the equivalent condition EC to the critical differentiability, other than by use of (5.2). It is important to mention that it was proved in [94, Sec. 3] that EC is strictly weaker than FDN, so the sufficiency part of Theorem 5.1.1 is not a vacuous extension of [95, Thm. 1.1]. Our proof of Theorem 5.1.1 can be used to show that at the endpoint $q = \infty$ the claim is equivalent with ellipticity of $\mathbb{B}$ alone. In view of [201, Open Prob. 8.3], we do not know whether the critical case $q = 1$ can be achieved, except in the case $\mathbb{B} = D$. It is known that the embedding

$$\dot{W}^{1,1}(\mathbb{R}^n, \mathbb{R}^N) \hookrightarrow L^{n/(n-1),1}(\mathbb{R}^n, \mathbb{R}^N)$$

holds [73, 176, 188], and our method implies the following fact which we could not trace in the literature, but is probably known to experts::

Proposition 5.1.2. *Let $u \in \text{BV}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^N)$. Then u is $L^{n/(n-1),1}$ -differentiable at $\mathcal{L}^n$ -a.e. $x \in \mathbb{R}^n$.*

5.1.2. A new L^∞ -embedding The result of Theorem 5.1.1 extends to higher order homogeneous operators, by which we mean

$$\mathbb{B}u := A(D^k u) \tag{5.3}$$

for smooth maps $u: \mathbb{R}^n \rightarrow V$ and $A \in \text{Lin}(V \odot^k \mathbb{R}^n, W)$. Here $V \odot^k \mathbb{R}^n$ denotes the vector space of symmetric, V -valued, k -linear maps on $\mathbb{R}^n$. More precisely, in order to investigate the L^p -differentiability properties of the derivatives of maps in $\text{BV}_{\text{loc}}^{\mathbb{B}}$, we will use the following inequality for EC operators

$$\|D^{k-j}u\|_{L^{n/(n-j)}} \lesssim \|\mathbb{B}u\|_{L^1} \tag{5.4}$$

for $u \in C_c^\infty(\mathbb{R}^n, V)$. This follows by [201, Thm. 1.3] and iterative application of the Gagliardo–Nirenberg–Sobolev Inequality *only* for $j = 1 \dots \min\{k, n-1\}$. If $k \geq n$, the boundedness of $D^{k-n}u$ cannot be inferred from (5.4) for $j = n-1$ since $\dot{W}^{1,n}$ does not embed into L^∞ , but BMO. This phenomenon is already observed in the case $\mathbb{B} = D^n$, in which case one has to use a different method¹ to show that $\dot{W}^{n,1}$ does indeed embed in L^∞ (C_0 , even). In more generality, it was shown by BOUSQUET and VAN SCHAFTINGEN in [43, Thm. 1.3] that if $k \geq n$ and $\mathbb{B}$ is EC, then

$$\|D^{k-n}u\|_{L^\infty} \lesssim \|\mathbb{B}u\|_{L^1} \tag{5.5}$$

for all $u \in C_c^\infty(\mathbb{R}^n, V)$. The definition of EC is formally the same as in the first order case, with the obvious modification of the definition of the symbol map $\mathbb{B}[\xi]v := A(v \otimes^k \xi)$. Here $v \otimes^k \xi := v \otimes \xi \otimes \xi \otimes \dots \otimes \xi$, where the exterior product is taken k times.

¹Simply the Fundamental Theorem of Calculus in all coordinate directions, in this case.

So far, little is known about the necessity of the EC condition for (5.5). Since for our L^p -differentiability claims ellipticity is necessary (see Lemmas 5.5.2, 5.6.2), we will assume it. As for cancellation, it may not be necessary, as the simple example $\mathbb{B} = D$ if $n = 1$ presented in [43] suggests. One of the main results of this chapter is to show that, somewhat surprisingly, (5.5) is equivalent to a substantially weaker new condition:

Theorem 5.1.3. *Suppose that $\mathbb{B}$ as in (5.3) is elliptic, $k \geq n \geq 1$. Then*

$$\|D^{k-n}u\|_{L^\infty(\mathbb{R}^n, V \odot^{k-n}\mathbb{R}^n)} \lesssim \|\mathbb{B}u\|_{L^1(\mathbb{R}^n, V)}$$

holds for all $u \in C_c^\infty(\mathbb{R}^n, V)$ if and only if

$$\mathcal{L}w := \int_{\mathbb{S}^{n-1}} \mathbb{B}^\dagger[\xi]w \otimes^{k-n} \xi \, d\mathcal{H}^{n-1}(\xi) = 0 \quad \text{for all } w \in \bigcap_{\xi \in \mathbb{S}^{n-1}} \text{im } \mathbb{B}[\xi], \quad (\text{WC})$$

where $\mathcal{L} \in \text{Lin}(W, V \odot^{k-n}\mathbb{R}^n)$ is defined on the entire W by the above formula.

We use the notation $\mathbb{B}^\dagger[\xi] := (\mathbb{B}^*[\xi]\mathbb{B}[\xi])^{-1}\mathbb{B}^*[\xi]$ for $\xi \neq 0$ and $:= 0$ otherwise, where “ $*$ ” denotes the adjoint. $\mathbb{B}^\dagger[\cdot]$ is well-defined by the ellipticity assumption.

The importance of Theorem 5.1.3 for L^1 -estimates is that, under the ellipticity assumption, it settles the investigation of Sobolev estimates of the type (5.4) with the endpoint case $j = n$. In the absence of ellipticity, there is no systematic method to obtain even weak-type estimates, which should be a simple baseline if $j < n$; only few examples have been considered, see e.g., [201, Prop. 5.4] or [152].

We next discuss condition (WC), which is obviously weaker than the canceling condition, thereby recovering the result in [43, Thm. 1.3]. By a simple homogeneity consideration, one immediately notes that condition (WC) is automatically satisfied if n is odd (in which case $\mathcal{L} \equiv 0$). In Section 5.4.3, we will give examples to show that in both even and odd dimensions, condition (WC) is strictly weaker than the canceling condition. In Proposition 5.4.3, we will give an analytic characterization which highlights why the discrepancy is possible. On the other hand, Proposition 5.4.3 reveals a common theme shared by (5.4) and (5.5), which is in sharp contrast to Ornstein’s Non-inequality. We will expand on this point in Section 5.4.2.

We next compare Theorem 5.1.3 with other instances when quantitative restrictions on the data of a partial differential equation imply everywhere continuity of the solution. An observation in this direction was made by STEIN in [174], where it is shown, as a recovery of the fact that $\dot{W}^{1,n} \not\hookrightarrow C_0$, that

$$Du \in L^{n,1}(\mathbb{R}^n, \mathbb{R}^n) \implies u \in C_0(\mathbb{R}^n),$$

up to the choice of a representative. In the same vein, one can show that for each $l = 1, \dots, n$, we have

$$D^l u \in L^{n/l,1}(\mathbb{R}^n, \mathbb{R}^n) \implies u \in C_0(\mathbb{R}^n), \quad (5.6)$$

(see for instance [115], where a detailed analysis of this and related phenomena is performed; cp. [54, 55, 56]). The case when $l < n$ follows by Hölder’s inequality and duality of Lorentz spaces (see Lemma 5.2.1), but for $l = n$, one cannot argue

in the same way and should use (5.5) for $\mathbb{B} = D^n$, as $L^{1,1} = L^1$ (of course, the inequality (5.5) in this case was known before, see, e.g., [152]). Should one try to generalize this to linear elliptic operators of order l on $\mathbb{R}^n$, i.e.,

$$\mathbb{B}u \in L^{n/l,1}(\mathbb{R}^n, W) \implies u \in C_0(\mathbb{R}^n), \quad (5.7)$$

the question is not very interesting for $l < n$ since, by boundedness of singular integrals (again, see Lemma 5.2.1), we have that (5.7) reduces to (5.6). For $l = n$, the characterization of elliptic $\mathbb{B}$ satisfying (5.6) is provided by Theorem 5.1.3. We expand on this point in a functional framework in Section 5.4.1. In the future, we hope to be able to tackle generalizations to non-linear problems with L^1 -data (cp. [125, 9]).

As it stands, condition (WC) depends on the choice of Euclidean structure on the domain and target spaces. In Remark 5.4.5, we will show that (WC) is independent of this choice, as is ellipticity and the estimate (5.5). We chose to present Theorem 5.1.3 in this form since, represented in coordinates, condition (WC) is computable, whereas its invariant form in (5.18) is implicit.

We conclude this section with the minor remark that Theorem 5.1.3 enables us to consistently add the case $n = 1$ to VAN SCHAFTINGEN's theory [201].

5.1.3. Higher order operators We have the following $\mathcal{L}^n$ -a.e.-generalization of DORRONSORO's [73, Thm. 1], concerning critical L^p -differentiability of BV^k -maps:

Theorem 5.1.4. *Let $\mathbb{B}$ be as in (5.3). Then:*

(a) *If $1 \leq j \leq \min\{k, n-1\}$, the following are equivalent:*

(i) *For all $u \in BV_{\text{loc}}^{\mathbb{B}}(\mathbb{R}^n)$, we have that*

$$D^{k-j}u \in \mathfrak{t}^{j,n/(n-j)}(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

(ii) *$\mathbb{B}$ is elliptic and canceling.*

(b) *If $k \geq n$, the following are equivalent:*

(i) *For all $u \in BV_{\text{loc}}^{\mathbb{B}}(\mathbb{R}^n)$, we have that*

$$D^{k-n}u \in \mathfrak{t}^{n,\infty}(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

(ii) *$\mathbb{B}$ is elliptic and satisfies condition (WC).*

In particular, in the former case, u has a k -th order $L^{n/(n-k)}$ -Taylor expansion a.e.; in the latter case u is k times differentiable a.e. (in the classical sense).

The spaces $\mathfrak{t}^{k,p}(x)$ and $\mathsf{T}^{k,p}(x)$ of k -times differentiable functions in the L^p -sense at x were introduced by CALDERÓN and ZYGMUND. We recall their definitions in Section 5.2.1. We also extend the definition of $u \in BV^{\mathbb{B}}$ if $u \in L^1$ and $\mathbb{B}u \in \mathcal{M}$ (an obvious extension to higher order operators).

As it stands, Theorem 5.1.4 is a generalization of [73, Thm. 1(i)] only. In the future, we intend to refine the results of Theorem 5.1.4 to lower dimensional exceptional sets, i.e., in the spirit of [73, Thm. 1(ii),(iii)].

The proof of Theorem 5.1.4 is a natural extension of the ideas used to prove Theorem 5.1.1. To prove (a), we again use [201, Thm. 1.3], as formulated in (5.4). To prove (b), we use Theorem 5.1.3.

The necessity of ellipticity of $\mathbb{B}$ for the conclusions of Theorem 5.1.4 is somewhat surprising, particularly since it is not necessary for the estimate (5.4) used in the proof of (a), for $u \in C_c^\infty$, *unless* $j = 1$ (see [201, Sec. 5.1] for detail). This is to say that ellipticity is not necessary to give critical estimates for lower derivatives, but, as we will see below in Theorem 5.1.5, ellipticity is necessary to get sub-critical L^p -differentiability in our setting.

With the machinery used to prove Theorem 5.1.4 in place, it is easy to prove the sub-critical version, which is characteristic of elliptic operators. We state it here for comparison with the critical case above and for completeness of this work.

Theorem 5.1.5. *Let $\mathbb{B}$ be as in (5.3). Then $\mathbb{B}$ is elliptic if and only if either of the following holds:*

- (a) *If $1 \leq j \leq \min\{k, n-1\}$, $1 < p < n/(n-j)$, for all $u \in BV_{\text{loc}}^{\mathbb{B}}$ we have*

$$D^{k-j}u \in \mathfrak{t}^{j,p}(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

- (b) *If $j = n \leq k$, $1 < p < \infty$, for all $u \in BV_{\text{loc}}^{\mathbb{B}}$ we have*

$$D^{k-j}u \in \mathfrak{t}^{j,p}(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

- (c) *If $n < k$, for all $u \in BV_{\text{loc}}^{\mathbb{B}}$ we have*

$$D^{k-n-1}u \in \mathfrak{t}^{n+1,\infty}(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

In particular, in the third case, u is k times classically differentiable a.e..

The reader is invited to formulate the analogous Lorentz variants of Theorems 5.1.4, 5.1.5 in the spirit of Theorem 5.1.1 or consult the Appendix (Section 5.7).

We next connect our results to overdetermined elliptic linear systems with L^1 -data. In [50, Thm. 1], CALDERÓN and ZYGMUND showed that, for $1 < p < \infty$, a k -th order linear elliptic system

$$\mathbb{B}u = f$$

for which it is assumed that $u \in W^{k,p}$, the pointwise regularity $f \in \mathfrak{t}^{m,p}(x)$ can be improved to $D^{k-j}u \in \mathfrak{t}^{m+j,q}(x)$ for critical q which can be computed using the Sobolev embedding in an obvious way. Their results rely on boundedness of singular integrals on L^p and cannot be extended to the limiting case $p = 1$. In this case, by Ornstein's Non-inequality, it is not reasonable to assume that $u \in W^{k,1}$, but we can extend the definition of $\mathfrak{t}^{k,1}$ in a natural way, so that a Radon measure lies in $\mathfrak{t}^{0,1}(x)$ for $\mathcal{L}^n$ -a.e. x by Lebesgue's Differentiation Theorem. If we then consider the system

$$\mathbb{B}u = \mu,$$

where μ is a Radon measure, Theorem 5.1.5 implies that $D^{k-j}u \in \mathfrak{t}^{j,q}(x)$ for $\mathcal{L}^n$ -a.e. x for any sub-critical q if and only if $\mathbb{B}$ is elliptic. In the range $1 \leq k < n$, Theorem 5.1.4 implies that the critical exponent q can be achieved if and only if, in addition, $\mathbb{B}$ is canceling. Finally, if $k \geq n$, condition (WC) is equivalent with critical regularity. Of course, our results also imply that, if $\mu \in \text{BV}_{\text{loc}}^m$, we obtain $D^{k-j}u \in \mathfrak{t}^{m+j,q}(x)$ for $\mathcal{L}^n$ -a.e. x with a similar discussion for the exponent q .

This chapter is organized as follows: In Section 5.2 we present general notation, the function spaces used, background facts about elliptic and EC operators, and the proof of sufficiency of ellipticity for Theorem 5.1.5. In Section 5.3 we prove Theorem 5.1.3, whereas its consequences for embeddings of function spaces are presented in Section 5.4.1. In Section 5.4.2 we give an invariant, analytic characterization of condition (WC) and examples to illustrate how it differs from cancellation. In Section 5.5 we prove Theorem 5.1.1. In Section 5.6 we prove Theorem 5.1.4; this includes the necessity of ellipticity for Theorem 5.1.5. In the Appendix (Section 5.7), we present all differentiability results on the full Lorentz scale.

5.2. Preliminaries

Throughout this chapter, we work on function spaces defined on $\mathbb{R}^n$, for $n \geq 1$. We denote the ball centred at $x \in \mathbb{R}^n$ of radius $r > 0$ by $B_r(x) \equiv B(x, r)$. We denote Lebesgue measure on $\mathbb{R}^n$ by $\mathcal{L}^n$ and the s -dimensional Hausdorff measure by $\mathcal{H}^s$. Averaged integrals, only taken with respect to Lebesgue measure ($\text{d}x$) throughout, are denoted by

$$\oint_{\Omega} := \frac{1}{\mathcal{L}^n(\Omega)} \int_{\Omega} =: (\cdot)_{\Omega}.$$

The space of linear maps between two vector spaces V, W is denoted by $\text{Lin}(V, W)$. The space $\mathcal{M}(\mathbb{R}^n, W)$ denotes the spaces of W -valued, bounded measures on $\mathbb{R}^n$ (i.e., of finite total variation), naturally equipped with the total variation norm $|\cdot|$. Inherently, $\mathcal{M}_{\text{loc}}$ denotes the space of Radon measures. We denote by C_c^{∞} the space of continuously supported maps, by $\mathcal{S}$ the space of Schwartz functions, by L^p (L_{loc}^p) the space of (locally) p -integrable functions. We denote by $\mathcal{D}'$ the space of distributions, the topological dual of C_c^{∞} , and by $\mathcal{S}'$ the space of tempered distributions, dual to the Schwartz space $\mathcal{S}$. Sobolev spaces are denoted by $W^{k,p}$. The Fourier transform is defined by

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \, \text{d}x \quad \text{for } \xi \in \mathbb{R}^n$$

for $u \in \mathcal{S}$ and extended for tempered distributions by duality.

We write $X(\Omega, V)$ for a space of (generalized) functions from some open $\Omega \subset \mathbb{R}^n$ into a vector space V . We abbreviate $X(\Omega) := X(\Omega, \mathbb{R})$, but, as a note of caution, we may suppress V altogether, when no confusion arises (also for spaces of vector-valued functions). This is mostly done for displayed estimates.

We also make the rather strange but very handy convention to ignore multiplicative non-zero (possibly complex) constant scalars. Such constants play no role in either the inequalities, or in the algebraic conditions we discuss. This can be formalized by redefining equality as a suitable equivalence relation. For example, under this relation the Fourier transform of $\partial_j u$ at ξ “equals” $\xi_j \hat{u}(\xi)$.

5.2.1. Function spaces In the spirit of [94, Sec. 2.1], we define for $\mathbb{B}$ as in (5.3),

$$BV_{\text{loc}}^{\mathbb{B}} := BV_{\text{loc}}^{\mathbb{B}}(\mathbb{R}^n) = \{u \in L_{\text{loc}}^1(\mathbb{R}^n, V) : \mathbb{B}u \in \mathcal{M}_{\text{loc}}(\mathbb{R}^n, W)\},$$

where $\mathcal{M}$ denotes the space of bounded measures. In general, we suppress the domain of maps defined locally or in full space. For each $u \in BV_{\text{loc}}^{\mathbb{B}}$, we denote the Radon–Nikodým decomposition of $\mathbb{B}u \in \mathcal{M}_{\text{loc}}$ by

$$\mathbb{B}u = \mathbb{B}^{ac}u \mathcal{L}^n + \mathbb{B}^s u = \frac{d\mathbb{B}u}{d\mathcal{L}^n} \mathcal{L}^n + \mathbb{B}^s u,$$

where $\mathbb{B}^s u \perp \mathcal{L}^n$.

We recall the spaces of k -times L^p -differentiable functions $T^{k,p}(x)$, $t^{k,p}(x)$, as introduced for $x \in \mathbb{R}^n$, $1 \leq p \leq \infty$ in [50, Def. 1–2] as the spaces of maps u defined in a neighbourhood of x for which there exists a polynomial $P =: P_x^k u$ of degree at most k (a k -th order L^p -Taylor expansion) such that

$$\left(\int_{B_r(x)} |u - P|^p dy \right)^{1/p} = O(r^k), \text{ respectively } o(r^k), \quad \text{as } r \downarrow 0. \quad (5.8)$$

If $p = \infty$, we replace the LHS by $\|u - P\|_{L^\infty(B_r(x))}$. We denote the *remainder* by $R_x^k u := u - P_x^k u$. It is easy to see that $T^{k,q}(x) \subset T^{k,p}(x)$ for $1 \leq p \leq q \leq \infty$; the same holds true for $t^{k,p}$. If $u \in t^{k,\infty}(x)$, then u has k classical derivatives at x . For more detail on these spaces, see ZIEMER's monograph [210, Ch. 3].

The Lorentz space $L^{p,q}$ consists, for $1 \leq p < \infty$, $1 \leq q \leq \infty$, of measurable functions u such that the quasi-norm

$$\begin{aligned} \|u\|_{L^{p,q}} &:= p^{1/q} \left(\int_0^\infty \left[\lambda \mathcal{L}^n(\{|u| > \lambda\})^{1/p} \right]^q \frac{d\lambda}{\lambda} \right)^{1/q} \quad \text{if } q < \infty \\ \|u\|_{L^{p,\infty}} &:= \sup_{\lambda > 0} \lambda \mathcal{L}^n(\{|u| > \lambda\})^{1/p} \end{aligned}$$

is finite. Lorentz spaces are a refinement of the Lebesgue scale, as can be seen, for example, since $L^{p,p} = L^p$ and $L^{p,\infty} = L_{\text{weak}}^p$. More precisely, $L^{p,q_1} \subset L^{p,q_2}$ if $q_1 \leq q_2$, and $L_{\text{loc}}^{p_1,\infty} \subset L_{\text{loc}}^{p_2,1}$ if $p_2 < p_1$.

5.2.2. Elliptic operators We begin with an important Green-type formula:

Lemma 5.2.1 ([43, Lem. 2.1]). *Let $\mathbb{B}$ as in (5.3) be elliptic. Then there exists a map $K^{\mathbb{B}} \in C^\infty(\mathbb{R}^n \setminus \{0\}, \text{Lin}(W, V)) \cap L_{\text{loc}}^1(\mathbb{R}^n, \text{Lin}(W, V))$ such that*

$$u(x) = \int_{\mathbb{R}^n} K^{\mathbb{B}}(x - y) \mathbb{B}u(y) dy = (K^{\mathbb{B}} \star \mathbb{B}u)(x) \quad (5.9)$$

for all $u \in C_c^\infty(\mathbb{R}^n, V)$ and $x \in \mathbb{R}^n$. Moreover, for all integers $l \geq \max\{0, k - n + 1\}$, the map $D^l K^{\mathbb{B}}$ is $(k - n - l)$ -homogeneous.

We next show that under the mild assumption of ellipticity, maps in $BV_{\text{loc}}^{\mathbb{B}}$ have appropriate local Sobolev regularity for their lower derivatives. The proof is reminiscent of the Deny–Lions Lemma, which covers the case $\mathbb{B} = D$ [64] (see also [178, Thm. 2.1] for the case $\mathbb{B} = \mathcal{E}$). Our proof is based on [101, Thm. 4.5.8].

Lemma 5.2.2. *Let $\mathbb{B}$ as in (5.3) be elliptic and $u \in \mathcal{D}'(\mathbb{R}^n, V)$ be such that $\mathbb{B}u \in \mathcal{M}_{\text{loc}}(\mathbb{R}^n, W)$. Then $u \in W_{\text{loc}}^{k-1,p}$ for $1 \leq p < n/(n-1)$.*

Proof. It is easy to see that $\mathbb{B}^*\mathbb{B}$ is elliptic. We write $E := K^{\mathbb{B}^*\mathbb{B}} \in C^\infty(\mathbb{R}^n \setminus \{0\}, \text{Lin}(V, V))$, as given by Lemma 5.2.1. In fact, the resulting E is constructed such that $\mathbb{B}^*[\xi]\mathbb{B}[\xi]\hat{E}(\xi)v = v$ for $\xi \in \mathbb{R}^n \setminus \{0\}$, hence $\mathbb{B}^*\mathbb{B}E = \text{Id}_V \delta_0$. Choose $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\rho = 1$ in an open cube $Q \subset \mathbb{R}^n$. It is then the case that

$$\rho u = E \star \mathbb{B}^*\mathbb{B}(\rho u) = \mathbb{B}E \star \mathbb{B}(\rho u),$$

where the latter inequality follows by integration by parts. It follows that

$$\begin{aligned} D^{k-1}(\rho u) &= D^{k-1}\mathbb{B}E \star \mathbb{B}(\rho u) \\ &= D^{k-1}\mathbb{B}E \star (\rho \mathbb{B}u) + \sum_{j=1}^k D^{k-1}\mathbb{B}E \star B_j(D^j \rho, D^{k-j}u) =: \mathbf{I} + \sum_{j=1}^k \mathbf{II}_j \end{aligned}$$

in the sense of distributions. Here B_j are bilinear pairings that depend on $\mathbb{B}$ only, as given by the Leibniz rule. By Lemma 5.2.1, we have that $D^{2k-1}E$ is $(1-n)$ -homogeneous (even for $n=1$, but this does not follow from Lemma 5.2.1, but (5.14)), hence so is $D^{k-1}\mathbb{B}E$ by k -homogeneity of $\mathbb{B}$. In particular, since E is smooth it follows that $|D^{k-1}\mathbb{B}E| \lesssim |\cdot|^{1-n}$, so $\mathbf{I} \in L_{\text{loc}}^p$ for $1 \leq p < n/(n-1)$ by boundedness of Riesz potentials [175, Thm. V.1] and inclusions of Lorentz spaces. To conclude, we claim that $\mathbf{II}_j$ are smooth in Q . This follows from an application of [101, Thm. 4.2.5], which implies that

$$\begin{aligned} \text{sing spt}(f \star g) &\subset \text{sing spt } f + \text{sing spt } g \\ &\subset \{0\} + \mathbb{R}^n \setminus Q = \mathbb{R}^n \setminus Q, \end{aligned}$$

where we wrote $f = D^{k-1}\mathbb{B}E$, $g = B_j(D^j \rho, D^{k-j}u)$ (the singular support $\text{sing spt } f$ of a distribution f is defined in [101, Def. 2.2.3] as the set of points x such that no restriction of f to a neighbourhood of x equals a smooth map). It follows that $D^{k-1}u \in L_{\text{loc}}^p(Q)$ for $1 \leq p < n/(n-1)$, so $D^{k-1} \in L_{\text{loc}}^p(\mathbb{R}^n)$ as Q is arbitrary. The conclusion follows either by iterating the above argument for $\mathbb{B} = D^{k-j}$, $j = 1, \dots, k$, or by standard theory for Sobolev Spaces [134, Sec. 1.1.11]. $\square$

The following Lemma is essentially due to CALDERÓN and ZYGMUND and covers the non-trivial statement that if a higher derivative of a Sobolev map u has L^p -derivatives at x , then the lower derivatives of u have more L^p -derivatives at x . More precisely, we show exchangeability of weak and L^p -derivatives

$$\nabla(D^{k-1}u) = \nabla^j(D^{k-j}u) \quad \mathcal{L}^n\text{-a.e.} \quad (5.10)$$

for $j = 2, \dots, k$, which is trivial if $u \in W_{\text{loc}}^{k,p}$, but less so if we only assume that $u \in W_{\text{loc}}^{k-1,p}$ and that the LHS of (5.10) is well defined. Here $\nabla^j u(x) = D^j P_x^j u(x)$.

Lemma 5.2.3. *Let $1 \leq p < n$, $u \in W_{\text{loc}}^{k-1,p}$, $x \in \mathbb{R}^n$ be such that $D^{k-1}u \in \mathfrak{t}^{1,p}(x)$. Then $D^{k-j}u \in \mathfrak{t}^{j,p}(x)$ for $j = 1 \dots k$.*

Note that due to Ornstein's Non-inequality, it is crucial that we do not assume that $u \in W_{\text{loc}}^{k,p}$, as we aim to apply this to $u \in \text{BV}_{\text{loc}}^{\mathbb{B}}$ (cp. Lemma 5.2.2).

Proof. We argue by induction on j , using the fact following from [50, Thm. 11.1], namely that if $f \in W_{\text{loc}}^{1,p}$ such that $Df \in t^{l,p}(x)$, then $f \in t^{l+1,p}$. The statement is true for $j = 1$. We apply the result mentioned above for $f = D^{k-j-1}u$ with $l = j = 1, \dots, k-1$ to get the conclusion. $\square$

The next Lemma is a consequence of the main result in [12] and follows from the first order case [95, Lem. 3.1] by replacing u with $D^{k-1}u$. As the result is crucial for Theorems 5.1.1, 5.1.4, 5.1.5, we include a complete proof.

Lemma 5.2.4. *Let $\mathbb{B}$ as in (5.3) be elliptic, $u \in BV_{\text{loc}}^{\mathbb{B}}$. Then for $\mathcal{L}^n$ -a.e. $x \in \mathbb{R}^n$, we have that $D^{k-1}u$ is L^p -differentiable at x , with (first) approximate gradient $\nabla^k u(x) := \nabla(D^{k-1}u)(x) \in V \odot^k \mathbb{R}^n$ such that*

$$\mathbb{B}^{ac}u(x) := \frac{d\mathbb{B}u}{d\mathcal{L}^n}(x) = A(\nabla^k u(x)), \quad (5.11)$$

where $1 \leq p < n/(n-1)$.

Proof. There is no loss of generality in assuming that $u \in BV^{\mathbb{B}}(\mathbb{R}^n)$, otherwise multiply u with a smooth cut-off function ρ that equals 1 in an arbitrarily large set. One then uses the Leibniz rule and Lemma 5.2.2 to show that $\mathbb{B}(\rho u) \in \mathcal{M}(\mathbb{R}^n, W)$. By Lemma 5.2.1 and a standard regularization argument, we have that $D^{k-1}u = D^{k-1}K^{\mathbb{B}} \star \mathbb{B}u$, where $D^{k-1}K^{\mathbb{B}}$ is $(1-n)$ -homogeneous. The L^p -differentiability statement follows from [12, Thm. 3.4].

It remains to prove (5.11). Let $x \in \mathbb{R}^n$ be a Lebesgue point of $D^{k-1}u$ and $\mathbb{B}^{ac}u$, and also a point of L^1 -differentiability of u . We also consider a sequence $(\eta_\varepsilon)_{\varepsilon>0}$ of standard mollifiers, i.e., $\eta_1 \in C_c^\infty(B_1(0))$ is radially symmetric and has integral equal to 1 and $\eta_\varepsilon(y) = \varepsilon^{-n}\eta_1(y/\varepsilon)$. Finally, we denote $u_\varepsilon := u \star \eta_\varepsilon$ and compute

$$\begin{aligned} D^k u_\varepsilon(x) &= \int_{B_\varepsilon(x)} D^{k-1}u(y) \otimes D_x \eta_\varepsilon(x-y) dy \\ &= - \int_{B_\varepsilon(x)} (P_x^1 D^{k-1}u)(y) \otimes D_y \eta_\varepsilon(y-x) dy \\ &\quad + \int_{B_\varepsilon(x)} (R_x^1 D^{k-1}u)(y) \otimes D_x \eta_\varepsilon(x-y) dy \\ &= \int_{B_\varepsilon(x)} \eta_\varepsilon(y-x) \nabla(D^{k-1}u)(x) dy + \\ &\quad + \int_{B_\varepsilon(x)} (R_x^1 D^{k-1}u)(y) \otimes D_x \eta_\varepsilon(x-y) dy \\ &= \nabla^k u(x) + \int_{B_\varepsilon(x)} (R_x^1 D^{k-1}u)(y) \otimes D_x \eta_\varepsilon(x-y) dy, \end{aligned}$$

where we used integration by parts to establish the third equality. Since

$$\|D_x \eta(x - \cdot)\|_\infty = \varepsilon^{-(n+1)} \|D\eta_1\|_\infty,$$

we have that $|D^k u_\varepsilon(x) - \nabla^k u(x)| \leq c(n, \eta_1) \varepsilon^{-1} (|R_x D^{k-1}u|)_{x,\varepsilon} = o(1)$ as $\varepsilon \downarrow 0$ since x is a point of L^1 -differentiability of $D^{k-1}u$. In particular, $D^k u_\varepsilon \rightarrow \nabla^k u$

$\mathcal{L}^n$ -a.e., so that $\mathbb{B}u_\varepsilon \rightarrow A(\nabla^k u)$ $\mathcal{L}^n$ -a.e. To conclude, we will also show that $\mathbb{B}u_\varepsilon \rightarrow \mathbb{B}^{ac}u$ $\mathcal{L}^n$ -a.e. We have that $\mathbb{B}u_\varepsilon = \mathbb{B}u \star \eta_\varepsilon$, so

$$\begin{aligned} \mathbb{B}u_\varepsilon(x) - \mathbb{B}^{ac}u(x) &= \mathbb{B}^{ac}u \star \eta_\varepsilon(x) - \mathbb{B}^{ac}u(x) + \mathbb{B}^s u \star \eta_\varepsilon(x) \\ &= \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y) (\mathbb{B}^{ac}u(y) - \mathbb{B}^{ac}u(x)) dy \\ &\quad + \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y) d\mathbb{B}^s u(y). \end{aligned}$$

Using the facts that $\|\eta_\varepsilon(x - \cdot)\|_\infty = \varepsilon^{-n} \|\eta_1\|_\infty$ and that x is a Lebesgue point of $\mathbb{B}^{ac}u$, we conclude that both integrals converge to zero as $\varepsilon \downarrow 0$. $\square$

Proof of sufficiency of ellipticity for Theorem 5.1.5. Assume that $\mathbb{B}$ is elliptic and let $u \in \text{BV}_{\text{loc}}^\mathbb{B}$. From Lemma 5.2.4, we have that for $\mathcal{L}^n$ -a.e. $x \in \mathbb{R}^n$, $D^{k-1}u \in \mathfrak{t}^{1,p}(x)$ for all $1 \leq p < n/(n-1)$. Apply [50, Thm. 11.1] inductively until (a) is proved. If $k \geq n$, apply [50, Thm. 11.1] once more to get (b). If $k > n$, apply [50, Thm. 11.2] to (b) once to get (c). $\square$

5.2.3. EC operators We give an analytic characterization of canceling operators:

Lemma 5.2.5. *Let $\mathbb{B}$ as in (5.3) be elliptic. Then $\mathbb{B}$ is canceling if and only if for $u \in \text{BV}_{\text{loc}}^\mathbb{B}$ and $w \in W$ such that*

$$\mathbb{B}u = \delta_0 w, \tag{5.12}$$

we necessarily have $w = 0$. Moreover, if $\mathbb{B}$ is elliptic and non-canceling, (5.12) has a solution $u_h \in C^\infty(\mathbb{R}^n \setminus \{0\}, V)$ such that $D^l u_h$ is $(k - n - l)$ -homogeneous for all $l \geq \max\{0, k - n + 1\}$.

Proof. Suppose that $\mathbb{B}$ is EC. By ellipticity, $\mathbb{B}$ has an exact annihilator (analogous to curl for $\mathbb{B} = D$), i.e., there exists a homogeneous differential operator $\mathcal{A}$ such that $\ker \mathcal{A}[\xi] = \text{im } \mathbb{B}[\xi]$ for all $\xi \neq 0$. An example would be the projection operator defined by

$$\mathcal{A}[\xi] := \det(\Delta_\mathbb{B}[\xi]) (\text{Id} - \mathbb{B}[\xi](\Delta_\mathbb{B}[\xi])^{-1} \mathbb{B}^*[\xi]),$$

where $\Delta_\mathbb{B} := \mathbb{B}^* \mathbb{B}$ (see [201, Rk. 4.1, Sec. 4.2]). Applying $\mathcal{A}$ to (5.12), we get that $\mathcal{A}(\delta_0 w) = 0$. We apply the Fourier transform to get that $\mathcal{A}[\xi]w = 0$ for each non-zero ξ . In particular,

$$w \in \bigcap_{\xi \in \mathbb{S}^{n-1}} \ker \mathcal{A}[\xi] = \bigcap_{\xi \in \mathbb{S}^{n-1}} \text{im } \mathbb{B}[\xi],$$

so that $w = 0$.

Conversely, suppose that $\mathbb{B}$ is elliptic, non-canceling so there exists a non-zero $w \in \text{im } \mathbb{B}[\xi]$ for any non-zero ξ . We define the map u_h via

$$u_h := K^\mathbb{B} w,$$

where, $K^{\mathbb{B}}$ is as in Lemma 5.2.1. Recall from the proof of [43, Lem. 2.1] that $K^{\mathbb{B}}$ is, in addition, a tempered distribution such that

$$\widehat{K^{\mathbb{B}}}(\xi) = \mathbb{B}^\dagger[\xi] \quad \text{for } \xi \neq 0,$$

where $\mathbb{B}^\dagger[\xi] = (\Delta_{\mathbb{B}}[\xi])^{-1} \mathbb{B}^*[\xi]$.

It is now clear that the smoothness and homogeneity properties of u_h follow from those of $K^{\mathbb{B}}$. The $BV^{\mathbb{B}}_{\text{loc}}$ -regularity follows from Lemma 5.2.2, as soon as we prove that $\mathbb{B}u_h = \delta_0 w$. This is so simply because

$$\widehat{\mathbb{B}u_h}(\xi) = \mathbb{B}[\xi] \widehat{u_h}(\xi) = \mathbb{B}[\xi] \mathbb{B}^\dagger[\xi] w = w,$$

where the last inequality follows from the definition of w and the elementary fact that $\mathbb{B}[\xi] \mathbb{B}^\dagger[\xi]$ is the orthogonal projection onto $\text{im } \mathbb{B}[\xi]$. Fourier inverting the previous equality gives (5.12). $\square$

In fact, while proving Lemma 5.2.5, we have proved the independently interesting fact that, for an elliptic operator $\mathbb{B}$ as in (5.3), we have

$$\bigcap_{\xi \in \mathbb{S}^{n-1}} \text{im } \mathbb{B}[\xi] = \{w \in W : \mathbb{B}u = \delta_0 w \text{ for some } u \in BV^{\mathbb{B}}_{\text{loc}}\}. \quad (5.13)$$

We will not make explicit use of this fact, but we will outline some consequences:

- (a) If $\mathbb{B}$ is determined ($\dim V = \dim W$), then $\mathbb{B}$ has a genuine fundamental solution E , by which we mean $\mathbb{B}E = \text{Id}_V \delta_0$ (cp. proof of Lemma 5.2.2).
- (b) If $\mathbb{B}$ is overdetermined ($\dim V < \dim W$, assuming that the latter is minimal), no such E exists. The kernel $K^{\mathbb{B}}$ from Lemma 5.2.1 is a significantly less flexible replacement.

This highlights a sharp difference between both the scalar and vectorial case (recall [74, 130]), as well as between the determined and overdetermined case. In particular, elliptic determined operators cannot be canceling, hence any inequality that holds for EC operators, must be proved without the use of a genuine fundamental solution. Moreover, the lack of fundamental solutions is implicitly necessary in the back-end of most linear L^1 -estimates. However, we so far lack a clear description of how the canceling condition can be seen at the level of the convolution kernel $K^{\mathbb{B}}$ from Lemma 5.2.1.

Proof of (a). We identify both W and V with $\mathbb{R}^N$, with orthonormal basis $\{e_i\}_{i=1}^N$. By (5.13), we can write u_i for a map that solves $\mathbb{B}u_i = \delta_0 e_i$. By writing E for the $N \times N$ matrix that with rows equal to u_i , we have that $\mathbb{B}E = \delta_0 \text{Id}_{\mathbb{R}^N}$. $\square$

The statement in (b) needs no proof as there can be no tensor-valued field E such that $\mathbb{B}E$ is valued in $\text{Lin}(V, V)$.

5.3. Proof of Theorem 5.1.3

We begin with a simple algebraic computation in Fourier Space. For u in $C_c^\infty(\mathbb{R}^n, V)$ and $\xi \in \mathbb{R}^n$, we have that

$$\begin{aligned}\mathbb{B}[\xi]\hat{u}(\xi) &= \widehat{\mathbb{B}u}(\xi) \\ \hat{u}(\xi) &= \mathbb{B}^\dagger[\xi]\widehat{\mathbb{B}u}(\xi) \\ \widehat{D^{k-n}u}(\xi) &= L[\xi]\widehat{\mathbb{B}u}(\xi) \\ D^{k-n}u &= \check{L} \star \mathbb{B}u,\end{aligned}$$

where $L[\xi] \in \text{Lin}(W, V \odot^{k-n} \mathbb{R}^n)$ is given by

$$L[\xi]w := \mathbb{B}^\dagger[\xi]w \otimes^{k-n} \xi$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$, $w \in W$. We have that $\check{L} \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L_{\text{loc}}^1(\mathbb{R}^n)$, valued in $\text{Lin}(W, V \odot^{n-k} \mathbb{R}^n)$, by the same reasoning as in [43, Lem. 2.1].

Essentially by [101, Thm. 7.1.20], we can write

$$\check{L} = H_0 + \log |\cdot| \int_{\mathbb{S}^{n-1}} L[\xi] d\mathcal{H}^{n-1}(\xi) = H_0 + \log |\cdot| \mathcal{L}, \quad (5.14)$$

where $H_0 \in C^\infty(\mathbb{R}^n \setminus \{0\}, \text{Lin}(W, V \odot^{n-k} \mathbb{R}^n))$ is zero-homogeneous. In particular, H_0 is essentially bounded in $\mathbb{R}^n$. We prove Formula (5.14) at the end of this section.

Proof of necessity. Let w fail condition (WC). Then $w \neq 0$, so there exists $u_h \in L_{\text{loc}}^1(\mathbb{R}^n, V)$ such that $\mathbb{B}u_h = \delta_0 w$ by Lemma 5.2.5. We then have by (5.14)

$$\|D^{k-n}u_h\|_{L^\infty} \geq \|H_0 w\|_{L^\infty} - \|\log |\cdot| \mathcal{L}w\|_{L^\infty},$$

which is clearly infinite (near 0) since $\mathcal{L}w \neq 0$ and H_0 is bounded. $\square$

We next show that condition (WC) is sufficient for the estimate (5.5). The ideas we use originate in [43, Sec. 2]. By (5.14), the triangle inequality, and Young's Convolution Inequality, we have that

$$\begin{aligned}\|D^{k-n}\|_{L^\infty} &\lesssim \|H_0 \star \mathbb{B}u\|_{L^\infty} + \|\log |\cdot| \star [\mathcal{L}\mathbb{B}u]\|_{L^\infty} \\ &\leq \|H_0\|_{L^\infty} \|\mathbb{B}u\|_{L^1} + \|\log |\cdot| \star [\mathcal{L}\mathbb{B}u]\|_{L^\infty},\end{aligned}$$

so it suffices to prove that

$$\|\log |\cdot| \star [\mathcal{L}\mathbb{B}u]\|_{L^\infty} \lesssim \|\mathbb{B}u\|_{L^1},$$

for $u \in C_c^\infty(\mathbb{R}^n, V)$. Equivalently, we want to show that for all $v \in V$, $\eta \in \mathbb{R}^n$ of unit length, we have that

$$\left| \int_{\mathbb{R}^n} \langle \log |y| v \otimes^{k-n} \eta, \mathcal{L}\mathbb{B}u(x-y) \rangle dy \right| \lesssim \|\mathbb{B}u\|_{L^1} \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n. \quad (5.15)$$

The proof of (5.15) will follow quite easily from the following Lemma:

Lemma 5.3.1. *Let $\mathbb{B}$ be elliptic and satisfying condition (WC). Then there exists an integer l such that for all $u \in C_c^\infty(\mathbb{R}^n, V)$ and all $\varphi \in C^\infty(\mathbb{R}^n \setminus \{0\}, \text{im } \mathcal{L}^*)$ such that $|\cdot|^j |D^j \varphi| \in L_{\text{loc}}^1(\mathbb{R}^n)$ for $j = 1, \dots, l$ we have that*

$$\left| \int_{\mathbb{R}^n} \langle \varphi, \mathbb{B}u \rangle dx \right| \lesssim \sum_{j=1}^l \int_{\mathbb{R}^n} |\mathbb{B}u| \cdot |\cdot|^j |D^j \varphi| dx.$$

Lemma 5.3.1 amounts to a minor, but crucial algebraic modification of [43, Lem. 2.2] (cp. [201, Prop. 8.9]; see also [201, Lem. 2.5]), ideas of which originate in [38, 198] (cp. [135, 42]).

Proof of Lemma 5.3.1. We denote by $\mathcal{A} = \sum_{|\alpha|=l} \partial^\alpha \mathcal{A}_\alpha$ an exact annihilator of $\mathbb{B}$, i.e., $\ker \mathcal{A}[\xi] = \text{im } \mathbb{B}[\xi]$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Such an operator exists by [201, Prop. 4.2].

Since $(\xi^\alpha)_{|\alpha|=l}$ is a basis for homogeneous polynomials of degree l , we have that $w_0 \in \ker \mathcal{A}[\xi]$ for all $\xi \neq 0$ is equivalent with w_0 lying in the kernel of the map $T: w \mapsto (\mathcal{A}_\alpha w)_{|\alpha|=l}$. By condition (WC), we have that $\text{im } \mathcal{L}^* \cap \bigcap_{\xi \in \mathbb{S}^{n-1}} \ker \mathcal{A}[\xi] = 0$, hence the restriction of T to $\text{im } \mathcal{L}^*$ is injective. Equivalently, this restriction is left-invertible, so there exist linear maps $K_\alpha \in \text{Lin}(W, \text{im } \mathcal{L}^*)$ such that

$$\sum_{|\alpha|=l} K_\alpha \mathcal{A}_\alpha \upharpoonright_{\text{im } \mathcal{L}^*} = \text{Id}_{\text{im } \mathcal{L}^*}.$$

The remainder of the proof follows exactly as in [43, pp.1426]. We reproduce the argument for the convenience of the reader.

Define the matrix-valued field

$$P(x) := \sum_{|\alpha|=l} \frac{x^\alpha}{\alpha!} K_\alpha^*,$$

which is essentially a right-inverse (integral) of $\mathcal{A}^*$, as

$$\mathcal{A}^* P = \sum_{|\alpha|=l} \mathcal{A}_\alpha^* \partial^\alpha P = \sum_{|\alpha|=l} \mathcal{A}_\alpha^* K_\alpha^* = \text{Id}_{\text{im } \mathcal{L}^*}.$$

By definition of φ , integration by parts, and $\mathcal{A} \circ \mathbb{B} \equiv 0$, we have that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \langle \varphi, \mathbb{B}u \rangle dx \right| &= \left| \int_{\mathbb{R}^n} \langle [\mathcal{A}^* P] \varphi, \mathbb{B}u \rangle dx \right| \\ &= \left| \int_{\mathbb{R}^n} \langle [\mathcal{A}^* P] \varphi - \mathcal{A}^* [P \varphi], \mathbb{B}u \rangle dx \right|. \end{aligned}$$

The presence of the “associator” ensures the elimination of the essentially inhomogeneous zero-th order term φ :

$$[\mathcal{A}^* P] \varphi - \mathcal{A}^* [P \varphi] = [\mathcal{A}^* P] \varphi - [\mathcal{A}^* P] \varphi - \sum_{j=1}^l B_j (D^j \varphi, D^{l-j} P),$$

where B_j are bilinear pairings on finite dimensional spaces that depend on $\mathcal{A}$ only. By l -homogeneity of P , the conclusion follows. $\square$

Proof of sufficiency. To prove (5.15), we need only apply Lemma 5.3.1 to the map

$$\varphi := \log |\cdot| \mathcal{L}^*(v \otimes^{k-n} \eta),$$

which has suitably homogeneous derivatives, to get that

$$\begin{aligned} \text{LHS}(5.15) &\lesssim \sum_{j=1}^l \int_{\mathbb{R}^n} |\mathbb{B}u(x-y)| |y|^j |D^j \log |y|| \|\mathcal{L}^*\| \, dy \\ &\lesssim \|\mathcal{L}^*\| \|\mathbb{B}u\|_{L^1}. \end{aligned}$$

The proof is complete. $\square$

Proof of Formula (5.14). It is shown in [101, Thm. 3.2.4, 7.1.18] that a $(-n)$ -homogeneous map $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ can be extended to a tempered distribution $f^\cdot$ satisfying a weakened homogeneity property [101, (3.2.24)']. The (inverse) Fourier Transform of $f^\cdot$ then equals $h + \log |\cdot| \int_{\mathbb{S}^{n-1}} f \, d\mathcal{H}^{n-1}$ by [101, (7.1.19)], where h is a 0-homogeneous map in $C^\infty(\mathbb{R}^n \setminus \{0\})$. We apply this to L component wise. $\square$

5.4. Remarks on and consequences of Theorem 5.1.3

5.4.1. Embeddings of $W^{\mathbb{B},1}$ and $BV^{\mathbb{B}}$ Throughout this Section, we restrict our attention to operators $\mathbb{B}$ on $\mathbb{R}^n$ of order $k = n$ and compare our result with the embeddings $W^{n,1}(\mathbb{R}^n) \hookrightarrow C_0$ (the space of continuous functions that vanish at infinity) and $BV^n(\mathbb{R}^n) \hookrightarrow L^\infty$. To this end, we consider the space

$$W^{\mathbb{B},1}(\mathbb{R}^n) := \{u \in L^1(\mathbb{R}^n, V) : \mathbb{B}u \in L^1(\mathbb{R}^n, W)\},$$

which is a Banach space when endowed with the obvious norm $\|u\|_{W^{\mathbb{B},1}} := \|u\|_{L^1} + \|\mathbb{B}u\|_{L^1}$ (see also [45, 94]). We have the following:

Theorem 5.4.1. *Let $\mathbb{B}$ be as in (5.3) be elliptic of order $k = n \geq 1$. The following are equivalent:*

- (a) $\mathbb{B}$ satisfies (WC).
- (b) $W^{\mathbb{B},1}(\mathbb{R}^n) \hookrightarrow C_0(\mathbb{R}^n, V)$.
- (c) $BV^{\mathbb{B}}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n, V)$.

Proof. Necessity of (WC) follows directly from Theorem 5.1.3. Assume now that $\mathbb{B}$ satisfies (WC), so Theorem 5.1.3 implies that

$$\|u\|_{L^\infty} \leq c \|\mathbb{B}u\|_{L^1} \tag{5.16}$$

for all $u \in C_c^\infty(\mathbb{R}^n, V)$. Since test functions are norm-dense in $W^{\mathbb{B},1}(\mathbb{R}^n)$ [45, Thm. 2.8], we have that $W^{\mathbb{B},1}(\mathbb{R}^n)$ embeds in the uniform closure of C_c^∞ , which is C_0 , so (b) is proved.

In the case of $BV^{\mathbb{B}}$, test functions are not norm-, but strictly-dense (more precisely, $\mathbb{B}$ -strictly dense [45, Thm. 2.8]) and it is easy to see that addition is not continuous in the strict topology on the space of bounded measures. In particular,

we cannot prove that $BV^{\mathbb{B}}$ embeds in C_0 (this is clearly visible if one looks at the indicator function of $(0, 1)$ in dimensions $n = 1$). Instead, one can prove (c): Let $u_j \in C_c^\infty(\mathbb{R}^n)$ be such that $u_j \rightarrow u$ $\mathcal{L}^n$ -a.e. and $|\mathbb{B}u_j|(\mathbb{R}^n) \rightarrow |\mathbb{B}u|(\mathbb{R}^n)$ (mollifications of u satisfy this). It then follows that

$$\|u\|_{L^\infty} \leq \liminf_{j \rightarrow \infty} \|u_j\|_{L^\infty} \leq c \liminf_{j \rightarrow \infty} |\mathbb{A}u_j|(\mathbb{R}^n) = c|\mathbb{B}u|(\mathbb{R}^n).$$

The proof of the equivalence is complete. $\square$

We compare our result with [152, Thm. 1.2], which states that if $n \geq 2$ and $u \in W^{1,1}(\mathbb{R}^n)$ and $\partial_1 \dots \partial_n u \in \mathcal{M}(\mathbb{R}^n)$, then u has a continuous representative. This strengthens an earlier result, attributable to TARTAR [152, Thm. 1.1], which states that if $n \geq 2$, $D^n u \in \mathcal{M}$ for $u \in W^{n-1,1}(\mathbb{R}^n)$, then u has a continuous representative. One can speculate that, in our case, if $n \geq 2$, $u \in W^{n-1,1}(\mathbb{R}^n)$ and $\mathbb{B}u \in \mathcal{M}$ for elliptic $\mathbb{B}$ of order n that satisfies (WC), then the embedding of $BV^{\mathbb{B}}$ in L^∞ can be improved to continuity. This is not true, as we illustrate with an example below. In particular, it must be that the results in [152] rely on a particular feature of the operator $\partial_1 \dots \partial_n$ on $\mathbb{R}^n$ (possibly, the availability of the Fundamental Theorem of Calculus in that case), rather than on a general self-improvement of the embedding of $BV^{\mathbb{B}}$ in L^∞ .

Example 5.4.2 ([158, pp.18-19]). Let $\mathbb{B} = \Delta \circ (\operatorname{div}, \operatorname{curl})$ on $\mathbb{R}^3$. Then $\mathbb{B}$ is elliptic (hence satisfies (WC)), non-canceling, and

$$\mathbb{B}u = \delta_0 e_1, \quad \text{where} \quad u(x) := \frac{x}{|x|}.$$

In particular, $u \in BV_{\operatorname{loc}}^{\mathbb{B}}(\mathbb{R}^3)$ (so that $u \in W_{\operatorname{loc}}^{2,1}(\mathbb{R}^3, \mathbb{R}^3)$).

Proof. All claims follow by direct computation, so that we can afford to present a streamlined proof. Writing $\mathbb{A} := (\operatorname{div}, \operatorname{curl})$, we note that $\mathbb{A}^* \circ \mathbb{B} = \Delta^2$, which has fundamental solution (proportional to) $|\cdot|$ in dimension 3. Ellipticity of $\mathbb{B}$ follows as composition of injective maps is injective. Non-cancellation follows (with intersection equal to $\mathbb{R}e_1$) by direct computation; alternatively, one can compute that $\mathbb{A}(x/|x|^3) = \delta_0 e_1$. We next write

$$u(x) = \mathbb{A}^* \circ \Delta^{-2}(\delta_0 e_1) = \mathbb{A}^*(|x|e_1) = D|x| = \frac{x}{|x|}.$$

The Sobolev regularity of u follows from Lemma 5.2.2. $\square$

In our setup ($\mathbb{B}$ elliptic of order n on $\mathbb{R}^n$), the ideas of this chapter are easily used to show that, in general, $BV_{\operatorname{loc}}^{\mathbb{B}}(\mathbb{R}^n) \subset C(\mathbb{R}^n, V)$ implies that $\mathbb{B}$ is canceling. We do not know whether the converse is true, but speculate that this is the case.

5.4.2. Characterization of condition (WC) When proving that condition (WC) is necessary for the embedding (5.5), we in fact showed that, if (WC) fails for w , there exists $u \in L_{\operatorname{loc}}^1(\mathbb{R}^n, V)$ such that $\mathbb{B}u = \delta_0 w$, but $D^{k-n}u$ is unbounded near zero. This property is actually equivalent with the failure of condition (WC) for elliptic operators:

Proposition 5.4.3. *Let $\mathbb{B}$ as in (5.3) be elliptic, $k \geq n$. Then $\mathbb{B}$ satisfies condition (WC) if and only if for $u \in \mathcal{S}'(\mathbb{R}^n, V)$, $w \in W \setminus \{0\}$ such that $\mathbb{B}u = \delta_0 w$, we have that $D^{k-n}u \in L_{\text{loc}}^\infty$.*

Proof. We need only prove one implication. Suppose that condition (WC) holds and let $0 \neq w \in W$ (if any exist) be such that $\mathbb{B}u = \delta_0 w$. We consider the map

$$f(\xi) := \mathbb{B}^\dagger[\xi]w \otimes^{k-n} \xi,$$

so f is smooth (rational) away from zero and $(-n)$ -homogeneous, hence a distribution in $\mathbb{R}^n \setminus \{0\}$. By [101, Thm. 3.2.4] and condition (WC), we have that f defines a $(-n)$ -homogeneous distribution in $\mathbb{R}^n$. By [101, Thm. 7.1.18], we have that $f \in \mathcal{S}'$. By [101, Thm. 7.1.16], we have that $\check{f}$ is 0-homogeneous and smooth away from zero. It is easy to see that then $\check{f} = D^{k-n}v$, where $v \in \mathcal{S}'$ is smooth away from zero and

$$\widehat{v}(\xi) = \mathbb{B}^\dagger[\xi]w,$$

so that $\mathbb{B}v = \delta_0 w$. So $u - v$ is $\mathbb{B}$ -free, hence, by ellipticity, u differs from v by a smooth map (analytic, even). Since $D^{k-n}v = \check{f}$ is bounded, it follows that $D^{k-n}u$ is locally bounded. $\square$

It seems relevant to compare the result of Proposition 5.4.3 to the corresponding result for VAN SCHAFTINGEN's embeddings (5.4), which was essentially already covered in Section 5.2.3. We give a another, very streamlined, proof.

Proposition 5.4.4 ([201, Prop. 5.5]). *Let $\mathbb{B}$ as in (5.3) be elliptic, $1 \leq j \leq \min\{k, n-1\}$. If there exists $u \in \mathcal{S}'(\mathbb{R}^n, V)$, $w \in W \setminus \{0\}$ such that $\mathbb{B}u = \delta_0 w$, then $D^{k-j}u \notin L_{\text{loc}}^{n/(n-j)}$.*

Proof. As above, we define $f(\xi) := \mathbb{B}^\dagger[\xi]w \otimes^{k-j} \xi$, which is $(-j)$ -homogeneous, so we can apply [101, Thm. 3.2.3, 7.1.16, 7.1.18]² to show that $\check{f} \in \mathcal{S}'$ is $(j-n)$ -homogeneous, so either $\check{f} \equiv 0$, or $\check{f} \notin L_{\text{loc}}^{n/(n-j)}$. As above, $\check{f} = D^{k-j}v$, where $u - v$ is $\mathbb{B}$ -free, hence a smooth map. Also $f \equiv 0$ implies $w = 0$, so the conclusion follows. $\square$

Although for $0 \leq j \leq n \leq k$ the inequalities with $u \in C_c^\infty(\mathbb{R}^n, V)$

$$\|D^{k-j}u\|_{L^{n/(n-j)}} \lesssim \|\mathbb{B}u\|_{L^1} \quad (5.17)$$

rely on different conditions for $1 \leq j \leq n-1$ and $j = n$, Propositions 5.4.3 and 5.4.4 reveal a phenomenological similarity: if the “worst” measures from the point of view of convolution with $D^{k-j}(\mathcal{F}^{-1}\mathbb{B}^\dagger[\cdot])$ do not lie in

$$\{\mathbb{B}u \text{ is a bounded measure}\},$$

then the embeddings hold. This is in sharp contrast with the case $j = 0$, when (5.17) holds only for trivial $\mathbb{B}$ by Ornstein's Non-inequality.

²It is important to mention that [101, Thm. 3.2.4], which we used to prove Proposition 5.4.3, is essentially the degenerate version of [101, Thm. 3.2.3], which we use here. In a nutshell, this is the basic difference between the canceling condition and condition (WC).

Also, if $\mathbb{B}$ is elliptic, satisfies condition (WC), but is not canceling (see Section 5.4.3 for examples), then the embedding (5.17) holds *only* for $j = n$. There exists a single $BV_{\text{loc}}^{\mathbb{B}}$ -map u such that $D^{k-j}u \notin L_{\text{loc}}^{n/(n-j)}$ for $j = 0, \dots, n-1$, but $D^{k-n}u \in L^\infty$.

Before moving on to examples, we use Proposition 5.4.3 to write out Theorem 5.1.3 in an invariant form, independent of the assumption made in the Introduction that $\mathbb{R}^n$, V , W have Euclidean structure:

Remark 5.4.5. *Assume that $\mathbb{R}^n$, V , W are (viewed as) normed finite dimensional vector spaces, i.e., no inner product is defined a priori. We claim that, for an elliptic operator $\mathbb{B}$ of order $k \geq n$, we have that*

$$\|D^{k-n}u\|_{L^\infty(\mathbb{R}^n, V \odot^{k-n} \mathbb{R}^n)} \lesssim \|\mathbb{B}u\|_{L^1(\mathbb{R}^n, W)}$$

if and only if

$$\mathbb{B}u = \delta_0 w \implies D^{k-n}u \in L_{\text{loc}}^\infty(\mathbb{R}^n, V \odot^{k-1} \mathbb{R}^n). \quad (5.18)$$

for $u \in \mathcal{S}'(\mathbb{R}^n, V)$, $w \in W \setminus \{0\}$.

Proof. We choose bases of $\mathbb{R}^n$, V , W . Defining inner products respectively such that these bases are orthonormal, we obtain new, equivalent norms and an equivalent Lebesgue measure on $\mathbb{R}^n$.

By the argument used to prove Proposition 5.4.3, we have that (5.18) is represented in the new coordinates as condition (WC). By the proof of Theorem 5.1.3 in Section 5.3, we have that the inequality holds, with respect to the new norms and Lebesgue measure. By finite dimensionality of $\mathbb{R}^n$, V , W , the inequality follows for the initial norms and Lebesgue measure. $\square$

5.4.3. Examples In this section, we will show by way of examples that the condition (WC) is strictly weaker than the canceling condition for elliptic operators. Throughout, with a slight abuse of notation, $\{e_j\}_j$ will denote the standard Euclidean basis in any finite dimensional space we may consider.

As noted in the Introduction, condition (WC) is automatically satisfied in odd dimensions for elliptic operators. We have:

Example 5.4.6. *Let $n = 2d + 1$. Then the operator $\mathbb{B} := \Delta^d \circ (\text{div}, \text{curl})$ on $\mathbb{R}^n$ from $\mathbb{R}^n$ to $\mathbb{R} \times \mathbb{R}^{n \times n}_{\text{asym}}$ is elliptic and non-canceling.*

Proof. We write $\mathbb{A} := (\text{div}, \text{curl})$. An elementary computation shows that, for non-zero ξ , $\text{curl}[\xi]v = 0$ implies that $v = \alpha\xi$ for some $\alpha \in \mathbb{R}$. If also $0 = \text{div}[\xi]v = \alpha|\xi|^2$, then $v = 0$, so $\mathbb{A}$ is elliptic. By set theory, so is $\mathbb{B}$. To see that $\mathbb{B}$ is non-canceling, note from the previous calculation that $\mathbb{B}[\xi]\xi = |\xi|^2 e_1$. $\square$

It may be that (5.5) holds for all elliptic operators. This is not the case by:

Example 5.4.7. *Let $n = 2d$. Then the operator $\mathbb{B} := \Delta^d$ on $\mathbb{R}^n$ from $\mathbb{R}$ to $\mathbb{R}$ is elliptic and fails condition (WC).*

Proof. Ellipticity is obvious. Failure of condition (WC) is equivalent to

$$\int_{\mathbb{S}^{n-1}} \frac{1}{|\xi|^n} d\mathcal{H}^{n-1}(\xi) \neq 0,$$

which is clearly true. $\square$

On the other hand, it may be that (WC) is not weaker than the canceling condition in even dimensions. This is also not the case by:

Example 5.4.8. *The operator*

$$\mathbb{B}_1 u := \begin{pmatrix} (\partial_1^2 - \partial_2^2)u_1 + 2\partial_1\partial_2 u_2 \\ -2\partial_1\partial_2 u_1 + (\partial_1^2 - \partial_2^2)u_2 \end{pmatrix}, \quad (5.19)$$

on $\mathbb{R}^2$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is elliptic, non-canceling, and satisfies condition (WC).

Proof. Ellipticity is in this case, equivalent to invertibility for $\xi \neq 0$ of the matrix

$$\mathbb{B}_1[\xi] = \begin{pmatrix} \xi_1^2 - \xi_2^2 & 2\xi_1\xi_2 \\ -2\xi_1\xi_2 & \xi_1^2 - \xi_2^2 \end{pmatrix},$$

which has determinant equal to $|\xi|^4$. Non-cancellation follows from ellipticity as $\text{im } \mathbb{B}_1[\xi] = \mathbb{R}^2$ for $\xi \neq 0$. To check condition (WC), we have that

$$\mathbb{B}_1^\dagger[\xi] = \mathbb{B}_1^{-1}[\xi] = \frac{1}{|\xi|^4} \begin{pmatrix} \xi_1^2 - \xi_2^2 & -2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_1^2 - \xi_2^2 \end{pmatrix},$$

so that one computes

$$\int_{\mathbb{S}^1} \xi_1^2 - \xi_2^2 \, d\mathcal{H}^1(\xi) = \int_0^{2\pi} \cos(2\theta) \, d\theta = 0, \quad \int_{\mathbb{S}^1} 2\xi_1\xi_2 \, d\mathcal{H}^1(\xi) = \int_0^{2\pi} \sin(2\theta) \, d\theta = 0$$

to conclude the proof. $\square$

So far, above or in [43], we have only displayed examples of $\mathbb{B}$ for which the inequality (5.5) holds which are either canceling or for which $\mathcal{L} \equiv 0$. The following example shows that condition (WC) cannot be simplified easily, as the interaction between $\mathcal{L}$ and the intersection of images can occur in a non-trivial way. We augment the previous example as follows:

Example 5.4.9. *The operator*

$$\mathbb{B}_2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} := \begin{pmatrix} \mathbb{B}_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \partial_1^2 u_3 \\ \sqrt{2}\partial_1\partial_2 u_3 \\ \partial_2^2 u_3 \end{pmatrix}$$

on $\mathbb{R}^2$ from $\mathbb{R}^3$ to $\mathbb{R}^5$ is elliptic, non-canceling, and satisfies condition (WC) with $\mathcal{L} \neq 0$. Here $\mathbb{B}_1$ is as in (5.19).

Proof. Consider, more generally, an operator

$$\mathbb{B}[\xi] := \begin{pmatrix} \mathbb{A}_1[\xi] & 0 \\ 0 & \mathbb{A}_2[\xi] \end{pmatrix},$$

where $\mathbb{A}_1$ is elliptic and square on $\mathbb{R}^n$ from V_1 to V_1 and $\mathbb{A}_2$ is elliptic and canceling on $\mathbb{R}^n$ from V_2 to W_2 . By matrix multiplication

$$\mathbb{B}^*[\xi]\mathbb{B}[\xi] = \begin{pmatrix} \mathbb{A}_1^*[\xi]\mathbb{A}_1[\xi] & 0 \\ 0 & \mathbb{A}_2^*[\xi]\mathbb{A}_2[\xi] \end{pmatrix}, \quad (5.20)$$

so that $\det(\mathbb{B}^*[\xi]\mathbb{B}[\xi]) = \det(\mathbb{A}_1^*[\xi]\mathbb{A}_1[\xi]) \det(\mathbb{A}_2^*[\xi]\mathbb{A}_2[\xi]) \neq 0$ for $\xi \neq 0$. In particular, $\mathbb{B}$ is elliptic. We also have that

$$\mathbb{B}[\xi] \begin{pmatrix} \mathbb{A}^{-1}[\xi]v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$$

for $0 \neq \xi \in \mathbb{R}^n$, $0 \neq v_1 \in V_1$, so $\mathbb{B}$ is non-canceling.

For later purposes we also compute $\bigcap_{\xi \in \mathbb{S}^{n-1}} \text{im } \mathbb{B}[\xi]$, which, by the above computation is contained in $V_1 \times \{0\}$. It is easy to see that equality holds: let

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \bigcap_{\xi \in \mathbb{S}^{n-1}} \text{im } \mathbb{B}[\xi],$$

so that, again by matrix multiplication, $v_2 \in \bigcap_{\xi \in \mathbb{S}^{n-1}} \text{im } \mathbb{A}_2[\xi] = \{0\}$.

We now choose $\mathbb{A}_1 := \mathbb{B}_1$ and

$$\mathbb{A}_2 := \begin{pmatrix} \partial_1^2 \\ \sqrt{2}\partial_1\partial_2 \\ \partial_2^2 \end{pmatrix}$$

on $\mathbb{R}^2$ from $\mathbb{R}$ to $\mathbb{R}^3$. The operator $\mathbb{B}_2$ thus defined is elliptic and non-canceling.

We next check that condition (WC) holds, to which end we record that

$$J := \bigcap_{\xi \in \mathbb{S}^{n-1}} \text{im } \mathbb{B}[\xi] = \mathbb{R}^2 \times \{0\}$$

from the above considerations. By (5.20), we have that $\mathbb{B}^*[\xi]\mathbb{B}[\xi] = |\xi|^4 \text{Id}$, so that $\mathbb{B}^\dagger[\xi] = |\xi|^{-4}\mathbb{B}^*[\xi]$. To check that $\mathcal{L}(J) = \{0\}$ is thus equivalent to

$$\int_{\mathbb{S}^1} \begin{pmatrix} \xi_1^2 - \xi_2^2 & -2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_1^2 - \xi_2^2 \end{pmatrix} d\mathcal{H}^1(\xi) = 0,$$

which we know to be true from Example 5.4.8.

To check that $\mathbb{R}^{3 \times 5} \ni \mathcal{L} \neq 0$, we look at

$$\mathcal{L}_{33} = \int_{\mathbb{S}^1} \frac{\xi_1^2}{|\xi|^4} d\mathcal{H}^1(\xi) > 0.$$

The proof is complete. □

5.5. Proof of Theorem 5.1.1

For clarity of exposition, we present the proofs for the order $k = 1$ case separately. We begin by showing sufficiency of EC for the Lorentz-differentiability. The outline of the following Lemma is that one can use the Sobolev-type embedding [201, Thm. 8.5] (i.e., (5.2) for Lorentz spaces) to boost the main result [12, Thm. 3.4], which gives L^1 -differentiability $\mathcal{L}^n$ -a.e. of maps in $BV^{\mathbb{B}}$ for elliptic $\mathbb{B}$.

Lemma 5.5.1. *Let $n > 1$, $\mathbb{B}$ as in (5.1) be EC, and $1 < q < \infty$. Then all maps in $BV_{\text{loc}}^{\mathbb{B}}$ are $L^{n/(n-1),q}$ -differentiable $\mathcal{L}^n$ -a.e.*

Proof. We begin by deriving the estimate

$$\|v\|_{L^{n/(n-1),q}(B_r(x))} \lesssim |\mathbb{B}v| \left(\overline{B_{2r}(x)} \right) + r^{-1} \|v\|_{L^1(B_{2r}(x))} \quad (5.21)$$

for all $v \in \text{BV}_{\text{loc}}^{\mathbb{B}}$. Consider a sequence v_j of mollifications of v (which converge $\mathbb{B}$ -strictly to v in $\text{BV}_{\text{loc}}^{\mathbb{B}}$ [45, Thm. 2.8]) and a positive cut-off function $\rho \in C_c^\infty(B_{2r}(x))$ such that $\rho = 1$ in $B_r(x)$ and $|\nabla \rho| \lesssim r^{-1}$. By Fatou's Lemma for Lorentz spaces and [201, Thm. 8.5] we have:

$$\begin{aligned} \text{LHS} &\leq \liminf_j \|\rho v_j\|_{L^{n/(n-1),q}(B_{2r}(x))} \\ &\lesssim \liminf_j \|\mathbb{B}(\rho v_j)\|_{L^1(B_{2r}(x))} \\ &\lesssim \liminf_j \left(\|\rho \mathbb{B}v_j\|_{L^1(B_{2r}(x))} + r^{-1} \|v_j\|_{L^1(B_{2r}(x))} \right) \\ &\leq \liminf_j \left(\|\mathbb{B}v_j\|_{L^1(B_{2r}(x))} + r^{-1} \|v_j\|_{L^1(B_{2r}(x))} \right) \\ &= \text{RHS}. \end{aligned}$$

We then fix $u \in \text{BV}_{\text{loc}}^{\mathbb{B}}$ and apply (5.21) to $v := R_x^1 u \in \text{BV}_{\text{loc}}^{\mathbb{B}}$. Recall that $R_x^1 u(y) = u(y) - u(x) - \nabla u(x)(y - x)$, where x is chosen as a Lebesgue point of $\mathbb{B}u$ and a point of L^1 -differentiability of u , with approximate gradient $\nabla u(x)$ (such points exist $\mathcal{L}^n$ -a.e. by Lemma 5.2.4 for $k = 1$). We obtain

$$\|R_x^1 u\|_{L^{n/(n-1),q}(B_r(x))} \lesssim |R_x^1 u| \left(\overline{B_{2r}(x)} \right) + r^{-1} \|R_x^1 u\|_{L^1(B_{2r}(x))}.$$

The second term equals $o(r^n)$ as $r \downarrow 0$ by L^1 -differentiability of u at x . The first term equals $|\mathbb{B}u - A(\nabla u(x))|(B_{2r}(x)) = o(r^n)$ as $r \downarrow 0$ by Lebesgue differentiation for Radon measures and (5.11). The proof is complete. $\square$

Lemma 5.5.2. *Let $\mathbb{B}$ be as in (5.1), $1 < p < \infty$, $1 \leq q \leq \infty$. Suppose that all maps in $\text{BV}_{\text{loc}}^{\mathbb{B}}$ are $L^{p,q}$ -differentiable $\mathcal{L}^n$ -a.e. Then $\mathbb{B}$ is elliptic.*

Proof. By making p smaller if necessary, we can assume that maps in $\text{BV}_{\text{loc}}^{\mathbb{B}}$ are in fact L^p -differentiable, with $p > 1$.

We assume that $\mathbb{B}$ is not elliptic, so there exist non-zero $\xi \in \mathbb{R}^n$, $v \in V$ such that $\mathbb{B}[\xi]v = 0$. For $f \in L_{\text{loc}}^1(\mathbb{R})$ and $u(x) := f(x \cdot \xi)v$ for $x \in \mathbb{R}^n$, a simple computation using (5.1) shows that $\mathbb{B}u = 0$ in $\mathcal{D}'$. We let $g \in L_{\text{loc}}^1(\mathbb{R})$ be a *positive* map that is not L^p -integrable in any neighbourhood of 0, e.g., $g(t) = |t|^{-1/p}$. We let $\{x_l\}_{l=1}^\infty$ be dense in $\mathbb{R}^n$ and define

$$u(x) := \sum_{l=1}^\infty 2^{-l} g((x - x_l) \cdot \xi) v, \quad (5.22)$$

so that we have $\mathbb{B}u = 0$. There is no loss of generality in assuming that $|v| = 1$ and $\xi = e_n$. For a cube Q with sides parallel to the coordinate axes, we have that

$$\|u\|_{L^1(Q)} \leq \sum_{l=1}^\infty 2^{-l} \int_{2Q(x_l)} g((x - x_l) \cdot e_n) dx$$

$$\begin{aligned}
&\leq \sum_{l=1}^{\infty} 2^{-l} \int_{2Q(0)} g(x \cdot e_n) \, dx \\
&= (2\ell(Q))^{n-1} \sum_{l=1}^{\infty} 2^{-l} \int_{-\ell(Q)}^{\ell(Q)} g(t) \, dt \\
&= (2\ell(Q))^{n-1} \int_{-\ell(Q)}^{\ell(Q)} g(t) \, dt,
\end{aligned}$$

where $\ell(Q)$ denotes the side length of Q and $2Q(y)$ denotes the cube centred at y of side length $2\ell(Q)$. In particular, $u \in BV_{\text{loc}}^{\mathbb{B}}$. On the other hand, in a neighbourhood $Q_r(x_k)$ we have, by positivity of g ,

$$\begin{aligned}
\|u\|_{L^p(Q_r(x_k))}^p &= \int_{Q_r(x_k)} \left| \sum_{j=1}^{\infty} g((x - x_j) \cdot e_n) \right|^p \, dx \\
&\geq \int_{Q_r(x_k)} g((x - x_k) \cdot e_n)^p \, dx \\
&= (2r)^{n-1} \int_{-r}^r g(t)^p \, dt,
\end{aligned}$$

so u is not p -integrable in any open set, hence cannot be L^p -differentiable. This contradiction completes the proof. $\square$

To complete the proof of Theorem 5.1.1, it remains to see that for elliptic first order operators $\mathbb{B}$, cancellation is necessary for $L^{n/(n-1),q}$ -differentiability. This will require the closer look at the canceling condition that we took in Lemma 5.2.5.

Lemma 5.5.3. *Let $\mathbb{B}$ as in (5.1) be elliptic, $1 \leq q < \infty$ and suppose that maps in $BV_{\text{loc}}^{\mathbb{B}}$ are $L^{n/(n-1),q}$ -differentiable $\mathcal{L}^n$ -a.e. Then $\mathbb{B}$ is canceling.*

Proof. We assume that $\mathbb{B}$ is not canceling, and let $u_h \in BV_{\text{loc}}^{\mathbb{B}}$ be as in the proof of Lemma 5.2.5, so $u_h \notin L^{n/(n-1),q}(B_r(0))$ for any $r > 0$.

We will formulate a Baire category argument to prove existence of a $BV^{\mathbb{B}}$ -map that is not in $L^{n/(n-1),q}$ in any open set, which clearly suffices to prove the claim. For a ball $B \subset \mathbb{R}^n$, let $\mathfrak{X} := BV^{\mathbb{B}}(B)$, which is a complete metric space under the metric induced by the obvious norm, let $\{x_l\}_{l=1}^{\infty}$ be a dense subset and consider the subsets

$$X_{l,m,L} := \{u \in \mathfrak{X} : \|u\|_{L^{n/(n-1),q}(B(x_l, m^{-1}) \cap B)} \leq L\}$$

for positive integers l, m, L . We also make the convention

$$X_{l,m,\infty} := \{u \in \mathfrak{X} : \|u\|_{L^{n/(n-1),q}(B(x_l, m^{-1}) \cap B)} < \infty\},$$

which is a vector subspace of $\mathfrak{X}$.

It is then easy to check that the sets $X_{l,m,L}$ are nowhere dense. Let $u_i \in X_{l,m,L}$ be such that $u_i \rightarrow u$ in $\mathfrak{X}$. Then u_i converges pointwisely to u on a subsequence which we do not relabel. By the Fatou property in Lorentz spaces we have that

$$\|u\|_{L^{n/(n-1),q}(B(x_l, m^{-1}) \cap B)} \leq \liminf_{i \rightarrow \infty} \|u_i\|_{L^{n/(n-1),q}(B(x_l, m^{-1}) \cap B)} \leq L,$$

so $X_{l,m,L}$ is closed. Since $X_{l,m,L} \subset X_{l,m,\infty}$, which is a proper subspace of $\mathfrak{X}$, as $u_h(\cdot - x_l) \notin L^{n/(n-1),q}(B(x_l, m^{-1}) \cap B)$, we have that $X_{l,m,L}$ have empty interior.

By the Baire's Category Theorem, $BV^{\mathbb{B}}(B)$, cannot be written as the (countable) union of $X_{l,m,L}$, which are closed and have empty interior (nowhere dense). In particular, there exists a map $u_0 \in BV^{\mathbb{B}}(B)$, which lies in no $X_{l,m,L}$, hence in no space $X_{l,m,\infty}$. It follows that the $L^{n/(n-1),q}$ -norm of u_0 in any ball contained in B is infinite, hence u_0 cannot be $L^{n/(n-1),q}$ -differentiable at any point in B . The proof is complete. $\square$

5.6. Proof of Theorem 5.1.4

We extend the ideas from Section 5.5. We begin with sufficiency of the algebraic conditions for the claims Theorem 5.1.4. We make the convention that $n/0 = \infty$.

Lemma 5.6.1. *Let $\mathbb{B}$ as in (5.3) be elliptic, $1 \leq j \leq \min\{n, k\}$. Suppose that*

- (a) *If $1 \leq j \leq \min\{n-1, k\}$ we have that $\mathbb{B}$ is canceling;*
- (b) *If $j = n \leq k$ we have that $\mathbb{B}$ satisfies condition (WC).*

Then for all $u \in BV_{\text{loc}}^{\mathbb{B}}$ we have that $D^{k-j}u \in \mathfrak{t}^{j,n/(n-j)}(x)$ for $\mathcal{L}^n$ -a.e. $x \in \mathbb{R}^n$.

Proof. We choose $x \in \mathbb{R}^n$ such that, by Lemmas 5.2.2, 5.2.3, 5.2.4, we have that $D^{k-j}u \in \mathfrak{t}^{j,1}(x)$ for $j = 1 \dots k$, where $D^{k-j}u \in L_{\text{loc}}^1$ are weak derivatives of u . In particular, we have a k -th order L^1 -Taylor Polynomial $P_x^k u$ of u at x . Uniqueness of L^p -Taylor polynomials implies that $D^{k-j}P_x^k u = P_x^j D^{k-j}u$ and that $\mathbb{B}P_x^k u = A(\nabla^k u(x))$. The set of such x has $\mathcal{L}^n$ -null complement.

We fix $1 \leq j \leq \min\{n-1, k\}$, so that $\mathbb{B}$ is canceling, and estimate:

$$\|D^{k-j}v\|_{L^{n/(n-j)}(B_r(x))} \lesssim |\mathbb{B}v|(\overline{B_{2r}(x)}) + \sum_{l=1}^k r^{-l} \|D^{k-l}v\|_{L^1(B_{2r}(x))}, \quad (5.23)$$

which holds for all $v \in BV_{\text{loc}}^{\mathbb{B}}$. This is obtained analogously to (5.21), using (5.4) instead of [201, Thm. 8.5]. We apply (5.23) to $R_x^k u := u - P_x^k u$ to get

$$\begin{aligned} \left(\int_{B_r(x)} |D^{k-j}R_x^k u|^{n/(n-j)} dy \right)^{(n-j)/n} &\lesssim |\mathbb{B}u - \mathbb{B}P_x^k u|(\overline{B_{2r}(x)}) \\ &+ \sum_{l=1}^k \frac{1}{r^l} \int_{B_{2r}(x)} |D^{k-l}u - D^{k-l}P_x^k u| dy \\ &= |\mathbb{B}u - A(\nabla^k u(x))|(\overline{B_{2r}(x)}) \\ &+ \sum_{l=1}^k \frac{1}{r^l} \int_{B_{2r}(x)} |D^{k-l}u - P_x^l D^{k-l}u| dy. \end{aligned}$$

To obtain the correct scaling, we multiply by r^{j-n} to get

$$\left(\int_{B_r(x)} |R_x^j D^{k-j}u|^{n/(n-j)} dy \right)^{(n-j)/n} \lesssim r^j \frac{|\mathbb{B}u - \mathbb{B}^{ac}u(x)|(\overline{B_{2r}(x)})}{r^n}$$

$$+ \sum_{l=1}^k r^{j-l} \int_{B_{2r}(x)} |R_x^l D^{k-l} u| dy.$$

As $r \downarrow 0$, the first term equals $o(r^j)$ by Lebesgue differentiation, whereas the averaged integrals equal $o(r^l)$ for all $l = 1 \dots k$. The first case is proved.

If $j = n \leq k$, we suppose that $\mathbb{B}$ satisfies condition (WC) and replace (5.23) with

$$\|D^{k-n} v\|_{L^\infty(B_r(x))} \lesssim |\mathbb{B}v|(\overline{B_{2r}(x)}) + \sum_{l=1}^k r^{-l} \|D^{k-l} v\|_{L^1(B_{2r}(x))} \quad (5.24)$$

for $v \in BV_{\text{loc}}^{\mathbb{B}}$. This, again, follows just as (5.21), by using Theorem 5.1.3 instead of (5.4). We proceed as above, with $v := R_x^k u$, so that we have

$$\|R_x^n D^{k-n} u\|_{L^\infty(B_r(x))} \lesssim r^n \frac{|\mathbb{B}u - \mathbb{B}^{ac}u(x)|(\overline{B_{2r}(x)})}{r^n} + \sum_{l=1}^k r^{n-l} \int_{B_{2r}(x)} |R_x^l D^{k-l} u|,$$

and we can conclude as in the previous case. $\square$

We next prove necessity of ellipticity for both Theorems 5.1.4, 5.1.5. This is contained in the following Lemma, which is proved by an inexpensive modification of the proof of Lemma 5.5.2.

Lemma 5.6.2. *Let $\mathbb{B}$ be as in (5.3). Suppose that there exist $1 \leq j \leq k$, $1 < p < \infty$ such that all $u \in BV_{\text{loc}}^{\mathbb{B}}$ are such that $D^{k-j}u \in \mathfrak{t}^{j,p}(x)$ for $\mathcal{L}^n$ -a.e. $x \in \mathbb{R}^n$. Then $\mathbb{B}$ is elliptic.*

Proof. Suppose that $\mathbb{B}$ is not elliptic, and let ξ, v be as in the proof of Lemma 5.5.2. We let $g(t) = |t|^{k-j-1/p}(\text{sgn}(t))^{k-j}$ for $t \in \mathbb{R}$, so $g \in L_{\text{loc}}^1(\mathbb{R})$, and define u by (5.22). It is clear that $\mathbb{B}u = 0$ and that $u \in L_{\text{loc}}^1$ (as in the proof of Lemma 5.5.2), so $u \in BV_{\text{loc}}^{\mathbb{B}}$. A simple computation shows that

$$\begin{aligned} D^{k-j}u(x) &= \sum_{l=1}^{\infty} \frac{d^{k-j}g}{dt^{k-j}}((x - x_l) \cdot \xi) v \otimes^{k-j} \xi \\ &= c(p, j, k) \sum_{l=1}^{\infty} |(x - x_l) \cdot \xi|^{-1/p} v \otimes^{k-j} \xi, \end{aligned}$$

which, by an argument as in the proof of Lemma 5.5.2, is nowhere L^p -integrable. The proof is complete. $\square$

We also generalize Lemma 5.5.3, to obtain necessity of cancellation for Theorem 5.1.4(a). Again, Lemma 5.2.5 will prove to be instrumental.

Lemma 5.6.3. *Let $\mathbb{B}$ as in (5.3) be elliptic. Suppose that for some $1 \leq j \leq \min\{k, n-1\}$ we have that all $u \in BV_{\text{loc}}^{\mathbb{B}}$ are such that $D^{k-j}u \in \mathfrak{t}^{j,n/(n-j)}(x)$ for $\mathcal{L}^n$ -a.e. $x \in \mathbb{R}^n$. Then $\mathbb{B}$ is canceling.*

Proof. We assume that $\mathbb{B}$ is not canceling and consider $u_h \in \text{BV}_{\text{loc}}^{\mathbb{B}}$ as in the proof of Lemma 5.2.5, so that $D^{k-j}u_h \notin L^{n/(n-j)}(B_r(0))$ for any $r > 0$. As before, we work in $\mathfrak{X} := \text{BV}^{\mathbb{B}}(\mathbb{R}^n)$, which is a complete metric space in the metric induced by the norm $\|\cdot\|_{\text{BV}^{\mathbb{B}}(\mathbb{R}^n)} := \|\cdot\|_{L^1(\mathbb{R}^n)} + |\mathbb{B} \cdot|(\mathbb{R}^n)$. Let $\{x_l\}_{l=1}^{\infty}$ be a dense subset of $\mathbb{R}^n$ and consider the subsets

$$Y_{l,m,L} := \{u \in \mathfrak{X} : \|D^{k-j}u\|_{L^{n/(n-j)}(B(x_l, m^{-1}))} \leq L\}$$

for positive integers l, m, L , and the subspace

$$Y_{l,m,\infty} := \{u \in \mathfrak{X} : \|D^{k-j}u\|_{L^{n/(n-j)}(B(x_l, m^{-1}))} < \infty\}.$$

We check that the sets $Y_{l,m,L}$ are nowhere dense. Let $u_i \in Y_{l,m,L}$ be such that $u_i \rightarrow u$ in $\mathfrak{X}$. By standard mollification arguments, we have that $C_c^\infty(\mathbb{R}^n, V)$ is $(\mathbb{B}-)$ strictly dense in $\mathfrak{X}$ [45, Thm. 2.8(b)], so that maps in $\mathfrak{X}$ can be represented by (5.9). In particular, Lemma 5.2.1 implies that

$$D^{k-j}v = D^{k-j}K^{\mathbb{B}} \star \mathbb{B}v$$

for $v \in \mathfrak{X}$, where $D^{k-j}K^{\mathbb{B}}$ is $(j-n)$ -homogeneous. Standard boundedness of Riesz potentials [175, Thm. V.1] and inclusion of Lorentz spaces imply that, for any $1 \leq p < n/(n-j)$ and any ball $B \subset \mathbb{R}^n$,

$$\|D^{k-j}v\|_{L^p(B)} \leq C(B)|\mathbb{B}u|(\mathbb{R}^n) \quad (5.25)$$

for any $v \in \mathfrak{X}$ (here, it is crucial that $0 < j < n$). Thus $D^{k-j}u_i$ converges $\mathcal{L}^n$ -a.e. to $D^{k-j}u$ on a subsequence which we do not relabel. By Fatou's Lemma we have:

$$\|D^{k-j}u\|_{L^{n/(n-j)}(B(x_l, m^{-1}))} \leq \liminf_{i \rightarrow \infty} \|D^{k-j}u_i\|_{L^{n/(n-j)}(B(x_l, m^{-1}))} \leq L,$$

so $Y_{l,m,L}$ is closed. Next, we consider cut-off functions $\rho_{l,m} \in C_c^\infty$ such that $\rho_{l,m} = 1$ in $B(x_l, m^{-1})$. It follows that $\rho_{l,m}u_h(\cdot - x_l) \in \text{BV}^{\mathbb{B}}(\mathbb{R}^n)$ by the regularity of u_h away from zero and the Leibniz rule. However, $D^{k-j}u_h(\cdot - x_l) \notin L^{n/(n-j)}(B(x_l, m^{-1}))$, so that $Y_{l,m,\infty}$ is a proper subspace of $\mathfrak{X}$. Therefore, the sets $Y_{l,m,L} \subset Y_{l,m,\infty}$ have empty interior, so we can conclude by Baire's Category Theorem. $\square$

It remains to establish necessity of condition (WC) for Theorem 5.1.4(b). The proof is, of course, similar to that of Lemma 5.6.3.

Lemma 5.6.4. *Let $\mathbb{B}$ as in (5.3) be elliptic, $k \geq n \geq 1$. Suppose we have that all $u \in \text{BV}_{\text{loc}}^{\mathbb{B}}$ are such that $D^{k-n}u \in \mathfrak{t}^{n,\infty}(x)$ for $\mathcal{L}^n$ -a.e. $x \in \mathbb{R}^n$. Then $\mathbb{B}$ satisfies condition (WC).*

Proof. If $n = 1$, condition (WC) follows by ellipticity. Let $n > 1$. We assume that $\mathbb{B}$ fails condition (WC) and consider $u_h \in \text{BV}_{\text{loc}}^{\mathbb{B}}$ as in the proof of necessity for Theorem 5.1.3 (see also Proposition (5.4.3)), so that $D^{k-n}u_h \notin L^\infty(B_r(0))$ for any $r > 0$. We slightly modify the proof of Lemma 5.6.3 by choosing, for $B := B(0, 1)$,

$$\mathfrak{X} := \{u \in \text{BV}^{\mathbb{B}}(\mathbb{R}^n) : \text{spt } u \subset 2\bar{B}\},$$

which is Banach if endowed with the subspace norm. Let $\{x_l\}_{l=1}^{\infty}$ be a dense subset of B and consider the subsets

$$Z_{l,m,L} := \{u \in \mathfrak{X} : \|D^{k-n}u\|_{L^\infty(B(x_l, m^{-1}))} \leq L\}$$

for positive integers $m > 1$, l, L , and the subspace

$$Z_{l,m,\infty} := \{u \in \mathfrak{X} : \|D^{k-n}u\|_{L^\infty(B(x_l, m^{-1}))} < \infty\}.$$

We check that the sets $Z_{l,m,L}$ are nowhere dense. Let $u_i \in Z_{l,m,L}$ be such that $u_i \rightarrow u$ in $\mathfrak{X}$. We collect from (5.25) with $j = n - 1 > 0$ that, for any $1 \leq p < n$ we have

$$\|D^{k-n+1}v\|_{L^p(B)} \leq C(B)|\mathbb{B}u|(\mathbb{R}^n) \quad (5.26)$$

for any $v \in \mathfrak{X}$. By Lemma 5.2.1 for $\mathbb{B} = D^{k-n+1}$ and $j = n - 1$, we have that for a $(1 - n)$ -homogeneous, smooth convolution kernel K

$$|D^{k-n}v| = |K \star D^{k-n+1}v| \lesssim I_1 |D^{k-n+1}v|$$

for $v \in \mathfrak{X}$. By [93, Lem. 7.12], for any $1 \leq q < np/(n - p)$ and (5.26) we have

$$\|D^{k-n}v\|_{L^q(B)} \leq C(B)\|D^{k-n+1}v\|_{L^p(B)} \leq C(B)|\mathbb{B}u|(\mathbb{R}^n)$$

for $v \in \mathfrak{X}$. In particular, $D^{k-n}u_i \rightarrow D^{k-n}u$ in L^s_{loc} for any $1 \leq s < \infty$, so $D^{k-n}u_i$ converges $\mathcal{L}^n$ -a.e. to $D^{k-n}u$ on a subsequence which we do not relabel. It follows that $\mathcal{L}^n$ -a.e. in $B(x_l, m^{-1})$, we have that $D^{k-n}u \leq L$, so $Z_{l,m,L}$ is closed.

Next, we consider cut-off functions $\rho_{l,m} \in C_c^\infty$ such that $\rho_{l,m} = 1$ in $B(x_l, m^{-1})$. It follows that $\rho_{l,m}u_h(\cdot - x_l) \in BV^{\mathbb{B}}(\mathbb{R}^n)$ by the regularity of u_h away from zero and the Leibniz rule (it is still the case that $u_h = K^{\mathbb{B}}w$, as in Lemma 5.2.5). Moreover, since $m \geq 2$ and $x_l \in B$, one can easily arrange that the support of $\rho_{m,l}$ is contained in $2\bar{B}$, so $\rho_{l,m}u_h(\cdot - x_l) \in \mathfrak{X}$. However, $D^{k-n}u_h(\cdot - x_l) \notin L^\infty(B(x_l, m^{-1}))$, so that $Z_{l,m,\infty}$ is a proper subspace of $\mathfrak{X}$. Therefore, the sets $Z_{l,m,L} \subset Z_{l,m,\infty}$ have empty interior, so we can conclude by Baire's Category Theorem. $\square$

5.7. Appendix

When presenting the main results on L^p -Taylor expansions in the Introduction we refrained from presenting them on the full Lorentz $L^{p,q}$ -scale in order not to overcomplicate the statements. However, presenting the results in this manner reveals, yet again, a very clear connection between L^p_{weak} -differentiability and weak-type estimates as well as $L^{p,q}$ -differentiability, $q < \infty$, and strong-type estimates.

We begin by extending the definition of the $\mathfrak{t}^{k,p}$ -spaces to fit the Lorentz scale:

Definition 5.7.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $x \in \Omega$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. A measurable map $u : \Omega \rightarrow V$ lies in $\mathfrak{t}^{k,(p,q)}(x)$ if there exists a polynomial $P_x^k u$ of degree at most k such that*

$$\|R_x^k u\|_{L^{p,q}(B_r(x))} = o\left(r^{k+n/p}\right) \quad \text{as } r \downarrow 0,$$

where $R_x^k u := u - P_x^k u$. If this is the case, we say that u has a k -th order $L^{p,q}$ -Taylor expansion $P_x^k u$ at x .

Here for $p = \infty$ we use the extension of the Lorentz scale, as introduced in [32], which is such that $L_{\text{loc}}^{\infty,1} \hookrightarrow L_{\text{loc}}^{\infty,q} \hookrightarrow L_{\text{loc}}^{\infty,\infty}$. Moreover,

$$L^{\infty,1} = L^{\infty} \quad \text{and} \quad L^{\infty,\infty} = L_{\text{weak}}^{\infty},$$

where the weak- L^{∞} space was introduced in [33]. We also recall the formal definition of the $L^{\infty,q}$ -spaces, which are not linear spaces in general (see also [131]).

Definition 5.7.2 ([32, Def. 2.1]). *Let $\Omega \subset \mathbb{R}^n$, $|\Omega| := \mathcal{L}^n(\Omega)$, $0 < q \leq \infty$. We say that $f \in L^{\infty,q}(\Omega)$ if $f: \Omega \rightarrow \mathbb{R}$ is measurable and the quantity*

$$\begin{aligned} \|f\|_{L^{\infty,q}(\Omega)} &:= \left(\int_0^{|\Omega|} [f^{**}(t) - f^*(t)]^q \frac{dt}{t} \right)^{1/q} \quad \text{for } q < \infty \\ \|f\|_{L^{\infty,\infty}(\Omega)} &:= \sup_{0 < t < |\Omega|} [f^{**}(t) - f^*(t)] \end{aligned}$$

is finite. Here f^* denotes the decreasing rearrangement of f ,

$$f^*(t) := \inf\{\lambda > 0: \mathcal{L}^n\{x \in \Omega: |f(x)| > \lambda\} \leq t\}$$

and f^{**} denotes the maximal function

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) \, ds$$

for $t \in (0, |\Omega|)$.

We make the convention $n/0 = \infty$. The methods of the present chapter are readily adapted to give the following unified extensions of Theorems 5.1.1, 5.1.4, 5.1.5, which, despite their rather intricate statements, tell the story of this chapter best:

Theorem 5.7.3 (Weak-type). *Let $\mathbb{B}$ be as in (5.3), $1 \leq j \leq \min\{k, n\}$. The following are equivalent:*

(a) *For all $u \in \text{BV}_{\text{loc}}^{\mathbb{B}}(\mathbb{R}^n)$, we have that*

$$D^{k-j}u \in \mathfrak{t}^{j, (n/(n-j), \infty)}(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

(b) *$\mathbb{B}$ is elliptic.*

Theorem 5.7.4 (Strong-type). *Let $\mathbb{B}$ be as in (5.3), $1 \leq q < \infty$. Then:*

(a) *If $1 \leq j \leq \min\{k, n-1\}$ and $1 < q < \infty$, the following are equivalent:*

(i) *For all $u \in \text{BV}_{\text{loc}}^{\mathbb{B}}(\mathbb{R}^n)$, we have that*

$$D^{k-j}u \in \mathfrak{t}^{j, (n/(n-j), q)}(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

(ii) *$\mathbb{B}$ is elliptic and canceling.*

(b) *If $k \geq n$ and $1 \leq q < \infty$, the following are equivalent:*

(i) For all $u \in BV_{\text{loc}}^{\mathbb{B}}(\mathbb{R}^n)$, we have that

$$D^{k-n}u \in \mathfrak{t}^{n,(\infty,q)}(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

(ii) $\mathbb{B}$ is elliptic and satisfies condition (WC).

The methods of the chapter can indeed be adapted in a straightforward manner, except for two aspects: necessity of condition (WC) for the strong-differentiability if $1 < q < \infty$ (which follows from the computable fact that $\log|\cdot| \notin L_{\text{loc}}^{\infty,q}$ for $1 \leq q < \infty$) and sufficiency of ellipticity for weak-differentiability if $j = n \leq k$, which follows easily from the following result, which is probably obvious for experts:

Proposition 5.7.5. *Let $\mathbb{B}$ as in (5.3) be elliptic, of order $k \geq n > 1$. Then*

$$\|D^{k-n}u\|_{L^{\infty,\infty}(\mathbb{R}^n, V \odot^{k-n}\mathbb{R}^n)} \lesssim \|\mathbb{B}u\|_{L^1(\mathbb{R}^n, W)}$$

for all $u \in C_c^\infty(\mathbb{R}^n, V)$.

The inequality follows from Formula (5.14) and the following bound:

$$\|I_n f\|_{L^{\infty,\infty}} \lesssim \|f\|_{L^1} \quad (5.27)$$

for $f \in C_c^\infty(\mathbb{R}^n)$. Here $I_n f := f \star \log|\cdot|$ is the analytic extension of the Riesz potential I_α , $0 < \alpha < n$, see e.g. [164, Ch. 2].

Proof of (5.27). By [32, Eq. (1.1)],

$$\|I_n f\|_{L^{\infty,\infty}} \lesssim \|DI_n f\|_{L^{n,\infty}} = \|I_{n-1} f\|_{L^{n,\infty}} \lesssim \|f\|_{L^1},$$

where for last inequality we applied the standard weak-type estimate for I_{n-1} . $\square$

The case $n = 1$ in Proposition 5.7.5 has to be ruled out if we want to use boundedness of I_{n-1} between L^1 and L_{weak}^n , but this is no restriction, since in that case Theorem 5.1.3 applies and the strong estimate is available (every elliptic operator satisfies condition (WC) in odd dimensions).

Remark 5.7.6. *We conclude by recalling [201, Open Prob. 8.3], which can be reformulated as, whether for elliptic operators $\mathbb{B}$, is it the case that cancellation implies*

$$\|D^{k-j}u\|_{L^{n/(n-j),1}} \lesssim \|\mathbb{B}u\|_{L^1}$$

for $u \in C_c^\infty(\mathbb{R}^n, V)$. Here $1 \leq j \leq \min\{n-1, k\}$. What the viewpoint we took in section indicates is that the result of Theorem 5.1.3 applies to the endpoint $j = n \leq k$, though we do not see a connection with the aforementioned problem.

CHAPTER 6

On limiting trace inequalities for differential operators

This chapter consists of joint work with F. Gmeineder and J. Van Schaftingen.

6.1. Introduction

The Sobolev inequality for $1 < p < n$,

$$\|u\|_{L^{\frac{np}{n-p}}} \leq C \|Du\|_{L^p} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n),$$

was proved in [173] essentially by showing that the Riesz Potential of order $\alpha \in (0, n)$, i.e.,

$$(\mathcal{I}_\alpha f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{\alpha-n}} dy \quad \text{for } x \in \mathbb{R}^n,$$

is bounded between $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$, where $q = \frac{np}{n-\alpha p}$. This result was extended by ADAMS in [2, 3] to prove trace embedding theorems on s -codimensional sets:

$$\|\mathcal{I}_\alpha f\|_{L^q(\Sigma; \mathcal{H}^{n-s})} \leq C \|f\|_{L^p(\mathbb{R}^n; \mathcal{L}^n)}, \quad (6.1)$$

where $q = \frac{(n-s)p}{n-\alpha p}$, provided that $\alpha p > s$. Here and in what follows, sets $\Sigma \subset \mathbb{R}^n$ satisfy the following upper density bound: there exists a constant $C = C(n) > 0$ such that

$$\mathcal{H}^{n-s}(B(x, r) \cap \Sigma) \leq C r^{n-s} \quad \text{for all } x \in \Sigma. \quad (6.2)$$

If the precise value of $s \in [0, \alpha p]$ is not specified, then it is clear from the context.

The extension of the Sobolev inequality in this case is the trace theorem for $\dot{W}^{1,p}(\mathbb{R}^n)$:

$$\|u\|_{L^q(\Sigma; \mathcal{H}^{n-s})} \leq C \|Du\|_{L^p(\mathbb{R}^n; \mathcal{L}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n), \quad (6.3)$$

where $q = \frac{(n-s)p}{n-p}$ for $s \in [0, p)$. In fact, one can allow for $s = p$ in (6.3) or for $s = \alpha p$ (6.1) (so that $q = p$ instead of $q > p$), provided that one replaces the L^p -norm on the right hand side of (6.3) is replaced by the endpoint Lorentz norm $L^{p,1}$, as was recently shown in [115, Thm. 1.2, Thm. 1.4].

As for the Sobolev inequality, if one chooses $p = 1$, the inequality (6.3) cannot follow by estimates on Riesz potentials in a straightforward manner. Instead, one can prove by the method in [210, Thm. 5.13.1] that

$$\|u\|_{L^{\frac{n-s}{n-1}}(\Sigma; \mathcal{H}^{n-s})} \leq C \|Du\|_{L^1(\mathbb{R}^n; \mathcal{L}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n), \quad (6.4)$$

for $s \in [0, 1)$. The methods of the present chapter show that this result also follows from (6.1) and embeddings of Sobolev spaces into Triebel–Lizorkin spaces. The limiting case $s = 1$ in (6.4) was proved earlier by GAGLIARDO in [90]. This result is also consistent with [115, Thm. 1.4] (for $k = n$) alluded to above, due to the fact that $L^{1,1} = L^1$.

Motivated partly by examples in Physics (e.g., [19, 178]), but mostly by complementing the theory of linear partial differential equations, one can investigate trace theorems such as (6.3) and (6.4) when the gradient Du in the right hand side is replaced by the pointwise weaker vector field $\mathbb{B}u$, where $\mathbb{B}$ is a homogeneous linear differential operator with constant coefficients (see (6.7) below). As will become transparent from the proofs in Section 6.2, it is necessary that we assume that $\mathbb{B}$ is (overdetermined) elliptic, the definition of which we recall below in (6.8). Under this assumption, one can employ boundedness of singular integrals to show that

$$\|D^k u\|_{L^p} \leq C \|\mathbb{B}u\|_{L^p} \quad \text{for all } u \in C_c^\infty, \quad (6.5)$$

where k is the order of $\mathbb{B}$ and $1 < p < \infty$. In particular, the study of trace inequalities

$$\|D^{k-1} u\|_{L^q(\Sigma; \mathcal{H}^{n-s})} \leq C \|\mathbb{B}u\|_{L^p(\mathbb{R}^n; \mathcal{L}^n)} \quad \text{for all } u \in C_c^\infty$$

in the regime $1 < p < n$ reduces to the study of (6.3).

The situation is dramatically different for $p = 1$, where (6.5) fails except from the trivial case $|D^k u| \sim |\mathbb{B}u|$ pointwise by Ornstein’s Non-inequality [148, 113]. Thus, the main focus of this chapter is to study the embeddings

$$\|D^{k-1} u\|_{L^{\frac{n-s}{n-1}}(\Sigma; \mathcal{H}^{n-s})} \leq C \|\mathbb{B}u\|_{L^1(\mathbb{R}^n; \mathcal{L}^n)} \quad \text{for all } u \in C_c^\infty, \quad (6.6)$$

where $s \in [0, 1]$ and $\Sigma \subset \mathbb{R}^n$ satisfies the regularity condition (6.2). The case $s = 0$ was tackled recently by VAN SCHAFTINGEN in [201], where it is shown that the corresponding generalization of the Gagliardo–Nirenberg–Sobolev Inequality holds if and only if the operator $\mathbb{B}$ is elliptic and canceling (see (C) below). In this case, the canceling condition is equivalent with existence of maps u such that $\mathbb{B}u = \delta_0 w$ for some vector $w \neq 0$, and can therefore be relevant only for the limiting case $p = 1$.

The main results of the present chapter are the following: For elliptic operators $\mathbb{B}$, we have that:

- (a) if $s \in (0, 1)$, then (6.6) holds for all suitable Σ if and only if $\mathbb{B}$ satisfies (C) (Theorem 6.1.1);
- (b) if $s = 1$ and (6.6) holds for all $(n - 1)$ -dimensional hyper-planes Σ , then $\mathbb{B}$ satisfies a strictly *stronger canceling* condition (SC) (Proposition 6.1.3);

(c) if $s = 1$ and $\mathbb{B}$ is of first order ($k = 1$), then (SC) implies (6.6).

For the remainder of the Introduction, we will present the statements of these results in detail. For this purpose we introduce a bit of notation: Differential operators will be of the form

$$\mathbb{B} = \sum_{|\alpha|=k} \partial^\alpha B_\alpha \quad \mathbb{B}[\xi] = \sum_{|\alpha|=k} \xi^\alpha B_\alpha \quad (6.7)$$

for linear maps $B_\alpha \in \text{Lin}(V, W)$ for all multi-indices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$. Here V, W are two finite dimensional real normed vector spaces. For simplicity, we endow these spaces with inner products, restriction which is easily removed. Recall that $\mathbb{B}$ is said to be (overdetermined) *elliptic* if

$$\mathbb{B}[\xi] \in \text{Lin}(V, W) \text{ is injective for all } \xi \in \mathbb{R}^n \setminus \{0\} \quad (6.8)$$

and *canceling* if and only if

$$\bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} \text{im } \mathbb{B}[\xi] = \{0\}. \quad (C)$$

We will sometimes use the shorthand *EC* for elliptic and canceling operators. Throughout, we assume that $n > 1$.

6.1.1. The case $s \in [0, 1)$: EC operators Our first result concerns the generalization of (6.4) to arbitrary homogeneous linear differential operators. In this case, ellipticity and cancellation is necessary and sufficient for non-critical lower dimensional trace embeddings:

Theorem 6.1.1. *Let $\mathbb{B}$ be as in (6.7), $0 \leq s < 1$. The following are equivalent:*

- (a) *For every subset $\Sigma \subset \mathbb{R}^n$ with (6.2), there exists a constant $C > 0$ such that for all $u \in C_c^\infty(\mathbb{R}^n, V)$ there holds*

$$\left(\int_{\Sigma} |D^{k-1}u|^{\frac{n-s}{n-1}} d\mathcal{H}^{n-s} \right)^{\frac{n-1}{n-s}} \leq C \int_{\mathbb{R}^n} |\mathbb{B}u| d\mathcal{L}^n.$$

- (b) *$\mathbb{B}$ is elliptic and satisfies (C).*

The case $s = 0$ of Theorem 6.1.1 follows at once from [201, Thm. 1.3]. The refinement is proved in Section 6.2 along the following lines: We use ADAMS' trace theorem (6.1) and classical elliptic estimates to prove that maps in the space $\{u: \mathbb{B}u \in \dot{F}^{-\theta, \frac{n}{n-\theta}}\}$ admit traces in $L^{\frac{n-s}{n-1}}(\Sigma; \mathcal{H}^{n-s})$, provided that $\frac{n(1-s)}{n-s} < \theta < 1$. This fact is proved in Proposition 6.2.1 and is the main tool in the proof of sufficiency of EC. At this stage, it is easy to use the estimate [201, Prop. 8.7] for co-canceling operators.

As a consequence of Theorem 6.1.1, we obtain the following new result for the space BD of maps of bounded deformation [178]:

Corollary 6.1.2. *Let $n > 1$, $s \in (0, 1)$, and $\Sigma \subset \mathbb{R}^n$ satisfy (6.2). Then there exists a well-defined trace operator $\text{Tr}: \text{BD}(\mathbb{R}^n) \rightarrow L^{\frac{n-s}{n-1}}(\Sigma, \mathbb{R}^n)$ that agrees with the classical trace on $C(\mathbb{R}^n, \mathbb{R}^n)$ and*

$$\|\text{Tr } u\|_{L^{\frac{n-s}{n-1}}(\Sigma; \mathcal{H}^{n-s})} \leq C |\mathcal{E}u|(\mathbb{R}^n)$$

for all $u \in \text{BD}(\mathbb{R}^n)$.

The corollary follows from Theorem 6.1.1 by setting $\mathbb{B}u = \frac{1}{2}(Du + (Du)^t)$ for $u: \mathbb{R}^n \rightarrow \mathbb{R}^n$. For $s = 0$, the result is due to STRAUSS [179], and for $s = 1$ the result was proved in [178] (see also [22, 45]).

6.1.2. The limiting case $s = 1$: first order operators As can be seen from the proof of Proposition 6.2.1 in Section 6.2, the methods used to prove Theorem 6.1.1 fail in the borderline case $s = 1$. This is no coincidence, as can be seen from the following:

Proposition 6.1.3. *Let $\mathbb{B}$ be as in (6.7). Suppose that for every $(n - 1)$ -dimensional hyper-plane $\Sigma \subset \mathbb{R}^n$ there exists a constant $C > 0$ such that for all $u \in C_c^\infty(\mathbb{R}^n, V)$ there holds*

$$\int_{\Sigma} |D^{k-1}u| d\mathcal{H}^{n-1} \leq C \int_{\mathbb{R}^n} |\mathbb{B}u| d\mathcal{L}^n. \quad (6.9)$$

Then $\mathbb{B}$ is elliptic and satisfies the stronger cancellation condition

$$\bigcap_{\xi \in H \setminus \{0\}} \text{im } \mathbb{B}[\xi] = \{0\} \quad \text{for all } H \leq \mathbb{R}^n \text{ with } \dim H = 2. \quad (\text{SC})$$

In particular, the statement of Theorem 6.1.1 fails in the limiting case $s = 1$, as we explain now. Obviously, Condition (SC) reduces to (C) if $n = 2$. For $n > 2$, to see that Condition (SC) is strictly stronger than cancellation in the class of elliptic operators, one considers an elliptic operator $\mathbb{B}_1$ on $\mathbb{R}^2$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ (e.g., $\mathbb{B}_1 = \Delta^{k/2}$ if k is even and $\mathbb{B}_1 = \Delta^{(k-1)/2} \circ (\text{div}, \text{curl})$ if k is odd) and sets

$$\mathbb{B} = \begin{pmatrix} \mathbb{B}_1 & 0 \\ 0 & \mathbb{B}_2 \end{pmatrix},$$

where $\mathbb{B}_2$ is a k -th order EC operator on $\mathbb{R}^{n-2}$ (e.g., the k -th gradient). We will prove Proposition 6.1.3 in Section 6.3.

As for the converse of Proposition 6.1.3, we focus on the case of first order operators ($k = 1$), where more tools are available [45]. We have:

Theorem 6.1.4. *Let $\mathbb{B}$ as in (6.7) be a first order operator. Then the following are equivalent:*

- (a) *For every $(n - 1)$ -dimensional hyper-plane $\Sigma \subset \mathbb{R}^n$ there exists a constant $C > 0$ such that for all $u \in C_c^\infty(\mathbb{R}^n, V)$ there holds*

$$\int_{\Sigma} |u| d\mathcal{H}^{n-1} \leq C \int_{\mathbb{R}^n} |\mathbb{B}u| d\mathcal{L}^n.$$

- (b) $\mathbb{B}$ is elliptic and (SC) holds.
- (c) For every $(n-1)$ -dimensional hyper-plane $\Sigma \subset \mathbb{R}^n$ there exists a constant $C > 0$ such that for all $u \in C_c^\infty(\mathbb{R}^n, V)$ there holds

$$\int_{\Sigma} |u| d\mathcal{H}^{n-1} \leq C \int_{\Sigma^+} |\mathbb{B}u| d\mathcal{L}^n.$$

Here Σ^+ is a half-space with boundary Σ .

- (d) $\mathbb{B}$ is $\mathbb{C}$ -elliptic, i.e., $\ker_{\mathbb{C}} \mathbb{B}[\xi] = 0$ for all $\xi \in \mathbb{C}^n \setminus \{0\}$.

The equivalence of (b) and (d) follows from Lemma 6.3.2 and an ad hoc argument that uses linearity of $\xi \mapsto \mathbb{B}[\xi]$ for first order operators (cp. [94, Lem. 3.5(b)]). The fact that (a) implies (b) follows from Proposition 6.1.3. The equivalence of (c) and (d) is essentially proved in [45] (see also Theorem 6.5.1). It is also clear that (c) implies (a). A complete proof of Theorem 6.1.4 will be given in Section 6.2.

6.1.3. The limiting case $s = 1$: higher order operators Considering the case $k > 1$, the method used to prove Theorem 6.1.4 fails due to the fact that for higher order operators $\mathbb{C}$ -ellipticity is *strictly* stronger than (SC). This can be seen from Lemma 6.3.2 and the simple example $\mathbb{B} = D^{k-1}(\text{div}, \text{curl})$, defined on $\mathbb{R}^2$.

It seems that at this stage we see a difference between exterior (boundary) traces and interior traces (restrictions). Firstly, one can employ a modification of the arguments in [45] to show that $\mathbb{C}$ -ellipticity of $\mathbb{B}$ is equivalent with the estimate

$$\int_{\Sigma} |D^{k-1}u| d\mathcal{H}^{n-1} dx \leq C \int_{\Sigma^+} |\mathbb{B}u| d\mathcal{L}^n \quad \text{for } u \in C_c^\infty(\mathbb{R}^n, V) \quad (6.10)$$

being true for $(n-1)$ -dimensional hyperplanes $\Sigma \leq \mathbb{R}^n$. Here, crucially, the integration on the right hand side is carried over a half-space Σ^+ with boundary Σ , so that RHS(6.10) is weaker than RHS(6.9) in general (see Theorem 6.5.1 in the Appendix).

From this point of view, it seems plausible that the interior trace estimate (6.9) holds for $\mathbb{B}$ satisfying the weaker condition (elliptic & (SC)). As vague evidence, in Theorem 6.1.1 we show that a cancellation condition suffices for integrability of interior traces. This would be consistent with Theorem 6.1.4, so that we postulate the converse of Proposition 6.1.3:

Open Problem 6.1.5 (Interior traces). *Let $\mathbb{B}$ as in (6.7) be elliptic and (SC). Is it the case that for every $(n-1)$ -dimensional hyper-plane $\Sigma \subset \mathbb{R}^n$ there exists a constant $C > 0$ such that*

$$\int_{\Sigma} |D^{k-1}u| d\mathcal{H}^{n-1} \leq C \int_{\mathbb{R}^n} |\mathbb{B}u| d\mathcal{L}^n \quad (6.11)$$

for all $u \in C_c^\infty(\mathbb{R}^n, V)$?

The reader is invited to compare this question with Theorem 6.5.1 in the Appendix, the method of which cannot yield necessity of $\mathbb{C}$ -ellipticity for (6.11).

With known techniques, it is not clear if one can tackle Open Problem 6.1.5 even in the “easiest” case $n = 2$, in which it is claimed that the estimate of traces on lines holds for EC operators $\mathbb{B}$. However, the algebraic proofs in this chapter suggest that the case $n = 2$ is particularly relevant.

6.2. Proofs of Theorems 6.1.1 and 6.1.4

We begin with an auxiliary inequality used to prove sufficiency of EC for the estimate in Theorem 6.1.1. We follow and refine the strategy used to prove [201, Prop. 4.6]. The elliptic estimate used, i.e., [201, Prop. 4.1] is replaced by the following possibly new trace embedding:

Proposition 6.2.1. *Let $\mathbb{B}$ be an elliptic operator of the form (6.7) and $s \in [0, 1)$. Then for any $s(n-1)/(n-s) < \gamma < 1$ there exists a constant $C = C(n, s, \gamma) > 0$ such that for all $u \in C_c^\infty(\mathbb{R}^n, V)$ there holds*

$$\left(\int_{\Sigma} |D^{k-1}u|^{\frac{n-s}{n-1}} d\mathcal{H}^{n-s} \right)^{\frac{n-1}{n-s}} \leq C \|\mathbb{B}u\|_{\dot{F}_2^{\gamma-1, \frac{n}{n+\gamma-1}}(\mathbb{R}^n, W)} \quad (6.12)$$

whenever Σ is an s -regular subset in the sense of (6.2).

Proof. Let $u \in C_c^\infty(\mathbb{R}^n, V)$ be arbitrary and $0 < \gamma < s(n-1)/(n-s)$. We define three Fourier multiplier operators by T_i , $i = 1, 2, 3$, by

$$\begin{aligned} T_1(F_1) &:= \mathcal{F}^{-1} [|\xi|^{1-k-\gamma} \widehat{F_1}(\xi) \otimes^{k-1} \xi], \quad F_1 \in C_c^\infty(\mathbb{R}^n, V), \\ T_2(F_2) &:= \mathcal{F}^{-1} [(\mathbb{B}^*[\xi] \mathbb{B}[\xi])^{-1} \mathbb{B}^*[\xi] |\xi|^k \widehat{F_2}(\xi)], \quad F_2 \in C_c^\infty(\mathbb{R}^n, W), \\ T_3(F_3) &:= \mathcal{F}^{-1} [|\xi|^{\gamma-1} \widehat{F_3}(\xi)] = \mathcal{I}_{1-\gamma} F_3, \quad F_3 \in C_c^\infty(\mathbb{R}^n, W). \end{aligned}$$

Here, for $0 < \theta < n$, we denote by $\mathcal{I}_\theta$ the θ -th Riesz potential, i.e.,

$$\mathcal{I}_\theta f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\theta}} dy \quad \text{for } x \in \mathbb{R}^n.$$

Then we have $T_1 \circ T_2 \circ T_3[\mathbb{B}u] = D^{k-1}u$. We note that the symbol map $\mathbb{R}^n \setminus \{0\} \ni \xi \mapsto (\mathbb{B}^*[\xi] \mathbb{B}[\xi])^{-1} \mathbb{B}^*[\xi] |\xi|^k \in \mathcal{L}(W, V)$ is homogeneous of degree zero, smooth away from zero and thus satisfies the assumptions of the Mihlin–Hörmander multiplier theorem; in consequence $T_2: L^r(\mathbb{R}^n, \mathcal{L}^n) \rightarrow L^r(\mathbb{R}^n, \mathcal{L}^n)$ boundedly for all $r \in (1, \infty)$.

By ADAMS’ trace embedding Theorem, [4, Thm. 7.2.2], if $n-s-(n-\gamma p)\frac{q}{p} = 0$, then $f \mapsto T_1 f$ extends to a bounded linear operator $L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n; \mu)$. This is due to the fact that $|T_1 f| \lesssim \mathcal{I}_\gamma |f|$ pointwisely and implies that

$$n-s-(n-\gamma p)\frac{q}{p} = 0 \iff \frac{n-s}{n-1} \stackrel{!}{=} q = p \frac{n-s}{n-\gamma p} \iff p = \frac{n}{n+\gamma-1}. \quad (6.13)$$

We then estimate:

$$\begin{aligned} \|D^{k-1}u\|_{L^q(\mathbb{R}^n; \mu, V \otimes^{k-1} \mathbb{R}^n)} &= \|T_1(T_2 T_3(\mathbb{B}u))\|_{L^q(\mathbb{R}^n; \mu, V)} \\ &\leq C \|T_2 T_3(\mathbb{B}u)\|_{L^p(\mathbb{R}^n; \mathcal{L}^n, V)} \\ &\leq C \|T_3(\mathbb{B}u)\|_{L^p(\mathbb{R}^n; \mathcal{L}^n, W)} \\ &= C \|\mathcal{I}_{1-\gamma}(\mathbb{B}u)\|_{L^p(\mathbb{R}^n; \mathcal{L}^n, W)}. \end{aligned} \quad (6.14)$$

Now we use [192, Thm. 5.2.3] and (6.13) to consequently obtain

$$\|\mathcal{I}_{1-\gamma}(\mathbb{B}u)\|_{L^p(\mathbb{R}^n; \mathcal{L}^n, W)} \leq C \|\mathbb{B}u\|_{\dot{F}_2^{\gamma-1, \frac{n}{n+\gamma-1}}(\mathbb{R}^n, W)},$$

and thus, by $q = \frac{n-s}{n-1}$ and (6.14), (6.12) follows and the proof is complete. $\square$

Proof of Theorem 6.1.1 (Sufficiency of EC). Let $u \in C_c^\infty(\mathbb{R}^n, V)$. We can employ [201, Prop. 4.2] to find an exact annihilator $L(D)$ of $\mathbb{B}$, which is thus co-cancelling, so we can apply [201, Prop. 8.7] to $f = \mathbb{B}u$, to get

$$\|D^{k-1}u\|_{L^{\frac{n-s}{n-1}}(\Sigma; \mathcal{H}^{n-s}, V \odot^{k-1} \mathbb{R}^n)} \leq C \|\mathbb{B}u\|_{\dot{F}_2^{\gamma-1, \frac{n}{n+\gamma-1}}(\mathbb{R}^n, W)} \leq C \|\mathbb{B}u\|_{L^1(\mathbb{R}^n, W)},$$

where in the first inequality we used Proposition 6.2.1. The proof is complete. $\square$

As for the necessity part, we begin by isolating two measure theoretic Lemmas. We first construct special regular sets of fractional dimension, which will be used as special choices of Σ .

Lemma 6.2.2. *Let $\mathcal{C} \subset \mathbb{R}^d$ be an open, positive cone with apex at 0, $d \geq 1$, and $\alpha \in (0, d]$. There exists a subset $\Sigma \subset \mathcal{C} \cap B(0, 1)$ and constants $0 < m \leq M < \infty$ such that*

$$mr^\alpha \leq \mathcal{H}^\alpha(B(0, r) \cap \Sigma) \leq Mr^\alpha$$

for all $0 \leq r \leq 1$.

Proof. We recall from [105] that for there exists a set $S \subset \mathbb{R}^d$ such that and constants $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda \leq \liminf_{r \downarrow 0} \frac{\mathcal{H}^\alpha(B(x, r) \cap S)}{r^\alpha} \leq \limsup_{r \downarrow 0} \frac{\mathcal{H}^\alpha(B(x, r) \cap S)}{r^\alpha} \leq \Lambda$$

for all $x \in S$. Since Hausdorff measure is translation invariant, we can assume that $0 \in S$, so that there exists $\varepsilon > 0$ such that

$$\frac{\lambda}{2} r^\alpha \leq \mathcal{H}^\alpha(B(0, r) \cap S) \leq 2\Lambda r^\alpha \quad \text{for all } 0 \leq r \leq \varepsilon. \quad (6.15)$$

Applying a stretch by ε^{-1} if necessary, we can assume that $\varepsilon = 1$ (the fact that uniform stretching by ε^{-1} changes $\mathcal{H}^\alpha$ -measure by a factor of $\varepsilon^{-\alpha}$ follows from the definition).

In particular, the problem is solved if $d = 1$, since in that case $\mathcal{C} \cap B(0, 1) = (-1, 1)$.

If $d \geq 2$, note that it suffices to solve the problem for $d - 1 < \alpha \leq d$, otherwise we apply the following construction in a hyper-plane of dimension $\lceil \alpha \rceil$ (we only need the statement with this restriction, anyway).

Write now explicitly $\mathcal{C} = \mathbb{R}\mathcal{S}$, where $\mathcal{S}$ is a spherical cap in $\mathbb{S}^{n-1}$. Say that the unit vector $e \in \mathbb{R}^n$ is the centre of the spherical cap $\mathcal{S}$. Let $\theta \in (0, \pi/2)$ be the angle e makes with $\partial\mathcal{C}$. Since we chose $\alpha > d - 1$, (6.15) holds with S replaced by $S \setminus e^\perp$. Write now H^+ for the open half-space bounded by $e^\perp$ that contains $\mathcal{S}$ and let $\mathcal{C}^+ = H^+ \cap \mathcal{C}$.

We will now define a bi-Lipschitz map L^+ between H^+ and $\mathcal{C}^+$ as follows: Let $x \in H^+$, so that $x \in |x|\mathbb{S}^{n-1}$. Consider the arc of angle θ_x in $|x|\mathbb{S}^{n-1}$ defined by $0, x, |x|e$. In this arc, there is a unique point y (in $|x|\mathcal{S}$) such that the angle defined by $0, y, |x|e$ equals $\theta_y = 2\theta_x\theta/\pi$. Define $L^+(x) = y$, so that the inverse of L^+ is given by $\theta_x = \pi\theta_y/(2\theta)$.

It is then clear by definition of the $\mathcal{H}^\alpha$ -measure that

$$\mathcal{H}^\alpha(L^+(S \cap H^+) \cap B(0, r)) \sim \mathcal{H}^\alpha((S \cap H^+) \cap B(0, r)), \quad (6.16)$$

for $0 \leq r \leq 1$, with constants given by the Lipschitz constants of L^+ and its inverse.

One then defines $L^-(x) = QL(Qx)$ for $x \in H^- = -H^+$, where Q is the reflection in $e^\perp$. Finally, it is now easy to see that $\Sigma = L^+(S \cap H^+) \cup L^-(S \cap H^-)$ satisfies the assumptions of the Lemma: By (6.15) and (6.16) we have that

$$\mathcal{H}^\alpha(\Sigma \cap B(0, r)) \sim \mathcal{H}^\alpha(S \setminus e^\perp \cap B(0, r)) \sim r^\alpha,$$

with implicit constants m, M depending on λ, Λ , the Lipschitz constants of L^+ and its inverse, and ε . $\square$

We next show that on the set Σ thus constructed, non-zero continuous $(-\alpha)$ -homogeneous functions are not $\mathcal{H}^\alpha$ -integrable.

Lemma 6.2.3. *Let Σ be as given by Lemma 6.2.2. Then*

$$\int_{\Sigma} |x|^{-\alpha} d\mathcal{H}^\alpha(x) = \infty.$$

Proof. We consider the following simple approximation from below:

$$|x|^{-\alpha} \geq \sum_{i=\lceil m^{-1} \rceil}^j r_{i,j}^{-\alpha} \chi_{A_{i,j}}(x) \quad \text{for } x \in B(0, 1),$$

where $A_{i,j}$ denotes the annulus $\{x \in \mathbb{R}^n : r_{i-1,j} \leq |x| < r_{i,j}\}$. Here, the radii $r_{i,j}$, are defined recurrently for fixed integer $j \geq m^{-1}$ by $r_{j,j} = 1$ and

$$r_{i-1,j} = r_{i,j} \left(\frac{m - i^{-1}}{M} \right)^{1/\alpha} \quad \text{for } i = \lceil m^{-1} \rceil, \lceil m^{-1} \rceil + 1, \dots, j$$

for constants $0 < m \leq M < \infty$ as in Lemma 6.2.2. We have that

$$\begin{aligned} \int_{\Sigma} |x|^{-\alpha} d\mathcal{H}^\alpha(x) &\geq \sum_{i=\lceil m^{-1} \rceil}^j r_{i,j}^{-\alpha} (\mathcal{H}^\alpha(B(0, r_{i,j})) - \mathcal{H}^\alpha(B(0, r_{i-1,j}))) \\ &\geq \sum_{i=\lceil m^{-1} \rceil}^j r_{i,j}^{-\alpha} (mr_{i,j}^\alpha - Mr_{i-1,j}^\alpha) = \sum_{i=\lceil m^{-1} \rceil}^j m - M \left(\frac{r_{i-1,j}}{r_{i,j}} \right)^\alpha \\ &= \sum_{i=\lceil m^{-1} \rceil}^j \frac{1}{i} \rightarrow \infty \quad \text{as } j \rightarrow \infty. \end{aligned}$$

The proof is complete. $\square$

Proof of Theorem 6.1.1 (Necessity of EC). By a standard regularization argument, the inequality of Theorem 6.1.1 holds whenever $\mathbb{B}u$ is a bounded measure.

Necessity of ellipticity. We initially cover the first order case $k = 1$. Assume that the estimate holds and $\mathbb{B}$ is not elliptic. Without loss of generality, we can write that there exists $0 \neq v \in V$ such that $\mathbb{B}[e_n]v = 0$, where (e_j) is an orthonormal basis of $\mathbb{R}^n$. We let $f(t) = |t|^{-\beta}$ for $t \in \mathbb{R}$ and $\beta = \frac{(1-s)(n-1)}{n-s}$, so that $f \in L^1_{\text{loc}}(\mathbb{R})$. We then define $u_f(x) = f(x \cdot e_n)v$ for $x \in \mathbb{R}^n$, such that $\mathbb{B}u_f = 0$ in $\mathcal{D}'(\mathbb{R}^n, V)$. However, to be able to assert this with confidence, we must first check that $u_f \in L^1_{\text{loc}}$; we write $x = (x', x_n)$ with respect to the basis $(e_j)_{j=1}^n$.

$$\begin{aligned} \int_{(-R,R)^n} |u_f| \, d\mathcal{L}^n &= \int_{(-R,R)^{n-1}} \int_{-R}^R f(x_n) |v| \, dx_n \, dx' \\ &= |v| (2R)^{n-1} \int_{-R}^R f(t) \, dt < \infty. \end{aligned}$$

It thus follows by the Leibniz rule that $\tilde{u}_f = \rho u_f$ for $\rho \in C_c^\infty(\mathbb{R}^n)$ satisfies $\mathbb{B}\tilde{u}_f \in L^1(\mathbb{R}^n, W)$, so the RHS of the estimate is finite.

To proceed, we let $S \subset \mathbb{R}$ be as in Lemma 6.2.2 with $d = 1$ and $\alpha = 1 - s$ and suppose that ρ as above satisfies $\rho = 1$ in $[0, 1]^n$. Then, with $q = \frac{n-s}{n-1}$ and $\Sigma = [0, 1]^{n-1} \times S$,

$$\int_{\Sigma} |u_f|^q \, d\mathcal{H}^{n-s} = \int_{(0,1)^{n-1}} \int_S f(x_n)^q |v|^q \, d\mathcal{H}^\alpha(x_n) \, dx' = |v|^q \int_S |t|^{-\alpha} \, d\mathcal{H}^\alpha(t).$$

To see that the last integral is infinite, one employs Lemma 6.2.3.

We next assume that the order k of $\mathbb{B}$ is arbitrary and keep all relevant notation from the $k = 1$ step. We now let $g(t) = |t|^{k-1-\beta}$ for $t \in \mathbb{R}$, which is clearly locally integrable, and let $u_g(x) = g(x_n)v$ for $x \in \mathbb{R}^n$, so that $u_g \in L^1_{\text{loc}}(\mathbb{R}^n, V)$, and, moreover, $\mathbb{B}u_g = 0$ in the sense of distributions. We also define $\tilde{u}_g = \rho u_g$, so that $\tilde{u}_g = u_g$ in $(0, 1)^n$.

We next claim that $\mathbb{B}\tilde{u}_g \in L^1(\mathbb{R}^n, W)$, which is slightly more involved than in the first order case. We first note that $D^{k-1}u_g \in L^1_{\text{loc}}(\mathbb{R}^n, V \odot^{k-1} \mathbb{R}^n)$ by computing

$$D^{k-1}u_g(x) = \frac{d^{k-1}g}{dt^{k-1}}(x_n)v \otimes^{k-1} e_n = \frac{(\text{sgn } x_n)^{k-1}}{(k-1)!} f(x_n)v \otimes^{k-1} e_n$$

and arguing as for the local integrability of u_f . By the Deny–Lyons Lemma [64], we have that $D^j u_g$ is locally integrable for $0 \leq j \leq k-1$. We then write

$$\mathbb{B}\tilde{u}_g = \sum_{j=0}^{k-1} B_j [D^j u, D^{k-j} \rho],$$

where B_j are bilinear pairings depending on $\mathbb{B}$ only. Since ρ is smooth with compact support, it immediately follows that $\mathbb{B}\tilde{u}_g \in \mathcal{M}(\mathbb{R}^n, W)$. With $\alpha = 1 - s$,

$$\int_{\Sigma} |D^{k-1}\tilde{u}_g|^q \, d\mathcal{H}^{n-s} = \left(\frac{|v \otimes^{k-1} e_n|}{(k-1)!} \right)^q \int_S f(t)^q \, d\mathcal{H}^\alpha(t) = \infty,$$

which concludes the proof by Lemma 6.2.3.

Necessity of cancellation. Suppose that $\mathbb{B}$ is elliptic and non-cancelling. By [155, Lem. 2.5], there exists a map $u \in C^\infty(\mathbb{R}^n \setminus \{0\}, V) \cap L^1_{\text{loc}}(\mathbb{R}^n, V)$ such that $\mathbb{B}u = \delta_0 w$ for some $0 \neq w \in W$ and $D^{k-1}u$ is homogeneous of degree $1-n$. Since $w \neq 0$, $D^{k-1}u$ must be non-zero at a point $x_0 \in \mathbb{S}^{n-1}$. By continuity, $D^{k-1}u$ must be bounded away from 0 in a relatively open neighbourhood $\mathcal{N}$ of x_0 in $\mathbb{S}^{n-1}$. By homogeneity, this implies, for some $c > 0$, the bound

$$|D^{k-1}u(x)|^{\frac{n-s}{n-1}} \geq c|x|^{s-n} \quad \text{for all } x \in \mathcal{C},$$

where $\mathcal{C}$ is the open cone $\{tn : t > 0, n \in \mathcal{N}\}$.

Let $\Sigma \subset \mathcal{C}$ be as in Lemma 6.2.2. Lemma 6.2.3 implies that $D^{k-1}u$ is not $L^{\frac{n-s}{n-1}}$ -integrable with respect to $\mathcal{H}^{n-s} \llcorner \Sigma$. This contradicts the inequality and concludes the proof of this case, as the measure $\mathbb{B}u$ is clearly bounded. $\square$

Proof of Theorem 6.1.4. The fact that (d) implies (c) is proved in more generality in the Appendix (Theorem 6.5.1). The fact that (c) implies (a) is obvious. We have that (a) implies (b) from Proposition 6.1.3.

It remains to prove that (b) implies (d). To this end, let $\mathbb{B}$ be elliptic and suppose further that $k = 1$. Assume for contradiction that $\mathbb{B}$ is not $\mathbb{C}$ -elliptic, so that there exist $\nu, \nu' \in \mathbb{R}^n$ and $0 \neq v, v' \in V$ such that

$$\begin{aligned} \mathbb{B}[\nu]v &= \mathbb{B}[\nu']v', \\ \mathbb{B}[\nu]v' &= -\mathbb{B}[\nu']v. \end{aligned}$$

We then write

$$\begin{aligned} \mathbb{B}[a\nu + b\nu'](av + bv') &= a^2\mathbb{B}[\nu]v + ab\mathbb{B}[\nu]v' + ab\mathbb{B}[\nu']v + b^2\mathbb{B}[\nu']v' \\ &= (a^2 + b^2)\mathbb{B}[\nu]v, \end{aligned}$$

so that, by ellipticity of $\mathbb{B}$,

$$\bigcap_{0 \neq \zeta \in \text{span}\{\nu, \nu'\}} \text{im } \mathbb{B}[\zeta] \ni \mathbb{B}[\nu]v \neq 0,$$

so that $\mathbb{B}$ fails (SC). The proof is complete. $\square$

As a minor remark to the proof of (b) $\implies$ (d) for $k = 1$ presented above, we note that we heavily rely on the linearity in ξ of the map $(\xi, v) \mapsto \mathbb{B}[\xi]v$, which does not hold for higher order operators.

6.3. Proofs of Proposition 6.1.3 and Lemma 6.3.2

We will soon prove Proposition 6.1.3, for which we use the following Lemma of Fourier Analysis:

Lemma 6.3.1. *Let $\mathbb{B}$ as in (6.7) be elliptic, $2 \leq d \leq n$ be an integer, and $H \leq \mathbb{R}^n$ be a d -dimensional subspace of $\mathbb{R}^n$. Suppose that $w \in \text{im } \mathbb{B}[\xi]$ for all $0 \neq \xi \in H$.*

Then there exists a map $u \in C^\infty(\mathbb{R}^n \setminus H^\perp, V)$ and a $(1-d)$ -homogeneous map $F \in C^\infty(H \setminus \{0\}, V \odot^{k-1} \mathbb{R}^n)$ such that

$$\mathbb{B}u = \left(\mathcal{H}^{n-d} \llcorner H^\perp \right) w \quad \text{and} \quad D^{k-1}u(g+h) = F(h)$$

for all $h \in H, g \in H^\perp$.

Proof. We define, for $h \in H$, $g \in H^\perp$

$$\begin{aligned} u(h+g) &= \mathcal{F}^{-1} \left[\left(\mathcal{H}^d \llcorner H \right) \mathbb{B}^\dagger[\cdot]w \right] (g+h) \\ &= \int_{\mathbb{R}^n} e^{i(g+h) \cdot \xi} \mathbb{B}^\dagger[\xi]w \, d(\mathcal{H}^d \llcorner H)(\xi) \\ &= \int_H e^{i h \cdot \xi} \mathbb{B}^\dagger[\xi]w \, d\mathcal{H}^d(\xi) \\ &= \mathcal{F}_H^{-1}(\mathbb{B}^\dagger[\cdot]w)(h), \end{aligned}$$

where $\mathbb{B}^\dagger[\xi] = (\mathbb{B}^* \mathbb{B})^{-1}[\xi] \mathbb{B}^*[\xi]$ for $\xi \neq 0$ and the right-hand side is a tempered distribution in $C^\infty(H^\perp \setminus \{0\}, V)$ by the proof of [43, Lem. 2.1]. Using the same ideas we can extrapolate that

$$D^{k-1}u(h+g) = \mathcal{F}_H^{-1}(\xi \mapsto \mathbb{B}^\dagger[\xi]w \otimes^{k-1} \xi)(h) =: F(h),$$

where F has the required regularity and we can further use [43, Lem. 2.1] to see that F is $(1-d)$ -homogeneous.

To complete the proof, note that

$$\begin{aligned} \hat{u}(\xi) &= (\mathcal{H}^d \llcorner H) \mathbb{B}^\dagger[\xi]w \\ \widehat{\mathbb{B}u}(\xi) &= (\mathcal{H}^d \llcorner H) \mathbb{B}[\xi] \mathbb{B}^\dagger[\xi]w. \end{aligned}$$

By recalling that $\mathbb{B}[\xi] \mathbb{B}^\dagger[\xi]$ is the orthogonal projection onto $\text{im } \mathbb{B}[\xi]$, the conclusion follows. $\square$

Proof of Proposition 6.1.3. To prove ellipticity of $\mathbb{B}$, one can choose $g(t) = |t|^{k-\frac{3}{2}}$ in the proof of Theorem 6.1.1 and define $\tilde{u}_g$ in the same manner. It is then clear that $D^{k-1}\tilde{u}_g$ admits no trace on $\Sigma = e_n^\perp$.

If $\mathbb{B}$ is elliptic, we assume that the embedding holds and that (SC) fails for some 2-dimensional subspace $H \leq \mathbb{R}^n$ and $0 \neq w \in \text{im } \mathbb{B}[\xi]$ for all $\xi \in H \setminus \{0\}$. We let u be as in Lemma 6.3.1 with $d = 2$. Then

$$D^{k-1}u(h+g) = F(h) = D^{k-1} \mathcal{F}_H^{-1}(\mathbb{B}^\dagger[\cdot]w)(h) \quad \text{for } h \in H, g \in H^\perp$$

is (-1) -homogenous and smooth in $H \setminus \{0\}$. Note that if $F \equiv 0$, then $\delta_0 w = \mathbb{B}u = 0$, so that there exists $\eta \in \mathbb{S}^{n-1} \cap H$ such that $F(\eta) \neq 0$.

Up to a change of coordinate, we can assume that $H = \text{span}\{e_1, e_2\}$, where $\eta = e_2$. We choose $\Sigma = e_1^\perp$. Let Q be the unit cube in the new coordinates. For a function $\rho \in C_c^\infty(\mathbb{R}^n)$ so that $\rho = 1$ in Q , we have by the argument in the proof of Theorem 6.1.1 that ρu is admissible for the estimate with $|\mathbb{B}(\rho u)|(\mathbb{R}^n) < \infty$. We then have

$$\int_{Q \cap \Sigma} |D^{k-1}u| \, d\mathcal{H}^{n-1} = \int_0^1 \int_{Q \cap H^\perp} |F(te_2)| \, d\mathcal{H}^{n-2} \, dt = |F(e_2)| \int_0^1 \frac{dt}{t} = \infty,$$

which concludes the proof. $\square$

We conclude the section with an algebraic proof of a strengthening of [94, Lem. 3.2]:

Lemma 6.3.2. *Let $\mathbb{B}$ as in (6.7) be $\mathbb{C}$ -elliptic. Then $\mathbb{B}$ satisfies (SC).*

Proof. Assume for contradiction that there exist $0 \neq w \in W$ and a subspace $H \leq \mathbb{R}^n$ with $\dim H = 2$ such that $w \in \operatorname{im} \mathbb{B}[\xi]$ for all $\xi \in H \setminus \{0\}$. We abbreviate $\mathbb{B}^\dagger[\xi] = (\mathbb{B}^*[\xi]\mathbb{B}[\xi])^{-1}\mathbb{B}^*[\xi]$, which is such that $\mathbb{B}^\dagger[\xi]\mathbb{B}[\xi] = \operatorname{Id}_V$ for $\xi \in \mathbb{R}^n \setminus \{0\}$. We record that $\mathbb{B}[\xi]\mathbb{B}^\dagger[\xi]$ is the orthogonal projection on $\operatorname{im} \mathbb{B}[\xi]$.

Let $e_i \in H$, $i = 1, 2$, be linearly independent. We choose $v \in V$ such that $\mathbb{B}_0[\cdot] \neq 0$, where we define

$$\mathbb{B}_0[\xi] = v \cdot \mathbb{B}^\dagger[\xi]w \quad \text{for } \xi \in H.$$

This is possible since $\mathbb{B}^\dagger[\xi]w$ is never 0 by ellipticity. Since H is a plane through 0 and $\mathbb{B}^\dagger[\cdot]$ is a $(-k)$ -homogeneous rational function, it follows that $\mathbb{B}_0[\cdot]$ is a $(-k)$ -homogeneous, rational function. Thus, we can write $\mathbb{B}_0[\xi] = P(\xi)/Q(\xi)$ for coprime, homogeneous polynomials P and Q on H .

We recall from [172] that there exists an integer $d \geq k$ and a homogeneous linear differential operator $\tilde{\mathbb{B}}$ such that

$$D^d = \tilde{\mathbb{B}} \circ \mathbb{B}.$$

In other words, all homogeneous polynomials of degree d can be written as multipliers of the symbol map $\mathbb{B}[\cdot]$. It follows that we can find scalar operators $\tilde{\mathbb{B}}_i$ such that

$$(\xi \cdot e_i)^d v^* = \tilde{\mathbb{B}}_i[\xi]\mathbb{B}[\xi], \quad i = 1, 2,$$

which then implies

$$\tilde{\mathbb{B}}_i[\xi]w = \tilde{\mathbb{B}}_i[\xi]\mathbb{B}[\xi]\mathbb{B}^\dagger[\xi]w = (\xi \cdot e_i)^d (v \cdot \mathbb{B}^\dagger[\xi]w) = (\xi \cdot e_i)^d P(\xi)/Q(\xi).$$

Therefore, $Q(\xi)\tilde{\mathbb{B}}_i[\xi]w = (\xi \cdot e_i)^d P(\xi)$ for all $\xi \in H$ from which it follows that Q divides both $(\xi \cdot e_i)^d$, $i = 1, 2$. Therefore Q is constant, which contradicts the $(-k)$ -homogeneity of $\mathbb{B}_0[\cdot] \neq 0$. $\square$

6.4. Function space application: maps of bounded deformation

In this section we prove Corollary 6.1.2, which states that maps of bounded deformation admit traces on s -codimensional sets for $s \in (0, 1)$. To this end, recall the spaces $\operatorname{BV}^\mathbb{B}$ of maps of bounded $\mathbb{B}$ -variation defined in [45, 94] as

$$\operatorname{BV}^\mathbb{B}(\Omega) = \{u \in L^1(\Omega, V) : \mathbb{B}u \in \mathcal{M}(\Omega, W)\},$$

which are Banach spaces when endowed with the norm $u \mapsto |\mathbb{B}u|(\Omega) + \|u\|_{L^1(\Omega)}$. Here $\Omega \subset \mathbb{R}^n$ denotes an open set. Examples of $\operatorname{BV}^\mathbb{B}$ spaces are BV (resp. BV^k , BD), in which case $\mathbb{B} = D$ (resp. $\mathbb{B} = D^k$, $\mathbb{B} = \mathcal{E}$).

Since maps in BV are more irregular than smooth- or $W^{1,1}$ -maps, to extend our main results to $\operatorname{BV}^\mathbb{B}$ -spaces requires careful density arguments. For subsets of co-dimension one, i.e., corresponding to Theorem 6.1.1, defining trace operators was done in [45, Sec. 4] and we do not repeat the argument here. Instead, we focus on sets of co-dimension $s \in (0, 1)$ and give a proof of a generalization of Corollary 6.1.2:

Lemma 6.4.1. *Let $\mathbb{B}$ be a first order differential operator of the form (6.7), $0 < s < 1$. Then $\mathbb{B}$ is elliptic and canceling if and only if for any $\Sigma \subset \mathbb{R}^n$ which satisfies (6.2) there exists a linear interior trace operator*

$$\text{Tr}_\Sigma: \text{BV}^\mathbb{B}(\mathbb{R}^n) \rightarrow L^{\frac{n-s}{n-1}}(\Sigma; \mathcal{H}^{n-s})$$

such that

$$\|\text{Tr}_\Sigma u\|_{L^{\frac{n-s}{n-1}}(\Sigma)} \leq C|\mathbb{B}u|(\mathbb{R}^n)$$

for all $u \in \text{BV}^\mathbb{B}(\mathbb{R}^n)$.

We restricted considerations to first order operators only to simplify notation. The proof can be immediately extended to k -th order operators by replacing u by $D^{k-1}u$.

Proof. Necessity of EC was proved in the course of proving Theorem 6.1.1. Suppose now that $\mathbb{B}$ is elliptic and canceling. Our first concern is to show that $\mathcal{H}^{n-s}$ -a.e. $x_0 \in \Sigma$ is a Lebesgue continuity point for u . To this end, we recall from [201, Thm. 8.1] that if $\mathbb{B}$ is EC, then for each $1 < p < \frac{n}{n-1}$ there exists constant $C > 0$ such that

$$\|u\|_{W^{\theta,p}(\mathbb{R}^n)} \leq C\|\mathbb{B}u\|_{L^1(\mathbb{R}^n, W)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n, V).$$

Here and in what follows, $\theta = \theta_p = n(\frac{1}{p} - 1) + 1$. This consequently yields $\text{BV}^\mathbb{B}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n, V)$: Let $u \in \text{BV}^\mathbb{B}(\mathbb{R}^n)$ and choose according to [45, Thm. 2.8] a sequence $(u_k) \subset C_c^\infty(\mathbb{R}^n, V)$ such that $u_k \rightarrow u$ in $L^1(\mathbb{R}^n, V)$ and $|\mathbb{B}u_k|(\mathbb{R}^n) \rightarrow |\mathbb{B}u|(\mathbb{R}^n)$. Passing to a non-relabelled subsequence, we can assume that $u_k \rightarrow u$ $\mathcal{L}^n$ -a.e., and thus, by Fatou's lemma, interpolation of L^p -spaces and [201, Thm. 1.3, Thm. 8.1]:

$$\begin{aligned} \|u\|_{W^{\theta,p}(\mathbb{R}^n, V)} &= \liminf_{k \rightarrow \infty} \|u_k\|_{L^p(\mathbb{R}^n, V)} + [\liminf_{k \rightarrow \infty} u_k]_{W^{\theta,p}(\mathbb{R}^n, V)} \\ &\leq \liminf_{k \rightarrow \infty} \left(\|u_k\|_{L^p(\mathbb{R}^n, V)} + [u_k]_{W^{\theta,p}(\mathbb{R}^n, V)} \right) \\ &\leq C \liminf_{k \rightarrow \infty} \left(\|u_k\|_{L^1(\mathbb{R}^n, V)} + \|u_k\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n, V)} + \|\mathbb{B}u_k\|_{L^1(\mathbb{R}^n, W)} \right) \\ &\leq C \liminf_{k \rightarrow \infty} \left(\|u_k\|_{L^1(\mathbb{R}^n, V)} + \|\mathbb{B}u_k\|_{L^1(\mathbb{R}^n, W)} \right) = C\|u\|_{\text{BV}^\mathbb{B}(\mathbb{R}^n)}. \end{aligned}$$

We choose $1 < p < \frac{n}{n-1}$ such that $0 < s < n - p(n-1)$. Then, by the above, $u \in W^{\theta,p}(\mathbb{R}^n)$ and thus

$$\dim_{\mathcal{H}} \left\{ x \in \mathbb{R}^n : \liminf_{r \downarrow 0} \int_{B_r(x)} |u(y) - (u)_{x,r}| dy > 0 \right\} \leq n - \theta p,$$

where $(\cdot)_{x,r}$ denotes the average on $B_r(x)$. In particular, $\mathcal{H}^{n-s}$ -a.e. $x_0 \in \Sigma$ is a Lebesgue point for u . Denoting $u^*(x_0)$ the precise representative of u at $x_0 \in \Sigma$, we may thus define the interior trace $\text{Tr}(u)(x_0)$ for $\mathcal{H}^{n-s}$ -a.e. $x_0 \in \Sigma$ by $\text{Tr}(u)(x_0) := u^*(x_0)$. Now let (ρ_ε) be a sequence of rescaled standard mollifiers.

Then $\mathbb{B}(\rho_\varepsilon \star u) \rightarrow \mathbb{B}u$ strictly as $\varepsilon \downarrow 0$ and, for any Lebesgue point x_0 of u , $(\rho_\varepsilon \star u)(x_0) \rightarrow u^*(x_0)$. Thus, by Theorem 6.1.1 (a) and Fatou's lemma,

$$\begin{aligned} \left(\int_{\Sigma} |\operatorname{Tr}(u)|^{\frac{n-s}{n-1}} d\mathcal{H}^{n-s} \right)^{\frac{n-1}{n-s}} &\leq \liminf_{\varepsilon \downarrow 0} \left(\int_{\Sigma} |\rho_\varepsilon \star u|^{\frac{n-s}{n-1}} d\mathcal{H}^{n-s} \right)^{\frac{n-1}{n-s}} \\ &\leq C \liminf_{\varepsilon \searrow 0} \int_{\mathbb{R}^n} |\rho_\varepsilon \star \mathbb{B}u| dx \leq C |\mathbb{B}u|(\mathbb{R}^n). \end{aligned}$$

The proof is complete. $\square$

We can see from the above proof that indeed the operator $\operatorname{Tr}_{\Sigma}$ does not detect the jumps of u and, of course, the proof fails when considering traces on sets of co-dimension one, i.e., $s = 1$. In that case, one can argue as in [18] and define one-sided limits of u across Σ . We only know how to formalize this type of argument for $\mathbb{C}$ -elliptic operators.

6.5. Appendix

We now prove a partial extension of the main result in [45]:

Theorem 6.5.1 (Exterior traces). *Let $\mathbb{B}$ be as in (6.7). Then $\mathbb{B}$ is $\mathbb{C}$ -elliptic if and only if there exists a constant $C > 0$ such that the estimate*

$$\int_{\Sigma} |D^{k-1}u| d\mathcal{H}^{n-1} \leq C \int_{\Sigma^+} |\mathbb{B}u| dx \quad \text{for } u \in C_c^\infty(B, V)$$

holds for all $(n-1)$ -dimensional $\Sigma \leq \mathbb{R}^n$. Here Σ^+ denotes a half-space bounded by Σ .

Proof. Assume that $\mathbb{B}$ is $\mathbb{C}$ -elliptic. The trace inequality follows by an extension of the ideas in [45], coupled with the Poincaré-type inequality [94, Prop. 4.2]. The presentation here is very streamlined and we refer the reader to [45, Sec. 3-4] and [94, Sec. 4] for more detail, particularly, the introduction of basic tools used in the following arguments.

We proceed by noting that there is no loss of generality in assuming that $\Sigma = \mathbb{R}^{n-1}$. We will use a coordinate notation $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. By a standard scaling argument, it suffices to prove that

$$\int_{\mathbb{R}^{n-1}} |D^{k-1}u| dx' \lesssim \int_{\mathbb{R}_+^n} |\mathbb{B}u| dx \quad \text{for all } u \in C_c^\infty(Q, V),$$

where $Q = (-1, 1)^n$ denotes the unit cube and $\mathbb{R}_+^n = \{x_n > 0\}$. We will write $e_n = (0, 1)$ for the unit normal to Σ .

We now set the geometric scene: For $j = 2, 3, \dots$, we write $\mathcal{S}^j$ for the strip $\{2^j \leq x_n \leq 2^{j+1}\}$. We cover $\mathcal{S}^j \cap Q$ with a collection of dyadic closed cubes $\{S_i^j\}_{i=1}^{2^{j(n-1)}}$ of side-length 2^{-j} which have faces parallel to Q and only intersect at faces. We write Q_i^j for the open cubes that are obtained as follows: Translate S_i^j by $-3 \cdot 2^{-j-1}e_n$ and dilate the interior of the resulting cube by a factor of $\frac{8}{7}$. In particular, the union $\mathcal{Q}^j = \bigcup_i Q_i^j$ covers $\Sigma \cap Q$ and the rectangle $R^j = \{x \in Q: -2^{-j-1} \leq x_n \leq 2^{j+1}\}$.

For each $j \geq 2$ we define partitions of unity $\{\varphi_i^j\}_i$ associated with $\{Q_i^j\}_i$ which satisfy $\sum_i \varphi_i^j = 1$ in R^j and $|D^l \varphi_i^j| \leq c 2^{jl}$ for $l = 0, 1, \dots, k$. To localize near Σ , we will use functions $\rho_j \in C^\infty(\mathbb{R}^n)$ such that $\rho_j = 1$ in Q^j , $\rho_j(x) = 0$ if $x_n \geq 2^{-j}$, and $|D^l \rho_j| \leq c 2^{jl}$ for $l = 0, 1, \dots, k$.

Let $\pi_C: L^2(C, V) \rightarrow \ker \mathbb{B}$ denote the L^2 -orthogonal projection onto $\ker \mathbb{B}$, where $C \subset \mathbb{R}^n$ is a cube. We abbreviate $\pi_i^j := \pi_{Q_i^j}$.

We can now consider the trace approximation operator

$$T_j u = \rho_j \sum_i \varphi_i^j \pi_i^j u + (1 - \rho_j) u.$$

As in [45, Sec. 4], we expect that $T_j u \rightarrow u$ is in several topologies, for which it is easier to first show that $(T_j u)_j$ is Cauchy. To this end, we record that

$$\begin{aligned} T_{j+1} u - T_j u &= (\rho_j - \rho_{j+1}) \left(u - \sum_i \varphi_i^j \pi_i^j u \right) + \rho_{j+1} \left(\sum_m \varphi_m^{j+1} \pi_m^{j+1} u - \sum_i \varphi_i^j \pi_i^j u \right) \\ &= (\rho_j - \rho_{j+1}) \sum_i \varphi_i^j (u - \pi_i^j u) + \rho_{j+1} \left(\sum_{i,m} \varphi_m^{j+1} \varphi_i^j (\pi_m^{j+1} u - \pi_i^j u) \right), \end{aligned}$$

where in the second equality we used the summability to 1 of both partitions of unity. We will use this identity near Σ , where $\rho_j = 1 = \rho_{j+1}$, so the first term vanishes.

The proof of the trace inequality is divided in a few steps:

Step I. We show that $(D^{k-1} T_j u)_j$ is Cauchy in $L^1(\Sigma)$. We note that in Q^{j+1} we have that

$$\begin{aligned} \mathbf{T} &= D^{k-1} (T_{j+1} u - T_j u) = \sum_{i,m} D^{k-1} [\varphi_m^{j+1} \varphi_i^j (\pi_m^{j+1} u - \pi_i^j u)] \\ &= \sum_{i,m} \sum_{l=0}^{k-1} \mathbf{B}_l [D^{k-l} (\varphi_m^{j+1} \varphi_i^j); D^l (\pi_m^{j+1} u - \pi_i^j u)], \end{aligned}$$

where $\mathbf{B}_l[\cdot; \cdot]$ are bi-linear pairings given by the Leibniz rule that depend only on $\mathbb{B}$. We can estimate

$$\begin{aligned} \|\mathbf{T}\|_{L^1(\Sigma)} &\leq c \sum_{i,m: Q_i^j \cap Q_m^{j+1} \neq \emptyset} 2^{j(k-1-l)} \|D^l (\pi_m^{j+1} u - \pi_i^j u)\|_{L^1(\Sigma \cap Q_i^j)} \\ &\leq c \sum_{i,m: Q_i^j \cap Q_m^{j+1} \neq \emptyset} 2^{j(k-l-n)} \|D^l (\pi_m^{j+1} u - \pi_i^j u)\|_{L^\infty(\Sigma \cap Q_i^j)} \\ &\leq c \sum_{i,m: Q_i^j \cap Q_m^{j+1} \neq \emptyset} 2^{j(k-l-n)} \|D^l (\pi_m^{j+1} u - \pi_i^j u)\|_{L^\infty(\bar{\Sigma} \cap Q_i^j)} \\ &\leq c \sum_{i,m: Q_i^j \cap Q_m^{j+1} \neq \emptyset} 2^{j(k-l)} \|D^l (\pi_m^{j+1} u - \pi_i^j u)\|_{L^1(\Sigma \cap Q_i^j)} \\ &\leq c \sum_{i,m: Q_i^j \cap Q_m^{j+1} \neq \emptyset} 2^{j(k-l)} \|D^l (\pi_m^{j+1} u - \pi_i^j u)\|_{L^1(S_i^j)} \end{aligned}$$

$$\leq c \|\mathbb{B}u\|_{L^1(\mathcal{S}^j)}.$$

In the fourth inequality we used the fact that on the space of polynomials of fixed maximal degree, all norms are equivalent, with appropriate scaling. For the penultimate inequality we used [94, Lem. 4.4]. The second inequality is simply Hölder's inequality in $\Sigma \cap Q_i^j$. We then have that

$$\|D^{k-1}(T_{j+s}u - T_ju)\|_{L^1(\Sigma)} \leq \|\mathbb{B}u\|_{L^1(2^{-j-s} \leq x_n \leq 2^{-j})}.$$

Since $\mathbb{B}u \in C_c^\infty(Q, W)$, the claim of Step I follows.

Step II. We show that $D^{k-1}T_ju \rightarrow D^{k-1}u$ in $L^1(\Sigma)$. We have that $(D^{k-1}T_ju)_j$ converges in $L^1(\Sigma)$, but it is not yet clear that the limit is $D^{k-1}u \llcorner \Sigma$. To prove that this is the case, we will show that $D^{k-1}T_ju \rightarrow u$ uniformly in $\bar{\Sigma}^+$. We write

$$D^{k-1}(u - T_ju) = \sum_i D^{k-1} \left[(\rho_j \varphi_i^j)(u - \pi_i^j u) \right].$$

By local finiteness of the cover $\{Q_i^j\}_i$, we have that

$$\begin{aligned} \|D^{k-1}(u - T_ju)\|_{L^\infty(\Sigma^+)} &\leq c \max_i \left\| D^{k-1} \left[(\rho_j \varphi_i^j)(u - \pi_i^j u) \right] \right\|_{L^\infty(Q_i^j)} \\ &\leq c \sum_{l=0}^{k-1} 2^{j(k-l-1)} \|D^l(u - \pi_i^j u)\|_{L^\infty(Q_i^j)} \\ &\leq c \sum_{l=0}^{k-1} 2^{j(k-l-1)} \left(\|D^l(u - \pi_{Q_i^j} u)\|_{L^\infty(Q_i^j)} \right. \\ &\quad \left. + \|D^l(\pi_{Q_i^j} u - \pi_{S_i^j} u)\|_{L^\infty(Q_i^j)} \right) \\ &\leq c 2^{-j} \|\mathbb{B}u\|_{L^\infty(Q_i^j)} \leq c 2^{-j} \|\mathbb{B}u\|_{L^\infty(\Sigma^+)} \rightarrow 0, \end{aligned}$$

which completes the proof of Step II. In the fourth inequality we used [94, Prop. 4.2] and [94, Lem. 4.4].

Step III. It remains to prove the trace inequality. In view of Step II, we have that

$$D^{k-1}u = L^1(\Sigma)_- \lim_{s \rightarrow \infty} \sum_{j=2}^s D^{k-1}(T_{j+1}u - T_ju),$$

so that

$$\begin{aligned} \|D^{k-1}u\|_{L^1(\Sigma)} &\leq \sum_{j \geq 2} \|D^{k-1}(T_{j+1}u - T_ju)\|_{L^1(\Sigma)} \leq \sum_{j \geq 2} \|\mathbb{B}u\|_{L^1(\mathcal{S}^j)} \\ &= c \|\mathbb{B}u\|_{L^1(\Sigma^+)}, \end{aligned}$$

which completes the proof of sufficiency of $\mathbb{C}$ -ellipticity for the trace inequality.

To complete the proof of the theorem, it remains to prove necessity of $\mathbb{C}$ -ellipticity. Necessity of ellipticity follows by a repetition of the arguments in the proof of necessity of ellipticity for Proposition 6.1.3. Assume now that $\mathbb{B}$ is not $\mathbb{C}$ -elliptic, so that there exist non-zero $\eta, \nu \in \mathbb{R}^n$ and $v \in V + iV$ such that

$\mathbb{B}[\eta + i\nu]v = 0$. There is no loss of generality in choosing $\Sigma^+ = \{x \in \mathbb{R}^n : x \cdot \nu > 0\}$. We choose a holomorphic branch of $\log : \mathbb{C} \setminus i(-\infty, 0] \rightarrow \mathbb{C}$ and let f to be such that $f^{(k-1)}(z) = z^{-1}$, so f is holomorphic in $\mathbb{C} \setminus i(-\infty, 0]$. Such a map exists by standard results of complex analysis. Define

$$u(x) = f(x \cdot \eta + i x \cdot \nu)v$$

Though u is complex-valued, which we do not per se allow for, the following argument follows also for whichever of $\Re u$ and $\Im u$ satisfies the final estimate (one of them must).

We note that $\mathbb{B}u(x) = 0$ for $x \in \Sigma^+$ by the proof of [94, Prop. 3.1], so ρu is admissible for the estimate, where ρ is a smooth cut-off function equal to one in the unit cube Q . Again by the proof of [94, Prop. 3.1], we have that

$$D^{k-1}u(x) = f^{(k-1)}(x \cdot \xi)v \otimes^{k-1} \xi \quad \text{for } x \in \Sigma^+.$$

Failure of the estimate follows since

$$\int_{\Sigma \cap Q} |D^{k-1}u| d\mathcal{H}^{n-1} \sim \int_{[0,1]^{n-2}} \int_0^1 x_1^{-1} dx_1 d\mathcal{H}^{n-2},$$

which is clearly infinite. □

Do note that the construction performed to obtain necessity of $\mathbb{C}$ -ellipticity in the proof of Theorem 6.5.1 cannot be performed to prove necessity of $\mathbb{C}$ -ellipticity for the estimate (6.9) in Proposition 6.1.3 precisely unless $k = 1$. This gives some weak evidence that Open Problem 6.1.5 could be settled in the positive.

CHAPTER 7

Embeddings for $\mathbb{B}$ -weakly differentiable functions on domains

This chapter consists of joint work with F. Gmeineder.

7.1. Introduction

A known principle in harmonic analysis is that strong-type L^1 -estimates are notoriously delicate to obtain. For example, singular integrals and Riesz potentials are only bounded from L^1 into a weak-type space, which contrasts the case of L^p -spaces, $p > 1$. To note that these L^1 -estimates of weak-type are sharp, one simply tests the inequalities with an approximation of the identity.

Historically, this discrepancy can be observed already from the original proof of the Sobolev inequality for $1 < p < n$ and $p^* = \frac{np}{n-p}$,

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c \|Du\|_{L^p(\mathbb{R}^n)} \quad (7.1)$$

for $u \in C_c^\infty(\mathbb{R}^n)$. The technique of proof in [173] resembles proving boundedness of the Riesz potential I_1 between L^p and $L^{np/(n-p)}$, but it is in no way adaptable to the $p = 1$ case. It was much later that GAGLIARDO [91] and NIRENBERG [147] independently showed with new methods that (7.1) holds also for $p = 1$, in particular showing that the specific vectorial structure of the gradient operator allows for a strong-type estimate, despite unboundedness of I_1 between L^1 and L^{1^*} .

Recalling the fact attributable to Calderón and Zygmund [49] that

$$\|D^k u\|_{L^p(\mathbb{R}^n)} \leq c \|\mathbb{B}u\|_{L^p(\mathbb{R}^n)} \quad (7.2)$$

for k -th order elliptic operators $\mathbb{B}$ and $1 < p < \infty$, it may seem possible that the vectorial structure of $\mathbb{B}$ may also compensate for unboundedness of singular integrals on L^1 . This fact was disproved by ORNSTEIN in [148] (see also [113]), where it is shown that the estimate (7.2) holds for $p = 1$ on if $|D^k u| \leq c |\mathbb{B}u|$ pointwisely for all $u \in C_c^\infty$.

However, it remained possible that strong-type L^1 -estimates for lower order derivatives can be deduced. Indeed, it was proved by STRAUSS in [179] that (7.1) holds for $p = 1$ when the L^1 -norm of Du is replaced by the weaker quantity $\|\mathcal{E}u\|_{L^1}$. Here $\mathcal{E}u = \frac{1}{2}(Du + (Du)^t)$ denotes the symmetrized gradient of u . More

recently, it was proved by BOURGAIN and BREZIS in [36, 38] that the estimate

$$\|Du\|_{L^{1*}} \leq c\|f\|_{L^1} \quad \text{for} \quad \Delta u = f$$

hold for divergence-free $f \in C_c^\infty(\mathbb{R}^n)$ if $n \geq 2$. This and substantial contributions in [39, 40, 41, 35, 37, 36, 38, 195, 196, 199, 200] led to the remarkable characterization by VAN SCHAFTINGEN [201] of all k -homogeneous linear differential operators $\mathbb{B}$ such that

$$\|D^{k-1}u\|_{L^{1*}(\mathbb{R}^n)} \leq c\|\mathbb{B}u\|_{L^1(\mathbb{R}^n)} \quad (7.3)$$

for all $u \in C_c^\infty$. The class of operators $\mathbb{B}$ for which (7.3) holds is that of *elliptic* and *canceling* operators (EC). Both of these assumptions are defined in terms of the symbol map of the operator $\mathbb{B}$, the definition of which we now recall. We will represent k -homogeneous linear differential operators with constant coefficients on $\mathbb{R}^n$ from V to W as

$$\mathbb{B}u = \sum_{|\alpha|=k} B_\alpha \partial^\alpha u, \quad u: \mathbb{R}^n \rightarrow V, \quad (7.4)$$

where $B_\alpha \in \text{Lin}(V, W)$ are fixed linear mappings between two finite dimensional normed real vector spaces V and W . The symbol map is defined as

$$\mathbb{B}[\cdot]: \mathbb{R}^n \rightarrow \mathcal{L}(V, W), \quad \mathbb{B}[\xi]v = \sum_{|\alpha|=k} \xi^\alpha B_\alpha v,$$

defined for $\xi \in \mathbb{R}^n$, $v \in V$. Algebraically, (overdetermined) *ellipticity* is defined by injectivity of the symbol map $\mathbb{B}[\xi]$ for all non-zero ξ , whereas *cancellation*, introduced in [201, Def. 1.2], is defined by

$$\bigcap_{\xi \neq 0} \text{im } \mathbb{B}[\xi] = \{0\}. \quad (7.5)$$

Analytically, ellipticity is equivalent to the classical estimate (7.2). Surprisingly and interestingly, cancellation is equivalent to non-admissibility for (7.3) of any approximation of the identity, in the sense that if $\mathbb{B}u_\varepsilon = \varphi_\varepsilon w$ for standard mollifiers $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ and $w \in W$, then $w \in \text{im } \mathbb{B}[\xi]$ for all $\xi \neq 0$.

An overarching overview of these and other recent developments on L^1 -estimates can be found in [202], where it is also asked in Open Problem 3 whether, under suitable complementing boundary conditions, one can develop global strong-type estimates on domains. It is implicitly conjectured that estimates on smooth domains $\Omega \subset \mathbb{R}^n$ such as

$$\|D^{k-1}u\|_{L^{1*}(\Omega)} \leq \|\mathbb{B}u\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)},$$

provided that $\mathbb{B}$ is EC and $u \in C^\infty(\bar{\Omega}, V)$ satisfy $\mathbb{B}_j u = 0$ on $\partial\Omega$, where $\mathbb{B}_j$ is a (finite collection of) linear differential operator(s) defined on $\partial\Omega$ that satisfy the Lopatinskiĭ–Shapiro Complementing Conditions. Such a result would provide a reasonable analogue of the results in [126, 7, 8, 100] to the case $p = 1$, in spite of Ornstein’s Non-inequality.

The aim of this chapter is to confirm this expectation in the case when $\mathbb{B}_j \equiv 0$ (“no boundary condition”) and Ω is a ball (whereas VAN SCHAFTINGEN’s result [201, Thm. 1.3] essentially deals with the antipodal case when $\mathbb{B}_j = \partial \nu^j$, $j = 0 \dots k-1$, i.e., “all boundary conditions”). We emphasize that in the present situation the geometry of $\partial\Omega$ is not the foremost problem, as is the extendibility of functions $u: \Omega \rightarrow V$ to some $v: \mathbb{R}^n \rightarrow V$ while ensuring that $\mathbb{B}v \in L^1(\mathbb{R}^n, V)$ boundedly. The reader can easily use our arguments to cover the case when Ω is a bounded Lipschitz domain.

The complementing conditions for smooth domains for an elliptic operator $\mathbb{B}$ and identically zero boundary conditions [102, Def. 20.1.1] can be rephrased as

$$\mathbb{B}[\xi] \in \text{Lin}(V + iV, W + iW) \quad \text{is injective for all } \xi \in \mathbb{C}^n \setminus \{0\}. \quad (7.6)$$

This condition, referred to as $\mathbb{C}$ -ellipticity in [45], and attributable to ARONSZAJN [20] (at least in the case of scalar-valued maps), was introduced to characterize operators $\mathbb{B}$ such that the local variant of (7.2) holds, i.e.,

$$\|D^k u\|_{L^p(B)} \leq c (\|\mathbb{B}u\|_{L^p(B)} + \|u\|_{L^p(B)}) \quad (7.7)$$

holds for $u \in C^\infty(\bar{B}, V)$ (here $1 < p < \infty$); see also [171, 172]. By Ornstein’s Non-inequality, no such estimate is possible for $p = 1$, but, inspired by VAN SCHAFTINGEN’s Theorem and Open Problem, we will prove that $\mathbb{C}$ -ellipticity of $\mathbb{B}$ is equivalent to the estimate

$$\|D^{k-1} u\|_{L^{n/(n-1)}(B)} \leq c (\|\mathbb{B}u\|_{L^1(B)} + \|u\|_{L^1(B)}) \quad (7.8)$$

for $u \in C_c^\infty(\bar{B}, V)$. In particular, we recover the Gagliardo–Nirenberg–Sobolev inequality on domains and the Korn–Sobolev inequality [178, Prop. 1.2], due to STRANG and TEMAM.

To formally state our results, we prefer to use the functional framework of $\mathbb{B}$ -weakly differentiable functions and define, in the spirit of [45], the space $W^{\mathbb{B},1}(B)$ as the space of $u \in L^1(B, V)$ such that $\mathbb{B}u \in L^1(B, W)$ with the obvious norm. For better readability of the chapter, we firstly state the result for first order operators:

Theorem 7.1.1. *Let $\mathbb{B}$ be as in (7.4), $k = 1$, $n > 1$. The following are equivalent:*

- (a) $\mathbb{B}$ is $\mathbb{C}$ -elliptic.
- (b) $W^{\mathbb{B},1}(B) \hookrightarrow L^{\frac{n}{n-1}}(B, V)$.
- (c) $W^{\mathbb{B},1}(B) \hookrightarrow L^p(B, V)$ for some $1 < p \leq \frac{n}{n-1}$.
- (d) $W^{\mathbb{B},1}(B) \hookrightarrow L^q(B, V)$ for all $1 \leq q < \frac{n}{n-1}$.
- (e) $W^{\mathbb{B},1}(B) \hookrightarrow L^1(B, V)$.

To prove this, a crucial step is to show that $\mathbb{C}$ -ellipticity implies cancellation. Then we extend to full-space and employ (7.3). The fact that the sub-critical embedding in (d) is compact generalizes the well-known result for BD (i.e., for

$\mathbb{B} = \mathcal{E}$; see [88, 178, 180]) and is achieved by a careful application of the Riesz–Kolmogorov criterion.

It is a priori far from obvious how can one connect the algebraic definitions of $\mathbb{C}$ -ellipticity (7.6) and cancellation (7.5). We, however, consider the analytic characterization of cancellation [201, Prop. 6.1] and a consequence of SMITH’s representation formulas [172] (the kernel of a $\mathbb{C}$ -elliptic operator is finite dimensional) to build a bridge in Lemma 7.3.2. In fact, it was already observed in [45] for first order operators that $\mathbb{C}$ -ellipticity of $\mathbb{B}$ is equivalent with

$$\dim\{u \in \mathcal{D}'(\mathbb{R}^n, V) : \mathbb{B}u = 0\} < \infty. \quad (\text{FDN})$$

In Proposition 7.3.1 we will give a short proof of the fact that $\mathbb{C}$ -ellipticity and FDN are equivalent for operators of arbitrary order. Henceforth, we will thus use “FDN” and “ $\mathbb{C}$ -elliptic” interchangeably.

On the other hand, we also show that the implication of EC by $\mathbb{C}$ -ellipticity is strict by considering the first order operator

$$\mathbb{B}u = (\partial_1 u_1 - \partial_2 u_2, \partial_2 u_1 + \partial_1 u_2, \partial_3 u_1, \partial_3 u_2) \quad \text{for } u: \mathbb{R}^3 \rightarrow \mathbb{R}^2.$$

In particular, for this operator, there are maps in $W^{\mathbb{B},1}(B)$ that are locally $\frac{n}{n-1}$ -integrable, but are *not* L^p -integrable up to the boundary for any $p > 1$. Nor do they admit traces in $L^1(\partial, B, V)$ by [45, Thm. 4.18]. We further expand on these points in Section 7.3.1.

In Section 7.3 we will give a more comprehensive comparison of the two conditions, depending on n , $N = \dim V$ and the order k of $\mathbb{B}$. We summarize our findings in a table:

	$N = 1$	$N \geq 2$
$n = 2$	$k = 1: \mathbb{E} \Rightarrow \text{FDN}$	$k = 1: \mathbb{E} \mathbb{C} \Rightarrow \text{FDN}$
	$k = 2: \mathbb{E} \mathbb{C} \Rightarrow \text{FDN}$	$k \geq 2: \mathbb{E} \mathbb{C} \not\Rightarrow \text{FDN}$
	$k \geq 3: \mathbb{E} \mathbb{C} \not\Rightarrow \text{FDN}$	
$n \geq 3$	$k = 1: \mathbb{E} \Rightarrow \text{FDN}$	$\mathbb{E} \mathbb{C} \not\Rightarrow \text{FDN}$
	$k \geq 2: \mathbb{E} \mathbb{C} \not\Rightarrow \text{FDN}$	

The matter of the extension is quite delicate, as we cannot formulate a simple proof as in [172, 109] due to lack of boundedness of singular integrals on L^1 (cp. Lemma 7.5.7). Instead, we resort to the involved technique introduced by JONES [107]. The ideas for adapting this construction for FDN operators instead originates in [45], where a related result was proved for first order operators. Namely, it was shown that maps in $W^{\mathbb{B},1}(B)$ admit L^1 -traces on ∂B if and only if $\mathbb{B}$ has FDN. In particular, their result and ours point in the direction that FDN is a suitable assumption for estimates near and on the boundary also in the case $p = 1$.

We collect the independently interesting, intermediate facts that we will prove in:

Theorem 7.1.2. *Let $\mathbb{B}$ be as in (7.4), $n > 1$. Then $\mathbb{B}$ has FDN if and only if $\mathbb{B}$ is $\mathbb{C}$ -elliptic. Moreover, if $\mathbb{B}$ has FDN, then $\mathbb{B}$ is cancelling and there exists a bounded, linear extension operator $E_B: W^{\mathbb{B},1}(B) \rightarrow W^{\mathbb{B},1}(\mathbb{R}^n)$.*

Using the tools from Theorem 7.1.2, we can refine our result on fractional scales, thereby obtaining the local versions of the embeddings in [201, Thm. 8.1, Thm. 8.4]:

Theorem 7.1.3. *Let $\mathbb{B}$ be as in (7.4), $s \in [k-1, k)$, $q \in (1, \infty)$. Then $\mathbb{B}$ has FDN if and only if there exists $c > 0$ such that*

$$\|u\|_{B_q^{s, \frac{n}{n-k+s}}(B, V)} \leq c \left(\|\mathbb{B}u\|_{L^1(B, W)} + \|u\|_{L^1(B, V)} \right)$$

for all $u \in C^\infty(\bar{B}, V)$.

Here, the Besov spaces on domains are defined as in [68, Sec. 2]. We obtain the embeddings $W^{\mathbb{B}, 1}(B) \hookrightarrow W^{s, n/(n-k+s)}(B, V)$ if we choose $q = n/(n-k+s)$ (cp. [201, Thm. 8.1]) and $W^{\mathbb{B}, 1}(B) \hookrightarrow W^{k-1, n/(n-1)}(B, V)$ if we further choose $s = k-1$ (cp. [201, Thm. 1.3]). The novelty of Theorems 7.1.1 and 7.1.3 comes from the fact that, up to our knowledge, there are only a few examples of L^1 -estimates near the boundary that go in the direction of [202, Open Prob. 3] (cp. [48]). The result of Theorem 7.1.3 is sharp on the fractional (or Besov) scale, in the sense that the parameter $s = k$ is ruled out by Ornstein's Non-inequality.

In view of Theorem 7.1.3, it is quite surprising that the Sobolev-type embedding of $W^{\mathbb{B}, 1}(B)$ holds for operators of arbitrary order whereas the techniques used to prove the trace embedding [45, Thm. 4.17–18] seem to be difficult to extend past first order operators. If $\mathbb{B}$ has FDN, we can use Theorem 7.1.3 to give a sub-optimal trace embedding

$$W^{\mathbb{B}, 1}(B) \hookrightarrow W^{s-\frac{1}{p}, p}(\partial B, V) \quad \text{for } s \uparrow k, \text{ so } p = \frac{n}{n-k+s} \downarrow 1, \quad (7.9)$$

using standard trace theory for Besov spaces. At this stage, it would seem reasonable to speculate that $W^{\mathbb{B}, 1}(B)$ -maps admit traces in $W^{k-1, 1}(\partial B)$. This is, however, not the case: the optimal embedding for $\mathbb{B} = \nabla^k$ was found by USPENSKIĬ in [193] and recently revisited by MIRONESCU and RUSS in [138]. They proved that the trace operator is continuous onto $B_1^{k-1, 1}$, which is in general strictly smaller than $W^{k-1, 1}$ (see [47, Rk. A.1]). In view of [45, Thm. 1.1], one would expect:

Open Problem 7.1.4. *An operator $\mathbb{B}$ as in (7.4) has FDN if and only if there exists a continuous, linear, surjective trace operator $\text{Tr}: W^{\mathbb{B}, 1}(B) \rightarrow B_1^{k-1, 1}(\partial B, V)$.*

A few remarks are in order. Necessity of FDN can be proved by a modification of the arguments in [45, Sec. 4.3]. Surjectivity is obvious, using [138, Thm. 1.3–4] and $W^{k, 1}(B, V) \hookrightarrow W^{\mathbb{B}, 1}(B)$. The difficulty stems from proving boundedness (hence, well-definedness) of the trace operator, which cannot be reduced to the situation in [138] by ORNSTEIN's Non-inequality, or to (7.9) by strict inclusion of Besov spaces. We do not see a way to merge the techniques in [45, 138] and intend to tackle the problem in the future.

This chapter is organized as follows: In Section 7.2 we collect preliminaries on function spaces, multi-linear algebra, harmonic analysis and give examples of operators. In Section 7.3 we give the proof of the first two statements in Theorem 7.1.2 and complete the comparison between EC and FDN, as well as the comparison between the embeddings (7.3) and (7.8). In Section 7.4 we construct the Jones-type extension and prove Theorems 7.1.1 and 7.1.3.

7.2. Preliminaries

Throughout this chapter we assume that $n > 1$.

7.2.1. Function spaces We define, reminiscent of [134], for $1 \leq p \leq \infty$ and open $\Omega \subset \mathbb{R}^n$,

$$\begin{aligned} W^{\mathbb{B},p}(\Omega) &:= \{u \in L^p(\Omega, V) : \mathbb{B}u \in L^p(\Omega)\}, \\ BV^{\mathbb{B}}(\Omega) &:= \{u \in L^1(\Omega, V) : \mathbb{B}u \in \mathcal{M}(\Omega, W)\}, \\ V^{\mathbb{B},p}(\Omega) &:= \{u \in W^{\mathbb{B},p}(\Omega) : \nabla^l u \in L^p(\Omega, V \odot^l \mathbb{R}^n), l = 1 \dots k-1\}, \end{aligned}$$

and the homogeneous spaces $\dot{W}^{\mathbb{B},p}$ as the closure of $C_c^\infty(\mathbb{R}^n, V)$ in the semi-norm $|u|_{\mathbb{B},p} := \|\mathbb{B}u\|_{L^p}$. In the case $\mathbb{B} = \nabla^k$, we write $W^{k,p}(\Omega, V)$, $V^{k,p}(\Omega, V)$. When it is clear from the context what the target space is, we abbreviate the L^p -norm of maps defined on Ω by $\|\cdot\|_{p,\Omega}$. We denote the space of V -valued polynomials of degree at most d in n variables by $\mathbb{R}_d[x]^V$. We recall the weighted Bergman spaces $A_\alpha^p(\mathbb{D})$ of holomorphic maps defined on the open unit disc $\mathbb{D} \subset \mathbb{C}$, that are p -integrable with weight $w_\alpha(z) = (1 - |z|^2)^\alpha$. It is well-known that these are Banach spaces under the $L_{w_\alpha}^p$ -norm for $1 \leq p < \infty$ and $-1 < \alpha < \infty$. We also recall, for $s > 0$, $1 \leq p, q < \infty$, the Besov space

$$B_q^{s,p}(\Omega) := \{u \in L^p(\Omega) : |u|_{B_q^{s,p}(\Omega)} < \infty\},$$

with an obvious choice of norm. Here, the Besov semi-norm is defined (see, e.g., [68, Sec. 2]) for integer $r > s$ by

$$|u|_{B_q^{s,p}(\Omega)} = \|u\|_{\dot{B}_q^{s,p}(\Omega)} := \left(\int_0^\infty \frac{\sup_{|h|<t} \|\Delta_h^r u\|_{L^p(\Omega)}^q}{t^{1+sq}} dt \right)^{\frac{1}{q}},$$

where the r -th finite difference $\Delta_h^r u$ is defined to be zero if undefined, i.e., if at least one of $x + jh$, $j = 1 \dots r$, falls outside Ω . We also define the homogeneous space $\dot{B}^{s,p}_q(\mathbb{R}^n)$ as the closure of $C_c^\infty(\mathbb{R}^n)$ in the Besov semi-norm.

We also collect the assumptions on our operators. As in Section 7.1, we say that $\mathbb{B}$ is

($\mathbb{C}$ -) *elliptic* if and only if the linear map $\mathbb{B}[\xi] : V(+iV) \rightarrow W(+iW)$ is injective for all non-zero $\xi \in \mathbb{R}^n(+i\mathbb{R}^n)$. Trivially, $\mathbb{C}$ -elliptic operators are elliptic. We say that $\mathbb{B}$ has *FDN* (finite dimensional null-space) if and only if the vector space $\{u \in \mathcal{D}'(\mathbb{R}^n, V) : \mathbb{B}u = 0\}$ is finite dimensional. Finally, $\mathbb{B}$ is *cancelling* if and only if $\bigcap_{\xi \in \mathbb{S}^{n-1}} \mathbb{B}[\xi](V) = \{0\}$.

7.2.2. Multi-linear algebra Let U, V be finite dimensional vector spaces and $l \in \mathbb{N}$. We write $\text{Lin}(U, V)$ for the space of linear maps $U \rightarrow V$ and $V \odot^l U$ for the space of V -valued symmetric l -linear maps on U . This is naturally the space of the l -th gradients, i.e., $D^l f(x) \in V \odot^l U$ for $f \in C^l(U, V)$, $x \in U$. For more detail, see [77, Ch. 1]. We also write $a \otimes b = (a_i b_j)$ (the usual tensor product) and $\otimes^l a := \otimes a \otimes \dots \otimes a$, where a appears l times on the right hand side. We single out the standard fact that $\widehat{\nabla^l f}(\xi) = \hat{f}(\xi) \otimes^l \xi \in V \odot^l U$ for $f \in \mathcal{S}(U, V)$, $\xi \in U$. We recall the pairing introduced in [45], $v \otimes_{\mathbb{B}} \xi := \mathbb{B}[\xi]v$, which is reminiscent of the

tensor product notation, i.e., if $\mathbb{B} = D$, we have $\otimes_{\mathbb{B}} = \otimes$. We have the following for $k = 1$:

$$\begin{aligned}\mathbb{B}(\rho u) &= \rho \mathbb{B}u + u \otimes_{\mathbb{B}} \nabla \rho & \text{for } u \in C^1(\mathbb{R}^n, V), \rho \in C^1(\mathbb{R}^n), \\ \mathbb{B}(\phi(w)) &= \phi'(w) \otimes_{\mathbb{B}} \nabla w & \text{for } \phi \in C^1(\mathbb{R}, V), w \in C^1(\mathbb{R}^n).\end{aligned}$$

The above can easily be checked by direct computation and will be used without mention.

7.2.3. Harmonic analysis Let $\mathbb{B}$ as in (7.4) be elliptic and $u \in \mathcal{S}(\mathbb{R}^n, V)$. We Fourier transform $\mathbb{B}u$ and apply the one-sided inverse

$$m_{\mathbb{B}}(\xi) := (\mathbb{B}^*[\xi]\mathbb{B}[\xi])^{-1}\mathbb{B}^*[\xi] \in \text{Lin}(W, V)$$

of $\mathbb{B}[\xi]$ to get that $\hat{u}(x) = m_{\mathbb{B}}(\xi)\widehat{\mathbb{B}u}(\xi)$ for $\xi \in \mathbb{R}^n$ (we omitted the complex multiplicative constant arising from Fourier transforming, as it can be absorbed in the definition of $m_{\mathbb{B}}$). We define the map $\mathbf{G}_{\mathbb{B}}$ as the inverse Fourier transform of the k -homogeneous map $m_{\mathbb{B}}$. Thus we have the Green's function representation $u = \mathbf{G}_{\mathbb{B}} \star \mathbb{B}u$. These considerations are formalized in [43, Lem. 2.1], an implication of which we recall below:

Lemma 7.2.1. *Let $\mathbb{B}$ as in (7.4) be elliptic. Then there exists a $(1 - n)$ -homogeneous map $\mathbf{K}_{\mathbb{B}} \in C^\infty(\mathbb{R}^n \setminus \{0\}, \text{Lin}(W, V \otimes^{k-1} \mathbb{R}^n))$ such that*

$$D^{k-1}u(x) = \int_{\mathbb{R}^n} \mathbf{K}_{\mathbb{B}}(x - y)\mathbb{B}u(y) \, dy = (\mathbf{K}_{\mathbb{B}} \star \mathbb{B}u)(x) \quad (7.10)$$

for all $u \in C_c^\infty(\mathbb{R}^n, V)$.

We also record standard facts regarding L^p -boundedness of Riesz potentials (see [175, Ch. V.1] and [93, Lem. 7.2]), which are defined by

$$I_\alpha f := |\cdot|^{-\alpha-n} \star f$$

for $\alpha \in [0, n)$ and measurable $f: \mathbb{R}^n \rightarrow \mathbb{R}$.

Theorem 7.2.2. *Let $1 \leq p, q \leq \infty$. We have that:*

- (a) I_α is bounded $L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$ for $0 < \alpha < n$, $1 < p < n/\alpha$, $q = \frac{np}{n-\alpha p}$;
- (b) I_α is bounded $L^p(\Omega) \rightarrow L^q(\Omega)$ for $0 < \alpha < n$, $0 \leq n(1/p - 1/q) < \alpha$ with

$$\|I_\alpha f\|_{L^q(\Omega)} \leq c(\text{diam } \Omega)^{\alpha-n(1/p-1/q)} \|f\|_{L^p(\Omega)}$$

for all $f \in L^p(\Omega)$.

In Theorem 7.2.2(b) we make the convention $1/\infty = 0$.

7.2.4. Examples We give examples of operators arising in conductivity, elasticity, plasticity, and fluid mechanics [85, 88, 137]. Let $\mathbb{B}$ be as in (7.4). The facts that we use without mention are the main Theorems 7.1.1, 7.1.2, and 7.1.3.

- (a) If $\mathbb{B} = \nabla^k$, we have that $\ker \mathbb{B} = \mathbb{R}_{k-1}[x]^V$, so $\mathbb{B}$ has FDN, hence is EC. This, of course, corresponds to the case of classical Sobolev spaces, but we highlight it here to stress that our generalization brings a new perspective on their study.
- (b) If $\mathbb{B}u = \mathcal{E}u := (\nabla u + (\nabla u)^\top)/2$ is the symmetrized gradient, it is easy to see that $\ker \mathbb{B}$ is the space of rigid motions, i.e., affine maps of anti-symmetric gradient, so $\mathbb{B}$ has FDN, hence is EC. In this case, we recover the inequality in [178, Prop. 1.2].
- (c) Let $\mathbb{B}u = \mathcal{E}^D u := \mathcal{E}u - (\operatorname{div} u/n)\mathbf{I}$, where $n \geq 2$ and $\mathbf{I}$ is the identity $n \times n$ matrix. If $n \geq 3$, we have from [159] that $\ker \mathbb{B}$ is the space of conformal Killing vectors, so $\mathbb{B}$ has FDN, hence is EC. If $n = 2$, we show in Counterexample 7.3.4 that $\mathbb{B}$ is elliptic. However, under the canonical identification $\mathbb{R}^2 \cong \mathbb{C}$, we can also identify $\mathcal{E}^D$ with the anti-holomorphic derivative $\bar{\partial}$, so that we can further identify $\ker \mathbb{B}$ with the space of holomorphic functions, so $\mathbb{B}$ does not have FDN. Neither is $\mathbb{B}$ cancelling: by ellipticity, we have that $\mathcal{E}^D[\xi](\mathbb{R}^2) = \mathbb{R}^2$. No critical embedding (7.11), (7.12) can hold in this case.
- (d) If $\mathbb{B} = \Delta$, which is clearly elliptic, we have that $\ker \mathbb{B}$ is the space of all harmonic functions, so $\mathbb{B}$ does not have FDN and since $\mathbb{B}[\xi](V) = (\xi_1^2 + \dots + \xi_n^2)\mathbb{R}^N = \mathbb{R}^N$ for $\xi \in \mathbb{R}^n \setminus \{0\}$, neither is $\mathbb{B}$ cancelling.
- (e) If $\mathbb{B}$ is elliptic, one can consider minimizers of the $\mathbb{B}$ -Dirichlet energy $u \mapsto \int_B |\mathbb{B}u|^2 dx$, which has Euler-Lagrange system $\mathbb{B}^* \mathbb{B}u = 0$. Then $\Delta_{\mathbb{B}} := \mathbb{B}^* \mathbb{B}$ is elliptic, as $\langle (\mathbb{B}^* \mathbb{B})[\xi]v, v \rangle = |\mathbb{B}[\xi]v|^2 \gtrsim |\xi|^{2k}|v|^2$, where the last inequality follows from $|\mathbb{B}[\xi]v| > 0$ on $\{|\xi| = 1, |v| = 1\}$ and homogeneity. Therefore $(\mathbb{B}^* \mathbb{B})[\xi](V) = V$ for all $\xi \neq 0$, so the Euler-Lagrange system above has infinite dimensional solution space (by Lemma 7.3.2).

7.3. EC Versus FDN

We begin by proving the first two statements in Theorem 7.1.2. Throughout, $n > 1$.

Proposition 7.3.1. *Let $\mathbb{B}$ be as in (7.4). Then $\mathbb{B}$ has FDN if and only if $\mathbb{B}$ is $\mathbb{C}$ -elliptic.*

Proof. From Theorem 7.5.3, we have that if $\mathbb{B}$ is $\mathbb{C}$ -elliptic, then $\ker \mathbb{B}$ consists of polynomials of degree at most $d = d(\mathbb{B})$. Suppose now that $\mathbb{B}$ is not $\mathbb{C}$ -elliptic, so that there exist non-zero $\xi \in \mathbb{C}^n$, $v \in V + iV$ such that $\mathbb{B}[\xi]v = 0$. We define $u_f(x) = f(x \cdot \xi)v$, for holomorphic $f: \mathbb{C} \rightarrow \mathbb{C}$. It can be shown by direct real differentiation of real and imaginary parts and use of the Cauchy-Riemann equations for f that $Du_f(x) = (\partial_1 f)(x \cdot \xi)v \otimes \xi$. Since $\partial_1 f$ is itself holomorphic, inductively we get that $D^l u_f(x) = (\partial_1^l f)(x \cdot \xi)v \otimes^l \xi$. We make the simple observation that there exists a linear map $A \in \operatorname{Lin}(V \odot^k \mathbb{R}^n, W)$ such that $\mathbb{B}u = A(D^k u)$, which can be viewed as a coordinate invariant (jet) definition

of $\mathbb{B}$. In this notation, by standard properties of the Fourier transform we get $\mathbb{B}[\eta]w = A(w \otimes^k \eta)$ for $\eta \in \mathbb{R}^n$, $w \in V$. It is then easy to see that $\mathbb{B}u_f(x) = (\partial_1^k f)(x \cdot \xi)A(v \otimes^k \xi) = 0$. In particular, $\Re u_f, \Im u_f \in \ker \mathbb{B}$, so $\mathbb{B}$ has infinite dimensional null-space. $\square$

The above result enables us to use FDN and $\mathbb{C}$ -ellipticity interchangeably. Note that to prove that FDN implies ellipticity, one can simply take real ξ , v and $f \in C^1(\mathbb{R})$. We next provide an instrumental ingredient for proving sufficiency of FDN for Theorem 7.1.3.

Lemma 7.3.2. *Let $\mathbb{B}$ be as in (7.4). If $\mathbb{B}$ has FDN, then $\mathbb{B}$ is cancelling.*

Proof. We use Lemma 7.5.1. Let $u \in C^\infty(\mathbb{R}^n, V)$ be such that $K := \text{spt } \mathbb{B}u$ is compact. Consider an open ball B containing K . Cover the complement of B with an increasing chain of overlapping open balls B_j such that $B^c \subset \bigcup_j B_j \subset K^c$. In particular, we have $\mathbb{B}u = 0$ in each B_j , so by Theorem 7.5.3, u must be a polynomial of degree at most $d(\mathbb{B})$ in each B_j . Since the pairs of balls overlap on a set of positive measure, we get that u equals a V -valued polynomial P (tacitly viewed as already extended to the entire $\mathbb{R}^n$) in B^c such that $\mathbb{B}P = 0$ in $\mathbb{R}^n$. To conclude, we elaborate on the notation introduced in the proof of Proposition 7.3.1. Put $m := \dim W$, so that we can write in coordinates $(A\mathcal{V})_l = A^l \cdot \mathcal{V}$ for fixed $A^l \in V \odot^k \mathbb{R}^n$, $l = 1 \dots m$, and all $\mathcal{V} \in V \odot^k \mathbb{R}^n$. For $l = 1 \dots m$, we integrate by parts to get

$$\begin{aligned} \int_{\mathbb{R}^n} (\mathbb{B}u)_l dx &= \int_B A^l \cdot D^k u dx = \int_{\partial B} A^l \cdot (D^{k-1} u \otimes \nu) d\mathcal{H}^{n-1} \\ &= \int_{\partial B} A^l \cdot (D^{k-1} P \otimes \nu) d\mathcal{H}^{n-1} = \int_B A^l \cdot D^k P dx = \int_B (\mathbb{B}P)_l dx = 0, \end{aligned}$$

where ν denotes the unit normal to ∂B . The proof is complete. $\square$

The converse of Lemma 7.3.2, however, is not true in general. In what follows, we complete the algebraic comparison of the FDN condition and VAN SCHAFTINGEN's EC condition. We write $N := \dim V$. The streamline here is that for $N = k = 1$, ellipticity alone implies FDN (rendering these cases rather uninteresting), whereas in higher dimensions or for higher orders, there are EC operators that are not FDN. Somewhat surprisingly, there are also a few instances in which ellipticity and $\mathbb{C}$ -ellipticity differ, but EC implies FDN. We give the details below.

Lemma 7.3.3. *Let $\mathbb{B}$ as in (7.4) be elliptic, $N = k = 1$. Then $\mathbb{B}$ has FDN.*

Proof. Since $N = 1$, it is clear that $\mathbb{B}$ is $\mathbb{F}$ -elliptic, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, if and only if the polynomials $(\mathbb{B}[\xi])_l$, $l = 1 \dots m$, have no common non-trivial zeroes in $\mathbb{F}$. Since we also assume $k = 1$, we have $\mathbb{B}[\xi] = A\xi$ for some $A \in \mathbb{R}^{m \times n}$. It is clear that all roots of the polynomials thus arising are real (in fact, $\mathbb{B}$ is $\mathbb{F}$ -elliptic if and only if $\ker_{\mathbb{R}} A = 0$). $\square$

If $n \geq 3$, EC turns out to be insufficient for FDN, even for scalar fields or first order operators.

Counterexample 7.3.4 (EC does *not* imply FDN). *Consider the operators*

$$\begin{aligned}\mathbb{A}_{k,n}u &:= \nabla^{k-1}(\partial_1 u_1 - \partial_2 u_2, \partial_2 u_1 + \partial_1 u_2, \partial_j u_i)_{(i,j) \notin \{1,2\} \times \{1,2\}} \quad \text{for } N \geq 2, \\ \mathbb{B}_{k,n}u &:= \nabla^{k-2}(\partial_1^2 u + \partial_2^2 u, \partial_j^2 u)_{j=3 \dots n} \quad \text{for } N = 1, k \geq 2.\end{aligned}$$

If $n \geq 3$ or $k \geq 2$, then $\mathbb{A}_{k,n}$ is elliptic and cancelling, but has infinite dimensional null-space. The same is true of $\mathbb{B}_{k,n}$ if $n \geq 3$ or $k \geq 3$.

Proof. The failure of FDN is clear: simply take

$$\begin{aligned}u_{\mathbb{A}}(x) &:= (\Re f(x_1 + i x_2), \Im f(x_1 + i x_2), 0, \dots, 0)^T \\ u_{\mathbb{B}}(x) &:= g(x_1, x_2)\end{aligned}$$

for holomorphic f and (scalar) harmonic g . We next show that $\mathbb{A}_{k,n} = \nabla^{k-1} \mathbb{A}_{1,n}$ is elliptic if $n, N \geq 2$. We can reduce to ellipticity of $\mathbb{A}_{1,n}$, since for non-zero ξ , we have that $0 = \mathbb{A}_{k,n}[\xi]v = (\mathbb{A}_{1,n}[\xi]v) \otimes^{k-1} \xi$, so $\mathbb{A}_{1,n}[\xi]v = 0$. Let $1 \leq j \leq n$ be such that $\xi_j \neq 0$. If $j \geq 3$, we clearly get $v = 0$. If $1 \leq j \leq 2$, we get that $v_i = 0$ for $3 \leq i \leq N$. The remaining equations are $\xi_1 v_1 - \xi_2 v_2 = 0 = \xi_2 v_1 + \xi_1 v_2$, with determinant $\xi_1^2 + \xi_2^2 > 0$, so $v_1 = 0 = v_2$. It remains to check that, under our assumptions, $\mathbb{A}_{k,n}$ is cancelling. The case $k > 1$ is easier, since the composition of operators $\mathbb{L}_1 \circ \mathbb{L}_2$ is cancelling if $\mathbb{L}_1$ is. This is simply due to the fact that $\text{im}(\mathbb{L}_1 \circ \mathbb{L}_2)[\xi] = \mathbb{L}_1[\xi](\text{im} \mathbb{L}_2[\xi]) \subseteq \text{im} \mathbb{L}_1[\xi]$. If $k = 1$ and $n \geq 3$ we can make a straightforward computation. Write $(w_l)_{l=1 \dots Nn-2} := \mathbb{A}_{1,n}[\xi]v$. For $w \in \bigcap_{\xi \neq 0} \mathbb{A}_{1,n}[\xi](V)$, we can essentially test with different values of $\xi \neq 0$. By choosing ξ to have exactly one non-zero entry, we obtain that $w_l = 0$ for $3 \leq l \leq Nn - 2$. Incidentally, when testing with ξ such that $\xi_1 = 0 = \xi_2$, we also obtain $w_1 = 0 = w_2$, so all properties are checked for $\mathbb{A}_{k,n}$. Ellipticity of $\mathbb{B}_{k,n}$ is obvious, whereas cancellation is established analogously. $\square$

The two specific cases that are not covered by Lemma 7.3.3 and Counterexample 7.3.4 reveal that the classes EC and FDN can coincide even if they are strictly smaller than the class of elliptic operators.

Lemma 7.3.5. *Let $n = 2$ and $\mathbb{B}$ be as in (7.4) be elliptic but not $\mathbb{C}$ -elliptic. If any of the following hold,*

- (a) $N = 1, k = 2$,
- (b) $N \geq 2, k = 1$,

then $\mathbb{B}$ is not cancelling.

Proof. Suppose that (a) holds. Since $N = 1$ and $\mathbb{B}$ is not $\mathbb{C}$ -elliptic, the homogeneous, quadratic, scalar polynomials $(\mathbb{B}[\xi])_l$, $l = 1 \dots m$, must have a common complex root. This root cannot be real, as $\mathbb{B}$ is real-elliptic. It follows that $(\mathbb{B}[\xi])_l$ are all multiples of the same quadratic polynomial $P : \mathbb{R} \rightarrow \mathbb{R}$, so that $\mathbb{B}[\xi]v = vP(\xi)w_0$ for all $v \in V \simeq \mathbb{R}$ and some $w_0 \in W \setminus \{0\}$. It is clear then that $\mathbb{B}[\xi](V) = \mathbb{R}w$ for all $\xi \neq 0$. We next assume that (b) holds. Since $\mathbb{B}$ is elliptic, there exist linearly independent $\xi, \eta \in \mathbb{R}^2$, $v, w \in \mathbb{R}^N$ such that $\mathbb{B}[\xi]v = \mathbb{B}[\eta]w$ and

$\mathbb{B}[\xi]w = -\mathbb{B}[\eta]v$. We also have that any $\zeta \in \mathbb{R}^2$ can be written as $\zeta = a\xi + b\eta$. We put $v_\zeta := av + bw$. It follows that

$$\mathbb{B}[\zeta]v_\zeta = \mathbb{B}[a\xi + b\eta](av + bw) = (a^2 + b^2)\mathbb{B}[\xi]v,$$

so that $\bigcap_{\zeta \in \mathbb{R}^2 \setminus \{0\}} \mathbb{B}[\zeta](V) \ni \mathbb{B}[\xi]v \neq 0$. $\square$

We conclude this section with a minor curiosity: we can append the proof above by taking $w_\zeta := bv - aw$ and obtain $\bigcap_{\zeta \neq 0} \mathbb{B}[\zeta](V) \supset \{\mathbb{B}[\xi]v, \mathbb{B}[\xi]w\}$. Therefore, if $n = 2$, $k = 1$, and $\mathbb{B}$ is elliptic but not cancelling, then

$$\dim \bigcap_{\zeta \neq 0} \mathbb{B}[\zeta](V) \geq 2.$$

This is no longer the case in higher dimensions, as can be seen by considering the first order operator $\mathbb{B}u = (\operatorname{div} u, \operatorname{curl} u)$ for $u: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $n \geq 3$.

7.3.1. Insufficiency of EC We next give examples of first order EC operators and domains $\Omega \subset \mathbb{R}^n$ for which the Sobolev-type embedding fails. Firstly, we pause to compare the embeddings in Theorem 7.1.1(b), (c) with VAN SCHAFTINGEN's homogeneous embedding $\dot{W}^{\mathbb{B},1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1}}(\mathbb{R}^n, V)$. For elliptic $\mathbb{B}$, the latter embedding is equivalent to

$$W_0^{\mathbb{B},1}(B) \hookrightarrow L^{\frac{n}{n-1}}(B, V) \quad (7.11)$$

(see Lemma 7.5.9 for a scaling argument) and it can easily be shown that, in the absence of cancellation, we can still prove by means of a Green's formula and boundedness of Riesz potentials that $W_0^{\mathbb{B},1}(B) \hookrightarrow L^p(B, V)$ for any $1 \leq p < n/(n-1)$ (see Lemma 7.5.10). Here $W_0^{\mathbb{B},p}(B)$ is defined as the closure of $C_c^\infty(B, V)$ in the (semi-)norm $u \mapsto \|\mathbb{B}u\|_{L^p}$. The situation is dramatically different as far as L^p -embeddings of $W^{\mathbb{B},1}(B)$ are concerned. By Theorem 7.1.1, if the critical embedding

$$W^{\mathbb{B},1}(B) \hookrightarrow L^{\frac{n}{n-1}}(B, V) \quad (7.12)$$

fails, then no uniform higher integrability estimate is possible. The difference can be even sharper: for EC, non-FDN operators there are maps in $W^{\mathbb{B},1}(B)$ that have no higher integrability, so the homogeneous embedding (7.11) can hold even if the inhomogeneous (7.12) fails completely. We highlight that the main difference between $W^{\mathbb{B},1}(B)$ and $W_0^{\mathbb{B},1}(B)$ lies in the traces, which are integrable if and only if $\mathbb{B}$ has FDN [45].

The existence of elliptic and canceling $\mathbb{B}$, domains Ω , and of maps $u \in W^{\mathbb{B},1}(\Omega)$ that are in no $L^p(\Omega, V)$, $p > 1$, follows from Counterexample 7.3.4 above and the next Lemma, which is a strengthened version of the strict inclusion of (weighted) Bergman spaces generalized to elliptic, non-FDN operators.

Lemma 7.3.6. *Let $k = 1$ and $\mathbb{B}$ as in (7.4) be elliptic but not have FDN, so there exist linearly independent $\eta_1, \eta_2 \in \mathbb{R}^n$ such that $\mathbb{B}[\eta_1 + i\eta_2]$ has non-trivial kernel in $V + iV$. Assume that η_1, η_2 are orthonormal. If any of the following holds:*

- (a) $\Omega := B_{\operatorname{span}\{\eta_1, \eta_2\}} \times [0, 1]^{n-2}$,

(b) $\Omega := B$,

then there exists smooth $u \in L^1 \setminus \bigcup_{p>1} L^p(\Omega, V)$ such that $\mathbb{B}u = 0$.

Proof. We write $\xi = \eta_1 + i\eta_2$, and write D for the unit disc in $\text{span}\{\eta_1, \eta_2\}$ (there is no risk of confusing the disc D with the total derivative in the course of this proof). We stress that each η_j must be non-zero by ellipticity of $\mathbb{B}$, so D is indeed a non-degenerate disc. We also know from the proof of Proposition 7.3.1 that there exist non-zero $v \in V + iV$ such that $\mathbb{B}[\xi]v = 0$, and one can show by direct computation that for any holomorphic function f we can define $u_f(x) := f(x \cdot \xi)v$, for which $\mathbb{B}\Re u_f = 0 = \mathbb{B}\Im u_f$. We have that

$$\begin{aligned} \int_{\Omega} |u_f(x)|^p dx &= \int_D \int_{(\eta + \{\eta_1, \eta_2\}^\perp) \cap \Omega} |f(\eta \cdot \xi)|^p |v|^p d\mathcal{H}^{n-2} d\mathcal{H}^2(\eta) \\ &= |v|^p \int_D |f(\eta \cdot \xi)|^p \mathcal{H}^{n-2} \left((\eta + \{\eta_1, \eta_2\}^\perp) \cap \Omega \right) d\mathcal{H}^2(\eta). \end{aligned}$$

We now make the case distinction. Assume (a) holds, so

$$\int_{\Omega} |u_f(x)|^p dx = |v|^p \int_D |f(\eta \cdot \xi)|^p d\mathcal{H}^2(\eta) = \int_{\mathbb{D}} |f(z)|^p d\mathcal{L}^2(z),$$

where $\mathbb{D}$ is the representation of D in $\mathbb{C}$. Assume (b), so

$$\begin{aligned} \int_{\Omega} |u_f(x)|^p dx &= c(n)|v|^p \int_D |f(\eta \cdot \xi)|^p (1 - |\eta|^2)^{\frac{n-2}{2}} d\mathcal{H}^2(\eta) \\ &= c(n)|v|^p \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^{\frac{n-2}{2}} d\mathcal{L}^2(z), \end{aligned}$$

where $c(n)$ denotes the volume of the $(n-2)$ -dimensional ball. By Lemma 7.3.7 below, we can choose $f \in A_\alpha^1(\mathbb{D}) \setminus \bigcup_{p>1} A_\alpha^p(\mathbb{D})$ for $\alpha = 0$ and $\alpha = (n-2)/2$ respectively, so that both $\Re u_f$ and $\Im u_f$ are in $L^1(B, V)$, but one of them is not in any other L^p . This proves the claim. $\square$

The following Lemma is also feasible by direct computation, but we prefer to give an abstract argument for the sake of brevity.

Lemma 7.3.7. *For all $1 \leq p < \infty$, $\alpha \geq 0$ the set $A_\alpha^p(\mathbb{D}) \setminus \bigcup_{q>p} A_\alpha^q(\mathbb{D})$ is non-empty.*

Proof. We abbreviate $A^p := A_\alpha^p(\mathbb{D})$. The proof relies on the strict inclusion $A^q \subsetneq A^p$ for $1 \leq p < q < \infty$ proved in [209, Cor. 68] and a Baire category argument. Assume that the result is false, so that by Hölder's Inequality we can find a sequence $q_j \downarrow p$ such that $A^p = \bigcup_j A^{q_j}$. For natural l , we define the sets $F_l^j := \{f \in A^{q_j} : \|f\|_{A^{q_j}}^{q_j} \leq l\}$, which we claim to be closed in A^p . Let $f_m \in F_l^j$ converge to f in A^p . By completeness of A^p , we have, by Fatou's Lemma on a pointwise convergent, not relabelled subsequence that

$$\int_{\mathbb{D}} |f|^{q_j} w_\alpha d\mathcal{L}^2 \leq \liminf_{m \rightarrow \infty} \int_{\mathbb{D}} |f_m|^{q_j} w_\alpha d\mathcal{L}^2 \leq l,$$

so that indeed $f \in F_l^j$. Since A^{q_j} is a proper subspace of A^p , it follows that the sets F_l^j are nowhere dense in A^p . It remains to notice that then $A^p = \bigcup_{j,l} F_l^j$, which contradicts completeness of A^p by Baire's Theorem. $\square$

7.3.2. Comparison to the Bourgain–Brezis condition We recall here the assumptions on $\mathbb{B}$ (sufficient for EC) under which a general inequality of the type (7.3) was first proved in [38], in the case $k = 1$ and $V = \mathbb{R}^n$. In their notation, we write $(\mathbb{B}u)_s = \langle L^{(s)}, \nabla u \rangle$ for matrices $L^{(s)} \in \mathbb{R}^{n \times n}$, $s = 1 \dots m$. It is shown in [38, Thm. 25], that if an operator $\mathbb{B}$ is elliptic such that $\det L^{(s)} = 0$ for $s = 1 \dots m$, then (7.3) holds. It is clear (either by [201, Thm. 1.3] or by direct computation) that such operators are cancelling. By Lemma 7.3.5, if $n = 2$, we have that such $\mathbb{B}$ also has FDN, and thus satisfies (7.12). However, if $n \geq 3$, we show that $\mathbb{A}_{1,n}$ as in Counterexample 7.3.4 with $N = n$ satisfies the BOURGAIN–BREZIS condition, but do not have FDN. We explicitly write down the matrices $L^{(s)}$ if $n = 3$, the general case being a simple exercise:

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

By the reasoning in Section 7.3.1, with $\mathbb{B} = \mathbb{A}_{1,n}$, we have that $\dot{W}^{\mathbb{B},1}(\mathbb{R}^n) \hookrightarrow L^{n/(n-1)}(\mathbb{R}^n)$, but there are maps in $W^{\mathbb{B},1}(B)$ that have no higher integrability.

7.4. The Sobolev-type Embedding on Domains

7.4.1. A Jones-type Extension In this section we complete the proof of Theorem 7.1.2 with the following generalization:

Theorem 7.4.1. *Let $\mathbb{B}$ as in (7.4) have FDN, $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there exists a bounded, linear extension operator*

$$E_\Omega: W^{\mathbb{B},p}(\Omega) \rightarrow V^{\mathbb{B},p}(\mathbb{R}^n).$$

To prove this result we use JONES' method of extension developed in the celebrated paper [107]. Recall that JONES's original idea was to decompose a small neighbourhood of $\partial\Omega$ into small cubes and assign suitable polynomials of degree at most $k - 1$ to each cube. Inspired by [45, Sec. 4.1-2], we assign elements of $\ker \mathbb{B}$ to such cubes, as explained below. We stress that the fact that $\ker \mathbb{B}$ consists of a finite dimensional space of polynomials is essential for the construction to work. With this crucial modification, the streamlined proof that we include below mostly follows the same lines as in [107, Sec. 2-3], where all the details we omit can be found. What deserves some special attention is a Poincaré-type inequality, which is interesting in its own right, as it implies that $W^{\mathbb{B},p}(B) \simeq V^{\mathbb{B},p}(B)$ for FDN operators (see Lemma 7.5.6). We present it below and mention that it is a generalization of the results in [134, Sec. 1.1.11] and [45, Sec. 3]. We extend the notation presented in Theorem 7.5.3 by $\pi_\Omega u := \Pi \mathcal{P}u$, where Π denotes the L^2 -orthogonal projection of $\mathbb{R}_d[x]^V$ onto $\ker \mathbb{B}$.

Proposition 7.4.2 (Poincaré-type inequality). *Let $\mathbb{B}$ as in (7.4) have FDN, $1 \leq p \leq \infty$, $0 \leq l < k$, and $\Omega \subset \mathbb{R}^n$ be a star-shaped domain with respect to a*

ball. Then there exists $c > 0$ such that

$$\|\nabla^l(u - \pi_\Omega u)\|_{p,\Omega} \leq c(\text{diam } \Omega)^{k-l} \|\mathbb{B}u\|_{p,\Omega} \quad (7.13)$$

for all $u \in C^\infty(\bar{\Omega}, V)$.

Interestingly, $\mathbb{B}$ having FDN is not necessary for the estimate (7.13) to hold, as can be seen from [87]. We believe that ellipticity alone is sufficient for the estimate to hold and intend to pursue this in future work.

Proof. We start with $\|\nabla^l(u - \pi_\Omega u)\|_{p,\Omega} \leq \|\nabla^l(u - \mathcal{P}u)\|_{p,\Omega} + \|\nabla^l(\mathcal{P}u - \pi_\Omega u)\|_{p,\Omega}$, and estimate both terms. We have by the growth conditions on K from Theorem 7.5.3 that

$$\begin{aligned} \|\nabla^l(u - \mathcal{P}u)\|_{p,\Omega} &= \left(\int_\Omega \left| \int_\Omega \nabla_x^l K(x, y) \mathbb{B}u(y) \, dy \right|^p \, dx \right)^{\frac{1}{p}} \\ &\lesssim \left(\int_\Omega \left(\int_\Omega \frac{|\mathbb{B}u(y)|}{|x - y|^{n+l-k}} \, dy \right)^p \, dx \right)^{\frac{1}{p}}. \end{aligned}$$

Now consider the case $n + l > k$, so that we can estimate the RHS using Theorem 7.2.2(b):

$$\|I_{k-l}(|\mathbb{B}u|)\|_{p,\Omega} \lesssim (\text{diam } \Omega)^{k-l} \|\mathbb{B}u\|_{p,\Omega}.$$

If, on the other hand, $n + l \leq k$, with $R = \text{diam } \Omega$, we perform the elementary estimation

$$\begin{aligned} \left(\int_\Omega \left(\int_\Omega |\mathbb{B}u(y)| |x - y|^{k-l-n} \, dy \right)^p \, dx \right)^{\frac{1}{p}} &\leq R^{k-l-n} \left(\int_\Omega |\mathbb{B}u(y)| \, dy \right) \left(\int_\Omega \, dx \right)^{\frac{1}{p}} \\ &\leq R^{k-l-n} \cdot R^{n(p-1)/p} \cdot \|\mathbb{B}u\|_{p,\Omega} \cdot R^{n/p} \\ &= R^{k-l} \|\mathbb{B}u\|_{p,\Omega}, \end{aligned}$$

where obvious modifications have to be made if $p = \infty$.

We then note that $P \mapsto \|\nabla^l(P - \Pi P)\|_{p,\Omega}$ and $P \mapsto \|\mathbb{B}P\|_{p,\Omega}$ respectively define a semi-norm and a norm on the finite dimensional vector space $\mathbb{R}_d[x]^V / \ker \mathbb{B}$, so that the second term $\|\nabla^l(\mathcal{P}u - \pi_\Omega u)\|_{p,\Omega} \lesssim \|\mathbb{B}\mathcal{P}u\|_{p,\Omega}$, with a domain dependent constant. We recall from the original proof of Theorem 7.5.3 that $\mathcal{P}u$ is the averaged Taylor polynomial

$$\mathcal{P}u(x) = \int_\Omega \sum_{|\alpha| \leq d} \frac{\partial_y^\alpha ((y - x)^\alpha w(y))}{\alpha!} u(y) \, dy = \int_\Omega \sum_{|\alpha| \leq d} \frac{(x - y)^\alpha}{\alpha!} w(y) \partial^\alpha u(y) \, dy,$$

where the weight w is a smooth map supported in the ball with respect to which Ω is star-shaped such that $\int w = 1$. One can show by direct computation that averaged Taylor polynomials “commute” with derivatives, in the sense that

$$\mathbb{B}\mathcal{P}u = \int_\Omega \sum_{|\beta| \leq d-k} \frac{\partial_y^\beta ((y - \cdot)^\beta w(y))}{\alpha!} \mathbb{B}u(y) \, dy.$$

It is then obvious that $\|\mathbb{B}\mathcal{P}u\|_{p,\Omega} \lesssim \|\mathbb{B}u\|_{p,\Omega}$. The precise dependence of the constant on the domain follows by standard scaling arguments. $\square$

We next introduce the framework required to prove Theorem 7.4.1. We use the same Whitney coverings as in [107], which we recall for the reader's convenience. Firstly recall the Decomposition Lemma introduced in [205], that any open subset $\Omega \subset \mathbb{R}^n$ can be covered with a countable collection $\mathcal{W}_1 := \{S_j\}$ of closed dyadic cubes satisfying

$$(D_1) \quad \ell(S_j)/4 \leq \ell(S_l) \leq 4\ell(S_j) \text{ if } S_j \cap S_l \neq \emptyset,$$

$$(D_2) \quad \text{int } S_j \cap \text{int } S_l = \emptyset \text{ if } j \neq l,$$

$$(D_3) \quad \ell(S_j) \leq \text{dist}(S_j, \partial\Omega) \leq 4\sqrt{n}\ell(S_j) \text{ for all } j,$$

where $\ell(Q)$ denotes the side-length of a cube Q . We henceforth assume that Ω is as in the statement of Theorem 7.4.1. We further consider a Whitney decomposition $\mathcal{W}_2 := \{Q_l\}$ of $\mathbb{R}^n \setminus \bar{\Omega}$, and further define $\mathcal{W}_3 := \{Q \in \mathcal{W}_2 : \ell(Q) \leq \varepsilon\delta/(16n)\}$. We reflect each cube $Q \in \mathcal{W}_3$ to a non-unique interior cube $Q^* \in \mathcal{W}_1$ such that

$$(R_1) \quad \ell(Q) \leq \ell(Q^*) \leq 4\ell(Q),$$

$$(R_2) \quad \text{dist}(Q, Q^*) \leq C\ell(Q),$$

where above and in the following C denotes a constant depending on $k, p, n, \varepsilon, \delta$ only; additional dependencies will be specified. The non-uniqueness causes no issues, as one can show that

$$(R_3) \quad \text{For any two choices } S_1, S_2 \text{ of } Q^*, \text{ we have } \text{dist}(S_1, S_2) \leq C\ell(Q),$$

$$(R_4) \quad \text{For any } S \in \mathcal{W}_1, \text{ there are at most } C \text{ cubes } Q \in \mathcal{W}_3 \text{ such that } S = Q^*,$$

$$(R_5) \quad \text{For any adjacent } Q_1, Q_2 \in \mathcal{W}_3, \text{ we have } \text{dist}(Q_1^*, Q_2^*) \leq C\ell(Q_1).$$

For detail on these basic properties of the reflection see [107, Lem. 2.4-7]. We conclude the presentation of the decomposition by quoting the following:

Lemma 7.4.3 ([107, Lem. 2.8]). *For any adjacent cubes $Q_1, Q_2 \in \mathcal{W}_3$, there is a chain $\mathcal{C}(Q_1^*, Q_2^*) := \{Q_1^* =: S_1, S_2, \dots, S_m := Q_2^*\}$ of cubes in $S_j \in \mathcal{W}_1$, i.e., such that S_j and S_{j+1} touch for all j , and $m \leq C$.*

We proceed to define the extension operator

$$E_\Omega u := \begin{cases} u & \text{in } \Omega \\ \sum_{Q \in \mathcal{W}_3} \varphi_Q \pi_{Q^*} u & \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \end{cases}$$

where $\{\varphi_Q\}_{Q \in \mathcal{W}_3} \subset C^\infty(\mathbb{R}^n)$ is a partition of unity such that for all $Q \in \mathcal{W}_3$ we have

$$(P_1) \quad 0 \leq \varphi_Q \leq 1 \text{ and } \sum_{Q \in \mathcal{W}_3} \varphi_Q = 1 \text{ in } \bigcup \mathcal{W}_3,$$

$$(P_2) \quad \text{spt } \varphi_Q \subset 17/16Q, \text{ where } \lambda Q \text{ denotes the homothety of } Q \text{ by } \lambda \text{ about its centre,}$$

$$(P_3) \quad |\nabla^l \varphi_Q| \leq C\ell(Q)^{-l} \text{ for all } 0 \leq l \leq k.$$

Our proof mostly follows the lines of the original proof. We first prove an estimate on chains in $\mathcal{W}_1$, then suitably bound the norms of the derivatives in the exterior domains, and we conclude by showing that the extension has weak derivatives in full-space. We warn the reader that in the remainder of this section we may use the properties of the decomposition, reflection and partition of unity without mention.

Lemma 7.4.4 ([107, Lem. 3.1]). *Let $\mathcal{C} := \{S_1, \dots, S_m\} \subset \mathcal{W}_1$ be a chain. Then for $0 \leq l < k$ we have*

$$\|\nabla^l(\pi_{S_1}u - \pi_{S_m}u)\|_{p,S_1} \leq C(m)\ell(S_1)^{k-l}\|\mathbb{B}u\|_{p,\cup\mathcal{C}}$$

for all $u \in C^\infty(\bar{\Omega}, V)$.

Proof. We remark that L^p -norms of polynomials of degree at most d on adjacent cubes in $\mathcal{W}_1$ are comparable (see, e.g., [107, Lem. 2.1]). We get

$$\begin{aligned} \text{LHS} &\leq \sum_{j=1}^{m-1} \|\nabla^l(\pi_{S_{j+1}}u - \pi_{S_j}u)\|_{p,S_1} \\ &\leq C(m) \sum_{j=1}^{m-1} \|\nabla^l(\pi_{S_{j+1}}u - \pi_{S_j \cup S_{j+1}}u)\|_{p,S_{j+1}} + \|\nabla^l(\pi_{S_j \cup S_{j+1}}u - \pi_{S_j}u)\|_{p,S_j} \\ &\leq C(m) \sum_{j=1}^{m-1} \left(\|\nabla^l(\pi_{S_{j+1}}u - u)\|_{p,S_{j+1}} + 2\|\nabla^l(u - \pi_{S_j \cup S_{j+1}}u)\|_{p,S_j \cup S_{j+1}} \right. \\ &\quad \left. + \|\nabla^l(u - \pi_{S_j}u)\|_{p,S_j} \right) \end{aligned}$$

and we can use the Poincaré-type inequality, Proposition 7.4.2, to conclude. $\square$

Lemma 7.4.5 ([107, Prop. 3.4]). *For $1 \leq p \leq \infty$, we have $\|E_\Omega u\|_{V^{\mathbb{B},p}(\mathbb{R}^n \setminus \bar{\Omega})} \leq C\|u\|_{W^{\mathbb{B},p}(\Omega)}$ for all $u \in C^\infty(\bar{\Omega}, V)$.*

Proof. We estimate on each cube in $\mathcal{W}_2$, distinguishing between small and large cubes. We also distinguish between $\mathbb{B}$ and the derivatives of order less than k . Let $Q_0 \in \mathcal{W}_3$. Then, since φ_Q sum to one in Q_0 and $\mathbb{B}\pi_{Q_0^*}u \equiv 0$, we have

$$\begin{aligned} \|\mathbb{B}E_\Omega u\|_{p,Q_0} &= \left\| \mathbb{B} \sum_{Q \in \mathcal{W}_3} \varphi_Q(\pi_{Q^*}u - \pi_{Q_0^*}u) \right\|_{p,Q_0} \\ &\leq \left\| \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \mathbb{B}(\varphi_Q(\pi_{Q^*}u - \pi_{Q_0^*}u)) \right\|_{p,Q_0} \\ &\leq C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \sum_{j=0}^{k-1} \|\nabla^{k-j} \varphi_Q\| \|\nabla^j(\pi_{Q^*}u - \pi_{Q_0^*}u)\|_{p,Q_0} \\ &\leq C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \sum_{j=0}^{k-1} \ell(Q_0)^{j-k} \|\nabla^j(\pi_{Q^*}u - \pi_{Q_0^*}u)\|_{p,Q_0^*} \end{aligned}$$

$$\leq C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \|\mathbb{B}u\|_{p, \cup \mathcal{C}(Q_0^*, Q^*)},$$

where the last inequality follows from Lemma 7.4.4. With a similar reasoning we obtain, for $0 \leq l \leq k-1$, that

$$\|\nabla^l E_\Omega u\|_{p, Q_0} \leq C \left(\|\nabla^l u\|_{p, Q_0^*} + \ell(Q_0)^{k-l} \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \|\mathbb{B}u\|_{p, \cup \mathcal{C}(Q_0^*, Q^*)} \right).$$

We move on to the case $Q_0 \in \mathcal{W}_2 \setminus \mathcal{W}_3$, so if $Q \cap Q_0 \neq \emptyset$, then $\ell(Q) \geq \ell(Q_0)/4 \geq \varepsilon\delta/(64n) \geq C$. Let $0 \leq l \leq k-1$, so that

$$\begin{aligned} \|\nabla^l E_\Omega u\|_{p, Q_0} &\leq \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \|\nabla^l(\varphi_Q \pi_{Q^*} u)\|_{p, Q_0} \\ &\leq C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \sum_{j=1}^l \ell(Q_0)^{j-l} \|\nabla^j \pi_{Q^*} u\|_{p, Q_0} \\ &\leq C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \sum_{j=1}^l \ell(Q_0)^{j-l} \|\nabla^j \pi_{Q^*} u\|_{p, Q^*} \\ &\leq C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \sum_{j=1}^l \ell(Q_0)^{j-l} (\|\nabla^j(\pi_{Q^*} u - u)\|_{p, Q^*} + \|\nabla^j u\|_{p, Q^*}) \\ &\leq C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \|u\|_{V^{l,p}(Q^*, V)} + \ell(Q_0)^{k-l} \|\mathbb{B}u\|_{p, Q^*}. \end{aligned}$$

As, above, we similarly show that $\|\mathbb{B}E_\Omega u\|_{p, Q_0} \leq C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \|u\|_{V^{l,p}(Q^*)}$. There is no loss in assuming that $\ell(Q_0) \leq 1$ for any $Q_0 \in \mathcal{W}_2$, so that we can collect the estimates to obtain

$$\|E_\Omega u\|_{V^{l,p}(Q_0)} \leq C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \|u\|_{V^{l,p}(\mathcal{C}(Q_0^*, Q^*))}.$$

It remains to use local finiteness of the partition of unity (see, e.g., [107, Eqn. (3.1-4)]) and Lemma 7.5.6 to conclude. $\square$

Proof of Theorem 7.4.1. We firstly show that $E_\Omega u$ has weak derivatives in $\mathbb{R}^n$, for which it suffices (by Lemma 7.5.5) to show that E_Ω maps $u \in V^{k,\infty}(\bar{\Omega}, V)$ into $V^{k,\infty}(\mathbb{R}^n, V)$. This we do in two steps. First, we show that the obvious candidate $(\nabla^l u)\chi_{\bar{\Omega}} + (\nabla^l E_\Omega u)\chi_{\mathbb{R}^n \setminus \bar{\Omega}}$ is bounded for all $0 \leq l \leq k$. We need only prove this for $l = k$, the other cases being dealt with in Lemma 7.4.5 for $p = \infty$. As before, we first take $Q_0 \in \mathcal{W}_3$, where

$$\begin{aligned} |\nabla^k E_\Omega u| &\leq |\nabla^k \pi_{Q_0^*} u| + \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} |\nabla^k(\varphi_Q(\pi_{Q^*} u - \pi_{Q_0^*} u))| \\ &\leq C \left(|\nabla^k \pi_{Q_0^*} u| + \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \|\nabla^k u\|_{\infty, \mathcal{C}(Q_0^*, Q^*)} \right). \end{aligned}$$

Clearly, $P \mapsto \|\nabla^k P\|_{\infty, Q_0^*}$ is a norm on $\mathbb{R}_d[x]^V / \mathbb{R}_{k-1}[x]$, whereas $P \mapsto \|\nabla^k \Pi P\|_{\infty, Q_0^*}$ is a semi-norm. We therefore get that $\|\nabla^k \pi_{Q_0^*} u\|_{\infty, Q_0^*} \leq C \|\nabla^k \mathcal{P}_{Q_0^*} u\|_{\infty, Q_0^*} \leq C \|\nabla^k u\|_{\infty, Q_0^*}$, where the latter inequality is given by the stability of averaged Taylor polynomials (for more detail, the reader can revisit the proof of Proposition 7.4.2, which contains a very similar argument). Now consider the other case, when $Q_0 \in \mathcal{W}_2 \setminus \mathcal{W}_3$, and recall that then $\ell(Q_0) \geq C$. We have

$$\begin{aligned} |\nabla^l E_\Omega u| &\leq \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} |\nabla^k(\varphi_Q \pi_{Q^*})| \leq C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \sum_{j=1}^k \ell(Q)^{j-k} |\nabla^j \pi_\Omega u| \\ &\leq C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \sum_{j=1}^k \ell(Q_0)^{j-k} |\nabla^j \pi_\Omega u| \leq C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \sum_{j=1}^k |\nabla^j \pi_\Omega u|, \end{aligned}$$

so we can conclude as in the previous step.

The second step is to show that $\nabla^l E_\Omega u$ is continuous for $0 \leq l < k$. To this end, it suffice to show that

$$\|\nabla^l E_\Omega u - (\nabla^l u)_{Q_0^*}\|_{\infty, Q_0} \rightarrow 0 \quad \text{as } \ell(Q_0) \rightarrow 0$$

for $Q_0 \in \mathcal{W}_3$. Here $(\cdot)_S$ denotes the average with respect to Lebesgue measure on S . By the triangle inequality and properties of the partition of unity, we get

$$\begin{aligned} \|\nabla^l E_\Omega u - (\nabla^l u)_{Q_0^*}\|_{\infty, Q_0} &\leq \left\| \nabla^l \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \varphi_Q (\pi_{Q^*} - \pi_{Q_0^*}) \right\|_{\infty, Q_0} \\ &\quad + \|\nabla^l \pi_{Q_0^*} u - (\nabla^l u)_{Q_0^*}\|_{\infty, Q_0} = \mathbf{I} + \mathbf{II}. \end{aligned}$$

By an estimation which is by now routine (see the proof of Lemma 7.4.5) we have that

$$\mathbf{I} \leq C \ell(Q_0)^{k-l} \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \|\mathbb{B}u\|_{\infty, \cup \mathcal{C}(Q_0^*, Q^*)},$$

which tends to zero as $\ell(Q_0) \rightarrow 0$ since $k > l$. For the second term, we have by [107, Lem. 2.1] and closeness of Q_0 and Q_0^* that

$$\begin{aligned} \mathbf{II} &\leq C \|\nabla^l \pi_{Q_0^*} u - (\nabla^l u)_{Q_0^*}\|_{\infty, Q_0^*} \leq C \|\nabla^l \pi_{Q_0^*} u - \nabla^l u\|_{\infty, Q_0^*} \\ &\quad + C \|\nabla^l u - (\nabla^l u)_{Q_0^*}\|_{\infty, Q_0^*} \leq C \ell(Q_0) \|\mathbb{B}u\|_{\infty, Q_0^*} + C \ell(Q_0) \|\nabla^{l+1} u\|_{\infty, Q_0^*}, \end{aligned}$$

where the last inequality we used Proposition 7.4.2 and the fact that $\nabla^l u$ is Lipschitz as $l < k$.

We next note that by density of smooth functions in $W^{\mathbb{B}, p}(\Omega)$ (Lemma 7.5.5), it suffices to prove boundedness of the extension for maps in $u \in C^\infty(\bar{\Omega}, V)$. By the argument above, we have that $E_\Omega u \in V^{k, \infty}(\mathbb{R}^n, V)$, so that

$$\begin{aligned} \nabla^l E_\Omega u &= (\nabla^l u) \chi_{\bar{\Omega}} + (\nabla^l E_\Omega u) \chi_{\mathbb{R}^n \setminus \bar{\Omega}} \quad \text{for } 0 \leq l \leq k-1, \\ \mathbb{B} E_\Omega u &= (\mathbb{B} u) \chi_{\bar{\Omega}} + (\mathbb{B} E_\Omega u) \chi_{\mathbb{R}^n \setminus \bar{\Omega}}. \end{aligned}$$

It follows that

$$\|E_\Omega u\|_{V^{\mathbb{B}, p}(\mathbb{R}^n)} \leq \|u\|_{V^{\mathbb{B}, p}(\Omega)} + \|E_\Omega u\|_{V^{\mathbb{B}, p}(\mathbb{R}^n \setminus \bar{\Omega})} \leq C \|u\|_{W^{\mathbb{B}, p}(\Omega)},$$

where in the last inequality we used Lemmas 7.4.5 and 7.5.6. The proof is complete. $\square$

7.4.2. Proofs of the main results We now begin the proof of Theorem 7.1.1. It is clear that (b) implies (c) and that (d) implies (e). We first prove that (a) implies (b) in full generality.

Proof of Theorem 7.1.3 (sufficiency of FDN). Since $\mathbb{B}$ has FDN, by Theorem 7.1.2, there exists a bounded, linear extension operator $E_B: W^{\mathbb{B},1}(B) \rightarrow V^{\mathbb{B},1}(\mathbb{R}^n)$. A close inspection of the proof of Theorem 7.4.1 reveals that E_B maps restrictions to the ball B of $C^\infty(\mathbb{R}^n, V)$ -functions into $C_c^\infty(\tilde{B}, V)$ for a larger ball $\tilde{B} \supset B$, which depends on B only. We write $p := n/(n - k + s)$ and use Hölder's Inequality to get that

$$\begin{aligned} \|u\|_{B_q^{s,p}(B,V)} &\leq \|E_B u\|_{B_q^{s,p}(\mathbb{R}^n,V)} = \|E_B u\|_{L^p(\tilde{B},V)} + \|E_B u\|_{\dot{B}_q^{s,p}(\mathbb{R}^n,V)} \\ &\lesssim \|E_B u\|_{L^{\frac{n}{n-1}}(\tilde{B},V)} + \|E_B u\|_{\dot{B}_q^{s,p}(\mathbb{R}^n,V)} \\ &\lesssim \|\nabla^{k-1} E_B u\|_{L^{\frac{n}{n-1}}(\tilde{B},V)} + \|E_B u\|_{\dot{B}^{s,p}_q(\mathbb{R}^n,V)} \end{aligned}$$

where the last estimate follows from Poincaré's Inequality with zero boundary values. We conclude by [201, Thm. 1.3, Thm. 8.4] and boundedness of E_B . $\square$

We will complete the proof of Theorem 7.1.3 (i.e., show necessity of FND) at the end of this section. Returning to Theorem 7.1.1, to see that (b) implies (d), we prove the following:

Theorem 7.4.6. *Let $\mathbb{B}$ be as in (7.4) with $k = 1$. Suppose that $W^{\mathbb{B},1}(B) \hookrightarrow L^{\frac{n}{n-1}}(B, V)$. Then $W^{\mathbb{B},1}(B) \hookrightarrow L^q(B, V)$ for all $1 \leq q < \frac{n}{n-1}$.*

The proof of Theorem 7.4.6 relies on the Riesz-Kolmogorov criterion and the following Nikolskiĭ-type estimate:

Lemma 7.4.7 (Nikolskiĭ-type Estimate). *Let $\mathbb{B}$ be an elliptic operator of the form (7.4), $k = 1$. Fix $R > 0$. Then for every $0 < s < 1$ there exists a constant $c = c(s, R) > 0$ such that if $u \in W^{\mathbb{B},1}(\mathbb{R}^n)$ vanishes identically outside $B(0, R)$, then there holds*

$$\int_{\mathbb{R}^n} |u(x+y) - u(x)|^p dx \leq c \|\mathbb{B}u\|_{L^1(B(0,R),W)}^p |y|^{sp}.$$

whenever $p < n/(n - 1 + s)$.

Note that by ORNSTEIN's Non-Inequality, $s = 1$ is not allowed in the lemma. A more general, sharp version, can be found in [201, Prop. 8.22].

Proof of Theorem 7.4.6. Recall that by the Riesz-Kolmogorov Theorem on relatively compact subsets of L^p -spaces [46, Thm. 4.26] on $\Omega \subset \mathbb{R}^n$ open and $1 \leq p < \infty$, a subset $\mathcal{F} \subset L^p(\Omega, V)$ is relatively compact in $L^p(\Omega, V)$ if and only if

- (i) $\mathcal{F}$ is a bounded set in $L^p(\Omega, V)$ and
- (ii) for all $\varepsilon > 0$ there exists $\delta > 0$ such that there holds

$$\|\bar{f}(\cdot + y) - \bar{f}(\cdot)\|_{L^p(\mathbb{R}^n, V)} < \varepsilon, \tag{7.14}$$

for all $f \in \mathcal{F}$ and all $y \in \mathbb{R}^n$ with $|y| < \delta$

Here $\bar{f}$ is the trivial extension of $f \in \mathcal{F}$ to $\mathbb{R}^n$.

Let $1 \leq q < 1^*$ and $\mathcal{F}$ be the unit ball in $W^{\mathbb{B},1}(B)$. The embedding $W^{\mathbb{B},1}(B) \hookrightarrow L^{n/(n-1)}(B, V)$ implies that $\mathcal{F}$ is bounded in $L^q(B, V)$ which shows condition (i) of the Riesz–Kolmogorov criterion. As to (ii), let $\varepsilon > 0$ be arbitrary. Given $\varrho > 0$ sufficiently small (to be determined later on), let $\tilde{\rho}_\varrho: [0, 1] \rightarrow [0, 1]$ be the Lipschitz function given by

$$\rho_\varrho(t) := \begin{cases} 1 & \text{if } 0 < t < 1 - 2\varrho, \\ -\frac{1}{\varrho}t + \frac{1-\varrho}{\varrho} & \text{if } 1 - 2\varrho < t < 1 - \varrho, \\ 0 & \text{if } 1 - \varrho < t \leq 1, \end{cases}$$

and put $\rho_\varrho(x) := \tilde{\rho}_\varrho(|x|)$, $x \in \mathbb{R}^n$, and finally set, for given $f \in \mathcal{F}$, $f_\varrho := \rho_\varrho f$. Denoting $B_t := B(0, t)$ for $t > 0$, we note that if $|y| < \varrho$, then $f(\cdot + y) - f(\cdot)$ and $f_\varrho(\cdot + y) - f_\varrho(\cdot)$ coincide on $B_{1-3\varrho}$. Let $f \in W^{\mathbb{B},1}(B)$ be arbitrary. We split

$$\begin{aligned} \int_{\mathbb{R}^n} |\bar{f}(x+y) - \bar{f}(x)|^q dx &= \left(\int_{\mathbb{R}^n \setminus B_{1-3\varrho}} + \int_{B_{1-3\varrho}} \right) |\bar{f}(x+y) - \bar{f}(x)|^q dx \\ &=: \mathbf{I}_\varrho + \mathbf{II}_\varrho, \end{aligned}$$

with an obvious definition of $\mathbf{I}_\varrho$ and $\mathbf{II}_\varrho$.

Ad $\mathbf{I}_\varrho$. As $|y| < \varrho$, if $x \in \mathbb{R}^n \setminus B_{1-3\varrho}$, then $x+y \in \mathbb{R}^n \setminus B_{1-4\varrho}$. Therefore, we obtain with a constant $c > 0$ independent of $f \in \mathcal{F}$

$$\begin{aligned} \mathbf{I}_\varrho &\leq c \int_{B_1 \setminus B_{1-4\varrho}} |f(z)|^q dx \leq c \left(\int_B |f|^{\frac{n}{n-1}} dx \right)^{\frac{(n-1)q}{n}} \mathcal{L}^n(B_1 \setminus B_{1-4\varrho})^{\frac{n-q(n-1)}{n}} \\ &\leq c \mathcal{L}^n(B_1 \setminus B_{1-4\varrho})^{\frac{n-q(n-1)}{n}} \end{aligned}$$

and we may hence record that there exists $\delta_1 > 0$ such that if $0 < \varrho < \delta_1$, then $\mathbf{I}_\varrho < \varepsilon/3$.

Ad $\mathbf{II}_\varrho$. Firstly, since $1 \leq q < n/(n-1)$, we find and fix $0 < s < 1$ such that $q < n/(n-1+s)$. By Lemma 7.5.8, $W^{\mathbb{B},1}(B) \hookrightarrow L^{1^*}(B, V)$ implies that $\mathbb{B}$ is elliptic so that we are in position to suitably apply Lemma 7.4.7. Since $f(\cdot + y) - f(\cdot)$ equals $f_\varrho(\cdot + y) - f_\varrho(\cdot)$ on $B_{1-3\varrho}$ and f_ϱ is compactly supported in $\mathbb{R}^n$ with supports in a sufficiently large fixed ball, we find with a constant $c > 0$ independent of $f \in \mathcal{F}$

$$\begin{aligned} \mathbf{II}_\varrho &= \int_{B_{1-3\varrho}} |f_\varrho(x+y) - f_\varrho(x)|^q dx \leq \int_{\mathbb{R}^n} |f_\varrho(x+y) - f_\varrho(x)|^q dx \\ &\leq c \|\mathbb{B} f_\varrho\|_{L^1(B_R, W)}^q |y|^{sq} \\ &\leq c (\|\rho_\varrho \mathbb{B} f\|_{L^1(B_R, W)}^q + \|f \otimes_{\mathbb{B}} D\rho_\varrho\|_{L^1(B_R, W)}^q) |y|^{sq} \\ &\leq c (\|\mathbb{B} f\|_{L^1(B_R, W)}^q + \|f \otimes_{\mathbb{B}} D\rho_\varrho\|_{L^1(B_R, W)}^q) |y|^{sq} \end{aligned} \tag{7.15}$$

Pick $\delta_2 > 0$ such that if $|y| < \delta_2$, then $c \sup_{f \in \mathcal{F}} \|\mathbb{B} f\|_{L^1(B_R, W)}^q |y|^{sq} < \varepsilon/3$. Finally, we note because of $|D\rho_\varrho| \leq 4/\varrho$ by definition of ρ_ϱ ,

$$\left(\int_{\mathbb{R}^n} |D\rho_\varrho|^n dx \right)^{\frac{1}{n}} \leq \frac{c}{\varrho} ((1-2\varrho)^n - (1-3\varrho)^n)^{\frac{1}{n}} = 1 + \mathcal{O}(\varrho) \tag{7.16}$$

and so, by $W^{\mathbb{B},1}(B) \hookrightarrow L^{1^*}(B, V)$ and since $0 < \varrho \leq 1$,

$$\begin{aligned} \|D\rho_\varrho \otimes_{\mathbb{B}} f\|_{L^1(B_R, W)}^q |y|^{sq} &\leq \left(\int_{\mathbb{R}^n} |D\rho_\varrho|^n dx \right)^{\frac{q}{n}} \sup_{f \in \mathcal{F}} \|f\|_{W^{\mathbb{B},1}(B)}^q |y|^{sq} \\ &\leq C \sup_{f \in \mathcal{F}} \|f\|_{W^{\mathbb{B},1}(B)}^q |y|^{sq}, \end{aligned}$$

and from here it is evident that there exists $\delta_3 > 0$ such that if $|y| < \delta_3$, then $\|D\rho_\varrho \otimes_{\mathbb{B}} f\|_{L^1(B_R, W)}^q |y|^{sq} < \varepsilon/3$ and so, by (7.15), $\mathbf{II}_\delta < 2\varepsilon/3$ for all $f \in \mathcal{F}$. Now let $0 < \delta < \varrho := \min\{\delta_1, \delta_2, \delta_3\}$. Collecting estimates, we see that (ii) is satisfied and thus we can conclude that the compact embedding $W^{\mathbb{B},1}(B) \hookrightarrow L^q(B, V)$ holds. $\square$

With an inexpensive modification of the proof of Theorem 7.4.6, one can show that (c) implies that $W^{\mathbb{B},1}(B) \hookrightarrow L^q(B, V)$ for all $1 \leq q < p$, which trivially then implies (e).

Proof of Theorem 7.1.1. It remains to see that (e) implies (a), which is now a simple consequence of the Equivalence Lemma 7.5.2. We choose $E_1 = W^{\mathbb{B},1}(B)$, $E_2 = L^1(B, W)$, $E_3 = L^1(B, V)$, and $A := \mathbb{B} \in \mathcal{L}(W^{\mathbb{B},1}(B), L^1(B, W))$, whereas $B := \iota$ is the embedding operator $\iota: W^{\mathbb{B},1}(B) \hookrightarrow L^1(B, V)$. It is then clear that $\|u\|_{W^{\mathbb{B},1}(B)} = \|u\|_*$, so the equivalence lemma yields that $\mathbb{B}$ has finite dimensional null-space. $\square$

Proof of Theorem 7.1.3 (necessity of FDN). Assume that the embedding holds. By standard embeddings of Besov spaces, we have that $W^{\mathbb{B},1}(B) \hookrightarrow W^{k-1,p}(B, V)$ for some $p > 1$. If $k = 1$, we use Theorem 7.1.1, (c) implies (a), to see that $\mathbb{B}$ has FDN. Otherwise, we give the following simple argument: assume that $\mathbb{B}$ is not FDN, so that the maps $u_j(x) = \exp(jx \cdot \xi)v$ lie in $\ker \mathbb{B}$ for some non-zero complex ξ, v . We traced this example back to [172], but it was likely known before (cp. [20, Eq. (3.2)]). The assumed embedding and Hölder's Inequality give

$$\begin{aligned} j^{k-1} \left(\int_B |\exp(jx \cdot \xi)|^p dx \right)^{\frac{1}{p}} &\lesssim \|u_j\|_{W^{k-1,p}(B, V)} \lesssim \|u_j\|_{L^1(B, V)} \\ &\lesssim \left(\int_B |\exp(jx \cdot \xi)|^p dx \right)^{\frac{1}{p}}, \end{aligned}$$

which leads to a contradiction as $j \rightarrow \infty$. Here constants depend on $\text{diam } B, p, n$ only. $\square$

7.5. Appendix

7.5.1. Miscellaneous background The following relevant facts we quote without proof:

Lemma 7.5.1 ([201], Proposition 6.1). *Let $\mathbb{B}$ as in (7.4) be elliptic. Then $\mathbb{B}$ is cancelling if and only if we have that*

$$\int_{\mathbb{R}^n} \mathbb{B}u \, dx = 0$$

for all $u \in C^\infty(\mathbb{R}^n, V)$ such that the support of $\mathbb{B}u$ is compact.

Lemma 7.5.2 (Peetre–Tartar Equivalence Lemma, [188, Lem. 11.1]). *Let E_1 be a Banach space and let E_2, E_3 be two normed spaces (with corresponding norms $\|\cdot\|_i$, $i \in \{1, 2, 3\}$) and let $A \in \mathcal{L}(E_1, E_2)$ and $B \in \mathcal{L}(E_1, E_3)$ be two bounded linear operators such that B is compact and the norms $\|\cdot\|_1$ and $\|\cdot\|_* := \|A \cdot\|_2 + \|B \cdot\|_3$ are equivalent on E_1 . Then $\dim(\ker A) < \infty$.*

Theorem 7.5.3 ([108, Thm. 4]). *Let $\mathbb{B}$ as in (7.4) be $\mathbb{C}$ -elliptic, and $\Omega \subset \mathbb{R}^n$ be a star-shaped domain with respect to a ball. Then there exist an integer $d := d(\mathbb{B})$, a linear map $\mathcal{P} \in \text{Lin}(C^\infty(\bar{\Omega}, V), \mathbb{R}_d[x]^V)$ and a smooth map $K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta, \text{Lin}(W, V))$, where $\Delta = \{(x, x) : x \in \mathbb{R}^n\}$ such that $|D_x^\alpha D_y^\beta K(x, y)| \lesssim |x - y|^{k-n-|\alpha|-|\beta|}$ for all multi-indices α, β and all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, and*

$$u(x) = \mathcal{P}u(x) + \int_{\Omega} K(x, y) \mathbb{B}u(y) \, dy$$

for all $x \in \Omega$ and $u \in C^\infty(\bar{\Omega}, V)$. Therefore $\ker \mathbb{B} \subseteq \mathbb{R}_d[x]^V$.

7.5.2. Other facts about $W^{\mathbb{B}, p}$ We collect some complementary results that explain, e.g., our choice of definition for the $\mathbb{B}$ -Sobolev spaces and of extension technique for $p = 1$.

Definition 7.5.4. *A connected open set $\Omega \subset \mathbb{R}^n$ is called a*

- (a) C^0 -domain if for any $x \in \partial\Omega$ there exist a neighbourhood $\mathcal{N}$ of x relatively open in Ω , a coordinate system in $\mathbb{R}^n$ and a continuous function f such that, in the new coordinates (x', x_n) , $\mathcal{N} = \{(x', x_n) : 0 < x_n < f(x'), x' \in B_1(0)\}$.
- (b) $C^{0,1}$ - (or Lipschitz-) domain if Ω is a C^0 -domain and the function f above can be chosen to be Lipschitz.
- (c) domain with the cone property if for any $x \in \Omega$ there exists a cone $\mathcal{C}$ with apex at x and a coordinate system with respect to which, for some constants $c_i > 0$ we have $\mathcal{C} = \{(x', x_n) : |x'|^2 \leq c_1 x_n^2, 0 \leq x_n \leq c_2\}$.
- (d) star-shaped domain (with respect to a ball $B \subset \Omega$) if for all $x \in \Omega$, $y \in B$, and $0 \leq \theta \leq 1$ we have that $\theta x + (1 - \theta)y \in \Omega$.

We collect a few facts from [134, Sec. 1.1] on bounded domains, which will be used without mention in the sequel: any star-shaped domain is Lipschitz; Lipschitz domains have the cone property; domains with the cone property can be written as finite unions of star-shaped domains. The following density result closely mimics [134, Sec. 1.1.4-5]. We reproduce the proof here since, on one hand, it is very elegant and, on the other, it is crucial to prove the extension Theorem 7.4.1.

Lemma 7.5.5. *Let $\mathbb{B}$ be as in (7.4), $1 \leq p < \infty$, and $\Omega \subset \mathbb{R}^n$ be a bounded C^0 -domain. Then $C^\infty(\bar{\Omega}, V)$ is dense in $W^{\mathbb{B}, p}(\Omega)$. The same holds true for $V^{\mathbb{B}, p}(\Omega)$.*

Proof. We only prove the result for $V^{\mathbb{B}, p}$, the other case following in the same manner.

Step I. We first show that $C^\infty(\Omega, V) \cap V^{\mathbb{B},p}(\Omega)$ is dense in $V^{\mathbb{B},p}(\Omega)$. This step requires no regularity or boundedness assumption on Ω , other than that it is open in $\mathbb{R}^n$.

Consider a Whitney decomposition $\{Q_j\}_{j=1}^\infty$ of Ω (which is locally finite), and let $\varepsilon \in (0, 1/2)$ and $\rho_j \in C_c^\infty(Q_j)$ be a partition of unity associated with this decomposition. We denote by v_j a mollification of $\rho_j u$ such that

- (a) $\text{spt } v_j$ is also a locally finite cover of Ω ;
- (b) If $\text{diam } v_j = \lambda_j \text{diam } Q_j$, then $\lambda_j \downarrow 0$;
- (c) $\|\rho_j u - v_j\|_{V^{\mathbb{B},p}(\Omega)} \leq \varepsilon^j$ for all $j \geq 1$.

To make (c) plain, we recall, that mollification and weak derivatives are interchangeable. It follows that $v = \sum v_j \in C^\infty(\Omega, V)$ and $u = \sum \rho_j u$ in any compact subset of Ω . Moreover, due to the upper bound on ε , we obtain

$$\|u - v\|_{W^{\mathbb{B},p}(\Omega)} \leq \sum_{j=1}^\infty \|\rho_j u - v_j\|_{W^{\mathbb{B},p}(\Omega)} \leq \varepsilon(1 - \varepsilon)^{-1} \leq 2\varepsilon,$$

which proves that $v \in V^{\mathbb{B},p}(\Omega)$ and concludes the proof of this step.

Step II. To conclude the proof of the Lemma, by Step I, it suffices to show density of $C^\infty(\bar{\Omega}, V)$ in $C^\infty(\Omega, V) \cap V^{\mathbb{B},p}(\Omega)$.

We cover the boundary of Ω by open neighbourhoods $\{\mathcal{N}_x\}_{x \in \partial\Omega}$, where each $\mathcal{N}_x$ is the graph of a continuous function as in Definition 7.5.4(a). We extract a finite subcollection $\{\mathcal{N}_j\}_j$ that still covers $\partial\Omega$ and let $\{\rho_j\}_j$ be a partition of unity associated with $\{\mathcal{N}_j\} \cup \mathcal{N}$, where $\Omega \setminus \bigcup \mathcal{N}_j \Subset \mathcal{N} \Subset \Omega$. Since $u = \sum \rho_j u$ in Ω , it suffices to prove the claim for u and Ω relabelled by $\rho_j u$ and $\mathcal{N}_j \cap \Omega$, respectively. In coordinates (x', x_n) as in Definition 7.5.4(a), we choose $u_\varepsilon(x', x_n) = u(x', x_n - \varepsilon)$ for small enough $\varepsilon > 0$. Clearly $u_\varepsilon \in C^\infty(\bar{\Omega}, V)$ and

$$\|\partial^\alpha u - \partial^\alpha u_\varepsilon\|_{L^p(\Omega, V)} \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0 \text{ whenever } \partial^\alpha u \in L^p(\Omega, V),$$

which completes the proof. $\square$

Lemma 7.5.6. *Let $\mathbb{B}$ be as in (7.4) have FDN and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then $W^{\mathbb{B},p}(\Omega) \simeq V^{\mathbb{B},p}(\Omega)$, for each $1 \leq p \leq \infty$.*

Proof. One embedding is clear by definition. Conversely, we first prove the inequality under the extra assumption that Ω is star-shaped with respect to a ball. Let $u \in W^{\mathbb{B},p}(B)$. We use Proposition 7.4.2 to estimate, for $1 \leq l \leq k-1$,

$$\|\nabla^l u\|_{p,\Omega} \leq \|\nabla^l(u - \pi_\Omega u)\|_{p,\Omega} + \|\nabla^l \pi_\Omega u\|_{p,\Omega} \lesssim \|\mathbb{B}u\|_{p,\Omega} + \|\nabla^l \pi_\Omega u\|_{p,\Omega}$$

To estimate the latter term, we note that $P \mapsto \|\nabla^l P\|_{p,\Omega}$ defines a semi-norm on $\mathbb{R}_d[x]^V$, so it is controlled by the L^p -norm of P . We have that

$$\|\nabla^l \pi_\Omega u\|_{p,\Omega} \lesssim \|\pi_\Omega u\|_{p,\Omega} \leq \|u - \pi_\Omega u\|_{p,\Omega} + \|u\|_{p,\Omega} \lesssim \|\mathbb{B}u\|_{p,\Omega} + \|u\|_{p,\Omega},$$

where the last inequality follows by another application of Proposition 7.4.2. Altogether, we have proved that

$$\|u\|_{V^{\mathbb{B},p}(\Omega)} \leq C(\Omega) \|u\|_{W^{\mathbb{B},p}(\Omega)}. \quad (7.17)$$

We now assume just that Ω is Lipschitz, hence has the cone property. Hence there exist M sub-domains Ω_i that are star-shaped with respect to a ball and cover $\Omega = \bigcup_{i=1}^M \Omega_i$. We apply (7.17) in each Ω_i to get

$$\begin{aligned} \|u\|_{V^{\mathbb{B},p}(\Omega)} &\leq \sum_{i=1}^M \|u\|_{V^{\mathbb{B},p}(\Omega_i)} \leq \sum_{i=1}^M C(\Omega_i) \|u\|_{W^{\mathbb{B},p}(\Omega_i)} \\ &\leq \max_{1 \leq i \leq M} C(\Omega_i) \sum_{i=1}^M \|u\|_{W^{\mathbb{B},p}(\Omega_i)}. \end{aligned}$$

Let now $1 \leq p < \infty$. By concavity of the function $[0, \infty) \ni t \mapsto t^{1/p}$ and Jensen's Inequality, we obtain

$$\sum_{i=1}^M \|u\|_{W^{\mathbb{B},p}(\Omega_i)} \leq M^{1-1/p} \|u\|_{W^{\mathbb{B},p}(\Omega)}. \quad (7.18)$$

If $p = \infty$, we simply estimate $\|u\|_{W^{\mathbb{B},\infty}(\Omega_i)}$ by $\|u\|_{W^{\mathbb{B},\infty}(\Omega)}$ to note that (7.18) holds for $p = \infty$ as well (with the convention $1/p = \infty$). The proof is complete. $\square$

Lemma 7.5.7. *Let $\mathbb{B}$ as in (7.4) have FDN, $1 < p < \infty$, and $\Omega \subset \mathbb{R}^n$ be a star-shaped domain with respect to a ball. Then there exists a bounded, linear extension operator $E_\Omega: W^{\mathbb{B},p}(\Omega) \rightarrow V^{k,p}(\mathbb{R}^n, V)$.*

Proof. We use the extension suggested in [109], namely, in the notation of Theorem 7.5.3,

$$E_\Omega u(x) := \eta(x) \left(\mathcal{P}u(x) + \int_{\Omega} K(x, y) \mathbb{B}u(y) dy \right)$$

for $u \in C^\infty(\bar{\Omega}, V)$ and $x \in \mathbb{R}^n$. Here $\eta \in C_c^\infty(\mathbb{R}^n)$ is a smooth cut-off that equals 1 in a neighbourhood of Ω . We abbreviate $\mathcal{K}u = \int_{\Omega} K(\cdot, y) \mathbb{B}u(y) dy$. Let $0 \leq l \leq k$, and let B be a ball containing the support of η . Then, with domain dependent constants,

$$\|\nabla^l E_\Omega u\|_{p,B} \lesssim \sum_{j=0}^l \|\nabla^j (\mathcal{P}u + \mathcal{K}u)\|_{p,B} \leq \|\mathcal{P}u\|_{V^{l,p}(B,V)} + \sum_{j=0}^l \|\nabla^j \mathcal{K}u\|_{p,B}.$$

We note that $\|\cdot\|_{V^{l,p}(B,V)}$ and $\|\cdot\|_{L^p(\Omega,V)}$ both define norms on $\mathbb{R}_d[x]^V$, hence they are equivalent. We also remark that $\nabla^j \mathcal{K}u = \int_{\Omega} \nabla_x^j K(\cdot, y) \mathbb{B}u(y) dy$, so that

$$\|\nabla^j \mathcal{K}u\|_{p,B} \leq \|\nabla^j \mathcal{K}u\|_{p,\mathbb{R}^n} \lesssim \|\mathbb{B}u\|_{p,\Omega}. \quad (7.19)$$

If $0 \leq j < k$, the proof of (7.19) is presented in the proof of Proposition 7.4.2. If $j = k$, (7.19) follows from [175, Ch. II] and the growth bounds on $\nabla_x^k K$. Collecting, we get

$$\begin{aligned} \|\nabla^l E_\Omega u\|_{p,B} &\lesssim \|\mathcal{P}u\|_{p,\Omega} + \|\mathbb{B}u\|_{p,\Omega} \leq \|\mathcal{P}u + \mathcal{K}u\|_{p,\Omega} + \|\mathcal{K}u\|_{p,\Omega} + \|\mathbb{B}u\|_{p,\Omega} \\ &\lesssim \|u\|_{W^{\mathbb{B},p}(\Omega)}, \end{aligned}$$

where the last inequality is obtained from (7.19) with $j = 0$. $\square$

Lemma 7.5.8. *Let $\mathbb{B}$ be as in (7.4) and $\Omega \subset \mathbb{R}^n$ be a bounded, open set. If $W^{\mathbb{B},1}(\Omega) \hookrightarrow W^{k-1,p}(\Omega, V)$ for some $p > 1$, then $\mathbb{B}$ is elliptic.*

Proof. Suppose $\mathbb{B}$ is not elliptic, such that there exist $\xi \in \mathbb{S}^{n-1}$, $v \in V \setminus \{0\}$ such that $\mathbb{B}[\xi]v = 0$. Consider open cubes Q_1, Q_2 in $\mathbb{R}^n$ such that ξ is normal to one of their faces and $\bar{Q}_1 \subset \Omega \subset Q_2$, of side-lengths $2l_1, 2l_2$, respectively. We put $u(x) = f((x - x_0) \cdot \xi)v$ for x_0 the centre of Q_1 and $f(t) = |t|^{k-1-1/p}$ if $t \in \mathbb{R} \setminus \{0\}$. We have that $\mathbb{B}u = 0$ in $\mathbb{R}^n$ and

$$\int_{\Omega} |u| \, dx \leq \int_{Q_2} |u| \, dx = \int_{Q_2} |f((x - x_0) \cdot \xi)| |v| \, dx = l_2^{n-1} |v| \int_{-l_2}^{l_2} |t|^{k-1-1/p} \, dt < \infty,$$

so that $u \in W^{\mathbb{B},1}(\Omega)$. On the other hand,

$$\begin{aligned} \int_{\Omega} |D^{k-1}u|^p \, dx &\geq \int_{Q_1} |D^{k-1}u|^p \, dx = \int_{Q_1} |f^{(k-1)}((x - x_0) \cdot \xi)|^p |v \otimes^{k-1} \xi|^p \, dx \\ &= l_1^{n-1} |v \otimes^{k-1} \xi|^p \int_{-l_1}^{l_1} |t|^{-1} \, dt = \infty, \end{aligned}$$

so that $u \notin W_{\text{loc}}^{k-1,p}(\Omega, V)$. The proof is complete. $\square$

Lemma 7.5.9. *Let $\mathbb{B}$ be as in (7.4). If $W^{\mathbb{B},1}(B) \hookrightarrow W^{k-1,n/(n-1)}(B, V)$, then $\mathbb{B}$ is elliptic and cancelling.*

Proof. Necessity of ellipticity follows via Lemma 7.5.8. We next show that our assumed embedding implies $\dot{W}^{\mathbb{B},1}(\mathbb{R}^n) \hookrightarrow \dot{W}^{k-1,n/(n-1)}(\mathbb{R}^n, V)$ by a scaling argument, so that cancellation follows by the necessity part of [201, Thm. 1.3]. Let $u \in C_c^\infty(\mathbb{R}^n, V)$ be such that $\text{spt } u \subset B_r := B(0, r)$. Then $u_r(x) := u(rx)$ for $x \in \mathbb{R}^n$ is also a test function, with $\text{spt } u_r \subset B := B(0, 1)$. We estimate, with constants independent of r :

$$\begin{aligned} \|D^{k-1}u\|_{L^{\frac{n}{n-1}}} &= \left(\int_{B_r} |D^{k-1}u(x)|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} = \left(\int_B r^{\frac{n(k-1)}{n-1}} |D^{k-1}u_r(y)|^{\frac{n}{n-1}} r^n \, dy \right)^{\frac{n-1}{n}} \\ &= r^{n-k} \left(\int_B |D^{k-1}u_r(y)|^{\frac{n}{n-1}} \, dy \right)^{\frac{n-1}{n}} \\ &\leq cr^{n-k} \int_B |\mathbb{B}u_r(y)| + |u_r(y)| \, dy \\ &= c \int_{B_r} |\mathbb{B}u(x)| \, dx + cr^{-k} \int_{B_r} |u(x)| \, dx \\ &\leq c \int_{B_r} |\mathbb{B}u(x)| \, dx = c \|\mathbb{B}u\|_{L^1}, \end{aligned}$$

where the last inequality follows from a change of variable and the Poincaré-type inequality with zero boundary values (for elliptic operators)

$$\|v\|_{L^1(\Omega, V)} \leq c(\text{diam } \Omega)^k \|\mathbb{B}v\|_{L^1(\Omega, W)} \quad (7.20)$$

for all $v \in C_c^\infty(\Omega, V)$. The proof is complete. $\square$

The inequality (7.20) follows from an iteration of Poincaré's Inequality, the Green-type Formula (7.10), and Theorem 7.2.2(b), in the following way:

$$\begin{aligned} \|v\|_{L^1(\Omega)} &\leq c(\text{diam } \Omega)^{k-1} \|D^{k-1}v\|_{L^1(\Omega)} = c(\text{diam } \Omega)^{k-1} \|\mathbb{K}_{\mathbb{B}} \star (\mathbb{B}v)\|_{L^1(\Omega)} \\ &\leq c(\text{diam } \Omega)^{k-1} \|I_1|\mathbb{B}v|\|_{L^1(\Omega)} \leq c(\text{diam } \Omega)^k \|\mathbb{B}v\|_{L^1(\Omega)} \end{aligned}$$

A similar, straightforward argument also gives the inequality in Lemma 7.5.10 below.

Lemma 7.5.10. *Let $\mathbb{B}$ as in (7.4) be elliptic, $k = 1$. Then for each $1 \leq p < n/(n-1)$, there exists $c > 0$ such that*

$$\|u\|_{L^p(B,V)} \leq c \|\mathbb{B}u\|_{L^1(B,W)}$$

for all $u \in C_c^\infty(B, V)$.

CHAPTER 8

Epilogue

Few aspects of this thesis are reasonably optimal, although we hope that we persuaded the reader that much fewer are crudely sub-optimal. There is, of course, much more to do, so we conclude the thesis with a list of open problems that were mentioned in the text and are in the scope of the author.

Firstly, as one of the main results of this thesis, it was shown in Chapter 2 that for any operator¹ $\mathcal{A}$ of constant rank there exists another operator $\mathbb{B}$ such that

$$\ker \mathcal{A}(\xi) = \operatorname{im} \mathbb{B}(\xi) \quad \text{for all } \xi \neq 0. \quad (8.1)$$

This algebraic equality in Fourier space can then be used to show that $\mathcal{A}w = 0$ implies that $w = \mathbb{B}u$ for some u *provided that* w is a $(0, 1)^n$ -periodic function with zero average (Lemma 2.3.1). Without assumptions on the boundary conditions, it does not in general follow that

$$\mathcal{A}w = 0 \text{ in } \Omega \implies w = \mathbb{B}u. \quad (8.2)$$

This can be seen by taking the trivial example of an elliptic operator $\mathcal{A}$, i.e., $\operatorname{LHS}(8.1) = 0$, so $\mathbb{B} \equiv 0$, but there clearly are non-zero $\mathcal{A}$ -free fields. This is easily seen, e.g., if $\mathcal{A} = D^2$. The first open problem we suggest, which is admittedly quite vague, consists of the speculation that a relation such as (8.2) may hold up to an error:

Open Problem 8.0.1. *Let $\mathcal{A}$ and $\mathbb{B}$ be operators of constant rank satisfying (8.1) and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then each $w \in \mathcal{M}(\Omega, W)$ such that $\mathcal{A}w = 0$ can be written as*

$$w = \mathbb{B}u + v,$$

where $u \in \operatorname{BV}^{\mathbb{B}}(\Omega)$ and v is controlled suitably.

Alternatively, one can hope for a suitable Helmholtz-type decomposition

$$w = \mathbb{B}u + \mathcal{A}^*g + v,$$

¹As was the case in the entire thesis, by “operator”, we mean a linear homogeneous differential operator with constant coefficients.

with good control on each term. Do note that such a decomposition is not difficult to prove with $v \equiv 0$ and Schwartz functions u, g , provided that $w \in C_c^\infty(\Omega, W)$.

Examples when Open Problem 8.0.1 is well understood are $\mathcal{A} = \text{curl}$ (with $\mathbb{B} = D$ and $v \equiv 0$), $\mathcal{A} = \text{curl curl}$ (with $\mathbb{B} = \mathcal{E}$ and $v \equiv 0$) and $\mathcal{A}$ elliptic (when $\mathbb{B} = 0$ and v analytic). Settling this problem would likely bring the $\mathcal{A}$ -free and $\mathbb{B}$ -frameworks even closer (see Sections 1.1.1, 2.1, 1.2), in the sense that a bridge could likely be made directly at the level of the variational problems; recall that the results of this thesis show that there is no difference between the two frameworks at the level of $\mathcal{A}$ -quasiconvexity and $\mathcal{A}$ -free Young measures.

The main difficulty in solving Open Problem 8.0.1 stems from the fact that the we cannot alter measures w that are $\mathcal{A}$ -free in Ω in a way such that the exact relation (8.1) can be used. Either by truncating inside Ω , or by attempting to extend outside Ω , the behaviour of w near the boundary should be understood. So far, we see no way to use what information is readily available in this direction.

A problem that easily comes to mind and is worth working on is to remove the technical assumption of Theorem 3.1.1:

Open Problem 8.0.2. *Let $\nu = ((\nu_x), \lambda, (\nu_x^\infty))$ be a generalized Young measure with $\lambda(\partial\Omega) = 0$ and barycentre $w = \bar{\nu}_x \mathcal{L}^n + \bar{\nu}_x^\infty \lambda$ and let $\mathcal{A}$ be a constant rank operator. Suppose that $\mathcal{A}w = 0$ in Ω and that*

$$\langle f, \nu_x \rangle + \langle f^\infty, \nu_x^\infty \rangle \frac{d\lambda}{d\mathcal{L}^n}(x) \geq f\left(\bar{\nu}_x + \bar{\nu}_x^\infty \frac{d\lambda}{d\mathcal{L}^n}(x)\right)$$

holds for $\mathcal{L}^n$ -a.e. $x \in \Omega$ and all $\mathcal{A}$ -quasiconvex f of linear growth. Then ν is an $\mathcal{A}$ -free Young measure, i.e., is generated by an asymptotically $\mathcal{A}$ -free sequence.

Here the barycentre w is allowed to be an $\mathcal{A}$ -free measure in $\mathcal{M}(\Omega, W)$, whereas the partial result in Theorem 3.1.1 was obtained under the additional assumption that $w \in L^1(\Omega, W)$. Details on this issue were already given in Sections 1.1.2, 1.3.

The open problems discussed above pertain to the first part of the thesis, on variational problems. In the realm of linear L^1 -estimates, there are several challenging problems for which the author lacks significant evidence. However, his intuition points in the direction that they could be settled in the positive.

Open Problem 8.0.3 ([201, Sec. 8]). *Let $\mathbb{B}$ be an elliptic and canceling operator of order k on $\mathbb{R}^n$ and $s \in (0, n)$. Does either of the inequalities*

$$\|D^{k-1}u\|_{L^{\frac{n}{n-1},1}(\mathbb{R}^n)} \leq c\|\mathbb{B}u\|_{L^1(\mathbb{R}^n)} \quad \text{or} \quad \|u\|_{\dot{B}_1^{k-s,\frac{n}{n-s}}(\mathbb{R}^n)} \leq c\|\mathbb{B}u\|_{L^1(\mathbb{R}^n)}$$

hold for all $u \in C_c^\infty$?

The above is perhaps the most challenging problem in this field and little is known to support the claim. Firstly, the weaker inequalities under $L^{n/(n-1),q_-}$ and $\dot{B}_q^{k-s,n/(n-s)}$ -norms hold true for $1 < q < \infty$, as proved by VAN SCHAFTINGEN in [201, Sec. 8]. Secondly, it may be interesting to know that the latter estimate is stronger, so the Lorentz estimate may be more approachable. Lastly, the author

feels that the question can be settled in the positive, but only due to his failure of finding a stronger necessary condition than cancellation on an elliptic operator $\mathbb{B}$ that satisfies the inequalities.

Another interesting question stemmed from the author's analysis of the inclusion

$$BV^{\mathbb{B}}(\mathbb{R}^n) \subset L^\infty \quad (8.3)$$

for n -th order elliptic operators $\mathbb{B}$ (see Sections 1.4, 5.1.2, 5.3, 5.4). One of the main results of this thesis was to show that the inclusion (8.3) is equivalent to a newly introduced *weakly canceling* condition (WC), which is strictly weaker than VAN SCHAFTINGEN's canceling condition (C). On the other hand, it is known that $BV^n(\mathbb{R}^n) \subset C$ if $n \geq 2$ [152]. Somewhat surprisingly, one can show that, in this context,

$$BV^{\mathbb{B}}(\mathbb{R}^n) \subset C \quad (8.4)$$

implies the canceling condition, so there are elliptic operators of $\mathbb{B}$ of order n for which (8.3) holds, but (8.4) fails. We speculate that the following can be settled in the positive:

Open Problem 8.0.4. *Let $\mathbb{B}$ be an elliptic and canceling operator of order n on $\mathbb{R}^n$. Is it the case that $BV^{\mathbb{B}}(\mathbb{R}^n) \subset C$, i.e., if $\mathbb{B}u$ is a measure, is it the case that u is continuous?*

So far, it seems that a good place to start would be to try to prove the claim for $\mathbb{C}$ -elliptic operators first, where more tools (e.g., from Chapter 7) are available.

The question of trace embeddings on lower dimensional manifolds was studied in Chapter 6. In particular, the question of Sobolev regularity of traces on 1-co-dimensional surfaces for higher order operators remains open:

Open Problem 8.0.5. *Let $\mathbb{B}$ be an elliptic operator of order $k > 1$ on $\mathbb{R}^n$ that satisfies (SC). Is it the case that*

$$\|D^{k-1}u\|_{L^1(\Sigma)} \leq c\|\mathbb{B}u\|_{L^1(\mathbb{R}^n)}$$

hold for all $u \in C_c^\infty$ and all hyperplanes $\Sigma \subset \mathbb{R}^n$ of co-dimension one?

The claim is true for $\mathbb{C}$ -elliptic operators, as shown in Theorem 6.5.1. Sadly, the class of $\mathbb{C}$ -elliptic operators is, in general, strictly larger than the newly introduced class of *strongly canceling* (SC) operators (Lemma 6.3.2). In fact, the two classes coincide only for first order operators, in which case Open Problem 8.0.5 can be settled in the positive (Theorem 6.1.4). The only evidence the author has that the problem can be settled in the positive is that the proof of necessity of (SC) for the estimate seems very natural and conceptually similar to the proof of necessity of (C) for [201, Thm. 1.3] (Proposition 6.1.3).

The boundary estimate proved in Chapter 7 is conceptually pleasing particularly when viewed as a generalization of the Gagliardo–Nirenberg–Sobolev inequality on domains. However, as far as more general boundary estimates go, it would be desirable to prove an L^1 -version of Theorem 1.1.13. With this in mind, we formulate the following simplified statement:

Open Problem 8.0.6 ([202, Open Prob. 3]). *Let $\mathbb{B}$ be an elliptic operator of order k on $\mathbb{R}^n$ and $\nu \in \mathbb{R}^n$ be a unit vector, normal to a hyperplane Σ . Suppose that $\ker_{\mathbb{C}} \mathbb{B}(\xi + i\nu) = \{0\}$ for all $\xi \in \mathbb{R}^n$. Is it the case that either of the estimates*

$$\|D^{k-1}u\|_{L^{\frac{n}{n-1}}(\Sigma^+)} \leq c\|\mathbb{B}u\|_{L^1(\Sigma^+)} \quad \text{or} \quad \|D^{k-1}u\|_{L^1(\Sigma)} \leq c\|\mathbb{B}u\|_{L^1(\Sigma^+)}$$

hold for all $u \in C_c^\infty$? Here Σ^+ is a half-space bounded by Σ .

The claim seems more than plausible, considering the known estimates for canceling operators (Theorem 1.1.20), the L^p -version (1.1.13), and the author's contribution in Theorem 7.1.3. However, properly tackling this problem requires quite a bit of technical knowledge that the author has not acquired yet.

The trace inequalities suggested in Open Problems 8.0.5 and 8.0.6 amount to proving the Sobolev regularity $W^{k-1,1}$ at the boundary. However, this can be refined on the scale of Besov spaces to $B_1^{k-1,1}$, as is the case for BV^k [138].

Open Problem 8.0.7. *Let $\mathbb{B}$ be a $\mathbb{C}$ -elliptic operator of order $k > 1$ and let $B \subset \mathbb{R}^n$ be the unit ball. Is it the case that*

$$\|u\|_{B_1^{k-1,1}(\partial B)} \leq c \left(\|\mathbb{B}u\|_{L^1(B)} + \|u\|_{L^1(B)} \right)$$

for all u smooth in a neighbourhood of B ?

The author suspects that a suitable refinement of the estimates in the proof of Theorem 6.5.1 is possible and will pursue this idea in the future.

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At the end of the road, the author would like to thank the interested reader for embarking on this journey into uncharted facts on simple differential operators and peculiarities of L^1 . It is his hope that the dedication with which this work has been derived and collected transpired from the manuscript. It is also the author's hope that aspects of this work were worth pondering about for the reader, if not useful, or perhaps beautiful. We conclude by making a case for this final Chapter, that this document is in fact the start of a much longer road on which the curious reader is invited to embark on.

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Time to slip in that zone till' I find myself
Inside the realm of the unknown and boldly go
In the waters where nobody else has gone before
Or willing to go, uncharted
Eminem