Essays on Strategic Competition in Financial Economics and Political Economy



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Abstract

This thesis is a collection of essays analyzing strategic competition in complex environments. We analyze cross-shareholding in research and development (R&D) races, vote splitting in elections, and underpricing and token retention in Initial Coin Offerings.

In the first chapter, we model R&D races using all-pay auctions. There are two industries, each consisting of two firms. There are spillovers in cost reduction R&D from both a firm's own industry and its complementary or substitute industry. We first derive the symmetric equilibrium investing strategy for each firm and show that the expected profit of each firm is zero in equilibrium. Then we analyze the effects of cross-industry mergers on each firm's expected investment level and profit. We show that cross-industry mergers will lead to zero expected profit for each merged firm and the same total expected R&D investment as before merging. Furthermore, we study cross-shareholding, in which each firm's major investor also holds shares in all other firms. We prove an impossibility theorem: there is no symmetric cross-shareholding structure that can simultaneously increase a firm's R&D investment and its major investor's expected profit.

In the second chapter, we study vote splitting in elections. Vote splitting refers to the phenomenon that the participation of candidates with little chance of winning has the effect of splitting votes from more popular candidates. We develop a model of an electoral game with two parties, each consisting of one strong candidate and some weak candidates. We analyze strong candidates' incentives to give side transfers to weak candidates to manipulate their campaign participation decisions. We then derive the winning probabilities of the strong candidates in equilibrium. We show that with one weak candidate in each party, when candidates from the two parties have symmetric support rates, the side transfers that one weak candidate receives are independent of the campaign cost and bias of the other weak candidate. We also study the role of uncertain campaign costs of weak candidates. In particular, we show that if the campaign cost of one party's weak candidate is uncertain when strong candidates decide side transfers, then the strong candidate from the other party will be more likely to win than in the case with full information. Therefore, the lack of information

makes a strong candidate win suboptimally too often.

In the third chapter, we develop a signaling theory to explain the empirically observed phenomena of underpricing and token retention in Initial Coin Offerings (ICOs). When a project's type is the entrepreneur's private information, a good entrepreneur can credibly signal his type through a low price in the ICO because he can recoup the loss from a more favorable price in the final sale owing to a high probability of early investors staying as users. In contrast, a bad entrepreneur would find this strategy too costly. This signaling theory can also explain airdrops in ICOs, where a developer distributes free tokens to potential early investors. We then extend our model to make the late community formation probability depend on the result of initial community formation, and show that this leads to another separating equilibrium, where a good developer can signal his quality through token retention. We show that this equilibrium is mutually exclusive with the underpricing equilibrium, and provide welfare comparison results when both separating and pooling equilibria exist.

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Chapter 1

R&D Contests with Spillovers and Cross-Shareholdings Across Industries

Abstract

In this paper, we model R&D races using all-pay auctions. There are two industries, each consisting of two firms. There are spillovers in cost reduction R&D from both a firm's own industry and its complementary or substitute industry. We first derive the symmetric equilibrium investing strategy for each firm and show that the expected profit of each firm is zero in equilibrium. Then we analyze the effects of cross-industry mergers on each firm's expected investment level and profit. We show that cross-industry mergers will lead to zero expected profit for each merged firm and the same total expected R&D investment as before merging. Furthermore, we study cross-shareholding, in which each firm's major investor also holds shares in all other firms. We prove an impossibility theorem: there is no symmetric cross-shareholding structure that can simultaneously increase a firm's R&D investment and its major investor's expected profit.

1.1 Introduction

In research and development (R&D) races, firms compete by spending money to conduct research on technologies in order to either create new products or reduce production costs. The winner who develops a better product or reduces more costs could often take the entire market while all firms need to pay their R&D costs. Previous studies that use all-pay auction to model R&D races often assume the winning prize to be fixed, but in many cases, it is more appropriate to model the winning prize as endogenously determined by the firm's own investment level as well as spillovers from other firms.

In this paper, we systematically study all-pay contests in which there are two industries each consisting of two firms. We model R &D races that aim to reduce production costs rather than creating new products. In an all-pay contest, each firm simultaneously chooses an investment level, and the firm that chooses a higher level takes the entire market. We use the terms "investment" and "effort" interchangeably throughout the paper. We take into account spillovers from both a firm's own industry and its complementary or substitute industry. In our model, there are no complementarity or substitutability in demands, but they rather refer to the spillover effects that R&D investments of firms in these industries have on other firms' final marginal costs of production.

We first derive the symmetric equilibrium investing strategy for each firm, and then we show that the expected profit of each firm is zero when using the strategy. Afterwards, we study the extent to which expected investment level depends on the degree of complementarity or substitutability. Then we analyze the effects of cross-industry merger, in which two firms from different industries merge, on expected investment and maximum potential investment. We show that when two firms merge and the two resulting merged firms run an all-pay contest where the winner captures both markets, the expected investment of a merged firm equals the sum of expected investments of the two original firm, but the maximum potential investment in equilibrium of the merged firm is higher than that of the sum of the original firms.

In addition, we study the effects of cross-shareholding on firms' effort and profit. An investor who controls a firm (i.e., owns more than 50% of its stock) can diversify his investment portfolio by investing in other firms in the same industry or other industries. For instance, as discussed by Gilo, Moshe, and Spiegel (2006), GM, who controlled National Car Rental, held a 25% stake in Avis, who was National Car Rental's rival during the 1990s. A firm can also acquire stocks of its competitors as passive investment. For example, Microsoft once held approximately 18.1 million shares in Apple. We show that each firm's major investor holding shares in its rival firm as well as firms in the substitute industry will reduce the firm's expected R&D investment while increasing his expected profit. More importantly, we prove an impossibility theorem:

symmetric cross-shareholding can never increase a firm's expected R&D investment while giving its major investor a positive expected profit.

In our model, four firms from two related industries hold shares in each other. Such complex cross-shareholding structures are most often seen in Japanese *keiretsus*. A *keiretsu* is a group of companies with strategic cooperations and cross-shareholdings. *Keiretsus* emerged after World War II from the dissolution of *zaibatsus*, which are family-owned conglomerates. For example, the Mitsubishi Group is centered around the Mitsubishi UFJ Bank, but also includes Mitsubishi Heavy Industries, Mitsubishi Motors, and other companies. As another example, Toyota group is centered around Toyota Motor, but also includes suppliers of automotive parts like Denso, as well as other companies in the automobile industry like Daihatsu Motor Co., Ltd. Toyota Motors hold about 25% shares in Denso, while Denso holds around 1% of shares in Toyota Motors. In addition, Mitsubishi Group holds around 3% of shares in Toyota Motors. Such complex cross-shareholding relationships have important effects on firms' innovative incentives and profits, and yet few papers have done theoretical work taking into account both spillovers and complex cross-shareholding structures. We now conduct a detailed review of relevant papers.

1.2 Literature Review

Our paper introduces endogenous rewards and cross-shareholding into the all-pay contests model. Baye, Kovenock, and de Vries (1996) derive symmetric equilibria in all-pay auctions with multiple fixed prizes, linear costs, and complete information. Their seminal paper serves as a foundation for subsequent works. Bos and Ranger (2014) make rewards endogenous by modelling it as increasing in one's own effort. They characterize the unique equilibrium in mixed strategies in a game between two players when the they have different valuation functions of the reward. Their result is built on the work of Siegel (2009, 2010), which introduces asymmetry among players into the all-pay contest model. These papers claim that the all-pay auction is suitable for modelling R&D contests, but they usually assume a fixed prize. A main innovation of our paper is introducing a microeconomic foundation for the prize through dividing the game into an R&D stage and a subsequent Bertrand competition stage. A new paper written by Cho (2026) published after our first draft uses a similar setup.

Lee and Kang (1998) study externalities in all-pay contests. They model externalities by assuming each player's cost of effort depends on the aggregate effort. They argue that, for example, when there are more lobbyists in Washington, DC, better hotels will be built. Our paper differ from theirs in that they use Tullock contests, in which the player who exerts higher effort wins with a higher probability rather than for sure,

and also that our sources of externalities are not only from the total effort in the players' own industry, but also from the total effort in the complementary or substitute industry.

Baye, Kovenock, and de Vries (2009) study two player all-pay contests in which the reward has an exogenous component as well as an endogenous component as a linear function of the effort of each player. Our payoff structure is similar to theirs. Betto and Thomas (2024) generalize the framework of Baye, Kovenock, and de Vries (2009) to allow asymmetry between the two players. They derive equilibrium bidding strategies under asymmetry and general payoff structures. Compared to these papers, one of our main innovations is introducing a new form of spillover, in which each player's payoff is affected by the result of another all-pay contest of which he is not a participant.

We study the effects of cross-industry mergers on the total expected R&D investment and profit. Beggs (1994) analyzes a game with two groups each consisting of two firms. Firms sell products that are complementary within its group but substitute across the two groups. Remarkably, Beggs (1994) shows that mergers between firms in the same group may be unprofitable. Salant, Switzer, and Reynolds (1983) also shows that mergers may be unprofitable under Cournot competitions. Church (2004) studies the effects of vertical and conglomerate mergers on competition. In our model, we show that firms expect zero profit both before and after cross-industry mergers despite having spillovers.

Another main focus of our paper is cross-shareholding. In their seminal paper, Reynolds and Snapp (1986) study an industry of multiple firms holding shares in each other while playing a modified Cournot competition. They show that this partial cross-investing reduces output and makes prices higher. Gilo, Moshe, and Spiegel (2006) analyze the anti-competitive and collusive effects of cross-investing between rival firms in an infinitely repeated Bertrand model. They give conditions under which tacit collusion between firms through partial ownership agreements can exist. Fu and Lu (2013) study all-pay contests with cross-shareholding. In their model, there is no spillover from the rival firm or other industries. Our model generalize their setting as we allow spillovers from two sources, making our equilibria more complex and our shareholding problem more intricate.

More recently, Schmalz (2021) gives a systematic review of empirical studies of common ownership that confirm its anti-competitive effect. Despite this well-documented effect, Antón et al (2021) show that common ownership can encourage innovation if technological spillovers are large enough to outweigh product market spillovers. Our result confirms their finding as we show that vertical cross-shareholding can increase firms' R&D investment, but we also highlight the impossibility of also having a higher profit for each major investor when higher R&D investments are made. Antón et al (2023) theoretically study the effect of common

ownership on managerial incentives, and empirically show that firms with more common ownership tend to have less performance-sensitive managerial incentives.

1.3 Model

1.3.1 The Baseline Model

There are two industries, 1 and 2. Firms a and b are in industry 1 and c and d are in industry 2. The timing of the model is as follows. In time 1, each industry simultaneously runs a R&D contest, where firm i chooses investment level $e_i \in [0, \infty)$. In time 2, each firm's investment level is revealed and then they compete in Bertrand duopoly with different marginal costs. In industry I, the aggregate demand is

$$Q_I = 1 - P_I,$$

where P_I is the lower of the two prices chosen by the firms.

If $e_i > e_j$, the marginal cost of firm i in industry I is

$$c_i = \underline{c}_I - \tilde{\alpha}(e_i + e_j) - \tilde{\beta}(e_k + e_l),$$

and the marginal cost of firm j is

$$c_j = \bar{c}_I.$$

In the event of a tie, both get the lower marginal cost with probability $\frac{1}{2}$. The expression $\tilde{\alpha}(e_i + e_j)$ captures intra-industry spillover while $\tilde{\beta}(e_k + e_l)$ captures inter-industry spillovers from R&D investments. We assume $\tilde{\alpha} > 0$ while $\tilde{\beta}$ can be either positive or negative, and we will give more restrictions on them later.

In the expressions above, $\underline{c}_I$ and $\bar{c_I}$ are constants. We assume $\underline{c}_I \leq \bar{c}_I < \frac{1}{2}$, as this assumption enables us to focus on the interesting case where $\bar{c}_I$ is less than the monopoly price of the firm with a lower marginal cost for all possible equilibrium investment levels of each firm. The derivation is the following. The winning firm's monopoly price is $\frac{1+c_i}{2}$. Here c_i is not fixed, so we could pick the tightest bound for c_I to ensure that the winning firm can never price like a monopoly for any $c_i > 0$. So the assumption becomes $\bar{c_I} < \frac{1}{2}$.

We solve the model backwards. The following result from Blume (2003) gives a Nash equilibrium of the game in time 2.

Proposition 1: When $c_i < c_j$, a Nash equilibrium in undominated strategies in Bertrand duopoly with different marginal costs is such that firm i charges $\bar{c}_I$ and firm j randomizes uniformly over $[\bar{c}_I, \bar{c}_I + \epsilon]$ for some small positive ϵ.

Proof. Please see Blume (2003). □

Intuitively, the firm with the higher marginal cost does not want to deviate because it can never get a positive profit by deviating. The firm with the lower marginal cost also does not want to deviate because charging a lower price will make it win with the same probability, which is 1, but yields a lower profit; deviating to a higher price will make it lose the market with a positive probability.

Therefore,

$$\Pi_i(e_i, e_j, e_k, e_l) = \begin{cases} (1-\bar{c}_I)[\bar{c}_I - \underline{c}_I + \tilde{\alpha}(e_i + e_j) + \tilde{\beta}(e_k + e_l)] - e_i & \text{if } e_i > e_j \\ \frac{1}{2}(1-\bar{c}_I)[\bar{c}_I - \underline{c}_I + \tilde{\alpha}(e_i + e_j) + \tilde{\beta}(e_k + e_l)] - e_i & \text{if } e_i = e_j \\ -e_i & \text{if } e_i < e_j, \end{cases}$$

where firm j is the rival firm in the same in same industry as i, and k and l are firms in the complementary industry. To simplify expressions, we denote

$$V_I = (1-\bar{c}_I)(\bar{c}_I - \underline{c}_I),$$
$$\alpha_I = (1-\bar{c}_I)\tilde{\alpha},$$
$$\beta_I = (1-\bar{c}_I)\tilde{\beta}.$$

To make the model more tractable, we assume $\bar{c}_I = \bar{c}_J$ in the rest of the paper, while $\underline{c}_I$ can be different from $\underline{c}_J$. Thus we have $\alpha_I = \alpha_J \equiv \alpha$ and $\beta_I = \beta_J \equiv \beta$, while V_I can differ from V_J. These notations will be used throughout the rest of the paper. The profit function of firm i in industry $I \in \{1, 2\}$,
$\Pi_i(e_i, e_j, e_k, e_l) : [0, \infty)^4 \to \mathbb{R}$, is given by

$$\Pi_i(e_i, e_j, e_k, e_l) = \begin{cases} V_I + \alpha(e_i + e_j) + \beta(e_k + e_l) - e_i & \text{if } e_i > e_j \\ \frac{1}{2}[V_I + \alpha(e_i + e_j) + \beta(e_k + e_l)] - e_i & \text{if } e_i = e_j \\ -e_i & \text{if } e_i < e_j. \end{cases}$$

In the expression above, V_I is the exogenous reward for winning the contest and taking the market, $\alpha(e_a + e_b)$

captures the increase in reward from the firm's own effort as well as the positive externalities from the rival firm's effort, and $\beta(e_c + e_d)$ captures the externalities from the other industry. Therefore, β is the inter-industry complementarity/substitutability factor. $\beta > 0$ means that the two industries have complementarity in cost reduction R&D while $\beta < 0$ means there are substitutability effects. We assume $0 < |\beta| < \alpha < 0.5$. The reason behind this assumption is that if the marginal reward from increasing one's own R&D investment is too high, it could make the optimal investment infinity when coupled with the complementarity from the other industry, whose members also face such utility function. If β is too negative, the negative spillover may make firms unwilling to spend any money on R&D in equilibrium.

We additionally assume that $(1-\alpha)V_1 + \beta V_2 > 0$, which means that even when $\beta < 0$, the negative spillover from the substitute industry is not too strong. This assumption will ensure the existence of an equilibrium where each firm makes positive expected investment.

Theorem 1: Under the assumptions above, there is no pure strategy subgame perfect equilibrium. A symmetric subgame perfect equilibrium is as follows. In the R&D stage, firms a and b randomize according to the c.d.f.

$$G_1(x) = \frac{1}{\alpha}\left\{1 - \left[\frac{(1-\alpha)^2 V_1 + \beta(1-\alpha)V_2}{(1-\alpha)^2 V_1 + \beta(1-\alpha)V_2 + 2\alpha[(1-\alpha)^2 - \beta^2]x}\right]^{\frac{1}{2}}\right\},$$

over the interval

$$\left[0, \frac{(2-\alpha)[(1-\alpha)V_1 + \beta V_2]}{2(1-\alpha)[(1-\alpha)^2 - \beta^2]}\right],$$

and firms c and d randomize using the c.d.f.

$$G_2(x) = \frac{1}{\alpha}\left\{1 - \left[\frac{(1-\alpha)^2 V_2 + \beta(1-\alpha)V_1}{(1-\alpha)^2 V_2 + \beta(1-\alpha)V_1 + 2\alpha[(1-\alpha)^2 - \beta^2]x}\right]^{\frac{1}{2}}\right\},$$

over the interval

$$\left[0, \frac{(2-\alpha)[(1-\alpha)V_2 + \beta V_1]}{2(1-\alpha)[(1-\alpha)^2 - \beta^2]}\right].$$

In the production stage, firms use strategies described in **Proposition 1**. Moreover, each firm expects zero profit in equilibrium:

$$\mathbb{E}[\Pi_i({e_i}^*, {e_{-i}}^*)] = 0$$

for all i, where e_i^* is the equilibrium investing strategy of firm i.

Proof. The claim that there is no pure strategy equilibrium in the R&D stage follows from the standard argument in the all-pay auction literature. Suppose firm i chooses a positive effort e_i and earns a positive profit. Then firm j can always deviate to $e_i + \epsilon$ for sufficiently small ϵ and capture the entire prize. But then firm i would prefer 0 to e_i, and yet clearly both choosing 0 effort is not an equilibrium.

Now we derive the mixed strategy subgame perfect equilibrium. We start by defining the expected winning prize $\tilde{V_I}$ for a firm in industry I when it takes into account the spillover from its complementary industry as firms in that industry are using the equilibrium investing strategy:

$$\tilde{V_1} = \mathbb{E}[V_1 + \beta(e_c^* + e_d^*)],$$

and

$$\tilde{V_2} = \mathbb{E}[V_2 + \beta(e_a^* + e_b^*)].$$

Now we show that firms a and b both investing using a mixed strategy corresponding to the c.d.f.

$$G_1(x) = \frac{1}{\alpha}\left[1 - \Big(\frac{\tilde{V_1}}{\tilde{V_1} + 2\alpha x}\Big)^{\frac{1}{2}}\right]$$

over the interval $[0, \frac{(2-\alpha)\tilde{V_1}}{2(1-\alpha)^2}]$ is a Nash equilibrium in the game between firms a and b, and that firms c and d investing

$$G_2(x) = \frac{1}{\alpha}\left[1 - \Big(\frac{\tilde{V_2}}{\tilde{V_2} + 2\alpha x}\Big)^{\frac{1}{2}}\right]$$

over the interval $[0, \frac{(2-\alpha)\tilde{V_2}}{2(1-\alpha)^2}]$ is a Nash equilibrium for their game. Let $g_1(x)$ and $g_2(x)$ be p.d.f.s of $G_1(x)$ and $G_2(x)$ respectively. We first focus on the game between a and b. The right bound is derived by solving

$$\frac{1}{\alpha}\left[1 - \Big(\frac{\tilde{V_1}}{\tilde{V_1} + 2\alpha x}\Big)^{\frac{1}{2}}\right] = 1,$$

as the upper limit of the c.d.f. is 1. To show that it is an equilibrium, suppose firms b, c, and d are following

these strategies while firm a deviates to investing $s \in [0, \frac{(2-\alpha)\tilde{V_1}}{2(1-\alpha)^2}]$. Then firm a's expected profit is

$$\begin{aligned}
\mathbb{E}[\Pi_a(s, e_b^*, e_c^*, e_d^*)] &= \int_0^s \int_0^{\frac{(2-\alpha)\tilde{V_2}}{2(1-\alpha)^2}} \int_0^{\frac{(2-\alpha)\tilde{V_2}}{2(1-\alpha)^2}} [V_1 + \beta(e_c + e_d) + (\alpha-1)s + \alpha e_b] \\
&\qquad g_1(e_b)g_2(e_c)g_2(e_d)\, de_c\, de_d\, de_b + \int_s^{\frac{(2-\alpha)\tilde{V_1}}{2(1-\alpha)^2}} -s g_1(e_b)\, de_b \\
&= \int_0^s [\tilde{V_1} + (\alpha-1)s + \alpha e_b] g_1(e_b)\, de_b + \int_s^{\frac{(2-\alpha)\tilde{V_1}}{2(1-\alpha)^2}} -s g_1(e_b)\, de_b \\
&= \int_0^s [\tilde{V_1} + (\alpha-1)s + \alpha e_b] \frac{\tilde{V_1}^{\frac{1}{2}}}{(\tilde{V_1} + 2\alpha x)^{\frac{3}{2}}}\, de_b + \int_s^{\frac{(2-\alpha)\tilde{V_1}}{2(1-\alpha)^2}} -s \frac{\tilde{V_1}^{\frac{1}{2}}}{(\tilde{V_1} + 2\alpha e_b)^{\frac{3}{2}}}\, de_b \\
&= 0,
\end{aligned}$$

which is constant for all $s \in [0, \frac{(2-\alpha)\tilde{V_1}}{2(1-\alpha)^2}]$. Deviating to $s > \frac{(2-\alpha)\tilde{V_1}}{2(1-\alpha)^2}$ will make firm a win with probability 1 but the resulting expected utility is lower. Thus randomizing using $G_1(x)$ is optimal.

By symmetry, randomizing using $G_1(x)$ is optimal for firm b as well. Similar calculations show that randomizing using $G_2(x)$ is a symmetric equilibrium investing strategy for firms c and d.

Now we find $\tilde{V_1}$ and $\tilde{V_2}$. The expected investment of firms a and b is

$$\begin{aligned}
\mathbb{E}[e_a^*] = \mathbb{E}[e_b^*] &= \int_0^{\frac{(2-\alpha)\tilde{V_1}}{2(1-\alpha)^2}} 1 - G_1(x)\, dx \\
&= \int_0^{\frac{(2-\alpha)\tilde{V_1}}{2(1-\alpha)^2}} 1 - \frac{1}{\alpha}\left[1 - \Big(\frac{\tilde{V_1}}{\tilde{V_1} + 2\alpha x}\Big)^{\frac{1}{2}}\right] dx \\
&= \frac{\tilde{V_1}}{2(1-\alpha)}.
\end{aligned}$$

Similarly, for firms c and d,

$$\mathbb{E}[e_c^*] = \mathbb{E}[e_d^*] = \int_0^{\frac{(2-\alpha)\tilde{V}_2}{2(1-\alpha)^2}} 1 - G_2(x)\,dx$$

$$= \int_0^{\frac{(2-\alpha)\tilde{V}_2}{2(1-\alpha)^2}} 1 - \frac{1}{\alpha}\left[1 - \Big(\frac{\tilde{V}_2}{\tilde{V}_2 + 2\alpha x}\Big)^{\frac{1}{2}}\right] dx$$

$$= \frac{\tilde{V}_2}{2(1-\alpha)}.$$

In the definition of $\tilde{V_1}$, notice that V_1 is constant, so

$$\tilde{V_1} = \mathbb{E}[V_1 + \beta(e_c^* + e_d^*)]$$

$$= V_1 + \beta\mathbb{E}[e_c^* + e_d^*]$$

$$= V_1 + \frac{\beta\tilde{V}_2}{1-\alpha}.$$

Similarly, we have

$$\tilde{V_2} = \mathbb{E}[V_2 + \beta(e_a^* + e_b^*)]$$

$$= V_2 + \beta\mathbb{E}[e_b^* + e_b^*]$$

$$= V_2 + \frac{\beta\tilde{V}_1}{1-\alpha}.$$

Solving this system of equations, we get the desired results:

$$\tilde{V_1} = \frac{V_1 + \frac{\beta}{1-\alpha}V_2}{1 - (\frac{\beta}{1-\alpha})^2}$$

$$= \frac{(1-\alpha)^2 V_1 + \beta(1-\alpha)V_2}{(1-\alpha)^2 - \beta^2},$$

and

$$\tilde{V_2} = \frac{V_2 + \frac{\beta}{1-\alpha}V_1}{1 - (\frac{\beta}{1-\alpha})^2}$$

$$= \frac{(1-\alpha)^2 V_2 + \beta(1-\alpha)V_1}{(1-\alpha)^2 - \beta^2}.$$

Plugging these expressions back into the equilibrium investing strategies, we get that the symmetric equi-

librium investing strategy for each firm in industry 1 is mixing using

$$G_1(x) = \frac{1}{\alpha}\left\{1 - \left[\frac{(1-\alpha)^2 V_1 + \beta(1-\alpha)V_2}{(1-\alpha)^2 V_1 + \beta(1-\alpha)V_2 + 2\alpha[(1-\alpha)^2 - \beta^2]x}\right]^{\frac{1}{2}}\right\},$$

and the investing interval is

$$\left[0, \frac{(2-\alpha)[(1-\alpha)V_1 + \beta V_2]}{2(1-\alpha)[(1-\alpha)^2 - \beta^2]}\right].$$

The symmetric equilibrium investing strategy for firms in industry 2 is

$$G_2(x) = \frac{1}{\alpha}\left\{1 - \left[\frac{(1-\alpha)^2 V_2 + \beta(1-\alpha)V_1}{(1-\alpha)^2 V_2 + \beta(1-\alpha)V_1 + 2\alpha[(1-\alpha)^2 - \beta^2]x}\right]^{\frac{1}{2}}\right\},$$

over the interval

$$\left[0, \frac{(2-\alpha)[(1-\alpha)V_2 + \beta V_1]}{2(1-\alpha)[(1-\alpha)^2 - \beta^2]}\right].$$

□

Corollary 1: Let $G_i(x)$ be the equilibrium investing strategy of a firm in industry i with parameters α and β. Let and $\tilde{G}_i$ be the equilibrium strategy when holding α constant and changing β to $\tilde{\beta} < \beta$. Then $G_i(x)$ first order stochastically dominates $\tilde{G}_i(x)$.

Proof. When $\beta > \tilde{\beta}$,

$$(1-\alpha)^2 V_1 + \beta(1-\alpha)V_2 > (1-\alpha)^2 V_1 + \tilde{\beta}(1-\alpha)V_2,$$

and

$$2\alpha[(1-\alpha)^2 - \beta^2] < 2\alpha[(1-\alpha)^2 - \tilde{\beta}^2].$$

Then we have

$$\left[\frac{(1-\alpha)^2V_1+\beta(1-\alpha)V_2}{(1-\alpha)^2V_1+\beta(1-\alpha)V_2+2\alpha[(1-\alpha)^2-\beta^2]x}\right]^{\frac{1}{2}}$$
$$>\left[\frac{(1-\alpha)^2V_1+\tilde{\beta}(1-\alpha)V_2}{(1-\alpha)^2V_1+\tilde{\beta}(1-\alpha)V_2+2\alpha[(1-\alpha)^2-\tilde{\beta}^2]x}\right]^{\frac{1}{2}}.$$

Thus

$$G_1(x)=\frac{1}{\alpha}\left\{1-\left[\frac{(1-\alpha)^2V_1+\beta(1-\alpha)V_2}{(1-\alpha)^2V_1+\beta(1-\alpha)V_2+2\alpha[(1-\alpha)^2-\beta^2]x}\right]^{\frac{1}{2}}\right\}$$
$$<\frac{1}{\alpha}\left\{1-\left[\frac{(1-\alpha)^2V_1+\tilde{\beta}(1-\alpha)V_2}{(1-\alpha)^2V_1+\tilde{\beta}(1-\alpha)V_2+2\alpha[(1-\alpha)^2-\tilde{\beta}^2]x}\right]^{\frac{1}{2}}\right\}=\tilde{G}_1(x)$$

over the interval of potential investment levels. The same reasoning holds for firms in industry 2. □

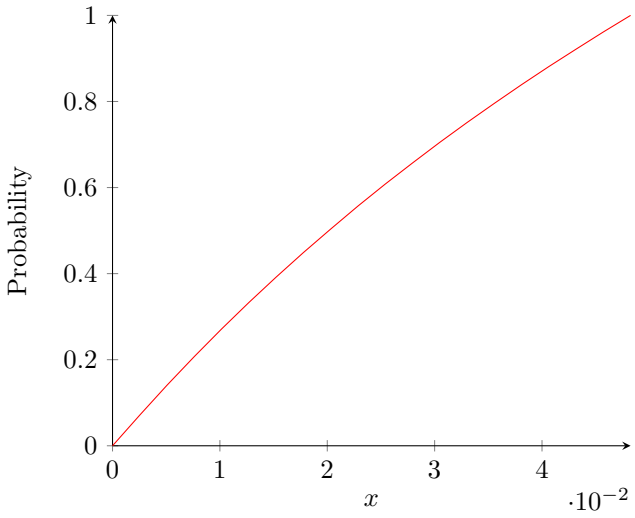
Now we give a numerical example of equilibrium investing strategies. The parameters are chosen to satisfy all assumptions made earlier as well as the condition that the duopoly market yields a higher expected social welfare than monopoly. The condition is derived in the section **Discussion and Welfare Analysis**. Consider the case where the two industries are identical, with the following parameters:

$$\bar{c_I}=\bar{c_J}=0.4$$
$$\underline{c}_I=\underline{c}_J=0.35$$
$$V=0.6\cdot(0.4-0.35)=0.03$$
$$\alpha=0.2$$
$$\beta=0.1$$
$$\tilde{\alpha}=\frac{1}{3}$$
$$\tilde{\beta}=\frac{1}{6}.$$

Then by **Theorem 1**, the equilibrium investing strategy is

$$G(x)=5-5\Big[\frac{0.0216}{0.0216+0.252x}\Big]^{0.5},$$

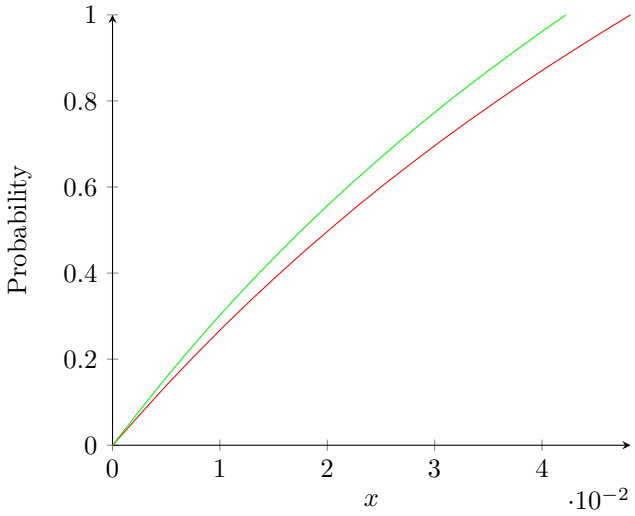
over the interval $[0, 0.04825]$. This mixed strategy's c.d.f. is the red curve in the graph below.



Now we change β from 0.1 to $\underline{\beta} = 0$, so there is no inter-industry spillover. The new equilibrium investing strategy is

$$\underline{G}(x) = 5 - 5\Big[\frac{0.0192}{0.0192 + 0.256x}\Big]^{0.5}$$

over the interval $[0, 0.04219]$. This strategy is the green curve in the graph below. Notice that it is first-order stochastically dominated by the original equilibrium strategy when there is inter-industry spillover, which is consistent with **Corollary 1**.



We now focus on the case where the complementarity factor $\beta > 0$ and analyze the effect of β on each firm's expected investment level. Intuition suggests that when the level of complementarity is higher, firms exert higher effort because the spillover makes the prize of winning larger. The following result formally shows this. In addition, it highlights that expected investment is convex in the complementarity factor.

Proposition 2: An increase in the strength of positive inter-industry spillover increases one's expected investment: $\frac{\partial \mathbb{E}[e_i^*]}{\partial \beta} > 0$. Furthermore, **feedback effect** leads to the following result: $\frac{\partial^2 \mathbb{E}[e_i^*]}{\partial \beta^2} > 0$ for all i.

Proof. When $\beta > 0$, for firm a,

$$\begin{aligned}\frac{\partial \mathbb{E}[e_a^*]}{\partial \beta} &= \frac{\partial}{\partial \beta}\left\{\frac{1}{2(1-\alpha)} \cdot \frac{V_1 + \frac{\beta}{1-\alpha} \cdot V_2}{1 - (\frac{\beta}{1-\alpha})^2}\right\} \\ &= \frac{(1-\alpha)\beta V_1 + \frac{V_2}{2}[\beta^2 + (1-\alpha)^2]}{[(1-\alpha)^2 - \beta^2]^2} > 0\end{aligned}$$

Now, observe that $\alpha + \beta < 1$ implies $(1-\alpha)^2 - \beta^2 > 0$, so

$$\frac{\partial}{\partial \beta}\left\{\frac{1}{[(1-\alpha)^2 - \beta^2]^2}\right\} = \frac{4\beta}{[(1-\alpha)^2 - \beta^2]^3} > 0.$$

And similarly

$$\begin{aligned}\frac{\partial}{\partial \beta}\left\{\frac{\beta}{[(1-\alpha)^2 - \beta^2]}\right\} &= \frac{(1-\alpha)^2 + \beta^2}{[(1-\alpha)^2 - \beta^2]^3} > 0, \\ \frac{\partial}{\partial \beta}\left\{\frac{\beta^2}{[(1-\alpha)^2 - \beta^2]}\right\} &= \frac{2\beta(1-\alpha)^2}{[(1-\alpha)^2 - \beta^2]^3} > 0.\end{aligned}$$

This implies that when we take the second order derivative from the first order derivative, each component will give us a positive result, so

$$\frac{\partial^2 \mathbb{E}[e_a^*]}{\partial \beta^2} > 0.$$

Performing the same calculation for other firms leads to the desired conclusion. □

The second result is interesting because it means that although the reward is linear in the sum of investments of the other industry, the second order partial derivative of a firm's own expected investment with respect to the complementarity factor is positive. The intuition behind this result is feedback effect: when $\beta > 0$, when β increases, firm a wants to invest more, because the prize of winning becomes higher even when firms c and d both play the original strategy. This then leads to firms c and d both exerting higher efforts because the prize of winning becomes higher for them, which again makes firm a wants to choose an even higher investment level.

1.3.2 Cross-Industry Mergers

Now we consider the effect of cross-industry mergers on R&D incentives. We focus on the case where after the merger, the reward is still increasing in one's own effort, and the merged firm also internalizes the inter-industry complementarity or substitutability. We analyze the game where a and c merge into firm A while b and d merge into firm B. In the new game between A and B, the profit of each firm is given by

$$\Pi_i(e_i) = \begin{cases} V_1 + V_2 + (\alpha+\beta)(e_i+e_j) - e_i & \text{if } e_i > e_j \\ \frac{1}{2}[V_1 + V_2 + (\alpha+\beta)(e_i+e_j)] - e_i & \text{if } e_i = e_j \\ -e_i & \text{if } e_i < e_j. \end{cases}$$

Proposition 3: The symmetric equilibrium in the merged game is for each firm to invest

$$H(x) = \frac{1}{\alpha+\beta}\left[1 - \left(\frac{V_1+V_2}{V_1+V_2+2(\alpha+\beta)x}\right)^{\frac{1}{2}}\right]$$

over the interval $[0, \frac{(2-\alpha-\beta)(V_1+V_2)}{2(1-\alpha-\beta)^2}]$. In this game, each merged firm expects zero profit.

Proof. The proof is similar to the proof of **Theorem 1**. □

Notice that each firm will be indifferent between merging or not, because they would expect zero expected utility in both cases. Now we calculate expected efforts in each contest.

Theorem 2: The expected investment of each merged firm is the same as the sum of the two firms before merging. However, when $\beta > 0$, the sum of maximum possible investment levels of two original firms is lower than that of a merged firm; when $\beta < 0$, the opposite is true.

Proof. As calculated before, without merging, the expected efforts of firms a and b both equal

$$\mathbb{E}[e_a^*] = \mathbb{E}[e_b^*] = \frac{1}{2(1-\alpha)} \cdot \frac{V_1 + \frac{\beta}{1-\alpha}V_2}{1 - (\frac{\beta}{1-\alpha})^2}.$$

Moreover, the expected efforts of firms c and d equal

$$\mathbb{E}[e_c^*] = \mathbb{E}[e_d^*] = \frac{1}{2(1-\alpha)} \cdot \frac{V_2 + \frac{\beta}{1-\alpha}V_1}{1 - (\frac{\beta}{1-\alpha})^2}.$$

In the merged game, the expected effort of each firm is

$$\mathbb{E}[e_A^*] = \mathbb{E}[e_B^*] = \int_0^{\frac{(2-\alpha-\beta)(V_1+V_2)}{2(1-\alpha-\beta)^2}} 1 - H(x)\,dx$$

$$= \int_0^{\frac{(2-\alpha-\beta)(V_1+V_2)}{2(1-\alpha-\beta)^2}} 1 - \frac{1}{\alpha+\beta}\left[1 - \Big(\frac{V_1+V_2}{V_1+V_2+2(\alpha+\beta)x}\Big)^{\frac{1}{2}}\right] dx$$

$$= \frac{(V_1+V_2)}{2(1-\alpha-\beta)}$$

Then for all α and β,

$$\mathbb{E}[e_a^*] + \mathbb{E}[e_c^*]$$
$$= \frac{1}{2(1-\alpha)} \cdot \frac{V_1 + \frac{\beta}{1-\alpha} \cdot V_2}{1 - (\frac{\beta}{1-\alpha})^2} + \frac{1}{2(1-\alpha)} \cdot \frac{V_2 + \frac{\beta}{1-\alpha} \cdot V_1}{1 - (\frac{\beta}{1-\alpha})^2}$$
$$= \frac{(V_1+V_2)}{2(1-\alpha-\beta)} = \mathbb{E}[e_A^*].$$

Now we compare maximum potential investments in equilibrium. The sum of maximum potential investment levels of firms a and c is given by

$$\frac{(2-\alpha)[(1-\alpha)V_1 + \beta V_2]}{2(1-\alpha)[(1-\alpha)^2 - \beta^2} + \frac{(2-\alpha)[(1-\alpha)V_2 + \beta V_1]}{2(1-\alpha)[(1-\alpha)^2 - \beta^2]}$$
$$= \frac{(V_1+V_2)(2-\alpha)}{2(1-\alpha)(1-\alpha-\beta)}.$$

From **Proposition 3**, the maximum potential investment level of the merged firm A is

$$\frac{(2-\alpha-\beta)(V_1+V_2)}{2(1-\alpha-\beta)^2}.$$

Notice that when $\beta > 0$,

$$\frac{(V_1+V_2)(2-\alpha)}{2(1-\alpha)(1-\alpha-\beta)} = \frac{V_1+V_2}{2(1-\alpha-\beta)} \cdot \frac{2-\alpha}{1-\alpha} < \frac{V_1+V_2}{2(1-\alpha-\beta)} \cdot \frac{2-\alpha-\beta}{1-\alpha-\beta},$$

which is the maximum investment level of a merged firm. When $\beta < 0$, the opposite is true:

$$\frac{V_1+V_2}{2(1-\alpha-\beta)} \cdot \frac{2-\alpha}{1-\alpha} > \frac{V_1+V_2}{2(1-\alpha-\beta)} \cdot \frac{2-\alpha-\beta}{1-\alpha-\beta}.$$

□

This result complements a main result by Fang, Noe, and Strack (2020), who suggest that putting multiple all-pay contests into one (say, considering the case $V_1 = V_2 = V$ and letting four firms simultaneously run one all-pay contest in which the top two firms with the most efforts win V each and other firms win nothing) would discourage each player's effort in expectation. Here we show that another form of combining multiple contests into one, which is merging, actually leads to the same expected effort.

1.4 Cross-Shareholding

In this section, we study the effects of cross-shareholding on investments and profits of firms. Ownership structure can influence firms' competition. There are two types of cross-shareholding. An investor who owns the majority of the shares of a firm investing in its rival firm in the same industry is called horizontal cross-shareholding, while investing in firms in its complementary/substitute industry is called vertical cross-shareholding. We focus on the symmetric case where investor i who owns $1 - p - 2q > 0.5$ share in firm i also owns $p \in [0, 0.5)$ cross share in its rival firm in the same industry and $q \in [0, 0.25)$ cross share in each of the firms in its complementary industry. Investor i appoints a manager in firm i who makes R&D decisions to maximize the expected profit of investor i. This means that investor i's investments in other firms are passive.

Consider the problem of investor i who owns the majority of shares of firm i in industry I. Firm i's rival firm in I is j, and its complementary/substitute industry is J consisting of firms k, l. If i wins in the contest against firm j, then investor i will have an $1 - p - 2q$ share of the payoff $V_I + \alpha(e_i + e_j) + \beta(e_k + e_l) - e_i$, plus a p share of the the losing cost $-e_j$. Suppose firm k wins in industry J's contest, then investor i has a q share of the payoff $V_J + \alpha(e_k + e_l) + \beta(e_i + e_j) - e_k$ and another q share of the losing cost $-e_l$ from investing in firm l. Then the total payoff from his investment in industry J is $q[V_J + \alpha(e_k + e_l) + \beta(e_i + e_j) - e_k - e_l]$. Notice that firm i's payoff from investing in its complementary industry will be the same when firm l rather than firm k wins. If firm i loses in the contest, then its major investor will get an $1 - p - 2q$ share of the losing cost $-e_i$, a p share of the winning payoff $V_I + \alpha(e_i + e_j) + \beta(e_k + e_l) - e_i$, and his return from investing in the other industry will still be $q[V_J + \alpha(e_k + e_l) + \beta(e_i + e_j) - e_k - e_l]$. If there is a tie between firms i and j, then investor i expects $\frac{1}{2}$ of the sum of his winning and losing payoffs.

So investor i's profit is

$$\Pi_i(e_i,e_j,e_k,e_l)=\begin{cases}(1-p-2q)[V_I+\alpha(e_i+e_j)+\beta(e_k+e_l)-e_i]-pe_j & \\ +q[V_J+\alpha(e_k+e_l)+\beta(e_i+e_j)-e_k-e_l] & \text{if } e_i>e_j \\ \frac{1}{2}\{(1-p-2q)[V_I+\alpha(e_i+e_j)+\beta(e_k+e_l)-e_i]-pe_j\} & \\ +q[V_J+\alpha(e_k+e_l)+\beta(e_i+e_j)-e_k-e_l] & \text{if } e_i=e_j \\ (1-p-2q)(-e_i)+p[V_I+\alpha(e_i+e_j)+\beta(e_k+e_l)-e_j] & \\ +q[V_J+\alpha(e_k+e_l)+\beta(e_i+e_j)-e_k-e_l] & \text{if } e_i<e_j.\end{cases}$$

Rearranging terms, we get

$$\Pi_i(e_i,e_j,e_k,e_l)=\begin{cases}(1-p-2q)V_I+qV_J+[(1-p-2q)\beta+q(\alpha-1)](e_k+e_l) & \\ -[(1-p-2q)(1-\alpha)-q\beta]e_i-[p-(1-p-2q)\alpha-q\beta]e_j & \text{if } e_i>e_j \\ \frac{1}{2}\{(1-p-2q)[V_I+\alpha(e_i+e_j)+\beta(e_k+e_l)-e_i]-pe_j\} & \\ +q(V_J+\alpha(e_k+e_l)+\beta(e_i+e_j)-e_k-e_l) & \text{if } e_i=e_j \\ pV_I+qV_J+[p\beta+q(\alpha-1)](e_k+e_l)-[(1-p-2q)-p\alpha-q\beta]e_i & \\ -[p(1-\alpha)-q\beta]e_j & \text{if } e_i<e_j.\end{cases}$$

To focus on the interesting case where there exists an equilibrium where each firm makes positive R&D investments, we need two more assumptions. When $\beta>0$, we need

$$(1-2p-2q)(1-\alpha-\beta)+p(1-\alpha)-q\beta>0, \tag{A1}$$

which means that $p(1-\alpha)$ cannot be much smaller than $q\beta$. This condition is always satisfied when $\beta<0$. It will be clear from our later calculations that this assumption is critical for each firm's expected R&D investment to be positive. When $\beta<0$, we need another restriction to ensure that firms want to exert positive efforts:

$$\begin{aligned}&\big\{(1-p-2q)V_I+qV_J+[(1-p-2q)\beta+q(\alpha-1)]\mathbb{E}[{e_k}^*+{e_l}^*]\big\}\\&-\big\{pV_I+qV_J+[p\beta+q(\alpha-1)]\mathbb{E}[e_k^*+e_l^*]\big\}>0.\end{aligned}$$

This restriction states that the expected winning prize minus the expected losing prize needs to be positive.

Notice that this is automatically satisfied when $\beta > 0$. From our calculations below, it follows that the condition is

$$V_I > \frac{\beta \tilde{V}_I}{[(1-p-2q)(1-\alpha)-q\beta]} \tag{A2}$$

for all $I \in \{1,2\}$, where $\tilde{V}_I$ is defined in **Theorem 3.** Roughly speaking, the two assumptions require that $|\beta|$ is relatively low, that is, the degree of inter-industry spillover cannot be too high.

The following main result characterizes the symmetric investing equilibrium in this game.

Theorem 3: Suppose assumptions (A1) and (A2) are satisfied, then the symmetric equilibrium investing strategy for firms a and b is mixing according to the c.d.f.

$$B_1(x) = \frac{[(1-p-2q)-p\alpha-q\beta]}{(1-2p-2q)\alpha}\left[1-\Big(\frac{\tilde{V}_1}{\tilde{V}_1+2(1-2p-2q)\alpha x}\Big)^{\frac{1}{2}}\right]$$

over the interval

$$\left[0, \frac{\tilde{V}_1}{2(1-2p-2q)\alpha}\left\{\Big[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\Big]^2-1\right\}\right],$$

where

$$\tilde{V}_1 = \frac{(1-2p-2q)V_1 + \frac{(1-2p-2q)\beta}{(1-p-2q)(1-\alpha)-q\beta}(1-2p-2q)V_2}{1-\left[\frac{(1-2p-2q)\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2}.$$

The symmetric equilibrium investing strategy for firms c and d is the c.d.f.

$$B_2(x) = \frac{[(1-p-2q)-p\alpha-q\beta]}{(1-2p-2q)\alpha}\left[1-\Big(\frac{\tilde{V}_2}{\tilde{V}_2+2(1-2p-2q)\alpha x}\Big)^{\frac{1}{2}}\right]$$

over the interval

$$\left[0, \frac{\tilde{V}_2}{2(1-2p-2q)\alpha}\left\{\Big[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\Big]^2-1\right\}\right],$$

where

$$\tilde{V}_2 = \frac{(1-2p-2q)V_2 + \frac{(1-2p-2q)\beta}{(1-p-2q)(1-\alpha)-q\beta}(1-2p-2q)V_1}{1-\left[\frac{(1-2p-2q)\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2}.$$

Proof. Please see the **Appendix**. □

The key economic trade-off from cross-shareholding is that holding shares in the other industry reduces each investor's direct share of the endogenous winning reward in his own firm when investing heavily in R&D, but his effort is compensated with a direct share of the winning reward in the other industry, which also depends on his effort.

1.4.1 Effects of Cross-Shareholding on R&D Investments and Profits

We now analyze how cross-shareholding affects firms' incentives to make R&D investments. In our analysis below, we assume $V_1 = V_2 = V$ in order to simplify expressions. We denote $\tilde{V_1} = \tilde{V_2} = \tilde{V}$, ${V_1}^w = {V_2}^w = V^w$, and ${V_1}^l = {V_2}^l = V^l$.

Proposition 4: Cross-investing increases each firm's expected effort level if and only if

$$q\beta > p(1-\alpha).$$

Proof. When $V_1 = V_2 = V$, $\tilde{V}$ becomes

$$\begin{aligned}\tilde{V} &= \frac{(1-2p-2q)V + \frac{(1-2p-2q)\beta}{(1-p-2q)(1-\alpha)-q\beta}(1-2p-2q)V}{1-\left[\frac{(1-2p-2q)\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2} \\ &= \frac{[(1-p-2q)(1-\alpha)-q\beta](1-2p-2q)V}{(1-p-2q)(1-\alpha)-q\beta-(1-2p-2q)\beta}.\end{aligned}$$

When there is no cross-shareholding, each firm's expected effort is

$$\frac{V}{2(1-\alpha-\beta)}.$$

With cross-shareholding, the expected effort of firm i becomes

$$\begin{aligned}\mathbb{E}[e_i^*] &= \frac{\tilde{V}}{2[(1-p-2q)(1-\alpha)-q\beta]} \\ &= \frac{(1-2p-2q)V}{2[(1-p-2q)(1-\alpha)-q\beta-(1-2p-2q)\beta]} \\ &= \frac{(1-2p-2q)V}{2\{(1-2p-2q)(1-\alpha-\beta)+[p(1-\alpha)-q\beta]\}},\end{aligned}$$

Then each firm's expected investment level with cross-shareholding is higher than that without cross-

shareholding if and only if

$$\frac{(1-2p-2q)V}{2\{(1-2p-2q)(1-\alpha-\beta)+[p(1-\alpha)-q\beta]\}} > \frac{V}{2(1-\alpha-\beta)},$$

which is equivalent to

$$q\beta > p(1-\alpha).$$

□

We pay special attention to cases where a firm only invests in its rival firm or complementary firms. The following observations discuss the effects of cross-shareholding on each firm's incentives to exert effort. **Observation 1:** When there is no vertical cross-shareholding, horizontal cross-investing reduces each firm's expected R&D investment level. When there is no horizontal cross-shareholding, vertical cross-investing increases each firm's expected R&D investment level when the two industries are complements but decreases investment levels when they are substitutes.

Proof. This result follows immediately from **Proposition 4.** □

The intuition behind **Observation 1** is rather clear: holding shares in one's rival firm makes competition less intense, because hurting its rival will hurt itself. The real world implication is that with horizontal cross-shareholding, each firm's manager's compensation would be less performance-based.

While horizontal shared ownership reduces incentives to compete and thus reduces the level of expected R&D effort, vertical shared ownership encourages each firm to make higher R&D investments when $\beta > 0$ because it enables its major investor to share the reward in the complementary industry regardless of winning or not in its own contest, and this reward is increasing in its own effort. When $\beta < 0$, vertical cross-investing reduces R&D investments because investing in the substitute industry is similar to investing in the firm's rival, which also leads to tacit collusion.

Now we analyze the effect of cross-shareholding on the profit of each firm.

Proposition 5: The expected profit of each investor is

$$\mathbb{E}[\Pi_i(e_i^*, e_{-i}^*)] = \frac{[p(1-\alpha)-q\beta]V}{2[(1-2p-2q)(1-\alpha-\beta)+p(1-\alpha)-q\beta]}.$$

Therefore, cross-investing is profitable if and only if

$$p(1-\alpha) > q\beta.$$

Proof. Let V^w and V^l be the expected winning and losing prizes of a firm respectively. Then

$$\begin{aligned} V^w &= (1-p-q)V + 2[(1-p-2q)\beta + q(\alpha-1)]\mathbb{E}[{e_i}^*] \\ &= (1-p-q)V + \frac{[(1-p-2q)\beta + q(\alpha-1)](1-2p-2q)V}{(1-p-2q)(1-\alpha) - q\beta - (1-2p-2q)\beta} \end{aligned}$$

and

$$\begin{aligned} V^l &= (p+q)V + 2[p\beta + q(\alpha-1)]\mathbb{E}[{e_i}^*] \\ &= (p+q)V + \frac{[p\beta + q(\alpha-1)](1-2p-2q)V}{(1-p-2q)(1-\alpha) - q\beta - (1-2p-2q)\beta}. \end{aligned}$$

Then investor i's expected profit is

$$\begin{aligned} \mathbb{E}[\Pi_i(e_i^*, e_{-i}^*)] &= \frac{[p(1-\alpha) - q\beta]V^w - [p - (1-p-2q)\alpha - q\beta]V^l}{(1-2p-2q)\alpha} \\ &+ \frac{2\tilde{V}[(1-p-2q)(1-\alpha) - q\beta][p(1-\alpha) - q\beta]}{[2(1-2p-2q)\alpha]^2}\left\{1 - \left[\frac{(1-p-2q) - p\alpha - q\beta}{(1-p-2q)(1-\alpha) - q\beta}\right]^2\right\} \\ &= \frac{p(1-\alpha) - q\beta}{(1-2p-2q)\alpha} \cdot \left\{(1-p-q)V + \frac{[(1-p-2q)\beta + q(\alpha-1)](1-2p-2q)V}{(1-p-2q)(1-\alpha) - q\beta - (1-2p-2q)\beta}\right\} \\ &- \frac{p - (1-p-2q)\alpha - q\beta}{(1-2p-2q)\alpha} \cdot \left\{(p+q)V + \frac{[p\beta + q(\alpha-1)](1-2p-2q)V}{(1-p-2q)(1-\alpha) - q\beta - (1-2p-2q)\beta}\right\} \\ &+ \frac{[(1-p-2q)(1-\alpha) - q\beta](1-2p-2q)V}{(1-p-2q)(1-\alpha) - q\beta - (1-2p-2q)\beta} \cdot \frac{2[(1-p-2q)(1-\alpha) - q\beta][p(1-\alpha) - q\beta]}{[2(1-2p-2q)\alpha]^2} \\ &\cdot \left\{1 - \left[\frac{(1-p-2q) - p\alpha - q\beta}{(1-p-2q)(1-\alpha) - q\beta}\right]^2\right\} \\ &= \frac{[p(1-\alpha) - q\beta]V}{2[(1-2p-2q)(1-\alpha-\beta) + p(1-\alpha) - q\beta]}. \end{aligned}$$

Thus, cross-investing is profitable if and only if

$$p(1-\alpha) > q\beta.$$

□

From our calculations above, we have the following observation on the profitability of cross-investing:

Observation 2: When there is no vertical cross-shareholding, horizontal cross-shareholding gives each firm a positive expected profit. When there is no horizontal cross shareholding, vertical cross-investing gives each firm a negative expected profit when the two industries are complements and a positive expected profit when they are substitutes.

Proof. This result follows immediately from **Proposition 5.** □

1.4.2 An Impossibility Theorem

Now we present the main result in the model with cross-shareholding.

Theorem 4: There is no cross-shareholding structure that can simultaneously increase a firm's R&D investment and its major investor's expected profit.

Proof. This result follows from noticing that the condition under which each firm's expected R&D effort will be higher with cross-shareholding is precisely the opposite of the condition under which each firm's major investor will get a positive expected profit. □

As an illustration, recall parameters in the previous example, and now we add cross-shareholding parameters:

$$p = 0.2$$

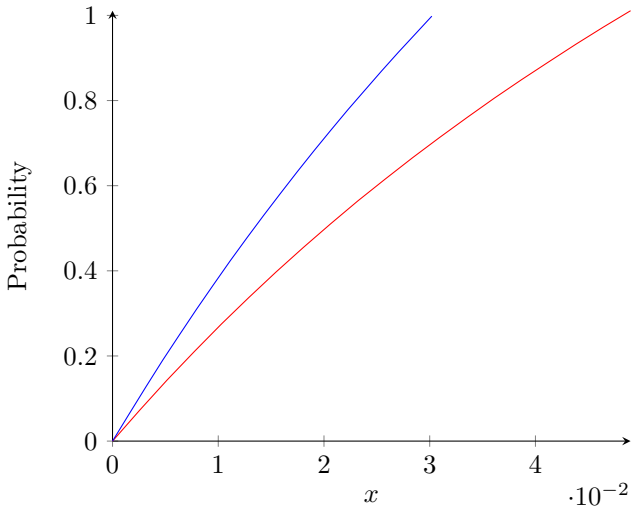
$$q = 0.1$$

$$\tilde{V} = 0.0131.$$

The resulting equilibrium investing strategy becomes

$$B(x) = 6.875 - 6.875\Big[\frac{0.0131}{0.0131 + 0.16x}\Big]^{0.5}$$

over the interval $[0, 0.0302]$. This c.d.f. is the blue curve in the graph below. We compare it with the original strategy, which is the red one below.



In the graph above, $G(x)$ first-order stochastically dominates $B(x)$, so firms tend to spend less on R&D with the cross shareholding structure specified above. This is because they are holding so many shares of their rivals' stocks that the anti-competitive effect dominates the effect that their holdings of shares of the complementary industry enable them to directly share the complementary industry's benefit from their own investment. This is consistent with the condition in **Proposition 4**:

$$p(1-\alpha) = 0.16$$
$$> q\beta = 0.01.$$

Future work can try to make a complete characterization of conditions under which changing a parameter will lead to first-order stochastic dominant equilibrium investing strategies.

1.5 Discussion and Welfare Analysis

Our results have important policy implications. Institutional investors own about seventy percent of the US stock market, and many of them practice horizontal or vertical cross-shareholding. As a result, whether cross-shareholding or common ownership violates antitrust regulations has been in the center of debates. Nigro et al (2016) state that section 1 of the Sherman Act requires "evidence that a shareholder facilitated an agreement among competing firms," so cross-ownership alone would not be considered a violation. They further claim that section 7 of the Clayton Act, which prohibits stock acquisitions that could significantly reduce competition between firms, "contains an exemption for acquisitions solely for investment, meaning that a plaintiff would need to show that the institutional investor actively sought to influence management of the company to lessen competition" (Nigro et al 2016).

From a theoretical perspective, our results reaffirm that horizontal cross-shareholding has anti-competitive effects. On the other hand, we show vertical cross-shareholding has different effects depending on whether the two industries are complementary or substitute. When they are complements, vertical cross-shareholding will not reduce competition at all; it actually encourages firms to better take advantage of the positive inter-industry spillover by spending more on R&D investments. However, vertical cross-shareholding with substitute industries induce tacit collusion and make firms invest less in R&D.

Now we analyze welfare in this model. Recall that $Q_I = 1 - P_I$ be the aggregate demand in industry I. If $e_i > e_j$, the marginal cost of firm i in industry I is

$$c_i = \underline{c}_I - \tilde{\alpha}(e_i + e_j) - \tilde{\beta}(e_k + e_l),$$

and the marginal cost of firm j is

$$c_j = \bar{c}_I.$$

The winning firm charges price $p^* = \bar{c}_I$ and produces $q^* = 1 - \bar{c}_I$. **Theorem 1** states that

$$\mathbb{E}[\Pi_i({e_i}^*, {e_{-i}}^*)] = 0$$

Thus expected social welfare in industry I is

$$\begin{aligned}\text{Expected Social Welfare in Industry I} &= \text{Producer Surplus} - \text{Expected Total R\&D Costs} + \text{Consumer Surplus} \\ &= 0 + \frac{1}{2}(1 - \bar{c}_I)^2 \\ &= \frac{1}{2}(1 - \bar{c}_I)^2.\end{aligned}$$

Therefore, the total social welfare across the two industries is

$$\begin{aligned}&\text{Total Expected Social Welfare} \\ &= \frac{1}{2}(1 - \bar{c}_I)^2 + \frac{1}{2}(1 - \bar{c}_J)^2.\end{aligned}$$

Notice that this expected social welfare is exactly the same as the one without R&D if we assume that both firms have $\bar{c_I}$ as their original marginal costs. This is slightly counter-intuitive because in most cases competition improve social welfare.

Consider the case of each industry being controlled by a monopolist in this setting. The monopolist in industry I can spend ϵ in R&D and capture marginal cost $\underline{c}_I$, and since ϵ is very small, we can ignore its effect on inter-industry spillovers. Then the special welfare in industry I would be

$$\begin{aligned}\text{Social Welfare in Industry I with Monopoly} &= \text{Producer Surplus} + \text{Consumer Surplus} \\ &= \frac{1}{2}(1-\underline{c}_I)^2 + \frac{1}{4}(1-\underline{c}_I)^2 \\ &= \frac{3}{8}(1-\underline{c}_I)^2.\end{aligned}$$

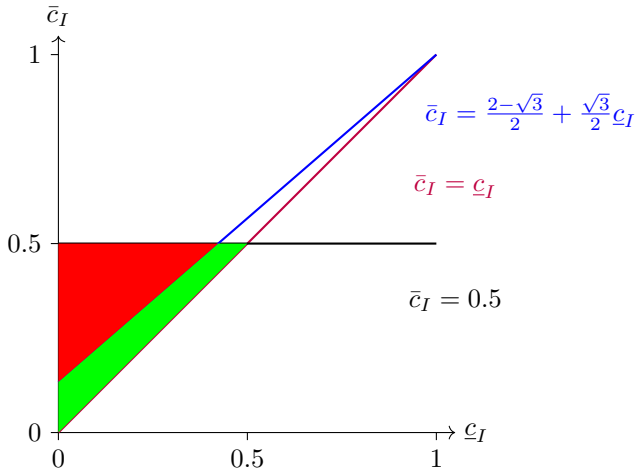
Then the condition under which the duopoly industry leads to a higher social welfare is

$$\frac{3}{8}(1-\underline{c}_I)^2 < \frac{1}{2}(1-\bar{c}_I)^2,$$

which is equivalent to

$$\bar{c}_I < \frac{2-\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\underline{c}_I.$$

In the graph below, three constraints are displayed: the positive prize constraint $\bar{c}_I > \underline{c}_I$; the non-drastic difference constraint $\bar{c}_I < 0.5$; and finally the constraint that guarantees duopoly yielding higher social welfare $\bar{c}_I < \frac{2-\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\underline{c}_I$. The green area is the set of parameters $\bar{c}_I$ and $\underline{c}_I$ that simultaneously satisfy all conditions, which is the area where duopoly leads to higher social welfare. The red area is the set of parameters that satisfy the first two but not the last. This means the red area is where monopoly leads to higher social welfare. The intuition is that we need the two marginal costs not to be far away, since the winning duopolist firm chargers the higher marginal cost while the monopolist optimizes using the low marginal cost.



From the graph, it is clear that the chosen set of parameters in the illustrative example, $\bar{c}_I = 0.4$ and $\underline{c}_I = 0.35$, is in the green area.

For a complete characterization of the conditions under which monopolization can improve social welfare, please read a recent paper by Cowan (2024). Total expected social welfare will be higher from R&D competition if we assume that the original marginal cost is higher than $\bar{c}_I$ (i.e., if they both exert 0 effort, then they remain at the status quo, but as long as some efforts are exerted even the losing firm will have a lower marginal cost). In this case, the equilibrium will be the same because the equilibrium mixed strategies attach 0 probability of exerting 0 effort.

In our model, despite intra- and inter-industry spillovers, the all-pay auction setup and the afterward Bertrand competition will always make producer surplus minus total R&D costs equal to 0. In the presence of cross-shareholding, each firm can earn a positive profit if it invests relatively heavily in its rival. In our model, increasing firms' profits is the same as increasing social welfare. The Impossibility Theorem therefore implies that the welfare-improving cross-shareholding structures are precisely the ones that lower each firm's expected R&D investments. This counter-intuitive result might help explain why cross-shareholding are often not regulated strictly. It also implies that the social planner should adopt different policies when he seeks to maximize welfare versus aggregate innovative efforts. Future studies might investigate whether similar conclusions will be reached when modelling competition with the Cournot model or Tullock contests.

1.6 Conclusion

In this paper, we modelled R&D contests using all-pay auctions to analyze firms' R&D investment decisions in the presence of both intra-industry and inter-industry spillovers. We developed a microeconomic foundation through a two-stage game to the all-pay auction framework of modelling R&D competition. We showed that cross-industry mergers led to the same total expected investment. Moreover, we studied symmetric cross-shareholding in this environment, where each investor who owns the majority of shares of one firm invests in all other firms. We proved an impossibility theorem: there is no cross-shareholding structure that can simultaneously increase a firm's R&D investment and its major investor's expected profit.

A natural way to extend our model is to study symmetric cross-shareholding with multiple firms in each industry and see whether the impossibility theorem holds in this case. Another extension is to analyze the scenario when the size of the winning reward does not have an endogenous component in a firm's own investment but is still affected by spillover from its rival. On the other hand, it is technically challenging to find equilibria in these extensions when using the all-pay auction model. It might be easier to find analytic

solutions using the Tullock model instead. Finally, another interesting direction is to endogenize the levels and structures of cross-shareholding.

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1.8 Appendix

Proof of **Theorem 3:**

Proof. We denote the expected winning prize of a firm in industry I as V_I^w and the expected losing prize as V_I^l:

$$
\begin{aligned}
{V_1}^w &= \mathbb{E}\{(1-p-2q)V_1 + qV_2 + [(1-p-2q)\beta + q(\alpha-1)](e_c^* + e_d^*)\},\\
{V_2}^w &= \mathbb{E}\{(1-p-2q)V_2 + qV_1 + [(1-p-2q)\beta + q(\alpha-1)](e_a^* + e_b^*)\},\\
{V_1}^l &= \mathbb{E}\{pV_1 + qV_2 + [p\beta + q(\alpha-1)](e_c^* + e_d^*)\},\\
{V_2}^l &= \mathbb{E}\{pV_2 + qV_1 + [p\beta + q(\alpha-1)](e_a^* + e_b^*)\}.
\end{aligned}
$$

Now we define the difference between a firm in industry I's expected winning and losing prizes:

$$\tilde{V_I} = {V_I}^w - {V_I}^l.$$

We show that the symmetric equilibrium investing strategy for firms a and b is the c.d.f.

$$B_1(x) = \frac{[(1-p-2q) - p\alpha - q\beta]}{(1-2p-2q)\alpha}\left[1 - \Big(\frac{\tilde{V_1}}{\tilde{V_1} + 2(1-2p-2q)\alpha x}\Big)^{\frac{1}{2}}\right]$$

over the interval

$$\left[0, \frac{\tilde{V_1}}{2(1-2p-2q)\alpha}\left\{\Big[\frac{(1-p-2q) - p\alpha - q\beta}{(1-p-2q)(1-\alpha) - q\beta}\Big]^2 - 1\right\}\right],$$

and the equilibrium investing strategy for firms c and d is the c.d.f.

$$B_2(x) = \frac{[(1-p-2q) - p\alpha - q\beta]}{2(1-2p-2q)\alpha}\left[1 - \Big(\frac{\tilde{V_2}}{\tilde{V_2} + 2(1-2p-2q)\alpha x}\Big)^{\frac{1}{2}}\right]$$

over the interval

$$\left[0, \frac{\tilde{V_2}}{2(1-2p-2q)\alpha}\left\{\Big[\frac{(1-p-2q) - p\alpha - q\beta}{(1-p-2q)(1-\alpha) - q\beta}\Big]^2 - 1\right\}\right].$$

Let $b_1(x)$ and $b_2(x)$ be p.d.f.s of $B_1(x)$ and $B_2(x)$ respectively. The upper bound of the mixed strategy is

found by solving

$$\frac{[(1-p-2q)-p\alpha-q\beta]}{(1-2p-2q)\alpha}\left[1-\Big(\frac{\tilde{V_1}}{\tilde{V_1}+2(1-2p-2q)\alpha x}\Big)^{\frac{1}{2}}\right]=1,$$

which gives

$$x=\frac{\tilde{V_1}}{2(1-2p-2q)\alpha}\left\{\Big[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\Big]^2-1\right\}.$$

Suppose firms b, c, and d are following these strategies while firm a deviates to investing $s\in\left[0,\frac{\tilde{V_1}}{2(1-2p-2q)\alpha}\left\{\Big[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\Big]^2-1\right\}\right]$.
Then investor a, who owns the majority of firm a's shares, gets an expected profit

$$\begin{aligned}\mathbb{E}[\Pi_a(s,e_b^*,e_c^*,e_d^*)]=&\int_0^s\int_0^{\frac{\tilde{V_2}}{2(1-2p-2q)\alpha}\left\{\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2-1\right\}}\int_0^{\frac{\tilde{V_2}}{2(1-2p-2q)\alpha}\left\{\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2-1\right\}}\\&\Big\{(1-p-2q)V_1+qV_2+[(1-p-2q)\beta+q(\alpha-1)](e_c+e_d)\\&-[(1-p-2q)(1-\alpha)-q\beta]s-[p-(1-p-2q)\alpha-q\beta]e_b\Big\}b_1(e_b)b_2(e_c)b_2(e_d)\,de_c\,de_d\,de_b\\+\int_s^{\frac{\tilde{V_1}}{2(1-2p-2q)\alpha}\left\{\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2-1\right\}}&\int_0^{\frac{\tilde{V_2}}{2(1-2p-2q)\alpha}\left\{\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2-1\right\}}\int_0^{\frac{\tilde{V_2}}{2(1-2p-2q)\alpha}\left\{\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2-1\right\}}\\&\Big\{pV_1+qV_2+[p\beta+q(\alpha-1)](e_c+e_d)-[(1-p-2q)-p\alpha-q\beta]s\\&-[p(1-\alpha)-q\beta]e_b\Big\}b_1(e_b)b_2(e_c)b_2(e_d)\,de_c\,de_d\,de_b\end{aligned}$$

$$= \int_0^s {V_1}^w - [(1-p-2q)(1-\alpha) - q\beta]s - [p-(1-p-2q)\alpha - q\beta]e_b b_1(e_b)\, de_b$$

$$+ \int_s^{\frac{\tilde{V_1}}{2(1-2p-2q)\alpha}\left\{\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2-1\right\}} {V_1}^l - [(1-p-2q)-p\alpha-q\beta]s - [p(1-\alpha)-q\beta]e_b b_1(e_b)\, de_b$$

$$= \frac{2\tilde{V_1}[(1-p-2q)(1-\alpha)-q\beta][p(1-\alpha)-q\beta]}{[2(1-2p-2q)\alpha]^2}\left\{1-\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2\right\}$$

$$+\frac{[p(1-\alpha)-q\beta]{V_1}^w - [p-(1-p-2q)\alpha-q\beta]{V_1}^l}{(1-2p-2q)\alpha}.$$

Notice that the above expression is independent of s, so firm a has no incentive to deviate from $B_1(x)$. Deviating to $s > \frac{\tilde{V_1}}{2(1-2p-2q)\alpha}\left\{\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2-1\right\}$ will make firm a win with probability 1 but lowers investor a's expected payoff. Thus randomizing using $B_1(x)$ is optimal. Performing the same calculation for firms b, c, and d leads to the desired conclusion.

The next step is to find $\tilde{V_1}$ and $\tilde{V_2}$. To do so, we compute the expected R&D investment of each firm. For firms a and b,

$$\mathbb{E}[e_a^*] = \mathbb{E}[e_b^*]$$

$$= \int_0^{\frac{\tilde{V_1}}{2(1-2p-2q)\alpha}\left\{\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2-1\right\}} 1 - B_1(x)\, dx$$

$$= \int_0^{\frac{\tilde{V_1}}{2(1-2p-2q)\alpha}\left\{\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2-1\right\}} 1 - \frac{(1-p-2q)-p\alpha-q\beta}{(1-2p-2q)\alpha}\left[1-\left(\frac{\tilde{V_1}}{\tilde{V_1}+2(1-2p-2q)\alpha x}\right)^{\frac{1}{2}}\right] dx$$

$$= -\frac{\tilde{V_1}}{(1-2p-2q)\alpha} + \frac{2[(1-p-2q)(1-\alpha)-q\beta]\tilde{V_1}}{[2(1-2p-2q)\alpha]^2}\left\{\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2-1\right\}$$

$$= \frac{\tilde{V_1}}{2[(1-p-2q)(1-\alpha)-q\beta]}.$$

Similarly, the expected efforts of firms c and d equal

$$\mathbb{E}[e_c^*] = \mathbb{E}[e_d^*]$$

$$= \int_0^{\frac{\tilde{V}_2}{2(1-2p-2q)\alpha}\left\{\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2-1\right\}} 1 - B_2(x)\,dx$$

$$= \int_0^{\frac{\tilde{V}_2}{2(1-2p-2q)\alpha}\left\{\left[\frac{(1-p-2q)-p\alpha-q\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2-1\right\}} 1 - \frac{(1-p-2q)-p\alpha-q\beta}{(1-2p-2q)\alpha}\left[1-\left(\frac{\tilde{V}_2}{\tilde{V}_2+2(1-2p-2q)\alpha x}\right)^{\frac{1}{2}}\right] dx$$

$$= \frac{\tilde{V}_2}{2[(1-p-2q)(1-\alpha)-q\beta]}.$$

Then we can express $\tilde{V_1}$ in terms of V_1 and $\tilde{V_2}$.

$$\begin{aligned}\tilde{V_1} &= \mathbb{E}\{(1-p-2q)V_1 + qV_2 + [(1-p-2q)\beta + q(\alpha-1)](e_c^* + e_d^*)\} - \mathbb{E}\{pV_1 + qV_2 + [p\beta + q(\alpha-1)](e_c^* + e_d^*)\} \\ &= (1-2p-2q)V_1 + (1-2p-2q)\beta\frac{\tilde{V_2}}{(1-p-2q)(1-\alpha)-q\beta},\end{aligned}$$

For $\tilde{V_2}$, we have

$$\begin{aligned}\tilde{V_2} &= \mathbb{E}\{(1-p-2q)V_2 + qV_1 + [(1-p-2q)\beta + q(\alpha-1)](e_c^* + e_d^*)\} - \mathbb{E}\{pV_1 + qV_2 + [p\beta + q(\alpha-1)](e_c^* + e_d^*)\} \\ &= (1-2p-2q)V_2 + (1-2p-2q)\beta\frac{\tilde{V_1}}{(1-p-2q)(1-\alpha)-q\beta}.\end{aligned}$$

Now we can solve this system to express $\tilde{V_1}$ and $\tilde{V_2}$ in terms of V_1 and V_2:

$$\begin{aligned}\tilde{V_1} &= \frac{(1-2p-2q)V_1 + \frac{(1-2p-2q)\beta}{(1-p-2q)(1-\alpha)-q\beta}(1-2p-2q)V_2}{1-\left[\frac{(1-2p-2q)\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2}, \\ \tilde{V_2} &= \frac{(1-2p-2q)V_2 + \frac{(1-2p-2q)\beta}{(1-p-2q)(1-\alpha)-q\beta}(1-2p-2q)V_1}{1-\left[\frac{(1-2p-2q)\beta}{(1-p-2q)(1-\alpha)-q\beta}\right]^2}.\end{aligned}$$

This yields the desired result. □

Chapter 2

Strategic Vote Splitting in Elections

Abstract

We study vote splitting in elections. Vote splitting refers to the phenomenon that the participation of candidates with little chance of winning has the effect of splitting votes from more popular candidates. We develop a model of an electoral game with two parties, each consisting of one strong candidate and some weak candidates. We analyze strong candidates' incentives to give side transfers to weak candidates to manipulate their campaign participation decisions. We then derive the winning probabilities of the strong candidates in equilibrium. We show that with one weak candidate in each party, when candidates from the two parties have symmetric support rates, the side transfers that one weak candidate receives are independent of the campaign cost and bias of the other weak candidate. We also study the role of uncertain campaign costs of weak candidates. In particular, we show that if the campaign cost of one party's weak candidate is uncertain when strong candidates decide side transfers, then the strong candidate from the other party will be more likely to win than in the case with full information. Therefore, the lack of information makes a strong candidate win suboptimally too often.

2.1 Introduction

In elections, the participation of candidates with little chance of winning has the effect of splitting votes from more popular candidates. This phenomenon is known as the "spoiler effect."

As an example, in the 1992 U.S. presidential election, Ross Perot ran as an independent candidate and won over 19.7 million votes, an 18.9% share of the popular vote, despite not getting any electoral votes from any state. His participation might have helped George Bush (although the result was unsuccessful), because according to a survey, 51% of Perot voters preferred Clinton as a second choice compared to 42% for Bush (Lavelle and Thomas 2023). So Ross Perot helped George Bush split away votes that would have given to Bill Clinton if Ross Perot had not run. The Ross Perot case is not the only one where vote splitting changes the winner. In fact, vote splitting is such a severe concern in the US Presidential Elections that Poundstone (2013) claims that at least 5 of these were won by the second most popular candidate due to the presence of spoilers.

Recognizing this effect, a strong candidate can strategically spoil elections by supporting spoilers. He can support weak candidates from the other party by funding their campaigns. As an example, in the recent 2024 U.S. Presidential Election, Rissman (2024) claims that Democrats hope to "siphon votes from Trump by pushing to get anti-abortion third-party candidate on ballot." They helped Randell Terry, a pro-life candidate, by funding his ballot access. As another example, in 2002, New Mexico's Republican Party chair, John Dendahl, proposed to give over $100,000 to the party in exchange for Green Party candidates running for election in two New Mexico congressional districts. Since Green Party's supporters tend to be left, this attempt could split votes away from Democrats.

To formalize the strategic tension, we develop a model of an electoral game with two parties, each consisting of one strong candidate and one weak candidate. Strong candidates give money to weak candidates from the opposite party to encourage them to run and also give money to weak candidates from his own party to ask them to stay out of the election. This then creates a contest between strong candidates in influencing each weak candidate's election participation decision. If a weak candidate decides to run, the strong candidate then decides whether to fund his campaign. The weak candidate will only publicly announce his participation decision if the strong candidate from the other party funds his campaign. We characterize equilibria in a variety of settings that differ in the number of weak candidates and their support rates. Furthermore, we analyze the role of information, where strong candidates may not know the exact amount of campaign cost of weak candidates at the time of sending side transfers.

2.2 Literature Review

Arrow (1951) identifies a fundamental dilemma in voting theory called "Arrow's Impossibility Theorem." The theorem suggests that any reasonable voting rule that satisfies unanimity and non-dictatorship suffers from the violation of the Independence of Irrelevant Alternatives axiom, which means the addition or removal of weak candidates can change results of elections. This is precisely the spoiler effect. Dutta et al. (2001) call the phenomenon that candidates run for elections not to win but just to change the winner "strategic candidacy." They conclude that all reasonable voting procedures suffer from this. Ishihara and Miura (2016) derive conditions under which minor candidates are kingmakers in a three-candidate model. Poundstone (2013) writes a book on election manipulation, giving detailed examples of historical incidents of vote splitting in US elections as well as examples where politicians strategically use spoilers to weaken their rivals.

Early game-theoretic works on election participation and competition include Osborne and Slivinski (1996) and Besley and Coate (1997). In their models, citizens choose endogenously whether to run for election and entry is costly. Building upon these models, Felli and Merlo (2007) analyze a game with two candidates in an election as well as two lobbyists from opposite parties choosing whose campaign to fund. They show that in equilibrium each lobbyist will fund the campaign of the candidate whose policy preferences are more aligned with his own, but will also make transfers to the elected candidate when the candidate from the other party wins in the election in order to compromise his policy. Our model is related to their seminal work, but differs from theirs in that in our model each strong candidate funds the weak candidate from the other party in order to reduce the number of votes his strong rival can get, and also gives money to the weak candidate from his own party to encourage him to stay away from the election. Some other early theoretical works on political lobbying use all-pay auctions as the model (Baye, Kovenock, and de Vries (1993), Che and Gale (1998)), but we use Tullock contests instead.

Klumpp and Polborn (2006) develop a model in which two candidates decide how to allocate campaign expenditures across different districts. In each district, a candidate's probability of success as a function of both candidates' expenditures follows a generalized Tullock contest success function. The candidate who wins in the majority of districts wins the election. They show that when contests in all districts are simultaneous, then in equilibrium each candidate spends an identical amount of money in each district. However, there can be significant asymmetric expenditure allocations when the contests are sequential. Compared to their model, in our model candidates take part in simultaneous contests to influence the election participation decisions of weak candidates, and the strong candidates also need to decide whether or not to fund the campaign of the weak candidate from his rival party if he agrees to run for election. We also consider

the effect of bias or imperfect side transfer technology on equilibrium side transfers and strong candidates' winning probabilities.

Dekel et al. (2008) analyze elections where candidates buy votes in a sequential bidding game through upfront binding payments or campaign promises, assuming these activities are legal. Our paper analyzes a mechanism similar to vote buying, but in our model candidates "buy votes" from other candidates rather than voters directly.

We also study the case where one weak candidate's campaign cost is common knowledge while the other's is uncertain at the time of giving side transfers. This is an application of Tullock contests with asymmetric information. Warneryd (2003) studies this problem with a more general contest success function. Interestingly, he shows that the uninformed agent may be more likely to win than the informed agent. Warneryd (2003) studies the same information asymmetry problem with all-pay auctions, and he shows that in this case, both the informed agent and the uninformed agent will win with $\frac{1}{2}$ probability.

2.3 Model

2.3.1 The Baseline Model

In this subsection, we study the case where there is one weak candidate in each party. There are four candidates in an election, S_1, W_1, S_2, and W_2. The first two are in party 1 and the last two are in party 2. The spectrum below is an illustration.

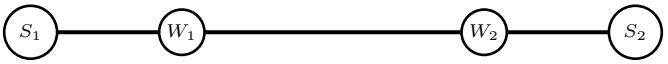


Figure 1

Let $\mathcal{P}$ be the set of strict preferences over the candidates. $S_1 \succ W_1 \succ W_2 \succ S_2$ means the voter will vote for S_1 if S_1 is in, otherwise will vote for W_1 if he is in, otherwise will vote for W_2 if he is in, and otherwise will vote for S_2 if he is in. Let $\mu(\cdot) : \mathcal{P} \to [0, 1]$ be the function that maps each preference from $\mathcal{P}$ to the proportion of the population with such a preference. Assuming that $\alpha > 0.25$ and restricting our attention

to single-peaked preferences, we have

$$\begin{cases} \mu(S_1 \succ W_1 \succ W_2 \succ S_2) = \alpha \\ \mu(W_1 \succ S_1 \succ W_2 \succ S_2) = 0.5 - \alpha \\ \mu(S_2 \succ W_2 \succ W_1 \succ S_1) = \alpha \\ \mu(W_2 \succ S_2 \succ W_1 \succ S_1) = 0.5 - \alpha \end{cases}$$

Here S_1 and S_2 are the strong candidates in their parties while W_1 and W_2 are the weak candidates. The candidate who gets the most votes wins the election, and if there is a tie, each tied candidate wins with probability $\frac{1}{2}$.

Let $\mathbb{C}$ be the set of candidates that decide to participate in the election. The reward of winning the election is V for each player. S_1 and S_2 are deep-pocketed and face campaign costs c_{s1} and c_{s2} respectively. W_1 and W_2 face campaign costs c_{w1} and c_{w2} respectively. Weak candidates find the election costly because they have no chance of winning as long as some strong candidates also run, but they need to spend resources to gain support. We assume c_{wi} to be common knowledge and $c_{wi} < \frac{V}{2}$ for all i. The timeline of the game is as follows:

Time 1: Each strong candidate decides whether to run for election or not. If neither decides to run, then each gets payoff 0. If, say, S_i runs but S_j does not, then S_i wins and gets $V_s - c_{si}$ while S_j gets 0.

Time 2: If both decide to run, strong candidate S_i gives transfer t_{ij} to each weak candidate W_j.

Time 3: Weak candidate W_j weighs side transfers t_{ij} and t_{jj} and decides whether to run or not according to a Tullock contest success function with bias. Let $S_{wi} = \{P, N\}$ be the set of actions for W_i, where P means participating in the election, and N denotes not participating. The decision rule for the weak candidate W_i is the following:

$$\begin{cases} \mathbb{P}(s_{wi} = P) = \frac{\delta_i t_{ji}}{t_{ii} + \delta_i t_{ji}} \\ \mathbb{P}(s_{wi} = N) = \frac{t_{ii}}{t_{ii} + \delta_i t_{ji}} \end{cases}$$

if $(t_{ii}, t_{ji}) \neq (0, 0)$, and

$$\mathbb{P}_{wi}(s_{wi} = P) = 0$$

otherwise. In the contest success function above, $0 < \delta_i \leq 1$ is W_i's bias against the transfer given by S_j,

which is a measure of his loyalty to his party. An alternative interpretation is that the side transfer to a candidate from the rival party induces some costs in order to be kept secret (i.e., the side transfer technology is imperfect), which resembles a common assumption in principal-agent models with collusion as analyzed by, for example, Tirole (1992). If $\delta_i = 1$, then W_i has no bias or the side transfer technology is perfect. W_j tells his decision to S_i and S_j but not the public. Furthermore, this decision rule can be viewed as a version of the standard Luce's choice rule in decision theory (Luce 1959).

Time 4: If W_j tells S_i from the rival party that he is willing to run, S_i decides whether or not to fund W_j's campaign by giving an additional c_{wj} to W_j. If he funds, then W_j will publicly announce that he will run; if he does not, then W_j will publicly announce that he will not run.

Time 5: The election takes place. Each strong player's payoff is as follows.

The payoff of S_i when he gives side transfers t_{ii} and t_{ij} to W_i and W_j respectively is

$$u_{si}(t_{ii}, t_{ij}) = \begin{cases} V - c_{si} - t_{ii} - t_{ij} & \text{if } \mathbb{C} = \{S_i, W_j, S_j\} \\ \frac{1}{2}V - c_{si} - t_{ii} - t_{ij} & \text{if } \mathbb{C} = \{S_i, S_j\} \text{ or } \{S_i, W_i, W_j, S_j\} \\ -c_{si} - t_{ii} - t_{ij} & \text{if } \mathbb{C} = \{S_i, W_i, S_j\}. \end{cases}$$

Let $\pi_{si}(t_{ii}, t_{ij}) : [0, \infty)^2 \to [0, 1]$ be S_i's probability of winning when he gives side transfers t_{ii} and t_{ij}.

Let us consider the simplest case as an example. Suppose there is no bias and weak candidates have no campaign costs. Then the game becomes a two-battle Tullock contest. S_1's payoff can be written as follows:

$$\begin{aligned} U_{s1}(t_{11}, t_{12}) = V \cdot \left(\frac{t_{11}}{t_{11} + t_{21}}\right) \cdot \left(\frac{t_{12}}{t_{12} + t_{22}}\right) \\ + \frac{V}{2}\left(\frac{t_{21}}{t_{11} + t_{21}}\right) \cdot \left(\frac{t_{12}}{t_{12} + t_{22}}\right) \\ + \frac{V}{2} \cdot \left(\frac{t_{11}}{t_{11} + t_{21}}\right) \cdot \left(\frac{t_{22}}{t_{12} + t_{22}}\right) - c_{s1} - t_{11} - t_{12} \end{aligned}$$

S_2's payoff is the following:

$$\begin{aligned} U_{s2}(t_{21}, t_{22}) = V \cdot \left(\frac{t_{21}}{t_{11} + t_{21}}\right) \cdot \left(\frac{t_{22}}{t_{12} + t_{22}}\right) \\ + \frac{V}{2}\left(\frac{t_{21}}{t_{11} + t_{21}}\right) \cdot \left(\frac{t_{12}}{t_{12} + t_{22}}\right) \\ + \frac{V}{2} \cdot \left(\frac{t_{11}}{t_{11} + t_{21}}\right) \cdot \left(\frac{t_{22}}{t_{12} + t_{22}}\right) - c_{s2} - t_{21} - t_{22} \end{aligned}$$

Solving the first-order conditions gives

$$t_{ij} = \frac{V}{8}$$

for all i and j. So each strong candidate expects to win with probability $\frac{1}{2}$ and expects a payoff of $\frac{V}{4}$, while each weak candidate also gets a total transfer $\frac{V}{4}$. Now we characterize the equilibrium in the general setting.

Theorem 1: When the individual rationality conditions are satisfied, there is a unique subgame perfect equilibrium where both strong candidates run for election. In this equilibrium, side transfers t_{ij}^* satisfy

$$\frac{t_{ii}}{t_{ji}} = \frac{V}{V - 2c_{wi}} \geq 1$$

for all i and j.

Each strong candidate funds the campaign cost of the weak candidate from the rival party if he is willing to run. The equilibrium chance of winning for S_1 is

$$\pi_{s1} = \frac{1}{2}\left[\frac{V}{V + \delta_1(V - 2c_{w1})} + \frac{\delta_2(V - 2c_{w2})}{V + \delta_2(V - 2c_{w2})}\right].$$

The equilibrium chance of winning for S_2 is

$$\pi_{s2} = \frac{1}{2}\left[\frac{\delta_1(V - 2c_{w1})}{V + \delta_1(V - 2c_{w1})} + \frac{V}{V + \delta_2(V - 2c_{w2})}\right].$$

Proof. Please see **Appendix A**. □

The equilibrium in **Theorem 1** has some interesting features highlighted in the following observations. The first one is an independence observation.

Observation 1: The side transfers given to one weak candidate are independent of the campaign cost or bias of the other weak candidate.

The second observation gives an intuition for equilibrium side transfers' ratios.

Observation 2: the ratio of equilibrium side transfers is precisely the ratio of "net rewards."

To see this, notice that

$$\frac{t_{ii}}{t_{ji}} = \frac{V}{V - 2c_{wi}} = \frac{\frac{V}{2}}{\frac{V}{2} - c_{wi}}.$$

When W_1 and W_2 have the same support rate, it is as if we can regard the entire bribing contest as two separate contests where winning the support of each weak candidate is worth $\frac{V}{2}$. For S_1, the net reward for winning the support of W_1 is $\frac{V}{2}$, while for S_2 the net reward is $\frac{V}{2} - c_{w1}$ because he needs to fund W_1's campaign.

We refer to this observation as the "net reward property."

The third observation compares strong candidates' chances of winning.

Observation 3: When $\delta_1 = \delta_2$, if $c_{w1} < c_{w2}$, then S_2's chance of winning in equilibrium is higher than S_1's.

Proof. The equilibrium chance of winning for S_1 is

$$\begin{aligned} &\frac{1}{2}\left[\frac{V}{V+\delta(V-2c_{w1})} + \frac{\delta(V-2c_{w2})}{V+\delta(V-2c_{w2})}\right] \\ <\ &\frac{1}{2}\left[\frac{V}{V+\delta(V-2c_{w2})} + \frac{\delta(V-2c_{w2})}{V+\delta(V-2c_{w2})}\right] \\ =\ &\frac{1}{2}. \end{aligned}$$

□

Intuitively, if a weak candidate has a low campaign cost, then the strong candidate from the other party can devote a large amount of money in influencing the weak candidate's participation decision, knowing that he would not need to pay much funding afterwards.

2.3.2 Asymmetric Support Rates

In this subsection, we consider the case where the support rates of strong candidates differ. Let the population have the following masses of preferences:

$$\begin{cases} \mu(S_1 \succ W_1 \succ W_2 \succ S_2) = \alpha \\ \mu(W_1 \succ S_1 \succ W_2 \succ S_2) = 0.5 - \alpha \\ \mu(S_2 \succ W_2 \succ W_1 \succ S_1) = \beta \\ \mu(W_2 \succ S_2 \succ W_1 \succ S_1) = 0.5 - \beta \end{cases}$$

Without loss of generality, let $\alpha > \beta$. This means S_1 is more popular than S_2, while W_2 is more popular than W_1. An analytical solution for the general model is hard to obtain, but the following proposition gives it under some simplifying assumptions.

Proposition 1: If the individual rationality constraints are satisfied, there exists a unique equilibrium in which the "net reward" property is satisfied for side transfers to W_1 but not W_2:

$$\frac{t_{11}}{t_{21}} = \frac{V}{V - 2c_{w1}}$$

while t_{22} may be lower than t_{22}. In the absence of weak candidates' campaign costs, if the individual rationality constraints are satisfied, then the equilibrium transfers are further simplified to the following:

$$\begin{cases} t_{11}^* = \frac{V}{16} \\ t_{12}^* = \frac{3V}{16} \\ t_{21}^* = \frac{V}{16} \\ t_{22}^* = \frac{3V}{16}. \end{cases}$$

S_1 wins with probability $\frac{5}{8}$ and S_2 wins with probability $\frac{3}{8}$.

Proof. Please see **Appendix A**, where we also give a complete characterization of equilibrium side transfers in the general case. □

Since it would have been a tie without spoilers, this result suggests that a strong candidate who is more popular than his major rival can use vote splitting to gain an additional probability of winning of around 12.5%. Notice that in the previous model where the two strong candidates are equally popular, when other parameters are also symmetric, then if they both strategically split votes away from each other, they would still win with 50% probability in equilibrium. The key difference driving this advantage is that for S_1, winning one contest and losing the other no longer always yields a tie: if both strong candidates fail to persuade the weak candidate in their own party, then S_1 will win when $\alpha > \beta$ while they would be tied when $\alpha = \beta$.

Moreover, the intuition behind the result that the"net reward property" holds for side transfers to W_1 is the following: for S_1, winning the support of W_1 is worth $\frac{V}{2}$ if he loses W_2 and nothing otherwise; for S_2, winning it is worth $\frac{V}{2} - c_{w1}$ if S_1 loses W_2 and nothing otherwise. Therefore, the ratio of their side transfers equals the ratio of *conditional* net rewards.

Furthermore, it is not always true that $t_{22} \geq t_{12}$. If, for example, $c_{w2} \to 0$ while c_{w1} is relatively high, then the leading strong candidate S_1 will give W_2 an amount higher than that of the trailing strong candidate. The intuition is that if S_1 does not need to save much money to fund W_2's campaign, then knowing that

persuading W_2 will secure himself a win, he will be willing to give a very generous side transfer to W_2. But for S_2, winning the support of W_2 does not secure a win: it could be a tie if W_1 does not run. Thus it is possible that $t_{12} > t_{22}$. A sufficient condition for $t_{22} > t_{12}$ is $c_{w1} = c_{w2} > 0$.

2.3.3 Asymmetric Distribution of Weak Candidates

In this subsection, we consider the case where there are two weak candidates in one party while the other party only has one. Party 1 has one weak candidate W_1, whose support rate is α, while party 2 has two weak candidates W_2 and W_3, whose support rates sum to α. W_1 has a campaign cost c_{w1}, W_2 has a campaign cost kc_{w2} ($k \leq \frac{1}{2}$), and W_3 has a campaign cost $(1-k)c_{w2}$. As before, we assume $\frac{V}{2} > c_{wi}$ for all i. The masses of single-peaked preferences are the following:

$$\begin{cases} \mu(S_1 \succ W_1 \succ W_2 \succ W_3 \succ S_2) = \alpha \\ \mu(W_1 \succ S_1 \succ W_2 \succ W_3 \succ S_2) = 0.5 - \alpha \\ \mu(S_2 \succ W_3 \succ W_2 \succ W_1 \succ S_1) = \alpha \\ \mu(W_2 \succ S_2 \succ W_3 \succ W_1 \succ S_1) = \gamma(0.5 - \alpha) \\ \mu(W_3 \succ S_2 \succ W_2 \succ W_1 \succ S_1) = (1-\gamma)(0.5 - \alpha), \end{cases}$$

where $\gamma \leq 0.5$.

The following spectrum illustrates the situation.

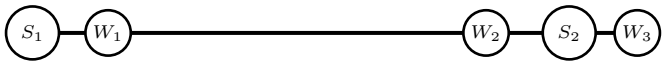


Figure 2

As before, let t_{ij} be the side transfer that S_i gives to W_j. From S_1's perspective, if W_1 is not willing to run while W_2 is willing to run, then to guarantee a win, he only needs to fund W_2's campaign regardless of whether W_3 is willing to run. This is clearly his optimal strategy in this case. Funding W_2's campaign will give him $V - kc_{w2} - t_{12} - t_{13} - t_{11} - c_{s1}$. If W_1 and W_2 are both unwilling to run while W_3 is willing, then by funding W_3's campaign, S_1 will get $V - (1-k)c_{w2} - t_{12} - t_{13} - t_{11} - c_{s1}$. By not funding it, he gets $\frac{V}{2} - t_{12} - t_{13} - t_{11} - c_{s1}$. Notice $V - (1-k)c_{w2} - t_{12} - t_{13} - t_{11} - c_{s1} > \frac{V}{2} - t_{12} - t_{13} - t_{11} - c_{s1}$, so funding W_3's campaign is optimal in this case. By the same logic, if W_1 is willing to run, S_1 will fund both W_2's and W_3's campaign if they are both willing to run, while he will fund neither of their campaigns otherwise because funding one of these will still give him a guaranteed loss. S_1 will get $\frac{V}{2}$ if none of the weak candidates runs.

Therefore, S_1's expected utility function is

$$\begin{aligned}
U_{s1}(t_{11}, t_{12}, t_{13}) = &\left(V - kc_{w2}\right) \cdot \left(\frac{t_{11}}{t_{11}+t_{21}}\right) \cdot \left(\frac{t_{12}}{t_{12}+t_{22}}\right) \\
&+\left(V - (1-k)c_{w2}\right) \cdot \left(\frac{t_{11}}{t_{11}+t_{21}}\right) \cdot \left(\frac{t_{22}}{t_{12}+t_{22}} \cdot \frac{t_{13}}{t_{13}+t_{23}}\right) \\
&+\frac{V}{2} \cdot \left(\frac{t_{11}}{t_{11}+t_{12}}\right) \cdot \left(\frac{t_{22}}{t_{12}+t_{22}} \cdot \frac{t_{23}}{t_{13}+t_{23}}\right) \\
&+\left(\frac{V}{2} - c_{w2}\right) \cdot \left(\frac{t_{21}}{t_{11}+t_{21}}\right) \cdot \left(\frac{t_{12}}{t_{12}+t_{22}} \cdot \frac{t_{13}}{t_{13}+t_{23}}\right) \\
&-t_{11} - t_{12} - t_{13} - c_{s1}
\end{aligned}$$

For S_2, if W_1 is willing to run for election and is funded while and W_2 and W_3 do not both run, then S_2 will win for sure and get $V - c_{w2} - t_{22} - t_{23} - t_{21} - c_{s2}$. S_2 will get at most $\frac{V}{2} - t_{22} - t_{23} - t_{21} - c_{s2}$ if he does not fund W_1's campaign. Since $\frac{V}{2} > c_{w1}$, it is optimal for S_2 to fund it. If W_1 is willing to run while both W_2 and W_3 run (in this case W_2's and W_3's campaigns will both be funded by S_2), then it is optimal for S_2 to fund W_1's campaign and get $\frac{V}{2} - c_{w1} - t_{22} - t_{23} - t_{21} - c_{s2}$, because by not funding he will get $-t_{22} - t_{23} - t_{21} - c_{s2}$. S_2 will get $\frac{V}{2}$ if none of the weak candidates runs for election. S_2 will lose for sure in other cases. Therefore, the expected utility of S_2 is

$$\begin{aligned}
U_{s2}(t_{21}, t_{22}, t_{23}) = &\left(V - c_{w2}\right) \cdot \left(\frac{t_{21}}{t_{11}+t_{21}}\right) \cdot \left(1 - \frac{t_{12}}{t_{12}+t_{22}} \cdot \frac{t_{13}}{t_{13}+t_{23}}\right) \\
&+\frac{V}{2} \cdot \left(\frac{t_{11}}{t_{11}+t_{21}}\right) \cdot \left(\frac{t_{22}}{t_{12}+t_{22}} \cdot \frac{t_{23}}{t_{13}+t_{23}}\right) \\
&+\left(\frac{V}{2} - c_{w2}\right) \cdot \left(\frac{t_{12}}{t_{12}+t_{22}}\right) \cdot \left(\frac{t_{12}}{t_{12}+t_{22}} \cdot \frac{t_{13}}{t_{13}+t_{23}}\right) \\
&-t_{21} - t_{22} - t_{23} - c_{s2} \\
= &\left(\frac{V}{2} - c_{w1}\right) \cdot \left(\frac{t_{21}}{t_{11}+t_{21}}\right) \\
&+\frac{V}{2} \cdot \left(\frac{t_{21}}{t_{11}+t_{21}}\right) \cdot \left(1 - \frac{t_{12}}{t_{12}+t_{22}} \cdot \frac{t_{13}}{t_{13}+t_{23}}\right) \\
&+\frac{V}{2} \cdot \left(\frac{t_{11}}{t_{11}+t_{21}}\right) \cdot \left(\frac{t_{22}}{t_{12}+t_{22}} \cdot \frac{t_{23}}{t_{13}+t_{23}}\right) \\
&-t_{21} - t_{22} - t_{23} - c_{s2}
\end{aligned}$$

To solve for the equilibrium, we take first-order conditions to the expected utility functions. There are six of them, forming a system of equations. A closed-form solution is hard to attain in this case due to computational complexity. The following proposition gives the analytical solution when we make some

simplifying assumptions.

Proposition 2: In the absence of campaign costs for weak candidates and biases, if the individual rationality constraints are satisfied, then the equilibrium transfers are the following:

$$\begin{cases} t_{11}^* = \frac{3V}{16} \\ t_{12}^* = \frac{V}{16} \\ t_{13}^* = \frac{V}{16} \\ t_{21}^* = \frac{3V}{16} \\ t_{22}^* = \frac{V}{16} \\ t_{23}^* = \frac{V}{16}. \end{cases}$$

Proof. This result follows from taking first-order conditions of the above expected utility functions when letting $c_{w1} = c_{w2} = 0$. It is straightforward to check that the second-order conditions are satisfied. □

Notice that here S_1 and S_2 send an equal amount of money to each weak candidate in party 1, and they both win with probability 0.5. Also notice that W_1 collects more money than the sum of that of W_2 and W_3. This is because his participation can more easily change the result of the election than that of W_2 or W_3.

Now we show that adding identical campaign costs to weak candidates will break the symmetry. Let us assign the following values to our parameters:

$$V = 100,$$

$$c_{w1} = c_{w2} = 3,$$

$$c_{s1} = c_{s2} = 5$$

$$k = 0.1.$$

Then equilibrium side transfers are the following:

$$\begin{cases} t_{11}^* = 18.7214. \\ t_{12}^* = 6.2040 \\ t_{13}^* = 5.8896 \\ t_{21}^* = 17.9921 \\ t_{22}^* = 6.2535 \\ t_{23}^* = 6.2451 \end{cases}$$

The equilibrium expected utilities of the strong candidates are

$$\begin{cases} U_{s1}^* = 13.7469, \\ U_{s2}^* = 12.7102. \end{cases}$$

So the individual rationality conditions are satisfied. The winning probabilities are

$$\begin{cases} \pi_{s1} = 0.5033, \\ \pi_{s2} = 0.4967. \end{cases}$$

There are three interesting observations in this equilibrium. First, S_1, who is the strong candidate with only one weak candidate in his party, is slightly more likely to win the election. The intuition for this is that when W_1 is not willing to run while one or more of W_2 or W_3 are willing, S_1 can save money by only funding the cheaper campaign while still guaranteeing a win. Knowing that he can save money in this scenario, he can then use some of the saved money as an extra boost in the side transfer in the earlier stage, which makes him more likely to win.

Second, the strongest weak candidate, W_1, collects more money than the sum of transfers given to both W_2 and W_3. This phenomenon is also present in the previous case when $c_{w1} = c_{w2} = 0$. Thus it is profitable to be the monopolist among weak candidates.

Lastly, although W_3's campaign cost is 9 times that of W_2, they receive similar amounts of influence money, with W_2 getting marginally more. This is because sending money to W_3 can be wasteful: when W_1 is not willing to run and both W_2 and W_3 are willing to run, his willingness will be neglected by S_1. On the other hand, sending money to him is still important because, in slightly less than half of the time, W_1 will be

willing to run, and then W_3' willingness becomes quite important.

Moreover, one interesting result in the baseline model is that the campaign cost of the weak candidate in one party does not affect the side transfers that the weak candidate in the other party collects. However, this result fails to hold in the asymmetric model.

Observation 4: The independence observation fails to hold in this setting.

Proof. Let us change c_{w2} from 3 to 4. The new equilibrium side transfers become

$$\begin{cases} t_{11}^* = 18.7153 \\ t_{12}^* = 6.2074 \\ t_{13}^* = 5.8899 \\ t_{21}^* = 17.7365 \\ t_{22}^* = 6.2548 \\ t_{23}^* = 6.2453 \end{cases}$$

Clearly, all side transfers change as a result. Therefore, we conclude that the independence observation fails to hold in the multiple weak candidates case. □

2.3.4 The Role of Information

In this section, we introduce uncertainty about the weak candidates' campaign costs. To see how information about weak candidates harms or benefits strong candidates, we consider a few cases below.

In the first case, the set of candidates and their respective support rates are the same as in the baseline model. The campaign cost of W_1 is c_{w1}, which is common knowledge. The campaign cost of W_2 is a random variable $\tilde{c}_{w2} \sim F(\cdot)$ over the range $[0, \bar{c}_{w2}]$, where $F(\cdot)$ is its c.d.f. function that is continuous and differentiable with support $[0, \bar{c}_{w2}]$, where $\bar{c}_{w2}$ is taken to be small enough to satisfy the individual rationality constraint of S_1. We assume that the campaign cost of W_2 is not known to anyone in **Time 2** when S_1 and S_2 decide side transfers, but will be realized in **Time 4**. Let c_{w2} be the realization of $\tilde{c}_{w2}$. The interpretation is that the weak candidate perhaps has no intention of running for election before, so it takes some time for him to calculate his campaign cost after being influenced to run. It is worth noting that nothing changes if we assume that S_2 observes the realization of $\tilde{c}_{w2}$ in **Time 2** since his optimization problem does not involve

c_{w2}. Let

$$\pi_{si}^{u} = \int_{0}^{\bar{c}_{w2}} \pi_{si}(t_{ii}, t_{ij}) dc_{w2}$$

be S_i's expected probability of winning when c_{w2} is realized in **time 4**, where u indicates "uncertain." and let

$$\pi_{si}^{f} = \int_{0}^{\bar{c}_{w2}} \pi_{si}(t_{ii}(c_{w2}), t_{ij}(c_{w2})) dc_{w2}$$

be S_i's expected winning probability when c_{w2} is realized in **Time 1** over all possible c_{w2}'s, where "f" indicates "full information."

Proposition 3: The equilibrium in this game is the same as the one in **Theorem 1** except that we replace c_{w2} with $\mathbb{E}[c_{w2}]$ in the characterization.

Proof. First notice that, as we discussed above, the optimization problem of S_2 is independent of c_{w2} as long as it does not break S_1's funding decision.

For S_1, his expected utility is

$$\begin{aligned}
U_{s1}(t_{11}, t_{12}) = &\int_{0}^{\bar{c}_{w2}} (V - c_{w2}) \cdot \left(\frac{t_{11}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left(\frac{\delta_2 t_{12}}{\delta_2 t_{12} + t_{22}}\right) \cdot f(c_{w2})\, dc_{w2} \\
&+ \int_{0}^{\bar{c}_{w2}} \left(\frac{V}{2} - c_{w2}\right)\left(\frac{\delta_1 t_{21}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left(\frac{\delta_2 t_{12}}{\delta_2 t_{12} + t_{22}}\right) \cdot f(c_{w2})\, dc_{w2} \\
&+ \frac{V}{2} \cdot \left(\frac{t_{11}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left(\frac{t_{22}}{\delta_2 t_{12} + t_{22}}\right) - c_{s1} - t_{11} - t_{12} \\
&= (V - \mathbb{E}[c_{w2}]) \cdot \left(\frac{t_{11}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left(\frac{\delta_2 t_{12}}{\delta_2 t_{12} + t_{22}}\right) \\
&+ \left(\frac{V}{2} - \mathbb{E}[c_{w2}]\right)\left(\frac{\delta_1 t_{21}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left(\frac{\delta_2 t_{12}}{\delta_2 t_{12} + t_{22}}\right) \\
&+ \frac{V}{2} \cdot \left(\frac{t_{11}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left(\frac{t_{22}}{\delta_2 t_{12} + t_{22}}\right) - c_{s1} - t_{11} - t_{12},
\end{aligned}$$

so his optimization problem is the same as in the proof of **Theorem 1** except that we change c_{w2} to $\mathbb{E}[c_{w2}]$. □

Our main result in this section is the following theorem, which suggests that uncertainty about the campaign cost of the weak candidate from the other party will make the strong candidate win more often.

Theorem 2: S_1's expected probability of winning when c_{w2} is drawn in **Time 4** is weakly higher than his ex-ante expected probability of winning when c_{w2} is drawn and revealed to everyone in **Time 1**.

Proof. The equilibrium expected chance of winning for S_2 with c_{w2} being realized at **Time 1** (and becoming common knowledge at **Time 1** too) is

$$\begin{aligned}\pi_{s2}^{f} &= \int_0^{\bar{c}_{w2}} \frac{1}{2} \cdot \left[\frac{\delta_1(V-2c_{w1})}{V+\delta_1(V-2c_{w1})} + \frac{V}{V+\delta_2(V-2c_{w2})}\right] \cdot f(c_{w2})\, dc_{w2} \\ &= \frac{1}{2} \cdot \frac{\delta_1(V-2c_{w1})}{V+\delta_1(V-2c_{w1})} + \int_0^{\bar{c}_{w2}} \frac{1}{2} \cdot \frac{V}{V+\delta_2(V-2c_{w2})} \cdot f(c_{w2})\, dc_{w2} \\ &\geq \frac{1}{2} \cdot \left[\frac{\delta_1(V-2c_{w1})}{V+\delta_1(V-2c_{w1})} + \frac{V}{V+\delta_2(V-2\mathbb{E}[c_{w2}])}\right] = \pi_{s2}^{u}.\end{aligned}$$

This follows from Jensen's inequality, noticing that $\frac{V}{V+\delta_2(V-2c_{w2})}$ is convex. Therefore, $\pi_{s1}^{f} \leq \pi_{s1}^{u}$. □

Observation 5: Suppose c_{w2} is drawn from a distribution such that $\mathbb{E}[c_{w2}] = c_{w1}$. Further assume there is no bias. Then S_1 and S_2 win with equal probability under uncertainty, but S_1 wins with weakly lower probability than S_2 under full information.

Proof. The fact that they win with equal probability can be seen from **Proposition 3**, which implies that with uncertainty, S_1 and S_2 both behave as if c_{w2} is fixed and equal to c_{w1}. But then **Theorem 2** shows that S_1 will have a winning probability weakly less than $\frac{1}{2}$ under full information. □

This result is interesting and counterintuitive because in common and interdependent value auctions, bidders tend to bid less aggressively due to uncertainty about other bidders' valuations to avoid the winner's curse. But here, uncertainty will make the strong candidate bribe more aggressively. We first give an example and then give an intuitive explanation for this result.

Let us again assign the following values to our parameters:

$$V = 100$$

$$c_{w1} = 5$$

$$\delta = 1$$

$$c_{s1} = c_{s2} = 5,$$

and let $\tilde{c}_{w2} \sim Uniform[0, 10]$. When there is uncertainty in c_{w2}, S_1 and S_2 each have a 50% chance of winning and have expected payoff 18.9474.

When c_{w2} is still drawn from $Uniform[0, 10]$ but is realized in **Time 1** and becomes common knowledge in that time, we have

$$\pi_{s1}^{f} = \int_0^{10} \frac{1}{2} \cdot \left[\frac{V}{V + \delta_1(V - 2c_{w1})} + \frac{\delta_2(V - 2c_{w2})}{V + \delta_2(V - 2c_{w2})} \right] \cdot \frac{1}{10} \, dc_{w2}$$

$$= 0.4998.$$

The equilibrium chance of winning for S_2 is

$$\pi_{s1}^{u} = \int_0^{10} \frac{1}{2} \cdot \left[\frac{\delta_1(V - 2c_{w1})}{V + \delta_1(V - 2c_{w1})} + \frac{V}{V + \delta_2(V - 2c_{w2})} \right] \cdot \frac{1}{10} \, dc_{w2}$$

$$= 0.5002.$$

The expected utility of S_1 with c_{w2} being realized in **Time 1** is 18.9819, and that of S_2 is 18.9858. Therefore, we can see that uncertainty about W_2's campaign cost actually makes S_1 slightly more likely to win at the expense of a lower expected payoff. S_1 wins sub-optimally too many times with uncertainty as it makes him spend too much on influence money. This uncertainty also harms S_2, although he does not need to fund W_2's campaign, because he only chooses one side transfer under uncertainty, while he can choose different transfers with different realizations of $\tilde{c}_{w2}$ with full information.

To see the intuition behind this result, consider the following discrete example. Let c_{w2} be 0 or $\frac{V}{2}$ each with probability $\frac{1}{2}$ and let $c_{w1} = 0$ be common knowledge. This is a slight departure from the assumption $c_{w2} < \frac{V}{2}$ because for the sake of conciseness the main analysis adopts this assumption to focus on interior solutions. Further assume there is no bias. Then if c_{w2} is revealed in **Time 1**, when $c_{w2} = \frac{V}{2}$, the equilibrium side transfers would be

$$\begin{cases} t_{11}^{*} = \frac{V}{8} \\ t_{12}^{*} = 0 \\ t_{21}^{*} = \frac{V}{8} \\ t_{22}^{*} = 0 \end{cases}$$

when $c_{w2} = \frac{V}{2}$. In this case, S_2 wins W_2's support for sure from the contest success function (i.e., W_2 will not run if neither gives him any side transfer). When $c_{w2} = 0$, we have

$$\begin{cases} t_{11}^* = \frac{V}{8} \\ t_{12}^* = \frac{V}{8} \\ t_{21}^* = \frac{V}{8} \\ t_{22}^* = \frac{V}{8} \end{cases}$$

In this full information case, S_1 wins with probability $\frac{1}{2}$ when $c_{w2} = 0$, and he wins with probability $\frac{1}{4}$ when $c_{w2} = \frac{V}{2}$, so his expected probability of winning is $\frac{3}{8}$.

Now, suppose S_1 chooses t_{12} without observing the realization of c_{w2}. Notice S_2 will fund W_2's campaign if he tells him he is willing to run and it costs $\frac{V}{2}$ because the side transfer is sunk. Then S_1's problem becomes

$$\begin{aligned} U_{s1}(t_{11}, t_{12}) = \frac{1}{2} \cdot \Bigg[& V \cdot \left(\frac{t_{11}}{t_{11} + t_{21}} \right) \cdot \left(\frac{t_{12}}{t_{12} + t_{22}} \right) \\ & + \frac{V}{2} \cdot \left(\frac{t_{21}}{t_{11} + t_{21}} \right) \cdot \left(\frac{t_{12}}{t_{12} + t_{22}} \right) \\ & + \frac{V}{2} \cdot \left(\frac{t_{11}}{t_{11} + t_{21}} \right) \cdot \left(\frac{t_{22}}{t_{12} + t_{22}} \right) \Bigg] \\ + \frac{1}{2} \Bigg[& \frac{V}{2} \cdot \left(\frac{t_{11}}{t_{11} + t_{21}} \right) \cdot \left(\frac{t_{12}}{t_{12} + t_{22}} \right) \\ + \frac{V}{2} & \cdot \left(\frac{t_{11}}{t_{11} + t_{21}} \right) \cdot \left(\frac{t_{22}}{t_{12} + t_{22}} \right) \Bigg] - c_{s1} - t_{11} - t_{12}. \end{aligned}$$

The expected payoff for S_2 is

$$\begin{aligned} U_{s2}(t_{21}, t_{22}) = & V \cdot \left(\frac{t_{21}}{t_{11} + t_{21}} \right) \cdot \left(\frac{t_{22}}{t_{12} + t_{22}} \right) \\ & + \frac{V}{2} \left(\frac{t_{21}}{t_{11} + t_{21}} \right) \cdot \left(\frac{t_{12}}{t_{12} + t_{22}} \right) \\ + \frac{V}{2} & \cdot \left(\frac{t_{11}}{t_{11} + t_{21}} \right) \cdot \left(\frac{t_{22}}{t_{12} + t_{22}} \right) - c_{s2} - t_{21} - t_{22}. \end{aligned}$$

Solving the system of FOCs, we get

$$\begin{cases} t_{11}^* = \frac{V}{8} \\ t_{12}^* = \frac{V}{18} \\ t_{21}^* = \frac{V}{8} \\ t_{22}^* = \frac{V}{9}. \end{cases}$$

In this case, S_1 wins the supports of both W_1 and W_2 with probability $\frac{1}{6}$, he loses both with probability $\frac{1}{3}$, and he wins one and loses one with probability $\frac{1}{2}$. This means his expected chance of winning is $\frac{5}{12}$. Then we have

$$\pi_{s1}^u - \pi_{s1}^f = \frac{5}{12} - \frac{3}{8} = \frac{1}{24}.$$

Thus, compared to the full information case, uncertainty makes S_1 win with an additional probability of around 4%. The intuition is the following: with uncertainty, S_1 wins with a much higher probability than full information when c_{w2} is high, but with a slightly lower probability when c_{w2} is low. The former outweighs the latter, causing the overall winning probability to increase to a suboptimally high level.

Another possibility of asymmetric information is where S_1 privately observes c_{w2} when he decides side transfers while S_2 does not observe c_{w2} at that time. This means a strong candidate is better informed about the campaign cost of a weak candidate from the other party than the strong candidate from that weak candidate's own party. We believe this case is less likely to be true in practice than the case studied above, so we solve the equilibrium in Appendix B.

2.4 Discussion

Our analysis above shows that the spoiler effect can significantly influence the outcome of an election. Admittedly, our results rely on the assumption that voters vote for their favourite candidate. This may not be true in practice when voters who support weak candidates know that their votes will not make the candidates win and thus will be wasted. This is called the wasted vote effect, which contributes to Duverger's Law: plurality voting tends to create two-party systems. Anticipating that their votes would be wasted if they vote for weak candidates, these voters might vote for the strong candidate in their party instead (Durand et al. 2024).

Is there a voting rule that is more immune to the spoiler effect? Durand et al. (2024) argue that the answer is approval voting, in which a voter can approve of (vote for) as many candidates as he would like and the candidate with the highest support wins. In theoretical modelling, this could correspond to setting a threshold utility, and each voter approves of all candidates whose policies would yield utilities above that threshold. In our model, suppose all voters approve of all candidates that are in his party. Then the two strong candidates would always be tied, which means vote splitting is irrelevant. Indeed, although approval voting still does not satisfy the Independence of Irrelevant Alternatives axiom, it can always select the Condorcet winner (a candidate who can defeat everyone else in one-to-one voting competitions) in three-candidate elections and generate a greater welfare than plurality voting (Durand et al. 2024). Gehrlein and Lepelley (1998) reach a similar conclusion that approval voting has a greater Condorcet efficiency than plurality voting.

Furthermore, we assume that weak candidates enter only to split votes away from strong candidates. In reality, they may enter even knowing that they have little chances of winning because they hope to build experience for future runs. Moreover, Fox and Lawless (2005) argue that a person's decision to run for office is closed related to his general sense of efficacy as a candidate and political upbringing. Nevertheless, our theory provides a theoretical framework for future work on the exploitation of the spoiler effect in elections.

2.5 Conclusion

In this paper, we studied vote splitting in elections with different distributions of support rates and information structures. We showed that when support rates are symmetric, the ratio of equilibrium side transfers given to a weak candidate equals the ratio of the net rewards of winning the support of that candidate. This property fails to hold when one party's strong candidate is more popular than the other party's strong candidate. Moreover, when party 1 has 1 weak candidate and party 2 has 2, then the strong candidate in party 1 has an advantage because he has the extra flexibility of not funding a weak candidate in party 2's campaign when it becomes unnecessary for him to win, and thus he can spend extra money on bribing them. Furthermore, if the strong candidate does not know the campaign cost of the weak candidate from the other party, then he will over-spend in equilibrium and win suboptimally too often.

Future work may consider alternative ways of modelling the situation. For example, the decisions of weak candidates in our model are probabilistic rather than strategic. Some alternative ways of modelling the bribing dynamic include all-pay auctions and regular second-price auctions. Another possibility is to use numerical methods to study more complicated settings where there are more parties or more than 2 weak

candidates in each party. Finally, it might be interesting to generalize our two-party setting to multiple parties.

2.6 References

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2.7 Appendix A

2.7.1 Proof of Theorem 1

Proof. We first consider the optimal choices of t_{11} and t_{12} by S_1. S_1 will win for sure if W_1 does not run for the election while W_2 runs with funding from S_1, because this means he will get $\frac{1}{2}$ of all votes while S_2 will get α. S_1 gets $V - c_{w2} - c_{s1} - t_{11} - t_{12}$ by funding W_2's campaign, and $\frac{V}{2} - c_{w2} - c_{s1} - t_{11} - t_{12}$ by not funding. Since $c_{w2} < \frac{V}{2}$,

$$V - c_{w2} - c_{s1} - t_{11} - t_{12} > \frac{V}{2} - c_{s1} - t_{11} - t_{12},$$

which means it is optimal to fund W_2's campaign if he tells S_1 he is willing to run. This case happens with probability

$$\left(\frac{t_{11}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left(\frac{\delta_2 t_{12}}{\delta_2 t_{12} + t_{22}}\right).$$

He will get $\frac{V}{2} - c_{w2} - c_{s1} - t_{11} - t_{12}$ if both W_1 and W_2 run and he funds the campaign of W_2. If he does not fund W_2, he will get $-c_{s1} - t_{11} - t_{12}$. Since

$$\frac{V}{2} - c_{w2} - c_{s1} - t_{11} - t_{12} > -c_{s1} - t_{11} - t_{12},$$

S_1 will also fund W_2's campaign in this case. This happens with probability

$$\left(\frac{\delta_1 t_{21}}{t_{11} + t_{21}}\right) \cdot \left(\frac{\delta_2 t_{12}}{\delta_2 t_{12} + t_{22}}\right).$$

He will get $\frac{V}{2} - c_{s1} - t_{11} - t_{12}$ if neither W_1 nor W_2 runs. This happens with probability

$$\left(\frac{t_{11}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left(\frac{t_{22}}{\delta_2 t_{12} + t_{22}}\right).$$

He will lose for sure if W_1 runs while W_2 does not. Therefore, his expected utility is

$$\begin{aligned} U_{s1}(t_{11}, t_{12}) = (V - c_{w2}) \cdot \left(\frac{t_{11}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left(\frac{\delta_2 t_{12}}{\delta_2 t_{12} + t_{22}}\right) \\ + \left(\frac{V}{2} - c_{w2}\right)\left(\frac{\delta_1 t_{21}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left(\frac{\delta_2 t_{12}}{\delta_2 t_{12} + t_{22}}\right) \\ + \frac{V}{2} \cdot \left(\frac{t_{11}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left(\frac{t_{22}}{\delta_2 t_{12} + t_{22}}\right) - c_{s1} - t_{11} - t_{12} \end{aligned}$$

The first-order condition for the optimality of t_{11} yields

$$\frac{\partial U_{s1}}{\partial t_{11}} = (V - c_{w2}) \cdot \left[\frac{\delta_1 t_{21}}{(t_{11} + \delta_1 t_{21})^2}\right] \cdot \left(\frac{\delta_2 t_{12}}{\delta_2 t_{12} + t_{22}}\right)$$
$$+\left(\frac{V}{2} - c_{w2}\right)\left[\frac{-\delta_1 t_{21}}{(t_{11} + \delta_2 t_{21})^2}\right] \cdot \left(\frac{\delta_2 t_{12}}{\delta_2 t_{12} + t_{22}}\right) + \frac{V}{2} \cdot \left[\frac{\delta_1 t_{21}}{(t_{11} + \delta_1 t_{21})^2}\right] \cdot \left(\frac{t_{22}}{\delta_2 t_{12} + t_{22}}\right)$$
$$-1 = 0,$$

which simplifies to

$$\frac{V}{2} \cdot \frac{\delta_1 t_{21}}{(t_{11} + \delta_1 t_{21})^2} = 1,$$

so the equilibrium level of transfer t_{11}^* satisfies

$$t_{11}^* = \left(\frac{V}{2}\delta_1 t_{21}\right)^{\frac{1}{2}} - \delta_1 t_{21}. \tag{2.1}$$

The second-order condition that guarantees the optimality of t_{11}^* is

$$-V \cdot \left[\frac{\delta_1 t_{21}}{(t_{11} + \delta_1 t_{21})^3}\right] < 0.$$

The first-order condition for the optimality of t_{12} yields

$$\frac{\partial U_{s1}}{\partial t_{12}} = (V - c_{w2}) \cdot \left[\frac{\delta_2 t_{22}}{(\delta_2 t_{12} + t_{22})^2}\right] \cdot \left(\frac{t_{11}}{t_{11} + \delta_1 t_{21}}\right)$$
$$+\left(\frac{V}{2} - c_{w2}\right) \cdot \left[\frac{\delta_2 t_{22}}{(\delta_2 t_{12} + t_{22})^2}\right] \cdot \left(\frac{\delta_1 t_{21}}{t_{11} + \delta_1 t_{21}}\right) + \frac{V}{2} \cdot \left[\frac{-\delta_2 t_{22}}{(\delta_2 t_{12} + t_{22})^2}\right] \cdot \left(\frac{t_{11}}{t_{11} + \delta_1 t_{21}}\right)$$
$$-1 = 0,$$

which simplifies to

$$\left(\frac{V}{2} - c_{w2}\right) \cdot \left[\frac{\delta_2 t_{22}}{(\delta_2 t_{12} + t_{22})^2}\right] = 1,$$

so in equilibrium,

$$t_{12}^* = \frac{1}{\delta_2}\left\{\left[\left(\frac{V}{2} - c_{w2}\right)\delta_2 t_{22}\right]^{\frac{1}{2}} - t_{22}\right\}. \tag{2.2}$$

The second-order condition requires

$$\left(\frac{V}{2}-c_{w2}\right)\cdot\left[\frac{-2\delta_2^2 t_{22}}{(\delta_2 t_{12}+t_{22})^3}\right]<0.$$

With the same logic, we find that the expected utility function for S_2 is

$$\begin{aligned}U_{s2}(t_{21},t_{22}) &= (V-c_{w1})\cdot\left(\frac{\delta_1 t_{21}}{t_{11}+\delta_1 t_{21}}\right)\cdot\left(\frac{t_{22}}{\delta_2 t_{12}+t_{22}}\right)\\ &+\left(\frac{V}{2}-c_{w1}\right)\left(\frac{\delta_1 t_{21}}{t_{11}+\delta_1 t_{21}}\right)\cdot\left(\frac{\delta_2 t_{12}}{\delta_2 t_{12}+t_{22}}\right)\\ &+\frac{V}{2}\cdot\left(\frac{t_{11}}{t_{11}+\delta_1 t_{21}}\right)\cdot\left(\frac{t_{22}}{\delta_2 t_{12}+t_{22}}\right)-c_{s2}-t_{21}-t_{22}\end{aligned}$$

The first-order condition for the optimality of t_{21} is

$$\begin{aligned}\frac{\partial U_{s2}}{\partial t_{21}} &= (V-c_{w1})\cdot\left[\frac{\delta_1 t_{11}}{(t_{11}+\delta_1 t_{21})^2}\right]\cdot\left(\frac{t_{22}}{\delta_2 t_{12}+t_{22}}\right)\\ &+\left(\frac{V}{2}-c_{w1}\right)\left[\frac{\delta_1 t_{11}}{(t_{11}+\delta_1 t_{21})^2}\right]\cdot\left(\frac{\delta_2 t_{12}}{\delta_2 t_{12}+t_{22}}\right)\\ &+\frac{V}{2}\cdot\left[\frac{-\delta_1 t_{11}}{(t_{11}+\delta_1 t_{21})^2}\right]\cdot\left(\frac{t_{22}}{\delta_2 t_{12}+t_{22}}\right)-1=0,\end{aligned}$$

which simplifies to

$$\left(\frac{V}{2}-c_{w1}\right)\cdot\left[\frac{\delta_1 t_{11}}{(t_{11}+\delta_1 t_{21})^2}\right]=1,$$

so the equilibrium transfer t_{21}^* satisfies

$$t_{21}^*=\frac{1}{\delta_1}\left\{\left[\left(\frac{V}{2}-c_{w1}\right)\delta_1 t_{11}\right]^{\frac{1}{2}}-t_{11}\right\}. \tag{2.3}$$

The second-order condition is

$$\left(\frac{V}{2}-c_{w1}\right)\cdot\left[\frac{-2\delta_1^2 t_{11}}{(t_{11}+\delta_1 t_{21})^3}\right]<0.$$

Finally, the first order condition with respect to t_{22} is given by

$$\begin{aligned}\frac{\partial U_{s2}}{\partial t_{22}} = (V - c_{w1}) \cdot \left(\frac{\delta_1 t_{21}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left[\frac{\delta_2 t_{12}}{(\delta_2 t_{12} + t_{22})^2}\right] \\ + \left(\frac{V}{2} - c_{w1}\right)\left(\frac{\delta_1 t_{21}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left[\frac{-\delta_2 t_{12}}{(\delta_2 t_{12} + t_{22})^2}\right] \\ + \frac{V}{2} \cdot \left(\frac{t_{11}}{t_{11} + \delta_1 t_{21}}\right) \cdot \left[\frac{\delta_2 t_{12}}{(\delta_2 t_{12} + t_{22})^2}\right] \\ -1 = 0,\end{aligned}$$

which simplifies to

$$\frac{V}{2} \cdot \left[\frac{\delta_2 t_{12}}{(\delta_2 t_{12} + t_{22})^2}\right] = 1.$$

This yields

$$t_{22}^* = \left(\frac{V}{2}\delta_2 t_{12}\right)^{\frac{1}{2}} - \delta_2 t_{12}. \tag{2.4}$$

and the second order condition is

$$V \cdot \left[\frac{-\delta_2^2 t_{12}}{(\delta_2 t_{12} + t_{22})^3}\right] < 0.$$

Thus we have four equations, (1) to (4), and four variables:

$$\begin{cases} t_{11}^* = \left(\frac{V}{2}\delta_1 t_{21}^*\right)^{\frac{1}{2}} - \delta_1 t_{21}^* \\ t_{12}^* = \frac{1}{\delta_2}\left\{\left[\left(\frac{V}{2} - c_{w2}\right)\delta_2 t_{22}^*\right]^{\frac{1}{2}} - t_{22}^*\right\} \\ t_{21}^* = \frac{1}{\delta_1}\left\{\left[\left(\frac{V}{2} - c_{w1}\right)\delta_1 t_{11}^*\right]^{\frac{1}{2}} - t_{11}^*\right\} \\ t_{22}^* = \left(\frac{V}{2}\delta_2 t_{12}\right)^{\frac{1}{2}} - \delta_2 t_{12}^* \end{cases}$$

Solving this system of equations, we get

$$\begin{cases} t_{11}^* = \dfrac{V^2\left[\delta_1(V-2c_{w1})\right]}{2\left[V+\delta_1(V-2c_{w1})\right]^2} \\ t_{12}^* = \dfrac{V\left[\delta_2(V-2c_{w2})^2\right]}{2\left[V+\delta_2(V-2c_{w2})\right]^2} \\ t_{21}^* = \dfrac{V\left[\delta_1(V-2c_{w1})^2\right]}{2\left[V+\delta_1(V-2c_{w1})\right]^2} \\ t_{22}^* = \dfrac{V^2\left[\delta_2(V-2c_{w2})\right]}{2\left[V+\delta_2(V-2c_{w2})\right]^2}. \end{cases}$$

These values indeed satisfy the second-order conditions, so the only conditions remaining are the individual rationality conditions for the strong candidates. The individual rationality condition for S_1 is

$$U_{s1}(t_{11},t_{12}) = (\frac{V}{2} - c_{w2})\left[\frac{\delta_2(V-2c_{w2})}{V+\delta_2(V-2c_{w2})}\right] + \frac{V}{2}\left[\frac{V}{V+\delta_1(V-2c_{w1})}\right] - \frac{V^2\left[\delta_1(V-2c_{w1})\right]}{2\left[V+\delta_1(V-2c_{w1})\right]^2} - \frac{V\left[\delta_2(V-2c_{w2})^2\right]}{2\left[V+\delta_2(V-2c_{w2})\right]^2} - c_{s1} \geq 0,$$

so

$$c_{s1} \leq \frac{(V-2c_{w2})^3\delta_2^2}{2\left[V+\delta_2(V-2c_{w2})\right]^2} + \frac{V^3}{2\left[V+\delta_1(V-2c_{w1})\right]^2}.$$

For S_2, the individual rationality constraint is

$$U_{s2}(t_{21},t_{22}) = (\frac{V}{2} - c_{w1}) \cdot \left(\frac{\delta_1(V-2c_{w1})}{V+\delta_1(V-2c_{w1})}\right) + \frac{V}{2} \cdot \left(\frac{V}{\delta_2(V-2c_{w2})+V}\right) - \frac{V\left[\delta_1(V-2c_{w1})^2\right]}{2\left[V+\delta_1(V-2c_{w1})\right]^2} - \frac{V^2\left[\delta_2(V-2c_{w2})\right]}{2\left[V+\delta_2(V-2c_{w2})\right]^2} - c_{s2} \geq 0,$$

which simplifies to

$$c_{s2} \leq \frac{(V-2c_{w1})^3\delta_1^2}{2\Big[V+\delta_1(V-2c_{w1})\Big]^2} + \frac{V^3}{2\Big[V+\delta_2(V-2c_{w2})\Big]^2}.$$

In equilibrium, S_1 wins with probability

$$\begin{aligned}
&1\cdot\left(\frac{t_{11}^*}{t_{11}^*+\delta_1 t_{21}^*}\right)\cdot\left(\frac{\delta_2 t_{12}^*}{\delta_2 t_{12}^*+t_{22}^*}\right)\\
&+\frac{1}{2}\cdot\left(\frac{\delta_1 t_{21}^*}{t_{11}^*+\delta_1 t_{21}^*}\right)\cdot\left(\frac{\delta_2 t_{12}^*}{\delta_2 t_{12}^*+t_{22}^*}\right)\\
&+\frac{1}{2}\cdot\left(\frac{t_{11}^*}{t_{11}^*+\delta_1 t_{21}^*}\right)\cdot\left(\frac{t_{22}^*}{\delta_2 t_{12}^*+t_{22}^*}\right)\\
=&\frac{1}{2}\left[\frac{V}{V+\delta_1(V-2c_{w1})}+\frac{\delta_2(V-2c_{w2})}{V+\delta_2(V-2c_{w2})}\right].
\end{aligned}$$

S_2 wins with probability

$$\begin{aligned}
&1-\frac{1}{2}\left[\frac{V}{V+\delta_1(V-2c_{w1})}+\frac{\delta_2(V-2c_{w2})}{V+\delta_2(V-2c_{w2})}\right]\\
=&\frac{1}{2}\left[\frac{\delta_1(V-2c_{w1})}{V+\delta_1(V-2c_{w1})}+\frac{V}{V+\delta_2(V-2c_{w2})}\right].
\end{aligned}$$

□

2.7.2 Proof of Proposition 1

Proof. Let us first consider S_1's problem. S_1 will win for sure as long as W_2 runs. If he funds W_2's campaign, then he secures $V-c_{w2}$; not funding gives him at most $\frac{V}{2}$. Thus he will always fund W_2's campaign. If neither W_1 nor W_2 is willing to run, then S_1 will be tied with S_2. S_1 will lose for sure otherwise. Therefore,

$$\begin{aligned}
U_{s_1}(t_{11},t_{12}) &= (V-c_{w2})\cdot\left(\frac{\delta_2 t_{12}}{\delta_2 t_{12}+t_{22}}\right)\\
&+\frac{V}{2}\cdot\left(\frac{t_{11}}{t_{11}+\delta_1 t_{21}}\right)\cdot\left(\frac{t_{22}}{\delta_2 t_{12}+t_{22}}\right)-c_{s1}-t_{11}-t_{12}.
\end{aligned}$$

S_2 will win for sure if W_1 runs while W_2 does not, and clearly he will fund W_1's campaign if W_1 is willing to run. S_2 will be tied with S_1 if neither weak candidate runs, and he will lose for sure otherwise. Therefore,

$$U_{s2}(t_{21},t_{22}) = (V-c_{w1})\cdot\left(\frac{\delta_1 t_{21}}{t_{11}+\delta_1 t_{21}}\right)\cdot\left(\frac{t_{22}}{\delta_2 t_{12}+t_{22}}\right) + \frac{V}{2}\cdot\left(\frac{t_{11}}{t_{11}+\delta_1 t_{21}}\right)\cdot\left(\frac{t_{22}}{\delta_2 t_{12}+t_{22}}\right) - c_{s2} - t_{11} - t_{12}.$$

Then we have the following four first order conditions:

$$\frac{\partial U_{s1}(t_{11},t_{12})}{\partial t_{11}} = -\frac{t_{11}t_{22}V}{2(t_{11}+\delta_1 t_{21})^2(\delta_2 t_{12}+t_{22})} + \frac{t_{22}V}{2(t_{11}+\delta_1 t_{21})(\delta_2 t_{12}+t_{22})} - 1 = 0$$

$$\frac{\partial U_{s1}(t_{11},t_{12})}{\partial t_{12}} = \frac{t_{11}t_{22}\delta_2 V}{2(t_{11}+\delta_1 t_{21})(\delta_2 t_{12}+t_{22})^2} - \frac{t_{12}\delta_2^2(V-c_{w2})}{(\delta_2 t_{12}+t_{22})^2} + \frac{\delta_2(V-c_{w2})}{\delta_2 t_{12}+t_{22}} - 1 = 0$$

$$\frac{\partial U_{s2}(t_{21},t_{22})}{\partial t_{21}} = \frac{\delta_1^2 t_{21}t_{22}(V-c_{w1})}{(t_{11}+\delta_1 t_{21})^2(\delta_2 t_{12}+t_{22})} + \frac{\delta_1 t_{22}(V-c_{w1})}{(t_{11}+\delta_1 t_{21})(\delta_2 t_{12}+t_{22})} - \frac{\delta_1 t_{11}t_{22}V}{2(t_{11}+\delta_1 t_{21})^2(\delta_2 t_{12}+t_{22})} - 1 = 0$$

$$\frac{\partial U_{s2}(t_{21},t_{22})}{\partial t_{22}} = \frac{\delta_1 t_{21}t_{22}(V-c_{w1})}{(t_{11}+\delta_1 t_{21})(\delta_2 t_{12}+t_{22})^2} + \frac{\delta_1 t_{21}(V-c_{w1})}{(t_{11}+\delta_1 t_{21})(\delta_2 t_{12}+t_{22})} - \frac{t_{11}t_{22}V}{2(t_{11}+\delta_1 t_{21})(\delta_2 t_{12}+t_{22})^2} + \frac{t_{11}V}{2(t_{11}+\delta_1 t_{21})(\delta_2 t_{12}+t_{22})} - 1 = 0.$$

Solving them gives

$$\begin{cases} t_{11}^* = \frac{\delta_1(V-2c_{w1})V^2[4\delta_1(V-c_{w1})^2-2\delta_1V(V-c_{w1})+V^2]}{2[V+\delta_1(V-2c_{w1})]^2[4\delta_1(V-c_{w1})^2-2\delta_1V(V-c_{w1})+V^2-\delta_2V^2+4\delta_1\delta_2(V-c_{w1})(V-c_{w2})+2(1-\delta_1)V(V-c_{w2})]} \\ t_{12}^* = \frac{\delta_2[4\delta_1(V-c_{w1})^2-2\delta_1V(V-c_{w1})+V^2][-V^2+4\delta_1(V-c_{w1})(V-c_{w2})+2(1-\delta_1)V(V-c_{w2})]^2}{2[V+\delta_1(V-2c_{w1})][4\delta_1(V-c_{w1})^2-2\delta_1V(V-c_{w1})+V^2-\delta_2V^2+4\delta_1\delta_2(V-c_{w1})(V-c_{w2})+2(1-\delta_1)V(V-c_{w2})]^2} \\ t_{21}^* = \frac{\delta_1(V-2c_{w1})^2V[4\delta_1(V-c_{w1})^2-2\delta_1V(V-c_{w1})+V^2]}{2[V+\delta_1(V-2c_{w1})]^2[4\delta_1(V-c_{w1})^2-2\delta_1V(V-c_{w1})+V^2-\delta_2V^2+4\delta_1\delta_2(V-c_{w1})(V-c_{w2})+2(1-\delta_1)V(V-c_{w2})]} \\ t_{22}^* = \frac{\delta_2[4\delta_1(V-c_{w1})^2-2\delta_1V(V-c_{w1})+V^2]^2[-V^2+4\delta_1(V-c_{w1})(V-c_{w2})+2(1-\delta_1)V(V-c_{w2})]}{2[V+\delta_1(V-2c_{w1})][4\delta_1(V-c_{w1})^2-2\delta_1V(V-c_{w1})+V^2-\delta_2V^2+4\delta_1\delta_2(V-c_{w1})(V-c_{w2})+2(1-\delta_1)V(V-c_{w2})]^2}. \end{cases}$$

In equilibrium, we have

$$\frac{t_{11}}{t_{21}} = \frac{\delta_1(V-2c_{w1})V^2[4\delta_1(V-c_{w1})^2-2\delta_1V(V-c_{w1})+V^2]}{2[V+\delta_1(V-2c_{w1})]^2[4\delta_1(V-c_{w1})^2-2\delta_1V(V-c_{w1})+V^2-\delta_2V^2+4\delta_1\delta_2(V-c_{w1})(V-c_{w2})+2(1-\delta_1)V(V-c_{w2})]} \Bigg/ \frac{\delta_1(V-2c_{w1})^2V[4\delta_1(V-c_{w1})^2-2\delta_1V(V-c_{w1})+V^2]}{2[V+\delta_1(V-2c_{w1})]^2[4\delta_1(V-c_{w1})^2-2\delta_1V(V-c_{w1})+V^2-\delta_2V^2+4\delta_1\delta_2(V-c_{w1})(V-c_{w2})+2(1-\delta_1)V(V-c_{w2})]} = \frac{V}{V-2c_{w1}}.$$

So the net reward property holds for W_1. As we discussed in the main text, the intuition is that for S_1, winning the support of W_1 is worth $\frac{V}{2}$ if he loses W_2 and zero otherwise; for S_2, winning it is worth $\frac{V}{2}-c_{w1}$

if S_1 loses W_2 and zero otherwise. This can be seen more clearly when we rewrite S_2's expected utility function as follows:

$$U_{s2}(t_{21}, t_{22}) = \frac{V}{2} \cdot \left(\frac{t_{22}}{\delta_2 t_{12} + t_{22}} \right) + \left(\frac{V}{2} - c_{w1} \right) \cdot \left(\frac{t_{11}}{t_{11} + \delta_1 t_{21}} \right) \cdot \left(\frac{t_{22}}{\delta_2 t_{12} + t_{22}} \right) - c_{s2} - t_{11} - t_{12}.$$

On the other hand,

$$\begin{aligned} &\frac{t_{22}}{t_{12}} \\ &= \frac{\delta_2[4\delta_1(V-c_{w1})^2 - 2\delta_1 V(V-c_{w1}) + V^2]^2[-V^2 + 4\delta_1(V-c_{w1})(V-c_{w2}) + 2(1-\delta_1)V(V-c_{w2})]}{2[V+\delta_1(V-2c_{w1})][4\delta_1(V-c_{w1})^2 - 2\delta_1 V(V-c_{w1}) + V^2 - \delta_2 V^2 + 4\delta_1\delta_2(V-c_{w1})(V-c_{w2}) + 2(1-\delta_1)V(V-c_{w2})]^2} \\ &\Big/ \frac{\delta_2[4\delta_1(V-c_{w1})^2 - 2\delta_1 V(V-c_{w1}) + V^2][-V^2 + 4\delta_1(V-c_{w1})(V-c_{w2}) + 2(1-\delta_1)V(V-c_{w2})]^2}{2[V+\delta_1(V-2c_{w1})][4\delta_1(V-c_{w1})^2 - 2\delta_1 V(V-c_{w1}) + V^2 - \delta_2 V^2 + 4\delta_1\delta_2(V-c_{w1})(V-c_{w2}) + 2(1-\delta_1)V(V-c_{w2})]^2} \\ &= \frac{4\delta_1(V-c_{w1})^2 - 2\delta_1 V(V-c_{w1}) + V^2}{-V^2 + 4\delta_1(V-c_{w1})(V-c_{w2}) + 2(1-\delta_1)V(V-c_{w2})}, \end{aligned}$$

So this property does not hold for W_2. We compare the denominator and the numerator:

$$\begin{aligned} &[4\delta_1(V-c_{w1})^2 - 2\delta_1 V(V-c_{w1}) + V^2] - [-V^2 + 4\delta_1(V-c_{w1})(V-c_{w2}) + 2(1-\delta_1)V(V-c_{w2})] \\ &= 2Vc_{w2} + 4\delta_1(V-c_{w1})(c_{w2}-c_{w1}) \end{aligned}$$

Therefore, it is not always true that $t_{22} \geq t_{12}$. □

2.8 Appendix B

2.8.1 S_1 More Informed Than S_2

Here we derive the equilibrium for the case where S_1 privately observes c_{w2} when he decides side transfers while S_2 does not observe c_{w2} at that time.

S_1's strategy becomes the pair $(t_{11}(c_{w2}), t_{12}(c_{w2}))$ to reflect the fact that he can give different transfers for different c_{w2}'s while S_2's strategy is still (t_{21}, t_{22}). The expected payoff for S_1 becomes

$$U_{s1}(t_{11}(c_{w2}), t_{12}(c_{w2})) = (V - c_{w2}) \cdot \left(\frac{t_{11}(c_{w2})}{t_{11}(c_{w2}) + \delta_1 t_{21}}\right) \cdot \left(\frac{\delta_2 t_{12}(c_{w2})}{\delta_2 t_{12}(c_{w2}) + t_{22}}\right)$$
$$+\left(\frac{V}{2} - c_{w2}\right)\left(\frac{\delta_1 t_{21}}{t_{11}(c_{w2}) + \delta_1 t_{21}}\right) \cdot \left(\frac{\delta_2 t_{12}(c_{w2})}{\delta_2 t_{12}(c_{w2}) + t_{22}}\right)$$
$$+\frac{V}{2} \cdot \left(\frac{t_{11}(c_{w2})}{t_{11}(c_{w2}) + \delta_1 t_{21}}\right) \cdot \left(\frac{t_{22}}{\delta_2 t_{12}(c_{w2}) + t_{22}}\right) - c_{s1} - t_{11}(c_{w2}) - t_{12}(c_{w2}).$$

The optimization problem for S_2 is

$$U_{s2}(t_{21}, t_{22}) = \int_0^{\bar{c}_{w2}} \Bigg[(V - c_{w1}) \cdot \left(\frac{\delta_1 t_{21}}{t_{11}(c_{w2}) + \delta_1 t_{21}}\right) \cdot \left(\frac{t_{22}}{\delta_2 t_{12}(c_{w2}) + t_{22}}\right)$$
$$+\left(\frac{V}{2} - c_{w1}\right)\left(\frac{\delta_1 t_{21}}{t_{11}(c_{w2}) + \delta_1 t_{21}}\right) \cdot \left(\frac{\delta_2 t_{12}}{\delta_2 t_{12} + t_{22}}\right)$$
$$+\frac{V}{2} \cdot \left(\frac{t_{11}(c_{w2})}{t_{11}(c_{w2}) + \delta_1 t_{21}}\right) \cdot \left(\frac{t_{22}}{\delta_2 t_{12}(c_{w2}) + t_{22}}\right)\Bigg] f(c_{w2})\, dc_{w2} - c_{s2} - t_{21} - t_{22}.$$

The FOC of $U_{s1}(t_{11}(c_{w2}), t_{12}(c_{w2}))$ with respect to t_{11} is

$$\frac{\delta_1 t_{21}}{(t_{11}(c_{w2}) + \delta_1 t_{21})^2} = 1.$$

This implies that $t_{11}(c_{w2})$ will not be different for different c_{w2}'s, so we can simplify it to t_{11}. The FOC of $U_{s2}(t_{21}, t_{22})$ with respect to t_{21} is

$$\left(\frac{V}{2} - c_{w1}\right) \cdot \left[\frac{\delta_1 t_{11}}{(t_{11} + \delta_1 t_{21})^2}\right] = 1.$$

So in equilibrium, t_{11}^* and t_{21}^* are the same as in the baseline model:

$$\begin{cases} t_{11}^* = \dfrac{V^2\left[\delta_1(V-2c_{w1})\right]}{2\left[V+\delta_1(V-2c_{w1})\right]^2} \\ t_{21}^* = \dfrac{V\left[\delta_1(V-2c_{w1})^2\right]}{2\left[V+\delta_1(V-2c_{w1})\right]^2} \end{cases}$$

The FOC for $t_{12}(c_{w2})$ is

$$\left(\frac{V}{2} - c_{w2}\right) \cdot \left[\frac{\delta_2 t_{22}}{(\delta_2 t_{12}(c_{w2}) + t_{22})^2}\right] = 1,$$

so

$$t_{12}^*(c_{w2}) = \frac{1}{\delta_2}\left\{\left[\left(\frac{V}{2} - c_{w2}\right)\delta_2 t_{22}\right]^{\frac{1}{2}} - t_{22}\right\}. \tag{2.5}$$

The FOC for t_{22} is

$$\int_0^{\bar{c}_{w2}} \frac{V}{2} \cdot \left[\frac{\delta_2 t_{12}(c_{w2})}{(\delta_2 t_{12}(c_{w2}) + t_{22})^2}\right] f(c_{w2})\, dc_{w2} = 1.$$

Multiplying both sides by t_{22} and using the FOC for $t_{12}(c_{w2})$,

$$t_{22} = \int_0^{\bar{c}_{w2}} \left(\frac{V}{V - 2c_{w2}}\right) t_{12}(c_{w2}) f(c_{w2})\, dc_{w2}. \tag{2.6}$$

$t_{12}^*(c_{w2})$ and t_{22}^* are implicitly solved by equations (2.5) and (2.6).

Chapter 3

A Theory of Underpricing and Token Retention in Initial Coin Offerings

Abstract

We develop a signaling theory to explain the empirically observed phenomena of underpricing and token retention in initial coin offerings (ICOs). When the project's type is the entrepreneur's private information, a good entrepreneur can credibly signal his type through a low price in the ICO because he can recoup the loss from a more favorable price in the final sale owing to a high probability of early investors staying as users. In contrast, a bad entrepreneur would find this strategy too costly. This signaling theory can also explain airdrops in ICOs, where a developer distributes free tokens to potential early investors. We then extend our model to make the late community formation probability depend on the result of initial community formation, and show that this leads to another separating equilibrium, where a good developer can signal his quality through token retention. We show that this equilibrium is mutually exclusive with the underpricing equilibrium, and provide welfare comparison results when both separating and pooling equilibria exist.

3.1 Introduction

Traditionally, firms raise money through debt or equity financing. If a large firm needs money to develop new projects, it often considers selling stocks in an Initial Public Offering (IPO). In recent years, the development of blockchain technology has given rise to a new way of financing. If a developer needs money to finish developing a cryptocurrency platform or ecosystem, he can sell tokens, which are digital assets on blockchains, through an Initial Coin Offering (ICO). Increasingly popular, ICOs raised a total of $8.7 billion in 2024.

A widely acknowledged phenomenon in the IPO literature is underpricing, which refers to a significant increase in stock price from the initial offering price to the first-day closing price (Ibbotson 1975; Ibbotson and Jaffe 1975; Ritter 1984). As a recent example, consider the IPO of Circle, issuer of one of the world's most widely used stablecoins, USDC. It went public on the New York Stock Exchange on June 5th, 2025. The IPO offer price was $31, and the first-day closing price was $83.23, meaning that the company left $1.76 billion on the table. This was the seventh largest underpricing among all offerings since 1980 (Tully 2025).

Following the literature on IPO underpricing, recent papers define ICO underpricing as a significant increase in token price from the initial coin offering price to the first trading day closing price (Felix and von Eije 2019). A slight subtlety is that unlike stocks, cryptocurrencies are traded continuously, which means there is no true "closing price." As a result, they use the price from a specific exchange right before midnight UTC time. A notable example of ICO underpricing is Ethereum's ICO in 2014. The ICO price of Ethereum was $0.3, while the early trading price per Ethereum surged to over $2, leading to an underpricing of over 500%.

Although IPO underpricing has been widely studied both theoretically and empirically, ICO underpricing has been studied only empirically (Felix and von Eije 2019; Benedetti and Kostovetsky 2021). We seek to be the first paper to theoretical analyze this intriguing phenomenon. Existing theoretical literature's explanations for IPO underpricing center around information asymmetry. Information asymmetry between firms and investors is likely more pronounced in ICOs than IPOs, due to the lack of regulation and the typically early-stage nature of the underlying projects. As a result, we seek to fill this gap by showing that information asymmetry can lead to underpricing in ICOs. In doing so, we consider two main differences between IPOs and ICOs. Firstly, in IPOs, investors do not typically become users, while in ICOs, investors often decide whether or not to stay as users. Secondly, the valuations of stocks and tokens are fundamentally different. Stocks are claims to future cash flows, while tokens' values are (at least partially) determined by the number of users of the underlying product or platform.

In our paper, we consider entrepreneurs with private information about the qualities of their projects, either Good or Bad, seeking to finance a project by selling tokens. Since we do not model exchanges explicitly,

we define ICO underpricing as the token's ICO price being lower than the expected token value, conditional on investors' information and under perfect competition. We show that under some conditions, developers of Good cryptocurrency projects want to signal to early users their good prospects by selling their tokens at a low price, in order to gain a favorable estimate of final user size in later sales. A low ICO price is more costly for Bad developers because Good developers can expect to recoup their loss from the future sale more often than Bad developers, since a Good project is more likely to form a community from early investors, which will boost the token's price in the later sale. We then show that airdrops, which are another phenomenon often seen in cryptocurrency communities, can also be explained by our signaling theory. With another assumption on network effects, our model predicts that a good entrepreneur can also credibly signal his quality through token retention, which is another empirically confirmed phenomenon in ICOs.

3.2 Literature Review

Theoretical literature often identifies asymmetric information as the main explanation for IPO underpricing. Among a few others, three theories are popular. In the first theory, Rock (1986) assumes the existence of a group of investors who have better information than the firm and all other investors. The offering firm must underprice the shares in order to attract both informed and uninformed investors. In the second theory, Baron (1982) develops an IPO model based on the principal-agent theory, where the investment banker is better informed than the issuer. Underpricing happens in the optimal contract as the issuer compensates the banker for his superior information. The third theory relies on signaling. Allen and Faulhaber (1989) argue that good firms want to signal investors their good prospects through a low price in IPO in order to get better prices in future sales.

Apart from these theories, it is also widely acknowledged that underpricing in IPOs is more severe during "hot markets," where there are more firms hoping to go public (Ibbotson and Jaffe 1975). Khanna, Noe, and Sonti (2008) develop an elegant theory with a fixed labor supply capable of screening IPOs to explain this phenomenon.

In the cryptocurrency world, recent empirical papers (Felix and von Eije 2019; Benedetti and Kostovetsky 2021) confirm the phenomenon of underpricing in ICOs. Benedetti and Kostovetsky (2021) find "average returns of 179% from the ICO price to the first day's opening market price, over a holding period that averages just 16 days." Similarly, Felix and von Eije (2019) find "an average level of underpricing of ICOs of 123% in the USA and 97% in the other countries," both levels significantly higher than IPO underpricing at the beginning of the dot.com bubble. Yet there is no theoretical paper that seeks to explain ICO underpricing. We develop

a signaling theory for ICO underpricing and argue that it is a plausible explanation for three reasons. First, as Felix and von Eije (2019) point out, information asymmetry is greater in ICOs than IPOs. This is because firms that go public are usually much larger than firms selling tokens, which are typically start-up firms. This information asymmetry problem is worsened as ICOs are also much more unregulated than IPOs and thus more sensitive to fraud. Second, Baron (1982)'s theory based on information asymmetry is unlikely to extend to ICOs because there is no "investment banker" or similar well-informed financial intermediary here. Thus the underpricing theories that can potentially be extended to ICOs are the ones written by Rock (1986) and Allen and Faulhaber (1989). We discuss the possibility of extending Rock (1986)'s theory in the discussion section and focus on the signaling theory in our model. Lastly, we show that in the case of selling tokens, a necessary assumption for underpricing to be an equilibrium in Allen and Faulhaber (1989)'s IPO model is unnecessary for it to be an equilibrium in our ICO model. The reason behind this lies in the difference between valuation of equity and tokens.

Although there is no theoretical paper that analyzes underpricing in ICOs, there are a few recent theoretical papers that compare token and equity financing. Chod and Lyandres (2021) argue that compared to equity financing, token financing alleviates entrepreneurial effort underprovision but leads to underproduction. Malinova and Park (2023) show that the optimal token financing contract, which is a combination of an output pre-sale and a revenue-sharing agreement, gives the entrepreneur the same payoff as equity and debt. Shakhnov and Zaccaria (2020) argue that token financing allow firms to take advantage of network effects, but venture capital's value-adding services can improve product quality.

In our model, we consider network effect in tokenomics, which means that the price of a token is increasing in the number of users. This is consistent with, for example, the conclusion by Cong, Li, and Wang (2018). Admittedly, our modelling of token price is simpler than theirs to ensure tractability of our results.

Furthermore, we consider the possibility of using airdrops as an alternative signal of quality. Cryptocurrency start-ups sometimes distribute free tokens to different wallets before ICOs. There are a few papers that study airdrops but none of them develops a theoretical model. Allen, Berg, and Lane (2023) analyze 12 high-profile airdrops to conclude that the two main rationales are marketing and decentralization. Compared to underpricing, airdrops induce a greater immediate sacrifice in price, but if a good developer can successfully signal his type prior to ICO through airdropping, he can get a better price for both ICO and final sale. Milgrom and Roberts (1986) study the possibility of using price and costly and uninformative advertising as signals of product quality. Our work is related to theirs but we analyze strategies common in the cryptocurrency world.

In the extended model, we add another network effect to the model. In practice, if initial investors stay as users, then it would be more likely for late investors to stay; if no one stays after the ICO, then late users will be less likely to stay. When we consider this effect, other separating equilibria exist. Bad developers may prefer to sell all tokens in the ICO because his expected probability of late community formation is lower than that of early community formation. But then a Good developer can signal his type through selling just enough tokens to raise C. This signaling by token retention can be considered as the cryptocurrency version of the model by Leland and Pyle (1977). Leland and Pyle (1977) show that when the entrepreneur has private information about his project, he can credibly signal his good type by retaining shares of his own firm. More recently, Davydiuk et al. (2021) develop a signaling model of token retention and empirically show that developers who retain a larger fraction of their tokens are more successful in their funding efforts and are more likely to develop a working product. The difference between their model and our model is that they have a reduced form cost function of retaining tokens, which comes from the entrepreneur's inability to diversify his investments when holding a large fraction of his wealth in a single type of token. Our model, on the other hand, shows another type of cost: bad developers know that late users are unlikely to stay due to a high chance of failure in the initial community formation, which means the late sale price would be low.

3.3 Model

A developer has an innovation described in his white paper, and he needs to raise capital C from selling tokens to finish developing his project. We normalize the total token supply to 1.

Time 1: Nature determines the type of the innovation. It can be either Good (G) or Bad (B), and in the entire ICO market, a θ fraction of the projects are Good and $1-\theta$ are Bad. The type of the innovation is the developer's private information.

Time 2: The developer chooses to sell an α fraction of the tokens to early investors at a price p_0 per token, who then decide whether to become permanent users or not. In principle, he can raise more than C, but we assume that he only sells just enough to finance the cost unless it is strictly profitable to sell more in ICO compared to final sale. In the extended model we thoroughly analyze the case where the entrepreneur raises more than C. There is a continuum of investors, whose total mass is 1. Each investor demands 1 unit of token. So there will be a mass α of early investors, where $\alpha = \frac{C}{p_0}$. If the project is Good, then there is a probability p of every early investor staying as a permanent user and an $1-p$ probability that no one stays as a permanent user. If the project is Bad, then there is a probability q of everyone staying as a permanent user and an $1-q$ probability no one staying, where $q < p$. The number of early permanent users

then becomes public information, but there is no credible way for early adopters to convey the type of the project to outsiders. We denote a successful initial community formation as s and a failure as f.

Time 3: Late investors update the probability that the project is Good after observing the result of the early community's formation. The developer sells the remaining $1-\alpha$ fraction of tokens to the remaining $1-\alpha$ late investors, who also have unit demand. The probabilities that they stay are the same as early investors for both types of projects.

Time 4: The result of final community formation is realized.

Let u_e be the mass of early adopters and u_l be the mass of late adopters, so $u = u_e + u_l$. The fair market price in the ICO is $\mathbb{E}[A(u_e + u_l)]$, where A measures the network value created by each active user. The developer may choose to **underprice**, which means offering a price below this level. The price at which the developer sells the $1-\alpha$ token is $Au_e + \mathbb{E}[Au_l]$, so late investors compete until each has 0 expected profit.

A strategy for each type of developer is a price-fraction pair $a(i) : \{G, B\} \to [0, \infty) \times [0, 1], a(i) = (p_{0i}, \alpha_i)$. Let $U_i(p_{0i}, \alpha_i)$ be type i developer's expected profit when he chooses (p_{0i}, α_i). A **separating equilibrium** is a Perfect Bayesian Equilibrium where different types choose different price-fraction pairs, and a **pooling equilibrium** is one where they choose the same strategy. In this setting, a low p_0, despite leading to lower initial profit, has two benefits for a Good developer. First, a low p_0 can be used by Good to separate itself from Bad as a costly signal so that outsiders will attach a high expectation of u_l in the second sale even if the initial community failed to form. Second, a low p_0 means there are more tokens sold, which can translate to a higher price in the second sale when the initial community is formed successfully. We focus on the **least-cost separating equilibrium** (Riley 1979), which is the Pareto dominant separating equilibrium where a Good developer signals his type credibly by choosing the highest possible ICO price that makes B find too costly to mimic.

3.3.1 Separating Equilibria

We now proceed to find the conditions under which there exists a separating equilibrium in the game.

Theorem 1: There exists a separating equilibrium if the smaller solution to the equation

$$Aq - C = (1-\alpha)\Big[q \cdot A\alpha + p \cdot A(1-\alpha)\Big]$$

is within (0,1) and that solution

$$\alpha^* = \frac{A(2p-q) - \sqrt{4A^2pq - 3A^2q^2 - 4AC(p-q)}}{2A(p-q)}$$

satisfies

$$\alpha^* \leq 1 - (1 - \frac{C}{Aq})\Big[\frac{C}{Aq} + \frac{q}{p} \cdot (1 - \frac{C}{Aq})\Big].$$

In this equilibrium, the Good developer underprices in ICO while the Bad developer prices his tokens at Aq.

Proof. Let us verify that the equilibrium described in the theorem is indeed an equilibrium. The Bad developer offers ICO price Aq, and has a q probability of forming the initial community of mass $\frac{C}{Aq}$ and the market correctly believes that there is also a q probability of forming the late community of mass $1 - \frac{C}{Aq}$. It costs him C in the development. So he expects

$$\begin{aligned} U_B(Aq, \frac{C}{Aq}) &= Aq \cdot \frac{C}{Aq} + (1 - \frac{C}{Aq})Aq - C \\ &= Aq - C \end{aligned}$$

under this strategy. Intuitively, in this equilibrium there is no mispricing of his tokens, so he expects to sell his tokens at price Aq and developing the project costs C.

Mimicking a Good developer means the Bad developer will sell (p_0^*, α^*) in ICO and be perceived as having a p probability of forming the late community of mass $1 - \alpha^*$ in the final sale. But he also knows that he only has a q chance of forming the initial community successfully. Therefore, he expects

$$\begin{aligned} U_B(p_0^*, \alpha^*) &= \alpha^* p_0^* + (1 - \alpha^*)\Big[q \cdot A\alpha^* + p \cdot A(1 - \alpha^*)\Big] - C \\ &= (1 - \alpha^*)\Big[q \cdot A\alpha^* + p \cdot A(1 - \alpha^*)\Big], \end{aligned}$$

since

$$\alpha^* p_0^* = C.$$

For mimicking to be unprofitable, we need

$$Aq - C \geq (1-\alpha^*)\Big[q \cdot A\alpha^* + p \cdot A(1-\alpha^*)\Big]$$

The Good developer does not need to underprice more than necessary, so this condition is binding in equilibrium. α^* and p_0^* can be solved by the two equations above.

The Good type must find it worthwhile to send the underpricing signal, compared to offering $(Aq, \frac{C}{Aq})$ like Bad. Notice here we assign off-path beliefs to be $\mathbb{P}(G|p' > p_0^*) = 0$ and $\mathbb{P}(G|p' < p_0^*) = 1$.

Signaling by underpricing gives the market a correct pricing of the expected number of late adopters, while not signaling will make the market believe that there is a q probability that late adopters will stay in the community. The Good developer's expected payoff when he chooses to underprice is

$$\begin{aligned} U_G(p_0^*, \alpha^*) &= \alpha^* p_0^* + (1-\alpha^*)\Big[p \cdot A\alpha^* + p \cdot A(1-\alpha^*)\Big] - C \\ &= (1-\alpha^*)Ap. \end{aligned}$$

When he does not underprice, he will sell $\frac{C}{Aq}$ of the firm in the ICO, so he expects

$$U_G(Aq, \frac{C}{Aq}) = (1 - \frac{C}{Aq})\Big[p \cdot A(\frac{C}{Aq}) + q \cdot A(1 - \frac{C}{Aq})\Big].$$

Therefore, we need

$$(1-\alpha^*)Ap \geq (1 - \frac{C}{Aq})\Big[p \cdot A(\frac{C}{Aq}) + q \cdot A(1 - \frac{C}{Aq})\Big],$$

which is equivalent to

$$\alpha^* \leq 1 - (1 - \frac{C}{Aq})\Big[\frac{C}{Aq} + \frac{q}{p} \cdot (1 - \frac{C}{Aq})\Big].$$

□

The key insight from this theorem is that in our model, unlike Allen and Faulhaber (1989), we do not assume a type change between initial offering and final sale, which means a Good project has a probability of becoming a Bad project. This assumption is essential for a separating equilibrium to exist in their model, but unnecessary in our model. In their model, firms raise capital through an IPO, then have a high or

low dividend, afterwards sell all the remaining shares, and finally have another high or low dividend. A Good firm has a higher chance of having a high dividend than a Bad firm. If there is no type change, then suppose a Good firm can somehow signal its type by undrpricing. Investors will not update their belief after observing the first dividend after the IPO. So there is no direct shift of belief towards Bad after observing a low dividend. Then the Bad firm will always find it worthwhile to mimic the Good firm since the cost of doing so would be the same as the Good firm's cost. But having a type change means investors will have some suspicion after observing a low cashflow, which makes the Bad firm find signaling more costly than the Good firm.

Our ability to eliminate that assumption lies in the difference between valuations of stocks and tokens. A past high dividend cannot directly increase the current price of stocks, with the only benefit being optimism on future dividends. However, a past successful community formation both directly boosts the current price of a token and leads to optimism on future community formation prospects. Therefore, even if there is no type change, a Good developer will find it less costly to send the signal because he will more often recoup the loss than the Bad developer as his initial community is more likely to form, which will be directly rewarded in the final sale price.

Proposition 1: This Separating Equilibrium passes the Intuitive Criterion.

Proof. To see this result, consider any off-path price p_0' above G's ICO price choice p_0^*. Then clearly G would be better off if the belief is $\mathbb{P}(G|p_0') = 1$. But in this case, B always also finds it profitable to deviate to p_0', because by construction, p_0^* is the highest price that could deter B from mimicking. □

As a result, the separating equilibrium we described in **Theorem 1** is the least-cost separating equilibrium, i.e., the Riley outcome.

This result is in contrast to a one-period version of signaling in IPO underpricing by Giammarino and Lewis (1988), because in their paper the interesting semipooling equilibrium, where G always asks for a pooled price and B randomises over that price and a lower separating price, fails the Intuitive Criterion.

3.3.2 Pooling Equilibria

In this section we analyze pooling equilibria. If we assign off-path belief to be that any other price-fraction pair signals Bad quality for sure, then for all parameters we can construct a pooling equilibrium where both types pool at the expected fair market price. This is because deviating by choosing a higher price will result in no sale and deviating lower cannot signal Good quality. We now focus on finding conditions under which

we can construct a pooling equilibrium that passes the Intuitive Criterion.

Theorem 2: Under some conditions, there exists a pooling equilibrium that passes the Intuitive Criterion where both types of developers set ICO price

$$p_0^* = A[\theta p + (1-\theta)q]$$

and sell a fraction

$$\alpha^* = \frac{C}{A[\theta p + (1-\theta)q]}.$$

Proof. The equilibrium price above is the expected terminal fair value of the token. The key difference between this pooling equilibrium and the separating equilibrium is that there will be an updating of the beliefs after observing the result of the initial community formation.

$$\begin{aligned}
P_G^s \equiv \mathbb{P}(G|s) &= \frac{\theta p}{\theta p + (1-\theta)q}, \\
P_B^s \equiv \mathbb{P}(B|s) &= \frac{(1-\theta)q}{\theta p + (1-\theta)q}, \\
P_G^f \equiv \mathbb{P}(G|f) &= \frac{\theta(1-p)}{\theta(1-p) + (1-\theta)(1-q)}, \\
P_B^f \equiv \mathbb{P}(G|f) &= \frac{(1-\theta)(1-q)}{\theta(1-p) + (1-\theta)(1-q)}.
\end{aligned}$$

If the market observes s, then the market would price the expected network value created by late users as

$$p_1^s \equiv \mathbb{E}(Au_l|s) = A(1-\alpha^*)(pP_G^s + qP_B^s).$$

If the markets observes f, then it would price

$$p_1^f \equiv \mathbb{E}(Au_l|f) = A(1-\alpha^*)(pP_G^f + qP_B^f).$$

Notice that this network value does not incorporate the network value created by early users. So for G, if the initial community is successfully formed, then he can sell the remaining tokens at $A\alpha^* + p_1^s$; if it fails, he can sell at p_1^f.

Then in this candidate equilibrium, G expects

$$U_G(p_0^*, \alpha^*) = (1-\alpha^*)\left[pA\alpha^* + pp_1^s + (1-p)p_1^f\right].$$

B expects

$$U_B(p_0^*, \alpha^*) = (1-\alpha^*)\left[qA\alpha^* + qp_1^s + (1-q)p_1^f\right].$$

If G wishes to signal his type instead through underpricing, then the highest price he can set would be the highest price at which B finds too costly to mimic. Denote this price by p_0' and let $\alpha' = \frac{C}{p_0'}$. The off-path beliefs can be that $\mathbb{P}(G|p_0 \leq p_o') = 1$ and $\mathbb{P}(G|p_0 > p_o') = 0$, where α' solves

$$(1-\alpha^*)\left[qA\alpha^* + qp_1^s + (1-q)p_1^f\right] = (1-\alpha')\left[qA\alpha' + pA(1-\alpha')\right]$$

For the candidate pooling equilibrium to hold, we therefore need G to find his payoff when choosing (p_0^*, α^*) to be higher than deviating to (α', p_0'):

$$(1-\alpha^*)\left[pA\alpha^* + pp_1^s + (1-p)p_1^f\right] \geq (1-\alpha')Ap,$$

which is equivalent to requiring

$$\left[pA\alpha^* + pp_1^s + (1-p)p_1^f\right] \cdot \left[qA\alpha' + pA(1-\alpha')\right] \geq \left[qA\alpha^* + qp_1^s + (1-q)p_1^f\right] \cdot Ap$$

The intuition is that in this pooling equilibrium, B is taking advantage of G. For it to be an equilibrium, we therefore need G's equilibrium payoff (measured by the first term on the LHS) to be high enough while his deviation payoff to be low enough (second term on the RHS).

This equilibrium clearly passes the Intuitive Criterion. Consider an off-path price $p_0' < p_0'' < p^*$. Then G would benefit from charging this price when the belief is $\mathbb{P}(G|p_0'') = 1$. But then B would also find it optimal to charge p_0'' because the construction of p_0' implies p_0' is the highest price that makes B find it suboptimal to mimic. □

Example

It is worth noting that in our model, the separating and pooling equilibria are not mutually exclusive. This is clearly true when we disregard the Intuitive Criterion for pooling equilibrium and set all off-path actions

to result in investors believing that the entrepreneur is type B with probability 1. Below is an example where both equilibria exist even when we take the Intuitive Criterion into account. Let

$$p = 0.7,$$

$$q = 0.3,$$

$$A = 20,$$

$$C = 2,$$

$$\theta = 0.95.$$

Then in the separating equilibrium,

$$\alpha^* = 0.5746 < 1 - (1 - \frac{C}{Aq})\left[\frac{C}{Aq} + \frac{q}{p} \cdot (1 - \frac{C}{Aq})\right] = 0.5873.$$

This means that G sells a fraction of 0.5746 of the tokens at a price of 3.4806 per token in the ICO, and B sells a fraction of 0.3333 of the tokens at a price of 6 per token. Equipped with the belief that any price below 3.4806 per token is done by G with probability 1, and any price above 3.4806 per token is done by B with probability 1, these strategies form a separating equilibrium.

For the pooling equilibrium, the equilibrium price-fraction pair in the ICO is $(13.5962, 0.1471)$, and

$$P_G^s = 0.9779,$$

$$P_G^f = 0.0221,$$

$$P_B^f = 0.8906,$$

$$P_B^f = 0.1094,$$

$$p_1^s = 11.7907,$$

$$p_1^f = 11.1949,$$

$$\alpha' = 0.1720.$$

As a result,

$$(1 - \alpha^*)\left[pA\alpha^* + pp_1^s + (1 - p)p_1^f\right] = 11.6610 > (1 - \alpha')Ap = 11.5925.$$

Analytically, the inequality in the separating equilibrium is independent of θ, which means for some parameter

values, it is possible to find some θ that would make the inequality in the pooling equilibrium hold. Intuitively, the ranges of credible signals by G are different between the separating and pooling equilibria because B has different payoffs in these equilibria.

3.3.3 Airdrops Versus Underpricing

Now we consider the possibility of using airdrops as a signal of quality. Suppose the developer can choose whether or not to give a β fraction of tokens for free prior to the ICO. Can airdrops serve as a less costly signal than underpricing?

Theorem 3: Under some conditions, there exists a separating Perfect Bayesian Equilibrium where G airdrops some fraction of his tokens before the ICO and does not underprice in ICO, although he would underprice if airdropping is not an option. B does not airdrop or underprice.

To prove this result, we first make two observations.

Claim 1: In any separating equilibrium where a Good developer airdrops, a Bad developer never airdrops any token.

The reason is that airdropping means giving tokens away at zero price, which is lower than any market price, but in a separating equilibrium, B deviating to giving no airdrop will not induce a worse belief about its type but saves money.

Claim 2: In any separating equilibrium where a Good developer airdrops, he will never underprice in ICO.

The reason behind this claim is that in such an equilibrium G already reveals to the market his type, so he no longer needs to send another costly signal about his type.

Now we look for conditions under which there exists a separating equilibrium where Good developers give free tokens before ICO and do not underprice in ICO, although they would underprice if airdrop is not an option, while Bad developers do not airdrop. The probabilities of community formation for both types are same as before. Let $U_i(\beta_i, p_{0i}^s, \alpha_i^s, p_{0i}^f, \alpha_i^f)$ be type i developer's expected profit when he chooses airdrop fraction β_i and ICO price-fraction pair (p_{0i}^s, α_i^s) if the airdropped community successfully forms and (p_{0i}^f, α_i^f) if it fails to form.

For a Good developer, there is a p probability that the airdropped users successfully form a community, so during ICO, he can price his tokens at

$$p_0^s = A\beta + p \cdot A(1-\beta).$$

Then the fraction of the token he needs to sell to raise C is

$$\alpha^s = \frac{C}{A\beta + p \cdot A(1-\beta)}.$$

there is an $1-p$ probability that the airdropped users fail to form a community, so he will set ICO price

$$p_0^f = p \cdot A(1-\beta)$$

and sell

$$\alpha^f = \frac{C}{p \cdot A(1-\beta)}$$

fraction of tokens. Take an underpricing equilibrium when airdropping is not an option. If G airdrops a β fraction, then he expects

$$\begin{aligned}
&U_G(\beta, p_0^s, \alpha^s, p_0^f, \alpha^f) = \beta \cdot 0 \\
&+p \cdot \left\{ \frac{C}{A\beta + pA(1-\beta)} \cdot [A\beta + p \cdot A(1-\beta)] + \Big[1 - \frac{C}{A\beta + p \cdot A(1-\beta)} - \beta\Big] \cdot \Big[A\beta + p \cdot A(1-\beta)\Big] \right\} \\
&+(1-p)\left\{ \frac{C}{p \cdot A(1-\beta)} \cdot [p \cdot A(1-\beta)] + \Big[1 - \frac{C}{p \cdot A(1-\beta)} - \beta\Big] \cdot [p \cdot A(1-\beta)] \right\} - C \\
&= (1-\beta)Ap - C.
\end{aligned}$$

Intuitively, G sacrifices β fraction of his tokens to inform the market of his type and get priced at p probability of successful community formation for the remaining $1-\beta$ fraction, and the cost of development is C.

The optimal β is the smallest one that makes B find too costly to mimic G. If B mimics G, then he expects a q probability that the airdropped community of mass β will form, and makes the market misprice his tokens in both ICO and final sale. There is also a $1-q$ probability that the community fails to form, but he will still benefit from mispricing.

$$\begin{aligned}
U_B(\beta, p_0^s, \alpha^s, p_0^f, \alpha^f) &= \beta \cdot 0 + q\left\{ \alpha^s \cdot p_0^s + (1-\beta-\alpha^s)\Big[A\beta + qA\alpha^s + pA(1-\beta-\alpha^s)\Big] \right\} \\
&+(1-q)\left\{ \alpha^f \cdot p_0^f + (1-\beta-\alpha^f)\Big[qA\alpha^f + pA(1-\beta-\alpha^f)\Big] \right\} - C \\
= q(1-\beta-\alpha^s)\Big[A\beta + qA\alpha^s + pA(1-\beta-\alpha^s)\Big] &+ (1-q)(1-\beta-\alpha^f)\Big[qA\alpha^f + pA(1-\beta-\alpha^f)\Big]
\end{aligned}$$

Therefore, the optimal β^* makes B indifferent between mimicking and not:

$$q(1-\beta^*-\alpha^s)\Big[A\beta^* + qA\alpha^s + pA(1-\beta^*-\alpha^s)\Big] + (1-q)(1-\beta^*-\alpha^f)\Big[qA\alpha^f + pA(1-\beta^*-\alpha^f)\Big] = Aq - C.$$

For G to prefer airdropping to underpricing, we therefore need

$$(1-\beta^*)Ap - C \geq (1-\alpha^*)Ap.$$

Example

Now we give an example to show that there are some parameters that make airdropping preferable to underpricing. Let

$$p = 0.5,$$

$$q = 0.2,$$

$$A = 20,$$

$$C = 0.5.$$

Then

$$\alpha^* = \frac{A(2p-q) - \sqrt{4A^2pq - 3A^2q^2 - 4AC(p-q)}}{2A(p-q)} = 0.5,$$

and the inequality holds:

$$\alpha^* = 0.5 < 1 - (1 - \frac{C}{Aq})\Big[\frac{C}{Aq} + \frac{q}{p}\cdot(1-\frac{C}{Aq})\Big] = 0.584.$$

So in the absence of airdrops, G selling half of his token supply at a price 1 per token while B selling $\frac{1}{8}$ of his token supply at price 4 per token is a separating equilibrium.

When airdropping becomes an option,

$$\alpha^s = 0.035,$$

$$\alpha^f = 0.088,$$

$$\beta^* = 0.43,$$

and then

$$(1-\beta)Ap - C = 5.2 \geq (1-\alpha^*)Ap = 5,$$

Thus airdropping is preferable to underpricing. Although this result might appear counterintuitive because airdropping means giving tokens for free, which seems like a greater loss of profit than underpricing, the intuition for this equilibrium is the following. Sometimes to signal its type, the underpricing equilibrium requires G to sell a large fraction of his tokens for very low price: the fair market price for G's token is 10 per token but he optimally sells at 1 per token in ICO for half of his total token supply. But to signal his type through airdropping, he only sacrifices 0.43 of his token supply but would gain a fair price in ICO, which means selling less than 0.1 fraction of his token supply even if the airdropped community fails to form.

3.3.4 Two-Stage ICOs

While in airdrops developers give away free tokens, in principle it can be viewed as a special case of a two-stage ICO with different prices in each round. The phenomenon that ICOs happen with multiple stages is common in practice. For example, a recent ICO to build the first decentralized exchange on the XRP ledger has hundreds of rounds selling the token $DXP, each with a slight price increase from the previous round. Now we study the optimal signaling strategy in two-stage ICOs. A separating equilibrium in this case would have the following form. In the first stage of the ICO, a Good developer offers (β, p_β), where p_β is the lower than the fair market value of the token. In the second stage of the ICO as well as the final sale, G's token would sell at the fair market prices (the fair prices vary depending on the success of earlier community formations). B will not underprice and will always price his tokens at Aq. We follow the same logic as before the derive the conditions under which this separating equilibrium can be sustained.

For a Good developer, there is a p probability that the first stage of the ICO successfully forms a community, so during the second stage, he can price his tokens at

$$p_0^s = A\beta + p \cdot A(1-\beta).$$

Then the fraction of the token he needs to sell to raise C is

$$\alpha^s = \frac{C - \beta p_\beta}{A\beta + p \cdot A(1-\beta)}.$$

there is an $1-p$ probability that the first stage investors fail to form a community, so he will set the second

stage ICO price

$$p_0^f = p \cdot A(1-\beta)$$

and sell

$$\alpha^f = \frac{C - \beta p_\beta}{p \cdot A(1-\beta)}$$

fraction of token. If G adopts the (p_β, β) strategy, then he expects

$$\begin{aligned}
&U_G(\beta, p_\beta, p_0^s, \alpha^s, p_0^f, \alpha^f) = \beta p_\beta \\
&+p \cdot \left\{ \frac{C - \beta p_\beta}{A\beta + pA(1-\beta)} \cdot [A\beta + p \cdot A(1-\beta)] + \Big[1 - \frac{C - \beta p_\beta}{A\beta + p \cdot A(1-\beta)} - \beta\Big] \cdot \Big[A\beta + p \cdot A(1-\beta)\Big] \right\} \\
&+(1-p)\left\{ \frac{C - \beta p_\beta}{p \cdot A(1-\beta)} \cdot [p \cdot A(1-\beta)] + \Big[1 - \frac{C - \beta p_\beta}{p \cdot A(1-\beta)} - \beta\Big] \cdot [p \cdot A(1-\beta)] \right\} - C \\
&= \beta p_\beta + (1-\beta)Ap - C.
\end{aligned}$$

G sells β fraction of his tokens at a low price p_β to signal his type in order to get the fair price for the remaining $1-\beta$ fraction.

The optimal β is the smallest one that makes B find too costly to mimic G. If B mimics G, then he benefits from mispricing in both the second stage of the ICO and the final sale, at the cost of lower chances of community formation:

$$\begin{aligned}
U_B(p_\beta, \beta, p_0^s, \alpha^s, p_0^f, \alpha^f) &= p_\beta \beta + q\left\{ \alpha^s \cdot p_0^s + (1-\beta-\alpha^s)\Big[A\beta + qA\alpha^s + pA(1-\beta-\alpha^s)\Big] \right\} \\
&+(1-q)\left\{ \alpha^f \cdot p_0^f + (1-\beta-\alpha^f)\Big[qA\alpha^f + pA(1-\beta-\alpha^f)\Big] \right\} - C \\
= q(1-\beta-\alpha^s)\Big[A\beta + qA\alpha^s + pA(1-\beta-\alpha^s)\Big] &+ (1-q)(1-\beta-\alpha^f)\Big[qA\alpha^f + pA(1-\beta-\alpha^f)\Big].
\end{aligned}$$

Therefore, the optimal β^* makes B indifferent between mimicking and not:

$$q(1-\beta^*-\alpha^s)\Big[A\beta^* + qA\alpha^s + pA(1-\beta^*-\alpha^s)\Big] + (1-q)(1-\beta^*-\alpha^f)\Big[qA\alpha^f + pA(1-\beta^*-\alpha^f)\Big] = Aq - C.$$

But here, unlike in the one-stage underpricing model, βp_β is not a fixed amount. Yet the equation above enables us to express the least costly β as a function of p_β. We should choose p_β^* to maximise G's expected

profit:

$$p^*_\beta = \underset{0 \le p_\beta \le Aq}{\arg\max}\, \beta(p_\beta)p_\beta + (1-\beta(p_\beta))Ap - C$$

To sustain the underpricing equilibrium, we need G to prefer underpricing in the first stage to mimicking B:

$$\beta^* p^*_\beta + (1-\beta^*)Ap \ge (1-\frac{C}{Aq})\Big[p \cdot A(\frac{C}{Aq}) + q \cdot A(1-\frac{C}{Aq})\Big].$$

One off-path belief that could pass the Intuitive Criterion is that any price below p^*_B is done by G with probability 1 and any price above it is done by B with probability 1.

3.4 Model Extension

In this section, we extend the baseline model by adding another kind of network effect. Consider, for example, the ICO of Filecoin, which provides a marketplace matching people who need storage space for data with people who have extra space. A successful ICO community formation means that there are honest sellers willing to share empty storage space and that the platform is likely reliable, which could make it more likely for late investors to stay. On the other hand, a failed ICO could make them less likely to stay. Suppose that for G(B), there is a probability $p(q)$ of forming the initial community like in the baseline model. If the initial community is formed, then late investors have a probability $\lambda_s p(\lambda_s q)$ of staying, where $\lambda_s > 1$. If the initial community fails to form, then late investors have a probability $\lambda_f p(\lambda_f q)$ of staying, where $\lambda_f < 1$. We assume $\lambda_s p < 1$.

In addition, we allow entrepreneurs to choose to either sell just enough tokens to raise C or sell all tokens in the ICO. With this new possibility, a new kind of separating equilibrium arises. Previously, no type of entrepreneur can strictly gain by selling all tokens in the ICO, because the late community formation probability is the same as early. Now, consider B's problem. B's expected probability of forming a community in the late sale is $q\cdot(\lambda_s q)+(1-q)\cdot(\lambda_f q) = q[\lambda_s q+\lambda_f(1-q)]$, which could be lower than q if $\lambda_s q+\lambda_f(1-q) < 1$. This means that if q is relatively low or λ_f is relatively high, B might wish to sell all his tokens in ICO. But then if G sells just enough tokens to finance the project (i.e., raising just C), then there might be separating equilibria even if G sells his tokens at the fair market price *without underpricing*. That is, G could signal his quality through *quantity* rather than *price*. This equilibrium does not arise in the baseline model because in the original separating equilibrium, it is not costly for B to sell tokens in the final sale. Let us call this separating equilibrium signaling by token retention.

Moreover, if B sells all tokens in the ICO, it may be tempting for him to mimic G if G charges the fair price. In this case, G might signal through a combination of underpricing and token retention, that is, G sells just enough tokens to raise C at a price below the market's expected price for a token issued by the good developer. We call such a separating equilibrium separating by both underpricing and token retention.

Before deriving the conditions under which signaling through underpricing, token retention, or both exist, we first prove that some of these equilibria are mutually exclusive.

Proposition 2: There cannot simultaneously exist a separating equilibrium where G signals through underpricing and one where G signals through either token retention or token retention and underpricing.

Proof. A separating equilibrium with token retention or both token retention and underpricing requires B to optimally sell all his tokens in the ICO, which requires $\lambda_s q + \lambda_f(1-q) < 1$. But a separating equilibrium with underpricing requires B to optimally only sell just enough to raise C, which requires $\lambda_s q + \lambda_f(1-q) \geq 1$. □

3.4.1 Signaling by Token Retention

Now we formally derive conditions under which this separating equilibrium with token retention can be supported.

Proposition 3: There exists a separating equilibrium where B sells all tokens at price Aq in the ICO and G sells just enough tokens to raise C at the fair price

$$p_0^{r^*} = \max\left\{p_0^r \middle| p_0^r = \mathbb{E}[Au|G] = Ap\frac{C}{p_0^r} + A\left(1 - \frac{C}{p_0^r}\right)p[\lambda_s p + \lambda_f(1-p)]\right\}$$

if

$$\left(1 - \frac{C}{p_0^{r^*}}\right)\left\{Aq\frac{C}{p_0^{r^*}} + Ap[\lambda_s q + \lambda_f(1-q)]\left(1 - \frac{C}{p_0^{r^*}}\right)\right\} + C \leq Aq \leq p_0^{r^*}.$$

Proof. In such a candidate equilibrium, B sells all tokens in ICO at price Aq per token, so

$$U_B(Aq, 1) = Aq - C.$$

Let G sell at price $p_0^{r^*}$ a fraction $\frac{C}{p_0^{r^*}}$ of his tokens in ICO. The "r" superscript indicates equilibrium price with token retention. His ICO pricing problem is more complex than in the baseline model. It is not simply Ap: the fair market price when the market knows his type is the expected total number of active users after the final sale multiplied by A, but the expected number of active users in the resale is not p times their mass.

Since in equilibrium the market must have rational expectation, $p_0^{r^*}$ is given by:

$$p_0^{r^*} = \max\left\{p_0^r \middle| p_0^r = \mathbb{E}[Au|G] = Ap\frac{C}{p_0^r} + A\left(1-\frac{C}{p_0^r}\right)p[\lambda_s p + \lambda_f(1-p)]\right\}.$$

The argument that the desired solution is the largest fixed point is as follows. In the equation above, if $\lambda_s p + \lambda_f(1-p) < 1$, Vieta's formula[1] suggests that it has only one positive root. If $\lambda_s p + \lambda_f(1-p) \geq 1$, then rearrange the equation to

$$p_0^{r^2} - Ap(\lambda_s p + \lambda_f(1-p))p_0^r - ApC\{1-[\lambda_s p + \lambda_f(1-p)]\} = 0.$$

Evaluating it at $p_0^r = Ap$ shows that the left hand side is negative, which implies that there is only one root above Ap. This root is our desired solution. This is because the late community formation probability is bigger than p, which means the fair ICO price should be higher than Ap.

In this equilibrium, G expects

$$\begin{aligned} U_G(p_0^{r^*}, \frac{C}{p_0^{r^*}}) &= p_0^* \frac{C}{p_0^{r^*}} + \left(1-\frac{C}{p_0^{r^*}}\right)\left\{Ap\frac{C}{p_0^{r^*}} + Ap[\lambda_s p + \lambda_f(1-p)]\left(1-\frac{C}{p_0^{r^*}}\right)\right\} - C \\ &= \left(1-\frac{C}{p_0^{r^*}}\right)\left\{Ap\frac{C}{p_0^{r^*}} + Ap[\lambda_s p + \lambda_f(1-p)]\left(1-\frac{C}{p_0^{r^*}}\right)\right\}. \end{aligned}$$

If B mimics G, there is a q probability that the initial community is formed and then the market will believe that the late community has a $\lambda_s p$ chance of successful formation probability. There is also a $1-q$ probability that the community is not formed, in which case the market will believe that the late community has a $\lambda_f p$ chance of formation. So the expected successful late community formation probability is $p[\lambda_s q + \lambda_f(1-q)]$. This inflated probability comes at a cost of not being able to sell everything at a potentially more profitable price Aq. So for mimicking to be suboptimal, we need

$$Aq - C \geq \left(1-\frac{C}{p_0^{r^*}}\right)\left\{Aq\frac{C}{p_0^{r^*}} + Ap[\lambda_s q + \lambda_f(1-q)]\left(1-\frac{C}{p_0^{r^*}}\right)\right\}$$

We assign off-path beliefs to be $\mathbb{P}(G|p_0 < p_0^{r^*}) = 1$ for all $p_0 < p_0^{r^*}$. Deviating to any price above $p_0^{r^*}$ will result in no sale. Then for G to find p_0 optimal, we simply need to check that he does not want to deviate to selling all his tokens at price Aq in ICO. So we need

$$Aq - C \leq \left(1-\frac{C}{p_0^{r^*}}\right)\left\{Ap\frac{C}{p_0^{r^*}} + Ap[\lambda_s p + \lambda_f(1-p)]\left(1-\frac{C}{p_0^{r^*}}\right)\right\}.$$

[1] See `https://en.wikipedia.org/wiki/Vieta\%27s_formulas` .

From the definition of $p_0^{r^*}$, this is equivalent to

$$Aq \leq p_0^{r^*}.$$

Notice that this condition is always satisfied when $\lambda_s p + \lambda_f(1-p) > 1$, because then $p_0^{r^*}$ would be higher than Ap, which is higher than Aq. To sum up, the separating equilibrium with underpricing can be supported when

$$\left(1 - \frac{C}{p_0^{r^*}}\right)\left\{Aq\frac{C}{p_0^{r^*}} + Ap[\lambda_s q + \lambda_f(1-q)]\left(1 - \frac{C}{p_0^{r^*}}\right)\right\} + C \leq Aq \leq p_0^{r^*}.$$

This equilibrium is the Riley outcome and clearly passes the Intuitive Criterion by a similar argument as in the baseline model. □

Example

The following parameters satisfy all the conditions:

$$A = 2,$$

$$p = 0.35,$$

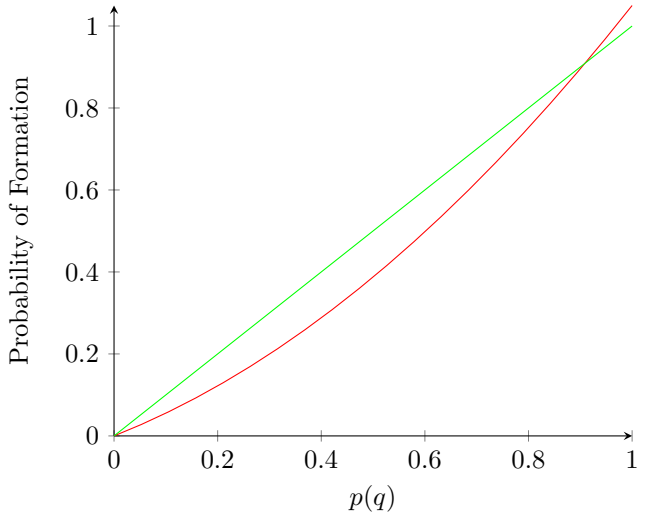
$$q = 0.3,$$

$$C = 0.4,$$

$$\lambda_s = 1.05,$$

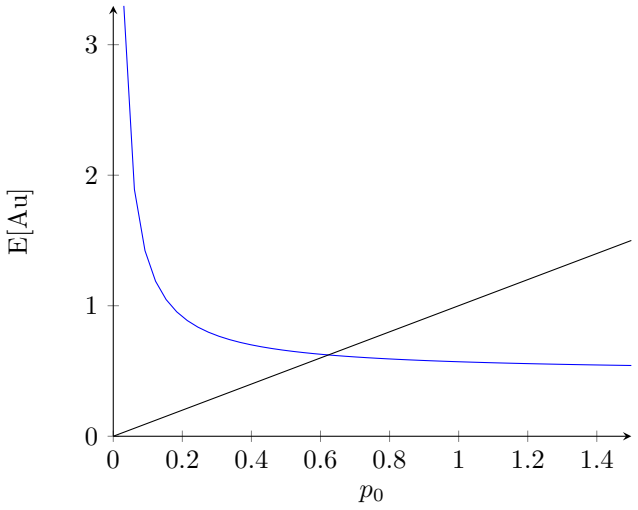
$$\lambda_f = 0.5.$$

In the graph below, the green line traces the probabilities of initial community formation for different p's and q's, and the red line traces the probabilities of late community formation for different p's and q's. In this example, both G and B have higher chances of making early investors stay in expectation, but B finds it more costly to limit sale in ICO.



Let us draw the function $f(p_0) = p_0$ (black) and RHS (blue) of the pricing equation for $p_0^{r^*}$:

$$p_0^{r^*} = \mathbb{E}[Au] = Ap\frac{C}{p_0^{r^*}} + A\left(1 - \frac{C}{p_0^{r^*}}\right)p[\lambda_s p + \lambda_f(1-p)].$$



$p_0^{r^*} = 0.62296$ is the desired solution.

Moreover,

$$\left(1 - \frac{C}{p_0^{r^*}}\right)\left\{Aq\frac{C}{p_0^{r^*}} + Ap[\lambda_s q + \lambda_f(1-q)]\left(1 - \frac{C}{p_0^{r^*}}\right)\right\} + C = 0.59751,$$

$$Aq = 0.6,$$

$$p_0^{r^*} = 0.62296,$$

so we have

$$\left(1-\frac{C}{p_0^{r^*}}\right)\left\{Aq\frac{C}{p_0^{r^*}}+Ap[\lambda_s q+\lambda_f(1-q)]\left(1-\frac{C}{p_0^{r^*}}\right)\right\}+C\leq Aq\leq p_0^{r^*}.$$

Moreover, clearly the projects have positive NPVs and entrepreneurs expect positive profits, so we have found a signaling by token retention equilibrium.

3.4.2 Signaling by Underpricing

Proposition 4: When $\lambda_s q+\lambda_f(1-q)\geq 1$, there can be a separating equilibrium with underpricing like the baseline model. In this equilibrium, B's ICO and final sale price would be the same, which we denote p_B^*. It is the largest fixed point to the rational expectation equation:

$$p_B^*=\max\left\{p_B\middle|p_B=Aq\frac{C}{p_B}+A\left(1-\frac{C}{p_B}\right)q[\lambda_s q+\lambda_f(1-q)]\right\}.$$

G's ICO strategy is to sell at price $p_0^{u^*}$ a fraction $\frac{C}{p_0^{u^*}}$ of tokens, where $p_0^{u^*}$ solves

$$\left(1-\frac{C}{p_0^{u^*}}\right)\left\{Aq\frac{C}{p_0^{u^*}}+Ap[\lambda_s q+\lambda_f(1-q)]\left(1-\frac{C}{p_0^{u^*}}\right)\right\}=p_B^*-C,$$

The equilibrium exists if

$$\begin{aligned}&\left(1-\frac{C}{p_0^{u^*}}\right)\left\{Ap\frac{C}{p_0^{u^*}}+Ap[\lambda_s p+\lambda_f(1-p)]\left(1-\frac{C}{p_0^{u^*}}\right)\right\}\\ \geq&\left(1-\frac{C}{p_B^*}\right)\left\{Ap\frac{C}{p_B^*}+Aq[\lambda_s q+\lambda_f(1-q)]\left(1-\frac{C}{p_B^*}\right)\right\}.\end{aligned}$$

Proof. In a separating equilibrium with underpricing, B sells his tokens at the fair market price. No arbitrage and perfect competition among investors imply that the ICO price and the final sale price should be the same. So we need

$$p_B=\mathbb{E}[Au|B]=Aq\frac{C}{p_B}+A\left(1-\frac{C}{p_B}\right)q[\lambda_s q+\lambda_f(1-q)]$$

To see that the desired p_B^* is the largest fixed point, rearrange the equation to

$$p_B^2-Aq[\lambda_s q+\lambda_f(1-q)]p_B-AqC\{1-[\lambda_s q+\lambda_f(1-q)]\}=0.$$

Evaluating it at $p_B=Aq$ shows that the left hand side is negative, which implies that there is only one root

above Aq, which is our desired solution.

Then B's expected payoff is

$$U_B(p_B^*, \frac{C}{p_B^*}) = \left(1 - \frac{C}{p_B^*}\right)\left\{Aq\frac{C}{p_B^*} + A\left(1 - \frac{C}{p_B^*}\right)q[\lambda_s q + \lambda_f(1-q)]\right\} = p_B^* - C$$

Consider G's candidate equilibrium ICO strategy $(p_0^{u^*}, \alpha^{u^*})$. Here we add "u" to indicate that the price is the underpricing equilibrium price. The optimal $p_0^{u^*}$ is the highest price that makes B find too costly to mimic. Thus $p_0^{u^*}$ solves

$$\left(1 - \frac{C}{p_0^{u^*}}\right)\left\{Aq\frac{C}{p_0^{u^*}} + Ap[\lambda_s q + \lambda_f(1-q)]\left(1 - \frac{C}{p_0^{u^*}}\right)\right\} = p_B^* - C$$

Let us assign off-path beliefs to be $\mathbb{P}(G|p' > p_0^{u^*}) = 0$ and $\mathbb{P}(G|p' < p_0^{u^*}) = 1$. If G deviates, the best choice would be to deviate to B's price, because deviating to a price lower than $p_0^{u^*}$ gives the market the same belief as with $p_0^{u^*}$ but lowers profit and deviating to any price above $p_0^{u^*}$ and below p_B^* would make the market perceive G as B but yields a profit less than deviating to p_B^*. If G deviates to p_B^*, then the expected late community formation probability from the market after the realization of initial community formation result is $q[\lambda_s p + \lambda_f(1-p)]$, because there is a p probability of a successful initial community formation, which will make the market believe the late community formation has a $\lambda_s q$ probability of formation due to the market perceiving G as B, and a $(1-q)$ probability that the market believes the late community has a $\lambda_f q$ probability of formation due to a failed initial community formation. Therefore, for G to prefer underpricing to deviating to B's price, we need

$$\begin{aligned}&\left(1 - \frac{C}{p_0^{u^*}}\right)\left\{Ap\frac{C}{p_0^{u^*}} + Ap[\lambda_s p + \lambda_f(1-p)]\left(1 - \frac{C}{p_0^{u^*}}\right)\right\}\\ \geq &\left(1 - \frac{C}{p_B^*}\right)\left\{Ap\frac{C}{p_B^*} + Aq[\lambda_s p + \lambda_f(1-p)]\left(1 - \frac{C}{p_B^*}\right)\right\}.\end{aligned}$$

□

3.4.3 Signaling by Both Underpricing and Token Retention

When it is tempting for B to sell all tokens in the ICO, i.e., $\lambda_s q + \lambda_f(1-q) < 1$, it is possible that simply holding some tokens and sell the rest at the fair price is not a sufficiently credible signal for G. In this case, G may choose to sell just enough tokens to raise C at a discounted price.

Proposition 5: When $\lambda_s q + \lambda_f(1-q) < 1$, there exists a separating equilibrium where B sells $(Aq, 1)$ in

the ICO, and G sells $(p_0^{ur^*}, \frac{C}{p_0^{ur^*}})$, such that

$$\left(1-\frac{C}{p_0^{ur^*}}\right)\left\{Aq\frac{C}{p_0^{ur^*}}+Ap[\lambda_s q+\lambda_f(1-q)]\left(1-\frac{C}{p_0^{ur^*}}\right)\right\}=Aq-C,$$

if

$$\begin{aligned}&\left(1-\frac{C}{p_0^{ur^*}}\right)\left\{Ap\frac{C}{p_0^{ur^*}}+Ap[\lambda_s p+\lambda_f(1-p)]\left(1-\frac{C}{p_0^{ur^*}}\right)\right\}\\ \geq &\max\left\{Aq-C,\left(1-\frac{C}{p_B^*}\right)\left\{Ap\frac{C}{p_B^*}+Aq[\lambda_s p+\lambda_f(1-p)]\left(1-\frac{C}{p_B^*}\right)\right\}\right\}.\end{aligned}$$

Proof. B's expected payoff when following the strategy above is

$$U_B(Aq,1)=Aq-C.$$

The least-cost credible $(p_0^{ur^*}, \frac{C}{p_0^{ur^*}})$ is the one that makes B indifferent between mimicking and not:

$$\left(1-\frac{C}{p_0^{ur^*}}\right)\left\{Aq\frac{C}{p_0^{ur^*}}+Ap[\lambda_s q+\lambda_f(1-q)]\left(1-\frac{C}{p_0^{ur^*}}\right)\right\}=Aq-C$$

Let us assign off-path beliefs to be $\mathbb{P}(G|p'>p_0^{ur^*})=0, \mathbb{P}(G|\alpha'>\frac{C}{p_0^{ur^*}})=0$, and $\mathbb{P}(G|p'<p_0^{u^*}, \alpha'=\frac{C}{p_0'})=1$. If G deviates, there are two possibly optimal deviations. First, he can deviate to selling everything at price Aq in the ICO, which gives him a payoff of $Aq-C$. Second, he can deviate to B's fair market price when selling just enough to raise C, which is

$$p_B^*=\max\left\{p_B\middle|p_B=\mathbb{E}[Au|B]=Aq\frac{C}{p_B}+A\left(1-\frac{C}{p_B}\right)q[\lambda_s q+\lambda_f(1-q)]\right\}.$$

and then selling the rest in final sale where the market prices the late community formation probability as $q[\lambda_s p+\lambda_f(1-p)]$. This subtlety is because there is a p probability that the market will believe the late community has a $\lambda_s q$ chance of formation and $(1-p)$ probability that the market will believe the late community formation probability is $\lambda_f q$. Therefore, to sustain the separating equilibrium with both underpricing and token retention, we need

$$\begin{aligned}&\left(1-\frac{C}{p_0^{ur^*}}\right)\left\{Ap\frac{C}{p_0^{ur^*}}+Ap[\lambda_s p+\lambda_f(1-p)]\left(1-\frac{C}{p_0^{ur^*}}\right)\right\}\\ \geq &\max\left\{Aq-C,\left(1-\frac{C}{p_B^*}\right)\left\{Ap\frac{C}{p_B^*}+Aq[\lambda_s p+\lambda_f(1-p)]\left(1-\frac{C}{p_B^*}\right)\right\}\right\}.\end{aligned}$$

□

3.4.4 Pooling Equilibria

Now we analyze the existence of pooling equilibria. The situation in the extended model is more complicated because the developers could pool at either both selling all tokens in the ICO or both selling just enough to raise C.

Consider a candidate pooling equilibrium where both types pool at selling all tokens. This equilibrium exists as long as no developer prefers to deviate to selling just enough tokens to raise C while being perceived as Bad, which may be profitable because of the new network effect. The pooling equilibrium where both types pool at selling just enough tokens to raise C exists as long as no one wants to deviate to selling all tokens in the ICO and being perceived as Bad. To make sure these pooling equilibria pass the Intuitive Criterion, we cannot simply assign off-path beliefs such that all off-path actions are perceived as Bad, so we need to check that credible deviations by Good (underpricing alone and underpricing together with token retention) are not profitable.

Proposition 6: Under some conditions, there exist two types of pooling equilibria that pass the Intuitive Criterion. G and B can either pool at selling just enough tokens to raise C at price

$$p_0^{I^*} = \max\left\{p_0^I \middle| p_0^I = A[\theta p + (1-\theta)q]\frac{C}{p_0^I} + A\left(1 - \frac{C}{p_0^I}\right)\{\theta p[\lambda_s p + \lambda_f(1-p)] + (1-\theta)q[\lambda_s q + \lambda_f(1-q)]\}\right\},$$

or pool at selling all tokens in the ICO at price

$$p_0^{II^*} = A[\theta p + (1-\theta)q].$$

Proof. In the first kind of pooling equilibrium, say both types pool at $(p_0^{I^*}, \alpha^*)$ in the ICO. we need to determine equilibrium fair market price and fraction.

Rational expectation for $p_0^{I^*}$ requires that the fair ICO price is the expected network value knowing that there is a θ probability that the project is Good and $(1-\theta)$ probability it is Bad. Thus, we can solve p_0^I from the following equation:

$$p_0^{I^*} = \max\left\{p_0^I \middle| p_0^I = A[\theta p + (1-\theta)q]\frac{C}{p_0^I} + A\left(1 - \frac{C}{p_0^I}\right)\{\theta p[\lambda_s p + \lambda_f(1-p)] + (1-\theta)q[\lambda_s q + \lambda_f(1-q)]\}\right\}$$

As in the baseline model, investors update the probability of a project being Good upon observing the result

of initial community formation.

$$
\begin{aligned}
P_G^s \equiv \mathbb{P}(G|s) &= \frac{\theta p}{\theta p + (1-\theta)q}, \\
P_B^s \equiv \mathbb{P}(B|s) &= \frac{(1-\theta)q}{\theta p + (1-\theta)q}, \\
P_G^f \equiv \mathbb{P}(G|f) &= \frac{\theta(1-p)}{\theta(1-p) + (1-\theta)(1-q)}, \\
P_B^f \equiv \mathbb{P}(G|f) &= \frac{(1-\theta)(1-q)}{\theta(1-p) + (1-\theta)(1-q)}.
\end{aligned}
$$

If the market observes s, then the market would price the expected network value created by late users as

$$
p_1^s = A\left(1 - \frac{C}{p_0^{I^*}}\right)\{p[\lambda_s p + \lambda_f(1-p)]P_G^s + q[\lambda_s q + \lambda_f(1-q)]P_B^s\}.
$$

If the markets observes f, then it would price

$$
p_1^f = A\left(1 - \frac{C}{p_0^{I^*}}\right)\{p[\lambda_s p + \lambda_f(1-p)]P_G^f + q[\lambda_s q + \lambda_f(1-q)]P_B^f\}.
$$

The equilibrium fraction of tokens sold in the ICO is

$$
\alpha^{I^*} = \frac{C}{p_0^{I^*}}
$$

So G expects

$$
U_G(p_0^{I^*}, \alpha^{I^*}) = (1 - \alpha^{I^*})\left[pA\alpha^{I^*} + pp_1^s + (1-p)p_1^f\right].
$$

B expects

$$
U_B(p_0^{I^*}, \alpha^{I^*}) = (1 - \alpha^{I^*})\left[qA\alpha^{I^*} + qp_1^s + (1-q)p_1^f\right].
$$

If G wishes to signal his type instead through underpricing, then the least costly price he can set would be the highest price at which B finds too costly to mimic. Denote this price by p_0' and let $\alpha' = \frac{C}{p_0'}$. Let off-path beliefs be $\mathbb{P}(G|p_0 \leq p_0') = 1$ and $\mathbb{P}(G|p_0 > p_o') = 0$, where p_0' solves

$$
(1-\alpha^*)\left[qA\alpha^* + qp_1^s + (1-q)p_1^f\right] = \left(1 - \frac{C}{p_0'}\right)\left\{Aq\frac{C}{p_0'} + Ap[\lambda_s p + \lambda_f(1-p)]\left(1 - \frac{C}{p_0'}\right)\right\}
$$

We need G's payoff when playing $(p_0^{I^*}, \alpha^*)$ to be higher than deviating to (p_0', α'):

$$(1-\alpha^*)\left[pA\alpha^* + pp_1^s + (1-p)p_1^f\right] \geq \left(1-\frac{C}{p_0'}\right)\left\{Ap\frac{C}{p_0'} + Ap[\lambda_s p + \lambda_f(1-p)]\left(1-\frac{C}{p_0'}\right)\right\}.$$

If either type finds it costly to hold tokens, we additionally need to check that type prefers the pooling equilibrium to deviating to selling everything in ICO while being perceived as B. That is, suppose $\lambda_s q + \lambda_f(1-q) < 1$, we additionally need

$$(1-\alpha^*)\left[qA\alpha^* + qp_1^s + (1-q)p_1^f\right] \geq Aq - C.$$

If $\lambda_s p + \lambda_f(1-p) < 1$, we need

$$(1-\alpha^*)\left[pA\alpha^* + pp_1^s + (1-p)p_1^f\right] \geq Ap - C.$$

This equilibrium passes the Intuitive Criterion from our construction of off-path beliefs.

In the second kind of pooling equilibrium, both types sell all tokens in the ICO. In this case, the fair market price in ICO is

$$p_0^{II^*} = A[\theta p + (1-\theta)q].$$

Both types of developers get

$$U_G(p_0^{II^*}, 1) = U_B(p_0^{II^*}, 1) = p_0^{II^*} - C.$$

If G wants to signal his type, he can do so by token retention or both token retention and underpricing. Such signaling strategies are possible when B finds it costly to hold tokens after ICO, i.e., $\lambda_s q + \lambda_f(1-q) < 1$. Clearly if the set of parameters rule out the possibility of signaling by token retention and underpricing, it automatically rules out the possibility of signaling by underpricing.

Consider G's deviation to $(p_0^{ur'}, \frac{C}{p_0^{ur'}})$. It can credibly signal his type if B does not want to mimic, and the least-cost price is

$$\left(1-\frac{C}{p_0^{ur'}}\right)\left\{Aq\frac{C}{p_0^{ur'}} + Ap[\lambda_s q + \lambda_f(1-q)]\left(1-\frac{C}{p_0^{ur'}}\right)\right\} = p_0^{II^*} - C.$$

To pass the Intuitive Criterion, we need to set off-path beliefs to $\mathbb{P}(G|p' < p_0^{ur'}, \alpha' < 1) = 1$ and assign all

other deviations with the belief that it is done by B with probability 1. But suppose we can find such a solution, then this would mean that G's deviation payoff

$$\begin{aligned}&\left(1-\frac{C}{p_0^{ur'}}\right)\left\{Ap\frac{C}{p_0^{ur'}}+Ap[\lambda_s p+\lambda_f(1-p)]\left(1-\frac{C}{p_0^{ur'}}\right)\right\}\\ >&\left(1-\frac{C}{p_0^{ur'}}\right)\left\{Aq\frac{C}{p_0^{ur'}}+Ap[\lambda_s q+\lambda_f(1-q)]\left(1-\frac{C}{p_0^{ur'}}\right)\right\}=p_0^{II^*}-C\end{aligned}$$

So it would always be a profitable deviation. Thus the condition is that

$$\left(1-\frac{C}{p_0^{ur'}}\right)\left\{Aq\frac{C}{p_0^{ur'}}+Ap[\lambda_s q+\lambda_f(1-q)]\left(1-\frac{C}{p_0^{ur'}}\right)\right\}=p_0^{II^*}-C,$$

does not have a solution. In this case, we can assign off-path beliefs to be that any off-path action is done by B for sure.

Failing to satisfy this condition does not break the pooling PBE; it just means the PBE does not pass the Intuitive Criterion, since we can still support this PBE by assigning even these credible deviations to be done by B with probability 1. □

In our analysis above, there are three cases regarding Good and Bad developers' success probabilities: $\lambda_s p+\lambda_f(1-p)>\lambda_s q+\lambda_f(1-q)\geq 1$; $\lambda_s p+\lambda_f(1-p)\geq 1>\lambda_s q+\lambda_f(1-q)$; and $1\geq\lambda_s p+\lambda_f(1-p)>\lambda_s q+\lambda_f(1-q)$. In case one, both types pooling at selling a fraction of tokens welfare dominates selling all tokens: selling tokens later gives a higher chance of community formation for both types, which improves social welfare. In case three, both types pooling at selling all tokens in ICO welfare dominates selling a fraction of tokens, because selling tokens later leads to a lower chance of community formation for both types. In case two, welfare dominance depends on parameters.

3.4.5 Welfare Analysis

Let us define expected social welfare to be the expected profit of the developer plus the sum of expected profits of all investors. That is,

$$\mathbb{E}[\text{Social Welfare}]=\theta\mathbb{E}[U_G]+(1-\theta)\mathbb{E}[U_B]+\int_0^1\mathbb{E}\Pi(i)di,$$

where $\Pi(i)$ is investor i's profit. Although the direct calculation of welfare is complex, notice that the total expected welfare created by a project is the expected final value of the tokens minus the fixed cost. To see this, suppose the project is Good and the ICO strategy is (p_0,α) and the final sale price and fraction pair

is $(p_1^s, 1-\alpha)$ if the initial community formation succeeds and $(p_1^f, 1-\alpha)$ is it fails. Then the entrepreneur receives $p_0\alpha + pp_1^s(1-\alpha) + (1-p)p_1^f(1-\alpha)$ in total, and pays C to finish the project, so his expected payoff is

$$p_0\alpha + pp_1^s(1-\alpha) + (1-p)p_1^f(1-\alpha) - C.$$

Early investors pay a total of $p_0\alpha$ and late investors pay a total of $pp_1^s(1-\alpha)+(1-p)p_1^f(1-\alpha)$ in expectation to receive a mass one of tokens worth $\mathbb{E}[Au|G]$, so the sum of all investors' expected profits is

$$\mathbb{E}[Au|G] - [p_0\alpha + pp_1^s(1-\alpha) + (1-p)p_1^f(1-\alpha)].$$

Therefore, the expected social welfare from a Good project is

$$\begin{aligned} &p_0\alpha + pp_1^s(1-\alpha) + (1-p)p_1^f(1-\alpha) - C + \mathbb{E}[Au|G] - [p_0\alpha + pp_1^s(1-\alpha) + (1-p)p_1^f(1-\alpha)] \\ &= \mathbb{E}[Au|G] - C. \end{aligned}$$

Considering the project can be Good or Bad, we have

$$\mathbb{E}[\text{Social Welfare}] = \theta\mathbb{E}[Au|G] + (1-\theta)\mathbb{E}[Au|B] - C.$$

In the baseline model, all equilibria yield the same social welfare because all projects are financed and these equilibria differ only in the distribution of surplus among Good and Bad developers and investors. In the pooling equilibrium, investors expect zero profit due to perfect competition; in the separating equilibrium, investors expect zero profit when they finance the Bad developer's project, but earn positive profit when they finance the Good developer's project because he "leaves money on the table" for the investors to credibly signal his type.

As we have shown that separating and pooling equilibria can coexist in the baseline model, the same is true in the extended model. This can be seen if we choose λ_s and λ_f arbitrarily close to 1. On the other hand, separating and pooling equilibria no longer have the same welfare in the extended model.

Theorem 4: When both separating equilibrium and pooling equilibrium exist:

I. Expected social welfare is higher in the pooling equilibrium with token retention than the separating equilibrium with underpricing when both types' late community formation probabilities are higher than early.

II. Expected social welfare is higher in the separating equilibrium with token retention than any pooling equilibrium when G's late community formation probability is higher than early while B's early community formation probability is higher than late.

III. Expected social welfare is higher in the pooling equilibrium with no token retention than any separating equilibrium when both types' late community formation probabilities are lower than early.

Proof. Part I:

When $\lambda_s p + \lambda_f(1-p) > \lambda_s q + \lambda_f(1-q) > 1$, in the pooling equilibrium with token retention, in ICO B sells tokens at price

$$p_0^* = \max\left\{p_0 \middle| p_0 = A[\theta p + (1-\theta)q]\frac{C}{p_0} + A\left(1 - \frac{C}{p_0}\right)\{\theta[\lambda_s p + \lambda_f(1-p)] + (1-\theta)[\lambda_s q + \lambda_f(1-q)]\}\right\},$$

which is higher than the separating equilibrium's price

$$p_B^* = \max\left\{p_B \middle| p_B = \mathbb{E}[Au|B] = Aq\frac{C}{p_B} + A\left(1 - \frac{C}{p_B}\right)q[\lambda_s q + \lambda_f(1-q)]\right\}.$$

To see this analytically, if we define

$$F(x) = A[\theta p + (1-\theta)q]\frac{C}{x} + A(1 - \frac{C}{x})\{\theta[\lambda_s p + \lambda_f(1-p)] + (1-\theta)[\lambda_s q + \lambda_f(1-q)]\}$$

and

$$G(x) = Aq\frac{C}{x} + A\left(1 - \frac{C}{x}\right)q[\lambda_s q + \lambda_f(1-q)],$$

we have

$$\begin{aligned} & F(x) - G(x) \\ = \theta & \left\{\left\{Ap\frac{C}{x} + A\left(1 - \frac{C}{x}\right)p[\lambda_s p + \lambda_f(1-p)]\right\} - \left\{Aq\frac{C}{x} + A\left(1 - \frac{C}{x}\right)q[\lambda_s q + \lambda_f(1-q)]\right\}\right\} \\ & > 0 \end{aligned}$$

for all $x > 0$. Suppose towards contradiction that $p_B^* > p_0^*$. Then

$$F(p_B^*) > G(p_B^*) = p_B^*.$$

But $F(x)$ and $G(x)$ are less than x for large x:

$$\lim_{x\to\infty} F(x) = A\{\theta[\lambda_s p + \lambda_f(1-p)] + (1-\theta)[\lambda_s q + \lambda_f(1-q)]\}.$$

So we can find $\hat{p}$ such that

$$F(\hat{p}) < \hat{p}.$$

But then Bolzano's Intermediate Value Theorem suggests that we can find $p_0^* < p_0^{**} < \hat{p}$ such that

$$F(p_0^{**}) = p_0^{**},$$

which is a contradiction to p_0^* being the largest fixed point.

The intuition behind this result is clear: in the separating equilibrium B's tokens are not overpriced because the market has no uncertainty about its type.

Then we have $\frac{C}{p_0^*} < \frac{C}{p_B^*}$, which means B sells less tokens in the pooling equilibrium. In this case, selling more tokens in the final sale improves social welfare because the late community is more likely to be formed. So the pooling equilibrium yields a higher expected social welfare than the separating equilibrium when the developer is B. When the developer is G, the separating equilibrium's ICO price $p_0^{u^*}$, which is the fixed point solution to

$$\left(1 - \frac{C}{p_0^{u^*}}\right)\left\{Aq\frac{C}{p_0^{u^*}} + Ap[\lambda_s p + \lambda_f(1-p)]\left(1 - \frac{C}{p_0^{u^*}}\right)\right\} = p_B^* - C$$

must be lower than p_B^*, because otherwise B will always mimic. Then

$$\frac{C}{p_0^{u^*}} > \frac{C}{p_0^*}.$$

Thus in the separating equilibrium G also generates less social welfare as selling more tokens in ICO is not socially efficient. Therefore, in the separating equilibrium with token retention, social welfare must be lower than the poling equilibrium with token retention.

Part II:

When $\lambda_s p + \lambda_f(1-p) > 1 > \lambda_s q + \lambda_f(1-q)$, if the developer is G, the separating equilibrium generates the a higher expected social welfare than the pooling equilibrium with token retention. This is because

$p_0^{r^*} > p_0^*$ by a similar argument to that in Part I, which means $\frac{C}{p_0^{r^*}} < \frac{C}{p_0^*}$. This means G sells less tokens in the separating equilibrium, which improves social welfare because he expects a bigger community formation probability in the final sale. The pooling equilibrium also welfare dominates the pooling equilibrium with no token retention because it lowers social welfare when G does not sell any token in the late sale.

If the developer is B, the separating equilibrium also generates a higher expected social welfare than the pooling equilibrium with token retention, because B's welfare maximizing choice is to sell all tokens in the ICO. the separating equilibrium generates the same expected social welfare as the pooling equilibrium with no token retention because B sells all tokens in ICO in both cases and the difference in prices represents a difference in the distribution of surplus between the developer and investors rather than a difference in welfare.

Taking expectations, we conclude that the separating equilibrium generates a higher expected social welfare than all pooling equilibria.

Part III:
When $1 > \lambda_s p + \lambda_f(1-p) > \lambda_s q + \lambda_f(1-q)$, the welfare-maximizing choice is for both types of developers to sell all tokens in the ICO. This is the case in the pooling equilibrium but not the case in the separating equilibrium with token retention or in the separating equilibrium with both underpricing and token retention, because in such equilibria G holds tokens to signal his type. □

An immediate consequence of the theorem above is that any separating equilibrium that involves underpricing is never welfare-maximizing. The intuition is that credible underpricing makes a Good developer sell too many tokens early to raise enough fund, which lowers the total expected number of users because more buyers would have stayed as users if some of these tokens were sold in the final sale. This result has deep financial and policy implications due to the prevalence of underpricing in ICOs in practice.

3.5 Discussion

Our analysis above centered around the assumption that both types of projects have positive NPVs. This assumption might not be true in real life due to the existence of rug pulls in the cryptocurrency world. If bad projects have negative NPV, then our baseline model predicts the non-existence of any separating equilibrium with underpricing. The reasoning is the following. Suppose there exists such an equilibrium. Then G's ICO strategy (p^*, α^*) would raise C. In this equilibrium, B's type is revealed so his project is not financed. But if he mimics G, he will be able to raise C and have $1 - \alpha^*$ fraction of tokens left, which is a

strict gain if $\alpha^* < 1$. This logic also implies the non-existence of separating equilibrium with underpricing in the extended model.

Besides our signaling theory, there are other plausible reasons behind ICO underpricing. Rock (1986)'s theory of IPO underpricing might be able to explain ICO underpricing as well. Suppose entrepreneurs with private information about their projects' qualities price their coins at the expected market price. Informed investors avoid overpriced low-quality projects and only invest in high-quality projects, so to make sure uninformed investors break even in the long run, entrepreneurs need to underprice instead of pricing at the expected market price. This is plausible because arguably the degree of information asymmetry is higher in initial coin offering since the projects are very early-stage, while companies about to go public have much more public information.

Another possible explanation is that entrepreneurs use underpricing to to attract attention through the unusually high return when the coin is listed on exchanges. This would create anxiety from investors, i.e., FOMO (fear of missing out), which could then attract a large number of retail investors.

Lastly, due to the lack of regulation, some ICOs are fraudulent. Dowlat (2018) reports that about 11% of total money raised through ICO were given to scammers and 14% went to failed projects. Since many investors are risk averse, to make them stay in the market, they have to be compensated by risk premium possibly in the form of ICO underpricing.

3.6 Conclusion

In this paper, we developed a signaling theory to explain the empirically confirmed phenomena of underpricing and token retention in Initial Coin Offerings. When the project's type is the entrepreneur's private information, a Good entrepreneur can credibly signal his type by choosing a low price in the ICO because he can recoup the loss from future sales owing to the high probability of successful community formation while a Bad entrepreneur cannot. We also derived conditions under which the Good entrepreneur can use airdrop instead of underpricing as a signal, which can be viewed as a special case of two-stage ICOs. We then added another type of network effect to the model: the late community is more likely to be formed successfully if the early community is successfully formed. We proved that with this new element, under some conditions there exists another type of separating equilibrium, where the Good developer can use token retention as the signal of quality. Finally, we gave a welfare dominance result when both pooling and separating equilibria exist.

Future works might develop models that seek to explain underpricing from other theories that we discussed

in the discussion section. It might also be worth studying the optimal design of multi-stage ICOs in the presence of asymmetric information and moral hazard.

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