

# Moving planes for domain walls in a coupled system

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## Abstract

The system leading to phase segregation in two-component Bose-Einstein condensates can be generalized to hyperfine spin states with a Rabi term coupling. This leads to domain wall solutions having a monotone structure for a non-cooperative system. We use the moving plane method to prove monotonicity and one-dimensionality of the phase transition solutions. This relies on totally new estimates for a type of system for which no Maximum Principle a priori holds. We also derive that one dimensional solutions are unique up to translations. When the Rabi coefficient is large, we prove that no non-constant solutions can exist.

## 1 Introduction

We study the monotonicity and uniqueness of a non-cooperative system coming from the physics of two-component Bose-Einstein condensates which displays partial phase transition. We are able to use the moving plane device in a case where a priori bounds do not come from standard techniques. The particularity of this system with respect to classical two-component segregated Bose-Einstein condensates is to couple both linearly and nonlinearly the two equations due to spin coupling of the hyperfine states

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added to the intercomponent coupling. Our system for the functions  $u$  and  $v$  is the following:

$$\begin{cases} \Delta u = -u(1 - u^2 - v^2) - v(\omega - \alpha uv) =: P(u, v) & \text{in } \mathbb{R}^N, \\ \Delta v = -v(1 - u^2 - v^2) - u(\omega - \alpha uv) =: Q(u, v) & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where  $u$  and  $v$  are the wave functions for each component,  $\alpha$  is a positive parameter such that  $1 + \alpha$  is the intercomponent coupling, and  $\omega$  is a positive parameter, which is the Rabi frequency of the one body coupling between the two internal states. We would like to study the phase transition solutions, so we will impose boundary conditions at infinity in one direction, namely:

$$\begin{cases} u(x', x_N) \rightarrow b, & v(x', x_N) \rightarrow a, & \text{as } x_N \rightarrow +\infty, \\ u(x', x_N) \rightarrow a, & v(x', x_N) \rightarrow b, & \text{as } x_N \rightarrow -\infty, \end{cases} \quad (1.2)$$

the limit being uniform in  $x' \in \mathbb{R}^{N-1}$ , where  $(a, b)$  is the solution to

$$a^2 + b^2 = 1, \quad ab = \frac{\omega}{\alpha}, \quad 0 < a < b. \quad (1.3)$$

The existence of such solutions requires, in addition to the positivity of  $\alpha$  and  $\omega$ , the condition

$$\frac{\omega}{\alpha} < \frac{1}{2}. \quad (1.4)$$

The study of domain wall solutions in coupled Gross-Pitaevskii equations or segregation patterns has been the subject of many papers concerning existence, uniqueness, monotonicity of asymptotic behaviour [2, 4, 19, 31]. It corresponds to the case  $\omega = 0$  and  $\alpha > 0$ .

Here, we would like to address a different physical background, that of a two-component Bose-Einstein condensates representing two different hyperfine states, and coupled by their spin, to take into account a one body coherent Rabi coupling, which corresponds to  $\omega > 0$ . In this case, segregation is not complete and this leads to what is called Rabi oscillations which have been experimentally observed in [28]. The ground states and excited states have been studied in [1, 15, 27, 29, 32]. The system (1.1)-(1.2) for  $N = 1$  has been analyzed in [3] where the existence and asymptotic properties of one dimensional domain wall solutions are derived in the case  $\omega/\alpha$  of order 1 and  $\alpha$  large and small. The properties and structures found in [3] have led us to investigate the monotonicity and uniqueness of solutions. Let us point out that this system has a heteroclinic structure and derives from the minimization of the Gross-Pitaevskii energy. Understanding the solutions of (1.1)-(1.2) is a first step in a future mathematical analysis of the specific vortex patterns appearing in the Rabi

coupled condensates called multidimer bound states [1, 13, 26]: this is a state where a pair of vortices of different components binds together as a molecule. These molecules then interact in a nontrivial manner with other molecules, with the result that a rich hierarchy of patterns, such as honeycombed, triangular, and square, is formed. It is the possibility to have non segregation at infinity with these positive limits which is at the origin of this specific pattern.

Here is our main result:

**Theorem 1.1.** *Assume  $\alpha > 0$ ,  $\omega > 0$ , and (1.4) holds. Then all positive solutions to (1.1)-(1.2) depend only on  $x_N$ , satisfy  $\partial_N u > 0$  and  $\partial_N v < 0$  and are translations of one another along the  $x_N$  direction.*

The proof relies on the moving plane device of [5, 7, 21, 22, 30] to prove monotonicity and one-dimensionality of the phase transition solutions and the sliding method [6, 9, 10] to prove uniqueness. We point out that it is not clear a priori that these methods can be applied as our system does not satisfy the usual monotonicity known for cooperative systems. Therefore, as a first step, we need to prove key estimates that will allow us to perform these methods and provide a structure to the system. These estimates also hold for solutions of the system without any conditions at infinity:

**Proposition 1.2.** *Assume  $\alpha > 0$ ,  $\omega > 0$ , and (1.4) holds. Let  $(u, v)$  be a positive regular solution of (1.1), then*

$$u^2 + v^2 \leq 1 \text{ and } uv \geq \frac{\omega}{\alpha}. \quad (1.5)$$

Moreover, either  $(u, v) \equiv (a, b)$  or  $(u, v) \equiv (b, a)$  or

$$a < u, v < b. \quad (1.6)$$

The bounds (1.5) provide a special structure to the problem and come at various stages in our proof. In the case  $\omega = 0$ , the bound is only  $u^2 + v^2 \leq 1$  and is obtained by Kato's inequality. Indeed, the lower bound is reduced to  $uv \geq 0$ . In this paper, it becomes  $uv \geq \omega/\alpha$ , and the upper and lower bounds have to be proved together in a way which is similar to an attractor set. It is not a simple application of comparison principles or Maximum Principle. Proving these bounds requires some non-trivial extra-work, to which we devote the most part of Section 2.

In order to start the moving plane method, we need a sign for the difference between the solution and its reflection with respect to a hyperplane close to infinity in one direction. Since for our non-cooperative system, we do not have asymptotic estimates of solutions at infinity, we show that the moving plane procedure can be started by obtaining decrease estimates of the  $L^2$  norm of the gradients of these

differences on large balls. This is inspired by [19] and relies on a Lemma in [18]. In our case, we need in a crucial way a lower bound on  $uv$  and the proof is not a mere adaptation of previous works since the estimates are more involved.

To continue the moving plane procedure, or to apply the sliding method, we need to be able to apply the strong Maximum Principle and the Hopf lemma. This relies again on our lower bound for  $uv$  and the special structure it provides to the system. Let us point out that once the bounds (1.5) are known, the equations lead to (1.6), but without a strict sign. Therefore, this provides signs to the right hand side of the system (1.1), namely the system gets a structure of the type

$$\begin{cases} -\Delta u = uf_1(u, v) + vf_2(u, v) & \text{in } \mathbb{R}^N, \\ -\Delta v = vf_1(v, u) + uf_2(v, u) & \text{in } \mathbb{R}^N, \end{cases} \quad (1.7)$$

with  $f_1$  positive, symmetric, decreasing in both variables and  $f_2$  negative, symmetric, decreasing in both variables. This structure of the system allows to apply the Strong Maximum Principle and the Hopf Lemma. One has to point out that this structure is not assumed from the beginning but is a consequences of our bounds. What our analysis emphasizes is that in order to obtain Maximum Principle or moving planes for non-cooperative systems, it requires both bounds from above for  $u^2 + v^2$  or  $-f_1(u, v)$  and below for  $uv$  or  $-f_2(u, v)$ .

The uniqueness proof for one dimensional solutions is based on the sliding method [6, 9, 10] and first requires an asymptotic behaviour for the ode's. Then the strong Maximum Principle relying on our lower bounds allows to apply the device and get uniqueness. We also refer to [23, 24, 25] where some related discussions on uniqueness of 1D solutions are made for a related system arising in the Landau-de Gennes theory for nematic liquid crystals.

We note that the existence of solutions to (1.1)-(1.2) can be obtained by approximation by solutions on finite intervals as proved in Proposition 4.1.

In the case  $\omega = 0$ , then the solution to (1.3) is given by  $a = 0$  and  $b = 1$ , and therefore domain wall solutions stay between 0 and 1. In the one-dimensional case the existence of a monotone solution has been derived in [4] while uniqueness has been derived in [2], based on a continuation argument and estimates of the linearized operator, under the assumption that either  $u' > 0$  or  $v' < 0$ . Asymptotic estimates in the large  $\alpha$  limit, still in one dimension, have been also obtained in [2], and rely on properties for a simpler outer system studied by many authors and for instance by [8, 11, 17, 20]. The other limit, that of weak segregation where  $\alpha$  tends to 0 has been analyzed in [31]. The proof that  $N$ -dimensional solutions are in fact one-dimensional and monotone is made in [19] and relies on the moving plane device. The combination of the results of [19] and [2] then leads to the uniqueness of the solution, up to translations.

We also get interesting results using similar estimates in the case where (1.4) does not hold. We prove that solutions to (1.1) are constant without any boundary condition at infinity.

**Theorem 1.3.** *Assume  $\alpha > 0$ , and*

$$\frac{\omega}{\alpha} \geq \frac{1}{2} \tag{1.8}$$

*holds. Let  $(u, v)$  be a positive regular solution of (1.1). Then  $(u, v)$  is identically equal to  $(c, c)$  with  $c = \sqrt{\frac{1+\omega}{2+\alpha}}$ .*

The paper is organized as follows: firstly, we prove our key a priori bounds, rough upper and lower bounds as well as the refined bounds of Proposition 1.2. Then we make the moving plane device work to get monotonicity and the one dimensional property. Lastly, we study the properties of the one-dimensional solutions: existence, uniqueness up to translations and exponential decay to constant at infinity.

## 2 A priori bounds. Proofs of Proposition 1.2 and Theorem 1.3.

The proofs of Proposition 1.2 and Theorem 1.3 follow from the same treatment. Note that the conclusion that  $u \equiv v \equiv c$  in Theorem 1.3 is equivalent to the pair of inequalities that  $u^2 + v^2 \leq 2c^2$  and  $uv \geq c^2$ , which resembles (1.5).

The rough idea of the proof is to bound the supremum of  $u^2 + v^2$  in terms of the infimum of  $uv$  and vice versa in such a way that the bounds self-bootstrap to the desired bounds.

### 2.1 A non-degeneracy estimate

We start with a result which gives positive lower and upper bounds for  $u + v$  for positive solutions to (1.1).

**Lemma 2.1.** *Let  $(u, v)$  be a positive regular solution of (1.1). Then*

$$\left[ \frac{1 + \omega}{\max(\frac{2+\alpha}{4}, 1)} \right]^{1/2} \leq u + v \leq \left[ \frac{1 + \omega}{\min(\frac{2+\alpha}{4}, 1)} \right]^{1/2} \text{ in } \mathbb{R}^N.$$

*In particular  $u, v \in L^\infty(\mathbb{R}^N)$ .*

*Proof.* Let  $w = u + v$ . Then

$$\Delta w = w(u^2 + v^2 + \alpha uv - 1 - \omega). \quad (2.1)$$

Thus, by the strong maximum principle, we have  $w > 0$  in  $\mathbb{R}^N$ .

In view of (2.1), we have

$$\Delta w \geq w \left( \min\left(\frac{2+\alpha}{4}, 1\right) w^2 - 1 - \omega \right) = \min\left(\frac{2+\alpha}{4}, 1\right) w(w^2 - \gamma^2) \quad (2.2)$$

where  $\gamma = \left[ \frac{1+\omega}{\min(\frac{2+\alpha}{4}, 1)} \right]^{1/2}$ .

By Kato's inequality, this implies that

$$\Delta(w - \gamma)_+ \geq \min\left(\frac{2+\alpha}{4}, 1\right) (w - \gamma)_+^3$$

which further implies  $(w - \gamma)_+ \equiv 0$  (see [12, Lemma 2]), i.e.

$$w \leq \gamma \text{ in } \mathbb{R}^N. \quad (2.3)$$

Returning to (2.1), we see that

$$\Delta w \leq w \left( \max\left(\frac{2+\alpha}{4}, 1\right) w^2 - 1 - \omega \right). \quad (2.4)$$

and thus the rescaled function  $\tilde{w}(x) = \max(\frac{(2+\alpha)^{1/2}}{2}, 1)(1+\omega)^{-1/2} w((1+\omega)^{-1/2} x)$  is a positive supersolution of the Allen-Cahn equation, i.e., it satisfies

$$-\Delta \tilde{w} \geq \tilde{w}(1 - \tilde{w}^2) \quad \text{in } \mathbb{R}^N. \quad (2.5)$$

To proceed, we need the following lemma.

**Lemma 2.2.** *There exists  $R_0 > 0$  such that, for  $R \geq R_0$ , the functional*

$$I[\varphi] = \int_{B_R} \left[ \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{4} (1 - \varphi^2)^2 \right] dx$$

*admits a non-trivial minimizer  $\psi$  in  $H_0^1(B_R)$ . Furthermore,  $\psi$  is a smooth function in  $\overline{B_R}$  satisfying  $0 < \psi < 1$  in  $B_R$ .*

Let us assume the above lemma for the moment and continue with the proof of Lemma 2.1. Let  $R_0$  be the constant in Lemma 2.2 and  $m := \min_{\overline{B_{R_0}}} \tilde{w} > 0$  then, for every  $\varepsilon \in (0, \min\{m, 1\})$ , the function  $\psi_\varepsilon := \varepsilon \psi$  satisfies

$$-\Delta \psi_\varepsilon \leq \psi_\varepsilon (1 - \psi_\varepsilon^2) \quad \text{in } \mathbb{R}^N,$$

and

$$\psi_\varepsilon \leq \tilde{w} \quad \text{in } B_{R_0}, \quad \psi_\varepsilon = 0 \quad \text{on } \partial B_{R_0}.$$

The sliding method (see [6, Lemma 3.1]), then gives that  $\tilde{w} \geq \varepsilon$  in  $\mathbb{R}^N$  and so  $w \geq \delta$  in  $\mathbb{R}^N$ , for some  $\delta > 0$ .

Now, from (2.4) we get

$$-\Delta w \geq w \left( 1 + \omega - \max\left(\frac{2+\alpha}{4}, 1\right) w^2 \right) \geq \delta \max\left(\frac{2+\alpha}{4}, 1\right) (\vartheta^2 - w^2) \quad (2.6)$$

where  $\vartheta = \left[ \frac{1+\omega}{\max(\frac{2+\alpha}{4}, 1)} \right]^{1/2}$ . Hence, by Kato's inequality we have

$$\Delta(\vartheta - w)_+ \geq \delta \max\left(\frac{2+\alpha}{4}, 1\right) [(\vartheta - w)_+]^2 \quad (2.7)$$

which implies  $(\vartheta - w)_+ \equiv 0$  (see [12, Lemma 2]), i.e.

$$w \geq \left[ \frac{1+\omega}{\max(\frac{2+\alpha}{4}, 1)} \right]^{1/2} \text{ in } \mathbb{R}^N. \quad (2.8)$$

□

*Proof of Lemma 2.2.* We have that  $I[0] = \frac{1}{4}|B_R|$ . Suppose that  $R > 1$  and consider the function

$$\varphi(x) = \min(R - |x|, 1) \text{ for } x \in B_R.$$

We have

$$I[\varphi] = \int_{B_R \setminus B_{R-1}} \left[ \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{4} (1 - \varphi^2)^2 \right] dx \leq \frac{3}{4} |B_R \setminus B_{R-1}|.$$

Clearly, for  $R$  sufficiently large,  $I[\varphi] < I[0]$  and so  $I$  possesses a non-trivial minimizer  $\psi$  in  $H_0^1(B_R)$ . Replacing  $\psi$  by  $\min\{|\psi|, 1\}$ , if necessary, we may assume that  $0 \leq \psi \leq 1$  in  $\overline{B_R}$ . Therefore  $\psi$  is a weak solution of the Allen-Cahn equation in  $\overline{B_R}$  and the remaining part of the claim follows by standard elliptic regularity and by the strong maximum principle. □

## 2.2 Proof of Proposition 1.2

*Proof.* We have  $u, v > 0$ , and, by Lemma 2.1, there is some  $p > 1$  such that

$$u + v \geq \frac{1}{p}. \quad (2.9)$$

Let  $A = u^2 + v^2$ ,  $B = -\ln(uv)$  and

$$m = \sup A, \quad m_* = \max(m, 1), \quad n = \sup B, \quad \text{and} \quad n_* = \max(n, -\ln \frac{\omega}{\alpha}). \quad (2.10)$$

1. We prove that

$$m \leq \frac{1}{2}(1 + \sqrt{1 - 8s_*}) \text{ where } s_* = \min_{t \geq e^{-n_*}} (\alpha t^2 - \omega t). \quad (2.11)$$

In particular, as  $s_* \geq \frac{-\omega^2}{4\alpha}$ , we have

$$m \leq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{2\omega^2}{\alpha}} \right) < \infty. \quad (2.12)$$

We have

$$\begin{aligned} \Delta A &\geq 2uP(u, v) + 2vQ(u, v) = 2u^4 + 2v^4 + 4(\alpha + 1)u^2v^2 - 2u^2 - 2v^2 - 4\omega uv \\ &= 2A^2 - 2A + 4\alpha e^{-2B} - 4\omega e^{-B}. \end{aligned}$$

Note that  $\alpha e^{-2B} - \omega e^{-B} \geq s_*$ , and, as  $n_* \geq -\ln \frac{\omega}{\alpha}$ ,  $s_* \leq 0$ . It follows that

$$\Delta A \geq 2A^2 - 2A + 4s_* = 2 \left( A - \frac{1}{2}(1 + \sqrt{1 - 8s_*}) \right) \left( A - \frac{1}{2}(1 - \sqrt{1 - 8s_*}) \right),$$

and so, by Kato's inequality,

$$\Delta \left( A - \frac{1}{2}(1 + \sqrt{1 - 8s_*}) \right)_+ \geq 2A^2 - 2A + 4s_* = 2 \left( A - \frac{1}{2}(1 + \sqrt{1 - 8s_*}) \right)_+^2.$$

Estimate (2.11) follows from [12, Lemma 2].

2. We prove that

$$n \leq -\ln \frac{\omega}{\alpha + 2 - \frac{2}{m_*}}. \quad (2.13)$$

Equivalently,

$$uv \geq \frac{\omega}{\alpha + 2 - \frac{2}{m_*}}. \quad (2.14)$$

We have

$$\begin{aligned} \Delta B &\geq -\frac{1}{u}P(u, v) - \frac{1}{v}Q(u, v) = -(\alpha + 2)(u^2 + v^2) + 2 + \frac{\omega(u^2 + v^2)}{uv} \\ &= -(\alpha + 2)A + 2 + \omega A e^B = 2 \left( 1 - \frac{A}{m_*} \right) + \left( \alpha + 2 - \frac{2}{m_*} \right) A \left( \frac{\omega}{\alpha + 2 - \frac{2}{m_*}} e^B - 1 \right). \end{aligned}$$



Using Kato's inequality, the inequality  $e^x - 1 \geq \frac{1}{2}x^2$  for  $x \geq 0$  and recalling (2.9), we get

$$\Delta(B - \ln \frac{\alpha + 2 - \frac{2}{m_*}}{\omega})_+ \geq \frac{\alpha + 2 - \frac{2}{m_*}}{4p^2} (B - \ln \frac{\alpha + 2 - \frac{2}{m_*}}{\omega})_+^2.$$

We deduce that  $(B - \ln \frac{\alpha + 2 - \frac{2}{m_*}}{\omega})_+ \equiv 0$ , again thanks to [12, Lemma 2]. This proves (2.13).

3. We prove that  $m_* = 1$  or  $n_* = -\ln \frac{\omega}{\alpha}$ .

Assume by contradiction that the above does not hold. Then  $m = m_* > 1$  and  $n = n_* > -\ln \frac{\omega}{\alpha}$ .

Let

$$h_1(t) = \frac{1}{2} \left( 1 + \sqrt{1 - 8\alpha t^2 + 8\omega t} \right), \quad (2.15)$$

$$h_2(t) = \frac{\omega t}{(\alpha + 2)t - 2}. \quad (2.16)$$

We claim that

$$m \leq h_1(h_2(m)). \quad (2.17)$$

Case (i):  $\alpha \geq 2$ . From (2.13), we have  $n \leq -\ln \frac{\omega}{\alpha+2} \leq -\ln \frac{\omega}{2\alpha}$ . It follows that

$$s_* = \min_{t \geq e^{-n}} (\alpha t^2 - \omega t) = \alpha e^{-2n} - \omega e^{-n}.$$

Plugging this into (2.11) yields

$$m \leq \frac{1}{2} (1 + \sqrt{1 - 8\alpha e^{-2n} + 8\omega e^{-n}}) = h_1(e^{-n}).$$

Since  $h_1$  is decreasing in  $[\frac{\omega}{2\alpha}, \infty)$ , this together with (2.13) implies (2.17).

Case (ii):  $\alpha < 2$ . As  $\alpha < 2$ ,  $\omega < \frac{\alpha}{2} < \frac{2\alpha}{2-\alpha}$  and so, by (2.12),

$$m \leq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{2\omega^2}{\alpha}} \right) < \frac{1}{2} \left( 1 + \sqrt{1 + \frac{8\alpha}{(2-\alpha)^2}} \right) = \frac{2}{2-\alpha}.$$

Inserting this into (2.13) yields  $n \leq -\ln \frac{\omega}{2\alpha}$ . We can now repeat the proof of Case (i) to reach (2.17).

We now compute

$$\begin{aligned} \frac{d}{dt}h_1(h_2(t)) &= h'_1(h_2(t))h'_2(t) = \frac{2(2\alpha h_2(t) - \omega)}{\sqrt{1 - 8\alpha h_2(t)^2 + 8\omega h_2(t)}} \frac{2\omega}{[(\alpha + 2)t - 2]^2} \\ &= \frac{1}{\sqrt{1 - 8\alpha h_2(t)^2 + 8\omega h_2(t)}} \frac{4\omega^2(\alpha t - 2t + 2)}{(\alpha t + 2t - 2)^3}. \end{aligned} \quad (2.18)$$

Note that  $h_2$  is decreasing in  $(\frac{2}{2+\alpha}, \infty)$  and so  $h_2(t) < \frac{\omega}{\alpha}$  for  $t > 1$ . Hence, for  $t > 1$ , we have  $\sqrt{1 - 8\alpha h_2(t)^2 + 8\omega h_2(t)} > 1$ ,  $\alpha t - 2t + 2 < \alpha t + 2t - 2$  and  $4\omega^2 < \alpha^2 < (\alpha t + 2t - 2)^2$  and so

$$\frac{d}{dt}h_1(h_2(t)) < 1 \text{ for all } t > 1.$$

As  $h_1(h_2(1)) = 1$ , this implies that the inequality equation  $t \leq h_1(h_2(t))$  has no solution in  $(1, \infty)$ . Therefore, (2.17) implies that  $m = 1$ . This finishes Step 3.

4. We prove (1.5). By Step 3, we have  $m_* = 1$  or  $n_* = -\ln \frac{\omega}{\alpha}$ .

If  $m_* = 1$ , then, in view of (2.13),  $n \leq -\ln \frac{\omega}{\alpha}$ , and so  $A \leq 1$  and  $B \geq -\ln \frac{\omega}{\alpha}$ , which give (1.5).

On the other hand, if  $n_* = -\ln \frac{\omega}{\alpha}$ , then, by (2.11),  $m \leq 1$ . Again we obtain  $A \leq 1$  and  $B \geq -\ln \frac{\omega}{\alpha}$ , which also give (1.5) as desired.

5. Finally, we prove the trichotomy that either  $(u, v) \equiv (a, b)$  or  $(u, v) \equiv (b, a)$  or (1.6) holds.

Suppose that  $(u, v) \not\equiv (a, b)$  and  $(u, v) \not\equiv (b, a)$ . From (1.5), we have

$$a \leq u, v \leq b. \quad (2.19)$$

We note that

$$P(b, v) = b(b^2 + v^2 - 1) + v(\alpha bv - \omega) \stackrel{(2.19)}{\geq} b(b^2 + a^2 - 1) + v(\alpha ba - \omega) = 0.$$

Hence the constant function  $b$  satisfies

$$\Delta b = 0 \leq P(b, v).$$

Since  $u \leq b$ , and  $\Delta u = P(u, v)$ , the strong maximum principle implies either  $u < b$  or  $u \equiv b$ . If the latter case holds, the second equation of (1.1) implies that  $v \equiv a$ , which contradicts our assumption that  $(u, v) \not\equiv (b, a)$ . We thus have  $u < b$ .

The remaining inequalities in (1.6) are shown similarly using

$$P(a, v) = a(a^2 + v^2 - 1) + v(\alpha av - \omega) \stackrel{(2.19)}{\leq} a(a^2 + b^2 - 1) + v(\alpha ab - \omega) = 0,$$

$$Q(u, a) = a(u^2 + a^2 - 1) + u(\alpha ua - \omega) \stackrel{(2.19)}{\leq} a(b^2 + a^2 - 1) + u(\alpha ba - \omega) = 0,$$

$$Q(u, b) = b(u^2 + b^2 - 1) + u(\alpha ub - \omega) \stackrel{(2.19)}{\geq} b(a^2 + b^2 - 1) + u(\alpha ab - \omega) = 0.$$

We omit the details. □

### 2.3 Proof of Theorem 1.3

*Proof.* We adapt the proof of Proposition 1.2, as the conclusion is equivalent to the following pair of inequalities:

$$u^2 + v^2 \leq \frac{2(1+\omega)}{2+\alpha} \text{ and } uv \geq \frac{1+\omega}{2+\alpha}.$$

By Lemma 2.1, (2.9) holds. Let  $A = u^2 + v^2$ ,  $B = -\ln(uv)$  and

$$\begin{aligned} m &= \sup A, \quad \tilde{m}_* = \max\left(m, \frac{2(1+\omega)}{2+\alpha}\right), \\ n &= \sup B, \quad \text{and} \quad \tilde{n}_* = \max\left(n, -\ln \frac{1+\omega}{2+\alpha}\right). \end{aligned}$$

(Note the difference between the definition of  $\tilde{m}_*$  and  $\tilde{n}_*$  and that of  $m_*$  and  $n_*$  in the proof of Proposition 1.2.)

1. We prove that

$$m \leq \frac{1}{2} \min \left( 1 + \sqrt{1 - 8s_1}, 1 + \omega + \sqrt{(1+\omega)^2 - 8\alpha s_2} \right), \quad (2.20)$$

where  $s_1 = \min_{t \geq e^{-\tilde{n}_*}} (\alpha t^2 - \omega t) \leq 0$  and  $s_2 = e^{-2\tilde{n}_*} \leq \frac{(1+\omega)^2}{8\alpha}$ .

As  $\tilde{n}_* \geq -\ln \frac{1+\omega}{2+\alpha} \geq -\ln \frac{\omega}{\alpha}$  (due to  $\omega \geq \frac{1}{2}\alpha$ ),  $s_1 \leq 0$ . Also  $s_2 \leq \frac{(1+\omega)^2}{(2+\alpha)^2} \leq \frac{(1+\omega)^2}{8\alpha}$ .

The proof of the inequality  $m \leq \frac{1}{2}(1 + \sqrt{1 - 8s_1})$  follows from the differential inequality

$$\Delta A \geq 2A^2 - 2A + 4\alpha u^2 v^2 - 4\omega uv,$$

exactly as in the proof of (2.11). To obtain  $m \leq \frac{1}{2}(1 + \omega + \sqrt{(1+\omega)^2 - 8\alpha s_2})$ , we use the inequality  $v_1 v_2 \leq \frac{1}{2}A$  in the above differential inequality:

$$\begin{aligned} \Delta A &\geq 2A^2 - 2(1+\omega)A + 4\alpha e^{-2B} \geq 2A^2 - 2(1+\omega)A + 4\alpha s_2 \\ &= 2 \left( A - \frac{1}{2}(1+\omega + \sqrt{(1+\omega)^2 - 8\alpha s_2}) \right) \left( A - \frac{1}{2}(1+\omega - \sqrt{(1+\omega)^2 - 8\alpha s_2}) \right). \end{aligned}$$

By Kato's inequality, this leads to

$$\Delta \left( A - \frac{1}{2}(1+\omega + \sqrt{(1+\omega)^2 - 8\alpha s_2}) \right)_+ \geq 2 \left( A - \frac{1}{2}(1+\omega + \sqrt{(1+\omega)^2 - 8\alpha s_2}) \right)_+^2$$

and so, by [12, Lemma 2],

$$A \leq \frac{1}{2}(1 + \omega + \sqrt{(1 + \omega)^2 - 8\alpha s_2}).$$

We have thus proved (2.20).

2. As in the proof of Proposition 1.2, we have

$$n \leq -\ln \frac{\omega}{\alpha + 2 - \frac{2}{\tilde{m}_*}}. \quad (2.21)$$

3. We show that  $\tilde{m}_* = \frac{2(1+\omega)}{2+\alpha}$  or  $\tilde{n}_* = -\ln \frac{1+\omega}{2+\alpha}$ . Assume by contradiction that this does not hold, so that  $m = \tilde{m}_* > \frac{2(1+\omega)}{2+\alpha}$  and  $n = \tilde{n}_* > -\ln \frac{1+\omega}{2+\alpha}$ .

Case (a):  $\frac{1+\omega}{2+\alpha} \geq \frac{\omega}{2\alpha}$  (i.e. either  $\alpha \geq 2$  or  $0 < \alpha < 2$  and  $\omega \leq \frac{2\alpha}{2-\alpha}$ ).

In this case, the argument in Step 3 of the proof of Proposition 1.2 gives

$$m \leq h_1(h_2(m)), \quad (2.22)$$

where  $h_1$  and  $h_2$  are defined in (2.15)-(2.16)

Now note that  $h_1(h_2(\frac{2(1+\omega)}{2+\alpha})) = \frac{2(1+\omega)}{2+\alpha}$ . Thus in order to obtain a contradiction, it suffices to show that

$$\frac{d}{dt}h_1(h_2(t)) < 1 \text{ for all } t > \frac{2(1+\omega)}{2+\alpha}. \quad (2.23)$$

To see this, recall formula (2.18) for the derivative of  $h_1 \circ h_2$ :

$$\frac{d}{dt}h_1(h_2(t)) = \frac{1}{\sqrt{1 - 8\alpha h_2(t)^2 + 8\omega h_2(t)}} \frac{4\omega^2(\alpha t - 2t + 2)}{[(\alpha + 2)t - 2]^3}.$$

Now if  $t > \frac{2(1+\omega)}{2+\alpha}$ , then as  $h_2$  is decreasing in  $(\frac{2}{2+\alpha}, \infty)$ , we have  $\frac{\omega}{2+\alpha} = h_2(\infty) < h_2(t) < h_2(\frac{2(1+\omega)}{2+\alpha}) = \frac{1+\omega}{2+\alpha} < \frac{\omega}{\alpha}$  (thanks to  $\alpha < 2\omega$ ), and so

$$1 - 8\alpha h_2(t)^2 + 8\omega h_2(t) > 1.$$

Also, for  $t > \frac{2(1+\omega)}{2+\alpha} > 1$ , we have  $\alpha t - 2t + 2 < \alpha t + 2t - 2$  and  $4\omega^2 < (\alpha t + 2t - 2)^2$ . (2.23) hence follows. This concludes Case (a).

Case (b):  $\frac{1+\omega}{2+\alpha} < \frac{\omega}{2\alpha}$  (i.e.  $0 < \alpha < 2$  and  $\omega > \frac{2\alpha}{2-\alpha}$ ).

We start by showing that

$$m \leq \tilde{h}_1(h_2(m)), \quad (2.24)$$

where  $h_2$  is defined in (2.16) and  $\tilde{h}_1$  is defined by

$$\tilde{h}_1(t) = \frac{1}{2}(1 + \omega + \sqrt{(1 + \omega)^2 - 8\alpha t^2}).$$

Indeed, By (2.21),  $n \leq -\ln h_2(m)$ . Plugging this into (2.20), we get  $m \leq \tilde{h}_1(e^{-n}) \leq \tilde{h}_1(h_2(m))$ , as  $\tilde{h}_1$  is decreasing in  $(0, \infty)$ .

Next, a direct computation gives  $\tilde{h}_1(h_2(\frac{2(1+\omega)}{2+\alpha})) = \frac{2(1+\omega)}{2+\alpha}$ . Thus, as in Case (a), it suffices to show that

$$\frac{d}{dt}\tilde{h}_1(h_2(t)) < 1 \text{ for all } t > \frac{2(1+\omega)}{2+\alpha}. \quad (2.25)$$

We compute

$$\begin{aligned} \frac{d}{dt}h_1(h_2(t)) &= h'_1(h_2(t))h'_2(t) = \frac{4\alpha h_2(t)}{\sqrt{(1+\omega)^2 - 8\alpha h_2(t)^2}} \frac{2\omega}{[(\alpha+2)t - 2]^2} \\ &= \frac{1}{\sqrt{(1+\omega)^2 - 8\alpha h_2(t)^2}} \frac{8\alpha\omega h_2(t)}{[(\alpha+2)t - 2]^2}. \end{aligned}$$

Now for  $t > \frac{2(1+\omega)}{2+\alpha}$ , we have  $\frac{\omega}{2+\alpha} < h_2(t) < \frac{1+\omega}{2+\alpha}$  as in the previous case thanks to the monotonicity of  $h_2$ . It follows that

$$\frac{d}{dt}h_1(h_2(t)) < \frac{1}{\sqrt{(1+\omega)^2 - 8\alpha \frac{(1+\omega)^2}{(2+\alpha)^2}}} \frac{8\alpha\omega \frac{1+\omega}{2+\alpha}}{4\omega^2} = \frac{2\alpha}{(2-\alpha)\omega} < 1 \text{ for all } t > \frac{2(1+\omega)}{2+\alpha}.$$

This proves (2.25), and so finishes Case (b). Step 3 is concluded.

4. Finally, we show that  $u \equiv v \equiv \sqrt{\frac{1+\omega}{2+\alpha}}$ .

By Step 3, we have  $\tilde{m}_* = \frac{2(1+\omega)}{2+\alpha}$  or  $\tilde{n}_* = -\ln \frac{1+\omega}{2+\alpha}$ .

If  $\tilde{m}_* = \frac{2(1+\omega)}{2+\alpha}$ , then, in view of (2.13),  $n \leq -\ln \frac{1+\omega}{2+\alpha}$ , and so  $A \leq \frac{2(1+\omega)}{2+\alpha}$  and  $B \geq -\ln \frac{1+\omega}{2+\alpha}$ , which give  $u \equiv v \equiv \sqrt{\frac{1+\omega}{2+\alpha}}$ .

On the other hand, if  $\tilde{n}_* = -\ln \frac{1+\omega}{2+\alpha}$ , then, by (2.11),  $m \leq \frac{2(1+\omega)}{2+\alpha}$ . Again we obtain  $A \leq \frac{2(1+\omega)}{2+\alpha}$  and  $B \geq -\ln \frac{1+\omega}{2+\alpha}$ , which then give  $u \equiv v \equiv \sqrt{\frac{1+\omega}{2+\alpha}}$ . We conclude the proof.  $\square$

## 3 Moving plane device

### 3.1 Monotonicity with respect to $x_N$

We are going to show that if  $(u, v)$  is a solution to (1.1)-(1.2), then it is monotone with respect to  $x_N$ . This relies strongly on the estimates (1.5)-(1.6).

**Proposition 3.1.** *Under the assumptions of Theorem 1.1, since (1.5)-(1.6) hold, we have*

$$\partial_N u > 0 \quad \text{and} \quad \partial_N v < 0 \quad \text{in } \mathbb{R}^N. \quad (3.1)$$

The proof is based on the moving planes method, in a version developed by [19]. Nevertheless our system requires some major adjustments which rely on new bounds, and in particular the bounds from below for the product  $uv$ , as we will point out.

For  $\lambda \in \mathbb{R}$ , we set

$$u_\lambda(x', x_N) := u(x', 2\lambda - x_N), \quad v_\lambda(x', x_N) := v(x', 2\lambda - x_N) \quad \text{and} \quad \Sigma_\lambda := \{x_N > \lambda\}.$$

We want to prove that

$$u_\lambda(x) \leq u(x) \quad \text{and} \quad v_\lambda(x) \geq v(x) \quad \forall x \in \Sigma_\lambda, \quad \forall \lambda \in \mathbb{R}. \quad (3.2)$$

This and the strong Maximum Principle will yield Proposition 3.1.

In order to prove that (3.2) holds, we will show that

$$\Lambda := \{\lambda \in \mathbb{R} : u_\mu \leq u \text{ and } v_\mu \geq v \text{ in } \Sigma_\mu \text{ for every } \mu \geq \lambda\} = \mathbb{R}. \quad (3.3)$$

In order to start the moving plane device, we will start by proving that  $\Lambda \neq \emptyset$ :

**Lemma 3.2.** *There exists  $\bar{\lambda} \in \mathbb{R}$  sufficiently large such that*

$$u \geq u_\lambda \quad \text{and} \quad v \leq v_\lambda \quad \text{in } \Sigma_\lambda$$

for any  $\lambda \geq \bar{\lambda}$ . In other words,  $\Lambda \supset [\bar{\lambda}, \infty)$ .

The pair  $(u_\lambda, v_\lambda)$  solves

$$\begin{cases} -\Delta u_\lambda = g(u_\lambda, v_\lambda) + v_\lambda(\omega - \alpha u_\lambda v_\lambda) \\ -\Delta v_\lambda = g(v_\lambda, u_\lambda) + u_\lambda(\omega - \alpha u_\lambda v_\lambda) \\ a < u_\lambda, v_\lambda < b \end{cases} \quad (P_\lambda)$$

where  $g(u, v) = u(1 - u^2 - v^2)$  and where we have used (1.6).

Let  $\varphi_R$  be a standard  $\mathcal{C}^1$  cut-off function on  $\mathbb{R}^N$  such that  $\varphi_R = 1$  in  $B_R$ ,  $\varphi_R = 0$  outside  $B_{2R}$  and  $|\nabla \varphi_R| \leq 2/R$  on  $\mathbb{R}^N$ . We subtract the equations for  $u_\lambda$  and  $u$ , and multiply by the test function

$$(u_\lambda - u)^+ \varphi_R^2 \mathbf{1}_{\Sigma_\lambda}.$$

We find

$$\int_{\Sigma_\lambda} |\nabla(u_\lambda - u)^+|^2 \varphi_R^2 = -2 \int_{\Sigma_\lambda} \varphi_R (u_\lambda - u)^+ \nabla(u_\lambda - u)^+ \cdot \nabla \varphi_R + I_1 + I_2, \quad (3.4)$$

where

$$I_1 = \int_{\Sigma_\lambda} (g(u_\lambda, v_\lambda) - g(u, v))(u_\lambda - u)^+ \varphi_R^2, \quad (3.5)$$

$$I_2 = \int_{\Sigma_\lambda} (v_\lambda(\omega - \alpha u_\lambda v_\lambda) - v(\omega - \alpha uv))(u_\lambda - u)^+ \varphi_R^2. \quad (3.6)$$

Let us point out that, in the above expressions, the term  $\omega - \alpha uv$  is non-negative by our lower bound on  $uv$ . We will soon see that this sign of  $\omega - \alpha uv$  plays an essential role in our argument.

We proceed similarly by subtracting the equations for  $v$  and  $v_\lambda$  and multiplying by

$$(v - v_\lambda)^+ \varphi_R^2 \mathbb{1}_{\Sigma_\lambda}$$

and get

$$\int_{\Sigma_\lambda} |\nabla(v - v_\lambda)^+|^2 \varphi_R^2 = -2 \int_{\Sigma_\lambda} \varphi_R (v - v_\lambda)^+ \nabla(v - v_\lambda)^+ \cdot \nabla \varphi_R + I_3 + I_4 \quad (3.7)$$

where

$$I_3 = \int_{\Sigma_\lambda} (g(v, u) - g(v_\lambda, u_\lambda))(v - v_\lambda)^+ \varphi_R^2, \quad (3.8)$$

$$I_4 = \int_{\Sigma_\lambda} (u(\omega - \alpha uv) - u_\lambda(\omega - \alpha u_\lambda v_\lambda))(v - v_\lambda)^+ \varphi_R^2. \quad (3.9)$$

Let

$$\begin{aligned} \mathcal{L}_\lambda(R) &:= \int_{\Sigma_\lambda \cap B_R} \left[ |\nabla(u_\lambda - u)^+|^2 + |\nabla(v - v_\lambda)^+|^2 \right], \\ \mathcal{J}_\lambda(R) &:= \int_{\Sigma_\lambda} \left[ ((u_\lambda - u)^+)^2 + ((v - v_\lambda)^+)^2 \right] \varphi_R^2. \end{aligned}$$

We deduce from (3.4)-(3.7) that for any  $\vartheta \in (0, 1)$ ,

$$\mathcal{L}_\lambda(R) \leq \vartheta \mathcal{L}_\lambda(2R) + \frac{4}{\vartheta R^2} \mathcal{J}_\lambda(R) + I_1 + I_2 + I_3 + I_4. \quad (3.10)$$

We want to estimate  $I_1, I_2, I_3, I_4$  in terms of  $\mathcal{J}_\lambda(R)$ .

*Estimate of  $I_2$  and  $I_4$ :* We deduce from (3.6) that

$$I_2 = \int_{\Sigma_\lambda} \left[ (v - v_\lambda)(\alpha u(v + v_\lambda) - \omega)(u_\lambda - u)^+ - \alpha v_\lambda^2 ((u_\lambda - u)^+)^2 \right] \varphi_R^2. \quad (3.11)$$

We recall from Proposition 1.2 that  $\alpha uv - \omega \geq 0$ , which is a crucial estimate. So that in the first term in the square bracket on the right hand side of (3.11), we can keep only  $(v - v_\lambda)^+$  in the upper bound and find

$$I_2 \leq \int_{\Sigma_\lambda} \left[ (\alpha uv - \omega)(v - v_\lambda)^+(u_\lambda - u)^+ + \alpha uv_\lambda(v - v_\lambda)^+(u_\lambda - u)^+ - \alpha v_\lambda^2((u_\lambda - u)^+)^2 \right] \varphi_R^2. \quad (3.12)$$

A similar computation for  $I_4$  yields

$$I_4 \leq \int_{\Sigma_\lambda} \left[ (\alpha u_\lambda v_\lambda - \omega)(v - v_\lambda)^+(u_\lambda - u)^+ + \alpha uv_\lambda(v - v_\lambda)^+(u_\lambda - u)^+ - \alpha u^2((v - v_\lambda)^+)^2 \right] \varphi_R^2. \quad (3.13)$$

We sum the two estimates (3.12) and (3.13), use that

$$2uv_\lambda(v - v_\lambda)^+(u_\lambda - u)^+ - v_\lambda^2((u_\lambda - u)^+)^2 - u^2((v - v_\lambda)^+)^2 \leq 0$$

to find

$$I_2 + I_4 \leq \int_{\Sigma_\lambda} \left[ (v - v_\lambda)^+(u_\lambda - u)^+((\alpha u_\lambda v_\lambda - \omega) + (\alpha uv - \omega)) \right] \varphi_R^2. \quad (3.14)$$

Because of (1.2), for  $\lambda$  large enough, in  $\Sigma_\lambda$ ,  $u$  tends to  $b$ ,  $v$  tends to  $a$  and  $uv$  tends to  $ab = \omega/\alpha$ . Moreover in the support of  $(v - v_\lambda)^+$ ,  $v \geq v_\lambda \geq a$ , so  $v_\lambda$  also tends to  $a$ , and in the support of  $(u_\lambda - u)^+$ ,  $b \geq u_\lambda \geq u$ , so  $u_\lambda$  tends to  $b$ . This implies that for  $\lambda$  large enough, in  $\Sigma_\lambda$ ,  $u_\lambda v_\lambda$  also tends to  $ab = \omega/\alpha$ . Therefore, for some small  $\varepsilon > 0$  which will be fixed later, there exists  $\bar{\lambda}$  large enough, so that for  $\lambda \geq \bar{\lambda}$ , in  $\Sigma_\lambda \cap S$ , where  $S$  is the intersection of the supports of  $(u_\lambda - u)^+$  and  $(v - v_\lambda)^+$ ,

$$|(\alpha u_\lambda v_\lambda - \omega) + (\alpha uv - \omega)| \leq \varepsilon$$

and

$$I_2 + I_4 \leq \frac{\varepsilon}{2} \int_{\Sigma_\lambda} \left[ ((v - v_\lambda)^+)^2 + ((u_\lambda - u)^+)^2 \right] \varphi_R^2. \quad (3.15)$$

*Estimate of  $I_1$  and  $I_3$ :* By the mean value theorem, there exist  $\xi_1(x) \in (u(x), u_\lambda(x))$  and  $\xi_2(x) \in (v_\lambda(x), v(x))$  such that

$$g(u_\lambda, v_\lambda) - g(u, v) = \frac{\partial g}{\partial u}(\xi_1, v_\lambda)(u_\lambda - u) + \frac{\partial g}{\partial v}(u, \xi_2)(v_\lambda - v).$$

Since  $g(u, v) = u(1 - u^2 - v^2)$ , we have  $\frac{\partial g}{\partial v}(u, \xi_2) \leq 0$ . Hence

$$(g(u_\lambda, v_\lambda) - g(u, v))(u_\lambda - u)^+ \leq \frac{\partial g}{\partial u}(\xi_1, v_\lambda)((u_\lambda - u)^+)^2 + \left| \frac{\partial g}{\partial v}(u, \xi_2) \right| (v - v_\lambda)^+(u_\lambda - u)^+.$$



Moreover,

$$\frac{\partial g}{\partial u}(\xi_1, v_\lambda) = 1 - 3\xi_1^2 - v_\lambda^2, \quad \frac{\partial g}{\partial v}(u, \xi_2) = -2u\xi_2.$$

Next, note that, as  $\lambda \rightarrow \infty$ ,  $u$  and  $u_\lambda$  tend to  $b$  in  $\Sigma_\lambda \cap \{u_\lambda > u\}$  and  $v$  and  $v_\lambda$  tend to  $a$  in  $\Sigma_\lambda \cap S$ . Hence, we have

$$\begin{aligned} & \text{in } \Sigma_\lambda \cap \{u_\lambda > u\}, \limsup_{\lambda \rightarrow \infty} \frac{\partial g}{\partial u}(\xi_1, v_\lambda) \leq -2b^2, \\ & \text{in } \Sigma_\lambda \cap S, \lim_{\lambda \rightarrow \infty} \frac{\partial g}{\partial v}(u, \xi_2) = -2ab. \end{aligned}$$

Therefore, in view of (3.5) and by enlarging  $\bar{\lambda}$  if necessary we have for  $\lambda \geq \bar{\lambda}$  that

$$I_1 \leq \int_{\Sigma_\lambda} \left[ (-2b^2 + \varepsilon)((u_\lambda - u)^+)^2 + (2ab + \varepsilon)(v - v_\lambda)^+(u_\lambda - u)^+ \right] \varphi_R^2. \quad (3.16)$$

We argue similarly to  $I_1$  for  $I_3$  and find

$$I_1 + I_3 \leq \int_{\Sigma_\lambda} (-(a - b)^2 + 2\varepsilon) \left[ ((u_\lambda - u)^+)^2 + (v - v_\lambda)^+ \right] \varphi_R^2. \quad (3.17)$$

Recall that  $(a - b)^2$  is positive as soon as (1.4) holds. In the sequel, we assume that  $(a - b)^2 > 3\varepsilon$ . Inserting (3.15) and (3.17) into (3.10), for  $\varepsilon > 0$ , there exists  $\bar{\lambda} > 0$ , such that we for  $\lambda > \bar{\lambda}$  and large  $R$  that

$$\mathcal{L}_\lambda(R) \leq \vartheta \mathcal{L}_\lambda(2R) + \left( -(a - b)^2 + 3\varepsilon \right) \mathcal{J}_\lambda(R). \quad (3.18)$$

We fix  $\vartheta := 2^{-(N+1)}$  and some  $\varepsilon > 0$  such that  $(a - b)^2 - 3\varepsilon > 0$ . Then for  $\lambda > \bar{\lambda}$  and large  $R$ ,

$$\mathcal{L}_\lambda(R) \leq \vartheta \mathcal{L}_\lambda(2R). \quad (3.19)$$

The conclusion will follow from a lemma proved and used in [18] which allows to show that  $\mathcal{L}_\lambda(R)$  is identically zero.

**Lemma 3.3** ([18, Lemma 2.1]). *Let  $\vartheta > 0$  and  $\gamma > 0$  such that  $\vartheta < 2^{-\gamma}$ . Moreover let  $R_0 > 0$ ,  $C > 0$  and*

$$\mathcal{L} : (R_0, +\infty) \rightarrow \mathbb{R}$$

*be a non-negative and non-decreasing function such that*

$$\begin{cases} \mathcal{L}(R) \leq \vartheta \mathcal{L}(2R) + G(R) & \forall R > R_0, \\ \mathcal{L}(R) \leq CR^\gamma & \forall R > R_0, \end{cases}$$

where  $G : (R_0, +\infty) \rightarrow \mathbb{R}^+$  is such that

$$\lim_{R \rightarrow +\infty} G(R) = 0.$$

Then

$$\mathcal{L}(R) = 0 \quad \forall R > R_0.$$

We have that  $\mathcal{L}_\lambda(R) \leq CR^N$  since  $|\nabla u|, |\nabla v| \in L^\infty(\mathbb{R}^N)$  by elliptic estimates and the  $L^\infty$  bound of Lemma 2.1. Lemma 3.3 then yields that

$$\mathcal{L}_\lambda(R) = 0 \text{ for all } \lambda > \bar{\lambda} \text{ and large } R.$$

Recalling that  $u = u_\lambda$  and  $v = v_\lambda$  on  $\partial\Sigma_\lambda$ , we reach the conclusion of Lemma 3.2.

We now have to prove that  $\tilde{\lambda} := \inf \Lambda$  (with  $\Lambda$  defined in (3.3)) is  $-\infty$  to complete the proof of Proposition 3.1:

**Lemma 3.4.** *We have  $\tilde{\lambda} = -\infty$ .*

*Proof.* Assume by contradiction that  $\tilde{\lambda}$  is finite. Then,  $\Lambda = [\tilde{\lambda}, +\infty)$ , and there exist sequences  $(\lambda_i)$  with  $\lambda_i \in (-\infty, \tilde{\lambda})$  and  $(x^i)$  with  $x^i \in \Sigma_{\lambda_i}$  such that  $\lambda_i \rightarrow \tilde{\lambda}$  as  $i \rightarrow \infty$ , and at least one of the two holds:

$$u_{\lambda_i}(x^i) > u(x^i) \quad \text{for every } i, \quad \text{or} \quad (3.20a)$$

$$v_{\lambda_i}(x^i) < v(x^i) \quad \text{for every } i. \quad (3.20b)$$

Assume that (3.20a) holds; the other case can be treated similarly. We claim that the sequence  $(x_N^i) \subset \mathbb{R}$  is bounded. If not, as  $x_N^i > \lambda_i$  and  $\lambda_i$  is bounded, up to a subsequence  $x_N^i \rightarrow +\infty$  as  $i \rightarrow \infty$ . It follows that  $2\lambda_i - x_N^i \rightarrow -\infty$ , and in light of assumption (1.2) we obtain

$$\lim_{i \rightarrow \infty} u_{\lambda_i}(x^i) = \lim_{i \rightarrow \infty} u((x^i)', 2\lambda_i - x_N^i) = a \quad \text{and} \quad \lim_{i \rightarrow \infty} u(x^i) = b,$$

in contradiction with (3.20a) for  $i$  sufficiently large. Hence the claim is proved and, up to a subsequence,  $x_N^i \rightarrow \bar{x}_N$  as  $i \rightarrow \infty$ .

Let us set

$$u^i(x) := u((x^i)' + x', x_N) \quad \text{and} \quad v^i(x) := v((x^i)' + x', x_N).$$

Since  $(u, v)$  is bounded (in view of (1.5)), by standard elliptic estimates  $|\nabla^k u|, |\nabla^k v| \in L^\infty(\mathbb{R}^N)$ , for  $k = 1, 2, \dots$ . Thus, after extracting a subsequence if necessary,  $(u^i, v^i)$  converges in  $\mathcal{C}_{loc}^2(\mathbb{R}^N)$  to a limit  $(\bar{u}, \bar{v})$ , still solution of (1.1).

We wish to show that  $\bar{x}_N = \tilde{\lambda}$ . From equation (3.20a), we obtain

$$\begin{aligned}\bar{u}_{\tilde{\lambda}}(0', \bar{x}_N) &= \bar{u}(0', 2\tilde{\lambda} - \bar{x}_N) = \lim_{i \rightarrow \infty} u((x^i)', 2\lambda_i - x_N^i) \\ &= \lim_{i \rightarrow \infty} u_{\lambda_i}(x^i) \geq \lim_{i \rightarrow \infty} u(x^i) = \bar{u}(0', \bar{x}_N).\end{aligned}\tag{3.21}$$

On the other hand, we observe that  $((x^i)' + x', x_N) \in \Sigma_{\tilde{\lambda}}$  whenever  $(x', x_N) \in \Sigma_{\tilde{\lambda}}$ , and by definition  $u_{\tilde{\lambda}} \leq u$  in  $\Sigma_{\tilde{\lambda}}$ . Consequently, by the convergence of  $u^i$  to  $\bar{u}$  we deduce that

$$\begin{aligned}\bar{u}_{\tilde{\lambda}}(x', x_N) &= \lim_{i \rightarrow \infty} u^i(x', 2\tilde{\lambda} - x_N) = \lim_{i \rightarrow \infty} u((x^i)' + x', 2\tilde{\lambda} - x_N) \\ &\leq \lim_{i \rightarrow \infty} u((x^i)' + x', x_N) = \lim_{i \rightarrow \infty} u^i(x', x_N) = \bar{u}(x', x_N)\end{aligned}$$

for every  $(x', x_N) \in \Sigma_{\tilde{\lambda}}$ . Analogously, as  $v_{\tilde{\lambda}} \geq v$  in  $\Sigma_{\tilde{\lambda}}$ , we have  $\bar{v}_{\tilde{\lambda}} \geq \bar{v}$  in  $\Sigma_{\tilde{\lambda}}$ .

Now

$$\begin{cases} -\Delta(\bar{u} - \bar{u}_{\tilde{\lambda}}) + c(x)(\bar{u} - \bar{u}_{\tilde{\lambda}}) &= (\bar{v}_{\tilde{\lambda}} - \bar{v})(\alpha\bar{u}\bar{v} - \omega + \alpha\bar{u}_{\tilde{\lambda}}\bar{v}_{\tilde{\lambda}} + \bar{u}_{\tilde{\lambda}}(\bar{v}_{\tilde{\lambda}} + \bar{v})) \\ \bar{u} - \bar{u}_{\tilde{\lambda}} \geq 0 &\text{in } \Sigma_{\tilde{\lambda}} \\ \bar{u} - \bar{u}_{\tilde{\lambda}} = 0 &\text{on } \partial\Sigma_{\tilde{\lambda}}, \end{cases}\tag{3.22}$$

with  $c \in C^0(\Sigma_{\tilde{\lambda}})$  defined by

$$c(x) = \alpha\bar{v}(x)\bar{v}_{\tilde{\lambda}}(x) + \bar{v}^2(x) + \bar{u}^2(x) + \bar{u}\bar{u}_{\tilde{\lambda}}(x) + \bar{u}_{\tilde{\lambda}}^2(x) - 1.$$

Because of (1.5), the right hand side on the first line of (3.22) is nonnegative, hence, the strong Maximum Principle implies that necessarily  $\bar{u} - \bar{u}_{\tilde{\lambda}} > 0$  in  $\Sigma_{\tilde{\lambda}}$ , and a comparison with (3.21) reveals that  $\bar{x}_N = \tilde{\lambda}$ , as desired.

At this point we are ready to reach a contradiction. On the one hand, by the absurd assumption (3.20a)

$$0 < u_{\lambda_i}(x^i) - u(x^i) = u^i(0', 2\lambda_i - x_N^i) - u^i(0', x_N) = 2\partial_N u^i(0', \xi^i)(\lambda_i - x_N^i) \quad \forall i,$$

for some  $\xi^i \in (2\lambda_i - x_N^i, x_N^i)$ . As  $\lambda_i < x_N^i$  for every  $i$  this implies  $\partial_N u^i(x', \xi_N^i) < 0$  for every  $i$ , and passing to the limit we infer that

$$\partial_N \bar{u}(0', \tilde{\lambda}) \leq 0,\tag{3.23}$$

where we used the fact that  $\lambda_i \leq \xi^i \leq x_N^i$  with  $\lambda_i, x_N^i \rightarrow \tilde{\lambda}$ .

On the other hand, thanks to (3.22) and the fact that  $\bar{u} - \bar{u}_{\tilde{\lambda}} > 0$  in  $\Sigma_{\tilde{\lambda}}$ , the Hopf Lemma implies that

$$-2\partial_N \bar{u}(0', \tilde{\lambda}) = \partial_{-e_N}(\bar{u}(0', \tilde{\lambda}) - \bar{u}_{\tilde{\lambda}}(0', \tilde{\lambda})) < 0,$$

in contradiction with (3.23).

The above argument establishes that (3.20a) cannot occur. With minor changes, we can show that also (3.20b) cannot be verified, and in conclusion  $\tilde{\lambda}$  cannot be finite.  $\square$

*Proof of Proposition 3.1.* By (3.3), we directly deduce that  $\partial_N u \geq 0$  and  $\partial_N v \leq 0$  in  $\mathbb{R}^N$ . Since

$$\begin{cases} -\Delta(\partial_N u) + (3u^2 + (\alpha + 1)v^2 - 1)\partial_N u = (\omega - 2(\alpha + 1)uv)\partial_N v \geq 0 & \text{in } \mathbb{R}^N \\ -\Delta(\partial_N v) + (3v^2 + (\alpha + 1)u^2 - 1)\partial_N v = (\omega - 2(\alpha + 1)uv)\partial_N u \leq 0 & \text{in } \mathbb{R}^N, \end{cases}$$

the strict inequality follows by the strong Maximum Principle.  $\square$

### 3.2 One-dimensional symmetry

We extend the monotonicity in  $x_N$  to all the directions of the open upper hemisphere  $\mathbb{S}_+^{N-1} := \{\nu \in \mathbb{S}^{N-1} : \langle \nu, e_N \rangle > 0\}$ . We follow the structure of proof in [19], introduced in [16] and in [20], though the specificity of our system requires new estimates.

**Proposition 3.5.** *For every  $\nu \in \mathbb{S}_+^{N-1}$ , we have*

$$\partial_\nu u > 0 \quad \text{and} \quad \partial_\nu v < 0 \quad \text{in } \mathbb{R}^N.$$

*In particular,  $u$  and  $v$  depend only on  $x_N$ .*

We divide the proof into several steps.

**Lemma 3.6.** *Let  $\sigma > 0$  be arbitrarily chosen. There exists an open neighborhood  $\mathcal{O}_{e_N}$  of  $e_N$  in  $\mathbb{S}^{N-1}$  such that*

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu}(x) < 0 \quad \forall x \in \overline{S_\sigma}, \quad \forall \nu \in \mathcal{O}_{e_N},$$

where  $S_\sigma := \mathbb{R}^{N-1} \times (-\sigma, \sigma)$ .

*Proof.* Let  $\sigma > 0$  be arbitrarily chosen. Firstly, we claim that there exists  $\varepsilon = \varepsilon(\sigma) > 0$  such that

$$\partial_N u(x) \geq \varepsilon \quad \text{and} \quad \partial_N v(x) \leq -\varepsilon \quad \forall x \in \overline{S_\sigma}. \quad (3.24)$$

By contradiction, assume that there exists a sequence  $(x^i)$ , with  $x^i \in S_\sigma$ , such that at least one of the two following equalities holds :

$$\lim_{i \rightarrow +\infty} \partial_N u(x^i) = 0 \quad (3.25a)$$

$$\text{or } \lim_{i \rightarrow +\infty} \partial_N v(x^i) = 0. \quad (3.25b)$$

We define

$$u^i(x) := u(x + x^i) \quad \text{and} \quad v^i(x) := v(x + x^i).$$

The sequence  $\{(u^i, v^i)\}$  is uniformly bounded in  $W^{1,\infty}(\mathbb{R}^N)$ , and hence by elliptic regularity  $(u^i, v^i) \rightarrow (\bar{u}, \bar{v})$  in  $\mathcal{C}_{\text{loc}}^2(\mathbb{R}^N)$  up to a subsequence, where  $(\bar{u}, \bar{v})$  is still a solution to (1.1)-(1.2), which also satisfies  $\partial_N \bar{u} \geq 0$  and  $\partial_N \bar{v} \leq 0$  on  $\mathbb{R}^N$ . The strong maximum principle and the condition at infinity (1.2) then imply that  $\partial_N \bar{u} > 0$  and  $\partial_N \bar{v} < 0$  on  $\mathbb{R}^N$ , and this contradicts (3.25a) or (3.25b). This completes the proof of claim (3.24).

Now we claim that

$$\text{The map } \nu \mapsto (\partial_\nu u, \partial_\nu v) \text{ is in } \mathcal{C}^{0,1}(\mathbb{S}^{N-1}, (\mathcal{C}^0(\mathbb{R}^N))^2). \quad (3.26)$$

This is a simple consequence of the global Lipschitz continuity of  $(u, v)$ , which implies that

$$\left| \frac{\partial u}{\partial \nu_1}(x) - \frac{\partial u}{\partial \nu_2}(x) \right| + \left| \frac{\partial v}{\partial \nu_1}(x) - \frac{\partial v}{\partial \nu_2}(x) \right| \leq 2C|\nu_1 - \nu_2|$$

for every  $x \in \mathbb{R}^N$ .

Combining (3.24) and (3.26), the conclusion follows.  $\square$

**Lemma 3.7.** *The function  $u$  is strictly increasing and  $v$  is strictly decreasing with respect to all unit vectors in an open neighborhood of  $e_N$  in  $\mathbb{S}^{N-1}$ .*

*Proof.* Firstly, we write down the equations satisfied by the directional derivatives  $u_\nu = \partial_\nu u$  and  $v_\nu = \partial_\nu v$ :

$$\begin{cases} -\Delta u_\nu + u_\nu(3u^2 + (\alpha + 1)v^2 - 1) + v_\nu(2(\alpha + 1)uv - \omega) = 0 \\ -\Delta v_\nu + v_\nu(3v^2 + (\alpha + 1)u^2 - 1) + u_\nu(2(\alpha + 1)uv - \omega) = 0 \end{cases} \quad \text{in } \mathbb{R}^N. \quad (3.27)$$

Fix some  $\sigma > 0$  for the moment and let  $\mathcal{O}_{e_N}$  be the neighborhood of  $e_N$  given by Lemma 3.6. We will show that  $u_\nu \geq 0$  and  $v_\nu \leq 0$  for all  $\nu \in \mathcal{O}_{e_N}$  by applying Lemma 3.3 to the quantity

$$\mathcal{I}_R := \int_{\mathcal{C}_R} [|\nabla u_\nu^-|^2 + |\nabla v_\nu^+|^2],$$

where  $\mathcal{C}_R := \Sigma_\sigma \cap B_R$  and  $\Sigma_\sigma := \{x_N > \sigma\}$ . The conclusion then follows from the strong maximum principle.

We test the first equation in (3.27) with  $u_\nu^- \varphi_R^2$  where  $\varphi_R$  is chosen exactly as in Lemma 3.2. Using the bounds (1.5) and the fact that  $u_\nu \geq 0$  on  $\{x_N = \sigma\}$  (due to

Lemma 3.6), we obtain

$$\begin{aligned}
\int_{\mathcal{C}_R} |\nabla u_\nu^-|^2 &\leq -2 \int_{\mathcal{C}_{2R}} u_\nu^- \varphi_R \nabla u_\nu^- \cdot \nabla \varphi_R \\
&\quad - \int_{\mathcal{C}_{2R}} (3u^2 + (\alpha + 1)v^2 - 1)(u_\nu^- \varphi_R)^2 + \int_{\mathcal{C}_{2R}} \varphi_R^2 (2(\alpha + 1)uv - \omega) u_\nu^- v_\nu^+ \\
&\leq \vartheta \int_{\mathcal{C}_{2R}} |\nabla u_\nu^-|^2 + \int_{\mathcal{C}_{2R}} (u_\nu^- \varphi_R)^2 \left( \frac{4}{\vartheta R^2} + \sup_{\Sigma_\sigma} (-3u^2 - (\alpha + 1)v^2 + 1) \right) \\
&\quad + \int_{\mathcal{C}_{2R}} \varphi_R^2 (2(\alpha + 1)uv - \omega) u_\nu^- v_\nu^+,
\end{aligned}$$

where  $0 < \vartheta < 2^{-N}$ . In a similar way, we find for  $v_\nu^+$

$$\begin{aligned}
\int_{\mathcal{C}_R} |\nabla v_\nu^+|^2 &\leq \vartheta \int_{\mathcal{C}_{2R}} |\nabla v_\nu^+|^2 + \int_{\mathcal{C}_{2R}} (v_\nu^+ \varphi_R)^2 \left( \frac{4}{\vartheta R^2} + \sup_{\Sigma_\sigma} (-3v^2 - (\alpha + 1)u^2 + 1) \right) \\
&\quad + \int_{\mathcal{C}_{2R}} \varphi_R^2 (2(\alpha + 1)uv - \omega) u_\nu^- v_\nu^+.
\end{aligned}$$

We notice that if  $\sigma > 0$  is sufficiently large, since  $u$  tends to  $b$  and  $v$  tends to  $a$  for  $x_N$  large, then, in  $\Sigma_\sigma$ ,

$$\begin{aligned}
2(\alpha + 1)uv - \omega &\rightarrow 2(\alpha + 1)ab - \omega, \quad 3u^2 + (\alpha + 1)v^2 - 1 \rightarrow 2b^2 + \alpha a^2, \\
&\text{and } 3v^2 + (\alpha + 1)u^2 - 1 \rightarrow 2a^2 + \alpha b^2.
\end{aligned}$$

Thus, for any small  $\delta > 0$ , we can choose  $\sigma$  and  $R$  sufficiently large so that

$$\begin{aligned}
\mathcal{I}_R &\leq \vartheta \mathcal{I}_{2R} - \int_{\mathcal{C}_{2R}} \left[ (2b^2 + \alpha a^2 - \delta)(u_\nu^-)^2 + (2a^2 + \alpha b^2 - \delta)(v_\nu^+)^2 \right. \\
&\quad \left. - 2(2(\alpha + 1)ab - \omega + \delta)|u_\nu^-||v_\nu^+| \right] \varphi_R^2. \quad (3.28)
\end{aligned}$$

We point out that

$$(2b^2 + \alpha a^2 - \delta)(u_\nu^-)^2 + (2a^2 + \alpha b^2 - \delta)(v_\nu^+)^2 \geq 2\sqrt{(2b^2 + \alpha a^2 - \delta)(2a^2 + \alpha b^2 - \delta)}|u_\nu^-||v_\nu^+|.$$

and that, by (1.3) and (1.4),

$$\begin{aligned}
&(2b^2 + \alpha a^2)(2a^2 + \alpha b^2) - (2(\alpha + 1)ab - \omega)^2 \\
&= 2\alpha(a^2 + b^2)^2 - 3\alpha(\alpha + 4)a^2b^2 + 4\omega(\alpha + 1)ab - \omega^2 = 2\alpha\left(1 - \frac{4\omega^2}{\alpha^2}\right) > 0.
\end{aligned}$$

Hence, by choosing first small  $\delta$  and then large  $\sigma$  from the start, we have for all sufficiently large  $R$  that the integral on the right hand side of (3.28) is non-negative. As a consequence, we infer that

$$\mathcal{I}_R \leq \vartheta \mathcal{I}_{2R} \text{ for all } R \text{ sufficiently large.} \quad (3.29)$$

We can now apply Lemma 3.3 to find that  $\mathcal{I}_R = 0$  for all large  $R$ . It follows that  $u_\nu \geq 0$  and  $v_\nu \leq 0$  in  $\Sigma_\sigma = \{x_N > \sigma\}$ . Arguing exactly in the same way, we can show that the same conditions are satisfied in  $\{x_N < -\sigma\}$ . By Lemma 3.6, we deduce that  $u_\nu \geq 0$  and  $v_\nu \leq 0$  in  $\mathbb{R}^N$  for every  $\nu \in \mathcal{O}_{e_N}$ , with both  $u_\nu \not\equiv 0$  and  $v_\nu \not\equiv 0$ . In view of (1.5) and (3.27), the conclusion follows from the strong maximum principle.  $\square$

*Proof of Proposition 3.5.* Here we can essentially apply the same argument used in step 4 of Proposition 6.1 in [20]. We report the details for completeness. Let  $\Omega$  be the set of the directions  $\nu \in \mathbb{S}_+^{N-1}$  for which there exists an open neighborhood  $\mathcal{O}_\nu \subset \mathbb{S}_+^{N-1}$  of  $\nu$  such that

$$\partial_\mu u > 0 \quad \text{and} \quad \partial_\mu v < 0 \quad \text{in } \mathbb{R}^N, \quad \forall \mu \in \mathcal{O}_\nu.$$

The set  $\Omega$  is open by definition, and contains  $e_N$  by Lemma 3.7. Since  $\mathbb{S}_+^{N-1}$  is arc-connected, if we can show that  $\partial\Omega \cap \mathbb{S}_+^{N-1} = \emptyset$ , then we conclude that  $\Omega = \mathbb{S}_+^{N-1}$ , as desired. Thus, let us suppose by contradiction that  $\bar{\nu} \in \partial\Omega \cap \mathbb{S}_+^{N-1}$  (notice in particular that  $\langle e_N, \bar{\nu} \rangle > 0$ ). By definition, there exists  $(\nu_n) \subset \Omega$  such that  $\nu_n \rightarrow \bar{\nu}$ . As

$$\partial_{\nu_n} u > 0 \quad \text{and} \quad \partial_{\nu_n} v < 0 \quad \text{in } \mathbb{R}^N, \quad \forall n,$$

by continuity

$$\partial_{\bar{\nu}} u \geq 0 \quad \text{and} \quad \partial_{\bar{\nu}} v \leq 0 \quad \text{in } \mathbb{R}^N.$$

By the strong maximum principle, recalling that  $(u_{\bar{\nu}}, v_{\bar{\nu}})$  solves (3.27), either  $u_{\bar{\nu}} \equiv 0$  or  $u_{\bar{\nu}} > 0$  in  $\mathbb{R}^N$ , and analogously either  $v_{\bar{\nu}} \equiv 0$  or  $v_{\bar{\nu}} < 0$  in  $\mathbb{R}^N$ . The alternatives  $u_{\bar{\nu}} \equiv 0$  and  $v_{\bar{\nu}} \equiv 0$  are in contradiction with assumption (1.2), since  $\bar{\nu}$  is not orthogonal to  $e_N$ , and hence

$$\partial_{\bar{\nu}} u > 0 \quad \text{and} \quad \partial_{\bar{\nu}} v < 0 \quad \text{in } \mathbb{R}^N. \quad (3.30)$$

Having established (3.30), it is possible to adapt the same proof of Lemmas 3.6 and 3.7, with  $\bar{\nu}$  instead of  $e_N$ , to deduce that  $u_\nu > 0$  and  $v_\nu < 0$  in  $\mathbb{R}^N$  in all the directions of an open neighborhood  $\mathcal{O}_{\bar{\nu}}$  of  $\bar{\nu}$  in  $\mathbb{S}_+^{N-1}$ . Thus, we have that  $\bar{\nu} \in \Omega \cap \partial\Omega$ , in contradiction with the openness of  $\Omega$ . This shows that  $\partial\Omega \cap \mathbb{S}_+^{N-1} = \emptyset$  which, as already observed, implies  $\Omega = \mathbb{S}_+^{N-1}$ .

Finally, the fact that  $\Omega = \mathbb{S}_+^{N-1}$  implies that both  $\partial_\tau u \equiv 0$  and  $\partial_\tau v \equiv 0$  for every  $\tau \in \mathbb{S}^{N-1}$  orthogonal to  $e_N$ , which proves the last assertion.  $\square$

## 4 Existence and uniqueness of positive 1D solutions when $2\omega < \alpha$ . Proof of Theorem 1.1.

In this section, we assume that  $0 < \omega < \frac{1}{2}\alpha$  unless otherwise stated. By Proposition 3.5, positive solutions to (1.1)-(1.2) depend only on  $x_N$  and are monotone. To conclude the proof of Theorem 1.1, it remains to prove the uniqueness up to translations of such one-dimensional solutions.

We are led to consider on  $\mathbb{R}$  the system

$$\begin{cases} u'' &= P(u, v), \\ v'' &= Q(u, v), \end{cases} \quad (4.1)$$

subject to

$$(u, v) \rightarrow (a, b) \text{ at } -\infty \text{ and } (u, v) \rightarrow (b, a) \text{ at } \infty. \quad (4.2)$$

The main result of this section is:

**Proposition 4.1.** *Suppose that  $0 < \omega < \frac{1}{2}\alpha$ . Then there exist positive solutions to (4.1)-(4.2), and these solutions are translations of one another, i.e. if  $(u, v)$  and  $(\bar{u}, \bar{v})$  both satisfy (4.1)-(4.2), then there is a constant  $T$  such that*

$$\bar{u}(x) = u(x + T) \text{ and } \bar{v}(x) = v(x + T).$$

Furthermore,  $u' > 0$  and  $v' < 0$  in  $\mathbb{R}$ .

*Proof of Theorem 1.1.* The result is a consequence of Propositions 3.5 and 4.1.  $\square$

The proof of the ‘uniqueness’ part in Proposition 4.1 uses the sliding method (cf. [6, 9, 10]) with the help of the bounds (1.5) as well as the following lemma on the asymptotic behavior of solutions.

Let

$$\lambda_{\pm} := \sqrt{\frac{1}{2} \left( (\alpha + 2) \pm \sqrt{(\alpha - 2)^2 + \frac{32\omega^2}{\alpha}} \right)}, \quad (4.3)$$

$$\mu := \frac{2\omega(\alpha + 2)}{\alpha \left( (\alpha - 2) \sqrt{1 - \frac{4\omega^2}{\alpha^2}} + \sqrt{(\alpha - 2)^2 + \frac{32\omega^2}{\alpha}} \right)} > 0, \quad (4.4)$$

which are related to the eigenvalues and eigenvectors of the linearized operator associated with (4.1) near the critical point  $(a, b)$ . We refer to Appendix A for a brief discussion on the origin of these constants.



**Lemma 4.2.** *Suppose that  $0 < \omega < \frac{1}{2}\alpha$ . Let  $(u, v)$  be a positive solution of (4.1)-(4.2) and  $\lambda_-$  and  $\mu$  be defined by (4.3)-(4.4). Then the limits*

$$\ell_1 := \lim_{x \rightarrow \infty} (b - u(x))e^{\lambda_- x} \text{ and } \ell_2 := \lim_{x \rightarrow \infty} (v(x) - a)e^{\lambda_- x} \text{ exist,} \quad (4.5)$$

and satisfy

$$\ell_2 = \mu \ell_1 > 0. \quad (4.6)$$

It should be noted that the asymptotic behavior of solutions changes somewhat when  $\omega = 0$ . See Lemma A.2 in the appendix.

An easy variational argument gives existence of the positive solutions to (4.1) on finite intervals. The sliding method can be adapted to this case yielding:

**Lemma 4.3.** *Suppose that  $0 < \omega < \frac{1}{2}\alpha$ . For every  $R \in (0, \infty)$ , there exists a unique positive solution to (4.1) in  $(-R, R)$  satisfying*

$$(u(-R), v(-R)) = (a, b) \text{ and } (u(R), v(R)) = (b, a). \quad (4.7)$$

Furthermore,  $u' > 0$  and  $v' < 0$  in  $(-R, R)$ .

## 4.1 Proof of Proposition 4.1

Let us assume Lemmas 4.2 and 4.3 for the moment and proceed with the proof of Proposition 4.1. Lemma 4.3 will be proved in the next subsection. Lemma 4.2 follows from a routine asymptotic analysis near a hyperbolic critical point for ODEs. Its proof is postponed to Appendix A.

*Proof.* 1. We prove the existence of a solution to (4.1)-(4.2) by sending  $R \rightarrow \infty$  in Lemma 4.3, where some care is needed to show that the solutions on finite intervals do not flatten to the constant solutions  $(a, a)$  or  $(b, b)$ .

For  $n = 1, 2, \dots$ , let  $(u_n, v_n)$  be the positive solution to (4.1) obtained in Lemma 4.3 with  $R = n$ . Fix  $x_n \in (-n, n)$  such that  $u_n(x_n) = \frac{1}{2}(a + b)$ . Define  $l_n = -n - x_n$ ,  $r_n = n - x_n$ , and

$$(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x + x_n), v_n(x + x_n)) \text{ for } x \in [l_n, r_n].$$

By Lemma 4.3,  $a \leq u_n, v_n \leq b$ . Using elliptic estimates on unit closed subintervals of  $[l_n, r_n]$ , we have

$$|\tilde{u}_n^{(k)}| + |\tilde{v}_n^{(k)}| \leq C \quad \text{in } [l_n, r_n] \quad \text{for } k = 0, 1, 2, 3, \quad (4.8)$$

where  $C$  is a positive constant independent of  $n$  and  $k$ . Then, passing to a subsequence if necessary, we may assume that  $l_n \rightarrow l_* \in [-\infty, 0]$ ,  $r_n \rightarrow r_* \in [0, \infty]$  (where  $l_*$  and

$r_*$  cannot be simultaneously finite),  $(\tilde{u}_n, \tilde{v}_n)$  converges in  $\mathcal{C}_{\text{loc}}^2(l_*, r_*)$  to some  $(u_*, v_*)$  satisfying (4.1),  $u'_* \geq 0$ ,  $v'_* \leq 0$  in  $(l_*, r_*)$ .

Note that for each  $n$ , the Hamiltonian

$$h_n := \frac{1}{2}(|\tilde{u}'_n|^2 + |\tilde{v}'_n|^2) - \frac{1}{4}(1 - \tilde{u}_n^2 - \tilde{v}_n^2)^2 - \frac{\alpha}{2}(\tilde{u}_n \tilde{v}_n - \frac{\omega}{\alpha})^2$$

is constant in  $[l_n, r_n]$ . As  $(\tilde{u}(R), \tilde{v}(R)) = (b, a)$ , it follows that  $h_n \geq 0$  and so

$$h_* := \frac{1}{2}(|u'_*|^2 + |v'_*|^2) - \frac{1}{4}(1 - u_*^2 - v_*^2)^2 - \frac{\alpha}{2}(u_* v_* - \frac{\omega}{\alpha})^2 \geq 0.$$

As said above, one has that  $l_* = -\infty$  or  $r_* = \infty$  (or both). We will only treat the case that  $r_* = \infty$ ; the other case can be dealt with similarly. In this case we have  $u_*(x) \geq \frac{1}{2}(a+b)$  for every  $x > 0$ , since  $x > 0 \geq l_*$  implies  $\tilde{u}_n(x) \geq \tilde{u}_n(0) = \frac{a+b}{2}$  for large  $n$ . Then, by the monotonicity of  $u_*$  and  $v_*$ , as  $x \rightarrow \infty$ ,  $(u_*(x), v_*(x))$  has a limit, say  $(\tilde{b}, \tilde{a})$ , which satisfies  $\frac{1}{2}(a+b) \leq \tilde{b} \leq b$  and  $a \leq \tilde{a} \leq b$ . By (4.1),  $(u''_*(x), v''_*(x))$  tends to  $(P(\tilde{b}, \tilde{a}), Q(\tilde{b}, \tilde{a}))$  as  $x \rightarrow \infty$ . Applying the mean value theorem to  $u|_{[n, n+1/2]}$ , we can find  $\xi_n \in (n, n+1/2)$  such that  $u'_*(\xi_n) \rightarrow 0$ . Likewise, there exist  $\eta_n \in (\xi_n, \xi_{n+1})$  such that  $u''_*(\eta_n) \rightarrow 0$ . It follows that  $P(\tilde{b}, \tilde{a}) = 0$ . This then implies that  $\sup_{[\xi_n-2, \xi_n+2]} u'_* \rightarrow 0$ , and so  $u'_*(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Similarly,  $Q(\tilde{b}, \tilde{a}) = 0$  and  $v'_*(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

Now, note that the equation  $P(x, y) = Q(x, y) = 0$  has three solutions in the positive quadrant, namely  $(a, b)$ ,  $(b, a)$  and  $(c, c)$  where  $c = \sqrt{\frac{1+\omega}{2+\alpha}} \in (a, b)$ . Also, as  $(u'_*(x), v'_*(x)) \rightarrow 0$  as  $x \rightarrow \infty$ ,

$$0 \leq h_* = -\frac{1}{4}(1 - \tilde{b}^2 - \tilde{a}^2)^2 - \frac{\alpha}{2}(\tilde{b}\tilde{a} - \frac{\omega}{\alpha})^2$$

and so  $h_* = 0$ ,  $\tilde{b}^2 + \tilde{a}^2 = 1$  and  $\tilde{b}\tilde{a} = \frac{\omega}{\alpha}$ . As  $\frac{1}{2}(a+b) \leq \tilde{b} \leq b$ , we thus have  $(\tilde{b}, \tilde{a}) = (b, a)$ , i.e.  $(u_*(x), v_*(x)) \rightarrow (b, a)$  as  $x \rightarrow \infty$ .

Now if  $l_*$  is finite, (4.8) yields

$$|\tilde{u}_*^{(k)}| + |\tilde{v}_*^{(k)}| \leq C \quad \text{in } (l_*, r_*) \quad \text{for } k = 0, 1, 2, \quad (4.9)$$

where  $C$  is a positive constant independent of  $k$ , and so  $u_*$  extends to a  $\mathcal{C}^1$  function in  $[l_*, r_*)$ . Now, since for every  $x \in (l_*, r_*)$ , we have  $x \in [l_n, r_n]$  for large  $n$ , from (4.8) we also get that

$$\begin{aligned} |u_*(x) - a| &= \lim_{n \rightarrow \infty} |\tilde{u}_n(x) - a| = \lim_{n \rightarrow \infty} |\tilde{u}_n(x) - \tilde{u}_n(l_n)| \\ &\leq \limsup_{n \rightarrow \infty} C|x - l_n| = C|x - l_*| \end{aligned}$$

which leads to  $u_*(l_*) = a$ . A similar argument gives  $v_*(l_*) = b$ . Now, by the strong maximum principle and the Hopf lemma (see the argument in Step 5 of the proof of Proposition 1.2), we have  $u'_*(x) > 0$  and  $v'_*(x) < 0$  for every  $x \in [l_*, r_*)$ , and using those properties with  $x = l_*$  we get  $h_* > 0$ , which contradicts the previous conclusion that  $h_* = 0$ . Hence  $l_* = -\infty$ . As above, this implies that  $(u_*(x), v_*(x)) \rightarrow (a, b)$  as  $x \rightarrow -\infty$ .

We have thus shown that  $(u_*, v_*)$  is a positive and strictly monotone solution to (4.1)-(4.2), as desired.

2. We use the sliding method to show that positive solutions to (4.1)-(4.2) are translations of one another.

Let  $(u, v)$  and  $(\bar{u}, \bar{v})$  be two positive solutions to (4.1)-(4.2). By Lemma 4.2, the limits

$$\begin{aligned} \ell_1^+ &:= \lim_{x \rightarrow \infty} (b - u(x))e^{-\lambda-x}, & \bar{\ell}_1^+ &:= \lim_{x \rightarrow \infty} (b - \bar{u}(x))e^{-\lambda-x}, \\ \ell_2^+ &:= \lim_{x \rightarrow \infty} (v(x) - a)e^{-\lambda-x}, & \bar{\ell}_2^+ &:= \lim_{x \rightarrow \infty} (\bar{v} - a)e^{-\lambda-x}, \\ \ell_1^- &:= \lim_{x \rightarrow -\infty} (u(x) - a)e^{\lambda-x}, & \bar{\ell}_1^- &:= \lim_{x \rightarrow -\infty} (\bar{u}(x) - a)e^{\lambda-x}, \\ \ell_2^- &:= \lim_{x \rightarrow -\infty} (b - v(x))e^{\lambda-x}, & \bar{\ell}_2^- &:= \lim_{x \rightarrow -\infty} (b - \bar{v})e^{\lambda-x} \end{aligned}$$

exist and are positive. Thus, in view of (1.6), there is some large  $T_0$  such that

$$u(x - T_0) \leq \bar{u}(x) \leq u(x + T_0) \text{ and } v(x - T_0) \geq \bar{v}(x) \geq v(x + T_0) \text{ for all } x \in \mathbb{R}.$$

Let

$$T = \inf\{t \in [-T_0, T_0] : \bar{u}(x) \leq u(x+t) \text{ and } \bar{v}(x) \geq v(x+t) \text{ for all } t \leq s \leq T_0, x \in \mathbb{R}\}.$$

Set  $\tilde{u}(x) = u(x+T)$  and  $\tilde{v}(x) = v(x+T)$ . Then  $\tilde{u} \geq \bar{u}$  and  $\tilde{v} \leq \bar{v}$ . The result will follow once we show that  $\tilde{u} \equiv \bar{u}$  and  $\tilde{v} \equiv \bar{v}$ . Assume by contradiction that this does not hold.

a. We show that

$$\tilde{u} > \bar{u} \text{ and } \tilde{v} < \bar{v}. \tag{4.10}$$

Note that  $\tilde{u} \geq \bar{u}$ ,  $\tilde{v} \leq \bar{v}$ , and  $(\tilde{u}, \tilde{v})$  is also a solution to (4.1) satisfying (4.7). Also, we have

$$\begin{aligned} P(\tilde{u}, \tilde{v}) - P(\tilde{u}, \bar{v}) &= (\tilde{v} - \bar{v})[(1 + \alpha)\tilde{u}(\tilde{v} + \bar{v}) - \omega] \\ &\leq (\tilde{v} - \bar{v})[(1 + \alpha)\tilde{u}\tilde{v} - \omega] \\ &\stackrel{(1.5)}{\leq} (\tilde{v} - \bar{v})[(1 + \alpha)\frac{\omega}{\alpha} - \omega] \\ &\leq 0, \end{aligned}$$

and

$$\begin{aligned}
Q(\tilde{u}, \tilde{v}) - Q(\bar{u}, \bar{v}) &= (\tilde{u} - \bar{u})[(1 + \alpha)\tilde{v}(\tilde{u} + \bar{u}) - \omega] \\
&\geq (\tilde{u} - \bar{u})[(1 + \alpha)\tilde{v}\tilde{u} - \omega] \\
&\stackrel{(1.5)}{\geq} (\tilde{u} - \bar{u})[(1 + \alpha)\frac{\omega}{\alpha} - \omega] \\
&\geq 0.
\end{aligned}$$

So we have

$$\begin{aligned}
\tilde{u}'' &\leq P(\tilde{u}, \tilde{v}), & \bar{u}'' &= P(\bar{u}, \bar{v}) \\
\tilde{v}'' &\geq Q(\bar{u}, \tilde{v}), & \bar{v}'' &= Q(\bar{u}, \bar{v}).
\end{aligned}$$

Assertion (4.10) thus follows from the strong maximum principle.

b. We proceed to deduce a contradiction.

Define

$$\begin{aligned}
\tilde{\ell}_1^+ &:= \lim_{x \rightarrow \infty} (b - \tilde{u}(x))e^{-\lambda-x}, & \tilde{\ell}_2^+ &:= \lim_{x \rightarrow \infty} (\tilde{v}(x) - a)e^{-\lambda-x}, \\
\tilde{\ell}_1^- &:= \lim_{x \rightarrow -\infty} (\tilde{u}(x) - a)e^{\lambda-x}, & \tilde{\ell}_2^- &:= \lim_{x \rightarrow -\infty} (b - \tilde{v}(x))e^{\lambda-x}.
\end{aligned}$$

Recall that, by Lemma 4.2,  $\ell_2^+ = \mu\ell_1^+$ ,  $\ell_1^- = \mu\ell_2^-$  and similar relations hold for the counterparts with bar and tilde on top. By the minimality of  $T$  and (4.10), we have that

$$\bar{\ell}_2^+ = \tilde{\ell}_2^+, \tag{4.11}$$

or

$$\bar{\ell}_2^- = \tilde{\ell}_2^-, \tag{4.12}$$

or both. Because of the unique correspondence of linearized solutions and nonlinear solutions near a hyperbolic critical point of ODEs [14, Chapter 13, Theorem 4.5] (with  $p = 1$ ) – see the argument leading to (A.10) – we then deduce that  $(\bar{u}, \bar{v}) \equiv (\tilde{u}, \tilde{v})$ , which contradicts (4.10). We conclude the proof.  $\square$

## 4.2 Proof of Lemma 4.3

The existence in Lemma 4.3 follows from an easy variational argument. As far as we are concerned with the application of Lemma 4.3 to the proof of Proposition 4.1, it is enough to show that  $u$  and  $v$  are monotone. This can be done as in Subsection 3.1. Here we provide an alternative proof which yields also uniqueness, which echoes the argument in Step 2 of the proof of Proposition 4.1.

We start with an adaptation of Proposition 1.2 for finite domains.

**Lemma 4.4.** *Suppose that  $0 < \omega < \frac{1}{2}\alpha$  and  $R \in (0, \infty)$ , and let  $(u, v)$  be a positive solution of (4.1) in  $(-R, R)$  satisfying (4.7). Then (1.5) and (1.6) hold in  $[-R, R]$ .*

*Proof.* The proof is similar to though easier than that of Proposition 1.2, thanks to the boundary condition (4.7). We will only give a sketch.

Let  $A = u^2 + v^2$ ,  $B = -\ln(uv)$  and define  $m = \max A$  and  $n = \max B$ .<sup>1</sup> If  $m$  is attained at the endpoints, we have  $m \leq 1$ . Otherwise,  $m = A(x_0)$  for some  $x_0 \in (-R, R)$ . We then have

$$\begin{aligned} 0 &\geq A''(x_0) \geq 2u(x_0)P(u(x_0), v(x_0)) + 2v(x_0)Q(u(x_0), v(x_0)) \\ &\geq 2A^2(x_0) - 2A(x_0) + 4s_* \text{ where } s_* := \min_{t \geq e^{-n}} (\alpha t^2 - \omega t). \end{aligned}$$

In either case, we obtain

$$m \leq \frac{1}{2}(1 + \sqrt{1 - 8s_*}). \quad (4.13)$$

Likewise, we have

$$n \leq -\ln \frac{\omega}{\alpha + 2 - \frac{2}{m}}. \quad (4.14)$$

We can now follow exactly the arguments in Steps 3 and 4 of the proof of Proposition 1.2 to reach the conclusion. We omit the details.  $\square$

*Proof of Lemma 4.3.* Note that (4.1) is the Euler-Lagrange equation for the functional

$$I[u, v] = \int_{-R}^R \left[ \frac{1}{2}(|u'|^2 + |v'|^2) + \frac{1}{4}(1 - u^2 - v^2)^2 + \frac{\alpha}{2}(uv - \frac{\omega}{\alpha})^2 \right] dx.$$

The existence of a positive solution to (4.1) satisfying (4.7) follows from a simple variational argument.

The uniqueness follows from the sliding method as we have seen earlier. Suppose that  $(u, v)$  and  $(\bar{u}, \bar{v})$  are positive solutions of (4.1) in  $(-R, R)$  satisfying (4.7). Extend  $(u, v)$  to the whole of  $\mathbb{R}$  by defining  $(u, v) \equiv (a, b)$  on  $(-\infty, -R)$  and  $(u, v) \equiv (b, a)$  on  $(R, \infty)$ . Let

$$\begin{aligned} T &= \inf \{t \in [0, 2R] : \bar{u}(x) \leq u(x + s) \\ &\quad \text{and } \bar{v}(x) \geq v(x + s) \text{ for all } x \in [-R, R], t \leq s \leq 2R\}. \end{aligned}$$

$T$  is well-defined thanks to Lemma 4.4. To conclude it suffices to show that  $T = 0$ .

---

<sup>1</sup>Note that in the notation of (2.10), we have  $m = m_*$  and  $n = n_*$  thanks to (4.7).

Set  $\tilde{u}(x) = u(x+T)$  and  $\tilde{v}(x) = v(x+T)$ . Note that  $\tilde{u} \geq \bar{u}$ ,  $\tilde{v} \leq \bar{v}$ ,  $(\tilde{u}, \tilde{v})$  is also a solution to (4.1) in the interval  $(-R, R-T)$ , and, in view of (1.5), we have as before that

$$\begin{aligned}\tilde{u}'' &\leq P(\tilde{u}, \tilde{v}), & \bar{u}'' &= P(\bar{u}, \bar{v}) \\ \tilde{v}'' &\geq Q(\tilde{u}, \tilde{v}), & \bar{v}'' &= Q(\bar{u}, \bar{v}).\end{aligned}$$

In particular, if  $T$  was positive, it would follow from the strong maximum principle and the Hopf lemma that there would exist some small  $\varepsilon > 0$  such that

$$\bar{u}(x) \leq u(x+s) \text{ and } \bar{v}(x) \geq v(x+s) \text{ for all } x \in [-R, R], T - \varepsilon \leq s \leq T,$$

which would contradict the definition of  $T$ . We hence have  $T = 0$ , as desired.  $\square$

## A Appendix: proof of Lemma 4.2.

We now prove of the exponential decay of solutions  $(u, v)$  to (4.1)-(4.2) to constants. This was needed in the proof of Proposition 4.1. We perform a standard asymptotic analysis near a hyperbolic critical point of ODEs.

We write  $u = b - \hat{u}$  and  $v = a + \hat{v}$ . The system (4.1) becomes

$$\hat{u}'' = -P(b - \hat{u}, a + \hat{v}) =: \hat{P}(\hat{u}, \hat{v}), \quad (\text{A.1})$$

$$\hat{v}'' = Q(b - \hat{u}, a + \hat{v}) =: \hat{Q}(\hat{u}, \hat{v}). \quad (\text{A.2})$$

The functions  $\hat{P}$  and  $\hat{Q}$  are polynomials and a direct computation gives

$$\frac{\partial(\hat{P}, \hat{Q})}{\partial(\hat{u}, \hat{v})}(0, 0) = \begin{bmatrix} 2b^2 + \alpha a^2 & -\frac{\omega(2+\alpha)}{\alpha} \\ -\frac{\omega(2+\alpha)}{\alpha} & 2a^2 + \alpha b^2 \end{bmatrix} =: \mathbf{A}.$$

In particular, we have, for large  $x$ , that

$$\hat{u}'' = (2b^2 + \alpha a^2)\hat{u} - \frac{\omega(2+\alpha)}{\alpha}\hat{v} + O(|\hat{u}|^2 + |\hat{v}|^2), \quad (\text{A.3})$$

$$\hat{v}'' = -\frac{\omega(2+\alpha)}{\alpha}\hat{u} + (2a^2 + \alpha b^2)\hat{v} + O(|\hat{u}|^2 + |\hat{v}|^2). \quad (\text{A.4})$$

The matrix  $\mathbf{A}$  has two positive eigenvalues  $\lambda_{\pm}^2$  (see (4.3)). For  $\omega > 0$ , an  $\mathbf{A}$ -eigenbasis of  $\mathbb{R}^2$  can be chosen as  $(1, \mu)$  and  $(-\mu, 1)$  (which correspond to the eigenvalues  $\lambda_-^2$  and  $\lambda_+^2$ , respectively) where  $\mu$  is defined in (4.4). Note that

$$\lim_{\omega \rightarrow 0} \mu = \begin{cases} 0 & \text{if } \alpha > 2, \\ \infty & \text{if } \alpha < 2, \\ 1 & \text{if } \alpha = 2. \end{cases}$$

This signifies some difference in the asymptotic behavior of  $(\hat{u}, \hat{v})$  for  $\omega = 0$  and for  $\omega > 0$ .

*Proof of Lemma 4.2.* 1. We prove (4.5) and the relation  $\ell_2 = \mu\ell_1$ .

As  $\hat{u}(x), \hat{v}(x) \rightarrow 0$  as  $x \rightarrow \infty$ , we have from (A.1)-(A.2) that  $\hat{u}''(x), \hat{v}''(x) \rightarrow 0$  as  $x \rightarrow \infty$ . By interpolation, this implies  $\hat{u}'(x), \hat{v}'(x) \rightarrow 0$  as  $x \rightarrow \infty$ . We now write  $\hat{v}_3 = \hat{u}', \hat{v}_4 = \hat{v}', \hat{\mathbf{v}} = (\hat{u}, \hat{v}, \hat{v}_3, \hat{v}_4)$  and recast (A.1)-(A.2) as a first order system

$$\hat{\mathbf{v}}' = \mathbf{M}\hat{\mathbf{v}} + \hat{\mathbf{f}}(\hat{\mathbf{v}}) \quad (\text{A.5})$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2b^2 + \alpha a^2 & -\frac{\omega(2+\alpha)}{\alpha} & 0 & 0 \\ -\frac{\omega(2+\alpha)}{\alpha} & 2a^2 + \alpha b^2 & 0 & 0 \end{bmatrix},$$

and  $\hat{\mathbf{f}}$  is a polynomial satisfying  $\hat{\mathbf{f}}(0) = 0$  and  $D\hat{\mathbf{f}}(0) = 0$ . Note that, as  $0 < \frac{\varepsilon}{\alpha} < \frac{1}{2}$ ,  $\mathbf{M}$  has real and nonzero eigenvalues  $\lambda_1 = -\lambda_+ < \lambda_2 = -\lambda_- < 0 < \lambda_3 = \lambda_- < \lambda_4 = \lambda_+$ . Hence the origin is a hyperbolic critical point of (A.5). As  $\hat{\mathbf{v}}(x) \rightarrow 0$  as  $x \rightarrow 0$ , we thus have that, for all large  $x$ ,  $\hat{\mathbf{v}}(x)$  belongs to the stable manifold of (A.5) at the origin. By the Stable Manifold Theorem (see e.g. [14, Chapter 13, Theorem 4.3]), we then have that  $\hat{\mathbf{v}}(x)$  converges exponentially to 0 as  $x \rightarrow \infty$  and the rate of convergence is  $O(e^{-\lambda x})$  for any  $0 < \lambda < |\lambda_2| = \lambda_-$ .

Set

$$X = \hat{u} + \mu\hat{v}, \quad (\text{A.6})$$

$$Y = -\mu\hat{u} + \hat{v}. \quad (\text{A.7})$$

We have

$$X'' = P(\hat{u}, \hat{v}) + \mu Q(\hat{u}, \hat{v}) = \lambda_-^2 X + O(|\hat{u}|^2 + |\hat{v}|^2), \quad (\text{A.8})$$

$$Y'' = -\mu P(\hat{u}, \hat{v}) + Q(\hat{u}, \hat{v}) = \lambda_+^2 Y + O(|\hat{u}|^2 + |\hat{v}|^2). \quad (\text{A.9})$$

Applying [14, Chapter 13, Theorem 4.5], we can find a constant  $k$  and some  $\delta > 0$  such that

$$X = ke^{-\lambda_- x} + O(e^{-(\lambda_- + \delta)x}) \text{ and } Y = O(e^{-(\lambda_- + \delta)x}). \quad (\text{A.10})$$

Assertion (4.5) and the relation  $\ell_2 = \mu\ell_1$  are readily seen.

2. We next show that  $\ell_1$  and  $\ell_2$  are positive.

Suppose by contradiction the assertion does not hold. As  $\ell_2 = \mu\ell_1$ , one has  $\ell_1 = \ell_2 = 0$ . Returning to (A.10) we have that

$$|X| + |Y| = O(e^{-(\lambda_- + \delta)x})$$

which implies

$$\limsup_{x \rightarrow \infty} \frac{\ln(|X| + |Y|)}{x} \leq -(\lambda_- + \delta).$$

Appealing again to [14, Chapter 13, Theorem 4.3], we thus have

$$\limsup_{x \rightarrow \infty} \frac{\ln(|X| + |Y|)}{x} \leq -\lambda_+.$$

This leads to

$$\lim_{x \rightarrow \infty} (b - u(x))e^{\lambda x} = \lim_{x \rightarrow \infty} (v(x) - a)e^{\lambda x} = 0 \text{ for all } 0 < \lambda < \lambda_+. \quad (\text{A.11})$$

As  $b\sqrt{2} < \lambda_+$ , this gives a contradiction to Lemma A.1 below and so concludes the proof.  $\square$

**Lemma A.1.** *Suppose that  $0 \leq \omega < \frac{1}{2}\alpha$  and let  $(u, v)$  be a positive solution of (4.1)-(4.2). There is some  $C > 0$  such that*

$$b - u(x) \geq \frac{1}{C}e^{-b\sqrt{2}x} \text{ for large } x. \quad (\text{A.12})$$

*Proof.* We note from (4.1) and (1.5) that

$$u'' \stackrel{(4.1), (1.5)}{\geq} u(u^2 + v^2 - 1) \stackrel{(1.5)}{\geq} u(u^2 - b^2). \quad (\text{A.13})$$

Now, take some  $R > 0$  such that  $u(R) \geq \frac{b}{\sqrt{3}}$  in  $(R, \infty)$ . Select  $x_0$  such that  $b \tanh(\frac{b}{\sqrt{2}}(R - x_0)) = u(R)$ . To prove (A.12), it suffices to show that

$$u(x) \leq b \tanh\left(\frac{b}{\sqrt{2}}(x - x_0)\right) \text{ in } (R, \infty).$$

To this end, we note that the function  $w_{1,c} = b \tanh(\frac{b}{\sqrt{2}}(x - x_0)) + c$  satisfies for  $c \geq 0$ ,

$$w_{1,c}'' = w_{1,0}(w_{1,0}^2 - b^2) \leq w_{1,c}(w_{1,c}^2 - b^2) \text{ in } (R, \infty), \quad (\text{A.14})$$

where we have used  $w_{1,0}(x) \geq w_{1,0}(R) = u(R) \geq \frac{b}{\sqrt{3}}$ .

Clearly there is some large  $c > 0$  such that  $w_{1,c} \geq u$  in  $[R, \infty)$ . Let

$$\underline{c} = \inf\{c \geq 0 : w_{1,c} \geq u \text{ in } [R, \infty)\}.$$

If  $\underline{c} > 0$ , then we have  $w_{1,\underline{c}} \geq u$  in  $[R, \infty)$ ,  $w_{1,\underline{c}}(R) > u(R)$ ,  $\lim_{x \rightarrow \infty} (w_{1,\underline{c}} - u) > 0$ , and there is some  $x_1 \in (R, \infty)$  such that  $w_{1,\underline{c}}(x_1) = u(x_1)$ , which gives a contradiction to the strong maximum principle, in view of (A.13) and (A.14). We thus have that  $\underline{c} = 0$ , which implies that  $w_{1,0} \geq u$  in  $[R, \infty)$ , which gives (A.12).  $\square$



The asymptotic behavior changes somewhat in the case  $\omega = 0$ , which we record here for comparison. (This is not used in the paper.)

**Lemma A.2.** *Suppose  $\alpha > 0$  and  $\omega = 0$ . Then,  $a = 0$  and  $b = 1$ . Let  $(u, v)$  be a positive solution of (4.1)-(4.2). Then the following statements hold.*

- (i)  $\tilde{\ell}_2 := \lim_{x \rightarrow \infty} (v(x) - a)e^{\sqrt{\alpha}x}$  exists and is positive.
- (ii) If  $\alpha > \frac{1}{2}$ , then  $\tilde{\ell}_1^+ := \lim_{x \rightarrow \infty} (b - u(x))e^{\sqrt{2}x}$  exists and is positive.
- (iii) If  $\alpha < \frac{1}{2}$ , then  $\tilde{\ell}_1^- := \lim_{x \rightarrow \infty} (b - u(x))e^{2\sqrt{\alpha}x}$  exists and is equal to  $\frac{(\alpha+1)\tilde{\ell}_2^2}{2(1-2\alpha)}$ .
- (iv) If  $\alpha = \frac{1}{2}$ , then  $\tilde{\ell}_1^* := \lim_{x \rightarrow \infty} (b - u(x))\frac{e^{\sqrt{2}x}}{x}$  exists and is equal to  $\frac{3\tilde{\ell}_2^2}{4\sqrt{2}}$ .

*Proof.* The proof is similar to that of Lemma 4.2 and is omitted.  $\square$

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