PARTIAL REGULARITY FOR BV MINIMIZERS

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ABSTRACT. We establish an ε-regularity result for the derivative of a map of bounded variation that minimizes a strongly quasiconvex variational integral of linear growth, and, as a consequence, the partial regularity of such BV minimizers. This result extends the regularity theory for minimizers of quasiconvex integrals on Sobolev spaces to the context of maps of bounded variation. Previous partial regularity results for BV minimizers in the linear growth set-up were confined to the convex situation.

1. INTRODUCTION

In this paper we investigate the local regularity properties of minimizers for variational integrals defined on Dirichlet classes of maps of bounded variation. In order to describe our set-up more precisely and why it is natural we consider a continuous real-valued function defined on $N \times n$ matrices, $F \colon \mathbb{R}^{N\times n} \to \mathbb{R}$, that we henceforth call an integrand. Assume that F is of linear growth, that is, for some constant $L > 0$ we have

$$|F(z)| \leq L\big(|z| + 1\big) \tag{1.1}$$

for all matrices $z \in \mathbb{R}^{N\times n}$. The reader is referred to Section 2 for undefined notation and terminology. For a bounded Lipschitz domain Ω in $\mathbb{R}^n$ and a given $\mathrm{W}^{1,1} = \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ Sobolev map $g \colon \Omega \to \mathbb{R}^N$ as boundary datum we seek to minimize

$$\int_\Omega F(\nabla v(x))\,\mathrm{d}x \tag{1.2}$$

over $v \in \mathrm{W}^{1,1}_g = \mathrm{W}^{1,1}_g(\Omega, \mathbb{R}^N)$, the $\mathrm{W}^{1,1}$ Dirichlet class determined by g. Here ∇v denotes the approximate Jacobi matrix that we recall coincides with the distributional derivative $Dv = \nabla v \mathscr{L}^n$ when v is a $\mathrm{W}^{1,1}$ Sobolev map, thus $\nabla v(x) := \big[\partial v_j / \partial x_i(x)\big]$, where j is the row number and i is the column number whereby ∇v is $\mathbb{R}^{N\times n}$-valued. The standard approach to the variational problem (1.2) is to let the functional set-up be dictated by the coercivity inherent to the problem. Under the linear growth hypothesis (1.1) the best one can hope for is that *all* minimizing sequences for (1.2) on $\mathrm{W}^{1,1}_g$ are bounded in the Sobolev space $\mathrm{W}^{1,1}$. Building on [22] we show in Proposition 3.1 below that this is equivalent to the existence of constants $c_1 > 0$, $c_2 \in \mathbb{R}$ such that

$$\int_\Omega F(\nabla v(x))\,\mathrm{d}x \geq \int_\Omega \bigg(c_1|\nabla v(x)| + c_2 \bigg)\,\mathrm{d}x \tag{1.3}$$

holds for all $v \in \mathrm{W}^{1,1}_g$. We express (1.3) by saying that F is mean coercive. In turn, Proposition 3.1 also establishes the equivalence between mean coercivity and the existence of a constant $\ell > 0$ such that $F - \ell E$ is *quasiconvex* at some $z_0 \in \mathbb{R}^{N\times n}$. Here $E \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is our *reference integrand* defined as

$$E(z) = \sqrt{1 + |z|^2} - 1. \tag{1.4}$$

We occasionally also refer to E as the *reduced area integrand*. By quasiconvexity we understand the notion introduced by MORREY in [63], and recall that for $n = 1$ or $N = 1$, quasiconvexity is just ordinary convexity, whereas in the multi-dimensional vectorial case $n,\ N > 1$, considered in this paper, many nonconvex quasiconvex integrands of linear growth exist. We refer to Subsection 2.5 for a discussion of this.

Now, the question of existence of minimizers can be successfully tackled if we assume that (1.3) holds and allow maps of bounded variation as minimizers. Indeed, a minimizing sequence (u_j) for the problem (1.2) is then bounded in $\mathrm{W}^{1,1}$ and so admits a subsequence (u_{j_k}) so that for some $u \in \mathrm{BV} = \mathrm{BV}(\Omega, \mathbb{R}^N)$ we have $u_{j_k} \to u$ in L^1 and of course still $\sup_k \int_\Omega |\nabla u_{j_k}| \,\mathrm{d}x < \infty$. We express this by writing $u_{j_k} \overset{*}{\rightharpoonup} u$ in BV and recall that $u \colon \Omega \to \mathbb{R}^N$ is of bounded variation, written $u \in \mathrm{BV}$, if it is L^1 and its distributional partial derivatives are measures: $Du = [\partial u_i / \partial x_j]$ is a bounded $\mathbb{R}^{N\times n}$-valued Radon measure on Ω. We must extend the functional (1.2) to such u in a meaningful way, which, in the present context, is done most conveniently by semicontinuity following a procedure used by LEBESGUE, SERRIN and for quasiconvex integrals of anisotropic growth MARCELLINI [55]:

$$(1.5) \qquad \mathscr{F}_g[u,\Omega] := \inf \left\{ \liminf_{j\to\infty} \int_\Omega F(\nabla u_j)\,\mathrm{d}x \colon\ (u_j) \subset \mathrm{W}^{1,1}_g(\Omega,\mathbb{R}^N),\ u_j \to u \text{ in } \mathrm{L}^1(\Omega,\mathbb{R}^N) \right\}$$

Building on the works by AMBROSIO & DAL MASO [7] and FONSECA & MÜLLER [33] an integral representation for the functional $\mathscr{F}_g[u,\Omega]$ was found in [52] under the assumptions of quasiconvexity, linear growth (1.1) and mean coercivity (1.3). For such integrands we define the *recession integrand* by

$$F^\infty(z) := \limsup_{t\nearrow\infty} \frac{F(tz)}{t}, \qquad z \in \mathbb{R}^{N\times n}.$$

Then F^∞ is quasiconvex and positively 1-homogeneous [66]. Given $u \in \mathrm{BV}$ we write the Lebesgue–Radon–Nikodým decomposition of the measure Du into its absolutely continuous and singular parts with respect to Lebesgue measure $\mathscr{L}^n$ as

$$Du = D^{ac}u + D^s u = \nabla u \mathscr{L}^n + \tfrac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}|D^s u|,$$

particularly recalling that u is approximately differentiable $\mathscr{L}^n$ almost everywhere and $D^{ac}u = \nabla u \mathscr{L}^n$. In such terms we have

$$(1.6) \qquad \mathscr{F}_g[u,\Omega] = \int_\Omega F(\nabla u)\,\mathrm{d}x + \int_\Omega F^\infty\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u| + \int_{\partial\Omega} F^\infty\big((g-u)\otimes\nu_\Omega\big)\,\mathrm{d}\mathcal{H}^{n-1},$$

where ν_Ω is the outward unit normal on $\partial\Omega$. The last term, akin to a penalization term for failure to satisfy the Dirichlet boundary condition, must be present because the trace operator is not weakly* continuous on BV. We shall use the shorthand

$$\int_\Omega F(Du) := \int_\Omega F(\nabla u)\,\mathrm{d}x + \int_\Omega F^\infty\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u|$$

for the first two terms on the right-hand side in (1.6). It turns out that this expression also coincides with an extension by (area-strict) continuity of the integral (1.2) initially defined on the Sobolev space $\mathrm{W}^{1,1}$ (see [52, Theorem 4] and Lemmas 2.1 and 2.2 below). Let us summarize, under the above assumptions on F we have that all minimizing sequences admit a weakly* convergent subsequence whose limit $u \in \mathrm{BV}$ is a minimizer for the functional defined at (1.6): $\mathscr{F}_g[u,\Omega] \leq \mathscr{F}_g[v,\Omega]$ holds for all $v \in \mathrm{BV}$. In particular, for $v \in \mathrm{BV}$ so that $u - v$ has compact support in Ω,

$$(1.7) \qquad \int_\Omega F(Du) \leq \int_\Omega F(Dv)$$

holds. It is clear that we should not expect that the minimality condition (1.7) under the above assumptions on F would entail regularity of u on the Schauder $\mathrm{C}^{k,\alpha}$ scale for a $k \geq 1$. For that we must evidently impose a stronger quasiconvexity condition on F, one that in particular ensures that F cannot be (rank one) affine on any open subset of matrix space $\mathbb{R}^{N\times n}$. In view of the above discussion it is natural to require that, for some fixed positive constant $\ell > 0$, $F - \ell E$ is quasiconvex at *all* $z \in \mathbb{R}^{N\times n}$. That this turns out to be sufficient for regularity is our main result:

Theorem 1.1. *Let ℓ, $L > 0$ be positive constants and suppose the integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies the following three hypotheses:*

$$\text{(H0)} \qquad F \;\text{ is }\; \mathrm{C}^{2,1}_{\mathrm{loc}}$$

$$\text{(H1)} \qquad |F(z)| \leq L(|z|+1) \quad \forall\, z \in \mathbb{R}^{N\times n}$$

$$\text{(H2)} \qquad z \mapsto F(z) - \ell E(z) \text{ is quasiconvex,}$$

where $E\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is the reference integrand defined in (1.4). Then for each $m > 0$ there exists $\varepsilon_m = \varepsilon_m(\ell/L, F'') > 0$ with the following property: If $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ is a minimizer in the sense of (1.7), and $B = B_r(x_0) \subset \Omega$ is a ball such that

$$|(Du)_B| := \left|\frac{Du(B)}{\mathscr{L}^n(B)}\right| < m \quad \text{and} \quad \frac{1}{\mathscr{L}^n(B)}\int_B E\big(Du - (Du)_B\mathscr{L}^n\big) < \varepsilon_m, \tag{1.8}$$

then u is $\mathrm{C}^{2,\alpha}$ on $B_{r/2}(x_0)$ for each $\alpha < 1$. More precisely, u is C^2 on $B_{r/2}(x_0)$ and there exists a constant $c = c(\alpha, \ell/L, F'')$ so that

$$\sup_{\substack{x,y\in B_{r/2}(x_0)\\ x\neq y}} \frac{|\nabla^2 u(x) - \nabla^2 u(y)|^2}{|x-y|^{2\alpha}} \leq \frac{c}{r^{n+2+2\alpha}} \int_{B_r(x_0)} E\big(Du - (Du)_{B_r(x_0)}\mathscr{L}^n\big). \tag{1.9}$$

In particular, it follows that the minimizer u is partially regular, in the sense that an open subset Ω_u of Ω exists such that $\mathscr{L}^n(\Omega \setminus \Omega_u) = 0$ and u is $\mathrm{C}^{2,\alpha}_{\mathrm{loc}}$ on Ω_u for each $\alpha < 1$.

It is important to note that without the smallness condition (1.8) we do not expect the minimizer to be regular in the sense of (1.9). This is a feature of the multi-dimensional vectorial case n, $N \geq 2$ rather than our assumptions, at least when $n \geq 3$, $N \geq 2$. Indeed, for dimensions $n \geq 3$, $N \geq 2$ there exists a regular variational integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ (meaning that F is C^∞ smooth, has bounded second derivative $|F''| \leq L$ and $z \mapsto F(z) - \ell|z|^2$ is convex) such that the corresponding variational integral admits a Lipschitz but non-C^1 minimizer, see [62]. In higher dimensions the minimizers of regular variational integrals can be more singular, for instance non-Lipschitz when $n \geq 3$, $N \geq 5$, and unbounded when $n \geq 5$, $N \geq 14$, see [74, 75]. When $n = 2$, $N \geq 2$ it is a result due to MORREY [64] that minimizers of regular variational integrals must be smooth, but at present it is still not clear precisely how big the singular set $\Omega \setminus \Omega_u$ can be when $n \geq 3$, $N \geq 2$. Higher differentiability and GEHRING's lemma [37] (in the adapted form [39, Proposition 5.1], see also [59, Proposition 3]) yield in combination with standard arguments from Potential Theory that, for minimizers of regular variational problems, the Hausdorff dimension of the singular set is strictly smaller than $n - 2$. We refer to [13, 40, 45, 60] for a comprehensive discussion of this and related matters. In the quasiconvex p-growth case these methods do not apply and the only result at present is [51]. There it is shown, under the additional assumption that the minimizer is Lipschitz, that its singular set is uniformly porous (and so in particular that it has an outer Minkowski dimension strictly smaller than n).

The underlying ideas for the proof of Theorem 1.1 have many sources and it is not easy to give proper credit. However, the proof strategy can be traced back to at least DE GIORGI [25] and ALMGREN [5, 6] in their works on minimal surfaces in the parametric context of geometric measure theory. The first ones to adapt their strategy to the nonparametric case seem to be GIUSTI & MIRANDA [46] and MORREY [65], who proved partial regularity for minimizers to regular variational problems and weak solutions to certain nonlinear elliptic systems, respectively. The key step in these proofs is to establish a so-called excess decay estimate, which amounts to an integral expression of Hölder continuity. This is achieved by use of the very robust excess decay estimates that hold for solutions to linear elliptic systems with constant coefficients. Indeed, these excess decay estimates are then transferred to the minimizer/weak solution by means of a linearization procedure and Caccioppoli inequalities. In the presence of convexity/monotonicity the required Caccioppoli inequalities are derived by suitable use of a difference-quotient method. This method cannot be applied in the quasiconvex case. The difficulty was overcome by EVANS [30] who adapted an argument used by WIDMAN [79] in another context to derive Caccioppoli inequalities of the second kind. Hereby he proved partial regularity of minimizers under controlled quadratic growth conditions (see [45] and [40] for the terminology). Shortly afterwards FUSCO & HUTCHINSON [35] and GIAQUINTA & MODICA [41] extended the result to minimizers of variational integrals with general integrands $F = F(x, u, \nabla u)$ of controlled p-growth in the ∇u variable for $p \geq 2$. This was further extended by ACERBI & FUSCO [1] to integrands of natural p-growth for $p \geq 2$. A more direct proof of this result was subsequently obtained by GIAQUINTA [42] who also established Caccioppoli inequalities for minimizers in the general case $F = F(x, u, \nabla u)$ with p-growth for $p \geq 2$. Let us note that the main role of the Caccioppoli inequalities in these proofs is to provide *compactness* in some suitable context dependent sense. This is clearly seen in the blow-up arguments used in for instance [30, 35, 1], and it was noticed by EVANS & GARIEPY [31] that it is possible to extract the necessary compactness information without explicitly going through a Caccioppoli inequality. Partial regularity in the general subquadratic case was established by CAROZZA, FUSCO & MINGIONE in [19], and many interesting extensions have followed since then, including [2, 3, 21, 34, 36, 23, 26, 28, 27, 47, 48, 70]. The monograph [45] gives a good summary of the situation around the mid 90s. All the above results concern the case of variational integrals that are coercive on a Sobolev space $\mathrm{W}^{1,p}$ for some $p > 1$ and do not concern the linear growth case. The only previous partial regularity results in the multi-dimensional vectorial case for minimizers of variational integrals of linear growth were based on a method proposed by ANZELLOTTI & GIAQUINTA in [10]. While this method has been adapted by SCHMIDT [71, 72] to cover some degenerate convex cases too, the method still crucially relies on convexity, and it cannot work for quasiconvex integrands. Further references on various interesting aspects of existence and regularity of minimizers in the BV context with a standard convexity assumption include [12, 14, 15, 38].

Remark 1.2. *The main point of Theorem 1.1 is that the smallness condition (1.8) under the hypotheses* (H0)*,* (H1)*,* (H2) *yields* C^1 *regularity of the minimizer near the point* x_0*. The fact that we obtain* $\mathrm{C}^{2,\alpha}$ *regularity on* $B_{r/2}(x_0)$ *for all* $\alpha < 1$ *is a standard outcome of this type of proof. In this connection we emphasize our hypothesis* (H0) *that is stronger than the usual assumption of* C^2 *which is normally used in this context. We invite the reader to check that our proof also yields* C^2 *regularity on* $B_{r/2}(x_0)$ *of minimizers under the smallness condition (1.8) when* (H0) *is relaxed to*

$$(\underline{\mathrm{H0}}) \qquad F \textit{ is } \mathrm{C}^{2,\beta}_{\mathrm{loc}}$$

for some $\beta > 1 - \frac{2}{n}$*. We are indepted to one of the referees for this observation. However, the proof does seem to require a local smoothness assumption on the integrand that is stronger than* C^2 *and it is even unclear if one can relax it beyond* ($\underline{\mathrm{H0}}$).

As indicated above, we prove Theorem 1.1 by adapting the linearization procedure and Caccioppoli inequalities to the linear growth BV scenario. In doing this there are a number of difficulties that must be overcome. The main difficulty turns out to be the linearization procedure where one cannot work in the natural energy space $\mathrm{W}^{1,2}$ for the linear elliptic system that corresponds to a suitable second Taylor polynomial of the integrand. This already happens in the case of subquadratic growth integrands on Sobolev space $\mathrm{W}^{1,p}$, but the situation in the linear growth case is more severe as it must be degenerate at infinity by its very nature. The usual ways for implementing this step do seem to require modification. Our variant consists in an explicit construction of a test map that delivers the required estimate upon use. We believe this approach could be a useful alternative in the standard p-growth case too, and we intend to return to this and other applications in future work. The Caccioppoli inequality of the second kind is established following the proof given by EVANS [30] and does not cause any problems. However, it is important to emphasize that in the linear growth case these Caccioppoli inequalities do not allow us to establish a reverse Hölder inequality for the gradient and so we cannot prove higher integrability by use of Gehring's Lemma. Indeed, such higher integrability is ruled out by easy counterexamples, for instance the sign function on $(-1, 1)$ which *does satisfy a Caccioppoli inequality* and is of bounded variation, but clearly not of Sobolev class $\mathrm{W}^{1,1}$. A brief discussion of the compactness that can be inferred from a Caccioppoli inequality of the second kind is contained in Remark 4.5 below. We refer the reader to [43] for more details on this, but emphasize that it is for this reason that we have not been able to treat the case of minimizers for the general linear growth case $F = F(x, u, \nabla u)$ so far.

Finally we note that the proof of Theorem 1.1 is fairly robust. However, in view of the failure of Korn's inequality in L^1, and its consequence that the space of maps of *bounded deformation* BD is strictly larger than BV, the extension of our results to a BD context under natural assumptions is not immediate. The main difficulty in transferring the proofs is that BD maps do not have an obvious *Fubini property* as do BV maps (see Lemma 2.3). Nevertheless, this obstacle can be overcome and the first author has extended some of the results presented here to BD in his DPhil thesis [43].

1.1. **Organization of the paper.** In Section 2 we clarify notation, collect basic facts about BV-functions and record various auxiliary estimates. We mention here in particular Subsection 2.5 on quasiconvexity that, besides recalling the relevant definitions and elementary facts, also makes explicit the very flexible and possibly *nonconvex* nature of integrands satisfying the hypotheses (H0), (H1), (H2). Section 3 contains the proof of Proposition 3.1 that, as mentioned above, clarifies the role of our strong quasiconvexity assumption (H2). The subsequent Section 4 is devoted to the proof of Theorem 1.1 that we spell out into 5 steps, each presented in a subsection. Probably the most interesting aspect of the proof is contained in Subsection 4.3 on approximation by harmonic maps, alias the linearization procedure. Finally, we end the paper by briefly indicating possible extensions and variants of Theorem 1.1 that can be easily established by suitable modifications of the proof given in Section 4.

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2. Preliminaries

2.1. **Basic notation.** Throughout the paper $|\cdot|$ has its usual meanings, including as the standard norm on the space $\mathbb{R}^{N\times n}$ of N-by-n matrices with real entries: for $z, w \in \mathbb{R}^{N\times n}$ we let $z \cdot w = \mathrm{trace}(z^{\dagger} w)$ and $|z| = \sqrt{z \cdot z}$. What is meant will be clear from the given context. The open ball with centre x_0 and radius $r > 0$ in $\mathbb{R}^n$ is denoted by $B_r(x_0)$, and if we require open balls in $\mathbb{R}^N$ or in $\mathbb{R}^{N\times n}$ we use a similar notation, but make it clear in the given context where the ball is given.

For an integrand $F \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ that is differentiable we let

$$F'(z)[w] := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} F(z + tw)$$

and, if it is twice differentiable,

$$F''(z)[w,w] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} F(z + tw)$$

and so on. The usual conventions apply, and we think of $F'(z)$ both as a linear function and as an N-by-n matrix, and of $F''(z)$ as a symmetric bilinear form.

2.2. **Functions of measures.** Here we fix the notation and recall background facts about measures. Our reference for measure theory is [8] whose notation and terminology we also follow. In particular, the Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathscr{L}^n$ and the d-dimensional Hausdorff measure by $\mathcal{H}^d$. Also for a measure μ and a measurable set A we write $\mu \llcorner A$ for the measure $B \mapsto \mu(A \cap B)$.

Let $\mathbb{H}$ be a finite dimensional Hilbert space and let μ be an $\mathbb{H}$–valued Radon measure on the open subset Ω of $\mathbb{R}^n$. Its total variation measure, denoted $|\mu|$ and defined using the norm of $\mathbb{H}$, is a nonnegative (possibly infinite) Radon measure on Ω. We say that μ is a bounded Radon measure if it has finite total variation on Ω: $|\mu|(\Omega) < \infty$. With respect to the n–dimensional Lebesgue measure $\mathscr{L}^n$ we have the Lebesgue–Radon–Nikodým decomposition of μ:

$$\mu = \mu^a + \mu^s = \frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}\mathscr{L}^n + \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|^s}|\mu|^s.$$

For a Borel function $f\colon \Omega\times\mathbb{H}\to\mathbb{R}$ satisfying for some constant $c\geq 0$ the linear growth, or 1-growth, condition $|f(x,z)|\leq c\big(|z|+1\big)$ for all $(x,z)\in\Omega\times\mathbb{H}$ we define the (upper) recession function as

$$f^\infty(x,z) := \limsup_{\substack{x'\to x,\, z'\to z\\ t\to\infty}} \frac{f(x',tz')}{t}, \qquad (x,z)\in\Omega\times\mathbb{H}. \tag{2.1}$$

Hereby $f^\infty\colon \Omega\times\mathbb{H}\to\mathbb{R}$ is Borel, satisfies the growth condition $|f^\infty(x,z)|\leq c|z|$ for all $(x,z)\in\Omega\times\mathbb{H}$ and is positively 1-homogeneous in its second argument: $f^\infty(x,tz)=tf^\infty(x,z)$ for $t\geq 0$. For μ, f as above we define the signed Radon measure $f(\cdot,\mu)$ by prescribing for each Borel set A whose closure is compact and contained in Ω that

$$\int_A f(\cdot,\mu) := \int_A f\left(\cdot,\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}\right)\,\mathrm{d}\mathscr{L}^n + \int_A f^\infty\left(\cdot,\frac{\mathrm{d}\mu}{\mathrm{d}|\mu^s|}\right)\,\mathrm{d}|\mu^s|.$$

When μ is a bounded Radon measure the above formula extends to all Borel sets $A\subseteq\Omega$ and we easily check that it hereby defines a bounded Radon measure on Ω. When, in addition to the above, f is assumed continuous and the limes superior in (2.1) is a limit for all (x,z), then we say that f admits a regular recession function. It is then easily seen that f^∞ must be continuous too (as a locally uniform limit of continuous functions). Note that the function $f = 1_\Omega\otimes E$ satisfies the above conditions and admits a regular recession function, $f^\infty = 1_\Omega\otimes|\cdot|$. In fact, as is easily seen, for any convex function $g\colon\mathbb{H}\to\mathbb{R}$ satisfying the above 1-growth condition, the integrand $f=1_\Omega\otimes g$ admits a regular recession function.

We apply in particular the above notation to functions that, as in the last two examples, do not depend explicitly on x, thus for $f\colon\mathbb{H}\to\mathbb{R}$, we write interchangably $f(\mu)(A)$ and $\int_A f(\mu)$ for the corresponding measure. This notation is consistent in the sense that for $f=|\cdot|$, $f(\mu)$ is simply the total variation measure of μ and for $f=E$, $f(\mu)+\mathscr{L}^n$ is the total variation measure of the $\mathbb{H}\times\mathbb{R}$-valued measure $(\mu,\mathscr{L}^n)$. It is well-known that these two functionals give rise to useful notions of convergence for sequences of Radon measures. For bounded $\mathbb{H}$-valued Radon measures on Ω we say that $\mu_j\to\mu$ strictly on Ω iff $\mu_j\overset{*}{\rightharpoonup}\mu$ in $\mathrm{C}_0(\Omega,\mathbb{H})^*$ and $|\mu_j|(\Omega)\to|\mu|(\Omega)$. A slightly stronger mode of convergence is E-strict or area-strict convergence on Ω: $\mu_j\overset{*}{\rightharpoonup}\mu$ in $\mathrm{C}_0(\Omega,\mathbb{H})^*$ and $\int_\Omega E(\mu_j)\to\int_\Omega E(\mu)$. Any Radon measure can be area-strictly approximated by smooth maps using mollification (see for instance [8, Theorem 2.2]) and a well-known result of RESHETNYAK [68] (see also [8, Theorem 2.39] or [52, Appendix]) states that for a continuous function $f\colon\Omega\times\mathbb{H}\to\mathbb{R}$ of 1-growth and admitting a regular recession function we have

$$\int_\Omega f(\cdot,\mu_j)\to\int_\Omega f(\cdot,\mu)$$

whenever $\mu_j\to\mu$ area-strictly on Ω. Finally we shall often use the short-hand

$$\int_\Omega f(\mu-z) := \int_\Omega f\big(\mu - z\mathscr{L}^n\big)$$

for $z\in\mathbb{H}$ and $\mathbb{H}$-valued Radon measures μ.

2.3. **Mappings of bounded variation.** Our reference for Sobolev maps and for maps of bounded variation is [8] and we follow the notation and terminology used there. Here we briefly recall a few definitions and background results.

Let Ω be a bounded, open subset of $\mathbb{R}^n$. We say that an integrable map $u\colon \Omega \to \mathbb{R}^N$ has bounded variation if its distributional gradient can be represented by a bounded $\mathbb{R}^{N\times n}$–valued Radon measure, that is, if

$$|Du|(\Omega) := \sup\left\{ \int_\Omega u\cdot \operatorname{div}(\varphi)\,\mathrm{d}x\colon\ \varphi\in \mathrm{C}^1_c(\Omega,\mathbb{R}^{N\times n}),\ |\varphi|\leq 1\right\} < \infty.$$

Here and in what follows, the divergence operator, div, applied to $\mathbb{R}^{N\times n}$–valued distributions is understood to act row–wise. In particular, $\operatorname{div}(\varphi)$ above is $\mathbb{R}^N$-valued. The space of maps of bounded variation is denoted by $\mathrm{BV}(\Omega,\mathbb{R}^N)$ and it is a Banach space under the norm $\|v\|_{\mathrm{BV}} := \|v\|_{\mathrm{L}^1} + |Dv|(\Omega)$. We shall use freely the results from [8] for such maps, including in particular Poincaré and Sobolev type inequalities.

We stress that we throughout the paper consider integrable maps in terms of their precise representatives that we define as follows. Assume $u\in \mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{H})$, where as in the previous subsection $\mathbb{H}$ denotes a finite dimensional Hilbert space. We say that u has approximate limit $y\in\mathbb{H}$ at $x_0\in\Omega$, and write

$$\operatorname{ap}\lim_{x\to x_0} u(x) = y$$

provided that

$$\lim_{r\searrow 0} ⨍_{B_r(x_0)} |u(x)-y|\,\mathrm{d}x = 0.$$

The set S_u of points in Ω where no such limit exists is the approximate discontinuity set for u: $S_u = \{x\in\Omega :\ u \text{ has no approximate limit at } x\}$. It is an $\mathscr{L}^n$ negligible Borel set and the precise representative is defined for each $x\in\Omega\setminus S_u$ by (a slight abuse of notation):

$$u(x) := \operatorname{ap}\lim_{x'\to x} u(x').$$

Then $u\colon \Omega\setminus S_u\to\mathbb{H}$ is Borel measurable, and when u is merely assumed locally integrable or if it is of $\mathrm{W}^{1,p}$ Sobolev class, it is not so important for the developments of this paper how we define the precise representative on the set S_u. However, we record that when u is of $\mathrm{W}^{1,p}$ Sobolev class, then always $\mathcal{H}^{n-1}(S_u)=0$, and in fact it is much smaller when $p>1$ (see [45]). The case of a BV map is more delicate and we adopt the definition of precise representative on the approximate jump discontinuity set $J_u\subseteq S_u$ given in [8, Remark 3.79 and Corollary 3.80] and recall that $\mathcal{H}^{n-1}(S_u\setminus J_u)=0$. Note that when $(\rho_\varepsilon)_{\varepsilon>0}$ is a standard smooth mollifier and $u\in\mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{H})$, then $u_\varepsilon=\rho_\varepsilon * u$ is C^∞ on $\Omega_\varepsilon=\{x\in\Omega :\ \operatorname{dist}(x,\partial\Omega)>\varepsilon\}$ and $u_\varepsilon(x)\to u(x)$ as $\varepsilon\searrow 0$ for each $x\in\Omega\setminus S_u$ (as well as locally in L^1 on Ω). When u is of bounded variation the above convergence holds in a stronger sense, though not in the BV norm defined above, see [8, Corollary 3.80]). Partly for this reason it is useful to consider other modes of convergence too. We say that a sequence (u_k) in $\mathrm{BV}(\Omega,\mathbb{R}^N)$ converges to $u\in\mathrm{BV}(\Omega,\mathbb{R}^N)$ in the weak*–sense if $u_k\to u$ strongly in $\mathrm{L}^1(\Omega,\mathbb{R}^N)$ and $Du_k\overset{*}{\rightharpoonup} Du$ in $\mathrm{C}_0(\Omega,\mathbb{R}^{N\times n})^*$ as $k\to\infty$. We further say that (u_k) converges to u in the BV strict sense on Ω if $u_k\overset{*}{\rightharpoonup} u$ and $|Du_k|(\Omega)\to|Du|(\Omega)$ as $k\to\infty$. Lastly, we say that (u_k) converges to u in the BV area–strict sense on Ω if $u_k\overset{*}{\rightharpoonup} u$ and

$$\int_\Omega E(Du_k)\to\int_\Omega E(Du)$$

as $k\to\infty$. We recall that smooth maps are dense in $\mathrm{BV}(\Omega,\mathbb{R}^N)$ in the BV area–strict sense, and more precisely:

Lemma 2.1. *Let $B=B_R(x_0)$ be a ball and $u\in\mathrm{BV}(B,\mathbb{R}^N)$. Then there exists a sequence (u_j) of C^∞ maps $u_j\colon B\to\mathbb{R}^N$, each of Sobolev class $\mathrm{W}^{1,1}(B,\mathbb{R}^N)$, satisfying $u_j|_{\partial B}=u|_{\partial B}$ and so $u_j\to u$ BV area–strictly on B.*

See for instance [14, Lemma B.2] or [53, Lemma 1] for a proof that works on general domains. The notation $u|_{\partial B}$ is self-explanatory and we shall interpret it both in the sense of trace of the map $u\colon B \to \mathbb{R}^N$ (briefly discussed below) and as a pointwise restriction of the precise representative of the map $u\colon B \to \mathbb{R}^N$ as in [8, Theorem 3.87]. The reader is referred to [8, Sections 3.7 and 3.8] for more background results and a fuller discussion. However, let us mention that when Ω is a Lipschitz domain and $g \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ we write $\mathrm{W}^{1,p}_g(\Omega,\mathbb{R}^N)$ for the Dirichlet class of $\mathrm{W}^{1,p}$ Sobolev maps $u\colon \Omega \to \mathbb{R}^N$ with $u|_{\partial\Omega} = g|_{\partial\Omega}$. Similarly for $\mathrm{BV}_g(\Omega,\mathbb{R}^N)$, where now $g \in \mathrm{BV}(\Omega,\mathbb{R}^N)$ is allowed, denotes the Dirichlet class of BV maps $u\colon \Omega \to \mathbb{R}^N$ with $u|_{\partial\Omega} = g|_{\partial\Omega}$.

Lemma 2.2. *Let $G\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be rank-one convex and of linear growth: $|G(z)| \leq c(|z|+1)$ for all $z \in \mathbb{R}^{N\times n}$. If u, $u_j \in \mathrm{BV}(\Omega,\mathbb{R}^N)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, and $u_j \to u$ BV area–strictly on Ω, then*

$$\int_\Omega G(Du_j) \to \int_\Omega G(Du) \quad \text{as} \quad j \to \infty.$$

We refer to [52, Theorem 4] for a proof.

The next result is the *Fubini property* for BV maps. In order to state it we first recall that a map v defined on a smooth compact $(n-1)$-dimensional manifold M, $v\colon M \to \mathbb{R}^N$, is BV (or $\mathrm{W}^{1,p}$) provided we can find finitely many local charts $\psi_i\colon B^{n-1}_1(0) \to M$, where $B^{n-1}_1(0)$ is the open unit ball in $\mathbb{R}^{n-1}$, $\bigcup_{i=1}^I \psi_i(B^{n-1}_1(0)) = M$ and each map $v \circ \psi_i\colon B^{n-1}_1(0) \to \mathbb{R}^N$ is BV (or $\mathrm{W}^{1,p}$) in the usual sense. We note that for a sphere $\partial B \simeq \mathbb{S}^{n-1}$ in $\mathbb{R}^n$ it suffices to use two local charts that correspond to stereographic projections from two antipodal points. We can now import the various results for BV and $\mathrm{W}^{1,p}$ Sobolev maps defined on (open subsets of) $\mathbb{R}^{n-1}$ to maps defined on an $(n-1)$-dimensional manifold M. We only require the results when M is an $(n-1)$-sphere, and in this situation we record that the *tangential approximate gradient* of a BV map $v\colon \partial B \to \mathbb{R}^N$ exists $\mathcal{H}^{n-1} \llcorner \partial B$ almost everywhere on ∂B and is denoted by $\nabla_\tau v$. The *tangential distributional derivative* of the BV map $v\colon \partial B \to \mathbb{R}^N$ is denoted by $D_\tau v$. Because ∂B is a submanifold of $\mathbb{R}^n$ it is natural to consider $D_\tau v$ as an $\mathbb{R}^{N\times n}$-valued bounded Radon measure. Its total variation measure $|D_\tau v|$ is defined using the norm from $\mathbb{R}^{N\times n}$, and we continue to use the notation

$$|D_\tau v|(A) = \int_A |D_\tau v|$$

for measurable subsets $A \subset \partial B$. Let us record that for $\mathrm{W}^{1,p}$ Sobolev maps $v\colon \partial B \to \mathbb{R}^N$ we have $D_\tau v = \nabla_\tau v\, \mathcal{H}^{n-1} \llcorner \partial B$.

Lemma 2.3. *For a ball $B = B_R(x_0)$ let $u \in \mathrm{BV}(B,\mathbb{R}^N)$. Then for $\mathscr{L}^1$ almost all radii $r \in (0,R)$ the pointwise restriction $u|_{\partial B_r}$ coincides with the traces from B_r and from $B \setminus \overline{B_r}$ of u and is BV on ∂B_r. Furthermore, given two radii $0 < r < s < R$ we can find a radius $t \in (r,s)$ such that $u|_{\partial B_t}$ is as above and its total variation over ∂B_t is bounded as*

$$\int_{\partial B_t} |D_\tau(u|_{\partial B_t})| \leq \frac{c}{s-r} \int_{B_s \setminus \bar{B}_r} |Du|, \tag{2.2}$$

where $c = c(n,N)$ is a constant.

Proof. We can assume that $x_0 = 0$. For a standard smooth mollifier $(\rho_\varepsilon)_{\varepsilon>0}$ we put $u_\varepsilon = \rho_\varepsilon * u$. Then $u_\varepsilon \in \mathrm{C}^\infty(B_{R-\varepsilon},\mathbb{R}^N)$ and we have $u_\varepsilon \to u$ BV strictly on $B_{s'} \setminus \overline{B}_{r'}$ for any radii $r \leq r' < s' \leq s$ with $|Du|(\partial B_{s'} \cup \partial B_{r'}) = 0$. Furthermore, $u_\varepsilon(x) \to u(x)$ for each $x \in B \setminus S_{\bar{u}}$ as $\varepsilon \searrow 0$.

The tangential derivative of u_ε at $x \in \partial B_t$ on the sphere ∂B_t is given by

$$\nabla_\tau(u_\varepsilon|_{\partial B_t})(x) = \nabla u_\varepsilon(x)\big(I - \tfrac{x\otimes x}{t^2}\big) \tag{2.3}$$

so by integration in polar coordinates and since $D_\tau(u_\varepsilon|_{\partial B_t}) = \nabla_\tau(u_\varepsilon|_{\partial B_t})\mathcal{H}^{n-1} \llcorner \partial B_t$ we get

$$\begin{aligned}\int_r^s \int_{\partial B_t} |D_\tau(u_\varepsilon|_{\partial B_t})|\,\mathrm{d}t &= \int_r^s \int_{\partial B_t} |\nabla_\tau(u_\varepsilon|_{\partial B_t})|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \\ &\leq \int_r^s \int_{\partial B_t} |\nabla u_\varepsilon|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \\ &= \int_{B_s \setminus B_r} |\nabla u_\varepsilon|\,\mathrm{d}x = \int_{B_s \setminus B_r} |Du_\varepsilon|.\end{aligned}$$

The set $M = \{t \in (r,s) : \mathcal{H}^{n-1}(S_u \cap \partial B_t) > 0\}$ is $\mathscr{L}^1$ negligible, and for $t \in (r,s) \setminus M$ we have that, as $\varepsilon \searrow 0$, $u_\varepsilon(x) \to u(x)$ for $\mathcal{H}^{n-1}$ a.e. $x \in \partial B_t$ and by the trace theorem (see [8] Th. 3.77) also in $\mathrm{L}^1(\partial B_t, \mathbb{R}^N)$. Next, Fatou's lemma and the strict convergence give for the radii $r \leq r' < s' \leq s$ selected above that

$$\int_{r'}^{s'} \liminf_{\varepsilon \searrow 0} \int_{\partial B_t} |\nabla_\tau(u_\varepsilon|_{\partial B_t})|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \leq \int_{B_s \setminus \bar{B}_r} |Du|,$$

and hence taking $r' \searrow r$ and $s' \nearrow s$ we get

$$\int_{r}^{s} \liminf_{\varepsilon \searrow 0} \int_{\partial B_t} |\nabla_\tau(u_\varepsilon|_{\partial B_t})|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \leq \int_{B_s \setminus \bar{B}_r} |Du|.$$

For each $t \in (r,s) \setminus M$ such that

$$\liminf_{\varepsilon \searrow 0} \int_{\partial B_t} |\nabla_\tau(u_\varepsilon|_{\partial B_t})|\,\mathrm{d}\mathcal{H}^{n-1} < \infty$$

which is $\mathscr{L}^1$ almost all t, we can find a subsequence $\varepsilon_j = \varepsilon_j(t) \searrow 0$ such that $u_{\varepsilon_j}|_{\partial B_t} \to u|_{\partial B_t}$ in $\mathrm{L}^1(\partial B_t, \mathbb{R}^N)$ and pointwise $\mathcal{H}^{n-1}$ a.e., and so

$$\lim_{j \to \infty} \int_{\partial B_t} |\nabla_\tau(u_{\varepsilon_j}|_{\partial B_t})|\,\mathrm{d}\mathcal{H}^{n-1} < \infty.$$

This implies that $u|_{\partial B_t} \in \mathrm{BV}(\partial B_t, \mathbb{R}^N)$. Finally, the last assertion follows because we can select $t \in (r,s) \setminus M$ so

$$\liminf_{\varepsilon \searrow 0} \int_{\partial B_t} |\nabla_\tau(u_\varepsilon|_{\partial B_t})|\,\mathrm{d}\mathcal{H}^{n-1} \leq \tfrac{2}{s-r} \int_{B_s \setminus \bar{B}_r} |Du|,$$

and then conclude by selecting a suitable subsequence $\varepsilon_j \searrow 0$ as above. It follows that the pointwise restriction of the precise representative $u|_{\partial B_t} \in \mathrm{BV}(\partial B_t, \mathbb{R}^N)$ coincides $\mathcal{H}^{n-1}$ a.e. with the traces of u from B_t and from $B_s \setminus \overline{B}_t$ and that (2.2) holds. □

For the statement of the next result we recall that for a ball $B = B_R(x_0)$ in $\mathbb{R}^n$, $s \in (0,1)$ and $p \in (1,\infty)$ the Sobolev-Slobodeckiĭ spaces $\mathrm{W}^{s,p}(B, \mathbb{R}^N)$ and $\mathrm{W}^{s,p}(\partial B, \mathbb{R}^N)$ consist of all integrable maps $u \colon B \to \mathbb{R}^N$, $v \colon \partial B \to \mathbb{R}^N$ for which the Gagliardo norm

$$\|u\|_{\mathrm{W}^{s,p}(B,\mathbb{R}^N)} = \Big(\|u\|^p_{\mathrm{L}^p(B,\mathbb{R}^N)} + |u|^p_{\mathrm{W}^{s,p}(B,\mathbb{R}^N)}\Big)^{\frac{1}{p}},$$

$$\|v\|_{\mathrm{W}^{s,p}(\partial B,\mathbb{R}^N)} = \Big(\|v\|^p_{\mathrm{L}^p(\partial B,\mathbb{R}^N)} + |v|^p_{\mathrm{W}^{s,p}(\partial B,\mathbb{R}^N)}\Big)^{\frac{1}{p}}$$

is finite, respectively. Here we define the corresponding semi-norms as, respectively,

$$|u|_{\mathrm{W}^{s,p}(B,\mathbb{R}^N)} = \left(\int_B \int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}}.$$

and

$$|v|_{\mathrm{W}^{s,p}(\partial B,\mathbb{R}^N)} = \left(\int_{\partial B} \int_{\partial B} \frac{|v(x)-v(y)|^p}{|x-y|^{n-1+sp}}\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}\mathcal{H}^{n-1}(y)\right)^{\frac{1}{p}}.$$

Lemma 2.4. *Assume the dimension $n \geq 3$. Let $B = B_R(x_0)$ be a ball and $v \in \mathrm{BV}(\partial B, \mathbb{R}^N)$. Then $v \in \mathrm{W}^{\frac{1}{n},\frac{n}{n-1}}(\partial B, \mathbb{R}^N)$ and*

$$\left(\fint_{\partial B}\int_{\partial B}\frac{|v(x)-v(y)|^{\frac{n}{n-1}}}{|x-y|^{n-1+\frac{1}{n-1}}}\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}\mathcal{H}^{n-1}(y)\right)^{1-\frac{1}{n}} \leq cR^{1-\frac{1}{n}}\fint_{\partial B}|D_\tau v|,$$

where $c = c(n, N)$ is a constant.

We refer to [16, Lemma D.2] for a proof that $\mathrm{BV}(\mathbb{R}^{n-1})$ for dimensions $n \geq 3$ embeds into $\mathrm{W}^{\frac{1}{n},\frac{n}{n-1}}(\mathbb{R}^{n-1})$. Lemma 2.4 can be recovered from this result by the usual arguments involving local coordinates and a partition of unity. Note that the embedding fails for dimension $n = 2$: an indicator function for a circular arc has bounded variation on ∂B but it is not of class $\mathrm{W}^{\frac{1}{2},2}(\partial B)$. In the two-dimensional case we instead have an embedding into the larger L^2 based Nikolskiĭ space that we in this context may define as

$$\mathrm{B}^{\frac{1}{2},2}_{\infty}(\partial B, \mathbb{R}^N) = \left\{ v \in \mathrm{L}^2(\partial B, \mathbb{R}^N):\ \sup_{0<|h|<R/2}\int_{\partial B}\left|v\left(\tfrac{x+h}{|x+h|}\right)-v(x)\right|^2\mathrm{d}\mathcal{H}^1(x)/|h| < \infty\right\}.$$

This definition is easily seen to be equivalent to the one obtained by transferring the usual definition on, say, the interval $(-1, 1)$ by local coordinates and a partition of unity. A proof of the aforementioned embedding can therefore be inferred from [77, Lemma 38.1]. In combination with a Sobolev embedding result (see [77, Lemma 22.2, (34.4) and Lemma 36.1] or [78, Theorem 4.6.1(a)]) we then deduce:

Lemma 2.5. *Assume the dimension $n = 2$. Let $B = B_R(x_0) \subset \mathbb{R}^2$ be a ball and $v \in \mathrm{BV}(\partial B, \mathbb{R}^N)$. Then $v \in \mathrm{W}^{1-\frac{1}{p},p}(\partial B, \mathbb{R}^N)$ for each $p \in (1, 2)$ and*

$$\left(\fint_{\partial B}\int_{\partial B}\frac{|v(x)-v(y)|^p}{|x-y|^p}\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y)\right)^{\frac{1}{p}} \leq cR^{\frac{1}{p}}\fint_{\partial B}|D_\tau v|,$$

where $c = c(N, p)$ is a constant.

When $u\colon \overline{\Omega} \to \mathbb{R}^N$ is continuous we denote by $\mathrm{Tr}_\Omega(u) = u|_{\partial\Omega}$ the primitive trace operator of u on $\partial\Omega$, and when we write $\mathrm{Tr}_\Omega(v)$ or $v|_{\partial\Omega}$ for more general Sobolev or BV mappings v below we understand as usual this trace as the extension by continuity of the primitive trace operator to the relevant space. We refer to [77, 78] for further background on Besov and Sobolev-Slobodeckiĭ spaces. However, for later reference we explicitly recall two instances of Gagliardo's trace theorem here, the first concerns the case of BV maps.

Lemma 2.6. *For bounded Lipschitz domains Ω in $\mathbb{R}^n$ the trace operator $u \mapsto u|_{\partial\Omega}$ extends from smooth maps on $\overline{\Omega}$ by strict continuity to a well-defined strictly continuous linear surjective operator*

$$\mathrm{Tr}_\Omega\colon\ \mathrm{BV}(\Omega, \mathbb{R}^N) \to \mathrm{L}^1(\partial\Omega, \mathbb{R}^N).$$

Furthermore, we already have $\mathrm{Tr}_\Omega\big(\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)\big) = \mathrm{L}^1(\partial\Omega, \mathbb{R}^N)$. In particular we have for a ball $B = B_R(x_0)$ and writing $\bar{u} = \mathrm{Tr}_B(u)$ for the trace of $u \in \mathrm{BV}(B, \mathbb{R}^N)$ that

$$\int_{\partial B}\left|\bar{u} - \fint_{\partial B}\bar{u}\,\mathrm{d}\mathcal{H}^{n-1}\right|\mathrm{d}\mathcal{H}^{n-1} \leq c\int_B |Du|, \tag{2.4}$$

where $c = c(n, N)$ is a constant.

The second is the extension result corresponding to $\mathrm{W}^{1,p}$ Sobolev maps for the exponent range $p \in (1, \infty)$.

Lemma 2.7. *For the unit ball $\mathbb{B} = B_1(0)$ there exists a bounded linear extension operator*

$$\mathrm{E}\colon\ \mathrm{W}^{1-\frac{1}{p},p}(\partial\mathbb{B}, \mathbb{R}^N) \to \mathrm{W}^{1,p}(\mathbb{B}, \mathbb{R}^N), \quad 1 < p < \infty.$$

More precisely, E *does not depend on* p *and is a bounded linear operator such that* $\mathrm{Tr}_{\mathbb{B}} \circ \mathrm{E}$ *is the identity on* $\mathrm{W}^{1-\frac{1}{p},p}(\partial\mathbb{B}, \mathbb{R}^N)$.

2.4. **Auxiliary estimates for the reference integrand E.** We write $E(z)$ for the reference integrand defined at (1.4) whenever $z \in \mathbb{H}$ and $\mathbb{H}$ is a finite dimensional Hilbert space. Firstly, elementary estimations yield

$$\begin{cases} (\sqrt{2}-1)\min\{|z|,|z|^2\} \leq E(z) \leq \min\{|z|,|z|^2\} \\ \\ E(az) \leq a^2 E(z) \quad \text{and} \quad E(z+w) \leq 2\big(E(z)+E(w)\big) \end{cases} \tag{2.5}$$

for all $z, w \in \mathbb{H}$ and $a \geq 1$. For the following, define for a measurable subset A of $\mathbb{R}^n$ and a $\mathbb{H}$–valued Radon measure μ on $\mathbb{R}^n$ the mean value $\mu_B := \mu(B)/\mathscr{L}^n(B)$.

Lemma 2.8. *Let* ϕ *be a bounded* $\mathbb{H}$*–valued Radon measure on an open ball* B *in* $\mathbb{R}^n$*. Then*

$$\int_B E(\phi - \phi_B) \leq 4 \int_B E(\phi - z) \tag{2.6}$$

for all $z \in \mathbb{H}$.

Proof. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a standard smooth mollifier and put $\phi_\varepsilon = \rho_\varepsilon * \phi$, where ϕ is extended to $\mathbb{R}^n \setminus B$ by defining $\phi(A) := \phi(A \cap B)$ for each Borel set $A \subset \mathbb{R}^n$. Clearly $\phi_\varepsilon \in \mathrm{L}^1(B, \mathbb{H})$, thus from (2.5) and Jensen's inequality we find for $z \in \mathbb{H}$,

$$\begin{aligned} \int_B E(\phi_\varepsilon - (\phi_\varepsilon)_B)\,\mathrm{d}x &\leq 2\int_B E(\phi_\varepsilon - z)\,\mathrm{d}x + 2\mathscr{L}^n(B)E((\phi_\varepsilon)_B - z) \\ &\leq 4\int_B E(\phi_\varepsilon - z)\,\mathrm{d}x. \end{aligned}$$

Now since $\phi(\partial B) = 0$ standard results allow us to conclude by taking $\varepsilon \searrow 0$ (see for instance [8, Theorem 2.2]). ☐

Lemma 2.9. *Let* ϕ *be a bounded* $\mathbb{H}$*–valued Radon measure on an open ball* B *in* $\mathbb{R}^n$*. Then*

$$\fint_B |\phi| \leq \sqrt{\mathscr{E}^2 + 2\mathscr{E}}, \quad \textit{where } \mathscr{E} = \fint_B E(\phi).$$

In particular, for $\mathscr{E} \leq 1$ *we have*

$$\fint_B |\phi|\,\mathrm{d}x \leq \sqrt{3\mathscr{E}}. \tag{2.7}$$

Proof. As in the proof of Lemma 2.8, using mollification, we reduce to the case $\phi \in \mathrm{L}^1(B, \mathbb{H})$. From Jensen's inequality

$$E\left(\fint_B |\phi|\,\mathrm{d}x\right) \leq \fint_B E(\phi)\,\mathrm{d}x = \mathscr{E},$$

and hence solving for the L^1 norm we easily conclude. ☐

2.5. **Estimates for Legendre-Hadamard elliptic systems.** The space of symmetric and real bilinear forms on $\mathbb{R}^{N\times n}$ is denoted by $\bigodot^2(\mathbb{R}^{N\times n})$ and equipped with the operator norm, denoted and defined for $\mathbb{A} \in \bigodot^2(\mathbb{R}^{N\times n})$ as $|\mathbb{A}| = \sup\{\mathbb{A}[z,w] : |z|, |w| \leq 1\}$. Observe that the precise meaning of $|\cdot|$ can be understood from the context, and for a matrix $z \in \mathbb{R}^{N\times n}$ we use it to denote the usual euclidean norm: $|z|^2 = \mathrm{trace}(z^t z)$. Fix $\mathbb{A} \in \bigodot^2(\mathbb{R}^{N\times n})$ and assume it satisfies the strong Legendre-Hadamard condition

$$\begin{cases} \alpha|y|^2|x|^2 \leq \mathbb{A}[y\otimes x, y\otimes x] \quad \forall y \in \mathbb{R}^N,\ \forall x \in \mathbb{R}^n \\ |\mathbb{A}| \leq \beta, \end{cases} \tag{2.8}$$

where $\alpha, \beta > 0$ are constants. Any $\mathbb{R}^N$-valued distribution u on Ω satisfying

$$-\mathrm{div}\,\mathbb{A}Du = 0 \quad \text{in the distributional sense on } \Omega$$

where div is understood to act row-wise, is called $\mathbb{A}$-harmonic, or simply harmonic when $\mathbb{A}$ is clear from the context.

The next lemma is a standard Weyl-type result and can for instance be proved using the difference-quotient method (see [40, 45, 61, 64]).

Lemma 2.10. *Let $\mathbb{A} \in \bigodot^2(\mathbb{R}^{N\times n})$ satisfy (2.8). Then there exists a constant $c = c(\frac{\beta}{\alpha}, n, N)$ with the following propeties. Let $B = B_R(x_0)$ be a ball in $\mathbb{R}^n$ and assume that $h \in \mathrm{W}^{1,1}(B, \mathbb{R}^N)$ is harmonic in B: $-\mathrm{div}\mathbb{A}\nabla h = 0$ in B. Then h is C^∞ on B and for any $z \in \mathbb{R}^{N\times n}$ we have*

$$\sup_{B_{\frac{R}{2}}} |\nabla h - z| + R \sup_{B_{\frac{R}{2}}} |\nabla^2 h| \leq c \fint_{B_R} |\nabla h - z| \,\mathrm{d}x.$$

Finally we state two basic existence and regularity results for inhomogeneous Legendre-Hadamard elliptic systems that are instrumental for our arguments below.

Proposition 2.11. *Let $\mathbb{A} \in \bigodot^2(\mathbb{R}^{N\times n})$ satisfy (2.8) and fix exponents $p \in (1,\infty)$ and $q \in [2,\infty)$. Denote $\mathbb{B} = B_1(0)$, the open unit ball in $\mathbb{R}^n$.*

(a) For each $g \in \mathrm{W}^{1-\frac{1}{p},p}(\partial\mathbb{B}, \mathbb{R}^N)$ there exists a unique solution $h \in \mathrm{W}^{1,p}(\mathbb{B}, \mathbb{R}^N)$ to the elliptic system

$$\begin{cases} -\mathrm{div}\mathbb{A}\nabla h = 0 & \text{in } \mathbb{B} \\ h|_{\partial\mathbb{B}} = g & \text{on } \partial\mathbb{B}, \end{cases} \tag{2.9}$$

and

$$\|h\|_{\mathrm{W}^{1,p}} \leq c\|g\|_{\mathrm{W}^{1-\frac{1}{p},p}}$$

where $c = c(n, N, p, \frac{\alpha}{\beta})$.

(b) For each $f \in \mathrm{L}^q(\mathbb{B}, \mathbb{R}^N)$ there exists a unique solution $w \in \mathrm{W}^{2,q}(\mathbb{B}, \mathbb{R}^N)$ to the elliptic system

$$\begin{cases} -\mathrm{div}\mathbb{A}\nabla w = f & \text{in } \mathbb{B} \\ w|_{\partial\mathbb{B}} = 0 & \text{on } \partial\mathbb{B}, \end{cases} \tag{2.10}$$

and

$$\|w\|_{\mathrm{W}^{2,q}} \leq c\|f\|_{\mathrm{L}^q},$$

where $c = c(n, N, q, \frac{\alpha}{\beta})$.

While these results are well-known we have been unable to find a precise reference. They can be inferred from more general results stated in [64], see in particular Theorems 6.4.8 and 6.5.5 there, and also from [57], Lemma 3.2 (taking the remark on page 106 into account). The last reference does not provide details for the general Legendre-Hadamard elliptic case, but the reader can find the nontrivial calculations and further background in the book [61]. All the above mentioned proofs rely on boundary layer methods, and the work on these is still ongoing with many interesting open questions remaining, see for instance [56]. However the proof of Proposition 2.11 need not be so sophisticated. An easier route goes via the elegant approach exposed by GIUSTI in [45, Chapter 10]. As stated there it builds on earlier works by STAMPACCHIA [73] and CAMPANATO [18], and derives L^p estimates from simple L^2 estimates, Schauder estimates and interpolation. For the convenience of the reader we provide a brief sketch along these lines.

Sketch of Proof. (a): By virtue of Gagliardo's trace theorem, as stated in Lemma 2.7, we can find an extension $\bar{g} \in \mathrm{W}^{1,p}(\mathbb{B}, \mathbb{R}^N)$ with $\bar{g}|_{\partial\mathbb{B}} = g$ and

$$\|\bar{g}\|_{\mathrm{W}^{1,p}} \leq c\|g\|_{\mathrm{W}^{1-\frac{1}{p},p}} \tag{2.11}$$

for a constant $c = c(n, N, p)$. If we put $\bar{h} = h - \bar{g}$, then by simple substitution we see that we can shift attention from (2.9) to the system

$$\begin{cases} -\mathrm{div}\mathbb{A}\nabla \bar{h} = -\mathrm{div}V & \text{in } \mathbb{B}, \\ \bar{h}|_{\partial\mathbb{B}} = 0 & \text{on } \partial\mathbb{B}, \end{cases} \tag{2.12}$$

where $V = \mathbb{A}\nabla\bar{g} \in \mathrm{L}^p(\mathbb{B}, \mathbb{R}^{N\times n})$. Now for $p \in [2,\infty)$ existence, uniqueness and L^p estimate all follow from [45, Theorem 10.15] and (2.11).

It remains to consider the subquadratic case $p \in (1,2)$. In this situation we take $V_j \in \mathrm{L}^2(\mathbb{B}, \mathbb{R}^{N\times n})$ so $\|V - V_j\|_{\mathrm{L}^p} \to 0$, and let $\bar{h}_j \in \mathrm{W}^{1,2}_0(\mathbb{B}, \mathbb{R}^N)$ be the unique solution to

$$\begin{cases} -\mathrm{div}\mathbb{A}\nabla\bar{h}_j = -\mathrm{div}V_j & \text{in } \mathbb{B}, \\ \bar{h}_j|_{\partial\mathbb{B}} = 0 & \text{on } \partial\mathbb{B}. \end{cases} \tag{2.13}$$

Note that $W_j = |\nabla\bar{h}_j|^{p-2}\nabla\bar{h}_j \in \mathrm{L}^{p'}(\mathbb{B}, \mathbb{R}^{N\times n})$, where $p' \in (2,\infty)$ is the Hölder conjugate exponent of p. Consequently, we infer from the above concluded superquadratic case that the elliptic system

$$\begin{cases} -\mathrm{div}\mathbb{A}\nabla\varphi_j = -\mathrm{div}W_j & \text{in } \mathbb{B}, \\ \varphi_j|_{\partial\mathbb{B}} = 0 & \text{on } \partial\mathbb{B} \end{cases} \tag{2.14}$$

admits a unique solution $\varphi_j \in \mathrm{W}^{1,p'}_0(\mathbb{B}, \mathbb{R}^N)$ with

$$\|\varphi_j\|_{\mathrm{W}^{1,p'}} \le c\|W_j\|_{\mathrm{L}^{p'}} = c\|\nabla\bar{h}_j\|^{p-1}_{\mathrm{L}^p} \tag{2.15}$$

where $c = c(n, N, p', \frac{\alpha}{\beta})$. If we test (2.13) with φ_j and use that $\mathbb{A}$ is symmetric, then $\|\nabla\bar{h}_j\|_{\mathrm{L}^p} \le c\|V_j\|_{\mathrm{L}^p}$ results. Now Poincaré's inequality and (2.11) easily allow us to conclude that there exists a solution to (2.9) satisfying the L^p estimate. It remains to prove uniqueness in the subquadratic case. To that end we assume $h \in \mathrm{W}^{1,p}_0(\mathbb{B}, \mathbb{R}^N)$ satisfies $-\mathrm{div}\mathbb{A}\nabla h = 0$ in $\mathbb{B}$. From Lemma 2.10 we know that $h \in \mathrm{C}^\infty(\mathbb{B}, \mathbb{R}^N)$ and so if we for $r \in (0,1)$ define $h_r(x) = h(rx)$, then clearly $h_r|_{\partial\mathbb{B}} \in \mathrm{W}^{\frac{1}{2},2}(\partial\mathbb{B}, \mathbb{R}^N)$ and $-\mathrm{div}\mathbb{A}\nabla h_r = 0$ in $\mathbb{B}$. By uniqueness of $\mathrm{W}^{1,2}$ solutions it follows from the above that $\|h_r\|_{\mathrm{W}^{1,p}} \le c\|h_r|_{\partial\mathbb{B}}\|_{\mathrm{W}^{1-\frac{1}{p},p}}$. But clearly $h_r \to h$ in $\mathrm{W}^{1,p}(\mathbb{B}, \mathbb{R}^N)$ as $r \nearrow 1$, so $\|h_r|_{\partial\mathbb{B}}\|_{\mathrm{W}^{1-\frac{1}{p},p}} \to 0$ as $r \nearrow 1$ by the continuity of trace, and thus $h = 0$.

(b): Since $q \in [2,\infty)$ the assertion follows directly from [45], see (10.60)–(10.63) on pp. 369–370. □

2.6. **Quasiconvexity.** We start by displaying MORREY's definition of quasiconvexity [63, 64]:

Definition 2.12. *A continuous integrand $G\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is quasiconvex at $z_0 \in \mathbb{R}^{N\times n}$ provided*

$$G(z_0) \le \int_{(0,1)^n} G(z_0 + \nabla\varphi(x))\,\mathrm{d}x$$

holds for all compactly supported Lipschitz maps $\varphi\colon (0,1)^n \to \mathbb{R}^N$. It is quasiconvex if it is quasiconvex at all $z_0 \in \mathbb{R}^{N\times n}$.

It is well-known (see [24, 64]) that quasiconvexity implies rank-one convexity, and that rank-one convexity and linear growth, say $|G(z)| \le L(|z|+1)$ for all $z \in \mathbb{R}^{N\times n}$, yield a Lipschitz bound that for C^1 integrands takes the form

$$|G'(z)| \le cL \quad \forall\, z \in \mathbb{R}^{N\times n}. \tag{2.16}$$

The proof in [11] gives (2.16) with the constant $c = \sqrt{\min\{n, N\}}$.

When a quasiconvex integrand G has linear growth it means that the quasiconvexity inequality can be tested by more general maps. We have from [52, Proposition 1]:

Lemma 2.13. *Assume $G\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is quasiconvex and of linear growth. If ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $\varphi \in \mathrm{BV}(\omega, \mathbb{R}^N)$ and $a\colon \mathbb{R}^n \to \mathbb{R}^N$ is affine, then*

$$\mathscr{L}^n(\omega)G(\nabla a) \le \int_\omega G(D\varphi) + \int_{\partial\omega} G^\infty\big((a-\varphi)\otimes\nu_\omega\big)\,\mathrm{d}\mathscr{H}^{n-1}$$

holds, where ν_ω is the outward unit normal on $\partial\omega$.

As explained in the Introduction, and substantiated in Section 3 below, our quasiconvexity assumption (H2), that we shall refer to as *strong quasiconvexity*, is very natural when compared to the minimal set of conditions that allows one to prove existence of a BV minimizer by use of the direct method. We emphasize that quasiconvexity is much more general than convexity and refer to [24] for a long list of examples of nonconvex quasiconvex integrands. That there exists nonconvex quasiconvex integrands of linear growth is also well-known (see [66, 80]). These integrands are not given by explicit and concrete formulas, but instead as the quasiconvex envelopes of suitable integrands. But once we know that such integrands do exist it is easy to show that there also exist nonconvex integrands satisfying our hypotheses (H0), (H1), (H2). Indeed, assume that G is a nonconvex quasiconvex integrand of linear growth and let $(\Phi)_{\varepsilon>0}$ be a standard smooth mollifier on $\mathbb{R}^{N\times n}$. Then $F=\Phi_\varepsilon * G+\varepsilon E$ satisfies (H0), (H1), (H2) for each $\varepsilon>0$, and for sufficiently small ε it will also be nonconvex. In fact, it is well-known that we may take the integrand G such that it is in addition nonconvex in the more severe sense that also the functional

$$v\mapsto \int_\Omega G(\nabla v)\,\mathrm{d}x \tag{2.17}$$

is nonconvex on the Dirichlet class $\mathrm{W}^{1,1}_g$. It is clear that this too will be inherited by F for small enough $\varepsilon>0$. However, these simple examples do not quite illustrate the abundance of nonconvex integrands F satisfying our hypotheses. A step in that direction is the following approximation result which is a variant of [50, Proposition 1.10].

Proposition 2.14. *Let $F\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ be quasiconvex and assume that for some exponent $p\in(1,\infty)$ and constant $L\geq 1$ we have the p-coercivity-growth condition:*

$$|z|^p\leq F(z)\leq L\big(|z|^p+1\big)\quad \forall\, z\in\mathbb{R}^{N\times n}.$$

Then there exists a sequence (F_j) of integrands $F_j\colon\mathbb{R}^{N\times n}\to\mathbb{R}$, each of class C^∞ and satisfying for some constants ℓ_j, $L_j>0$,

$$\begin{cases} |z|-2\leq F_j(z)\leq L_j(|z|+1) & \forall\, z\in\mathbb{R}^{N\times n}\\ z\mapsto F_j(z)-\ell_jE(z) \text{ is quasiconvex,} \end{cases}$$

such that $F_j(z)\nearrow F(z)$ as $j\nearrow\infty$ pointwise in $z\in\mathbb{R}^{N\times n}$.

Proof. We merely sketch the proof as it is a standard variant of [50, Proposition 1.10]. For $j\in\mathbb{N}$ and $z\in\mathbb{R}^{N\times n}$ we put

$$f_j(z):=\min\big\{F(z)-\tfrac{1}{j}E(z),\, j\big(|z|+1\big)\big\},$$

and let $g_j:=f_j^{\mathrm{qc}}$ be its quasiconvex envelope. We record that $g_j(z)\leq g_{j+1}(z)$ and since

$$F(z)-\frac{1}{j}E(z)\geq |z|^p-\tfrac{1}{j}|z|\geq |z|-1$$

for $j\geq j_p$ for some $j_p\in\mathbb{N}$, also $g_j(z)\geq|z|-1$ for $j\geq j_p$. We obviously have $g_j(z)\leq j(|z|+1)$, and therefore, by quasiconvexity, g_j is Lipschitz too. From (2.16) we have for some dimensional constant c that $\mathrm{lip}(g_j)\leq cj$, and hence for a standard mollifier $(\Phi_\varepsilon)_{\varepsilon>0}$:

$$g_j(z)-cj\varepsilon\leq(\Phi_\varepsilon * g_j)(z)\leq g_j(z)+cj\varepsilon.$$

Take $\varepsilon=\varepsilon_j:=\frac{1}{3cj}2^{-j}$, whereby

$$\begin{aligned}(\Phi_{\varepsilon_{j+1}} * g_{j+1})(z) &\leq g_{j+1}(z)+\tfrac{1}{3}2^{-j-1}\\ &\leq g_j(z)+\tfrac{1}{3}2^{-j-1}\\ &\leq (\Phi_{\varepsilon_j} * g_j)(z)+2^{-j-1},\end{aligned}$$

and so with $G_j(z) := (\Phi_{\varepsilon_j} * g_j)(z) - 2^{-j}$ we have that G_j is quasiconvex, C^∞, $|z|-1 \leq G_j(z) \leq j(|z|+1)$ and $G_j(z) \leq G_{j+1}(z)$. It therefore follows from the proof given in [50] that $G_j(z) \nearrow F(z)$ as $j \nearrow \infty$. Finally we check that

$$F_j(z) := (1-2^{-j})G_j(z) - 2^{-j} + 2^{-j}E(z)$$

does the job. Only the inequality $F_j(z) \leq F_{j+1}(z)$ remains to be checked. For this we note that $G_j(z) \geq E(z) - 1$ and have then

$$\begin{aligned} F_j(z) &= \big(1-2^{-1-j}\big)G_j(z) - 2^{-1-j}G_j(z) - 2^{-j} + 2^{-j}E(z) \\ &\leq \big(1-2^{-1-j}\big)G_{j+1}(z) - 2^{-1-j}\big(E(z)-1\big) - 2^{-j} + 2^{-j}E(z) \\ &= \big(1-2^{-1-j}\big)G_{j+1}(z) - 2^{-1-j} + 2^{-1-j}E(z) \\ &= F_{j+1}(z). \end{aligned}$$

□

Using Proposition 2.14 we can therefore for *any* quasiconvex integrand F satisfying the above p-coercivity-growth condition, any bounded subset B of matrix space $\mathbb{R}^{N\times n}$ and any $\varepsilon > 0$ find an integrand G satisfying the hypotheses (H0), (H1), (H2) of Theorem 1.1, such that

$$|F(z) - G(z)| < \varepsilon \quad \text{for all } z \in B.$$

Much more precise statements can be made, but the above suffices for showing that any kind of non-convexity that a quasiconvex integrand of p-growth can have on a bounded subset of matrix space can be exhibited too by an integrand satisfying (H0), (H1), (H2). We could therefore for instance apply the approximation result to the integrands constructed in [67] (or in fact directly use the flexible construction given there) to find integrands G satisfying (H0), (H1), (H2) so that the Euler-Lagrange system

$$-\mathrm{div}G'(\nabla v) = 0$$

admits, say compactly supported Lipschitz maps that are nowhere C^1 as weak solutions. For such integrands G it is clear that the corresponding functional

$$v \mapsto \int_\Omega G(Dv)$$

in particular must be nonconvex on the Dirichlet class $\mathrm{W}_0^{1,1}(\Omega,\mathbb{R}^N)$. In particular, in view of the nonconvex nature of the variational problems it becomes then relevant to investigate the regularity of various classes of local minimizers as done in the p-growth case in [54, 20, 76]. We leave this for future investigations and focus in the present paper entirely on absolute minimizers in the sense of (1.7).

2.7. **Extremality of minimizers.** The following result is closely related to [9, Theorem 3.7], but it concerns more general integrands that are not covered there.

Lemma 2.15. *Assume that $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is C^1, rank-one convex and that $|F(z)| \leq L(|z|+1)$ holds for all $z \in \mathbb{R}^{N\times n}$. Then for any local minimizer $u \in \mathrm{BV}(\Omega,\mathbb{R}^N)$ of the variational integral $\mathfrak{F}(v,\Omega) = \int_\Omega F(Dv)$ we have that $F'(\nabla u) \in \mathrm{L}^\infty(\Omega,\mathbb{R}^{N\times n})$ and*

$$-\int_\Omega F^\infty(D^s\varphi) \leq \int_\Omega F'(\nabla u)[\nabla\varphi]\,\mathrm{d}x \leq \int_\Omega F^\infty(-D^s\varphi) \tag{2.18}$$

holds for all $\varphi \in \mathrm{BV}_0(\Omega,\mathbb{R}^N)$. In particular, $F'(\nabla u)$ is row-wise divergence free.

Proof. First we recall that linear growth and rank-one convexity combine to give Lipschitz continuity (2.16), hence the matrix valued map $F'(\nabla u)$ is bounded. The Lipschitz continuity furthermore allows us to simplify the definition of the recession integrand given in (2.1) to

$$F^\infty(z) = \limsup_{t\to\infty} \frac{F(tz)}{t}.$$

From this it becomes clear that F^∞ is rank one convex too. Next, for $\varphi \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ with compact support in Ω and each $\varepsilon \geq 0$ we put $\mu := |D^s(u+\varepsilon\varphi)| + |D^s u| + |D^s\varphi|$. Then we may write

$$F^\infty\big(D^s(u+\varepsilon\varphi)\big) - F^\infty\big(D^s u\big) = \left(F^\infty\left(\frac{\mathrm{d}D^s u}{\mathrm{d}\mu} + \varepsilon\frac{\mathrm{d}D^s\varphi}{\mathrm{d}\mu}\right) - F^\infty\left(\frac{\mathrm{d}D^s u}{\mathrm{d}\mu}\right)\right)\mu.$$

Here we have according to [4] that

$$\mathrm{rank}\left(\frac{\mathrm{d}D^s\varphi}{\mathrm{d}\mu}\right) \leq 1 \quad \mu\text{–a.e.}$$

and thus, using that difference quotients of convex functions are non-decreasing (or see [49, Lemma 2.5]) and the assumptions on F we infer that

$$F^\infty\big(D^s(u+\varepsilon\varphi)\big) - F^\infty\big(D^s u\big) \leq \varepsilon F^\infty\big(D^s\varphi\big).$$

Consequently, by local minimality:

$$\begin{aligned} 0 &\leq \int_\Omega F\big(D(u+\varepsilon\varphi)\big) - \int_\Omega F\big(Du\big) \\ &\leq \int_\Omega\int_0^1 F'(\nabla u + t\varepsilon\nabla\varphi)[\varepsilon\nabla\varphi]\,\mathrm{d}t\,\mathrm{d}x + \varepsilon\int_\Omega F^\infty\big(D^s\varphi\big), \end{aligned}$$

and hence, invoking the Lipschitz bound and Lebesgue's dominated convergence theorem, we arrive at

$$0 \leq \int_\Omega F'(\nabla u)[\nabla\varphi]\,\mathrm{d}x + \int_\Omega F^\infty\big(D^s\varphi\big).$$

Finally, we extend the above inequality by continuity to hold for all $\varphi \in \mathrm{BV}_0(\Omega, \mathbb{R}^N)$, and conclude by using the bound with $-\varphi$ instead of φ. ☐

3. Boundedness of minimizing sequences and strong quasiconvexity

This section is intended to illustrate what the condition of strong quasiconvexity means more precisely, and our result extends the corresponding results of [22] about L^1 coercivity in two ways. First, and foremost, we show that coercivity in the given context is equivalent to $\mathrm{W}^{1,1}$ boundedness of all minimizing sequences. Secondly, we show that this in turn is equivalent to a strong quasiconvexity condition: for some $\ell > 0$ the integrand $F - \ell E$ is quasiconvex at some $z_0 \in \mathbb{R}^{N\times n}$. The corresponding results in [22] were stated in terms of the integrands $F - \ell|\cdot|$, and only gave the equivalence between coercivity and a quasiconvexity condition.

Proposition 3.1. *Assume $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is a continuous integrand of linear growth, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $g \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$. Then minimizing sequences for the variational problem*

$$\inf_{u \in \mathrm{W}^{1,1}_g(\Omega,\mathbb{R}^N)} \int_\Omega F(\nabla u)\,\mathrm{d}x \tag{3.1}$$

are all bounded in $\mathrm{W}^{1,1}$ if and only if there exist $\ell > 0$ and $z_0 \in \mathbb{R}^{N\times n}$ such that $F - \ell E$ is quasiconvex at z_0.

Proof. The *if* part follows from [22, Theorem 1.1] and to prove the *only if* part we must adapt the proofs from [22]. We proceed in three steps.
Step 1. Let X be a simplex in $\mathbb{R}^n$ and $z_0 \in \mathbb{R}^{N\times n}$. We show that if all minimizing sequences for the variational problem (3.1) in the special case $\Omega = X$ and $g(x) = z_0 x$ are $\mathrm{W}^{1,1}$ bounded, then we can find constants $\alpha > 0$, $\beta \in \mathbb{R}$ depending only on F, X, z_0, so

$$\int_X F(z_0 + \nabla\varphi)\,\mathrm{d}x \geq \int_X\Big(\alpha|\nabla\varphi| + \beta\Big)\,\mathrm{d}x \tag{3.2}$$

holds for all $\varphi \in \mathrm{W}^{1,1}_0(X, \mathbb{R}^N)$. We express this by saying that F is mean coercive, and recall from [22, Theorem 1.1] that this is a property of F (so that we have a bound like (3.2) for any bounded

Lipschitz domain Ω and any $g \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ with α, β now depending on F, Ω, g). Following [22] we consider the auxiliary function

$$\Theta(t) = \inf \left\{ \fint_X F(z_0 + \nabla\varphi)\,\mathrm{d}x : \varphi \in \mathrm{W}^{1,1}_0(X, \mathbb{R}^N),\ \fint_X |\nabla\varphi|\,\mathrm{d}x \geq t \right\} \quad (t \geq 0)$$

Because F has linear growth the $\mathrm{W}^{1,1}$ boundedness of minimizing sequences clearly implies that Θ is a real-valued non-decreasing function. According to [22, Proposition 3.2] it is also convex. Consequently, if Θ is bounded from above, then it must be constant: $\Theta(t) \equiv \theta$ for all $t \geq 0$, where $\theta \in \mathbb{R}$. But this is impossible as it leads to the existence of minimizing sequences for (3.1) that are not bounded in $\mathrm{W}^{1,1}$. Hence Θ is not bounded from above, and so by convexity we conclude that for some constants $\alpha > 0$, $\beta \in \mathbb{R}$ we must have $\Theta(t) \geq \alpha t + \beta$ for all $t \geq 0$. Unravelling the definitions we have shown that (3.2) holds.

Step 2. We show that if all minimizing sequences for (3.1) are $\mathrm{W}^{1,1}$ bounded, then F is mean coercive. For this it is easiest to argue by contradiction: Assume that all minimizing sequences for (3.1) are $\mathrm{W}^{1,1}$ bounded, but that F is not mean coercive. The former, taken together with the linear growth of F, means in particular that

$$m := \inf_{u \in \mathrm{W}^{1,1}_g(\Omega, \mathbb{R}^N)} \int_\Omega F(\nabla u)\,\mathrm{d}x \in \mathbb{R}.$$

The latter allows us by Step 1 to conclude that for any simplex X in $\mathbb{R}^n$ and any $z_0 \in \mathbb{R}^{N\times n}$, the variational problem (3.1) with $\Omega = X$ and $g(x) = z_0 x$ admits a minimizing sequence that is unbounded in $\mathrm{W}^{1,1}$. Fix a polygonal open subset $\Omega' \Subset \Omega$ and note that since F is continuous and of linear growth, the functional $v \mapsto \int_\Omega F(\nabla v)\,\mathrm{d}x$ is continuous on $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$. By density of piecewise affine maps in $\mathrm{W}^{1,1}$ (see for instance [29, Proposition 2.8]) we can therefore find a minimizing sequence (u_j) for (3.1) such that each restriction $u_j|_{\Omega'}$ is piecewise affine. Let τ_j be the regular and finite triangulation of Ω' so that u_j is affine on each simplex of τ_j. We apply the existence of $\mathrm{W}^{1,1}$ unbounded minimizing sequences for (3.1) for each $\Omega = X \in \tau_j$, $z_0 = \nabla u_j|_X$ to find $\varphi_{j,X} \in \mathrm{W}^{1,1}_0(X, \mathbb{R}^N)$ so

$$j\mathscr{L}^n(X) < \int_X |\nabla\varphi_{j,X}|\,\mathrm{d}x \text{ and } \int_X F(\nabla u_j + \nabla\varphi_{j,X})\,\mathrm{d}x \leq \int_X \left(F(\nabla u_j) + \frac{1}{j} \right)\mathrm{d}x.$$

Defining $v_j := u_j + \sum_{X \in \tau_j} \varphi_{j,X}$, where we extend each $\varphi_{j,X}$ by $0 \in \mathbb{R}^N$ off X, we have a $\mathrm{W}^{1,1}$ unbounded minimizing sequence for (3.1), a contradiction that finishes the proof of Step 2.

Step 3. Conclusion from (3.2). We fix a simplex X in $\mathbb{R}^n$ with $\mathscr{L}^n(X) = 1$ and take $z_0 = 0$.. Now $E(z) \leq |z|$ for $z \in \mathbb{R}^{N\times n}$, so if we take $\ell \in (0, \alpha)$, put $c = \alpha - \ell$ and $G = F - \ell E$, then (3.2) yields

$$\int_X G(\nabla\varphi)\,\mathrm{d}x \geq c \int_X |\nabla\varphi|\,\mathrm{d}x + \beta \tag{3.3}$$

for all $\varphi \in \mathrm{W}^{1,1}_0(X, \mathbb{R}^N)$. Recalling the Dacorogna formula for the quasiconvex envelope (see [24] and the discussion in [22]) we take a sequence (φ_j) in $\mathrm{W}^{1,1}_0(X, \mathbb{R}^N)$ so

$$\int_X G(\nabla\varphi_j)\,\mathrm{d}x \to G^{\mathrm{qc}}(0),$$

the quasiconvex envelope of G at 0. Obviously, $G^{\mathrm{qc}}(0) \geq \beta$, so G^{qc} is a real-valued quasiconvex integrand satisfying $G^{\mathrm{qc}} \leq G$. Because G has linear growth, so does G^{qc} (see [22]). The probability measures ν_j on $\mathbb{R}^{N\times n}$ defined for Borel sets $A \subset \mathbb{R}^{N\times n}$ by

$$\nu_j(A) := \mathscr{L}^n\Big(X \cap (\nabla\varphi_j)^{-1}(A) \Big)$$

all have centre of mass at 0 and uniformly bounded first moments:

$$c \int_{\mathbb{R}^{N\times n}} |z|\,\mathrm{d}\nu_k + \beta \leq \sup_j \int_X G(\nabla\varphi_j)\,\mathrm{d}x < \infty$$

for $k \in \mathbb{N}$. But then Banach-Alaoglu's theorem applied in $\mathrm{C}_0(\mathbb{R}^{N\times n})^*$ yields a subsequence (not relabelled) and $\nu \in \mathrm{C}_0(\mathbb{R}^{N\times n})^*$ such that $\nu_j \overset{*}{\rightharpoonup} \nu$ in $\mathrm{C}_0(\mathbb{R}^{N\times n})^*$. It is not hard to see that ν must again be a probability measure on $\mathbb{R}^{N\times n}$, and

$$\int_{\mathbb{R}^{N\times n}} |z|\,\mathrm{d}\nu \leq \liminf_{j\to\infty} \int_{\mathbb{R}^{N\times n}} |z|\,\mathrm{d}\nu_j < \infty.$$

Since $G - G^{\mathrm{qc}} \geq 0$ is continuous we get by routine means that

$$0 \leq \int_{\mathbb{R}^{N\times n}} \Big(G - G^{\mathrm{qc}} \Big)\,\mathrm{d}\nu \leq \liminf_{j\to\infty} \int_{\mathbb{R}^{N\times n}} \Big(G - G^{\mathrm{qc}} \Big)\,\mathrm{d}\nu_j = 0,$$

and thus $G = G^{\mathrm{qc}}$ on the support of ν, that is, $G = F - \ell E$ is quasiconvex at each z in the support of ν. This completes the proof. □

Remark 3.2. *It is not difficult to show that under assumptions* (H1), (H2) *we have a* principle of convergence of energies *in the sense that if* $u_j \overset{*}{\rightharpoonup} u$ *in* BV, $u_j|_{\partial\Omega} \to u|_{\partial\Omega}$ *in* $\mathrm{L}^1(\partial\Omega, \mathbb{R}^N)$ *and*

$$\int_\Omega F(Du_j) \to \int_\Omega F(Du),$$

then $u_j \to u$ *in the area-strict sense in* BV. *We do not give the details here, but intend to return to this and related questions in a more general framework elsewhere.*

4. Proof of Theorem 1.1

We split the proof into five steps, each of which is presented in a subsection.

4.1. **Bounds for shifted integrands.** For a C^2 integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ we define for each $w \in \mathbb{R}^{N\times n}$ the shifted integrand $F_w\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ by

$$\begin{aligned} F_w(z) &= F(z+w) - F(w) - F'(w)[z] \\ &= \int_0^1 (1-t) F''(w+tz)[z,z]\,\mathrm{d}t \end{aligned} \tag{4.1}$$

We use the same notation for shifted versions E_w of the reduced area integrand E, and record the following elementary result for later reference.

Lemma 4.1. *For* w, $z \in \mathbb{R}^{N\times n}$ *we have (with obvious interpretation for* $w = 0$ *or* $z = 0$*)*

$$E''(w)[z,z] = \frac{1 + |w|^2 - |w|^2 \Big(\frac{w}{|w|} \cdot \frac{z}{|z|} \Big)^2}{\big(1+|w|^2\big)^{\frac{3}{2}}} |z|^2 \tag{4.2}$$

and

$$E_w(z) \geq \tfrac{1}{4} \big(1+|w|^2\big)^{-\frac{3}{2}} E(z). \tag{4.3}$$

Proof. It is clear that (4.2) holds, and therefore

$$E_w(z) \geq |z|^2 \int_0^1 \frac{1-t}{\big(1+|w+tz|^2\big)^{\frac{3}{2}}}\,\mathrm{d}t$$

follows. Now $\big(1+|w+tz|^2\big)^{\frac{1}{2}} \leq \big(1+|w|^2\big)^{\frac{1}{2}} + t|z|$ so, writing $\langle w \rangle := (1+|w|^2)^{\frac{1}{2}}$, the integral can be estimated from below by

$$\int_0^1 \frac{1-t}{\big(\langle w\rangle + t|z|\big)^3}\,\mathrm{d}t = \frac{1}{2\langle w\rangle^2 \big(\langle w\rangle + |z|\big)},$$

and consequently,

$$E_w(z) \geq \frac{|z|^2}{2\langle w\rangle^2\big(\langle w\rangle + |z|\big)}.$$

Distinguishing cases $|z| \leq 1$, $|z| > 1$ and using (2.5) we deduce (4.3). □

Next, we record the following elementary properties that F_w inherits from F:

Lemma 4.2. *Suppose* $F\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ *satisfies* (H0), (H1), (H2). *For each* $m>0$ *there exists a constant* $c=c(m)\in[1,\infty)$ *with the following properties. Fix* $w\in\mathbb{R}^{N\times n}$ *with* $|w|\leq m$. *Then*

$$|F_w(z)|\leq cE(z),\qquad |F_w'(z)|\leq c\min\{|z|,1\},\tag{4.4}$$

$$|F_w''(0)z-F_w'(z)|\leq cE(z)\tag{4.5}$$

holds for all $z\in\mathbb{R}^{N\times n}$,

$$\int_B F_w(\nabla\varphi(x))\,\mathrm{d}x\geq \tfrac{1}{c}\int_B E(\nabla\varphi(x))\,\mathrm{d}x\tag{4.6}$$

holds for all $\varphi\in \mathrm{W}_0^{1,1}(B,\mathbb{R}^N)$ *and*

$$F''(w)[y\otimes x,y\otimes x]\geq \tfrac{1}{c}|y|^2|x|^2\tag{4.7}$$

holds for all $x\in\mathbb{R}^n$, $y\in\mathbb{R}^N$.

Proof. For the bounds (4.4) and (4.5) we distinguish the cases $|z|\leq 1$ and $|z|>1$. The bounds in (4.4) follow then easily from the definition of F_w and (2.16). We leave the details of this to the reader, and instead focus on (4.5). Here we have for $|z|\leq 1$ that

$$\begin{aligned}|F_w''(0)z-F_w'(z)| &\leq \int_0^1\big|F''(w)-F''(w+tz)\big|\,\mathrm{d}t|z|\\ &\leq \operatorname{lip}(F'',B_{m+1}(0))|z|^2\end{aligned}$$

and the latter is finite for each fixed m by hypothesis (H0). Next, for $|z|>1$ we use (2.16) to estimate:

$$\begin{aligned}|F_w''(0)z-F_w'(z)| &\leq |F''(w)||z|+cL\\ &\leq \big(\sup_{|v|\leq m}|F''(v)|+cL\big)|z|,\end{aligned}$$

and so we deduce (4.5) from (2.5). Finally we turn to the quasiconvexity condition (4.6). From (H2) we get

$$\int_B F_w(\nabla\varphi)\,\mathrm{d}x\geq \ell\int_B E_w(\nabla\varphi)\,\mathrm{d}x$$

and so from (4.3) we get (4.6) with $c=4(1+m^2)^{\frac{3}{2}}/\ell$. Finally, since quasiconvexity implies rank one convexity, (H2) yields in particular that

$$\begin{aligned}F''(w)[y\otimes x,y\otimes x] &\geq \ell E''(w)[y\otimes x,y\otimes x]\\ &\overset{(4.2)}{\geq} \frac{\ell}{(1+m^2)^{\frac{3}{2}}}|y|^2|x|^2,\end{aligned}$$

which of course implies (4.7). ☐

4.2. **Caccioppoli inequality of the second kind.** This is an important part of the proof, and in fact it is the only place where both the quasiconvexity and minimality assumptions are used. However, the proof in the considered linear growth case does not differ much from the usual ones as it follows that given by EVANS in [30] and relies crucially on WIDMAN's hole filling trick [79].

Proposition 4.3. *Suppose* $F\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ *satisfies* (H0), (H1), (H2) *and that* $u\in\mathrm{BV}(\Omega,\mathbb{R}^N)$ *is a minimizer. Then each* $m>0$ *there exists a constant* $c=c(m,n,N,\frac{L}{\ell})\in[1,\infty)$ *with the following property: For an affine map* $a\colon\mathbb{R}^n\to\mathbb{R}^N$ *with* $|\nabla a|\leq m$ *and a ball* $B_R(x_0)\subset\Omega$ *we have*

$$\int_{B_{\frac{R}{2}}(x_0)}E(D(u-a))\leq c\int_{B_R(x_0)}E\left(\frac{u-a}{R}\right)\,\mathrm{d}x.\tag{4.8}$$

Proof. Denote $\tilde{F} = F_{\nabla a}$ and $\tilde{u} = u - a$. Observe that $\tilde{u}$ is minimizing the integral functional corresponding to the shifted integrand $\tilde{F}$. Fix two radii r and s so that $\frac{R}{2} < r < s < R$, and let $\rho\colon \Omega \to \mathbb{R}$ be a Lipschitz cut-off function satisfying $\mathbf{1}_{B_r} \leq \rho \leq \mathbf{1}_{B_s}$ and $|\nabla\rho| \leq \frac{1}{s-r}$. Put $\varphi = \rho\tilde{u}$ and $\psi = (1-\rho)\tilde{u}$. For a standard smooth mollifier (ϕ_ε) we let $\varphi_\varepsilon = \rho(\phi_\varepsilon * \tilde{u})$, so that $\varphi_\varepsilon \in \mathrm{W}_0^{1,1}(B_s, \mathbb{R}^N)$. Hence by the consequence (4.6) of the quasiconvexity assumption (H2) we get

$$\frac{1}{c}\int_{B_s} E(\nabla\varphi_\varepsilon)\,\mathrm{d}x \leq \int_{B_s} \tilde{F}(\nabla\varphi_\varepsilon)\,\mathrm{d}x.$$

Observe that as $\varepsilon \searrow 0$, $\varphi_\varepsilon \to \varphi$ in L^1 and (since $\rho = 0$ on ∂B_s) that

$$\int_{B_s} E(\nabla\varphi_\varepsilon)\,\mathrm{d}x \to \int_{B_s} E(D\varphi).$$

We can therefore employ Lemma 2.2 to find, by taking $\varepsilon \searrow 0$ in the above inequality,

$$\frac{1}{c}\int_{B_s} E(D\varphi) \leq \int_{B_s} \tilde{F}(D\varphi).$$

Consequently, we have using minimality of $\tilde{u}$, (4.4), convexity of E and (2.5):

$$\begin{aligned}
\frac{1}{c}\int_{B_r} E(D\tilde{u}) &\leq \int_{B_s} \tilde{F}(D\tilde{u}) + \int_{B_s} \tilde{F}(D\varphi) - \int_{B_s} \tilde{F}(D\tilde{u}) \\
&\leq \int_{B_s} \tilde{F}(D\psi) + \int_{B_s} \tilde{F}(D\varphi) - \int_{B_s} \tilde{F}(D\tilde{u}) \\
&\leq c\int_{B_s\setminus B_r} E(D\tilde{u}) + c\int_{B_s\setminus B_r} E(\rho D\tilde{u} + \tilde{u}\otimes\nabla\rho) \\
&\quad + c\int_{B_s\setminus B_r} E((1-\rho)D\tilde{u} - \tilde{u}\otimes\nabla\rho) \\
&\leq 5c\int_{B_s\setminus B_r} E(D\tilde{u}) + 4c\int_{B_s} E\left(\frac{\tilde{u}}{s-r}\right)\mathrm{d}x.
\end{aligned}$$

We fill the hole whereby on denoting $\theta = 5c/(5c + \frac{1}{c}) \in (0,1)$ we arrive at

$$\begin{aligned}
\int_{B_r} E(D\tilde{u}) &\leq \theta\int_{B_s} E(D\tilde{u}) + \theta\int_{B_s} E\left(\frac{\tilde{u}}{s-r}\right)\mathrm{d}x \\
&\leq \theta\int_{B_s} E(D\tilde{u}) + \theta\int_{B_R} E\left(\frac{\tilde{u}}{s-r}\right)\mathrm{d}x.
\end{aligned}$$

The conclusion now follows in a standard way from the iteration Lemma 4.4 below. □

Lemma 4.4. *Let $\theta \in (0,1)$ and $R > 0$. Assume that Φ, $\Psi\colon (0,R] \to \mathbb{R}$ are nonnegative functions, that Φ is bounded, Ψ is decreasing with $\Psi(\sigma t) \leq \sigma^{-2}\Psi(t)$ for all $t \in (0,R]$, $\sigma \in (0,1]$ and that*

$$\Phi(r) \leq \theta\Phi(s) + \Psi(s-r) \tag{4.9}$$

holds for all r, $s \in [\frac{R}{2}, R]$ with $r < s$. Then there exists a constant $C = C(\theta) > 0$ such that

$$\Phi\left(\tfrac{R}{2}\right) \leq C\Psi(R). \tag{4.10}$$

The proof follows closely that of, for instance, [45, Lemma 6.1] and so we leave the details to the reader.

As mentioned in the Introduction it is not possible to use the Poincaré-Sobolev inequality to get a reverse Hölder inequality from which higher integrability can be deduced by use of Gehring's Lemma. The paper [17] gives some conditions that allow for a higher integrability result also in the L^1 context, but it is unclear how to verify these conditions in our case. However, the Caccioppoli inequality (4.8) still encodes some compactness as can be seen from the following remark that is stated in terms of Young measures and where we use the terminology from [53]. The reader can get a good overview of this general formalism and other related developments in the calculus of variations context in the recent monograph [69, Chapter 12].

Remark 4.5. *Let (u_j) be a sequence in $\mathrm{BV}(\Omega,\mathbb{R}^N)$ satisfying the Caccioppoli inequality (4.8) above uniformly: for each $m>0$ there exists a constant c_m (independent of j) such that for any affine map $a\colon \mathbb{R}^n\to\mathbb{R}^N$ with $|\nabla a|\leq m$ and any ball $B_R=B_R(x_0)\subset\Omega$ we have*

$$\int_{B_{\frac{R}{2}}(x_0)} E(D(u_j-a)) \leq c_m \int_{B_R(x_0)} E\left(\frac{u_j-a}{R}\right)\,\mathrm{d}x. \tag{4.11}$$

If (u_j) is bounded in $\mathrm{BV}(\Omega,\mathbb{R}^N)$, then (any subsequence) admits a subsequence (not relabelled) that converges weakly* in BV to a map $u\in\mathrm{BV}(\Omega,\mathbb{R}^N)$ and whose derivatives Du_j generate a Young measure $\nu=\big((\nu_x)_{x\in\Omega},\lambda,(\nu_x^\infty)_{x\in\overline{\Omega}}\big)$. The compactness encoded in (4.11) amounts to

$$\nu_x=\delta_{\nabla u(x)} \quad \mathscr{L}^n\text{-a.e.} \quad \text{and} \quad |D^s u|\leq \lambda \llcorner \Omega \leq c|D^s u|,$$

where $c=c(n)c_0$.

Proof. The existence of the subsequence with the asserted properties follows from [53, Theorem 8]. Thus we have for some subsequence (not relabelled), $u\in\mathrm{BV}(\Omega,\mathbb{R}^N)$ and Young measure ν that

$$u_j \overset{*}{\rightharpoonup} u \text{ in } \mathrm{BV} \text{ and } Du_j \overset{\mathrm{Y}}{\to} \nu.$$

By a result of CALDERÓN & ZYGMUND [8, Theorem 3.83], u is approximately differentiable $\mathscr{L}^n$ almost everywhere. Let $x_0\in\Omega$ be such a point and take $a(x)=u(x_0)+\nabla u(x_0)(x-x_0)$. With $m=|\nabla u(x_0)|$ we get from (4.11) on a ball $B_{2r}=B_{2r}(x_0)\subset\Omega$ after taking $j\nearrow\infty$:

$$\int_{B_r}\int_{\mathbb{R}^{N\times n}} E\big(\cdot-\nabla u(x_0)\big)\,\mathrm{d}\nu_x\,\mathrm{d}x+\lambda(B_r)\leq c_m\int_{B_{2r}} E\left(\frac{u-a}{r}\right)\,\mathrm{d}x.$$

Divide by $\mathscr{L}^n(B_r)$ and take $r\searrow 0$ to get by Lebesgue's differentiation theorem

$$\int_{\mathbb{R}^{N\times n}} E\big(\cdot-\nabla u(x_0)\big)\,\mathrm{d}\nu_{x_0}+\frac{\mathrm{d}\lambda}{\mathrm{d}\mathscr{L}^n}(x_0)\leq 0$$

for $\mathscr{L}^n$ almost all such x_0. But then both terms on the left-hand side must be 0, and so, using the strict convexity of E for the first term, we conclude that

$$\nu_{x_0}=\delta_{\nabla u(x_0)} \text{ for } \mathscr{L}^n\text{-a.e. } x_0 \text{ and } \lambda\perp\mathscr{L}^n.$$

We always have $|D^s u|\leq\lambda\llcorner\Omega$ (see for instance [53]). For the upper bound we fix an arbitrary ball $B_{2r}\subset\Omega$, take $a=u_{B_{2r}}$ above and pass to the limit whereby

$$\int_{B_r} E\big(\nabla u\big)\,\mathrm{d}x+\lambda(B_r)\leq c_0\int_{B_{2r}} E\left(\frac{u-u_{B_{2r}}}{r}\right)\,\mathrm{d}x$$

results. Using that $E(z)\leq|z|$ for all z and Poincaré's inequality on the right-hand side we get

$$\int_{B_r} E\big(\nabla u\big)\,\mathrm{d}x+\lambda(B_r)\leq c_0 c|Du|(B_{2r}).$$

Put $\mu=c_0c|Du|$; then $\lambda(B)\leq\mu(2B)$ for any ball B for which $2B\subset\Omega$. We reformulate this bound in terms of cubes as follows. For a closed ball $\overline{B}$ we let Q denote the largest closed cube with sides parallel to the coordinate axes that is contained in $\overline{B}$ and we let $\hat{Q}$ denote the smallest such cube that contains $s\overline{B}$ for some fixed $s>2$. Then Q and $\hat{Q}$ are concentric and the sidelengths satisfy $\ell(\hat{Q})=s\sqrt{n}\ell(Q)$. Clearly given a cube Q with sides parallel to the coordinate axes, the cube $\hat{Q}$ just described is uniquely determined and we must in particular have $\lambda(Q)\leq\mu(\hat{Q})$ for all cubes Q with $\hat{Q}\subset\Omega$. Now fix a closed cube $Q\subset\Omega$ and consider the system of its Q-dyadic subcubes at level $k\in\mathbb{N}$:

$$Q=\bigcup_{j=1}^{2^{nk}} Q_j^{(k)}.$$

For k sufficiently large we have that each $\hat{Q}_j^{(k)} \subset \Omega$ since $\ell(\hat{Q}_j^{(k)}) = s\sqrt{n}2^{-k}\ell(Q)$, $Q_j^{(k)} \subset \hat{Q}_j^{(k)} \cap Q \Subset \Omega$, and so $\lambda(Q_j^{(k)}) \leq \mu(\hat{Q}_j^{(k)})$, hence

$$\lambda(Q) \leq \sum_{j=1}^{2^{nk}} \mu(\hat{Q}_j^{(k)}) = \int \sum_{j=1}^{2^{nk}} \mathbf{1}_{\hat{Q}_j^{(k)}} \,\mathrm{d}\mu \leq c(n,s)\mu(Q_k),$$

where $Q_k = \bigcup_j \hat{Q}_j^{(k)}$ and we used that the family of cubes satisfies a uniform bounded overlap property. Taking $k \nearrow \infty$ we arrive at $\lambda(Q) \leq c(n,s)\mu(Q)$. Now since the cube $Q \subset \Omega$ was arbitrary and λ is singular the proof is complete. □

4.3. **Approximation by harmonic maps.** We turn to the announced approximation by harmonic maps. This step, where the minimizer is compared with the solution to a suitably linearized problem, is standard fare in partial regularity proofs and goes back to the works [5, 6, 25]. However, due to the L^1 set-up the usual ways of implementing this linearization (such as for instance [1, 2, 19, 28, 26, 45, 47]) do seem to require modification. Fortunately, our variant is quite straightforward and proceeds by explicit construction of a test map that yields the required estimate. In fact, we believe that when this construction is applied in the cases covered previously in the literature, it also offers a useful alternative argument there.

Because the approximation result is achieved by a linearization argument it is more natural if we also replace the key assumptions (H2) and minimality by their corresponding linearizations. More precisely, we shall replace the quasiconvexity hypothesis (H2) on the integrand F by its linearization, namely the corresponding weaker rank-one convexity hypothesis:

$$\text{(H2W)} \qquad z \mapsto F(z) - \ell E(z) \;\text{ is rank-one convex.}$$

Instead of assuming that u is a minimizer, we assume that $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ satisfies the extremality condition (2.18). We then have the following:

Proposition 4.6. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfy (H0), (H1) and (H2W) and assume that $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ satisfies (2.18). Fix a number $m > 0$. For any affine map $a\colon \mathbb{R}^n \to \mathbb{R}^N$ with $|\nabla a| \leq m$ and each ball $B = B_R(x_0) \subset \Omega$ so that $u|_{\partial B} \in \mathrm{BV}(\partial B, \mathbb{R}^N)$ and $|Du|(\partial B) = 0$ the elliptic system*

$$\text{(4.12)} \qquad \begin{cases} -\mathrm{div} F''(\nabla a)\nabla h = 0 & \text{in } B \\ h = u|_{\partial B} & \text{on } \partial B, \end{cases}$$

admits a unique solution $h \in \mathrm{W}^{1,1}(B, \mathbb{R}^N)$. This solution h satisfies

$$\text{(4.13)} \qquad \left(\fint_B |\nabla h - \nabla a|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \leq c \fint_{\partial B} |D_\tau (u-a)|$$

for exponents $p \in (1,2)$ when $n = 2$ and $p \in (1, \frac{n}{n-1}]$ when $n \geq 3$ and a corresponding constant $c = c(n, N, m, p, \frac{L}{\ell})$. Moreover, for each exponent $p \in (1, \frac{n}{n-1})$,

$$\text{(4.14)} \qquad \fint_B E\left(\frac{u-h}{R} \right) \mathrm{d}x \leq C \left(\fint_B E\big(D(u-a)\big) \right)^p,$$

where $C = C(m, n, N, p, L, \ell)$.

Proof. We give the details for the case $n \geq 3$ only and leave it to the reader to check that the same proof applies for $n = 2$, where the only difference is that Lemma 2.5 should be used instead of Lemma 2.4.

Let $x_0 \in \Omega$ and fix a number $m > 0$. By virtue of Lemma 2.3 $\mathscr{L}^1$ a.e. radii $R \in (0, \mathrm{dist}(x_0, \partial\Omega))$ have the property that $u|_{\partial B} \in \mathrm{BV}(\partial B, \mathbb{R}^N)$ and $|Du|(\partial B) = 0$, where we wrote $B = B_R(x_0)$. We fix such a radius R and write as already indicated $B = B_R(x_0)$. For an affine map $a\colon \mathbb{R}^n \to \mathbb{R}^N$ with

$|\nabla a| \leq m$ we put as in the previous subsection $\tilde{F} = F_{\nabla a}$ and $\tilde{u} = u - a$. Clearly, $\tilde{u}|_{\partial B} \in \mathrm{BV}(\partial B, \mathbb{R}^N)$ remains true. From (H0) and (H2W) we infer that

$$\tilde{F}''(0)[y \otimes x, y \otimes x] \geq \tfrac{1}{c}|y|^2|x|^2 \quad \forall y \in \mathbb{R}^N,\, \forall x \in \mathbb{R}^n \quad \text{and} \quad |\tilde{F}''(0)| \leq c, \tag{4.15}$$

where $c = c(m) > 0$ is a constant that as indicated depends on m.

As is customary in this context, we make use of (2.18) in a linearized form by rewriting it for $\varphi \in \mathrm{C}_c^\infty(B, \mathbb{R}^N)$ as

$$\begin{aligned} \int_B \tilde{F}''(0)[D\tilde{u}, \nabla\varphi] &= \int_B \tilde{F}''(0)[D^s\tilde{u}, \nabla\varphi] \\ &\quad + \int_B \langle \tilde{F}''(0)\nabla\tilde{u} - \tilde{F}'(\nabla\tilde{u}), \nabla\varphi\rangle \,\mathrm{d}x \\ &\overset{(4.5)}{\leq} c\int_B |D^s\tilde{u}||\nabla\varphi| \\ &\quad + c\int_B E(\nabla\tilde{u})|\nabla\varphi| \,\mathrm{d}x \\ &\leq c\int_B E(D\tilde{u})|\nabla\varphi|. \end{aligned}$$

It is at this stage we take advantage of the particular choice of radius R whereby $\tilde{u}|_{\partial B}$ is BV on ∂B and $|D\tilde{u}|(\partial B) = 0$. The latter ensures that we may extend the above bound by continuity to hold for all $\varphi \in (\mathrm{W}_0^{1,\infty} \cap \mathrm{C}^1)(B, \mathbb{R}^N)$. The former gives in combination with the embedding result of Lemma 2.4 that $\tilde{u}|_{\partial B} \in \mathrm{W}^{\frac{1}{n}, \frac{n}{n-1}}(\partial B, \mathbb{R}^N)$ and

$$\left(\fint_{\partial B}\int_{\partial B} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{\frac{n}{n-1}}}{|x-y|^{n-1+\frac{1}{n-1}}} \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}\mathcal{H}^{n-1}(y) \right)^{1-\frac{1}{n}} \leq cR^{1-\frac{1}{n}} \fint_{\partial B} |D_\tau \tilde{u}|$$

for a dimensional constant $c = c(n, N)$. In view of (4.15) and Proposition 2.11 we can then find a unique solution $\tilde{h} \in \mathrm{W}^{1,\frac{n}{n-1}}(B, \mathbb{R}^N)$ to the boundary value problem

$$\begin{cases} -\mathrm{div}\mathbb{A}\nabla\tilde{h} = 0 & \text{in } B \\ \tilde{h} = \tilde{u} & \text{on } \partial B, \end{cases} \tag{4.16}$$

where $\mathbb{A} = \tilde{F}''(0)$. In particular we record that

$$\int_B \mathbb{A}[\nabla\tilde{h}, \nabla\varphi] \,\mathrm{d}x = 0 \tag{4.17}$$

holds for all $\varphi \in \mathrm{W}_0^{1,n}(B, \mathbb{R}^N)$ and also that the integral estimate (4.13) holds. The reader will notice that the integral bound in Proposition 2.11 contains zero order terms too on the right-hand side. However, these can be absorbed in the right-hand side of (4.13) by subtracting the integrals average of $u - a$ over ∂B from $u - a$ and then employing a Poincaré inequality on ∂B. Clearly (4.13) is invariant under subtraction of constant vectors.

Put $\psi = \tilde{u} - \tilde{h}$ so that $\psi \in \mathrm{BV}_0(B, \mathbb{R}^N)$ and

$$\int_B \mathbb{A}[D\psi, \nabla\varphi] \,\mathrm{d}x \leq c\int_B E(D\tilde{u})|\nabla\varphi| \tag{4.18}$$

holds for all $\varphi \in (\mathrm{W}_0^{1,\infty} \cap \mathrm{C}^1)(B, \mathbb{R}^N)$, where $c = c(m, L)$. We extract information from (4.18) by constructing a suitable test map φ. It is convenient to change variables and refer everything to the open unit ball as follows: Put for $x \in \mathbb{B} := B_1(0)$

$$\Psi(x) = \frac{1}{R}\psi(x_0 + Rx), \quad \Phi(x) = \frac{1}{R}\varphi(x_0 + Rx), \quad U(x) = \frac{1}{R}\tilde{u}(x_0 + Rx).$$

Then (4.18) becomes

$$\int_{\mathbb{B}}\mathbb{A}[D\Psi,\nabla\Phi]\,\mathrm{d}x\leq c\int_{\mathbb{B}}E(DU)|\nabla\Phi|\quad\forall\,\Phi\in(\mathrm{W}_0^{1,\infty}\cap\mathrm{C}^1)(\mathbb{B},\mathbb{R}^N). \tag{4.19}$$

Denote by $T\colon\mathbb{R}^N\to\mathbb{R}^N$ the truncation mapping defined by

$$T(y)=\begin{cases} y & \text{if } |y|\leq 1\\ \frac{y}{|y|} & \text{if } |y|>1,\end{cases}$$

and consider the elliptic system

$$\begin{cases} -\mathrm{div}\,\mathbb{A}\nabla\Phi=T(\Psi) & \text{in } \mathbb{B}\\ \Phi=0 & \text{on } \partial\mathbb{B}.\end{cases} \tag{4.20}$$

Evidently the right-hand side is bounded and we have a unique solution $\Phi\in\mathrm{W}_0^{1,2}(\mathbb{B},\mathbb{R}^N)$. From Proposition 2.11 it follows that Φ is of Sobolev class $\mathrm{W}^{2,q}(\mathbb{B},\mathbb{R}^N)$ for each exponent $q\in(1,\infty)$ with bound

$$\int_{\mathbb{B}}|\nabla^2\Phi|^q\,\mathrm{d}x\leq C\int_{\mathbb{B}}|T(\Psi)|^q\,\mathrm{d}x \tag{4.21}$$

where $C=C(m,n,N,q,L,\ell)$ is a constant. If we take $p>n$, then we have that $\Phi\in\mathrm{C}^{1,1-\frac{n}{q}}(\mathbb{B},\mathbb{R}^N)$ and since $(\nabla\Phi)_{\mathbb{B}}=0$ it follows from Morrey's inequality (in the form [32, Sect. 4.5, Th. 3]) that

$$\|\nabla\Phi\|_{\mathrm{L}^\infty}\leq c\|\nabla^2\Phi\|_{\mathrm{L}^q}\leq c\|T(\Psi)\|_{\mathrm{L}^q}.$$

In particular, $\Phi\in(\mathrm{W}_0^{1,\infty}\cap\mathrm{C}^1)(\mathbb{B},\mathbb{R}^N)$ so that Φ indeed qualifies as a test map in (4.19) and then, in turn, by approximation, $\Psi\in\mathrm{BV}_0(\mathbb{B},\mathbb{R}^N)$ qualifies as a test map in (4.20). We also note that a simple estimation using (2.5) yields

$$\|T(\Psi)\|_{\mathrm{L}^q}\leq c\left(\int_{\mathbb{B}}E(\Psi)\,\mathrm{d}x\right)^{\frac{1}{q}},$$

and consequently

$$\|\nabla\Phi\|_{\mathrm{L}^\infty}\leq c\left(\int_{\mathbb{B}}E(\Psi)\,\mathrm{d}x\right)^{\frac{1}{q}} \tag{4.22}$$

holds for exponents $q\in(n,\infty)$ and corresponding constants $c=c(m,n,N,q,L,\ell)$. We plug this Φ into (4.19); recalling that Ψ can be used to test (4.20) and that $\mathbb{A}$ is symmetric the following string of inequalities results:

$$\begin{aligned}\int_{\mathbb{B}}\min\{|\Psi|^2,|\Psi|\}\,\mathrm{d}x &= \int_{\mathbb{B}}\langle\Psi,T(\Psi)\rangle\,\mathrm{d}x\\ &\overset{(4.20)}{=} \int_{\mathbb{B}}\langle\mathbb{A}\nabla\Phi,\nabla\Psi\rangle\,\mathrm{d}x\\ &= \int_{\mathbb{B}}\langle\mathbb{A}\nabla\Psi,\nabla\Phi\rangle\,\mathrm{d}x\\ &\overset{(4.19),(4.22)}{\leq} c\int_{\mathbb{B}}E(DU)\left(\int_{\mathbb{B}}E(\Psi)\,\mathrm{d}x\right)^{\frac{1}{q}},\end{aligned}$$

and thus (using again (2.5))

$$\left(\int_{\mathbb{B}}E(\Psi)\,\mathrm{d}x\right)^{1-\frac{1}{q}}\leq c\int_{\mathbb{B}}E(DU).$$

Hence we have shown that

$$\int_{\mathbb{B}}E(\Psi)\,\mathrm{d}x\leq C\left(\int_{\mathbb{B}}E(DU)\right)^p \tag{4.23}$$

where $p = q/(q-1) \in (1, \frac{n}{n-1})$ is the dual exponent and $C = C(m,n,N,p,L,\ell)$ is a constant. Finally, we change back variables $x \mapsto x_0 + Rx$ and recall that $\psi = \tilde{u} - \tilde{h}$ whereby (4.23) turns into (4.14) thus completing the proof. □

4.4. **Excess decay estimate.** For a map $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ and a ball $B_r(x_0) \subset \Omega$ the relevant excess functional is

$$\mathscr{E}(x_0, r) = \int_{B_r(x_0)} E\big(Du - (Du)_{B_r(x_0)}\big).$$

The goal of this subsection is the following excess decay estimate:

Proposition 4.7. *Suppose $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* (H0)*,* (H1)*,* (H2) *and that $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ is a minimizer. Then each $m > 0$ and $q \in (1, \frac{n}{n-1})$ there exists a constant $c = c(m, q, n, N, \frac{L}{\ell})$ with the following property. For a ball $B_R = B_R(x_0) \subset \Omega$ such that*

$$|(Du)_{B_R}| < m \tag{4.24}$$

and

$$\fint_{B_R} |Du - (Du)_{B_R}| \leq 1 \tag{4.25}$$

we have that

$$\mathscr{E}(x_0, \sigma R) \leq c \left(\sigma^{n+2} + \left(\frac{\mathscr{E}(x_0, R)}{\mathscr{L}^n(B_R(x_0))} \right)^{q-1} \right) \mathscr{E}(x_0, R) \tag{4.26}$$

holds for all $\sigma \in (0,1)$.

Proof. Again we give the details for the case $n \geq 3$ only and leave it to the reader to check that the same proof applies for $n = 2$, where the only difference is that Lemma 2.5 should be used instead of Lemma 2.4. As in the previous subsections we put $\tilde{u} = u - a$ and $\tilde{F} = F_{\nabla a}$, where a is the affine map $a(x) = u_{B_R} + (Du)_{x_{B_R}}(x - x_0)$, and remark that by virtue of our assumptions both results from subsections 3.2 and 3.3 are now available.

In view of Lemma 2.3 we can select $r \in (\frac{9}{10}R, R)$ such that $\tilde{u}|_{\partial B_r} \in \mathrm{BV}(\partial B_r, \mathbb{R}^N)$ and

$$\int_{\partial B_r} |D_\tau(\tilde{u}|_{\partial B_r})| \leq \frac{c}{R} \int_{B_R} |D\tilde{u}| \tag{4.27}$$

for some constant c. Now the harmonic map $\tilde{h}$ determined at (4.16) satisfies (4.13) and (4.14). Let $A\colon \mathbb{R}^n \to \mathbb{R}^N$ be the affine map $A(x) = \tilde{h}(x_0) + \nabla\tilde{h}(x_0)(x - x_0)$ and put $a_0 = a + A$. Then a_0 is clearly affine and in order to estimate $|\nabla a_0|$ we note that according to Lemma 2.10 we have for a constant $c = c(n, N, m, \frac{L}{\ell})$:

$$\begin{aligned}
|\nabla\tilde{h}(x_0)| &\leq \sup_{B_{\frac{r}{2}}} |\nabla\tilde{h}| \leq c \fint_{B_r} |\nabla\tilde{h}| \,\mathrm{d}x \\
&\leq c \left(\fint_{B_r} |\nabla\tilde{h}|^{\frac{n}{n-1}} \,\mathrm{d}x \right)^{\frac{n-1}{n}} \\
&\overset{(4.13)}{\leq} c \fint_{\partial B_r} |D_\tau(\tilde{u}|_{\partial B_r})| \\
&\overset{(4.27)}{\leq} \frac{c}{Rr^{n-1}} \int_{B_R} |D\tilde{u}| \\
&\leq c \fint_{B_R} |D\tilde{u}|.
\end{aligned}$$

In view of (4.25) we therefore have that

$$\begin{aligned} |\nabla a_0| &\leq |(Du)_{B_R}| + c\fint_{B_R} |Du - (Du)_{B_R}| \\ &< m + c(m) =: C_m \end{aligned}$$

holds. For $\sigma \in (0, \frac{1}{5})$ we have by (2.6)

$$\int_{B_{\sigma R}} E(Du - (Du)_{B_{\sigma R}}) \leq 12 \int_{B_{\sigma R}} E(D(u - a_0)).$$

Next, we estimate the right-hand side by use of the Caccioppoli inequality (4.2) on the ball $B_{\sigma R} = B_{\sigma R}(x_0)$ and with the affine map a_0 defined above:

$$\int_{B_{\sigma R}} E(D(u - a_0)) \leq c \int_{B_{2\sigma R}} E\left(\frac{u - a_0}{2\sigma R}\right) \mathrm{d}x$$

where $c = c(m)$ is the corresponding constant from Proposition 4.3. In view of the restrictions on σ and r we have $B_{2\sigma R} \subset B_{r/2}$, and so in particular (using also (2.5)) we get

$$\begin{aligned} \int_{B_{\sigma R}} E(Du - (Du)_{x_0,\sigma R}) &\leq C \int_{B_{2\sigma R}} \left(E\left(\frac{\tilde{u} - \tilde{h}}{\sigma R}\right) + E\left(\frac{\tilde{h} - A}{2\sigma R}\right)\right) \mathrm{d}x \\ &\leq \frac{c}{\sigma^2} \int_{B_r} E\left(\frac{\tilde{u} - \tilde{h}}{r}\right) \mathrm{d}x + c \int_{B_{2\sigma R}} E\left(\frac{\tilde{h} - A}{2\sigma R}\right) \mathrm{d}x. \end{aligned}$$

Here we have for the first term according to (4.14) for each exponent $q \in (1, \frac{n}{n-1})$ and constant $C = C(m, n, N, q, L, \ell)$ that

$$\int_{B_r} E\left(\frac{\tilde{u} - \tilde{h}}{r}\right) \mathrm{d}x \leq C \left(\fint_{B_r} E(D\tilde{u})\right)^q \mathscr{L}^n(B_R).$$

The second term is estimated using Lemma 2.10. Accordingly we have for $x \in B_{2\sigma R} \subset B_{\frac{r}{2}}$ and in view of our choice of the affine map A:

$$\begin{aligned} \frac{|\tilde{h}(x) - A(x)|}{\sigma R} &\leq c \sup_{x \in B_{2\sigma R}} \left(|\nabla^2 \tilde{h}(x)| \frac{|x - x_0|^2}{\sigma R} \right) \\ &\leq c \sup_{x \in B_{\frac{r}{2}}} |\nabla^2 \tilde{h}(x)| \sigma R \\ &\overset{\text{Lemma 2.10}}{\leq} c\sigma \fint_{B_r} |\nabla \tilde{h}| \,\mathrm{d}x \\ &\overset{(4.13),(4.27)}{\leq} c\sigma \fint_{B_R} |Du - (Du)_{B_R}| \\ &\overset{(4.25),(2.7)}{\leq} c\sigma \left(\fint_{B_R} E(Du - (Du)_{B_R})\right)^{\frac{1}{2}}, \end{aligned}$$

where (4.27) was used with $h = \tilde{h} + a$. Consequently we have

$$\begin{aligned} \int_{B_{2\sigma R}} E\left(\frac{\tilde{h} - A}{2\sigma R}\right) \mathrm{d}x &\leq c(\sigma R)^n E\left(\sigma \left(\fint_{B_R} E(Du - (Du)_{B_R})\right)^{\frac{1}{2}}\right) \\ &\leq c\sigma^{n+2} \int_{B_R} E(Du - (Du)_{B_R}), \end{aligned}$$

and hence we arrive upon collection of the bounds at (4.26). Increasing the constant c if necessary we see that the bound actually extends to hold for $\sigma \in [\frac{1}{5}, 1)$ too. The proof is complete. □

4.5. **Iteration and conclusion.** With the excess decay result of Proposition 4.7 at hand we can conclude in a standard manner. The first step is obtained by an iteration argument and is in terms of the normalized excess:

$$\Phi(x_0,r)=\frac{\mathscr{E}(x_0,r)}{\mathscr{L}^n(B_r(x_0))}=\fint_{B_r(x_0)}E\big(Du-(Du)_{B_r(x_0)}\big).$$

Proposition 4.8. *Suppose $F\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ satisfies* (H0), (H1), (H2) *and that $u\in \mathrm{BV}(\Omega,\mathbb{R}^N)$ is a minimizer. Let $\alpha\in(0,1)$ and $m>0$. Then there exist positive constants $c=c(n,N,\frac{L}{\ell},m)$ and $\varepsilon=\varepsilon(n,N,\frac{L}{\ell},m,\alpha)$ with the following property. If a ball $B_R(x_0)\subset\Omega$ satisfies*

$$|(Du)_{B_R(x_0)}|<m \tag{4.28}$$

and

$$\Phi(x_0,R)<\varepsilon, \tag{4.29}$$

then

$$\Phi(x_0,r)\le c\left(\frac{r}{R}\right)^{2\alpha}\Phi(x_0,R) \tag{4.30}$$

for all $r\in(0,R)$.

Proof. For ease of notation we write $B_r=B_r(x_0)$ and $\Phi(r)=\Phi(x_0,r)$. First recall from Lemma 2.9 that for $\Phi(r)\le 1$ we have

$$\fint_{B_r}|Du-(Du)_{B_r}|\le\sqrt{3\Phi(r)} \tag{4.31}$$

Consequently, if for a ball $B_r\subset\Omega$ we have $|(Du)_{B_r}|<m$ and $\Phi(r)\le\frac{1}{3}$, then Proposition 4.7 yields

$$\Phi(\sigma r)\le c\left(\sigma^2+\sigma^{-n}\Phi(r)^{q-1}\right)\Phi(r)$$

for $q\in(1,\frac{n}{n-1})$, $c=c(n,N,\frac{L}{\ell},m,q)$ and $\sigma\in(0,1)$. Fix $q\in(1,\frac{n}{n-1})$ and denote

$$C=c(n,N,\tfrac{L}{\ell},m+1,q) \tag{4.32}$$

where we emphasize that we take the constant corresponding to $m+1$ rather than to m. With this choice we then select $\sigma\in(0,1)$ satisfying $C\sigma^2<\frac{1}{2}\sigma^{2\alpha}$. For definiteness we fix

$$\sigma=(3C)^{-\frac{1}{2(1-\alpha)}}. \tag{4.33}$$

Next, take an $\varepsilon_0\in(0,\frac{1}{3})$ so $C\sigma^{-n}\varepsilon_0^{q-1}<\frac{1}{2}\sigma^{2\alpha}$, say

$$\varepsilon_0=\left(\frac{\sigma^{n+2\alpha}}{3C}\right)^{\frac{1}{q-1}}. \tag{4.34}$$

Observe that with these choices we have for any ball $B_r\subset\Omega$ satisfying $|(Du)_{B_r}|<m+1$ and $\Phi(r)<\varepsilon_0$ that

$$\Phi(\sigma r)\le\sigma^{2\alpha}\Phi(r). \tag{4.35}$$

We iterate this as follows. Let $\varepsilon\in(0,\varepsilon_0]$, further restrictions will be imposed below. For the remainder of the proof we fix a ball $B_R=B_R(x_0)\subset\Omega$ satisfying (4.28)–(4.29). We then have in particular that $\Phi(\sigma R)\le\sigma^{2\alpha}\varepsilon\le\varepsilon_0$. Also, in a standard manner we can estimate

$$\begin{aligned}|(Du)_{B_{\sigma R}}| &\le |(Du)_{B_R}|+|(Du)_{B_{\sigma R}}-(Du)_{B_R}|\\ &< m+\fint_{B_{\sigma R}}|Du-(Du)_{B_R}|\\ &\le m+\sigma^{-n}\fint_{B_R}|Du-(Du)_{B_R}|\\ &\overset{(2.7)}{\le} m+\sigma^{-n}\sqrt{3\varepsilon}.\end{aligned}$$

We require that $\sigma^{-n}\sqrt{3\varepsilon}\leq 1$, that is,

$$\varepsilon \leq \frac{\sigma^{2n}}{3}. \tag{4.36}$$

Thus in view of (4.35) we have shown that

$$\Phi(\sigma^j R) \leq \sigma^{2\alpha j}\Phi(R) \tag{4.37}$$

holds for $j = 1, 2$. Let $k \in \mathbb{N}$ and suppose that (4.37) holds for $j \in \{1, \ldots, k\}$. Then $\Phi(\sigma^j R) \leq \sigma^{2\alpha j}\Phi(R) < \sigma^{2\alpha j}\varepsilon < \varepsilon_0$ for each $j \leq k$ and as above we estimate

$$\begin{aligned} |(Du)_{B_{\sigma R}}| &\leq m + \sum_{j=1}^{k}\sigma^{-n}\sqrt{3\Phi(\sigma^{j-1}R)} \\ &\leq m + \sum_{j=1}^{k}\sigma^{-n}\sqrt{3\sigma^{2\alpha(j-1)}\varepsilon} \\ &< m + \frac{\sqrt{3\varepsilon}}{\sigma^n}\frac{1}{1-\sigma^\alpha}. \end{aligned}$$

We require that $\frac{\sqrt{3\varepsilon}}{\sigma^n}\frac{1}{1-\sigma^\alpha} \leq 1$. This is acheived if we take

$$\varepsilon = \min\{\varepsilon_0, \frac{(\sigma^n - \sigma^{n+\alpha})^2}{3}\}. \tag{4.38}$$

Thus with these choices we have for balls $B_R(x_0) \subset \Omega$ that satisfy (4.28)–(4.29) shown that (4.37) holds for all $j \in \mathbb{N}$. The conclusion follows in a standard manner from this. □

Using the excess decay estimate of Proposition 4.8 we conclude in a routine way with the following ε-regularity result that in view of Lebesgue's differentiation theorem also implies the last part of Theorem 1.1.

Theorem 4.9. *Suppose $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* (H0), (H1), (H2) *and that $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ is a minimizer. Then for each $m > 0$ there exists $\varepsilon_m = \varepsilon_m(F) \in (0, 1]$ with the following property. If the ball $B_R(x_0) \subset \Omega$ satisfies*

$$|(Du)_{B_R(x_0)}| < m \tag{4.39}$$

and

$$\Phi(x_0, R) < \varepsilon_m, \tag{4.40}$$

then u is $\mathrm{C}^{2,\alpha}_{\mathrm{loc}}$ on $B_{\frac{R}{2}}(x_0)$ for each $\alpha < 1$, and

$$\sup_{\substack{x,y\in B_{R/4}(x_0)\\ x\neq y}} \frac{|\nabla^2 u(x) - \nabla^2 u(y)|^2}{|x-y|^{2\alpha}} \leq c\frac{\Phi(x_0, R)}{R^{2+2\alpha}} \tag{4.41}$$

where $c = c(n, N, \frac{L}{\ell}, m, \alpha)$ is a constant.

Proof. We merely sketch the proof as it is essentially standard once the excess decay estimate from Proposition 4.8 has been established. Fix $m > 0$ and consider the corresponding

$$\tilde{\varepsilon} = \varepsilon(n, N, \tfrac{L}{\ell}, m+1, \tfrac{1}{2}) > 0$$

that was determined in Proposition 4.8. Note that we take the number that corresponds to $m+1$ rather than to m. Let $\varepsilon \in (0, \tilde{\varepsilon}]$ and assume that $B_R(x_0) \subset \Omega$ is a ball so that (4.28)–(4.29) hold. We shall determine ε in the course of the proof. Let $x \in B_{R/2}(x_0)$ and note that the ball $B_{R/2}(x) \subset B_R(x_0)$ satisfies

$$\Phi(x, \tfrac{R}{2}) \overset{\text{Lemma 2.8}}{\leq} 4\cdot 2^n\Phi(x_0, R) < 2^{n+2}\varepsilon$$

and, proceeding as above,

$$\begin{aligned}|(Du)_{B_{\frac{R}{2}}(x)}| &< m + 2^n \fint_{B_R(x_0)} |Du - (Du)_{B_R(x_0)}| \\ &\overset{(2.7)}{\leq} m + 2^n \sqrt{3\Phi(x_0, R)} \\ &< m + 2^n \sqrt{3\varepsilon}.\end{aligned}$$

Thus if we take $\varepsilon = \min\{\frac{\tilde{\varepsilon}}{2^{n+2}}, \frac{1}{3\cdot 2^{2n}}\}$, then Proposition 4.8 yields the bound

$$\Phi(x, r) \leq c\frac{r}{R}\Phi(x, \tfrac{R}{2}) \leq c_1 \frac{\Phi(x_0, R)}{R} r \quad \text{with} \quad c_1 = 2^{n+2}c$$

valid for all $x \in B_{R/2}(x_0)$ and all $r \in (0, \frac{R}{2})$. In view of Lemma 2.9 we can deduce a more familiar looking excess decay estimate:

$$\begin{aligned}\left(\fint_{B_r(x)} |Du - (Du)_{B_r(x)}|\right)^2 &\leq \Phi(x, r)^2 + 2\Phi(x, r) \\ &\leq c_1^2 \left(\tfrac{r}{R}\right)^2 \Phi(x_0, R)^2 + 2c_1 \tfrac{r}{R}\Phi(x_0, R) \\ &\leq c\frac{\Phi(x_0, R)}{R} r\end{aligned}$$

for all $x \in B_{R/2}(x_0)$ and $r \in (0, R/2)$. (Here $c = c_1^2 + 2c_1$ and we used that $\Phi(x_0, R) \leq 1$.) Using the Campanato-Meyers integral characterization of Hölder continuity we conclude that u is $\mathrm{C}^{1,\frac{1}{2}}$ on $B_{R/2}(x_0)$ and that we have

$$\sup_{\substack{x,y \in B_{R/2}(x_0) \\ x \neq y}} \frac{|\nabla u(x) - \nabla u(y)|^2}{|x - y|} \leq c\frac{\Phi(x_0, R)}{R}$$

for some constant $c = c(n, N, \frac{L}{\ell}, m)$. Finally, in order to boost the regularity of u we employ the difference-quotient method and elliptic Schauder estimates for linear Legendre-Hadamard elliptic systems. Put $B = B_{R/2}(x_0)$, let $\delta > 0$ be small and denote for increments $h \in \mathbb{R}$ with $|h| < \delta R$ the finite difference of ∇u in the j-th coordinate direction by $\Delta_{j,h}\nabla u(x) = \nabla u(x + he_j) - \nabla u(x)$, $x \in B' := B_{(1-\delta)R/2}(x_0)$. Define the x-dependent symmetric bilinear forms (for $x \in B'$, $|h| < \delta R$ and $1 \leq j \leq n$) by

$$Q(x)[z, w] = Q_{j,h}(x)[z, w] = \int_0^1 F''(\nabla u(x) + t\Delta_{j,h}\nabla u(x))[z, w] \,\mathrm{d}t \quad (z,\, w \in \mathbb{R}^{N\times n})$$

From (H0) and the above follows that $Q \in \mathrm{C}^{0,\frac{1}{2}}(B', \bigodot^2(\mathbb{R}^{N\times n}))$ with the corresponding Schauder norm of Q bounded uniformly in $|h| < \delta R$ and $1 \leq j \leq n$. By virtue of Lemma 4.2 the form Q is uniformly strongly Legendre-Hadamard elliptic: there exists a positive constant $c = c(n, N, \frac{L}{\ell}, m, \mathrm{diam}\Omega)$ such that for all $x \in B'$ and $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$,

$$Q(x)[a \otimes b, a \otimes b] \geq \tfrac{1}{c}|a|^2|b|^2 \quad \text{and} \quad |Q(x)| \leq c$$

hold. Freezing coefficients and using a partition of unity we establish the following Gårding inequality ($\alpha, \beta > 0$)

$$\int_{B'} Q(x)[\nabla\varphi, \nabla\varphi] \,\mathrm{d}x \geq \int_{B'} \left(\alpha|\nabla\varphi|^2 - \beta|\varphi|^2\right) \mathrm{d}x$$

valid for all $\varphi \in \mathrm{W}_0^{1,\infty}(B', \mathbb{R}^N)$, $|h| < \delta R$, $1 \leq j \leq n$. Using these bounds for the form Q and testing the Euler-Lagrange system by $\varphi = \Delta_{j,-h}\left(\rho^2\Delta_{j,h}u\right)$ for a suitable cut-off function ρ we find in a standard manner that $u \in \mathrm{W}^{2,2}_{\mathrm{loc}}(B, \mathbb{R}^N)$ and that for each direction $1 \leq j \leq n$,

$$\int_B F''(\nabla u)[\nabla D_j u, \nabla\varphi] \,\mathrm{d}x = 0 \quad \forall \varphi \in \mathrm{C}^1_c(B, \mathbb{R}^N) \tag{4.42}$$

It follows by Schauder estimates, see [40, Theorem 3.2], that $D_j u$ is $\mathrm{C}^{1,1/2}_{\mathrm{loc}}$ on B, and hence that u is $\mathrm{C}^{2,1/2}_{\mathrm{loc}}$ on B. But then the coefficients $F''(\nabla u)$ in the linear elliptic system (4.42) are locally Lipschitz and the desired regularity and bound (4.41) follow using Schauder estimates again (see [40, Theorem 3.3]). The proof is complete. □

5. Extensions

Let $F\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ be an integrand of linear growth (1.1) which is mean coercive (1.3), but possibly non-quasiconvex. Then for $v\in \mathrm{BV}(\Omega,\mathbb{R}^N)$ and a Lipschitz subdomain $O\subset\Omega$ we define a variant of the functional at (1.5) by relaxation from $\mathrm{W}^{1,1}$:

$$\mathscr{F}[v,O]=\inf\left\{\liminf_{j\to\infty}\int_O F(\nabla v_j)\,\mathrm{d}x\colon\ (v_j)\subset \mathrm{W}^{1,1}_v(O,\mathbb{R}^N),\ v_j\to v \text{ in } \mathrm{L}^1(O,\mathbb{R}^N)\right\}.$$

It turns out that an integral representation that is similar to (1.6), but with F replaced by its quasiconvex envelope F^{qc}, still holds (see [52, Theorem 1]):

$$\mathscr{F}[v,O]=\int_O F^{\mathrm{qc}}(Dv)$$

for each $v\in \mathrm{BV}(O,\mathbb{R}^N)$. We note that the functional $\mathscr{F}$ is area-strictly continuous on $\mathrm{BV}(O,\mathbb{R}^N)$ (see [52, Theorem 3]). In [6] Almgren extended the elliptic regularity theory in the parametric context for minimizers to various classes of *almost minimizers* allowing him to treat also variational problems with constraints. In the nonparametric context of quasiconvex variational integrals of p-growth for $p>1$ defined on $\mathrm{W}^{1,p}$ Sobolev maps a corresponding extension has been obtained by Duzaar, Grotowski & Kronz in [28]. Here we extend Theorem 1.1 to almost minimizers in the BV case of linear growth and at the same time localize the result in the spirit of Acerbi & Fusco [2] (and [10] in the convex case).

For an increasing continuous function $\omega\colon[0,\infty)\to\mathbb{R}$ with $\omega(0)=0$ we say that $u\in \mathrm{BV}(\Omega,\mathbb{R}^N)$ is an ω-almost minimizer for $\mathscr{F}$ provided that for each ball $B_r(x_0)\subset\Omega$ we have

$$\mathscr{F}[u,B_r(x_0)]\leq \mathscr{F}[v,B_r(x_0)]+\omega(r)\int_{B_r(x_0)}\Big(|Dv|+\mathscr{L}^n\Big) \tag{5.1}$$

whenever $v\in \mathrm{BV}(\Omega,\mathbb{R}^N)$ and $u-v$ is supported in $B_r(x_0)$.

Theorem 5.1. *Let $F\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ be globally Lipschitz and mean coercive (1.3). Suppose $u\in \mathrm{BV}(\Omega,\mathbb{R}^N)$ satisfies (5.1) for some function ω verifying $\limsup_{r\searrow 0}\omega(r)/r^{2\alpha}<\infty$, where $\alpha\in(0,1)$. Let $z_0\in\mathbb{R}^{N\times n}$ and assume that*

$$\fint_{B_r(x_0)} E(Du-z_0\mathscr{L}^n)\to 0 \text{ as } r\searrow 0.$$

If F is $\mathrm{C}^{2,1}$ near z_0 and if for some $\ell>0$ the integrand $z\mapsto F(z)-\ell E(z)$ is quasiconvex at z_0, then u is $\mathrm{C}^{1,\alpha}$ near x_0.

We are not giving the detailed proof for Theorem 5.1 here since it follows closely the proof from Section 4 of Theorem 1.1. In order to execute the modified proof one requires the following observation that is closely related to [2, Lemma 2.2]:

Lemma 5.2. *Let $F\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ be globally Lipschitz and mean coercive (1.3), and fix $z_0\in\mathbb{R}^{N\times n}$. If F is C^2 near z_0 and for some $\ell>0$ the integrand $z\mapsto F(z)-\ell E(z)$ is quasiconvex at z_0, then the quasiconvex envelope F^{qc} of F is real-valued, satisfies (1.3), $\mathrm{lip}(F^{\mathrm{qc}})=\mathrm{lip}(F)$, $z\mapsto F^{\mathrm{qc}}(z)-\ell E(z)$ is quasiconvex at z_0 and $F^{\mathrm{qc}}=F$ near z_0. In fact, $F-\frac{\ell}{2}E$ is quasiconvex and equals $F^{\mathrm{qc}}-\frac{\ell}{2}E$ near z_0.*

Proof. Since $F \geq F^{\mathrm{qc}} \geq (F-\ell E)^{\mathrm{qc}} + \ell E$ and equality holds at z_0 we infer that $F^{\mathrm{qc}} - \ell E$ is quasiconvex at z_0. In particular, F^{qc} is then a real-valued quasiconvex integrand. From [22, Lemma 3.1] we deduce that F^{qc} satisfies (1.3) with the same constants as F. That $\mathrm{lip}(F^{\mathrm{qc}}) = \mathrm{lip}(F)$ is a consequence of [58, Lemma 5.1, Corollary 5.2]. Finally, if$F$ is C^2 on the ball $B_r(z_0)$ and we assume, as we may, that $F(z_0)=0$, $F'(z_0)=0$, then

$$|F(z)| \leq c\Theta\big(|z-z_0|\big)E\big(|z-z_0|\big) \quad \forall z \in \mathbb{R}^{N\times n} \tag{5.2}$$

for some constant c and modulus of continuity Θ. We can arrange that $\Theta\colon [0,\infty) \to [0,1]$ is continuous, increasing, concave and $\Theta(0)=0$, $\Theta(1)=1$. The proof of (5.2) is implicit in the proof of Lemma 2.2 in [2] that we may also follow to conclude that $F^{\mathrm{qc}} = F$ on $B_{r/2}(z_0)$.

Finally, regarding the last statement of the lemma, we may apply the aforementioned argument to the integrand $F - \frac{\ell}{2}E$ to conclude that it is quasiconvex near z_0. Since also F^{qc} equals F near z_0 the proof is concluded. □

As we have dealt with the case of autonomous integrands in the main part of this paper, let us finish by briefly addressing the case of x-dependent integrands and explain how these can be handled. From a technical perspective, the way in which functions are applied to vectorial Radon measures is equally covered by Section 2.2. We focus here on a special case and merely state a result that can be made to follow from Theorem 5.1.

Corollary 5.3. *Let $F\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be continuous and assume that for some constants ℓ, $L>0$ and $\alpha \in (0,1)$ we have for x, x_1, $x_2 \in \Omega$ and $z \in \mathbb{R}^{N\times n}$,*

$$\begin{cases} \ell|z| \leq F(x,z) \leq L(|z|+1), \\ |F(x_1,z)-F(x_2,z)| \leq L\min\{1,|x_1-x_2|^{2\alpha}\}(|z|+1), \\ z \mapsto F(x,z) \text{ is } \mathrm{C}^3 \text{ and } \partial^3 F(x,z)/\partial z^3 \text{ is jointly continuous in } (x,z) \\ z \mapsto F(x,z) - \ell E(z) \text{ is quasiconvex.} \end{cases}$$

Suppose that $u \in \mathrm{BV}(\Omega,\mathbb{R}^N)$ is a minimizer in the sense that

$$\int_\Omega F(x,Du) \leq \int_\Omega F(x,Dv)$$

holds for all $v \in \mathrm{BV}(\Omega,\mathbb{R}^N)$ for which $v-u$ has compact support in Ω. Then there exists an open subset $\Omega_u \subset \Omega$ such that $\mathscr{L}^n(\Omega\setminus\Omega_u)=0$ and u is $\mathrm{C}^{1,\alpha}_{\mathrm{loc}}$ on Ω_u.

Finally we remark that all the above stated regularity results would extend if instead of the integrand $F=F(Du)$ (or $F=F(x,Du)$) we considered the integrand $F(Du)+f(x,u)$, where $f\colon \Omega\times\mathbb{R}^N \to \mathbb{R}$ is Carathéodory and satisfies the growth condition

$$0 \leq f(x,y) \leq c\big(|y|^{\frac{n}{n-1}}+1\big) \quad \forall (x,y) \in \Omega\times\mathbb{R}^N,$$

where $c>0$ is a constant (see [47] for general results in this spirit in the p-growth context). We could also cover the more general notions of almost minimizers considered in [71] for the purpose of treating some image restoration problems.

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