

## Bilinear Forms, Clifford Algebras, $q$ -Commutation Relations, and Quantum Groups

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Constructions are described which associate algebras to arbitrary bilinear forms, generalising the usual Clifford and Heisenberg algebras. Quantum groups of symmetries are discussed, both as deformed enveloping algebras and as quantised function spaces. A classification of the equivalence classes of bilinear forms is also given. © 2000 Academic Press

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### 1. INTRODUCTION

In this paper we shall show how to classify arbitrary non-degenerate bilinear forms on a finite-dimensional vector space, define associated Clifford and Grassmann algebras (where possible), and see how quantum groups arise naturally as symmetries. Although offering no new fundamental insights this approach does provide simple alternative constructions of various familiar algebras and quantum groups, as well as a source of other straightforward examples.

Let  $B$  be a non-degenerate bilinear form on a finite-dimensional vector space  $V$  over an algebraically closed field  $\mathbb{F}$ , of characteristic different from 2. (In the examples we shall always assume that  $\mathbb{F} = \mathbb{C}$ .) When  $B$  is a symmetric form one may construct the Clifford algebra, and when  $B$  is skew symmetric one has the Heisenberg or CCR (canonical commutation relation) algebra, but we shall explore what happens when  $B$  is not necessarily in either of these classes. (Besides the obvious mathematical interest, we recall that Einstein's unified field theory featured an asymmetric metric [5, Appendix II], and note that in the non-commutative Riemannian geometry of Brzeziński and Majid [3] asymmetric metrics also appear

naturally.) Our aim is to construct a quotient of the tensor algebra by an ideal generated by elements of the form  $P(u \otimes v) - B(u, v)1$ , where  $P$  is a linear operator on  $V \otimes V$ , but there is a consistency requirement which we now explore.

Each  $v \in V$  defines linear functionals,  $B_1v: u \in V \mapsto B(v, u)$  and  $B_2v: u \mapsto B(u, v)$ . The non-degeneracy of  $B$  means that  $B_1$  and  $B_2$  are isomorphisms from  $V$  to its dual, so that we may define a non-singular linear map  $T = B_1^{-1}B_2: V \rightarrow V$  satisfying

$$B(u, v) = (B_2v)(u) = (B_1Tv)(u) = B(Tv, u). \quad (1)$$

We shall see in Section 2 that  $T$  determines  $B$  up to equivalence. (This is not surprising, since when  $T = -1$  it is known that symplectic forms can always be put into Darboux form, whilst Sylvester's law of inertia tells us that when  $T = 1$  a non-degenerate real symmetric form can be expressed as a sum and difference of squares, and over  $\mathbb{C}$  one can write it just as a sum.) Since, by repetition  $B(u, v) = B(Tv, u) = B(Tu, Tv)$ , we see that  $T$  preserves  $B$ , and also, replacing  $v$  by  $T^{-1}v$ , we have  $B(u, T^{-1}v) = B(v, u)$ , providing an alternative formula for transposing the arguments of  $B$ . The last expression also gives  $B(u, T^{-1}v) = B(Tu, v)$ , and similarly we obtain  $B(T^{-1}u, v) = B(u, Tv)$ , showing that  $T^{-1}$  is the adjoint of  $T$ , on both sides. (For general forms  $B$  the adjoints on the two sides are usually different.)

We may also consider  $B$  as a linear functional on  $V \otimes V$ , and then, denoting the transposition  $u \otimes v \mapsto v \otimes u$  by  $\sigma$ , we have  $B(u \otimes v) = B((T \otimes 1)\sigma(u \otimes v))$ , so that  $B = B \circ \hat{T}$ , where  $\hat{T} = (T \otimes 1)\sigma$ . In other words  $B$  annihilates  $\text{im}[1 - \hat{T}]$ , and this means that the ideal should contain the image of  $P(1 - \hat{T})$ . This will impose the fewest constraints and give the largest possible algebra if we require  $P(1 - \hat{T}) = 0$ , and we shall henceforth assume this condition. We note that whenever  $P$  commutes with  $\hat{T}$ , we have  $(1 - \hat{T})P = 0$ , so that it projects onto  $\ker(1 - \hat{T})$ . This subspace is non-trivial as we shall now show.

We may define a dual form  $B^*$  on  $V^*$  by  $B^*(B_1u, B_1v) = B(u, v)$ , and regard  $B^*$  as an element of  $V \otimes V$ . It does not matter whether we use  $B_1$  or  $B_2$  in this definition, since  $B^*(B_2u, B_2v) = B^*(B_1Tu, B_1Tv) = B(Tu, Tv) = B(u, v)$ . Then  $B^*$  is the unique element of  $V \otimes V$  satisfying  $B^{(2)}(B^*, u \otimes v) = B^*(B_2u, B_2v) = B(u, v)$ . Using the fact that  $T$  and its inverse are adjoint with respect to  $B$ , we obtain

$$\begin{aligned} B^{(2)}(\hat{T}B^*, \hat{T}(u \otimes v)) &= B^{(2)}(B^*, u \otimes v) = B(u, v) = B(Tv, u) \\ &= B^{(2)}(B^*, \hat{T}(u \otimes v)), \end{aligned} \quad (2)$$

from which we deduce that  $\hat{T}B^* = B^*$  and  $B^* \in \ker(1 - \hat{T})$ .

The minimal polynomial  $m(x)$  of  $\hat{T}$  therefore has a factor of  $x - 1$  and can be written as  $m(x) = (x - 1)p(x)$ . This suggests that we should simply take  $P = p(\hat{T})$ , to ensure that  $P(1 - \hat{T}) = 0$ . (Indeed if we want  $P$  to be a function of  $\hat{T}$  then, up to multiples, this is the only non-trivial possibility.) The generalised Heisenberg–Clifford algebra  $A(B, P)$  is defined as the quotient of the tensor algebra  $\otimes V$  by the ideal generated by  $P(u \otimes v) - B(u, v)1$  with  $u, v \in V$ , that is,

$$A(B, P) = \otimes V / \{P(u \otimes v) - B(u, v)1 : u, v \in V\}. \quad (3)$$

We shall see later (Theorem 3.6) that when  $T$  has non-trivial Jordan blocks the kernel and image of  $1 - \hat{T}$  usually intersect in an isotropic subspace. However, when  $T$  is diagonalisable the intersection is trivial and  $V \otimes V$  is the direct sum of the two subspaces. In that case it is sufficient to factor out by the ideal generated by  $\{X - B(X)1 : X \in \ker(1 - \hat{T})\}$ .

It will be useful to note that  $P$  can be regarded as an orthogonal projection on  $V \otimes V$  with respect to the natural non-degenerate bilinear form  $B^{(2)} = B \otimes B$ . Using the fact that the two-sided adjoint of  $T$  is  $T^{-1}$ ; whilst  $\sigma$  is self-adjoint, the orthogonal complement of  $\text{im}[1 - \hat{T}]$  is

$$\ker[1 - \hat{T}^{-1}]. \quad (4)$$

(Were  $B$  to take values in a non-commutative algebra then  $\sigma$  might not be self-adjoint, but this could easily be accommodated within the same general framework.) We can further simplify things by noting that

$$[1 - \hat{T}^{-1}] = \hat{T}^{-1}[\hat{T} - 1], \quad (5)$$

so that  $\ker[1 - \hat{T}^{-1}] = \ker[1 - \hat{T}]$ .

EXAMPLE 1.1. If  $B$  is symmetric then  $T = 1$ , and  $P$  is the projection onto the kernel of  $1 - \sigma$ , which is given by  $P = \frac{1}{2}(1 + \sigma)$  or  $P(u \otimes v) = \frac{1}{2}(u \otimes v + v \otimes u)$ , so that the algebra is the algebra generated by  $V$  subject to the relations  $\frac{1}{2}(uv + vu) = B(u, v)$ , that is, the usual Clifford algebra. Similarly, if  $B$  is antisymmetric then  $T = -1$ ,  $P = \frac{1}{2}(1 - \sigma)$ , and the algebra is generated by  $V$  subject to the relations  $\frac{1}{2}(uv - vu) = B(u, v)$ , that is, the normal Heisenberg algebra for the commutation relations.

EXAMPLE 1.2. It is often useful to choose a basis  $e_1, e_2, \dots, e_n$  for  $V$  and identify elements of  $V \otimes V$  with matrices via the map  $u \otimes v \mapsto uv'$  (where prime denotes transpose). Then  $\sigma$  transposes the matrices, and the elements of  $\ker[1 - \hat{T}]$  are matrices  $X$  satisfying  $X = TX'$ . It is easy to check that  $T$  is given in terms of the Gram matrix  $G = (B(e_i, e_j))$  by  $T = (G')^{-1}G$ , so that the condition amounts to  $G'X = GX'$ . (We note that  $\det(T) = \det(G')^{-1} \det(G) = 1$ .) When  $n = 2$  and

$$G = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}, \quad (6)$$

we have

$$T = - \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}, \quad (7)$$

and unless  $q = \pm 1$  (the cases already considered) we have  $X$  a multiple of

$$\begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \quad (8)$$

corresponding to the tensor  $e_1 \otimes e_2 - qe_2 \otimes e_1$ . This gives the algebra generated by  $V$  subject to the relation

$$\frac{1}{2}(e_1e_2 - qe_2e_1) = 1. \quad (9)$$

This is a  $q$ -CCR algebra recently considered by many authors [1, 2, 5–10, 13], but especially [2].

We could have arrived at the same result in a basis-free way, by noting that if  $\hat{T}(X) = X$  then, writing  $T_1 = T \otimes 1$ ,  $(T_1 - 1)X = T_1(1 - \sigma)X$ . Now  $A = X - \sigma(X)$  is an antisymmetric tensor and, if 1 is not an eigenvalue of  $T$ , we have  $X = (T_1 - 1)^{-1}T_1A$ , so that  $X$  is determined by its antisymmetric part. The dimension of  $\ker(1 - \hat{T})$  is thus at most that of the space of antisymmetric tensors, which for two-dimensional  $V$  is just one-dimensional, so that  $\ker(1 - \hat{T})$  must be spanned by  $B^*$ . Since the antisymmetric tensor  $A$  has even rank,  $(T_1 - 1)$  must have even rank, which means that in dimension 2,  $T_1 - 1$  is either zero or onto, that is, if 1 is an eigenvalue of  $T$ , Then  $T = 1$ . Thus either  $T = 1$  and we are in the symmetric case or 1 is not an eigenvalue and  $\ker(1 - \hat{T})$  is a one-dimensional space spanned by  $B^*$ . It is easy to check that the Gram matrix of  $B^*$  with respect to the dual basis is  $G'^{-1}$ , from which we find that  $B^* = e_1 \otimes e_2 - qe_2 \otimes e_1$ .

In order to make the connection with other approaches clearer, it is useful to work with the crossed product algebra containing an element  $U$  such that for all  $v \in V$ ,  $U^{-1}vU = q^{-(1/2)}Tv$ . (By treating  $U$  as a group-like element, so that, in particular  $\Delta(U) = U \otimes U$ , we may extend the Hopf algebra structure to the crossed product.) We may then define  $\hat{e}_1 = e_1U$  and  $\hat{e}_2 = Ue_2$  to obtain a formula involving ordinary commutators,

$$\begin{aligned} [\hat{e}_1, \hat{e}_2] &= e_1U^2e_2 - Ue_2e_1U = e_1(qT^{-1}e_2)U^2 - qe_2e_1U^2 \\ &= (e_1e_2 - qe_2e_1)U^2 = 2U^2. \end{aligned} \quad (10)$$

In the physics literature  $e_1$  and  $e_2$  are taken to be adjoints and  $q^{-1}U^2$  is interpreted in terms of a number operator. One can also obtain a comulti-

plication by taking

$$\Delta(e_1) = e_1 \otimes U + U \otimes e_1, \quad \Delta(e_2) = e_2 \otimes U^{-1} + U^{-1} \otimes e_2. \quad (11)$$

Changing  $q$  to  $-q$  gives the  $q$ -Clifford algebra relation

$$\frac{1}{2}(e_1 e_2 + q e_2 e_1) = 1. \quad (12)$$

EXAMPLE 1.3. When studying the symmetries of the asymmetric tensor  $B$  the usual co-commutative comultiplication on linear transformations is inappropriate, and it should be replaced by more general  $\Delta$ . To preserve  $B$  a linear transformation  $S$  should satisfy  $B \circ \Delta(S) = B$ , or the infinitesimal (Lie algebra) version  $B \circ \Delta(S) = 0$ , which is the appropriate substitute for the usual condition that  $B(Su, v) + B(u, Sv) = 0$  when  $\Delta(S) = S \otimes 1 + 1 \otimes S$ .

The most obvious symmetry of  $B$  is provided by  $T$ , and in that case we can use  $\Delta(T) = T \otimes T$ , but if we define  $L_j v = B(e_j, v)e_j$  and set  $\Delta(L_j) = L_j \otimes 1 - T \otimes L_j$ , for  $j = 1, 2, \dots, n$ , then

$$\begin{aligned} B(\Delta(L_j)(u \otimes v)) &= B(L_j u, v) - B(Tu, L_j v) \\ &= B(L_j u, v) - B(L_j v, u) \\ &= B(e_j, u)B(e_j, v) - B(e_j, v)B(e_j, u) = 0, \end{aligned} \quad (13)$$

showing that each  $L_j$  is an infinitesimal symmetry. (When  $B$  is skew symmetric and  $T = -1$  we have the usual comultiplication  $\Delta(L_j) = L_j \otimes 1 + 1 \otimes L_j$ , and when  $B$  is symmetric,  $\Delta(L_j) = L_j \otimes 1 - 1 \otimes L_j$  is the same as the usual Clifford algebra comultiplication when one takes account of the grading of the tensor product.) Let us now specialise to the case of  $B$  and  $T$  defined by (6) and (7), where we also have

$$TL_1 T^{-1} = q^{-2} L_1, \quad TL_2 T^{-1} = q^2 L_2, \quad (14)$$

and, for  $v = v_1 e_1 + v_2 e_2$ ,

$$\begin{aligned} [L_1, L_2]v &= B(e_1, e_2)B(e_2, v)e_1 - B(e_2, e_1)B(e_1, v)e_2 \\ &= -q^{-1}v_1 e_1 + q^{-1}v_2 e_2, \end{aligned} \quad (15)$$

which is easily seen to be identical to  $(1 - q^2)^{-1}(T - T^{-1})v$ , so that the three elements  $T$ ,  $L_1$ , and  $L_2$  generate the quantum group  $\mathfrak{sl}_2(q^2)$ .

EXAMPLE 1.4. We know that

$$B((T - q)u, v) = B(Tv, (T - q)u) = B((1 - qT)v, u), \quad (16)$$

from which, taking  $u \in \ker(T - q)$ ,

$$\ker(T - q) \subseteq \operatorname{im}(1 - qT)^\perp \quad (17)$$

so that if  $q$  is an eigenvalue then  $(1 - qT)$  is not onto, and so  $q^{-1}$  is also an eigenvalue. In particular, in a two-dimensional space if  $T$  has an eigenvalue  $-q \neq \pm 1$ , then its other eigenvalue is  $-q^{-1}$ , and it is equivalent to

$$\begin{pmatrix} -q^{-1} & 0 \\ 0 & -q \end{pmatrix}, \quad (18)$$

which we already studied in Example 1.2.

We observed earlier that 1 is an eigenvalue if and only if  $T$  is the identity, which is the case  $q = -1$  above. The only other possibility is that  $T$  has the repeated eigenvalue  $-1$ . This can either happen with diagonal  $T$ , the case  $q = 1$  above, or (in two dimensions) with the indecomposable Jordan block

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad (19)$$

and  $B$  having Gram matrix equivalent to a multiple of

$$G = \begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix}. \quad (20)$$

This exceptional case turns out to be linked to the exceptional  $R$ -matrix of Demidov et al. [4] (see Section 5). For a two-dimensional space this exhausts the possibilities for  $B$  and  $T$ , and gives an indication of why the  $q$  deformations are natural in low dimensions.

EXAMPLE 1.5. A series of higher dimensional  $q$ -CCR algebras is easily constructed by taking  $V = W \otimes W^*$ , for some real vector space  $W$  with dual  $W^*$ , and setting

$$B((x, \xi), (y, \eta)) = \eta(x) - q^{-1}\xi(y), \quad (21)$$

for  $x, y \in W$  and  $\xi, \eta \in W^*$ . Then  $T(y, \eta) = -(q^{-1}y, q\eta)$ , and one can show that the algebra is generated by  $W$  and  $W^*$  subject to the relations

$$\xi y - qy\xi = 2\xi(y)1, \quad (22)$$

for all  $\xi \in W^*$  and  $y \in W$ . This is not quite the same as the usual  $q$ -commutation relations which consist of a number of mutually commuting copies of the two-dimensional  $q$ -commutation relations.

Nonetheless it is easy to find a representation of the above relations: We take for the representation space the tensor algebra of  $W$  and let  $x \in W$  act by left tensor multiplication by  $2x$ , and inductively define the action of  $\eta \in W^*$  to annihilate degree zero tensors and satisfy

$$\eta(w \otimes Z) = qw \otimes \eta(Z) + \eta(w)Z, \quad (23)$$

for  $Z \in \otimes^k W$  and  $w \in W$ . This definition ensures that

$$\eta x - qx\eta = 2\eta(x)1, \quad (24)$$

but there is no kind of commutativity between the action of distinct  $x$  and  $y \in W$ . Commutativity can be arranged by restricting the tensor product to symmetric tensors, and in two dimensions (with a suitable completion) this gives the usual Fock representation of the  $q$ -commutation relations [2].

EXAMPLE 1.6. A KMS state  $\phi$  on a von Neumann algebra  $\mathcal{A}$  provides a natural example of an asymmetric bilinear form  $B(a, b) = \phi(ab)$  for  $a, b \in \mathcal{A}$  [14]. The KMS condition gives  $\phi(ab) = \phi(b\Delta a\Delta^{-1})$ , where  $\Delta$  is the modular operator, and so  $T(a) = \Delta^{-1}a\Delta$ . For the Heisenberg algebra or Clifford algebra, generated by elements  $c(v)$  with  $v \in V$ , one also gets a bilinear form  $B(u, v) = \phi(c(u)c(v))$  on  $V$ , and often the KMS condition gives an explicit formula for  $T$  here as well. For example, a short calculation shows that the  $\beta$ -KMS state for the Clifford algebra of  $\mathbb{R}^2$  with evolution at time  $t$  given by rotation through  $\omega t$  gives rise to the above  $q$ -Clifford algebra with  $q = \exp(\pm\beta\omega)$ .

EXAMPLE 1.7. Finally we note that although one might construct generalised Grassmann (or exterior) quadratic algebras by factoring out the ideal generated by the image of  $P$ , there is an alternative more useful approach, which uses a square root  $S = T^{1/2}$ . The right adjoint  $S^*$  defined by  $B(Su, v) = B(u, S^*v)$  satisfies

$$B(v, S^{*2}Tu) = B(S^2v, Tu) = B(Tv, Tu) = B(v, u), \quad (25)$$

so that  $S^{*2} = T^{-1}$ . If  $S$  can be chosen to be a normal operator (so that  $S$  and  $S^*$  commute) then  $(S^*S)^2 = S^{*2}S^2 = 1$  so that  $I = S^*S$  is an involution. We also have

$$B(Sv, S^*u) = B(S^2v, u) = B(Tv, u) = B(u, v), \quad (26)$$

so that  $B(Sv, Iu) = B(Su, v)$ . Thus the form  $B(Su, v)$  is symmetric on the  $(+1)$ -eigenspace of  $I$  and antisymmetric on the  $(-1)$ -eigenspace. This suggests using the quotient of the tensor algebra  $\otimes V$  by the ideal generated by elements of the form  $Su \otimes v + Sv \otimes Iu$  or, more simply, by elements

$$u \otimes v + Sv \otimes S^*u. \quad (27)$$

For the form  $B((x, \xi), (y, \eta)) = \eta(x) - q^{-1}\xi(y)$  of Example 1.5, we have  $T(x, \xi) = -(q^{-1}x, q\xi)$ , so that we may take  $S(x, \xi) = \pm i(q^{-(1/2)}x, \pm q^{1/2}\xi)$ , with any combination of signs. A brief calculation shows that  $S^*(x, \xi) = \pm i(\pm q^{1/2}x, q^{-(1/2)}\xi)$ , where the signs inside and outside the bracket match those for  $S$ . The signs outside the bracket cancel in the formula for elements of the ideal which can be written as

$$(x, \xi) \otimes (y, \eta) - (q^{-(1/2)}y, \pm q^{1/2}\eta) \otimes (\pm q^{1/2}x, q^{-(1/2)}\xi), \quad (28)$$

where the signs match in the two factors of the middle term. This gives the algebra generated by  $W$  and  $W^*$  subject to the relations

$$xy = \pm yx, \quad \xi\eta = \pm \eta\xi, \quad x\eta = q\eta x, \quad (29)$$

for  $x, y \in W$ ,  $\xi, \eta \in W^*$ . According to the choice of upper or lower signs, these give deformations of the symmetric or exterior algebra. In general, for any involutions  $I_1$  and  $I_2$  on  $W$  and  $W^*$ , respectively, one sees that  $S(x, \xi) = i(q^{-(1/2)}I_1x, q^{1/2}I_2\xi)$  is a square root of  $T$ , but it is a normal operator only when  $I_1$  and  $I_2$  commute. We can, however, narrow the possibilities to those previously considered, and ensure that  $S$  is normal, by insisting that the square roots be in the algebra generated by  $T$ . In fact, whenever  $T$  has a quadratic minimal polynomial of the form  $T^2 - bT + 1 = 0$ , we readily check that the operators  $S = \pm(b + 2\epsilon)^{-(1/2)}(T + \epsilon)$ , with  $\epsilon^2 = 1$ , are square roots of  $T$ . (In the example where  $b = -(q + q^{-1})$ , these reduce to the square roots already given above. When  $T$  has the non-trivial Jordan form of Example 1.4 one must take  $\epsilon = -1$ .) We then have

$$\begin{aligned} S^* &= \pm(b + 2\epsilon)^{-(1/2)}(T^{-1} + \epsilon) = \pm(b + 2\epsilon)^{-(1/2)}\epsilon T^{-1}(T + \epsilon) \\ &= \epsilon T^{-1}S = \epsilon S^{-1}, \end{aligned} \quad (30)$$

so that  $SS^* = \epsilon$ .

Returning to the general case, unfortunately one cannot usually construct Clifford algebras by this approach. The obvious idea would be to factor the tensor algebra by the ideal generated by elements of the form  $u \otimes v + Sv \otimes S^*u - 2B(u, v)1$ , giving an algebra generated by  $V$  with relations

$$uv + (Sv)(S^*u) = 2B(u, v)1. \quad (31)$$

In the case of the standard example, defined by Eqs. (6) and (7), one obtains the algebra generated by  $W$  and  $W^*$  subject to the inhomogeneous version of the relations in (29),

$$xy = \pm yx, \quad \xi\eta = \pm \eta\xi, \quad x\eta - q\eta x = 2\eta(x)1, \quad (32)$$



for  $x, y \in W$ ,  $\xi, \eta \in W^*$ . From the third relation we deduce that  $xy\eta - qx\eta y = 2\eta(y)x$  and  $x\eta y - q\eta xy = 2\eta(x)y$ , and taking a linear combination

$$xy\eta - q^2\eta xy = 2\eta(y)x + 2q\eta(x)y. \quad (33)$$

Interchanging  $x$  and  $y$ , we see that

$$yx\eta - q^2\eta yx = 2\eta(x)y + 2q\eta(y)x. \quad (34)$$

The relation  $[x, y] = 0$  now gives

$$2\eta(y)x + 2q\eta(x)y = 2\eta(x)y + 2q\eta(y)x, \quad (35)$$

and, unless  $q = 1$ , we obtain

$$\eta(x)y = \eta(y)x. \quad (36)$$

When  $\dim(W) > 1$  we may take linearly independent  $x$  and  $y$ , and a linear functional  $\eta$  whose kernel contains  $x$  but not  $y$ , to deduce that  $x = 0$ , which contradicts the other relations. (The ideal generates the whole algebra, giving a trivial quotient.) For two-dimensional  $V$  where  $x$  and  $y$  must always be linearly dependent there is no problem and the above relations define the  $q^{-1}$  commutation relations. This example shows both the strengths and weaknesses of this approach: it gives more useful Grassmann algebras than the other approach, but does not give useful Clifford algebras in higher dimensions.

The remainder of the paper is organised as follows. In the next section we classify the possible forms in terms of  $T$ . (One would expect this to have been done long ago, but searches and enquiries have failed to find it in the literature.) Then in Section 3 we show how to construct the projection  $P$  when it exists. We indicate the close relation to Manin's work on quantised function algebras in Section 4, following that with a brief discussion of the relationship to the  $R$  matrix in Section 5, before showing how quantum enveloping algebras arise as symmetries of asymmetric forms in Section 6.

## 2. THE CLASSIFICATION OF NON-DEGENERATE BILINEAR FORMS

In this section we shall assume that  $V$  is a complex vector space, and classify the possible forms  $B$  in terms of the Jordan form of  $T$  or, equivalently, the primary decomposition of  $V$  as

$$V = \bigoplus_j \ker(T - q_j)^{k_j}, \quad (37)$$

where the minimal polynomial of  $T$  is  $\prod_j (x - q_j)^{k_j}$ . Each summand may be further broken down into cyclic subspaces, but these must be chosen carefully in relation to  $B$ . To this end we note that each  $T$ -invariant subspace  $U$  has a well defined orthogonal complement  $U^\perp$  since

$$\{v \in V : B(u, v) = 0, u \in U\} = \{v \in V : B(v, u) = 0, u \in U\}. \quad (38)$$

(If  $B(u, v) = 0$  for all  $u \in U$ , then  $B(v, u) = B(Tu, v) = 0$ , and similarly the other way around.) As usual, we call a subspace  $U$  isotropic if  $B|_{U \times U} = 0$ , and non-degenerate if  $U^\perp \cap U = 0$ . Our strategy will be to identify non-degenerate cyclic subspaces of the primary decomposition. Then one studies the orthogonal complement and an induction on the dimension gives a complete classification. The first step is the following generalisation of a well-known result.

**PROPOSITION 2.1.** *Let  $u \in \ker(T - \lambda)^k$  and  $v \in \ker(T - \mu)^m$ . Then  $B(u, v) = 0$  unless  $\lambda\mu = 1$ .*

*Proof.* Using the unitary of  $T$  with respect to  $B$ , we easily show by induction that  $B(T^k u, v) = B(u, T^{-k} v) = B(T^{1-k} v, u)$  for any positive integer  $k$ , and taking linear combinations of these identities we see that

$$B(r(T)u, v) = B(u, r(T^{-1})v) = B(\text{Tr}(T^{-1})v, u), \quad (39)$$

for any polynomial  $r$ . On the other hand

$$\begin{aligned} & (1 - \lambda\mu)^{k+m} B(u, v) \\ &= B(u, (\lambda(T - \mu) + (1 - \lambda T))^{k+m} v) \\ &= \sum_{r=-m}^k \binom{k+m}{k-r} \lambda^{m+r} B(u, (1 - \lambda T)^{k-r} (T - \mu)^{m+r} v), \end{aligned} \quad (40)$$

which, by our earlier remark, can be rewritten as

$$\begin{aligned} & (1 - \lambda\mu)^{k+m} B(u, v) \\ &= \sum_{r=-m}^k \binom{k+m}{k-r} \lambda^{m+r} B((T - \lambda)^{k-r} T^{-k+r} u, (T - \mu)^{m+r} v). \end{aligned} \quad (41)$$

One or another argument of  $B$  vanishes for each value of  $r$ , so that  $(1 - \lambda\mu)^{k+m} B(u, v) = 0$  and, unless  $\lambda\mu = 1$ , we deduce that  $B(u, v) = 0$ . ■

This means that the  $q$  and  $q^{-1}$  summands in the primary decomposition of  $V$  are orthogonal to all others, and must span a non-degenerate

subspace. This provides a  $T$ -invariant orthogonal decomposition, so that it suffices just to study these non-degenerate subspaces. We shall first show how to construct non-degenerate subspaces continuing just one or two indecomposable Jordan blocks for  $T$ , and then show how to put  $B$  into a canonical form within these subspaces.

We treat the case of  $q \neq \pm 1$  first, and suppose that its primary summand has minimal polynomial  $(T - q)^k$ . There must exist  $u \in V$  such that  $(T - q)^{k-1}u \neq 0$ , and then, by non-degeneracy, there also exists  $v \in V$  such that  $B((T - q)^{k-1}u, v) \neq 0$ . By the earlier comments on orthogonality we may as well take  $v \in \ker(T - q^{-1})^k$ , and without loss of generality we take  $B((T - q)^{k-1}u, v) = 1$ .

**PROPOSITION 2.2.** *For  $q \neq \pm 1$  the  $T$ -invariant subspace  $V_q$  generated by  $u$  and  $v$  is non-degenerate with respect to  $B$ .*

*Proof.* Since  $T$  is non-singular the vectors

$$u, (T - q)u, \dots, (T - q)^{k-1}u, v, (T^{-1} - q)v, \dots, (T^{-1} - q)^{k-1}v \quad (42)$$

form a basis for  $V_q$ , and so any element can be written in the form  $p(T - q)u + r(T^{-1} - q)v$  for some polynomials  $p$  and  $r$  of degree at most  $k - 1$ . If  $p(T - q)u + r(T^{-1} - q)v$  is orthogonal to the whole of  $V_q$  then we have

$$B(p(T - q)u + r(T^{-1} - q)v, (T^{-1} - q)^s v) = 0 \quad (43)$$

for all  $s$ . Using the earlier relations this gives

$$0 = B(p(T - q)u, (T^{-1} - q)^s v) = B((T - q)^s p(T - q)u, v). \quad (44)$$

Choosing first  $s = k - 1$  we see from the definition of  $u$  and  $v$  that  $p$  has vanishing constant term, and then inductively we check that all the other coefficients in  $p$  vanish, so that  $p = 0$ . Similarly  $r = 0$ , showing that  $V_q \cap V_q^\perp = 0$ , which is the non-degeneracy condition. ■

Preceding inductively, this shows that we can further decompose the space into a  $B$ -orthogonal direct sum of spaces of the form  $V_q \oplus V_q^\perp$ .

The following result plays an important role in the classification of  $B$ .

**PROPOSITION 2.3.** *For any non-zero  $q \in \mathbb{F}$  we have the identities*

$$T^{-1} - q^{-1} = -q^{-2}(T - q)(1 + q^{-1}(T - q))^{-1}, \quad (45)$$

and in the subspace  $\ker(T - q)^k$ , for  $s \geq 1$ ,

$$(T^{-1} - q^{-1})^s = \sum_{j=0}^{k-s-1} \binom{s+j-1}{j} (-1)^{s+j} q^{-2s-j} (T - q)^{s+j}. \quad (46)$$

*Proof.* The first result follows immediately from

$$\begin{aligned} T^{-1} &= q^{-1}(1 + q^{-1}(T - q))^{-1} \\ &= q^{-1}\left[1 - q^{-1}(T - q)(1 + q^{-1}(T - q))^{-1}\right]. \end{aligned} \quad (47)$$

Then, taking its  $s$ th power and expanding the inverse by the binomial theorem, we obtain the second identity. ■

We note that, in particular,

$$(T^{-1} - q^{-1})^{k-1} = (-1)^{k-1} q^{-2k+2} (T - q)^{k-1}, \quad (48)$$

$$\begin{aligned} (T^{-1} - q^{-1})^{k-2} &= (-1)^{k-2} q^{-2k+4} (T - q)^{k-2} \\ &\quad + (-1)^{k-1} (k-2) q^{-2k+3} (T - q)^{k-1}. \end{aligned} \quad (49)$$

**PROPOSITION 2.4.** *For any  $u, v \in V_q$  and natural numbers  $r$  and  $s$  one may express  $B((T - q)^r u, (T - q^{-1})^s v)$  in terms of  $B((T - q)^j u, v)$ , for  $j \geq r + s$ . When  $r + s \geq k$ ,  $B((T - q)^r u, (T - q^{-1})^s v) = 0$ .*

*Proof.* Proposition 2.3 shows that

$$\begin{aligned} B((T - q)^r u, (T - q^{-1})^s v) \\ = (-q^2)^{-s} B((T - q)^{r+s} (1 + q^{-1}(T - q))^{-s} u, v), \end{aligned} \quad (50)$$

and by expanding the negative power in a geometric series we obtain the result. (When  $r + s \geq k$  we use  $(T - q)^k u = 0$ .) ■

This result tells us that  $B$  is determined on  $V_q$ , by the values  $B((T - q)^j u, v)$ , for  $j = 0, \dots, k - 1$ . There is, however, still some flexibility as we might have chosen different vectors  $u$  and  $v$ . To investigate this another technical result is useful.

**PROPOSITION 2.5.** *For any vectors  $u_0, u' \in \ker(T - q)^k$  and  $v_0, v' \in \ker(T - q^{-1})^k$ , the vectors  $u_a = u_0 + a(T - q)^{k-r-1} u'$  and  $v_b = v_0 + b(T - q^{-1})^{k-r-1} v'$  satisfy*

$$\begin{aligned} B((T - q)^r u_a, v_b) &= aB((T - q)^{k-1} u', v_0) \\ &\quad + (-1)^{k-r-1} q^{-2(k-r-1)} bB((T - q)^{k-1} u_0, v') \\ &\quad + B((T - q)^r u_0, v_0), \end{aligned} \quad (51)$$

and, for  $s > r$

$$B((T - q)^s u_a, v_b) = B((T - q)^s u_0, v_0). \quad (52)$$

*Proof.* For any  $s \geq r > 0$  we have

$$\begin{aligned}
 B((T - q)^s u_a, v_b) &= aB((T - q)^{k+s-r-1} u', v_0) \\
 &\quad + bB((T - q)^s u_0, (T - q^{-1})^{k-r-1} v') \\
 &\quad + B((T - q)^s u_0, v_0) \\
 &\quad + abB((T - q)^{k+s-r-1} u', (T - q^{-1})^{k-r-1} v'),
 \end{aligned} \tag{53}$$

and when  $s > r$  this reduces to

$$B((T - q)^s u_a, v_b) = B((T - q)^s u_0, v_0). \tag{54}$$

For  $s = r$  we get the expression for  $B((T - q)^r u_a, v_b)$ ,

$$\begin{aligned}
 &aB((T - q)^{k-1} u', v_0) + bB((T^{-1} - q^{-1})^{k-r-1} (T - q)^r u_0, v') \\
 &\quad + B((T - q)^r u_0, v_0),
 \end{aligned} \tag{55}$$

whence the result follows using Proposition 2.3. ■

This gives us a way to modify some of the matrix elements in the subspace  $V_q$  without affecting others.

**PROPOSITION 2.6.** *The vectors  $u$  and  $v$  in  $V_q$  may be chosen so that  $B((T - q)^r u, v) = 0$  unless  $r = k - 1$ , and  $B((T - q)^{k-1} u, v) = 1$ .*

*Proof.* We take  $u' = u_0 = u$ ,  $v' = v_0 = v$  and  $r = k - 2$  in Proposition 2.5 to get

$$B((T - q)^{k-1} u_a, v_b) = B((T - q)^{k-1} u_0, v_0) \tag{56}$$

and

$$\begin{aligned}
 B((T - q)^{k-2} u_a, v_b) &= (a - q^{-2}b)B((T - q)^{k-1} u, v) \\
 &\quad + B((T - q)^{k-2} u_0, v_0),
 \end{aligned} \tag{57}$$

so that by choosing  $a - q^{-2}b = -B((T - q)^{k-2} u, v)$  we may ensure that  $B((T - q)^{k-2} u_a, v_b) = 0$ , whilst retaining  $B((T - q)^{k-1} u_a, v_b) = 1$ . Repeating this with lower values of  $r$  we may ensure that  $B((T - q)^r u, v) = 0$  unless  $r = k - 1$ . ■

The following result enables us to calculate the matrix element  $B((T - q^{-1})^s v, u)$ , for any vectors  $u$  and  $v$ , in terms of  $B((T - q)^s u, v)$  with  $s \geq r$ .

PROPOSITION 2.7. *For any  $u, v \in V$ , and any  $q \in \mathbb{F}$  one has*

$$B(v, u) - qB(u, v) = B((T - q)u, v). \quad (58)$$

*If  $B((T - q)^s u, v) = 0$  for all  $s > r$  (and so, in particular, when  $u \in \ker(T - q)^k$  and  $r = k - 1$ ), one has*

$$B((T - q)^r u, v) = (-1)^r q^{2r-1} B((T - q^{-1})^r v, u). \quad (59)$$

*Proof.* The first identity is immediate from the defining property of  $T$ . For the second we change  $u$  to  $(T - q)^r u$ , to get

$$qB((T - q)^r u, v) = B(v, (T - q)^r u) = B((T^{-1} - q)^r v, u). \quad (60)$$

Applying Proposition 2.3 this reduces to

$$qB((T - q)^r u, v) = (-1)^r q^{2r} B((T - q^{-1})^r v, u), \quad (61)$$

from which the result follows. ■

Combined with the previous information about decompositions this result tells us that  $B$  is determined up to equivalence on the direct sum of such non-degenerate  $T$ -invariant subspaces. We must now turn our attention to the case of the eigenvalues  $\pm 1$ . Within the subspace  $\ker(T - q)^k$  we can, as before, find a vector  $v$  such that  $(T - q)^{k-1}v \neq 0$ , and there must also be a vector  $u \in V$  such that  $B((T - q)^{k-1}v, u) = 1$ . When  $q = \pm 1$  there is a dichotomy at this point.

PROPOSITION 2.8. *If  $q = (-1)^{k-1}$  then there is a vector  $w \in \ker(T - q)^k$  such that  $B((T - q)^{k-1}w, w) = 1$ , and  $w$  generates a non-degenerate  $T$ -invariant subspace. Otherwise there are two isotropic subspaces of  $\ker(T - q)^k$  generated by vectors  $v$  and  $u$  whose direct sum is non-degenerate.*

*Proof.* Let  $u$  and  $v$  be chosen as in Proposition 2.6. Interpreting Proposition 2.7 when  $q^2 = 1$  and  $u = v$  we see that if  $B(T - q)^s v, v) = 0$  for all  $s > r$  then

$$B((T - q)^r v, v) = (-1)^r q^{-1} B((T - q)^r v, v), \quad (62)$$

from which it follows that either  $q = (-1)^r$  or  $B((T - q)^r v, v) = 0$ . In particular we see that if there is any vector  $w \in \ker(T - q)^k$  such that  $B((T - q)^{k-1}w, w) \neq 0$  then  $q = (-1)^{k-1}$ .

Next we use Proposition 2.5 with  $u_0 = v_0 = v$ ,  $u' = v' = u$ , and  $a = b$ ,  $w = u_a = v_b$  to see that, for  $s > r$ ,  $B((T - q)^s w, w) = B((T - q)^s v, v) = 0$ ,

$$\begin{aligned} B((T - q)^r w, w) &= aB((T - q)^{k-1}u, v) \\ &\quad + (-1)^{k-r-1} aB((T - q)^{k-1}v, u) \\ &\quad + B((T - q)^r v, v). \end{aligned} \quad (63)$$

Using Eq. (59) we can combine the first two terms to get

$$\begin{aligned} B((T - q)^r w, w) &= (1 + (-1)^r q) a B((T - q)^{k-1} u, v) \\ &\quad + B((T - q)^r v, v) \\ &= (1 + (-1)^r q) a + B((T - q)^r v, v). \end{aligned} \quad (64)$$

We shall now distinguish two cases.

(i) If  $q = (-1)^{k-1}$  then we can take  $r = k - 1$ , and we see that there are certainly values of  $a$  which give non-vanishing  $B((T - q)^{k-1} w, w)$ .

(ii) If  $q \neq (-1)^{k-1}$ , and the subspace generated by  $v$  is not already isotropic, let  $r < k - 1$  be the largest integer for which  $B((T - q)^r v, v) \neq 0$ . By our earlier observation we must have  $q = (-1)^r$ , so by the earlier equation

$$B((T - q)^r w, w) = 2a + B((T - q)^r v, v), \quad (65)$$

and we can take  $a = -\frac{1}{2}B((T - q)^r v, v)$  to reduce this to zero. The largest exponent  $s$  for which  $B((T - q)^s w, w) \neq 0$  is therefore less than  $r$  and we can use an inductive argument to reduce to the case where  $B((T - q)^r v, v) = 0$  for all exponents  $r$ , in which case the subspace generated by  $v$  is isotropic. One can similarly modify  $u$  to obtain another isotropic subspace generated by  $u$  which pairs with it to give a non-degenerate subspace in the same way that the  $q$  and  $q^{-1}$  subspaces paired before. ■

In the case of paired isotropic subspaces we can proceed exactly as before to find a canonical form for  $B$ . In the other case of a single non-degenerate cyclic subspace, we note that  $B((T - q)^r w, (T - q)^s w)$  can be expressed in terms of  $B((T - q)^j w, w)$  with  $j \geq r + s$ . However, in this indecomposable case one can go further than before. Setting  $q = 1$  and  $u = v = (T - q)^{(1/2)(k-3)} w$  in Eq. (59) gives  $B((T - q)^{(1/2)(k-1)} w, (T - q)^{(1/2)(k-3)} w) = 0$ , which means that  $B((T - q)^r w, (T - q)^s w)$  with  $r + s = k - 2$  are already determined by  $B((T - q)^{k-1} w, w)$ . When  $q = -1$  the dimension  $k$  is even and with  $u = v = (T - q)^{(1/2)(k-1)} w$  we have

$$\begin{aligned} 2B((T - q)^{(1/2)(k-1)} w, (T - q)^{(1/2)(k-1)} w) \\ &= B((T - q)^{(1/2)k} w, (T - q)^{(1/2)(k-1)} w) \\ &= (-1)^{(1/2)(k-1)} B((T - q)^{k-1} w, w), \end{aligned} \quad (66)$$

again showing that  $B((T - q)^r w, (T - q)^s w)$  with  $r + s = k - 2$  is determined by  $B((T - q)^{k-1} w, w)$ . Thus we cannot hope to follow the earlier strategy of choosing a new cyclic vector with vanishing  $B((T - q)^{k-2} w, w)$ . (The usual trick turns out not to change the value of this matrix element.)

We can, however, still use Proposition 2.5 with  $u_0 = v_0 = u' = v' = w$  and  $a = b$ ,  $w_a = u_a = v_b$  to obtain for  $s > r$

$$B((T - q)^s w_a, w_a) = B((T - q)^s w, w) \quad (67)$$

and

$$\begin{aligned} B((T - q)^r w_a, w_a) &= (1 + (-1)^{k-r-1})aB((T - q)^{k-1} w, w) \\ &\quad + B((T - q)^r w, w). \end{aligned} \quad (68)$$

Taking  $r = k - 3$  we can choose  $a$  so as to make  $B((T - q)^{k-3} w_a, w_a) = 0$ , and then, proceed inductively, to put the Gram matrix of  $B$  in a form where  $B((T - q)^{k-1-r} w, w) = 0$  for all even  $r < k - 1$ . Since these determine the values for odd  $r$ , this gives a canonical form.

We remark at this point that when  $T$  has two identical indecomposable Jordan blocks of size  $k$  with eigenvalues  $q = (-1)^{k-1}$  there is an apparent ambiguity, as these might either be paired isotropic subspaces or two orthogonal non-degenerate subspaces. However, the following argument shows that these two possibilities are equivalent. In the non-degenerate case, choose generating vectors  $v_1$  and  $v_2$  for the two non-degenerate subspaces as above, and let  $S$  map the basis elements  $T^r v_1$  and  $T^r v_2$  to  $\frac{1}{2}T^r((1 - i)v_1 + (1 + i)v_2)$  and  $\frac{1}{2}T^r((1 + i)v_1 + (1 - i)v_2)$ , respectively. By definition  $S$  commutes with  $T$ , but we easily check that  $B(ST^r v_j, ST^s v_j) = 0$  for  $j = 1, 2$  and all  $r, s = 0, \dots, k - 1$ , so that the images of the two subspaces under  $S$  are isotropic, and (since  $B$  is non-degenerate) they are paired.

This observation removes the last ambiguity in the determination of  $B$  from  $T$ . We have now dealt with all the possible indecomposable subspaces of  $V$  and have shown in each case how to construct a non-degenerate subspace on which  $B$  as well as  $T$  has a particular form. By induction on the dimension we arrive at the following result.

**THEOREM 2.9.** *The equivalence class of a non-degenerate bilinear form  $B$  over  $\mathbb{F}$  is uniquely determined by  $T$ .*

We can also describe completely the possible summands in the decomposition for  $V$  using, in particular, the condition that  $q = (-1)^{k-1}$  for a non-degenerate cyclic subspace. (For  $q = -1$ , part (ii) generalises the well-known fact that symplectic forms exist only in even dimensions.)



THEOREM 2.10. *The space  $V$  is a direct sum of non-degenerate subspaces, each of which is one of the following,*

- (i) *The direct sum of two indecomposable isotropic subspaces of the same dimension, on which the eigenvalues of  $T$  are reciprocals of each other.*
- (ii) *An even-dimensional non-degenerate indecomposable subspace on which  $T$  has the eigenvalue  $-1$ .*
- (iii) *An odd-dimensional non-degenerate indecomposable subspace on which  $T$  has the eigenvalue  $+1$ .*

### 3. ADJOINTS AND PROJECTIONS

For general non-degenerate forms there are two (usually distinct) adjoints defined by

$$B(Su, v) = B(u, S^*v) \quad B(u, Sv) = B(*Su, v), \quad (69)$$

for any linear transformation  $S$  of  $V$ .

PROPOSITION 3.1. *The adjoints are related by  $*ST = TS^*$ ,  $*(S^*) = S = (*S)^*$ , and  $T(S^{**})T^{-1} = S = T^{-1}(*S)T$ .*

*Proof.* We have

$$B(u, Sv) = B(Sv, T^{-1}u) = B(v, S^*T^{-1}u) = B(TS^*T^{-1}u, v), \quad (70)$$

so that  $*S = TS^*T^{-1}$ , and the next two identities are almost tautologies, and then  $TS^{**} = (*S^*)T = ST$ , and similarly for the remaining identity. ■

Self-adjointness is independent of which adjoint is chosen, since  $B(Su, v) = B(u, Sv)$  can be read both as  $S = S^*$  and as  $S = *S$ . Proposition 3.1 therefore tells us that self-adjoint  $S$  must commute with  $T$ , since

$$TS = TS^* = *ST = ST. \quad (71)$$

This notion of self-adjointness is therefore rather restricted, but it does have one very useful application (see Proposition 3.3).

The earlier identification of  $V \otimes V$  with matrices can more elegantly be replaced by defining the transformation  $L_X$  for each  $X \in V \otimes V$  by

$$B(L_X u, v) = B^{(2)}(X, u \otimes v), \quad (72)$$

and non-degeneracy means that this map gives an isomorphism between  $V \otimes V$  and the linear transformations of  $V$ .

PROPOSITION 3.2. *This correspondence satisfies  $L_{(S \otimes C)X} = CL_X S^*$  and*

$${}^*L_X = L_{\sigma(X)} T^{-1} = L_{\hat{T}(X)}. \quad (73)$$

*With respect to a basis  $e_1, e_2, \dots, e_n$  for  $V$  and dual basis  $f_1, f_2, \dots, f_n$  such that  $B(e_j, f_k) = \delta_{jk}$ , one has*

$$X = \sum e_j \otimes L_X f_j. \quad (74)$$

*Proof.* By definition

$$\begin{aligned} B^{(2)}((S \otimes C)X, u \otimes v) &= B^{(2)}(X, S^*u \otimes C^*v) = B(L_X S^*u, C^*v) \\ &= B(CL_X S^*u, v) \end{aligned} \quad (75)$$

whence  $L_{(S \otimes C)X} = CL_X S^*$ . Taking  $S = T$  and  $C = 1$ , and recalling that  $T^* = T^{-1}$ , we obtain  $L_{T_1 X} = L_X T^{-1}$ . We also have

$$\begin{aligned} B(u, L_X v) &= B(L_X v, T^{-1}u) \\ &= B^{(2)}(X, v \otimes T^{-1}u) \\ &= B^{(2)}(\sigma(X), T^{-1}u \otimes v) = B(L_{\sigma(X)} T^{-1}u, v), \end{aligned} \quad (76)$$

whence  ${}^*L_X = L_{\sigma(X)} T^{-1} = L_{\hat{T}(X)}$ . For  $X = e \otimes f$ , we have  $B(L_X u, v) = B(e, u)B(f, v)$ , and  $L_X u = B(e, u)f$ . Setting  $e = e_j$  and taking  $u = f_k$  gives  $L_X f_k = \delta_{jk}f$ , whence, in this case,  $X = e_j \otimes L_X f_j$ , and the general result follows by summation. ■

We note, in particular, that  $L_X = 1$  corresponds to  $X = \sum e_j \otimes f_j$ , and then  $\sigma(X) = \sum f_j \otimes e_j$  gives

$$L_{\sigma(X)} = {}^*L_X T = T. \quad (77)$$

For future use we note that in terms of the Gram matrix  $f_j = \sum_k G_{kj}^{-1} e_k$ , so that

$$\sum_j e_j \otimes f_j = \sum_{jk} G_{jk}^{-1} e_k \otimes e_j. \quad (78)$$

This is actually the bilinear form  $B^*$  on  $V^*$  dual to  $B$  under the isomorphism  $B_2$  (or under  $B_1$ ), since that isomorphism sends  $\{f_k\}$  to the basis  $\{e_k^*\}$  dual to  $\{e_k\}$ , whence  $B^*(e_j^*, e_k^*) = B(f_j, f_k) = G_{kj}^{-1}$ . Using this we may also write

$$X = (1 \otimes L_X) B^* = \sum_{jk} G_{jk}^{-1} e_k \otimes L_X e_j. \quad (79)$$

Combined with the following result this will show that the most general  $X$  has the form  $(1 \otimes L)B^*$  with  $L$  self-adjoint.

**PROPOSITION 3.3.** *The tensor  $X$  is in  $\ker(1 - \hat{T})$  if and only if  $L_X$  is self-adjoint, and  $X \in \text{im}(1 - \hat{T})$  if and only if there exists a linear transformation  $L$  with  $L_X = L - {}^*L$ . We also have  $B^{(2)}(X, Y) = \text{tr}(L_X^* L_Y)$ , for all tensors  $X, Y$ , and, in particular,  $B(Y) = \text{tr}(L_Y)$ .*

*Proof.* By the Proposition 3.2  $X = \hat{T}(X)$  if and only if  ${}^*L_X = L_{\hat{T}(X)} = L_X$ ; that is,  $L_X$  is self-adjoint. Similarly,  $X = Y - \hat{T}Y$  is equivalent to  $L_X = L_Y - {}^*L_Y$ . Since  $B(f_j, T^{-1}e_k) = B(e_k, f_j)$ , we see that  $\{f_j\}$  and  $\{T^{-1}e_k\}$  form dual bases, so that we may write

$$Y = \sum f_j \otimes L_Y T^{-1}e_j. \quad (80)$$

With the earlier identity  $X = \sum e_j \otimes L_X f_j$ , this gives

$$\begin{aligned} B^{(2)}(X, Y) &= \sum_{j,k} B(e_j, f_k) B(L_X f_j, L_Y T^{-1}e_k) \\ &= \sum_j B(L_X f_j, L_Y T^{-1}e_j) \\ &= \sum_j B(f_j, L_X^* L_Y T^{-1}e_j) \\ &= \sum_j B(TL_X^* L_Y T^{-1}e_j, f_j), \end{aligned} \quad (81)$$

which is  $\text{tr}(TL_X^* L_Y T^{-1}) = \text{tr}(L_X^* L_Y)$ . Then

$$B(X) = \sum_j B(e_j, L_X f_j) = \text{tr}(L_X) \quad (82)$$

gives the final identity. ■

Combining Proposition 3.3 and Eq. (71) we note that if  $X \in \ker(1 - \hat{T})$  then  $L_X$  commutes with  $T$ , and also, as any real polynomial in  $L_X$  is also self-adjoint it too corresponds to a tensor in the kernel.

**COROLLARY 3.4.** *For  $X \in \ker(1 - \hat{T})$  in two dimensions and antisymmetric  $B$ ,  $L_X$  is a multiple of the identity. The elements of  $\text{im}(1 - \hat{T})$  are the trace-free elements.*

*Proof.* In two dimensions there is only one antisymmetric form, which we denote by  $\epsilon$ , with Gram matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (83)$$

Now  $B(L_X u, v) - B(L_X v, u)$  is clearly antisymmetric in  $u$  and  $v$  so that  $B(L_X u, v) - B(L_X v, u) = \lambda \epsilon(u, v)$ . But if  $L_X = L_X^*$ , then the left-hand side can be rewritten as  $B(L_X u, v) - B(v, L_X u)$  which is antisymmetric in  $L_X u$  and  $v$  and so of the form  $\beta \epsilon(L_X u, v)$ , and  $\beta$  cannot vanish since  $B$  is not symmetric. Comparing the two expressions we have  $L_X = \beta^{-1} \lambda 1$ , as claimed. The image is the orthogonal complement of the kernel so it consists of elements whose trace inner product with the identity vanishes. ■

**THEOREM 3.5.** *For any  $T$  in two dimensions and more generally whenever  $1$  and  $-1$  are not eigenvalues of  $T$ , the kernel and image of  $1 - \hat{T}$  intersect trivially.*

*Proof.* An element  $X$  is in the image if and only if it is orthogonal to the kernel, so the intersection consists of those elements which are both in the kernel and orthogonal to every element of the kernel. Since  $L_X = L_X^*$ , we know that  $L_X$  commutes with  $T$ , and it will be sufficient to concentrate our attention on a subspace where

$$T = \begin{pmatrix} -q^{-1}1 & 0 \\ 0 & -q1 \end{pmatrix}. \quad (84)$$

There we readily check that  $L_X$  has the block diagonal form

$$L_X = \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}. \quad (85)$$

Moreover, to be in the intersection of the image and kernel we now require  $\text{tr}(L_X L_Y) = \text{tr}(L_X^* L_Y) = 0$  for all  $Y \in \ker(1 - \hat{T})$ , or equivalently  $\text{tr}(AB) = 0$  for all  $B$ , which means that  $A$  and also  $L_X$  vanish. It is worth noting that since  $L_X$  in the intersection must be orthogonal to itself we have  $\text{tr}(L_X^2) = 0$ , which means that  $L_X$  is nilpotent. In two dimensions we already saw that  $L_X$  is a multiple of the identity, and so nilpotent only if it vanishes. ■

**EXAMPLE 3.1.** In two dimensions it follows from the identity

$$L_X = (L_X - \tfrac{1}{2}\text{tr}(L_X)1) + \tfrac{1}{2}\text{tr}(L_X)1 \quad (86)$$

that  $L_{PX} = \tfrac{1}{2}\text{tr}(L_X)1 = \tfrac{1}{2}B(X)1$  so that  $PX = \tfrac{1}{2}B(X)\Sigma e_j \otimes f_j$ . The algebra is therefore generated by  $V$  subject to the single relation

$$\tfrac{1}{2}(e_1 f_1 + e_2 f_2) = 1. \quad (87)$$

By our earlier observation (following Proposition 3.2)  $f_1 = \det(G)^{-1}(G_{22}e_1 - G_{21}e_2)$  and  $f_2 = \det(G)^{-1}(-G_{12}e_1 + G_{11}e_2)$ , giving the single relation

$$\tfrac{1}{2}(G_{22}e_1^2 - G_{21}e_1e_2 - G_{12}e_2e_1 + G_{11}e_2^2) = \det(G)1. \quad (88)$$

It is easily checked that this agrees with our earlier formulae for the special cases.

EXAMPLE 3.2. Applying Eq. (88) to the exceptional two-dimensional form given by Eq. (20), with indecomposable Jordan matrix  $T$ , (19), gives an algebra generated by  $v_1$  and  $v_2$  subject to the relation

$$v_1^2 + 2(v_1v_2 - v_2v_1) = 8. \quad (89)$$

This can be represented by taking  $v_1$  to be multiplication by  $2t$ , and  $v_2$  to be  $(t^2 - 2)d/dt$ . It can also be obtained as a singular limit of the  $q$ -CCR algebra as  $q \rightarrow 1$ . More precisely, if  $e_1$  and  $e_2$  satisfy

$$e_1e_2 - qe_2e_1 = 2, \quad (90)$$

we set

$$v_1 = (q - 1)^{1/2}(e_1 - e_2), \quad v_2 = (q - 1)^{-(1/2)}(e_1 + e_2), \quad (91)$$

and note that

$$v_1^2 - (q - 1)^2v_2^2 + (q + 1)(v_1v_2 - v_2v_1) = 4(e_1e_2 - qe_2e_1) = 8, \quad (92)$$

which tends to the singular relation as  $q \rightarrow 1$ .

In two dimensions we can see from our explicit formulae that the kernel and image of  $1 - \hat{T}$  intersect trivially even when  $T$  is not semi-simple, but this is exceptional, as the following result shows.

THEOREM 3.6. *Except in the case of the two-dimensional block given in the Example 3.1, whenever  $T$  contains a non-trivial Jordan block the image and kernel of  $(1 - \hat{T})$  have a non-trivial intersection.*

*Proof.* Since  $\ker(1 - \hat{T}) = \text{im}(1 - \hat{T})^\perp$ , we also have

$$\text{im}(1 - \hat{T}) \cap \ker(1 - \hat{T}) = \ker(1 - \hat{T})^\perp \cap \ker(1 - \hat{T}). \quad (93)$$

The intersection therefore consists of those  $X$  such that  $L_X$  is self-adjoint and satisfies  $\text{tr}(L_X L) = 0$  for all self-adjoint  $L$ .

Let us first deal with the non-degenerate non-trivial indecomposable blocks with eigenvalue  $\pm 1$ . If the eigenvalue is 1, the block has odd dimension  $k > 1$ , and we take  $L_X = (T - 1)^{k-1}$ , which is self-adjoint by Eq. (59), and also nilpotent. Any self-adjoint  $L$  commutes with  $T$  and so is a polynomial in  $T - 1$  on this block. The product with  $L_X$  therefore has trace 0. If the eigenvalue is  $-1$ , the block has even dimension  $k$ , and we take  $(T + 1)^{k-2} + \frac{1}{2}(k - 2)(T + 1)^{k-1}$ , which is self-adjoint by Proposition 2.7. Provided that  $k > 2$  it is nilpotent and the argument proceeds as in the previous case.

In the other cases we have paired indecomposable blocks  $\ker(T - q)^k \oplus \ker(T - q^{-1})^k$ , and the general element commuting with  $T$  can be written in the form

$$L = (T - q)^k p(T^{-1} - q) + (T^{-1} - q)^k r(T - q), \quad (94)$$

for some polynomials  $p$  and  $r$ . It is self-adjoint if and only if  $p = r$ . In particular,  $L_X = (T - q)^k(T^{-1} - q)^{k-1} + (T^{-1} - q)^k(T - q)^{k-1}$  is self-adjoint, and nilpotent, so that we can argue, as before, to get a non-trivial intersection. ■

When  $V = W \oplus W^*$  with  $B((x, \xi), (y, \eta)) = \eta(x) - q^{-1}\xi(y)$  and  $T(x, \xi) = -(q^{-1}x, q\xi)$ , the elements commuting with  $T$  have the form  $Q \oplus R$ , and are self-adjoint if and only if  $R = Q'$ , the dual of  $Q$ . From this we see that for

$$L_X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (95)$$

we have

$$L_{PX} = \begin{pmatrix} \frac{1}{2}(A + D') & 0 \\ 0 & \frac{1}{2}(A' + D) \end{pmatrix}. \quad (96)$$

This gives the algebra discussed at the end of the Introduction, generated by  $V$  subject to the relations

$$\xi y - q^{-1}y\xi = 2\xi(y)1, \quad (97)$$

for all  $\xi \in W^*$  and  $y \in W$ . In terms of the tensors one calculates that

$$2P((x, \xi) \otimes (y, \eta)) = (x \otimes \eta - q\eta \otimes x) + (\xi \otimes y - q^{-1}y \otimes \xi). \quad (98)$$

#### 4. QUANTISED FUNCTION ALGEBRAS

We now consider symmetries of the form  $B$ . In Manin's treatment quantum groups appear as symmetries of quadratic algebras [11]. (Superspaces, which Manin allows, could also be incorporated into our framework.) Now deformed analogues of the symmetric and exterior algebras can be obtained in our setting by working with the algebras obtained by factoring out the ideal generated by elements of the form  $P(u \otimes v)$ . Suppose that a bialgebra  $\mathcal{A}$  (with multiplication  $\mu$  and comultiplication  $\Delta$ ) has a corepresentation on  $V$ , that is, a coaction  $\delta: V \rightarrow \mathcal{A} \otimes V$  satisfying  $(\Delta \otimes 1) \circ \delta = (1 \otimes \delta) \circ \delta$ . With respect to a basis  $\{e_j\}$  of  $V$  the coaction

can be written in the form

$$\delta(e_j) = a_{jp} \otimes e_p, \quad (99)$$

where  $a_{jp} \in \mathcal{A}$  (and the summation convention has been used).

We can define a map  $(\mu \otimes 1)\phi(\delta \otimes \delta)$  from  $V \otimes V$  to  $\mathcal{A} \otimes (V \otimes V)$ , where  $\phi: \mathcal{A} \otimes V \otimes \mathcal{A} \otimes V \rightarrow \mathcal{A} \otimes \mathcal{A} \otimes V \otimes V$  simply interchanges the second and third tensor factors. In order to obtain symmetries of the Clifford or Grassmann algebras it is natural to require that this map intertwines  $P$  and  $1 \otimes P$ , that is,

$$(\mu \otimes 1)\phi(\delta \otimes \delta)P = (1 \otimes P)(\mu \otimes 1)\phi(\delta \otimes \delta). \quad (100)$$

However, the relations  $P(u \otimes v) = 0$  for the deformed Grassmann algebra generated by  $V$  are also preserved by the coaction of the map intertwines  $\sigma \circ P$  and  $1 \otimes \sigma \circ P$ . The following result shows an interesting link between this condition and Manin's quantised function algebras.

**THEOREM 4.1.** *Let  $\dim V = 2$  and the coaction be defined by  $\delta(e_j) = a_{jp} \otimes e_p$ . Then*

$$(\mu \otimes 1)\phi(\delta \otimes \delta)\sigma \circ P = (1 \otimes \sigma \circ P)(\mu \otimes 1)\phi(\delta \otimes \delta) \quad (101)$$

*if and only if for some  $\lambda \in \mathcal{A}$  commuting with the matrix  $A = (a_{jp})$  we have  $AGA' = \lambda G$  and  $A'G^{-1}A = \lambda G^{-1}$ . When*

$$G = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (102)$$

*these are equivalent to Manin's conditions*

$$\begin{aligned} ab &= q^{-1}ba, & cd &= q^{-1}dc, & ac &= q^{-1}ca, & bd &= q^{-1}db, \\ bc &= cb, & ad - da &= q^{-1}bc - qcb, \end{aligned} \quad (103)$$

*with  $\lambda = ad - q^{-1}bc = da - qcb$ , the quantum determinant of  $A$ .*

*Proof.* We know that in a two-dimensional space,

$$\sigma \circ P(u \otimes v) = \frac{1}{2}B(u, v)(G^{-1})_{jk}e_j \otimes e_k, \quad (104)$$

so that

$$(\mu \otimes 1)\phi(\delta \otimes \delta)\sigma \circ P(e_r \otimes e_s) = \frac{1}{2}G_{rs}G_{jk}^{-1}a_{jp}a_{kq} \otimes (e_p \otimes e_q), \quad (105)$$

whilst

$$(1 \otimes \sigma \circ P)(\mu \otimes 1)\phi(\delta \otimes \delta)(e_r \otimes e_s) = \frac{1}{2}G_{jk}G_{pq}^{-1}a_{rj}a_{sk} \otimes (e_p \otimes e_q). \quad (106)$$

We therefore require

$$G_{rs}G_{jk}^{-1}a_{jp}a_{kq} = G_{jk}G_{pq}^{-1}a_{rj}a_{sk}, \quad (107)$$

which reduces to

$$G_{rs}(A'G^{-1}A)_{pq} = (AGA')_{rs}G_{pq}^{-1}, \quad (108)$$

and is clearly satisfied if and only if

$$AGA' = \lambda G, \quad A'G^{-1}A = \lambda G^{-1} \quad (109)$$

for some  $\lambda \in \mathcal{A}$ . With the standard form

$$G = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \quad (110)$$

and writing

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (111)$$

these two conditions become

$$\begin{pmatrix} ab - q^{-1}ba & ad - q^{-1}bc \\ cb - q^{-1}da & cd - q^{-1}dc \end{pmatrix} = \lambda \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \quad (112)$$

and

$$\begin{pmatrix} ca - qac & cb - qad \\ da - qbc & db - qbd \end{pmatrix} = \lambda \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix}. \quad (113)$$

These give Manin's conditions as stated and also  $\lambda = ad - q^{-1}bc = da - qcb$ , which is Manin's quantum determinant  $\text{DET}_q(A)$ . We note that by simplifying  $AGA'G^{-1}A$  in two different ways we get  $\lambda A = A\lambda$ , showing that the quantum determinant is central in the algebra generated by the matrix elements of  $A$  (cf. Takeuchi's general proof of the centrality of the quantum determinant, as given in [12, Theorem 4.6.1]). ■

*Remarks.* The first of the two conditions found above is equivalent to the condition that the coaction  $A$  should preserve  $B$  up to some multiple  $\lambda$ , for that condition can be written in the form  $(\mu \otimes B)\phi(\delta \otimes \delta) = B$ , which gives  $a_{jr}a_{ks}B(e_r, e_s) = \lambda B(e_j, e_k)$  or  $AGA' = \lambda G$ . This can also be interpreted as saying that  $GA'G^{-1}$  is the right inverse of  $A$ , and the other condition is equivalent to demanding that it be the left inverse too. (Since  $qG^2 = 1$  it could also be interpreted as the condition that  $A'$  preserves  $B$



as well as  $A$ .) The conditions are closely related to Tunstall's proof that  $A^n$  satisfies Manin's relations for parameter  $q^n$  [11, Section 4.2.9].

In higher dimensions more useful results are obtained from the algebras of Example 1.7. We therefore consider the condition that the operator  $(\sigma + S \otimes S^*)$  should intertwine the corepresentation map introduced above, that is,

$$\begin{aligned} & (\mu \otimes 1)\phi(\delta \otimes \delta)(\sigma + S \otimes S^*) \\ &= (1 \otimes (\sigma + S \otimes S^*))(\mu \otimes 1)\phi(\delta \otimes \delta). \end{aligned} \quad (114)$$

It follows from the earlier expressions for  $S$  that

$$\begin{aligned} (\sigma + S \otimes S^*)(x, \xi) \otimes (y, \eta) &= (y, \eta) \otimes (x, \xi) \\ &\quad - (x, q\xi) \otimes (y, q^{-1}\eta). \end{aligned} \quad (115)$$

We shall restrict ourselves to the case of  $B((x, \xi), (y, \eta)) = \eta(x) - q^{-1}\xi(y)$  on the space  $V = W \oplus W^*$ , using dual bases  $e_1, e_2, \dots, e_r$  of  $W$  and  $f_1, f_2, \dots, f_r$  of  $W^*$  so that  $B(e_j, f_k) = \delta_{jk}$ , and take the coaction in the block form

$$\delta(x, \xi) = (a.x + b.\xi)_j \otimes e_j + (c.x + d.\xi)_j \otimes f_j. \quad (116)$$

A lengthy calculation tells us that the matrix elements of  $a$  commute with each other, as do those of  $b$ ,  $c$ , and  $d$ , but that there are commutation relations

$$\begin{aligned} a_{jk}b_{rs} &= qb_{rs}a_{jk}, & c_{jk}d_{rs} &= qd_{rs}c_{jk}, \\ a_{jk}c_{rs} &= qc_{rs}a_{jk}, & b_{jk}d_{rs} &= qd_{rs}b_{jk}, \\ b_{jk}c_{rs} &= c_{rs}b_{jk}, \end{aligned} \quad (117)$$

$$a_{jk}d_{rs} - d_{rs}a_{jk} = qc_{rs}b_{jk} - q^{-1}b_{jk}c_{rs}$$

between the blocks. These correspond to even Manin relations with a single deformation parameter  $q$ . Due to the degeneracy of the eigenvalues of  $T$ , there is an obvious classical symmetric under the group  $GL(W)$  of linear transformations of  $W$ , which corresponds to the case of  $b = c = 0$  and  $ad' = 1$ .

## 5. THE $R$ MATRIX

By writing  $\sigma \circ P(e_j \otimes e_k) = P_{jk}^{rs}e_r \otimes e_s$  and  $\delta(e_j) = a_{jr}e_r$ , the intertwining property for the coaction of the last section becomes

$$a_{rp}a_{sq}P_{jk}^{rs} = P_{pq}^{rs}a_{jp}a_{kq}, \quad (118)$$

which is strikingly reminiscent of the  $R$ -matrix relation. Naturally this is no coincidence.

Whenever the  $R$  matrix  $\hat{R} = R\sigma$  satisfies a quadratic equation

$$(\hat{R} - \alpha)(\hat{R} - \beta) = 0 \quad (119)$$

the projection onto the  $\beta$ -eigenspace can be written as

$$(\beta - \alpha)^{-1}(\hat{R} - \alpha). \quad (120)$$

This projection determines  $\hat{R}$  completely, and its intertwining operators are exactly those of  $\hat{R}$ . In our case  $\hat{R}$  satisfies the Hecke relation

$$(\hat{R} - q)(\hat{R} + q^{-1}) = 0, \quad (121)$$

and  $P$  is the projection.

In fact one can even admit a spectral parameter  $\lambda$  such that  $\hat{R}_\lambda = R_\lambda \sigma$ , with spectral parameter  $\lambda$ , satisfies a quadratic equation

$$(\hat{R}_\lambda - 1)(\hat{R}_\lambda - 1 + \lambda(q + q^{-1})) = 0. \quad (122)$$

(Defining  $\hat{R} = q\hat{R}_{q^{-1}}$  gives a solution of the Hecke relation, so that is merely a special case.) One readily checks that our standard form with Gram matrix

$$\begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \quad (123)$$

lies in the  $1 - \lambda(q + q^{-1})$  eigenspace. The projection onto this eigenspace is  $(1 - \hat{R}_\lambda)/\lambda(q + q^{-1})$ . For two-dimensional  $V$  this eigenspace should be just one-dimensional, spanned by the above form, so that with respect to the basis  $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$ , we have

$$\frac{1 - \hat{R}_\lambda}{\lambda(q + q^{-1})} = \frac{1}{1 + q^{-2}} \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -q^{-1} & 0 \end{pmatrix}, \quad (124)$$

which gives the usual  $R$ -matrix

$$\hat{R}_\lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \lambda q & \lambda & 0 \\ 0 & \lambda & 1 - \lambda q^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (125)$$

The exceptional  $R$  matrix

$$\hat{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ q^{-1} - q & 0 & 1 & 0 \\ q - q^{-1} & 1 & 0 & 0 \\ q - q^{-1} & -1 & 1 & 1 \end{pmatrix}, \quad (126)$$

of Demidov et al. [4] has the exceptional form with Gram matrix

$$G = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}i & 0 \\ 0 & -i \end{pmatrix} \quad (127)$$

as  $-q^{-1}$ -eigenvector. This has

$$T = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}i & 0 \\ 0 & -i \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}i & 0 \\ 0 & -i \end{pmatrix}, \quad (128)$$

and is clearly equivalent to our exceptional form, given by Eqs. (19) and (20). The higher-dimensional  $R$  matrices for  $\mathfrak{sl}_q(n)$  similarly have our standard forms with Gram matrix

$$\begin{pmatrix} 0 & 1 \\ -q^{-1}1 & 0 \end{pmatrix} \quad (129)$$

as eigenvectors.

## 6. SYMMETRIES OF FORMS

The symmetries of this system can also be studied in the dual enveloping algebra approach. As noted in Example 1.3, we need a comultiplication  $\Delta$  on some suitable subset of linear transformations  $S$  of  $V$ , and look for those satisfying  $B \circ \Delta(S) = B$  or its infinitesimal version,  $B \circ \Delta(L) = 0$ .

To generalise Example 1.3 to higher-dimensional situations we note that if  $\Delta L = L \otimes 1 + K \otimes L$ , then the condition that it preserve  $B$  reduces to

$$0 = B(Lu, v) + B(Ku, Lv) = B(Lu, v) + B(*LK u, v), \quad (130)$$

so that we require  $L + *LK = 0$ . We shall demand that  $K$  is a unitary element, so that  $K^* = K^{-1}$  (and  $\Delta(K) = K \otimes K$  preserves  $B$ ), and this means that there is a neat expression in terms of tensors, for if  $L = L_X$  we have  $*LK = L_{K_1^{-1}\hat{T}(X)}$ , and the condition reduces to

$$K_1 X + \hat{T}(X) = K_1(X + K_1^{-1}\hat{T}(X)) = 0. \quad (131)$$

Thus we have proved the following result:

**PROPOSITION 6.1.** *For unitary  $K$  the element  $\Delta L_X = L_X \otimes 1 + K \otimes L_X$  annihilates  $B$  if and only if  $K_1 X = -\hat{T}(X)$ .*

One immediate consequence of this is the identity

$$(K \otimes K)X = -K_2 \hat{T}(X) = -\hat{T}(K_1 X) = \hat{T}(T_1 X) = (T \otimes T)\sigma(X). \quad (132)$$

Now  $KL_X K^{-1} = L_{(K \otimes K)X} = L_{(T \otimes T)\sigma(X)} = TL_{\sigma(X)} T^{-1}$  gives the conjugation action of  $K$  on  $L_X$ . To obtain the standard quantum groups we need  $X$  to be an eigenvector of  $K \otimes K$ , and so of  $(T \otimes T)\sigma$ .

This is readily achieved by taking  $V$  to be the direct sum of dual isotropic subspaces  $W \oplus W^*$ , with  $W$  having a basis  $e_1, e_2, \dots, e_r$ , satisfying

$$Te_j = -\tau_j e_j, \quad (133)$$

where  $\tau_j = q^{-2(r-j)-1}$ . With respect to the dual basis  $f_1, f_2, \dots, f_r$  of  $W^*$ , which satisfies  $B(e_j, f_k) = \delta_{jk}$ , and we have

$$Tf_j = -\tau_j^{-1} f_j. \quad (134)$$

We now define the unitary elements  $K_j$  by

$$K_j e_k = q^{-\delta_{jk} + \delta_{j+1k}} e_k, \quad (135)$$

from which it follows that

$$K_j f_k = q^{\delta_{jk} - \delta_{j+1k}} f_k. \quad (136)$$

We also introduce, for  $j = 1, 2, \dots, r-1$ ,

$$X_j = e_j \otimes f_{j+1} + q^{-1} \tau_j^{-1} f_{j+1} \otimes e_j \quad (137)$$

and

$$Y_j = e_{j+1} \otimes f_j + q^{-1} \tau_j^{-1} f_j \otimes e_{j+1}, \quad (138)$$

and note that

$$(K_j \otimes 1)X_j = q^{-1} e_j \otimes f_{j+1} + q^{-2} \tau_j^{-1} f_{j+1} \otimes e_j \quad (139)$$

is identical to

$$-\hat{T}X_j = -Tf_{j+1} \otimes e_j - q^{-1} \tau_j^{-1} Te_j \otimes f_{j+1} = \tau_{j+1}^{-1} f_{j+1} \otimes e_j + q^{-1} e_j \otimes f_{j+1}, \quad (140)$$

and similarly  $(K_j \otimes 1)Y_j = -(T \otimes 1)Y_j$ , so that these define infinitesimal symmetries of  $B$ . We complement these with  $X_r = e_r \otimes e_r$  and  $Y_r = f_r \otimes f_r$ , which also provide infinitesimal symmetries of  $B$ .

Finally we define the  $T$ -commutator to be

$$[X, Y]_T = XY - TYT^{-1}X. \quad (141)$$

This reduces to the usual formula for symmetric and antisymmetric forms where  $T = \pm 1$ , and also gives

$$[L_X, L_Y]_T = L_X L_Y - L_{(T \otimes T)Y} L_X \quad (142)$$

for operators of the kind we have been discussing, reflecting the interchange symmetry of  $B$ . (It will sometimes be convenient to abbreviate  $[L_X, L_Y]_T$  to  $[X, Y]_T$ .) We also have

$$B \circ \Delta([X, Y]_T) = B \circ \Delta X \Delta Y - B \Delta T \Delta Y \Delta(T^{-1}X) = 0. \quad (143)$$

(It would be possible to redefine things to give normal commutators using, for example, the identity  $[TX, Y]_T = T[X, Y]$ .)

**THEOREM 6.2.** *The elements  $K_j$ ,  $X_j$ , and  $Y_j$ , defined above, satisfy*

$$\begin{aligned} K_j X_k K_j^{-1} &= q^{-2\delta_{jk} + \delta_{1|j-k|}} X_k, & K_j Y_k K_j^{-1} &= q^{2\delta_{jk} - \delta_{1|j-k|}} Y_k, \\ [X_j, Y_k]_T &= \delta_{jk} \frac{K_j^2 - 1}{q^{-2} - 1}, \end{aligned} \quad (144)$$

for  $j = 1, \dots, r$ ,  $k = 1, \dots, r-1$ , and

$$\begin{aligned} K_j X_r K_j^{-1} &= q^{-2\delta_{jk} + 2\delta_{(j+1)r}} X_r, & K_j Y_r K_j^{-1} &= q^{2\delta_{jk} - 2\delta_{(j+1)r}} Y_r, \\ [X_j, Y_r]_T &= \delta_{jr} \frac{K_j^2 - 1}{q - q^{-1}}. \end{aligned} \quad (145)$$

*Proof.* As all the operators are known explicitly on  $V$  this result can be obtained by direct calculation. Apart from some rescaling these relations give the quantum enveloping algebra  $\mathfrak{sp}_q(2r)$ , as might seem appropriate as  $B$  gives a deformed symplectic structure. (We think of  $L_{X_j}$  as  $E_j$ , and  $Y_j K_j^{-1}$  as  $F_j$ .)

We should note that these relations do not merely rely on the particular representation on  $V$ . Since comultiplication is a homomorphism, we have (identifying the tensors and operators)

$$\begin{aligned} \Delta([X_j, Y_j]_T) &= [\Delta(X_j), \Delta(Y_k)]_T \\ &= \Delta(X_j)\Delta(Y_j) - \Delta(TY_j T^{-1})\Delta(X_j) \\ &= \Delta(X_j)\Delta(Y_j) - \Delta(K_j Y_j K_j^{-1})\Delta(X_j), \end{aligned} \quad (146)$$

or, more explicitly,

$$\begin{aligned} & [X_j, Y_k]_T \otimes 1 + K_j^2 \otimes [X_j, Y_j]_T + X_j K_j \otimes Y_j - K_j X_j \otimes K_j Y_j K_j^{-1} \\ & + K_j Y_j \otimes X_j - K_j Y_j \otimes X_j. \end{aligned} \quad (147)$$

The cancellation of the last two terms provides part of the motivation for the use of the  $T$ -commutator, but since  $X_j$  and  $Y_j$  give inverse eigenvalues the third and fourth terms also cancel, leaving us with

$$\Delta([X_j, Y_j]_T) = [X_j, Y_k]_T \otimes 1 + K_j^2 \otimes [X_j, Y_j]_T, \quad (148)$$

from which we readily see that  $[X_j, Y_j]_T$  is a multiple of  $(K_j^2 - 1)$ , with the representation determining which multiple. ■

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