ON CERTAIN PROBLEMS IN HOMOTOPY THEORY

A thesis submitted for the degree of

Doctor of Philosophy

in the University of Oxford

by

James Dillon Stasheff

Brasenose College

June 1960

ABSTRACT

John Milnor has given a construction [15] which associates with each topological group G a sequence of spaces GP(i) [our notation] which can be regarded as generalizations of the classical projective spaces $RP(i) = S^0P(i)$, $CP(i) = S^1P(i)$ and $QP(i) = S^3P(i)$. If one considers H-spaces instead of topological groups, one finds that additional homotopy structure is necessary to imitate this construction. Intuitively, an A_n-space X is an H-space with sufficient additional structure to permit the construction for $i \leq n$ of spaces XP(i) with properties similar to those of the spaces GP(i). An A_n-map can be thought of as a map from one A_n-space to another which up to homotopy respects the A_n-structure.

The principal goals of this study are 1) to investigate the multiplication on an A_n-space in terms of the geometry of the projective spaces and 2) to analyse the algebra of the homology of an A_n-space. In the first direction we observe that in the classical case there exist maps $RP(r) \times RP(s) \longrightarrow RP(r+s)$ and $CP(r) \times CP(s) \longrightarrow CP(r+s)$ which restricted to either factor are the usual imbeddings $RP(r) \subset RP(r+s)$ and $CP(r) \subset CP(r+s)$. Working with A_n-spaces we find that the existence of such maps is implied

by the multiplication's being itself an A_{r+s}-map. In particular, an H-space (X,m) is homotopy abelian if and only if there is a map of $SX \times SX \rightarrow XP(2)$ which is the inclusion on either factor.

The second direction is guided by William Massey's definition of cohomology products. The essential properties of the singular chain complex on an A_n-space are embodied in the concept of an $A(n)$-algebra. Other algebraic objects called <u>stacks</u> are defined which give rise to spectral sequences of topological significance. Applied to A_n-spaces, these spectral sequences yield homology operations which serve as criteria for A_n-maps, and are related to the homology of the projective spaces.

The two paths of study are re-united in application to $\Omega CP(3)$, the loop space on complex projective 3-space. We find that $\Omega CP(3)$ is homotopy abelian and that there exists a homotopy equivalence $f : S^1 \times \Omega S^7 \longrightarrow \Omega CP(3)$ which is an H-map but not an A_4-map.

Table of Contents

Page

INTRODUCTION

The present work involves a further study of objects introduced in [21], but except for results explicitly quoted can be read independently.

Familiarity with certain concepts is assumed in this introduction, but formal definitions of all but the most elementary are given in later sections.

John Milnor has given a construction [15] which associates with each topological group G a sequence of spaces GP(i) [our notation] which can be regarded as generalizations of the classical projective spaces:
$RP(i) = S^0P(i)$, $CP(i) = S^1P(i)$ and $QP(i) = S^3P(i)$. If one considers H-spaces instead of topological groups, one finds that additional homotopy structure is necessary to imitate this construction. Intuitively, an A_n-space X is an H-space with sufficient additional structure to permit the construction for $i \leq n$ of spaces XP(i) with properties similar to those of the spaces GP(i). An A_n-map can be thought of as a map from one A_n-space to another which up to homotopy respects the A_n-structure.

The principal goals of this study are 1) to investigate the multiplication on an A_n-space in terms of the geometry of the projective spaces and 2) to analyze the algebra of the homology of an A_n-space. In the first direction we observe that in the classical case there exist maps $RP(r) \times RP(s) \longrightarrow RP(r+s)$ and $CP(r) \times CP(s) \longrightarrow CP(r+s)$ which restricted to either factor are the usual imbeddings: $RP(r) \subset RP(r+s)$ and $CP(r) \subset CP(r+s)$. Working with A_n-spaces we find that the existence of such maps is implied by the multiplication being itself an A_{r+s}-map. In particular, an H-space (X,m) is homotopy abelian if and only if there is a map of $SX \times SX \longrightarrow XP(2)$ which is the inclusion on either factor.

The second direction is guided by William Massey's definition of cohomology products. Algebraic objects called <u>stacks</u> are defined which give rise to spectral sequences of topological significance. Applied to A_n-spaces, these spectral sequences yield homology operations which serve as criteria for A_n-maps.

The two paths of study are reunited in application to $\Omega\, CP(3)$, the loop space on complex projective 3-space. We find that $\Omega\, CP(3)$ is homotopy abelian and that there exists a homotopy equivalence $f : S^1 \times \Omega\, S^7 \longrightarrow \Omega\, CP(3)$ which is an H-map but not an A_4-map.

The work is divided into Chapters as follows:

Chapter I contains basic definitions and notation, including some remarks on H-spaces and on singular homology.

Chapter II contains the definition of A_n-spaces, an explicit construction of projective spaces, the definition of

A_n-maps and proofs of certain fundamental properties. The initial sections are intended primarily for reference.

Chapter III studies the existence of axial maps. Particular attention is given to axial maps of SX × SX into XP(2) and of SX × XP(2) into XP(3).

Chapter IV is algebraic in nature except for the applications. A brief discussion shows the relation of the work to that of Massey. The concept of an A(n)-algebra is introduced; it embodies the essential properties of the singular chain complex on an A_n-space. A detailed investigation is made of the homology stack and bar operations and of their relation to A_n-maps as well as to the homology of projective spaces.

Throughout the work numbers in square brackets refer to the bibliography. References to results within the same chapter are indicated by arabic numerals; references to results in other chapters are preceded by a roman numeral.

I am extremely grateful to Dr. Ioan James for his advice and criticism during the preparation of this thesis and to Dr. Michael Barratt for his supervision during my first year at Oxford. I am particularly indebted to Dr. James for calling to my attention the problems relating to axial maps, for suggesting Proposition III.1.11 and Lemma III.1.14, and for communicating his joint work with Emery Thomas concerning Lie groups.

CHAPTER I BASIC CONCEPTS AND NOTATION

1. Basic spaces and maps

All spaces will be assumed to have base points and all maps and homotopies will respect base points. The base point of a CW-complex will be assumed to be a 0-cell. All spaces will further be assumed to have the homotopy type of a countable CW-complex.

Let X,Y and Z be spaces with base points e_X, e_Y and e_Z. Let $f : Z \longrightarrow X$, $g : Z \longrightarrow Y$, and $j : Y \longrightarrow X$ be maps. The following spaces and maps will be of particular interest.

S^1, the topological group of complex numbers of unit norm, has base point 1.

ΩX, the space of loops on X, is the function space (with the compact-open topology) of all maps (respecting base points) $\lambda : S^1 \longrightarrow X$. The base point of ΩX is λ_e defined by $\lambda_e(S^1) = e_X$. We define loop addition in ΩX by

$$
\begin{aligned}
(\lambda + \mu)(s) &= \lambda(2s) && \text{for } 0 \leq 2s \leq 1 \\
&= \mu(2s-1) && \text{for } 0 \leq 2s - 1 \leq 1.
\end{aligned}
$$

For any map $f : X \longrightarrow Y$, we denote by $\Omega f : \Omega X \longrightarrow \Omega Y$ the map defined by $[\Omega f(\lambda)](s) = f(\lambda(s))$ for $\lambda \in \Omega X$, $s \in [0,1]$.

Note that Ωf is a homomorphism with respect to $+$:

$$
[(\Omega f)(\lambda+\mu)](s) = f((\lambda+\mu)(s)) = \begin{cases} f(\lambda(2s)) \\ f(\mu(2s-1)) \end{cases}
$$

$$
= [\Omega f(\lambda) + \Omega f(\mu)](s).
$$

We will use $\Omega_M X$ to denote the space of loops as defined by John C. Moore [18, p.30]. Each point of $\Omega_M X$ consists of a pair $\{\lambda, r\}$ where $\lambda : [0, r] \longrightarrow X$ is a map such that $\lambda(0) = \lambda(r) = e_X$. An associative addition $+$ on $\Omega_M X$ is defined by

$$\{\lambda, r\} + \{\mu, s\} = \{\lambda + \mu, r + s\}$$

where $(\lambda + \mu)(t) = \lambda(t)$ for $0 \leq t \leq r$

$\mu(t - r)$ for $r \leq t \leq r + s$.

$X \times Y$, the Cartesian product of X and Y, has base point (e_X, e_Y). The map $(f \times g) : Z \longrightarrow X \times Y$ is defined by $(f \times g)(z) = (f(z), g(z))$.

X^i is an abbreviation for $X \times X \times \cdots \times X$, the Cartesian product of i factors. The map $f^i : X^i \longrightarrow Z$ denotes $(f \times f \times \cdots \times f)$.

$X \vee Y$, the wedge of X and Y, is the union of X and Y with common base point, e_X being identified with e_Y. The map $f \vee j : Z \vee Y \longrightarrow X$ is defined by $(f \vee j)(z) = f(z)$, $(f \vee j)(y) = j(y)$.

$X^{[i]}$ is the subset of X^i consisting of points with at least one coordinate being e_X.

$X \# Y$, the "smash" product of X and Y, is the space obtained from $X \times Y$ by identifying $X \times e_Y$ and $e_X \times Y$ with (e_X, e_Y) which represents the base point of $X \# Y$.

$X^{(i)}$, the i-fold "smash" product of X, is $X \# \cdots \# X$.

$X * Y$, the (reduced) join of X with Y, is the space obtained from $X \times I \times Y$ by identifying a) $(x,0,y)$ with $x \in X$, b) $(x,1,y)$ with $y \in Y$, and c) (e_X,t,e_Y) with e_X and e_Y to form the base point of $X * Y$.

CX, the (reduced) cone on X, is the (reduced) join of a point with X.

$CX \cup_f Y$, where $f : X \longrightarrow Y$, is the mapping cone of f, the space obtained from CX and Y by identifying $(1,x)$ with $f(x)$. The base point is $(1,e_X) = (e_Y)$.

SX, the (reduced) suspension of X, is $CX \cup_\theta e_X$ where $\theta : X \longrightarrow e_X$. The map $Sf : SZ \longrightarrow SX$ is defined by $(Sf)(t,z) = (t,f(z))$.

<u>Remark</u>. If we restrict ourselves to the category of spaces having the homotopy type of countable CW-complexes, the above constructions stay within the category. This follows for all but ΩX from the usual theorems on products of CW-complexes and on spaces formed from CW-complexes by identifications. Milnor has shown that ΩX has the homotopy type of a countable CW-complex whenever X does [17].

Δ^n is the standard n-simplex, represented as the set of (n+1)-tuples $(t_0,\ldots,t_n)$ of non-negative real numbers such that $\sum_{i=0}^{n} t_i = 1$.

$\pi(X,Y)$ denotes the set of homotopy classes of maps of X into Y.

2. H-spaces

Definition 2.1. An H-space (X,m) consists of a space X with base point e_X and a map $m : X \times X \longrightarrow X$ such that $m(x,e_X) = x = m(e_X,x)$. We refer to m as a multiplication on X.

Definition 2.2. A map $f : X \to Y$ is an H-map of an H-space (X,m) into an H-space (Y,n) if the map $f \circ m$ is homotopic to the map $n \circ (f \times f)$.

We note that any map homotopic to an H-map is itself an H-map and that the composition of two H-maps is an H-map.

Definition 2.3. An H-space (X,m) is homotopy associative if the maps $m(1 \times m)$ and $m(m \times 1)$ taking $X \times X \times X$ into X are homotopic.

Definition 2.4. An H-space (X,m) is a monoid if m is associative, i.e. $m(m(x,y),z) = m(x,m(y,z))$.

Let (X,m) be an H-space.

The induced map $m_* : H_*(X \times X) \longrightarrow H_*(X)$ can be applied to $H_*(X) \otimes H_*(X)$ as imbedded in $H_*(X \times X)$. The resultant ring structure on $H_*(X)$ is called the Pontrjagin product. It is natural with respect to H-maps. It is associative if (X,m) is homotopy associative.

3. Singular homology

Singular homology is usually based on simplices or cubes. We refer to these spaces as the models for the singular theory. It is possible, however, to use other spaces as models, such as collections of acyclic regular cell complexes for which the appropriate notions of singular boundary and degeneracy can be defined. That such theories will produce the standard results follows from the theory of acyclic models [5]. For example, the collection of models could be arbitrary finite products of simplices.

If a space X is arcwise connected, we can generate the singular theory using only mappings of the models which carry the 0-skeleton to the base point (cf. [19 , p.444]).

Now let the collection of models $\{L_\alpha\}$ be closed under products, e.g. cubes. Given two singular models $u : L_\alpha \longrightarrow X$ and $v : L_\beta \to X$ we define a new singular model $u \times v : L_\alpha \times L_\beta \longrightarrow X \times X$ by $(u \times v)(s,t) = (u(s),v(t))$ for $s \in L_\alpha$, $t \in L_\beta$. This can be extended to define a map $C(X) \otimes C(X) \longrightarrow C(X \times X)$ which, as in the classical case, will be a chain equivalence.

CHAPTER II FUNDAMENTAL PROPERTIES OF A_n-SPACES

1. Definition of A_n-spaces

In order to define A_n-spaces we use special complexes K_i, which will satisfy conditions 1.1.1 - 1.1.3 below. The last of these is relatively unimportant for the present study, as is condition 1.2.c in the definition of an A_n-space. In [21], an explicit construction is given of the special complexes K_i ($i=2, 3, \ldots$); they have the following properties:

1.1.1. K_i is homeomorphic to the $(i-2)$-dimensional cube.

1.1.2. For $2 \leq r < i$ and $1 \leq k \leq r$, there are imbeddings (faces) $\varphi^k : K_r \times K_s \longrightarrow K_i$ ($s = i-r+1$) such that

a) $\varphi^k(K_r \times K_s) \subset \partial K_i$

b) $\varphi^k(\varphi^j(K_p \times K_q) \times K_s) =$

$$\varphi^j(\varphi^{k-q+1}(K_p \times K_s) \times K_q) \text{ for } j+q-1 < k$$

$$\varphi^j(K_p \times \varphi^{k-j+1}(K_q \times K_s)) \text{ for } j \leq k \leq j+q-1$$

$$\varphi^{j+s-1}(\varphi^k(K_p \times K_s) \times K_q) \text{ for } k < j.$$

1.1.3. There are maps $\Xi_j : K_i \to K_{i-1}$ (degeneracies) for $1 \leq j \leq i$ such that

a) the following maps are homeomorphisms:

i. $\Xi_1 \varphi^2(K_2 \times K_{i-1})$

$\Xi_i \varphi^1(K_2 \times K_{i-1})$

II. $\Xi_k\, \varphi^k(K_{i-1} \times K_2)$ $\qquad 1 \leq k < i$

$\Xi_k\, \varphi^{k-1}(K_{i-1} \times K_2)$ $\qquad 1 < k \leq i$

b) the following relations hold:

I. $\Xi_j\, \varphi^k(K_r \times K_s) = \varphi^k(K_r \times \Xi_{j-k+1}(K_s))$ for $k \leq j \leq k+s-1$

$= \varphi^k(\Xi_j(K_r) \times K_s)$ for $j < k$

$= \varphi^k(\Xi_{j-s+1}(K_r) \times K_s)$ for $k+s-1 < j$

II. $\Xi_j \Xi_k = \Xi_k \Xi_{j+1}$ for $j \geq k$.

<u>Definition</u> 1.2. An A_n-<u>space</u> $(X; \{h_i\})$ consists of a space X with base point e and maps $h_i : K_i \times X^i \longrightarrow X$ for $2 \leq i \leq n$ such that

a. $h_2(x,e) = h_2(e,x) = x$ for $x \in X$

b. $h_i(\varphi^k(\rho, \sigma), x_1, \ldots, x_i) =$

$h_r(\rho, x_1, \ldots, x_{k-1}, h_s(\sigma, x_k, \ldots, x_{k+s-1}), x_{k+s}, \ldots, x_i)$

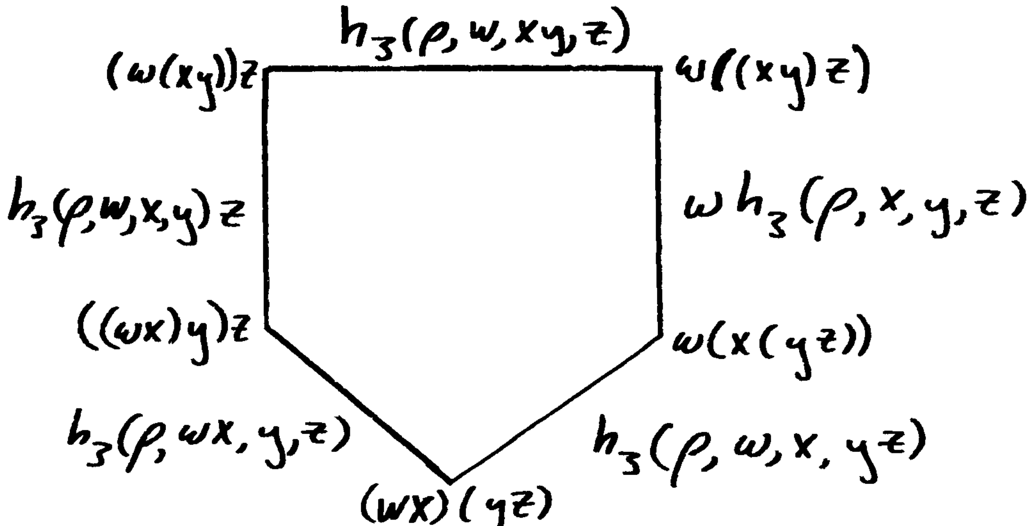
for $\rho \in K_r$, $\sigma \in K_s$, $r+s = i+1$.

c. $h_i(\rho, x_1, \ldots, x_{j-1}, e, x_{j+1}, \ldots, x_i) =$

$h_{i-1}(\Xi_j(\rho), x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_i)$ for $\rho \in K_i$.

For example, an A_2-space is just an H-space with multiplication h_2, while an A_3-space is a homotopy associative H-space since ∂K_3 consists of two points and the corresponding maps are $h_2(h_2 \times 1)$ and $h_2(1 \times h_2)$.

Now consider the five ways of associating four points. In an A_3-space each of these ways can be connected to two others by a single application of homotopy associativity, thus we have the following map of $S^1 \times X^4$ into X:



If we can extend to the (2-cell) $\times X^4$, we have an A_4-space.

<u>Definition</u> 1.3. An A_∞ -<u>space</u> $(X; \{h_i\})$ consists of a space X with base point e and maps $h_i : K_i \times X^i \longrightarrow X$ for $i = 2, \ldots$ such that $(X; \{h_i, 2 \le i \le n\})$ is an A_n-space for $n = 2, \ldots$.

A space of loops is an A_∞ -space.

Since we will often have occasion to use formulas like that of 1.2.b, we will abbreviate the right hand side as $h_r(\rho, x_1, \ldots, h_s(\sigma, x_k, \ldots, x_{k+s-1}), \ldots, x_i)$, it being understood that h_s is preceded by x_{k-1} and followed by x_{k+s}.

2. X-projective n-spaces

A major result of [21] is

Theorem 2.1. If $(X; \{h_i\})$ is an A_{n+1}-space, there exist spaces $XE(n)$ and $XP(n)$ and maps $f : X \to XE(n)$ and $p: XE(n) \to XP(n)$ such that

1) $XE(n)$ has the homotopy type of the $(n+1)$-fold join $X * \cdots * X$.

2) f is an imbedding and is contractible

3) pf is the constant map

4) $p_* : \pi_i(XE(n), X) \to \pi_i(XP(n), p(X))$ is an isomorphism for all i.

The X-projective n-space $XP(n)$ is defined if $(X; \{h_i\})$ is only an A_n-space; it can be described as follows:

Case 1. $(X; h_2)$ is a monoid (h_2 is associative). We write $h_2(x,y) = xy$.

$XP(1) = SX$, the base point is denoted by $*$.

$$XP(n) = \Delta^n \times X^n \cup_{\gamma(n)} XP(n-1)$$

where $\gamma(n) : \partial\Delta^n \times X^n \cup \Delta^n \times X^{[n]} \to XP(n-1)$ (see Section I.1 for the definition of $X^{[n]}$) is an attaching map defined as follows:

Let $\tau = (t_0,\ldots,t_n) \in \Delta^n$ and $\chi = (x_1,\ldots,x_n) \in X^n$.

Let τ_i denote any τ with $t_i = 0$.

Let $\hat{\tau}_i = (t_0,\ldots,t_{i-1},t_{i+1},\ldots,t_n)$.

Define $\gamma(n)(\tau_0,\chi) = (\hat{\tau}_0, x_2,\ldots,x_n)$

$$(\tau_i,\chi) = (\hat{\tau}_i,x_1,\ldots,x_i x_{i+1},\ldots,x_n) \quad 1 \le i < n$$

$$(\tau_n,\chi) = (\hat{\tau}_n,x_1,\ldots,x_{n-1}).$$

Further define

$$\gamma(n)(\tau,x_1,\ldots,x_{i-1},e,x_{i+1},\ldots,x_i) =$$

$$(t_0,\ldots,t_{i-1} + t_i,\ldots,t_n,x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n).$$

Since e is an exact unit and multiplication is associative, $\gamma(n)$ is well-defined,

Case 2. For an A_n-space, the construction of XP(i) for $i \le n$ will be similar to that in the associative case. We replace Δ^n by a suitable homeomorph ${}^n\Delta$.

<u>Definition</u> 2.2. An n-<u>string</u> is a subsequence $(i_1,\ldots,i_q)$ of $(0,\ldots,n)$. The <u>length</u> of the n-string is q.

The standard simplex Δ^n can be regarded as the cone on its boundary $\partial\Delta^n$ which is obtained by identification from $\bigcup_{n>q} \Delta_\alpha^{n-q}$ where α runs over all n-strings of length q. The homeomorph ${}^n\Delta$ will have a boundary $\partial\,{}^n\Delta$ which can be regarded as obtained by identification from a union of cells which are indexed in part by n-strings.

<u>Definition</u> 2.3. An n-string is <u>fraternal</u> if $i_{j+1} = i_j + 1$ for $1 \leq j < q$ or if $q = 1$.

It is sufficient, by induction, to describe the (n-1)-faces of ${}^n\Delta$ and how they fit together. The (n-1)-faces of ${}^n\Delta$ are indexed by fraternal strings $\alpha = (i, \ldots, i+q-1)$. The face indexed by α has the form ${}^{n-q}\Delta \times K_{q+1}$ and is regarded as a face of ${}^n\Delta$ by means of a map $\theta_\alpha : {}^{n-q}\Delta \times K_{q+1} \longrightarrow \partial\, {}^n\Delta$. For $0 < i < n-q+1$, θ_α is a homeomorphism on the interior of ${}^{n-q}\Delta \times K_{q+1}$. For $\alpha = (0, \ldots, q-1)$ or $(n-q+1, \ldots, n)$, θ_α can be factored through projection on to the first factor, the interior of which is mapped homeomorphically. The faces fit together as follows:

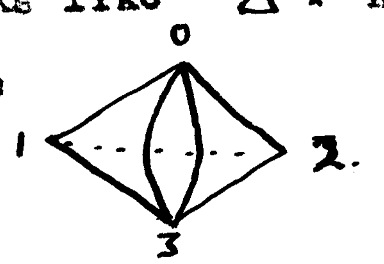
2.4. Let $\alpha = (i, \ldots, i+q-1)$, $\beta = (j, \ldots, j+u-1)$, and $\bar{\alpha} = (i-u, \ldots, i+q-u-1)$, $j \leq i$. We identify

$$\theta_\alpha(\theta_\beta({}^{n-q-u}\Delta \times K_{u+1}) \times K_{q+1}) \text{ with}$$
$$\theta_\beta(\theta_{\bar{\alpha}}({}^{n-q-u}\Delta \times K_{q+1}) \times K_{u+1}) \text{ in the}$$

obvious way.

2.5. We also identify $\theta_\alpha({}^{n-q}\Delta \times \varphi^k(K_r \times K_s))$ with $\theta_{\alpha''}(\theta_{\alpha'}({}^{n-q}\Delta \times K_r) \times K_s)$ where $\alpha' = (i, \ldots, i+k-2)$, $\alpha'' = (i+k-1, \ldots, i+k+s-1), \ldots,$

For example, ${}^2\Delta = \Delta^2$. As for ${}^3\Delta$ observe that $K_3 = I$ and that ${}^3\Delta$ has 4 2-simplices as faces and one 2-face which looks like ${}^1\Delta \times K_3$, i.e. like I^2. Thus ${}^3\Delta$ looks something like



We now define $XP(n)$ by induction.

$XP(1) = SX$

$XP(n) = {}^n\Delta \times X^n \smile_{\gamma(n)} XP(n-1)$ where

$\gamma(n) : \partial\, {}^n\Delta \times X^n \smile {}^n\Delta \times X^{[n]} \longrightarrow XP(n-1)$ is the attaching map defined as follows:

Let $\tau \in {}^{n-q}\Delta$, $\rho \in K_{q+1}$, and $\chi \in X^n$ with $\alpha = (i,\dots,i+q-1)$.

Define $\gamma(n)(\theta_\alpha(\tau,\rho),\chi)$ to be the point of $XP(n-q)$ represented by

(2.6) $(\tau, x_1,\dots,h_{q+1}(\rho, x_i,\dots,x_{i+q}),\dots,x_n)$

with the convention that the undefined expressions $h_{q+1}(\rho, x_0,\dots,x_q)$ and $h_{q+1}(\rho, x_{n-q+1},\dots,x_{n+1})$ are to be omitted, e.g. for $\alpha = (0,\dots,q-1)$, then $\gamma(n)(\theta_\alpha(\tau,\rho),\chi) = (\tau, x_{q+1},\dots,x_n)$.

In order to define $\gamma(n)$ in the associative case we used implicitly maps $q_i : \Delta^n \longrightarrow \Delta^{n-1}$ defined by

$q_i(t_0,\dots,t_n) = (t_0,\dots,t_{i-1}+t_i,\dots,t_n)$, $1 \le i \le n$.

We wish to define corresponding maps $r_i : {}^n\Delta \to {}^{n-1}\Delta$ satisfying the following relations in terms of the maps

θ_α and $\equiv_j$ (cf. 1.1.3):

for $\alpha = (k, \ldots, k+q-1)$,

(2.7)

a. $r_i r_j = r_{j-1} r_i$ for $i < j$

b. $r_i\, \theta_\alpha = \theta_{\alpha'}(r_i \times 1)$, $\alpha' = (k-1, \ldots, k+q-2)$ for $i < k$

c. $r_i\, \theta_\alpha = \theta_{\alpha'}(1 \times \equiv_{i-k+1})$, $\alpha' = (k, \ldots, k+q-2)$ for $k \leq i \leq k+q$

d. $r_i\, \theta_\alpha = \theta_\alpha (r_{i-q} \times 1)$ for $i > k+q$.

We note that equations (2.7) b-d are compatible with the identifications in ${}^n\Delta$ and hence by induction give a well-defined map of $\partial\, {}^n\Delta$ into ${}^{n-1}\Delta$. We extend to ${}^n\Delta$ by "taking the cone", i.e. regard ${}^j\Delta$ as the cone on $\partial\, {}^j\Delta$:

If $x \in \partial\, {}^n\Delta$ and $r_i(x) = (s, y)$ for $y \in \partial\, {}^{n-1}\Delta$, $s \in [0,1]$, then define

(2.7)e. $\quad r_i(t, x) = (ts, y)$.

By induction, one can verify (2,7)a on $\partial\, {}^n\Delta$. From (2.7)e it follows that (2.7)a holds on all of ${}^n\Delta$.

Let $\tau \in {}^{n}\Delta$. Define $\gamma(n)(\tau, x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_n)$ to be the point of $XP(n-1)$ represented by

$(r_i(\tau), x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

<u>Lemma</u> 2.8. $\gamma(n)$ is well-defined.

Proof. We must verify that if $\theta_\alpha(\tau, \rho)$ belongs to two distinct $(n-1)$-faces of ${}^{n}\Delta$, then using both definitions of $\gamma(n)$ we obtain the same point of $XP(n-1)$.

Case 1. Let $\alpha, \beta, \bar{\alpha}$ be as in 2.4.

Let $\tau \in {}^{n-q-u}\Delta$ and $\rho' \in K_{u+1}$, $\rho'' \in K_{q+1}$ with $\chi \in X^n$.

By (2.6), we see that

$\gamma(n)(\theta_\alpha(\theta_\beta(\tau, \rho'), \rho''), \chi)$ is represented by

(2.9) $(\theta_\beta(\tau, \rho'), x_1, \ldots, h_{q+1}(\rho'', x_i, \ldots, x_{i+q}), \ldots, x_n)$

while $\gamma(n)(\theta_\beta(\theta_{\bar{\alpha}}(\tau, \rho''), \rho'), \chi)$ is represented by

(2.10) $(\theta_{\bar{\alpha}}(\tau, \rho''), x_1, \ldots, h_{u+1}(\rho', x_j, \ldots, x_{j+u}), \ldots, x_n)$.

Applying (2.6) once again, we see that both (2.9) and (2.10) are identified with the point of $XP(n-q-u)$ represented by

(2.11) $(\tau, x_1, \ldots, h_{u+1}(\rho', x_j, \ldots, x_{j+u}), \ldots$
$\ldots, h_{q+1}(\rho'', x_i, \ldots, x_{i+q}), \ldots, x_n)$.

Case 2. Let $\alpha, \alpha', \alpha''$ be as in (2.5).

Let $\rho = \varphi^k(\rho', \rho'')$. While $\gamma(n)(\theta_\alpha(\tau, \rho), \chi)$ is represented by (2.6), we see that

$\gamma(n)\,(\theta_{\alpha''}(\theta_{\alpha'}(\tau,\rho'),\rho''),\chi)$ is represented by

(2.12) $(\theta_{\alpha'}(\tau,\rho'),x_1,\dots,h_s(\rho'',x_{i+k-1},\dots,x_{i+k+s-2}),\dots,x_n)$.

Using (2.6) again and condition 1.2.b we see that both points under consideration are identified with the point of XP(n-q) represented by

(2.13) $(\tau,x_1,\dots,h_r(\rho',x_i,\dots,h_s(\rho'',x_{i+k-1},\dots$
$\dots,x_{i+k+s-2}),\dots,x_{i+q}),\dots,x_n)$.

We must also verify that $\gamma(n)$ is uniquely defined on ${}^n\Delta \times X^{[n]}$, i.e. on points of the form $(\tau,x_1,\dots,e,\dots,e,\dots,x_n)$ with e in the i-th and j-th places; this follows from the first equation of (2.7). From the other equations and condition 1.2.c of the definition of an A_n-space, we see that $\gamma(n)$ is uniquely defined on $\partial\,{}^n\Delta \times X^n \cap {}^n\Delta \times X^{[n]}$.

This completes the proof of Lemma 2.8 and the construction of the spaces XP(n). Note that if Y is a sub-A_n-space of X, YP(n) is imbedded in XP(n).

3. A_n-maps

In the familiar case of topological groups, homomorphisms induce maps of classifying spaces, and in fact maps of (Milnor) projective n-spaces. Similarly if we make the following

Definition 3.1. A map $f : Y \to X$ is a homomorphism of an A_n-space $(Y; \{h_i\})$ into an A_n-space $(X; \{g_i\})$ if $f \circ h_i = g_i \circ (1 \times f \times \cdots \times f)$ (where $1: K_i \to K_i$ is the identity) for $2 \leq i \leq n$,

we see that an A_n-homomorphism induces maps $fP(i) : YP(i) \longrightarrow XP(i)$ for $i \leq n$. (Any map $f : Y \to X$ induces $1 \times f^i : {}^i\Delta \times Y^i \to {}^i\Delta \times X^i$. The homomorphic property is used to show $fP(i-1) \circ \gamma(i) = \gamma(i)(1 \times f^i) \mid \partial\, {}^i\Delta \times Y^i \cup {}^i\Delta \times Y^{[i]}$.)

Homomorphy is not properly a concept of homotopy theory. It may be weakened to the concept of an A_n-map, which is a generalization of the concept of an H-map.

We recall from [21] the definition of an A_n-map in terms of certain maps defined explicitly there. By an ordered partition ρ of an integer i we mean a sequence of integers $\rho = (p_1, \ldots, p_r)$ such that $\sum_j p_j = i$. Let $I_i = [1, i]$.

For each partition $\mathcal{P} = (p_0, \ldots, p_m)$ of i ..., let

$$\psi^{\mathcal{P}} : I_{p_1} \times \cdots \times I_{p_r} \times K_r \times K_{p_1} \times \cdots \times K_{p_r} \longrightarrow I_i \times K_i$$

denote the homeomorphism constructed in [21].

<u>Definition</u> 3.2. A map $f : Y \to X$ is an A_n-<u>map</u> of an A_n-space $(Y; \{h_i\})$ into an A_n-space $(X; \{g_i\})$ if there exist homotopies $G_i : I_i \times K_i \times Y^i \to X$ for $1 \leq i \leq n$ such that

a) $G_1 = f$,

b) $G_i \,|\, (1) \times K_i \times Y^i = f \circ h_i$

$G_i \,|\, (i) \times K_i \times Y^i = g_i \circ (1 \times f \times \cdots \times f)$

where $1 : K_i \to K_i$ is the identity,

c) Let $\rho \in K_r$, $\sigma \in K_s$, $\chi \in Y^i$ with $r + s = i + 1$, then $G_i(t, \varphi^k(\rho, \sigma), \chi) =$

(3.3) $G_r(t, \rho, y_1, \ldots, h_s(\sigma, y_k, \ldots, y_{k+s-1}), \ldots, y_i)$

for $t \in I_r$,

d) Let $\mathcal{P} = (p_1, \ldots, p_r)$ be a partition of i. .

Let $\rho \in K_r$, $t_j \in I_{p_j}$, $\sigma_j \in K_{p_j}$ for $j = 1, \ldots, r$.

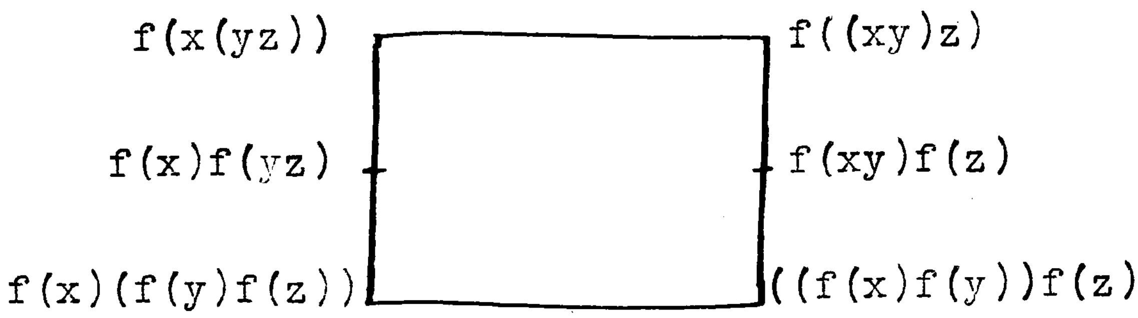
Let $t = (t_1, \ldots, t_r)$ and $\sigma = (\sigma_1, \ldots, \sigma_r)$, $\chi \in Y^i$, then $G_i(\psi^{\mathcal{P}}(t, \rho, \sigma), \chi) =$

(3.4) $g_r(\rho, G_{p_1}(t_1, \sigma_1, x_1, \ldots, x_{p_1}), \ldots$

$\ldots, G_{p_r}(t_r, \sigma_r, x_{i-p_r+1}, \ldots, x_i))$.

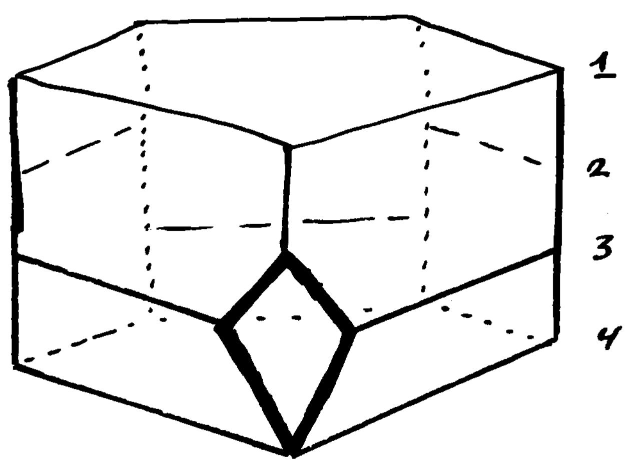
Remark 3.5. The maps $\psi^{\mathcal{P}}$ were constructed so that conditions a) - d) determine a well-defined map of $\partial(I_i \times K_i) \times Y^i$ into X.

Example 3.6. We can indicate the values of $G_3(I_3 \times K_3 \times (x,y,z))$ as follows



The sides of the square are defined by $f \circ h_3$, $G_2(t,xy,z)$, $G_2(t-1,x,y)f(z)$, $g_3(1 \times f \times f \times f)$, $f(x)G_2(t-1,yz)$, and $G_2(t,x,yz)$ in their respective positions.

Example 3.7. The following figure shows the decomposition of $I_4 \times K_4$; the heavily scored face is the one on which G_4 is defined in terms of $G_2(t,w,x)G_2(s,y,z)$.



The relation between A_n-maps and projective spaces is established in [21]. We have

Proposition 3.8. <u>If</u> $f : (Y; \{h_i\}) \longrightarrow (X; \{g_i\})$ <u>is an A_n-map, there are induced maps</u> $fP(i) : YP(i) \to XP(i)$ <u>for</u> $i \leq n$ <u>such that</u> $fP(i) = fP(j) \mid YP(i)$ <u>for</u> $i \leq j$ <u>and</u> $fP(1) = Sf : SY \to SX$.

We can regard the construction of the spaces $XP(n)$ and the maps $fP(n)$ as a functor from the category of A_n-spaces and A_n-maps to the category of spaces and maps. In the converse direction, we establish

Theorem 3.9. *If* $(Y; \{h_i\})$ *is an* A_n*-space and* (X, g_2) *is a monoid, then* $f: Y \to X$ *is an* A_n*-map if* $Sf: SY \to SX$ *extends to* $fP(n): YP(n) \to XP(n)$.

Proof. Suppose that $Sf: SY \to SX$ does extend to some map $fP(n): YP(n) \to XP(n)$. Consider the loop spaces $\Omega YP(n)$ and $\Omega XP(n)$ and the induced map $\Omega fP(n): \Omega YP(n) \to \Omega XP(n)$. Let $w_n: Y \to \Omega YP(n)$ be given by $w_n(y)(t) = (t, y) \in SY \subset YP(n)$. Consider the following diagram.

$$\begin{tikzcd}
Y \arrow[r, "w_n"] \arrow[d, "f"'] & YP(n) \arrow[d, "\Omega fP(n)"] & \\
X \arrow[r, "w_n"] & XP(n) \arrow[r] & XP(\infty).
\end{tikzcd}$$

The map $\Omega fP(n)$ is a homomorphism of the loop structure and hence an A_n-map for all n. Note that $w_n(Y)$ lies in ΩSY. Since $fP(n)$ is an extension of Sf, we have $\Omega fP(n)\, w_n = w_n f$; the diagram is commutative.

Theorem 2.1 enables one to show that the imbedding $X \to \Omega XP(\infty)$ is a homotopy equivalence and A_∞-map, [21, cf. 22, p.127]. Hence there is an A_∞-map $\Omega XP(\infty) \to X$ such that the composition

$$Y \to \Omega\, YP(n) \to \Omega\, XP(n) \to \Omega\, XP(\infty) \to X$$

is homotopic to f. That f is an A_n-map follows from the general

Proposition 3.10. <u>The map</u> $w_n : X \to \Omega XP(n)$ <u>is an</u> A_n-<u>map</u>.

Proof. We consider first the case n = 2 to illustrate the technique. The inclusion w_2 is an H-map if $\eta(2) : \partial I \times X^2 \to \Omega\, XP(2)$ can be extended to $I \times X^2$, $\eta(2)$ being given by

$$\eta(2)\,(0,x,y)(u) = (2u,x) \quad \text{for } 0 \le u \le \tfrac{1}{2}$$
$$= (2u-1,y) \quad \text{for } \tfrac{1}{2} \le u \le 1$$
$$\eta(2)\,(1,x,y)(u) = (u,xy) \quad \text{for } 0 \le u \le 1.$$

Equivalently we can consider the adjoint map $SX^2 \to XP(2)$, represented by $\zeta(2) : \partial(I \times I) \times X^2 \to XP(2)$ given by

$$\zeta(2)\,(u,\varepsilon,x,y) = \eta(2)(\varepsilon,x,y)(u) \quad \text{for } \varepsilon = 0 \text{ or } 1$$
$$\zeta(2)\,(\varepsilon,t,x,y) = * \quad \text{for } \varepsilon = 0 \text{ or } 1.$$

Since $\zeta(2)$ is constant for $u = 0$ or 1, we can define a map $\vartheta(2) : I \times I \longrightarrow \Delta^2$ so that

$$\gamma(2) = \zeta(2)(\vartheta(2)^{-1} \times 1 \times 1) \Big| \partial\Delta^2 \times X^2.$$

Since $\gamma(2)$ extends to $\Delta^2 \times X^2$, $\zeta(2)$ extends to $I^2 \times X^2$ and hence $\eta(2)$ extends to $I \times X^2$ as desired.

<u>The case</u> $n > 2$, h_2 <u>associative</u>. In this case we are looking for maps

$G_i : I_i \times K_i \times X^i \longrightarrow \Omega XP(n)$ for $2 \leq i \leq n$ such that

1) $G_i(1, \rho, \chi) = w_n(x_1 \cdots x_i)$

$G_i(i, \rho, \chi) = w_n(x_1) \cdots w_n(x_i)$

2) $G_i(t, \varphi^k(\rho, \sigma), \chi) =$

$= G_r(t, \rho, x_1, \ldots, x_k \cdots x_{k+s-1}, \ldots, x_i)$ for $t \in I_r$

3) $G_i(\psi^{\mu}(t, \rho, \sigma), \chi) =$

$= G_{\rho_1}(t_1, \sigma_1, x_1, \ldots, x_{\rho_1}) \cdots G_{\rho_r}(t_r, \sigma_r, x_{i-\rho_r+1}, \ldots, x_i)$

for $\mu, t, \rho, \sigma, \chi$ as in 3.2.d.

We regard points of $I \times I_i \times K_i$ as given by coordinates (u, t, ρ), $u \in I$, $t \in I_i$, $\rho \in K_i$. The adjoint maps to the maps G_i will be represented by maps

$\zeta(i) : I \times I_i \times K_i \times X^i \longrightarrow XP(n)$ such that

3.11. 1) $\zeta(i)(\varepsilon, t, \rho, \chi) = *$ $\qquad \varepsilon = 0$ or 1

2) $\zeta(i)(u, 1, \rho, \chi) = (u, x_1 \cdots x_i)$

$\zeta(i)(u, i, \rho, \chi) = (iu-j, x_{i+1})$ $\quad 0 \leq iu \dot{-} j \leq 1$

$0 \leq j < i.$

3) For $\rho \in K_r$, $\sigma \in K_s$ we have

$$\zeta(i)\,(u,t,\ q^{k}(\rho,\sigma),\ \chi) =$$
$$\zeta(r)\,(u,t,\rho,x_1,\dots,x_k\cdots x_{k+s-1},\dots,x_n)$$

4) For μ, t, ρ, σ and χ as in 3.2.d, we have

$$\zeta(i)\,(u,\ \psi^{\mu}(t,\rho,\sigma),\ \chi) =$$
$$= \zeta(\rho_j)\,(iu-k,\ t_j,\ \sigma_j,\ x_{k+1},\dots,x_{k+\rho_j})$$

where $k=\sum_{m=1}^{j-1}\rho_m$, $1\le j \le r$.

Let $\overline{\gamma(i)}$ denote the inclusion of $\Delta^i \times X^i$ into $XP(i)$. We wish to define a sequence of maps

$\vartheta^{G}(i) : I \times I_i \times K_i \longrightarrow \Delta^i$ for $i = 2,\dots,n$ such that if we set $\zeta(i) = \overline{\gamma(i)}\,(\vartheta(i) \times 1)$ (where $1 : X^i \longrightarrow X^i$ is the identity) then $\zeta(i)$ will satisfy conditions (3.11.1-3).

We specify $\vartheta(i)$ by giving, for each point $y = (u,t,\rho)$ of the boundary, functions $t_0(y),\dots,t_i(y)$ such that $\sum_j t_j(y) = 1$. Instead of using the notation $t_0(y) = \ ,\dots, t_i(y) = \ ,$ we find it more economical to use the notation $y \to t_0 = \ ,\dots, t_i = \ .$ Thus $\vartheta(i)$ is given by

$(0,t,\rho) \to t_0 = 1,\ t_m = 0$ for $m > 0$

$(1,t,\rho) \to t_i = 1,\ t_m = 0$ for $m < i$

$(u,1,\rho) \to t_0 = 1-u,\ t_i = u,\ t_m = 0$ for $0 < m < i$

for $0 \le iu - j \le 1$:

$(u,i,\rho) \to t_j = 1-iu+j,\ t_{j+1} = iu-j,\ t_m = 0$ for $m \ne j, j+1$.

for $t \in I_r$:

$$(u,t,\varphi^k(\rho,\sigma)) \longrightarrow t_m = t_m(u,t,\rho) \qquad 0 \le m < k$$

$$t_m = t_{m-s+1}(u,t,\rho) \qquad k+s-1 \le m \le i$$

$$t_m = 0 \qquad k \le m < k+s-1$$

for μ, t, ρ, σ as in 3.11.4:

Let $k = \sum_{m=0}^{j-1} p_m$. For $0 \le iu-k \le p_j$

$$(u, \psi^{\mu}(t,\rho,\sigma)) \longrightarrow t_m = t_{m-k}(iu-k, t_j, \sigma_j) \qquad k \le m < k+p_j$$

$$t_m = 0 \qquad 0 \le m < k \quad \text{or} \quad k+p_j \le m \le i.$$

It is somewhat tedious but straightforward to verify that the formulas above yield a well-defined map of $\partial(I \times I_i \times K_i)$ into Δ^i. We obtain $\vartheta(i)$ by extending radially to the interior.

Lemma 3.12. The map $\zeta(i)$ defined by

$$\zeta(i) = \overline{\gamma(i)}\,(\vartheta(i) \times 1)$$

satisfies conditions 3.11.1) - .3).

Proof. 1) If $u = 0$ or 1, then $t_0 = 1$ and $\gamma(i)\,(1,0,\ldots,0,\chi) = *$.

2) $t = 1$: $\gamma(i)\,(i-u,0,\dots,0,u,\chi)$ is identified with $(u,x_1\cdots x_i)\in SX$.

$t = 1$: For $0 \le iu-j \le 1$, $\gamma(i)\,(0,\dots,1-iu+j,iu-j,0,\dots,0,\chi)$ is identified with $(iu-j,x_{j+1})\in SX$.

3) $t\in I_r$: Since $t_m = 0$ for $k \le m < k+s-1$, $\gamma(i)\,(t_0,\dots,t_i,\chi)$ is identified with
$(t_0,\dots,t_{k-1},t_{k+s-1},\dots,t_i,x_1,\dots,x_k\cdots x_{k+s-1},\dots,x_i)$
which for the specified values of t_m is in fact
$\zeta(r)\,(u,t,\rho,x_1,\dots,x_k\cdots x_{k+s-1},\dots,x_i)$.

4) μ,t,ρ,σ as in 3.11.4: Let $k = \sum_{m=0}^{j-1} p_m$. For $0 \le iu-k \le p_j$, we have $t_m = 0$ for $0 \le m < k$ or $k+p_j \le m \le i$, hence $\gamma(i)\,(\tau,\chi)$ is identified with $(t_k,\dots,t_{k+p_j},x_{k+1},\dots,x_{k+p_j})$. For the specified values of t_m this is the same as
$\zeta(p_j)\,(iu-k,t_j,\sigma_j,x_{k+1},\dots,x_{k+p_j})$.

This completes the proof of Lemma 3.12. Since we have been able to define maps $\zeta(i)$ satisfying 3.11.1) -3) for $2 \le i \le n$, we can define the adjoint maps G_i and hence w_n is an A_n-map.

Case 2. Non-associative A_n-spaces. The corresponding map $\vartheta(i): I\times I_i \times K_i \to {}^i\Delta$ involves somewhat less collapsing. For example when $t\in I_r$, $(u,t,\partial^k(\rho,\sigma))$ is mapped to the appropriate point of $\theta_k({}^{i-s+1}\Delta \times K_s)$, $\alpha = (k,\dots,k+s-2)$. Otherwise the same procedure can be applied as in the associative case.

This completes the proof of Lemma 3.10. Theorem 3.9 now follws since the composition of A_n-maps is an A_n-map [21].

CHAPTER III AXIAL MAPS OF PROJECTIVE SPACES

1. Statement of the problem and main results

Let X,Y and Z be spaces with base points e_X, e_Y and e_Z.

Definition 1.1. The axes of a map $f: X \times Y \longrightarrow Z$ are the maps $g : X \longrightarrow Z$ and $h : Y \longrightarrow Z$ defined by $g(x) = f(x,e_Y)$ and $h(y) = f(e_X,y)$.

Definition 1.2. A map $f : X \times Y \longrightarrow Z$ is axial with respect to maps $g : X \longrightarrow Z$ and $h : Y \longrightarrow Z$ if the axes of f are g and h.

Convention 1.3. Let X and Y be subspaces of Z. A map $f : X \times Y \longrightarrow Z$ is an axial map if the axes are the inclusions $X \subset Z$ and $Y \subset Z$.

For example, the multiplication m of an H-space (X,m) is an axial map.

Definition 1.4. An H-space (X,m) is homotopy abelian if m is homotopic to mT where $T : X \times X \longrightarrow X \times X$ is given by $T(x,y) = (y,x)$.

In connection with their work on homotopy abelian Lie groups [11], Ioan James and Emery Thomas have proved the following theorem (unpublished):

Theorem 1.5 Let G be a topological group.[1] By means of the Milnor construction [15], identify SG with a subspace of B_G, the classifying space for G. The group G is homotopy

---

1. As usual, we restrict ourselves to topological groups which have the homotopy type of countable CW-complexes.

abelian if and only if there exists an axial map $f : SG \times SG \longrightarrow B_G$.

We will present the following generalization:

Theorem 1.6. Let (X,m) be an H-space. If (X,m) is homotopy abelian, there exists an axial map $f : SX \times SX \longrightarrow XP(2)$. The converse holds provided (X,m) is homotopy associative.

Define $p : S\Omega X \longrightarrow X$ by $p(t,\lambda) = \lambda(t)$ for $\lambda \in \Omega X$, i.e. $\lambda : [0,1] \longrightarrow X$, and $t \in [0,1]$. The method of proof of Theorem 1.6 will yield

Theorem 1.7. If there exists a map $f : S\Omega X \times S\Omega X \longrightarrow X$ which is axial with respect to p, then ΩX (with loop addition) is homotopy abelian.

Section 2 is devoted to proving Theorems 1.6 and 1.7, but first the remainder of this section will illustrate the usefulness of these results. In order to apply Theorem 1.7, we give

Definition 1.8. A bunch of spheres is a space which is homeomorphic to a union of spheres with a single point in common.

Theorem 1.9. If X has trivial Whitehead products and $S\Omega X$ has the homotopy type of a bunch of spheres, then ΩX is homotopy abelian.

Proof. Let Y denote the appropriate union of spheres and $g : Y \rightarrow S\Omega X$ the homotopy equivalence. The map $p : S\Omega X \rightarrow X$ combines with g to yield a map $h : Y \vee Y \longrightarrow X$.

Since all Whitehead products vanish in X, we can extend $h \mid S^p \vee S^q$ to $S^p \times S^q$ for each pair of spheres S^p, S^q contained in Y. Hence we can extend h to $Y \times Y$. Using a homotopy inverse to g, we obtain the desired map $S\Omega X \times S\Omega X \to X$.

<u>Definition</u> 1.10. Let (Z,n) be an H-space. For any two maps $f : X \to Z$ and $g : Y \to Z$ the <u>sum</u> $(f+g) : X \times Y \to Z$ is defined by $(f+g)(x,y) = f(x)g(y)$.

In order to apply Theorem 1.9, we establish

Proposition 1.11. <u>If</u> $\pi : E \to B$ <u>is a weak fibring</u> (i.e. $\pi_* : \pi_i(E,F) \to \pi_i(B)$ is an isomorphism for all i) <u>with</u> F <u>contractible in</u> E, <u>then there exists</u> $q : F \to \Omega B$ <u>such that</u> $(q + \Omega\pi) : F \times \Omega E \to \Omega B$ <u>is a homotopy equivalence</u>.

Proof. Since F is contractible in E, we can define a map $q : F \to \Omega B$ so as to obtain the following non-commutative diagram with $\partial\pi_*^{-1}\partial^{-1}q_* = 1$:

$$\begin{array}{ccccccc}
0 \longrightarrow \pi_i(E) \longrightarrow & \pi_i(E,F) & \xrightarrow{\partial} & \pi_{i-1}(F) & \longrightarrow 0 \\
& \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle q_*} & \\
& \pi_i(B) & \xrightarrow{\partial} & \pi_{i-1}(\Omega B) &
\end{array}$$

the horizontal arrows ∂ being the usual boundary operators [23, Prop.1, p.5]. Thus we can write

$$\pi_{i-1}(\Omega B) \approx \pi_{i-1}(F) + \pi_{i-1}(\Omega E).$$

The inclusion $\pi_{i-1}(\Omega E) \subset \pi_{i-1}(\Omega B)$ is just $(\Omega \pi)_*$. Since the usual addition in $\pi_{i-1}(\Omega B)$ coincides with that induced by the multiplication on ΩB [8, p.371], we see that $(q + \Omega \pi)_* : \pi_{i-1}(F \times \Omega E) \longrightarrow \pi_{i-1}(\Omega B)$ is an isomorphism. In the category we are considering, we can conclude that $(q + \Omega \pi)$ is a homotopy equivalence.

Note that F admits an H-structure $m : F \times F \longrightarrow F$ by Theorem 1 of [22, p.109] or because it is dominated by the H-space ΩB [12, (6.1)]. In general the injection $q : F \longrightarrow \Omega B$ need not be an H-map; e.g. $q : S^1 \longrightarrow \Omega S^2$, corresponding to the Hopf fibring of S^3 onto S^2, is not an H-map since, up to homotopy, S^1 admits only one H-structure, which is abelian, yet the Samelson product $\langle q, q \rangle$ in ΩS^2 is non-trivial since it corresponds to the Whitehead product $[\iota_2, \iota_2]$ in S^2.

Again even if $q : F \longrightarrow \Omega B$ is an H-map, the homotopy equivalence between ΩB and $F \times \Omega E$ need not be an H-map. We, in fact, establish

Theorem 1.12. _Let_ (X,m) _and_ (Y,m') _be_ H-_spaces. Let_ (Z,n) _be a homotopy associative_ H-_space. If_ $f : X \longrightarrow Z$ _and_ $g : Y \longrightarrow Z$ _are H-maps, then_ $(f + g) : X \times Y \longrightarrow Z$ _is an_ H-_map if and only if it is homotopic to_ $(g + f)T : X \times Y \longrightarrow Z$.

Proof. Let $F_2 : I \times X \times X \longrightarrow Z$ and $G_2 : I \times Y \times Y \longrightarrow Z$ be homotopies such that

$$F_2(0,x,x') = f(xx') \quad G_2(0,y,y') = g(yy')$$
$$F_2(1,x,x') = f(x)f(x') \quad G_2(1,y,y') = g(y)g(y').$$

Suppose there is a homotopy $H : I \times X \times Y \longrightarrow Z$ such that $H(0,x,y) = f(x)g(y)$ and $H(1,x,y) = g(y)f(x)$. Then $(f+g)$ can be shown to be an H-map by the following string of homotopies

$$(f+g)((x,y)\cdot(x',y')) = (f+g)(xx',yy')$$
$$= f(xx')g(yy') \overset{F_2 \cdot G_2}{\simeq} f(x)f(x')g(y)g(y')$$
$$\overset{H}{\simeq} f(x)g(y)f(x')g(y') = (f+g)(x,y)\cdot(f+g)(x',y').$$

We have omitted the parentheses for associating; strictly speaking we must use an associating homotopy several times to pass from $(f(x)f(x'))(g(y)g(y'))$ to $f(x)((f(x')g(y))g(y'))$, etc.

Conversely, if $(f+g)$ is an H-map, the map $(f(x)f(x'))(g(y)g(y'))$ is homotopic to $(f(x)g(y))(f(x')g(y'))$. By taking $x = e_X$, $y' = e_Y$, we see that $(f+g)$ is homotopic to $(g+f)T$.

We are now ready to apply Theorem 1.9.

Let $CP(3)$ denote complex projective 3-space. By Proposition 1.11 there is a homotopy equivalence

$$(q + \Omega\pi) : S^1 \times \Omega S^7 \longrightarrow \Omega CP(3).$$

<u>Lemma</u> 1.13. The map $q : S^1 \longrightarrow \Omega CP(3)$ is an H-map.

Proof. The obstructions to q being an H-map are classes in $H^i(S^1 \# S^1; \pi_i(\Omega CP(3)))$. However $H^i(S^1 \# S^1) = 0$ for any coefficient group unless $i = 2$, while $\pi_2(\Omega CP(3)) \approx \pi_3(CP(3)) \approx \pi_3(S^7) \approx 0$ so that all the obstructions vanish.

<u>Lemma</u> 1.14. All Whitehead products vanish in CP(3).

Proof. We can write $\pi_i(\Omega CP(3))$ as $\pi_i(S^1) + \pi_i(\Omega S^7)$. The injections of $\pi_i(S^1)$ and $\pi_i(\Omega S^7)$ are induced by H-maps, hence Samelson products are natural with respect to the injections. Since Samelson products vanish in S^1 and ΩS^7, the only possible non-trivial Whitehead products are of the form $[\alpha, \beta]$ or $[\beta, \alpha]$ with $\alpha \in \pi_1(S^1)$, $\beta \in \pi_i(S^7)$, a generator of $\pi_1(S^1)$ being identified with a generator $\iota_2 \in \pi_2(CP(3))$.

Now consider the fibring $p: CP(3) \longrightarrow S^4$ induced by regarding the complex plane as a quaternionic line [9]. The fibre is S^2. From the exact sequence

$$\cdots \longrightarrow \pi_i(S^2) \xrightarrow{i_*} \pi_i(CP(3)) \xrightarrow{p_*} \pi_i(S^4) \xrightarrow{\partial} \pi_{i-1}(S^2) \longrightarrow \cdots$$

and the naturality of the Whitehead product, we see that

$p_*[\iota_2, \beta] = 0$ hence $[\iota_2, \beta] = i_*\gamma$ for some class $\gamma \in \pi_{i+1}(S^2)$. However, $i_*(\pi_{i+1}(S^2)) = 0$ for $i > 1$ since the 4-cell of CP(3) is attached to S^2 by a generator of $\pi_3(S^2)$. Hence all Whitehead products are zero in CP(3). (All Whitehead products of the form $[\iota_2, \beta]$ for $\beta \in \pi_i(S^{4r+3})$ vanish in CP(2r+1) by the above argument with CP(2r+1) replacing CP(3), QP(r) replacing S^4.)

Applying Lemma 1.13 and Theorem 1.9 we obtain

Theorem 1.15. $\Omega\, CP(3)$ <u>is homotopy abelian</u>.

Proof. We need only observe that $S(\Omega\, CP(3))$ has the homotopy type of a bunch of spheres since

1) $S(S^1 \times \Omega\, S^{2r+1})$ has the homotopy type of $S^2 \vee S\,\Omega\, S^{2r+1} \vee S^2\,\Omega\, S^{2r+1}$, and

2) $S\,\Omega\, S^{2r+1}$ has the homotopy type of a bunch of spheres.

Statement 1) is a consequence of the general result that $S(X \times Y)$ has the homotopy type of $SX \vee SY \vee S(X \# Y)$ [7, p.111] and the identification of $S^1 \# Y$ with SY. Statement 2) follows from the results of James on reduced products [10, 16].

Finally we obtain

Corollary 1.16. The homotopy equivalence $(q + \Omega\pi)$: $S^1 \times \Omega\, S^7 \longrightarrow \Omega\, CP(3)$ is an H-map.

Proof. Since $\Omega\, CP(3)$ is homotopy abelian, $(f+g)$ is homotopic to $(g+f)T$ for any two maps $f : X \longrightarrow \Omega\, CP(3)$, $g : Y \longrightarrow \Omega\, CP(3)$. Hence $(q + \Omega\pi)$ is an H-map by Theorem 1.12. In IV.8 we will show that q is not an A_4-map and hence $(q + \Omega\pi)$ is not an A_4-map.

Next we turn our attention to axial maps of the form $XP(r) \times XP(s) \longrightarrow XP(r+s)$. (Theorem 1.6 concerns the case $r = s = 1$.)

In the classical case we have maps $RP(r) \times RP(s) \longrightarrow RP(r+s)$ and $CP(r) \times CP(s) \longrightarrow CP(r+s)$ for all r, s. More generally we will establish

Theorem 1.17. <u>Let</u> (X,n) <u>be a monoid with a multiplication</u> m <u>such that</u>

(1.18) $m(n(w,x), n(y,z)) = n(m(w,y), m(x,z))$. <u>Let</u> $XP(r)$ <u>for</u> $r = 1, \ldots$ <u>denote the</u> X-<u>projective spaces constructed with respect to</u> n. <u>Then there exist axial maps</u> $XP(r) \times XP(s) \longrightarrow XP(r+s)$ <u>for all</u> r,s.

It is possible to consider A_∞-spaces instead of monoids and to replace condition (1.18) by a homotopy condition, but the present formulation (suggested by J. C. Moore) is considerably neater and sufficiently general to indicate the phenomena of importance. An approximate converse to Theorem 1.17 is familiar. If $XP(\infty)$ admits a multiplication, then $\Omega XP(\infty)$ admits two multiplications, loop addition denoted by n and the induced multiplication m, i.e. $m(\lambda, \mu)(s) = \lambda(s)\mu(s)$ for $\lambda, \mu \in \Omega XP(\infty), s \in [0,1]$, satisfying (1.18). Alternatively we could use the Moore loop space, in which case n would be associative but condition (1.18) would be replaced by a homotopy condition.

By taking $m = n$, Theorem 1.17 implies

Corollary 1.19. If X admits the structure of an abelian monoid, there exist axial maps $XP(r) \times XP(s) \longrightarrow XP(r+s)$ for all r,s.

It is possible to obtain the classical case from Corollary 1.19 by identifying $RP(r)$ with $S^0P(r)$ and $CP(r)$ with $S^1P(r)$ as is done in [21].

Still more generally, using the concept of an A_n-map, we find it possible to give fairly general conditions which imply the existence of axial maps $XP(r) \times XP(s) \longrightarrow XP(r+s)$ for a limited range of $r+s$. In particular, let (X,m) be a monoid. Denote multiplication by juxtaposition, $m(x,y) = xy$. Suppose that m is homotopy abelian; i.e. there exists $c_s : X \times X \longrightarrow X$ such that $c_0(x,y) = xy$ and $c_1(x,y) = yx$. We call c_s a <u>commuting homotopy</u> for (X,m).

Define $\bar{c} : S^1 \to X^{X^3}$ by

$$\begin{aligned} \bar{c}(s)(x,y,z) &= xc_s(y,z) && \text{for } 0 \le s \le 1 \\ &= c_{s-1}(x,z)y && \text{for } 1 \le s \le 2 \\ &= c_{3-s}(xy,z) && \text{for } 2 \le s \le 3 \end{aligned}$$

where S^1 is parameterized from 0 to 3.

Theorem 1.20. <u>If for some commuting homotopy</u> c_s <u>for</u> X, <u>the map</u> $\bar{c}$ <u>defined as above is null-homotopic, then there exists an axial map</u> $XP(1) \times XP(2) \longrightarrow XP(3)$.

2. Proof of Theorem 1.6

Elsewhere [20] we have given a proof of Theorem 1.6 based on the methods used by James and Thomas to prove Theorem 1.5. For the purposes of generalization we present an alternative proof.

Let (X,m) be an H-space. We can give $X \times X$ the structure of an H-space by multiplying coordinatewise, i.e. $(w,x)\cdot(y,z) = (wy,xz)$. Hence it is meaningful to ask: Is $m : X \times X \longrightarrow X$ an H-map? We observe

Theorem 2.1. <u>The multiplication</u> $m : X \times X \longrightarrow X$ <u>is an</u> H-<u>map</u> <u>if and only if</u> m <u>is homotopy commutative and homotopy associative.</u>

Proof. If m is an H-map, the maps $g_0 : X^4 \longrightarrow X$ and $g_1 : X^4 \longrightarrow X$ defined by $g_0(w,x,y,z) = (wx)(yz)$ and $g_1(w,x,y,z) = (wy)(xz)$ must be homotopic. Setting $w = z = e$, we see that m is homotopy commutative. With x or $y = e$, we see that m is homotopy associative. Conversely, both homotopy properties can be used to give:

$$(2.2) \qquad (wx)(yz) \simeq w(x(yz)) \simeq w((xy)z) \simeq w((yx)z) \simeq w(y(xz)) \simeq (wy)(xz).$$

Now if m is an H-map there is an induced map $mP(2) : (X \times X)P(2) \longrightarrow XP(2)$. The inclusions $i_1, i_2 : X \rightarrow X \times X$ imbedding X as either factor are homomorphisms and hence induce maps $j_1, j_2 : XP(i) \longrightarrow (X \times X)P(i)$ for $i = 1, 2$.

By an abuse of language we also denote by j_1, j_2 the corresponding inclusions $XP(1) \subset (X \times X)P(2)$. Since m itself is axial, the composition $mP(2)j_i$ for $i = 1,2$ is the inclusion $SX = XP(1) \subset XP(2)$. Hence if $j_1 \vee j_2$ extends to $g : SX \times SX \longrightarrow (X \times X)P(2)$ then $mP(2)g$ is an axial map.

<u>Lemma</u> 2.3. The map $j_1 \vee j_2 : SX \vee SX \longrightarrow (X \times X)P(2)$ extends to $g : SX \times SX \longrightarrow (X \times X)P(2)$.

Proof. We represent points of SX as pairs (t,x) and points of $(X \times X)P(2) - S(X \times X)$ as 7-tuples.
$(u_0, u_1, u_2, (w,x), (y,z))$ with $u_0 + u_1 + u_2 = 1$.
Define g by

$$\begin{aligned} g((t,x),(s,y)) &= (s, s-t, t, (e,y), (x,e)) \quad \text{for } t \leq s \\ &= (t, t-s, s, (x,e), (e,y)) \quad \text{for } s \leq t. \end{aligned}$$

Notice that g is well defined; i.e. respects identification and when $s = t$ is uniquely determined as $(t,(x,y)) \in S(X \times X)$; and that g extends $j_1 \vee j_2$.

Notice further that to obtain the desired axial map we need only $mP(2) \mid g(SX \times SX)$. This image involves points in X^4 of the form (x,e,e,y) or (e,y,x,e). We need the homotopy between g_0 and g_1 only on the subset of X^4 consisting of points of the above forms. Now $g_0(x,e,e,y) = xy = g_1(x,e,e,y)$, but $g_0(e,x,y,e) = xy$ while $g_1(e,x,y,e) = yx$. Thus $mP(2) \mid g(SX \times SX)$ is defined if m is homotopy abelian.

As for the converse, let $f : SX \times SX \longrightarrow XP(2)$ be an axial map. In general we regard $SX \times SY$ as

$I^2 \times X \times Y \cup_\beta SX \vee SY$ where $\beta : \partial(I^2) \times X \times Y \cup I^2 \times (X \vee Y)$ is defined by

$$\beta(t, \varepsilon, x, y) = (t, x) \in SX \quad \text{for} \quad \varepsilon = 0 \text{ or } 1$$
$$\beta(\varepsilon, s, x, y) = (s, y) \in SY \quad \text{for} \quad \varepsilon = 0 \text{ or } 1$$
$$\beta(t, s, x, e) = (t, x), \quad \beta(t, s, e, y) = (s, y).$$

Let $X = Y$ and $f' = f\beta$, so that

$$(2.4) \quad \begin{aligned} f'(t, \varepsilon, x, y) &= (t, x) \in SX \quad \text{for} \quad \varepsilon = 0 \text{ or } 1 \\ f'(\varepsilon, s, x, y) &= (s, y) \in SX \quad \text{for} \quad \varepsilon = 0 \text{ or } 1 \end{aligned}$$

By identifying $\partial(I^2)$ with S^1, we can regard f' as a map of $S(X^2)$ into $XP(2)$. The adjoint will be a map of X^2 into $\Omega XP(2)$ which from (2.4) we see can be written as

(2.5) $(w_2(x) + w_2(y)) + (-w_2(x) - w_2(y))$

where $w_2 : X \longrightarrow \Omega XP(2)$ is the usual imbedding. Since w_2 is an H-map (II.3.10), the map described by (2.5) is homotopic to $w_2(xy) - w_2(yx)$. Since f' extends to $I^2 \times X^2$, the adjoint is null-homotopic, hence $w_2 m$ is homotopic to $w_2 mT$. If m is homotopy associative, there is a weak fibring $p : XE(2) \longrightarrow XP(2)$ with the fibre X contractible in XE(2) by Theorem II.2.1. Hence by Proposition 1.11 there is a retraction $r : \Omega XP(2) \longrightarrow X$, i.e. $rw_2 \simeq 1$. Applying r to $w_2 m$ and $w_2 mT$, we see that m is homotopy commutative.

As for Theorem 1.7, the latter discussion would apply, substituting X for $(\Omega X)P(2)$ throughout. The adjoint to f' is a map of $\Omega X \times \Omega X$ into ΩX. In this case for

$\lambda, \mu \in \Omega X$ we have

$$f'(t, \varepsilon, \lambda, \mu) = \lambda(t) \quad \text{for} \quad \varepsilon = 0 \text{ or } 1$$

$$f'(\varepsilon, s, \lambda, \mu) = \mu(s) \quad \text{for} \quad \varepsilon = 0 \text{ or } 1;$$

hence the adjoint can be written as m-mT. Since f' extends to $I^2 \times \Omega X \times \Omega X$, we see that m-mT is null-homotopic; that is, m is homotopy abelian.

3. The general case for a monoid.

Theorem 1.17 will follow from a study of axial maps $XP(r) \times XP(s) \longrightarrow XP(r+s)$ for general r, s. A partial result is the following:

Theorem 3.1. Let $(X; m)$ be a monoid and $XP(i)$ for $i = 1,2,\ldots$ the corresponding X-projective i-spaces. There exists an axial map $XP(r) \times XP(s) \longrightarrow XP(r+s)$ if m is an A_{r+s}-map (with respect to coordinatewise multiplication on $X \times X$).

Proof. If m is an A_{r+s}-map we have the map $mP(r+s) : (X \times X)P(r+s) \longrightarrow XP(r+s)$. As suggested by the proof of Theorem 1.6, we now obtain Theorem 3.1 from

Proposition 3.2. There exists $g_{rs} : XP(r) \times XP(s) \longrightarrow (X \times X)P(r+s)$ which is axial with respect to $j_1 : XP(r) \longrightarrow (X \times X)P(r+s)$ and $j_2 : XP(s) \longrightarrow (X \times X)P(r+s)$.

Proof. We have obviously extended our use of the symbols j_1 and j_2. We construct g_{rs} explicitly. We wish to map $\Delta^r \times X^r \times \Delta^s \times X^s$ into $\Delta^n \times X^{2n}$ for $r + s = n$ so as to respect the identifications imposed by $\gamma(r)$, $\gamma(s)$ and $\gamma(r+s)$.

For any two ordered simplicial complexes K and L, the product $K \times L$ can be subdivided so as to form a simplicial complex $\overline{K \times L}$ [6, p.67]. This subdivision is functorial, hence for any subcomplex $K' \subset K$, the product $K' \times L$ is subdivided as it would be as a subset of $K \times L$. For the particular case

$\Delta^r = K$, $\Delta^s = L$, the subdivision can be described as follows: Let Δ^r be the ordered simplex $(0,\dots,r)$ and Δ^s the ordered simplex $(0,\dots,s)$. The vertices of $\overline{\Delta^r \times \Delta^s}$ are all pairs (i,j) with $0 \le i \le r$, $0 \le j \le s$; partially order them as products, i.e. $(i,j) \le (i',j')$ if $i \le i'$ and $j \le j'$. The ordered n-simplices of $\overline{\Delta^r \times \Delta^s}$ are all simply ordered sequences of length $n+1$ of the vertices of $\Delta^r \times \Delta^s$. We map $\Delta^r \times \Delta^s$ onto Δ^n by an order preserving map of the vertices of each n-simplex of the decomposition.

Now we wish to describe the related map of $X^r \times X^s$ into X^{2n}.

<u>Definition</u> 3.3. A <u>shuffle</u> of the symbols $x_1,\dots,x_r$, $y_1,\dots,y_s$ is a permutation which is order preserving on $x_1,\dots,x_r$ and on $y_1,\dots,y_s$.

With each n-simplex σ of $\overline{\Delta^r \times \Delta^s}$ we associate a shuffle $\omega(\sigma)$ as follows: If $\sigma = (v_0,\dots,v_n)$, each vertex v_k must be a pair (i,j) with $i + j = k$. The vertex v_{k-1} must be either $(i-1,j)$ or $(i,j-1)$.

Write $\omega(\sigma)(x_1,\dots,x_r,y_1,\dots,y_s) = (z_1,\dots,z_{r+s})$ with

$$(3.4)\qquad \begin{aligned} z_k &= x_i \quad \text{if } v_k = (i,j),\ v_{k-1} = (i-1,j) \\ z_k &= y_j \quad \text{if } v_k = (i,j),\ v_{k-1} = (i,j-1). \end{aligned}$$

We can extend our definition of associated shuffles to the subdivision $\overline{K \times L}$ of any product of ordered simplicial complexes. If Δ^r is replaced by an r-simplex $(p_0,\dots,p_r)$ and Δ^s by $(q_0,\dots,q_s)$, then $v_k = (i,j)$ is replaced by $v_k = (p_i,q_j)$.

A simplex $\sigma = (v_0,\ldots,v_n)$ of $\overline{\Delta^r \times \Delta^s}$ intersects $(0,\ldots,i-1, i+1,\ldots,r) \times (0,\ldots,s)$ in an $(n-1)$-simplex σ' if and only if precisely one vertex v_k has the form (i,j) for some j. The face σ' is $(v_0,\ldots,v_{k-1},v_{k+1},\ldots,v_n)$. From formulas (3.4) it follows that if $\sigma = (v_0,\ldots,v_n)$ and σ' is obtained by omitting v_k, the only vertex with first coordinate p_i, then for

$\omega(\sigma)(x_1,\ldots,x_r,y_1,\ldots,y_s) = (z_1,\ldots,z_n)$

we have

$$\begin{aligned}
&\omega(\sigma')(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_r,y_1,\ldots,y_s) && \\
(3.5)\qquad &\quad = (z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_n) && \text{for } 0 < k \leq n \\
&\omega(\sigma')(x_2,\ldots,x_r,y_1,\ldots,y_s) && \\
&\quad = (z_2,\ldots,z_n) && \text{for } k = 0.
\end{aligned}$$

Now let $w_1,\ldots,w_{r+s}$ be points of X and let $x_i = (w_i,e)$ for $1 \leq i \leq r$, $y_j = (e,w_{j+r})$ for $1 \leq j \leq s$. Define $\bar{g} : \Delta^r \times X^r \times \Delta^s \times X^s \longrightarrow \Delta^n \times X^{2n}$ as follows: for $t \in \Delta^r$, $u \in \Delta^s$ and $(t,u) \in \sigma \subset \overline{\Delta^r \times \Delta^s}$,

$$\begin{aligned}
(3.6)\qquad &\bar{g}(t,w_1,\ldots,w_r;\ u,w_{r+1},\ldots,w_{r+s}) = \\
&\quad (\overline{(t,u)},\ \omega(\sigma)(x_1,\ldots,x_r,y_1,\ldots,y_s))
\end{aligned}$$

where $\overline{(t,u)}$ is the coordinate of (t,u) in σ.

<u>Lemma</u> 3.7. The function $\bar{g}$ induces a well defined map $g_{rs} : XP(r) \times XP(s) \longrightarrow (X \times X)P(r+s)$.

Proof. We <u>assume the</u> lemma proved for $r+s-1$. Two n-simplices of $\overline{\Delta^r \times \Delta^s}$ have an $(n-1)$-face in common if and only if they have the form

$\sigma = (v_0,\dots,v_n)$ and $\sigma' = (v_0,\dots,v'_{i+j},\dots,v_n)$ where $v_{i+j} = (i,j)$ and $v'_{i+j} = (i+1, j-1)$ (or vice versa).
Let $W_1 = (w_1,\dots,w_r)$, $W_2 = (w_{r+1},\dots,w_{r+s})$.
In this case $\bar{g}(t,W_1;\, u,W_2)$ is defined as (3.6) and as

(3.8) $\quad ((\overline{\overline{t,u}}),\ \omega(\sigma')(x_1,\dots,x_r,y_1,\dots,y_s))$, where $\overline{\overline{(t,u)}}$ is the coordinate of (t,u) in σ'. Since (t,s) belongs to both σ and σ', the coordinates $(\overline{t,u})$ and $(\overline{\overline{t,u}})$ describe the same point of the face $t_{i+j} = 0$ of Δ^n. Applying $\gamma(n)$ to (3.6) and (3.8) we obtain two points of $(X\times X)P(n-1)$ which have the same coordinate in Δ^{n-1}. As for the coordinate in X^{2n-2}, we note that $\omega(\sigma)$ and $\omega(\sigma')$ have the same effect except in the $(i+j)$-th and $(i+j+1)$-st position, i.e. $z_{i+j} = y_j$, $z_{i+j+1} = x_{i+1}$ while $z'_{i+j} = x_{i+1}$, $z'_{i+j+1} = y_j$. Since $t_{i+j} = 0$, $\gamma(n)$ multiplies the entries in the $(i+j)$-th and $(i+j+1)$-st positions. We have

$$z_{i+j}\, z_{i+j+1} = (w_{i+1}, w_{j+r}) = z'_{i+j}\, z'_{i+j+1}.$$

Similarly one verifies that g_{rs} is well defined on $\partial(\Delta^r\times\Delta^s)$. Suppose $(t,u)\in\Delta^r\times\Delta^s$ lies in a common $(n-1)$-simplex of $(0,\dots,i-1,i+1,\dots,r)\times(0,\dots,s)$ and an n-simplex $\sigma = (v_0,\dots,v_n)$ of $\overline{\Delta^r\times\Delta^s}$.
Applying $\gamma(n)$ to (3.6) we obtain a point of $XP(n-1)$ represented by

(3.9) $\quad (\tau, z_1,\dots,z_k\, z_{k+1},\dots,z_n)$ if $0<k<n$, $v_k = (i,j)$

where τ is the coordinate of $\overline{(t,u)}$ in Δ^{n-1} and $z_1,\ldots,z_n$ denotes the effect of the shuffle. Applying $\gamma(r)$ first and then (3.6) we have for $0 < k < n$

(3.10) $(\tau, \omega(\sigma')(x_1,\ldots,x_{i-1},x_i x_{i+1},\ldots,x_r,y_1,\ldots,y_s))$

where σ' is obtained from σ by omitting v_k, the only vertex of the form (i,j) for some j. Applying (3,5) we see that (3.9) and (3.10) represent the same point.

If $k = 0$ or n, we must change (3.9) to

(3.9') $(\tau, z_2,\ldots,z_n)$ or $(\tau, z_1,\ldots,z_{n-1})$

and (3.10) to

(3.10') $(\tau, \omega(\sigma')(x_2,\ldots,x_r,y_1,\ldots,y_s)$ or

$(\tau, \omega(\sigma')(x_1,\ldots,x_{r-1},y_1,\ldots,y_s)$.

Again applying (3.5) we see that the two points are the same. Using the symmetry of (3.6), the proof of Lemma 3.7 is complete.

<u>Lemma</u> 3.11. The induced map

$g_{rs} : XP(r) \times XP(s) \longrightarrow (X \times X)P(r+s)$ extends $j_1 \vee j_2$.

Proof: Let $t = (0,\ldots,1,\ldots,0) \in \Delta^r$. Under $\gamma(r)$, the point (t,W_1) is identified with the base point $*$ of $XP(r)$. Hence, to check that f extends j_2, we may look at $\bar{g}(t,W_1;\ u,\ W_2)$ for t of the above form. If 1 is in the i-th position, (t,u) belongs to the simplex

$\sigma = ((0,0),\ldots,(i,0),\ldots,(i,s),\ldots,(r,s))$.

The corresponding shuffle is

$x_1,\ldots,x_i,y_1,\ldots,y_s,x_{i+1},\ldots,x_r$. The point (t,u) lies on the face of σ obtained by omitting $v_0, \ldots, v_{i-1}, v_{i+s+1}, \ldots, v_n$, hence the point of $(X \times X)P(n)$ defined by (3.6) is identified with the point of $(X \times X)P(s)$ represented by $(u, y_1, \ldots, y_s)$, i.e. by $j_2(u, W_2)$. Using the symmetry, f also extends j_1.

This completes the proof of Theorem.3.1. In order to obtain Theorem 1.17, we observe that we have used only the fact that m is an A_{r+s}-map with respect to whatever monoid structure $n : X \times X \longrightarrow X$ is used to construct $XP(i)$. Thus Theorem 1.17 follows from

<u>Lemma</u> 3.12. If (X,n) is a monoid and $m : X \times X \longrightarrow X$ satisfies (1.18) then m is an A_∞-map (with respect to the associative structure).

Proof: The required homotopies $G_i : I_i \times K_i \times (X^2)^i \to X$ can be defined as

$$G_i(t, \rho, x_1, y_1, \ldots, x_i, y_i) = x_1y_1 \cdot x_2y_2 \cdot \cdots \cdot x_iy_i$$

where juxtaposition denotes multiplication by m and $\cdot$ denotes multiplication by n.

Theorem 3.1 can be strengthened. Suppose that m is an A_{r+s-1}-map so that we have

$$f : XP(r) \times XP(s-1) \cup XP(r-1) \times XP(s) \longrightarrow XP(r+s-1).$$

Can f be extended to $XP(r) \times XP(s)$?

Since m is an A_{r+s-1}-map, we have homotopies $G_i : I_i \times K_i \times X^{2i} \longrightarrow X$ for $i < r+s$ satisfying the usual conditions. We assume that f is obtained by composing the map $mP(r+s-1)$ which corresponds to the homotopies G_i with the appropriate maps

$g_{ij} : XP(i) \times XP(j) \longrightarrow (X \times X)P(i+j)$. Continue to let $n = r + s$. Using conditions II.3.2.b,c,d it is possible to define $\bar{G}_n : \partial (I_n \times K_n) \times X^{2n} \longrightarrow X$.

In general, if (X,m) is an H-space and $f : Y \times Z \longrightarrow X$ is a map, then there exists a corresponding map $\overset{o}{f} : Y \divideontimes Z \longrightarrow X$ defined as follows:

Let $\sigma : X \longrightarrow X$ denote an inverse for m; i.e. $m(1 \times \sigma)$ is null-homotopic [22,§4]. Define $f' : Y \times Z \longrightarrow X$ by $f'(y,z) = [f(y,z) \; \sigma f(y,e)] \; \sigma f(e,z)$. Clearly $f' | Y \vee Z$ is null-homotopic. Let $\overset{o}{f}$ be any map obtained by shrinking $f' | Y \vee Z$ to a point and applying the homotopy extension theorem.

Let $\alpha_n \in \pi(S^{n-2}(X^{2n}),X)$ denote the class of $\overset{o}{\bar{G}}_n : \partial (I_n \times K_n) \divideontimes X^{2n} \longrightarrow X$.

Since $\bar{G}_n | \partial (I_n \times K_n) \times e \times \cdots \times e = e$, the map $\overset{o}{\bar{G}}_n$ is null-homotopic if and only if $\bar{G}_n$ extends to $I_n \times K_n \times X^{2n}$. The class $\alpha_n = 0$ if and only if m is an A_{r+s}-map.

To obtain the axial map we seek, it is sufficient to have $mP(r+s) | g_{rs}(XP(r) \times XP(s))$. Let ω be a subset of $(1,\ldots,2n)$. The notation X_ω will mean the subset of X^{2n} consisting of points whose i-th coordinate is e unless i belongs to ω. From the definition of $\bar{g}$ (3.6) we see that we will have a suitable map of

$g_{rs}(XP(r) \times XP(s))$ into $XP(r+s)$ if we can extend $\bar{G}_i$ to $I_n \times K_n \times (\bigcup X_\omega)$ where ω runs over all subsets of $(1,\ldots,2n)$ consisting of r odd numbers and s even numbers.

Thus we have

Theorem 3.13. *Let* $i : \bigcup X_\omega \longrightarrow X^{2n}$ *denote the inclusion of* $\bigcup X_\omega$ (defined as above). *There exists an axial map* $XP(r) \times XP(s) \longrightarrow XP(r+s)$ *if* $i^*(\alpha_n) = 0$ in $\pi(S^{n-2}(\bigcup X_\omega), X)$.

We give substance to Theorem 3.13 by working out its implications in the case $r + s = 3$.

4. The case $r + s = 3$

By the general theory of the preceding section, we know that axial maps $f_{12} : SX \times XP(2) \longrightarrow XP(3)$ and $f_{21} : XP(2) \times SX \longrightarrow XP(3)$ exist if (X,m) is a monoid and m is an A_3-map. We first extend this result to A_3-spaces $(X; \{h_i\})$.

Next we turn to the map $\bar{c}$ defined in section 1. We define a corresponding map, also called $\bar{c}$, for an A_3-space and show f_{12} exists if $\bar{c}$ is null-homotopic. We investigate the corresponding obstruction $\underline{c}$ for f_{21} and its relation to $\bar{c}$.

Finally we show that if for some commuting homotopy c_s both $\bar{c}$ and $\underline{c}$ are null-homotopic, then there exist f_{12} and f_{21} which agree on $SX \times SX$. If further $(X; \{h_i\})$ is an A_4-space, we show that h_2 is an A_3-map.

We begin with

Lemma 4.1. Let $(X; \{h_i\})$ be an A_3-space. Then the map $g : SX \times SX \longrightarrow (X \times X)P(2)$ defined in Lemma 2.3 extends to an axial map

$g' : SX \times XP(2) \cup XP(2) \times SX \longrightarrow (X \times X)P(3)$.

Proof. By the symmetry of the definition of g we need define only $g' \big| SX \times XP(2)$. If (X,h_2) is a monoid, we have already defined g' by formula (3.6). We wish to make a similar construction for an A_3-space; we must define

${}^1\Delta \times {}^2\Delta \longrightarrow {}^3\Delta$ with appropriate properties. Since ${}^1\Delta = \Delta^1$ and ${}^2\Delta = \Delta^2$, we already have a subdivision of ${}^1\Delta \times {}^2\Delta$ into copies of Δ^3. Our construction of g' will rest upon a relative homeomorphism

$d : {}^3\Delta, \partial\, {}^3\Delta \longrightarrow \Delta^3, \partial \Delta^3$ which is specified on $\partial\, {}^3\Delta$ by the relations

(4.2) $\qquad d\, \theta_i = \bar{\theta}_i \qquad$ on $\quad {}^2\Delta = \Delta^2$

(4.3) $\qquad d\, \theta_{12} = \bar{\theta}_{12}\, p_1 \qquad$ on $\quad {}^1\Delta \times K_3 = \Delta^1 \times K_3$

where $\bar{\theta}_i$ is the imbedding of the face $t_i = 0$ in Δ^3, $\bar{\theta}_{12}$ is the imbedding of the edge $t_1 + t_2 = 0$, and $p_1 : \Delta^1 \times K_3 \longrightarrow \Delta^1$ is the projection onto the first factor. The inverse d^{-1} can be regarded as a multivalued function but $\gamma(3)(d^{-1} \times 1)\bar{g}$ is uniquely defined. This is obvious on all but the edge $t_1 + t_2 = 0$ of some 3-simplex σ of $\Delta^1 \times \Delta^2$.

Let z_1, z_2, z_3 be the corresponding shuffle of $(w_1, e), (e, w_2), (e, w_3)$. Let $\bar{h}_i$ denote the usual structure on $X \times X$, e.g. $\bar{h}_3(\rho, x_1, \ldots, x_6) = (h_3(\rho, x_1, x_3, x_5),\ h_3(\rho, x_2, x_4, x_6))$. If r denotes the coordinate on the edge in question, the set $\gamma(3)(d^{-1} \times 1)\bar{g}$ is the set of points $\{(r, \bar{h}_3(\rho, z_1, z_2, z_3)) \mid \rho \in K_3\}$. This reduces to the single point $(r, (w_1, w_2 w_3))$ since $\bar{h}_3(\rho, z_1, z_2, z_3) = (h_3(\rho, w_1, e, e),\ h_3(\rho, e, w_2, w_3)) = (w_1, w_2 w_3)$. The arguments of Lemma 3.7 and 3.11 can now be applied to show that we have a well-defined extension of g to $SX \times XP(2)$. This completes the proof of Lemma 4.1.

Using Lemma 4.1, we can prove

Theorem 4.4. <u>Let</u> $(X; \{h_i\})$ <u>be an</u> A_3<u>-space. If</u> h_2 <u>is an</u> A_3<u>-map, there exist axial maps</u> $f_{12} : SX \times XP(2) \longrightarrow XP(3)$ <u>and</u> $f_{21} : XP(2) \times SX \longrightarrow XP(3)$ <u>such that</u> $f_{12} \mid SX \times SX = f_{21} \mid SX \times SX$.

Proof. If h_2 is an A_3-map, the composition $h_2 P(3) g'$ gives the axial maps.

Again using the approach of Theorem 3.13, we can obtain an axial map under somewhat weaker hypotheses. Let $i_j : X \longrightarrow X^6$ denote the imbedding of X as the j-th factor, $j = 1, \dots, 6$.

<u>Lemma</u> 4.5. Let $(X; \{h_i\})$ be an A_3-space. Suppose $G_2 : I_2 \times X^4 \longrightarrow X$ is a homotopy showing that h_2 is an H-map. Let $f : SX \times SX \longrightarrow XP(2)$ be the corresponding axial map. Let $\alpha_3 \in \pi(S(X^6), X)$ be the class defined as in section 3. Then f extends to an axial map

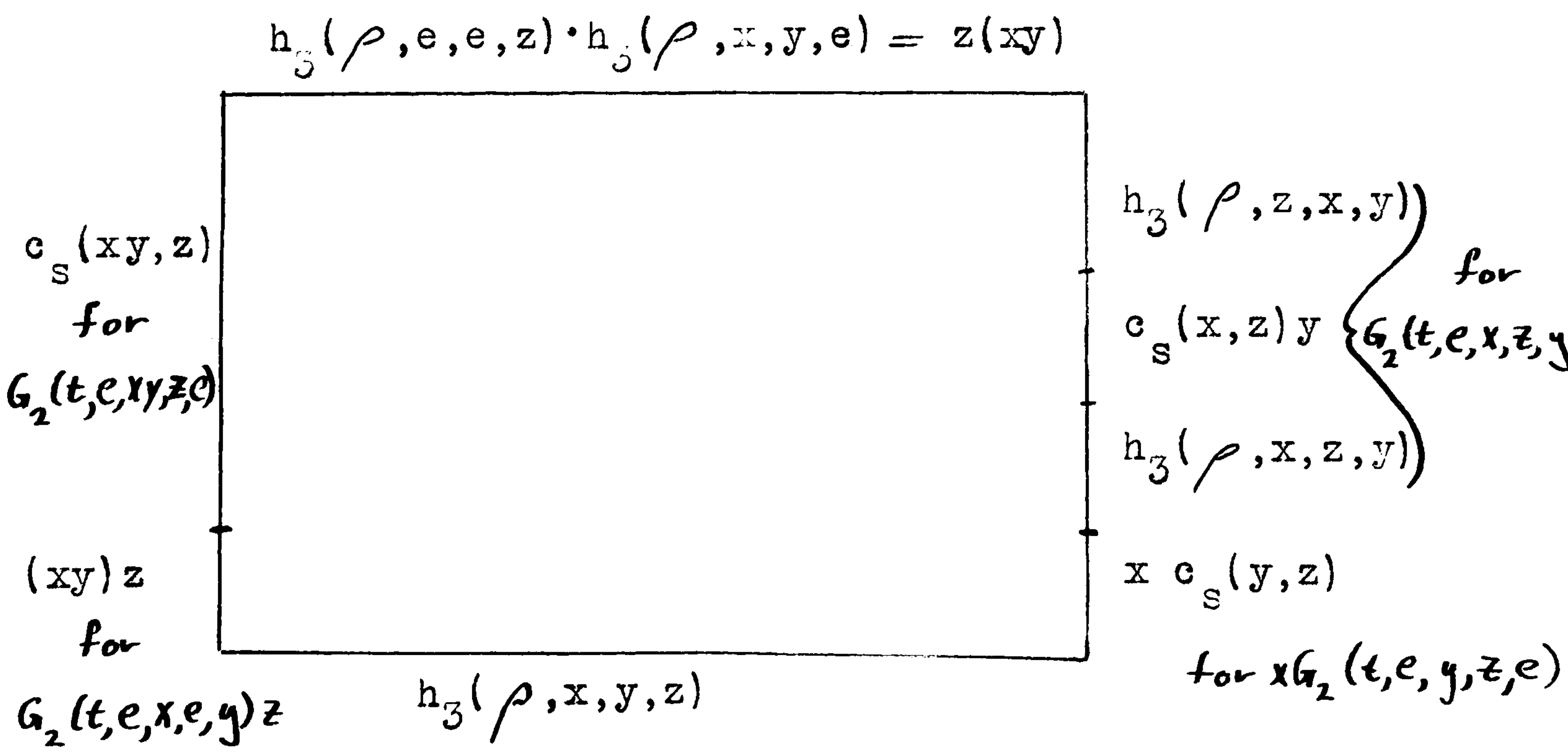
$f_{12} : SX \times XP(2) \longrightarrow XP(3)$ if $(i_2 \times i_4 \times i_5)^* \alpha_3 = 0$ in $\pi(S(X_{245}), X)$.

Proof. To obtain an axial map we need only extend $h_2 P(2)$ properly to $g'(SX \times XP(2)) \subset (X \times X)P(3)$. This we can do if we can extend $\bar{G}_3 : \partial(I_3 \times K_3) \times X^6 \longrightarrow X$ to $I_3 \times K_3 \times \bigcup X_\omega$ where $\bigcup X_\omega$ consists of points of X^6 of one of the following three forms:

X_{146} : x,e,e,y,e,z
X_{236} : e,x,y,e,e,z
X_{245} : e,x,e,y,z,e.

It is easy to define $G_3 \mid I_3 \times K_3 \times X_\omega$ for $\omega = 146$ or 236, e.g. $G_3(t, \rho, x,e,e,y,e,z) = x(yz)$ assuming (without loss of generality) that $G_2(t,e,y,e,z) = yz$, etc. It is only the subset X_{245} which presents problems; hence Lemma 4.5.

If h_2 is an H-map, then h_2 is homotopy abelian. Let $c_s : X \times X \longrightarrow X$ be a commuting homotopy for (X, h_2); i.e. $c_0(x,y) = xy$ and $c_1(x,y) = yx$ for $x,y \in X$. Suppose that $G_2 : I_2 \times X^4 \longrightarrow X$ is given by the string of homotopies (2.2). The map $\bar{G}_3 : \partial(I_3 \times K_3) \times X^6 \longrightarrow X$ corresponds to a map of $\partial(I_3 \times K_3)$ into X^{X^6}. At a particular point e,x,e,y,z,e, we can portray this map by the following diagram:



By identifying $\partial(I_3 \times K_3)$ with S^1 parameterized from 0 to 6 in a suitable manner, we can write this map more neatly as $\bar{c} : S^1 \longrightarrow X^{X^3}$ defined by

$$
\begin{aligned}
\bar{c}(s)(x,y,z) &= xc_s(y,z) && \text{for } 0 \le s \le 1 \\
&= h_3(s-1,x,z,y) && 1 \le s \le 2 \\
&= c_{s-2}(x,z)y && 2 \le s \le 3 \\
&= h_3(4-s,z,x,y) && 3 \le s \le 4 \\
&= c_{5-s}(xy,z) && 4 \le s \le 5 \\
&= h_3(6-s,x,y,z) && 5 \le s \le 6.
\end{aligned}
\tag{4.6}
$$

Thus Lemma 4.5 can be reworded as

Theorem 1.20'. Let $(X; \{h_i\})$ be an A_3-space. If there exists a commuting homotopy c_s such that the map $\bar{c}$ defined by (4.6) is null homotopic, then there exists an axial map $f_{12} : SX \times XP(2) \longrightarrow XP(3)$.

Theorem 1.20' reduces to Theorem 1.20 in the associative case.

The transpose. By symmetry we can deduce that the corresponding map $\underline{c}$ which represents the obstruction to constructing $f_{21} : XP(2) \times SX \to XP(3)$ can be written

$$
\begin{aligned}
\underline{c}(s)(x,y,z) &= c_s(x,y)z && \text{for } 0 \le s \le 1 \\
&= h_3(2-s,y,x,z) && 1 \le s \le 2 \\
&= y\, c_{s-2}(x,z) && 2 \le s \le 3 \\
&= h_3(s-3,y,z,x) && 3 \le s \le 4 \\
&= c_{5-s}(x,yz) && 4 \le s \le 5 \\
&= h_3(s-5,x,y,z) && 5 \le s \le 6
\end{aligned}
$$

Now from c_s there can be obtained another commuting homotopy c_s' given by $c_s'(x,y) = c_{1-s}(y,x)$. We call c_s' the <u>transpose</u> of c_s. Replacing c_s by c_s' in G_2, we would obtain a new homotopy $G_2': I_2 \times X^4 \longrightarrow X$ and consequently a new $\bar{G}_3' : \partial (I_3 \times K_3) \times X^6 \longrightarrow X$. Substituting c_s' for c_s in the definition of $\underline{c}$ we have the changes

$$\begin{aligned}
\underline{c}(s)(x,y,z) &= c_{1-s}(y,x)z && \text{for } 0 \le s \le 1 \\
&= y\, c_{3-s}(z,x) && 2 \le s \le 3 \\
&= c_{s-4}(yz,x) && 4 \le s \le 5.
\end{aligned}$$

Thus $\underline{c}(s)(x,yz)$ is the same as $\bar{c}(3-s)(y,z,x)$. Hence if $\bar{c}$ or $\underline{c}$ is null-homotopic, we will have axial maps

$f_{12} : SX \times XP(2) \longrightarrow XP(3)$ and $f_{21} : XP(2) \times SX \longrightarrow XP(3)$, though the restrictions of f_{12} and f_{21} to $SX \times SX$ need not agree. (They will be homotopic if c_s is homotopic, rel 0 and 1, to c_s', i.e. if (X,h_2) satisfies the H_2-condition of Kudo and and Araki [13].) However, if both $\bar{c}$ and $\underline{c}$ are null-homotopic then $\bar{G}_3$ can be extended to

$I_3 \times K_3 \times (X_{245} \cup X_{235})$; hence

Theorem 1.20". <u>Let</u> $(X; \{h_i\})$ <u>be an</u> A_3<u>-space. If there exists a commuting homotopy</u> c_s <u>such that</u> $\bar{c}$ <u>and</u> $\underline{c}$ <u>are null-homotopic, then there exist axial maps</u> $f_{12} : SX \times XP(2) \longrightarrow XP(3)$ <u>and</u> $f_{21} : XP(2) \times SX \longrightarrow XP(3)$ <u>such that</u> $f_{12} | SX \times SX = f_{21} | SX \times SX$.

Recall Theorem 4.4. Notice that if h_2 is an A_3-map, both $\bar{c}$ and $\underline{c}$ are null-homotopic. Moreover, Lemma 4.7. If h_2 is an A_3-map, there exists $h_4 : K_4 \times X^4 \longrightarrow X$ so as to make $(X; \{h_i, 2 \le i \le 4\})$ an A_4-space.

Proof: Let $G_3 : I_3 \times K_3 \times X^6 \longrightarrow X$ be the homotopy showing h_2 is an A_3-map. Consider $G_3 | I_3 \times K_3 \times X_{1356}$. With $x_2 = x_4 = e$ and G_2 as given by (2.2), with a slight change in parameters we have

$$
\begin{aligned}
G_3(1, \rho, x_1, \ldots, x_6) &= h_3(\rho, x_1, x_3, x_5)x_6 \\
G_3(3, \rho, x_1, \ldots, x_6) &= h_3(\rho, x_1, x_3, x_5x_6) \\
G_3(t, 0, x_1, \ldots, x_6) &= h_3(2-t, x_1, x_3x_5, x_6) & 1 \le t \le 2 \\
&= x_1h_3(3-t, x_3, x_5, x_6) & 2 \le t \le 3 \\
G_3(t, 1, x_1, \ldots, x_6) &= h_3(2-t, x_1x_3, x_5, x_6) & 1 \le t \le 2 \\
&= (x_1x_3)(x_5x_6) & 2 \le t \le 3.
\end{aligned}
$$

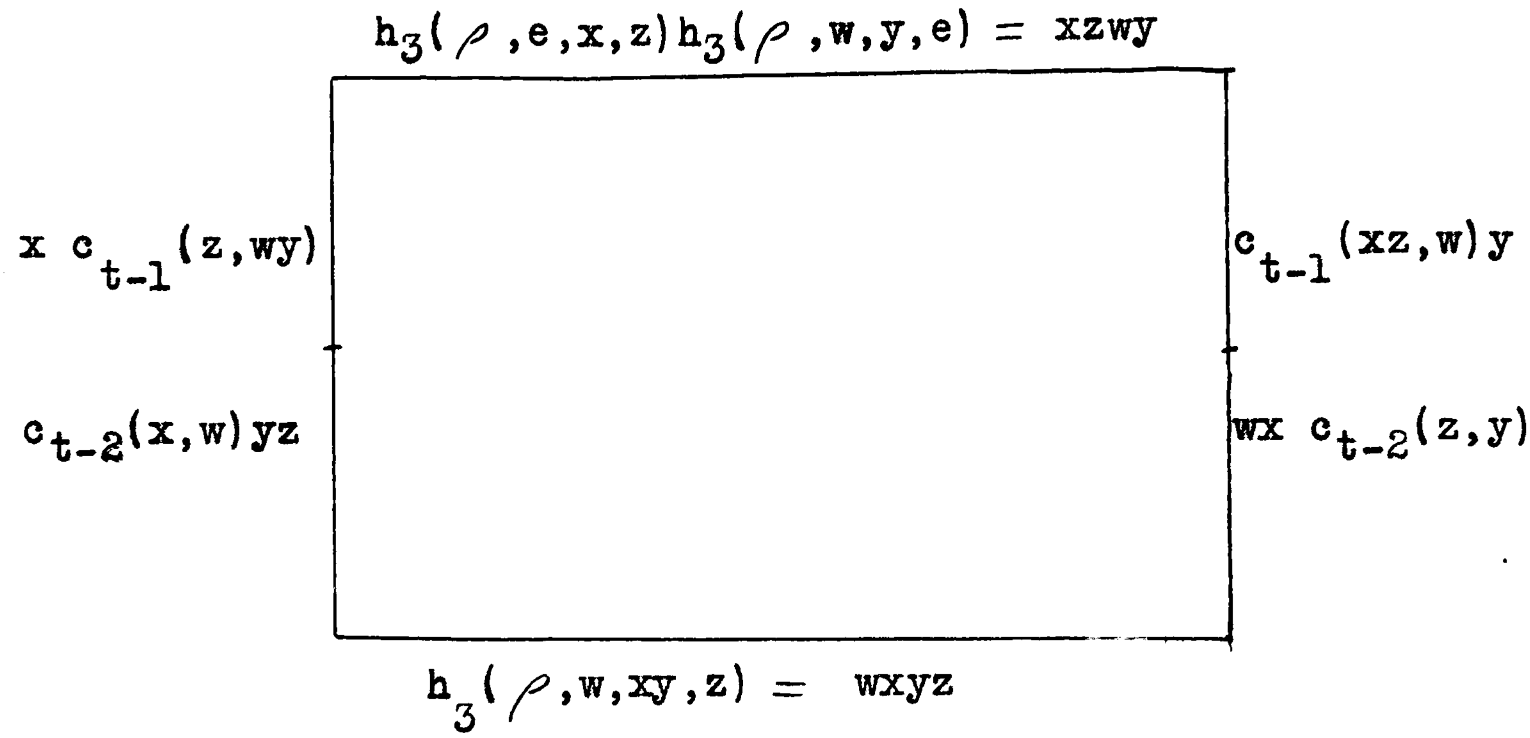
Hence for a suitable map $\varphi : I_3 \times K_3 \longrightarrow K_4$, we can define h_4 by

$$h_4(\varphi(t, \rho), w, x, y, z) = G_3(t, \rho, w, e, x, e, y, z).$$

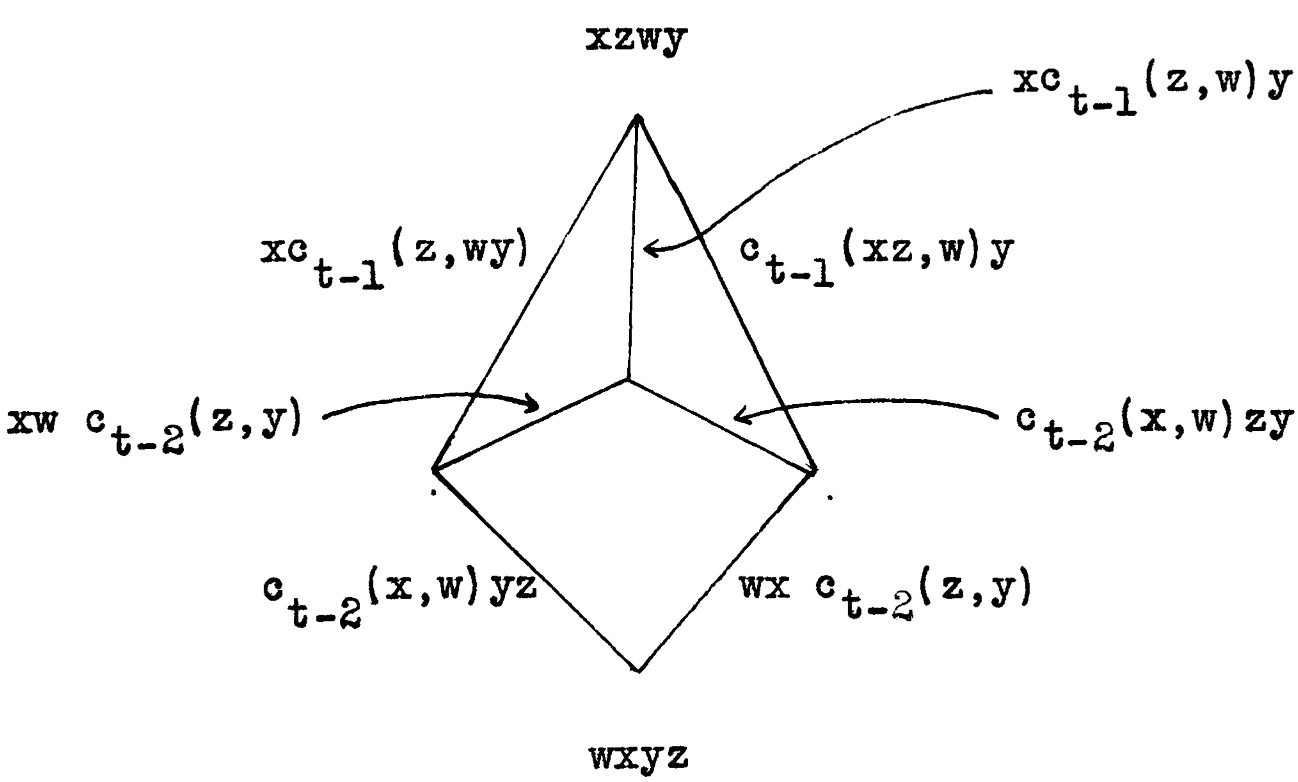
Conversely we show

Theorem 4.8. <u>Let $(X; \{h_i\})$ be an A_4-space. The multiplication h_2 is an A_3-map if there exists a commuting homotopy c_s such that $\bar{c}$ and $\underline{c}$ are null-homotopic.</u>

Proof. Case 1. (X, h_2) is a monoid. We can write $G_2(t, w, x, y, z) = w\, c_{t-1}(y, x)z$. Corresponding to $\bar{G}_3 : \partial (I_3 \times K_3) \times X^6 \longrightarrow X$, we have a map of $\partial(I_3 \times K_3)$ into X^{X^6} which at a point e,w,x,y,z,e can be depicted as



The next diagram shows how null-homotpies for $\bar{c}$ and $\underline{c}$ fit together with $c_t(x,w)\cdot c_s(z,y)$ to give an extension to $I_3 \times K_3$ and hence an extension $\tilde{G}_3 : I_3 \times K_3 \times X_{2345} \longrightarrow X$ of $\bar{G}_3$.

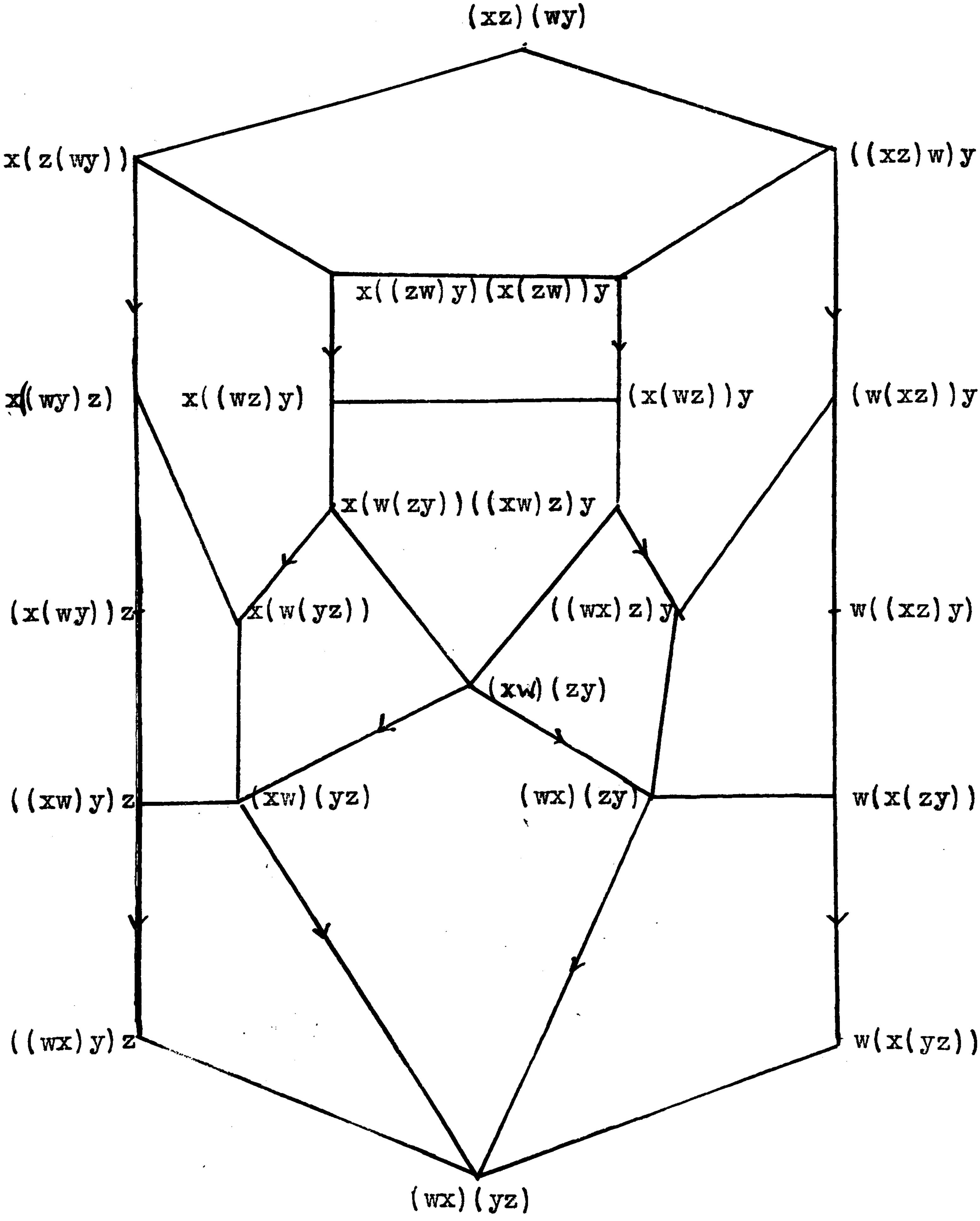


The triangles are filled in by using the null-homotopies for $\bar{c}$ and $\underline{c}$. We obtain G_3 by setting

$$G_3(t,\rho,x_1,\ldots,x_6) = x_1\tilde{G}_3(t,\rho,x_2,\ldots,x_5)x_6.$$

Case 2. (X, h_2) is not a monoid but $(X; \{h_i\})$ is an A_4-space.

The following more complicated diagram shows in a similar way how to obtain $\tilde{G}_3$.



We have labeled only the vertices but this is sufficient to determine the edges, since for those involving c_s we have indicated the direction of increasing s. The two hexagons are filled in using null-homotopies for $\bar{c}$ and $\underline{c}$, the pentagons by using the map h_4, and the quadilaterals by suitable combinations $c_s \cdot c_t$ or $h_3 \cdot c_s$ or $c_s \cdot h_3$. Thus we obtain $\tilde{G}_3 : I_3 \times K_3 \times X_{2345} \longrightarrow X$. To obtain G_3, we again multiply on left and right by x_1 and x_6 respectively, using the maps h_3 and h_4 to obtain the necessary association of factors.

CHAPTER IV HOMOLOGY OPERATIONS FOR A_n-SPACES

1. Massey operations in homology

In [14], Massey defines higher order cohomology products in terms of differentials in certain spectral sequences. We will construct corresponding invariants called <u>stack operations</u> in the homology of A_n-spaces. In particular these invariants will be defined in the space of loops ΩX on a space X. In this special case, it is possible to mimic Massey's procedure [14 , p.150-151] simply by changing a few words. Before giving the full details of the more general definition, we sketch these changes for the space of loops. Where Massey uses Γ^* "a suitable associative graded ring of cochains for X with coefficients in Λ," we substitute Γ_* "a suitable associative ring of chains for ΩX with coefficients in Λ." For example, we can take $\Omega_M X$ as defined by Moore [18] so that loop addition will be strictly associative and take Γ_* to be the ring of cubical singular chains on X (cubical since the product of cubes is a cube). Following Massey faithfully, we would obtain invariants (differentials in the spectral sequences constructed) which would be natural with respect to homomorphisms of the chain rings. Thus these invariants would be natural with respect to maps $f: \Omega_M X \to \Omega_M Y$ which are homomorphisms of loop addition; e.g. maps of the form $\Omega g: \Omega_M X \to \Omega_M Y$ for maps $g: X \to Y$. However, one of these invariants can be identified with the Pontrjagin product, which is natural with respect to H-maps. Is it possible that the higher order invariants are natural with respect to some subclass of H-maps, more general than homomorphisms?

To answer this question, we adapt Massey's procedure so as to define invariants in a more general situation. First we wish to depart from strict associativity.

We introduce non-associative chain algebras with additional structure, corresponding to the structure maps on A_n-spaces. Next we define algebraic objects called stacks which provide a suitable generalization of Massey's abstract sheaves. Standard techniques of homological algebra are used to obtain spectral sequences and to identify the term E^1. Applied to the chain algebra on an A_n-space, these spectral sequences yield the stack and bar operations. Again by translating homotopy conditions into algebraic conditions, we see that these operations are natural with respect to A_n-maps. By means of an explicit mapping, the E^∞ term of certain of the spectral sequences can be identified in terms of the homology of projective spaces. This enables us to give non-trivial examples of the operations.

2. A(n)-algebras

In the theory of H-spaces, it has proved useful to consider homotopy associative H-spaces and then homotopy associative H-spaces with further structure, the A_n-spaces of Part II. In the limit A_∞-spaces are equivalent to associative H-spaces. It seems worth while to provide a similar weakening of associativity in an algebraic setting.

Let Λ be a commutative ring. We assume familiarity with the concept of a differential graded augmented Λ-module (abbreviated: DGA Λ-module), [3 , p.205]. If (R,d) is a DGA Λ-module with $\Lambda \subset R_0$ as a sub DGA Λ-module, we will denote by $\bar{R}$ the DGA Λ-module R/Λ.

<u>Definition</u> 2.1. An A(n)-<u>algebra</u> $(R, d, \{h_i\})$ over a ring Λ is a DGA Λ-module $(R = \sum R_q, d)$ with functions

$$h_i : (\otimes^i \bar{R})_q \longrightarrow R_{q+i-2} \qquad \text{for } 2 \le i \le n$$

such that for $a = u_1 \otimes \cdots \otimes u_i \in \otimes^i \bar{R}$ we have

$$(2.2) \quad dh_i(a) = (-1)^i h_i\Big(\sum_{k=1}^{i} \operatorname{sgn} a_k\, u_1 \otimes \cdots \otimes du_k \otimes \cdots \otimes u_i\Big) + \sum_{\substack{r+s=i+1 \\ 1 \le k \le r \\ 2 \le r < i}} \operatorname{sgn} \bar{\varepsilon}(s,k,a)\, h_r(u_1 \otimes \cdots \otimes h_s(u_k \otimes \cdots \otimes u_{k+s-1}) \otimes \cdots \otimes u_i)$$

where $a_k = \sum_{j=1}^{k-1} \dim u_j$ and $\operatorname{sgn} x = (-1)^x$ and

$\bar{\varepsilon}(s,k,a) = k(s-1) + s(i + a_k)$.

Remark. If we adopt the convention $h_1 = d$, then equation (2.2) can be written

$$(2.3) \quad \sum_{\substack{r+s=i+1 \\ 1 \le k \le r}} \operatorname{sgn} \bar{\varepsilon}(s,k,a)\, h_r(1 \otimes \cdots \otimes h_s \otimes \cdots \otimes 1)(a) = 0,$$

h_s being in the k-th position in the general term.

Remark 2.4. If $(R, d, \{h_i,\ 2 \le i \le n\})$ is an $A(n)$-algebra, then $(R, d, \{h_i,\ 2 \le i \le n-1\})$ is an $A(n-1)$-algebra.

Definition 2.5. $(R, d, \{h_i,\ 2 \le i < \infty\})$ is an $A(\infty)$-algebra if for each n $(R, d, \{h_i,\ 2 \le i \le n\})$ is an $A(n)$-algebra.

Example 2.6. Any associative algebra R is an $A(\infty)$-algebra ($h_i = 0$ for $i > 2$).

Example 2.7. Any non-associative algebra R is an $A(2)$-algebra.

Example 2.8. We can describe an $A(3)$-algebra as a chain-homotopy-associative algebra. For example, the cubical singular chains on a homotopy associative H-space form an $A(3)$-algebra. More generally we have

Theorem 2.9. Let $(X; \{h_i\})$ be an A_n-space with structure maps $h_i : K_i \times X^i \to X$ for $2 \le i \le n$. Let $(C(X), d)$ denote the singular chain complex on X with coefficients in Λ for which the acyclic models are arbitrary finite products of the complexes K_i. Then the maps h_i induce functions $h_i : \otimes^i C(X) \to C(X)$ such that $(C(X), d, \{h_i\})$ is an $A(n)$-algebra

(There should be no confusion between the Λ-module maps h_i and the maps $h_i : K_i \times X^i \to X$ since the former will always appear with chains as arguments, the latter with points.)

Proof. Let $\kappa \in C(K_i)$ denote the singular (i-2)-chain corresponding to the identity map $1: K_i \to K_i$. We refer to this chain as the standard chain on K_i. For $u_j \in C(X)$, $j=1,\ldots,i$, define

(2.10) $h_i(u_1 \otimes \cdots \otimes u_i) = h_{i\#}(\kappa \times u_1 \times \cdots \times u_i)$.

If $\sum_{j=1}^{i} \dim u_j = q$, then $h_i(u_1 \otimes \cdots \otimes u_i)$ is in $C_{q+i-2}(X)$ as desired. We verify further

Lemma 2.11. Condition 2.2. is satisfied.

Proof. Recall that the boundary of K_i is decomposed into cells $\varphi^k(K_r \times K_s)$ where $r+s=i+1$, $1 \leq k \leq r$ and $\varphi^k: K_r \times K_s \to \partial K_i$ is an imbedding. We have

(2.12) $dh_i(u_1 \otimes \cdots \otimes u_i) = dh_{i\#}(\kappa \times u_1 \times \cdots \times u_i)$

$$= h_{i\#}(d\kappa \times u_1 \times \cdots \times u_i) + (-1)^i h_{i\#}(\kappa \times d(u_1 \times \cdots \times u_i))$$

$$= (-1)^i h_i\Big(\sum_k (-1)^{a_k} u_1 \otimes \cdots \otimes du_k \otimes \cdots \otimes u_i\Big)$$

$$+ \sum_{k,r} \omega(k,r)\, h_{i\#}(\varphi^k_{\#}(\mu \times \nu) \times u_1 \times \cdots \times u_i)$$

where μ and ν are the standard chains on K_r and K_s respectively and $\omega(k,r)$ is a sign determined as follows:

In [21], the complex K_i is given explicitly as a subset of I^{i-2}. The sign $\omega(k,r)$ is the sign associated with the corresponding face of the cube in the cubical singular theory [19 , p.440]. $\varphi^1(K_r \times K_s)$ corresponds to the face $t_{s+1}=0$, hence $\omega(1,r) = s+1$. For $k > 1$, $\varphi^k(K_r \times K_s)$ lies on the face $t_{k-1}=1$, hence $\omega(k,r)=k$ for $k > 1$.

By applying condition II, 1.2.b, we see that all the terms of (2.2) do, in fact, appear in dh_i. As for the signs, a few remarks about orientations are essential. We have already assumed with Serre that the cube is oriented by the ordering $(t_1,\ldots,t_i)$ of coordinates.

Let $f: X \times Y \longrightarrow Z$ and $g: Y \times X \longrightarrow Z$ be maps such that $f = gT$ where $T: X \times Y \longrightarrow Y \times X$ is given by $T(x,y) = (y,x)$. Let $u \in C_p(X)$, $v \in C_q(Y)$, then $f_{\#}(u \times v) = g_{\#} T_{\#}(u \times v) = (-1)^{pq} g_{\#}(v \times u)$.

In particular, $h_i \mid \partial K_i \times X^i$ is expressed in terms of a map of $K_r \times X^{k-1} \times K_s \times X^{i-k+1}$. We must now make use of φ^k as explicitly given in [21].

Case 1. $k=1$. $\varphi^k(\rho, \sigma) = (\sigma, \rho)$ and hence

$h_i(\sigma, \rho, x_1,\ldots,x_i) =$

$h_r(\rho, h_s(\sigma, x_1,\ldots,x_s), x_{s+1},\ldots,x_i)$. We obtain

$h_{i\#}(\varphi^k_{\#}(\mu \times \nu) \times u_1 \times \cdots \times u_i) =$

$(-1)^{sr} h_{r\#}(\mu \times h_{s\#}(\nu \times u_1 \times \cdots \times u_s) \times u_{s+1} \times \cdots \times u_i)$.

As desired, $\bar{\varepsilon}(s,1,a) = s-1+si$ is congruent mod 2 to $\omega(k,r) + sr = s-1+sr$ since $s(s+1)$ is always even and $r+s = i+1$.

Case 2. $k > 1$. Let $\sigma = (t_k,\ldots,t_{k+s-3}) \in K_s$ and $\rho = (t_1,\ldots,t_{k-2}, 2^{i-r}t_k\cdots t_{k+s-2}, t_{k+s-1},\ldots,t_{i-2}) \in K_r$ then $\varphi^k(\rho,\sigma) = (t_1,\ldots,t_{k-2},1,t_k,\ldots,t_{i-2})$.

and $h_i(\varphi^k(\rho,\sigma), x_1,\ldots,x_i) =$

$$h_r(\rho, x_1,\ldots,x_{k-1}, h_s(\sigma, x_k,\ldots,x_{k+s-1}), x_{k+s},\ldots,x_i).$$

Since σ must be moved past $x_1,\ldots,x_{k-1}$ and φ^k interchanges σ and $t_{k+s-2},\ldots,t_{i-2}$, we have

$$h_{i\#}(\varphi^k_{\#}(\mu\times\nu)\times u_1\times\cdots\times u_i) =$$
$$\operatorname{sgn} s(i-k-s+1+a_k)\, h_{r\#}(\mu\times u_1\times\cdots\times u_{k-1}\times h_{s\#}(\nu\times u_k\times\cdots$$
$$\cdots\times u_{k+s-1})\times u_{k+s}\times\cdots\times u_i).$$

As desired, $\bar{\varepsilon}(s,k,a) = k(s-1) + s(i+a_k)$ is congruent mod 2 to $k + s(i-k-s+1+a_k)$.

This completes the proof of Lemma 2.11 and hence the proof of Theorem 2.9.

3. Stacks

Recall that Massey defines his spectral sequences in terms of an abstract sheaf $\mathcal{G}$ on an abstract cell complex K. In order to carry out a similar definition for A(n)-algebras, we find we can dispense with much of the structure of an abstract cell complex.

<u>Definition</u> 3.1. A <u>graded set</u> L is an abstract set together with a decomposition $L = \bigcup_p L_p$ where p runs over the integers. If $x \in L_p$, we say $\deg x = p$. The dimension of L, dim L, is $\max \{p \mid L_p \neq \emptyset\}$.

<u>Definition</u> 3.2. A Λ -<u>stack</u> $\mathcal{G}$ on a graded set $L = \bigcup_p L_p$ consists of a triple function (G,D,I) such that

1) for $\sigma \in L$, $(G(\sigma), D(\sigma))$ is a DGA Λ-module

2) for each pair $\tau, \sigma \in L$ such that $\deg \tau \leq \deg \sigma$, $I(\tau, \sigma) : G(\sigma) \longrightarrow G(\tau)$ is a map of Λ -modules.

3) If $\deg \tau = \deg \sigma$, then

   a) $I(\sigma, \sigma) = D(\sigma)$

   b) $I(\tau, \sigma) = 0$ otherwise.

Define the r-dimensional chain group

$$\mathcal{C}_r(L; \mathcal{G}) = \sum_{\substack{p+q=r \\ \sigma \in L_p}} G(\sigma)_q .$$ Define

$\partial : \mathcal{C}_r(L; \mathcal{G}) \longrightarrow \mathcal{C}_{r-1}(L; \mathcal{G})$ by

$$\partial(a) = \sum I(\tau, \sigma)(a) \text{ for any } a \in G(\sigma) \tag{3.3}$$

the summation being over all $\tau \in L$ such that $\deg \tau \leq \deg \sigma$.

4) $\partial\partial = 0$.

Next we present certain important examples of Λ -stacks.

Definition 3.4. The Λ -stack $\mathcal{G}(L,R)$ on an ordered simplicial complex L canonically associated with an A(n)-algebra $(R, d, \{h_i\})$, $n \geq \dim L + 2$.

We can regard L as a graded set by taking L_p to be the set of p-simplices of L. Define $\mathcal{G}(L,R)$ as follows:

1) For any p-simplex σ of L,

$$(G(\sigma), D(\sigma)) = (\otimes^{p+2} R, (-1)^p d_{\otimes})$$

where $d_{\otimes}$ is the tensor product differential, i.e.

$$D(\sigma)(u_0 \otimes \cdots \otimes u_{p+1}) = \sum_{0 \leq k \leq p+1} (-1)^{p+b_k} u_0 \otimes \cdots \otimes du_k \otimes \cdots \otimes u_{p+1} \tag{3.5}$$

where $b_k = \sum_{j=0}^{k-1} \dim u_j$, $u_j \in R$.

2) For any ordered p-simplex $\sigma = (v_0, \ldots, v_p)$ and (p-q+1)-simplex $\tau = (v_0, \ldots, v_{k-1}, v_{k+q-1}, \ldots, v_p)$, the function $I(\tau, \sigma) : G(\sigma) \longrightarrow G(\tau)$ is given by

$$I(\tau, \sigma)(u_0 \otimes \cdots \otimes u_{p+1}) =$$

$$\operatorname{sgn} \varepsilon(q,k,a)\, u_0 \otimes \ldots \otimes h_q(u_k \otimes \cdots \otimes u_{k+q-1}) \otimes \cdots \otimes u_{p+1} \tag{3.6}$$

where $\varepsilon(q,k,a) = k(q-1) + q(p+b_k)$. This formula is extended to all of $\otimes^{p+2}R$ by linearity. For any other $(p-q+1)$-simplex τ, $I(\tau, \sigma) = 0$.

Theorem 3.7. $\mathcal{G}(L,R)$ <u>is a</u> Λ -<u>stack on</u> L.

Proof. We must verify that $\partial\partial = 0$. Let σ be a p-simplex of L and let $I(\tau, \sigma)$ be non-trivial, i.e. $\sigma = (v_0, \ldots v_p)$ and $\tau = (v_0, \ldots, v_{k-1}, v_{k+q-1}, \ldots, v_p)$. If $\sigma \neq \tau$ then $\partial(a) \mid G(\tau)$ is given by (2.6).

If $\sigma = \tau$, we have

$$\partial(a) \mid G(\sigma) = \sum (-1)^{p+b_k} u_0 \otimes \cdots \otimes h_1 u_k \otimes \cdots \otimes u_{p+1}$$

with the convention $h_1 = d$. (We omit limits of summation where they are fairly obvious.) Applying ∂ again we will be concerned with all simplices $\rho \in L$ such that there exists a simplex $\tau \in L$ with $I(\rho, \tau)$ and $I(\tau, \sigma)$ non-trivial. These are of two types. In each case ρ is obtained from τ by omitting m-1 consequtive vertices.

Type 1) ρ can be obtained from σ by omitting $v_i, \ldots, v_{i+m-2}, v_k, \ldots, v_{k+q-2}$ with $i \neq k+q-1$, $k \neq i+m-1$. Thus $\partial\partial(a) \mid G(\rho)$ has terms from $\partial(a) \mid G(\tau)$ and $\partial(a) \mid G(\tau')$ where $\tau = (v_0, \ldots, v_{k-1}, v_{k+q-1}, \ldots, v_p)$ and $\tau' = (v_0, \ldots, v_{i-1}, v_{i+m-1}, \ldots, v_p)$

Without loss of generality, we assume $i < k$. From $\partial(a) \mid G(\tau)$ we obtain

(3.8) $u_0 \otimes \cdots \otimes h_m(u_i \otimes \cdots \otimes u_{i+m-1}) \otimes \cdots \otimes h_q(u_k \otimes \cdots$

$\cdots \otimes u_{k+q-1}) \otimes \cdots \otimes u_{p+1}$

with coefficient $\operatorname{sgn} \varepsilon(m,i,b) \operatorname{sqn} \varepsilon(q,k,a)$

where $b = u_0 \otimes \cdots \otimes h_q (u_k \otimes \cdots \otimes u_{k+q-1}) \otimes \cdots \otimes u_{p+1}$

while from $\partial(a) \mid G(\tau')$ we obtain (3.8) with coefficient

$\operatorname{sgn} \varepsilon(q,k-m+1,c) \operatorname{sgn} \varepsilon(m,i,a)$ where

$c = u_0 \otimes \cdots \otimes h_m (u_i \otimes \cdots \otimes u_{i+m-1}) \otimes \cdots \otimes u_{p+1}$.

We have $\varepsilon(m,i,b) = i(m-1)+m(p-q+1+b_i)$,

$$\varepsilon(q,k,a) = k(q-1)+q(p+b_k),$$

$$\varepsilon(q+1,k-m,c) = (k-m+1)(q-1)+q(p-m+1+b_k+m-2)$$

since $\dim h_m (u_i \otimes \cdots \otimes u_{i+m-1}) = \sum_{j=i}^{i+m-1} \dim u_j + m-2$, and

$$\varepsilon(m,i,a) = i(m-1)+m(p+b_i).$$

The total coefficients differ by 1; the two occurrences of (3.8) cancel.

Type 2) ρ can be obtained from σ by omitting $v_i, v_{i+1}, \ldots, v_{i+q+m-3}$.

Now $\partial\partial(a) \mid G(\rho)$ has many terms, one from each of the terms $\partial(a) \mid G(\tau)$ where τ is any simplex obtained from σ by omitting $v_k, \ldots, v_{k+q-2}$ with $i \leq k < i+m$. We can write $\partial\partial(a) \mid G(\rho)$ as a sum of terms

(3.9) $u_0 \otimes \cdots \otimes h_m (u_i \otimes \cdots \otimes h_q (u_k \otimes \cdots \otimes u_{k+q-1}) \otimes \cdots$

$$\cdots \otimes u_{i+q+m-2}) \otimes \cdots \otimes u_{p+1}$$

each with coefficient $\operatorname{sgn} \varepsilon(m,i,b) \operatorname{sgn} \varepsilon(q,k,a)$ where b is as above.

We observe

$$(3.10)\quad \varepsilon(m,i,b) + \varepsilon(q,k,a)$$
$$= i(m-1) + m(p-q+1+b_i) + k(q-1) + q(p+b_k)$$
$$(3.11)\quad = i(m+q) + (m+q-1)(p+b_i) + p + b_i + (k-i)(q-1) + q(b_k-b_i) - m(q-1)$$

which mod 2 is congruent to

$$(3.12)\quad i(m+q-2) + (m+q-1)(p+b_i) + p-q-m+b_i + (k-i+1)(q-1) + q(m+q-1+b_k-b_i) + 1$$
$$= \varepsilon(m+q-1,i,a) + \varepsilon(1,i,u_o \otimes \cdots \otimes h_{m+q-1}(u_i \otimes \cdots \otimes u_{i+m+q-2}) \otimes \cdots \otimes u_{p+1}) + \bar{\varepsilon}(q,k-i+1,u_i \otimes \cdots \otimes u_{i+m+q-2}) + 1.$$

Hence if we sum all the terms of the form (3.9), we will obtain

$$\pm \sum \operatorname{sgn} \bar{\varepsilon}(q,k-i+1,u_i \otimes \cdots \otimes u_{i+m+q-2})\ (3.9).$$

Applying (2.5) we see that this is zero. Thus $\partial\partial = 0$.

<u>Definition</u> 3.13. The n-<u>plex</u> ∇^n is the sequence of integers $(-1,0,1,\ldots,n)$ regarded as a graded set, i.e. deg p = p.

<u>Definition</u> 3.14. <u>The</u> Λ -<u>stack</u> $\mathcal{G}(\nabla^n,R)$ <u>canonically associated with an</u> A(n+2)-<u>algebra</u> $(R,d,\{h_i\})$ is defined by:

1) $(G(p),D(p)) = (\otimes^{p+2} R, (-1)^p d_{\otimes})$

2) $I(p-q+1,p)(a) =$

$$(3.15)\quad \sum_{1\leq k\leq p-q+2} \operatorname{sgn} \varepsilon(s,k,a) u_0 \otimes \cdots \otimes h_q(u_k \otimes \cdots \otimes u_{k+q-1}) \otimes \cdots \otimes u_{p+1},$$

for $a = u_0 \otimes \cdots \otimes u_{p+1} \in G(p)$.

Theorem 3.16. $\mathcal{G}(\nabla^n, R)$ <u>is a</u> Λ -<u>stack on</u> ∇^n.

Proof. Let Δ^n denote the standard simplicial complex of the n-simplex. Consider $j : \Delta^n \longrightarrow \nabla^n$ defined by $j(\sigma) = \dim \sigma$ for $\sigma \in \Delta^n$. The function j induces a map $j = j(\mathcal{G}) : \mathcal{G}(\Delta^n, R) \to \mathcal{G}(\nabla^n, R)$, i.e. $j(\mathcal{G}) \mid G(\sigma^p)$ is the identity $1 : G(\sigma^p) \longrightarrow G(p)$. We see that

$D(\sigma)j = jD(\sigma)$ and $I(p+q-1,p)j =$

$$= \sum_{\tau^{p+q-1} \in \Delta^n} jI(\tau^{p+q-1}, \sigma^p), \text{ hence } \partial j = j\partial .$$

Since j is onto and $\partial\partial j = \partial j \partial = j\partial\partial = 0$, we conclude that $\partial\partial = 0$ in $\mathcal{G}(\nabla^n, R)$.

4. Definition of the spectral sequences

We will henceforth use C to denote a functor from the category of spaces and maps to the category of DGA Λ-modules and maps having the following properties:

A. If $(X; \{h_i\})$ is an arcwise connected A_n-space, the maps $h_i : K_i \times X^i \to X$ induce functions $h_i : \otimes^i \bar{C}(X) \longrightarrow \bar{C}(X)$ so as to form an A(n)-algebra $(\bar{C}(X), d, \{h_i\})$.

B. $H_*(C(X)) \approx H_*(X)$. [Λ being specified, we will use $H_*(X)$ to denote singular homology with coefficients in Λ.]

Example 4.1. By Theorem 2.9, an example of such a functor is the singular complex based on acyclic models which are arbitrary finite products of the complexes K_i. (The functions h_i pass to $\bar{C}$ because of condition II. 1.2.c.)

Example 4.2. $H_*(X)$ with trivial differential and the usual augmentation is a DGA Λ-module satisfying B. If $(X; \{h_i\})$ is an A_n-space, $\bar{H}_*(X)$ with the induced Pontrjagin multiplication is an A(n)-algebra (in fact, $\bar{H}_*(X)$ is associative for $n > 2$).

The chain groups $\mathcal{C}_r(L; \mathcal{G})$ defined in (3.2.3) can be given a bigrading when $\mathcal{G} = \mathcal{G}(L, \bar{C}(X))$, i.e.

$$\mathcal{C}_r(L; \mathcal{G}) = \sum_{p+q=r} \mathcal{C}_{p,q}(L; \mathcal{G}) \text{ where}$$

$$\mathcal{C}_{p,q}(L; \mathcal{G}) = \sum_{\sigma \in L_p} G(\sigma)_q .$$

We refer to q as the internal, p as the simplicial degree. Let $\partial_1 = \partial - D(\sigma)$. The DGA Λ-module $\mathcal{E}(\mathcal{G}(L,\bar{C}(X)))$ $= \sum_r \mathcal{E}_r(L; \mathcal{G}(L,\bar{C}(X)))$ is not a double complex in the usual sense for although ∂_1 and $D(\sigma)$ lower the total degree by 1 and $D(\sigma)$ lowers the internal degree by 1, ∂_1 does not lower the simplicial degree by a fixed amount. It is possible however to obtain one of the standard spectral sequences as follows:

Filter $\mathcal{E}(\mathcal{G}(L,\bar{C}(X)))$ by the simplicial degree:

$$F_p(\mathcal{E}(\mathcal{G}(L,\bar{C}(X))) = \sum_{i \leq p} \mathcal{E}_{i,j}(L; \mathcal{G}).$$

This filtration is compatible with the total differential ∂, hence by [4, ch. XV] there results a spectral sequence

$$\left\{E^i(X,L,\Lambda),\ d_i\right\},\ i \geq 1.$$

The successive differentials d_i will yield the homology invariants promised.

Remark. If X is an associative H-space, the above discussion reduces to one which parallels that given by Massey, as we have sketched in section 1.

5. Identification of the first term and definition of the invariants

We identify $E^1(X,L,\Lambda)$ in terms of $H_*(X)$ and the differential d_1 in terms of the Pontrjagin product.

__Definition__. 5.1. Let L be ∇^n or an ordered simplicial complex. Let $(X;\{h_i\})$ be an A_n-space with $n \geq \dim L + 2$. __The__ Λ __-stack__ $\mathcal{G}''(\bar{C}(X),L)$ __on__ L __specially associated with__ $\bar{C}(X)$ is defined by

1) For $\sigma \in L_p$, $(G''(\sigma), D''(\sigma)) = (\bar{C}(X^{(p+2)}), (-1)^p d)$ where d is the differential on $\bar{C}(X^{(p+2)})$.

2) Consider $\mathcal{G}(L,\bar{C}(X))$ with $I(\tau,\sigma)$ as given by (3.15) [respectively (3.6)]. $I''(\tau,\sigma)(a)$ for any $a \in \bar{C}(X^{(p+2)})$ is obtained by replacing

(5.2) $\operatorname{sgn} \varepsilon(q,k,a)\, u_0 \otimes \cdots \otimes h_q(u_k \otimes \cdots \otimes u_{k+q-1}) \otimes \cdots \otimes u_{p+1}$

by

(5.3) $\operatorname{sgn}(k+s(p+k))\quad f^k_{q\#}(\kappa \times a)$ where κ is the standard chain[1] of K_q and $f^k_q : X^{p+2} \to X^{p+3-q}$ is defined by

$$f^k_q(\rho, x_0, \ldots, x_{p+1}) =$$
$$(x_0, \ldots, x_{k-1}, h_q(\rho, x_k, \ldots, x_{k+q-1}), x_{k+q}, \ldots, x_{p+1})$$

for $\rho \in K_q$. Otherwise define $I(\tau,\sigma)$ to be the zero homomorphism.

1. I.e. a generator of $C_{q-2}(K_q)$ which under the isomorphism $H_*(C(K_q,\partial K_q)) \approx H_*(K_q,\partial K_q)$ corresponds to the class of $1: K_q \to K_q$.

Theorem 5.4. <u>$\mathcal{G}''(\bar{C}(X),L)$ is a Λ-stack.</u>

Proof. To verify that $\partial\partial = 0$ one need only repeat the proof of Theorem 3.7 substituting throughout (5.3) for (5.2), being careful to keep track of the changes in signs indicated in section 2.

Theorem 5.5. <u>Let L be ∇^n or an ordered simplicial complex. Let $(X; \{h_i\})$ be an arcwise connected A_n-space. Let $E^1(X,L,\Lambda)$ be the first term of the spectral sequence derived from $\mathcal{C}(\mathcal{G}(L,\bar{C}(X)))$ Then</u>

$$E^1(X,L,\Lambda) \approx \mathcal{C}(\mathcal{G}''(\bar{H}_*(X),L)).$$

[If Λ is a field, this is isomorphic with $\mathcal{C}(\mathcal{G}(L,\bar{H}_*(X)))$.]

Proof. Theorem 5.5 follows by using the chain equivalence between $\bar{C}(X^{(p+2)})$ and $\otimes^{p+2}\bar{C}(X)$ and applying certain facts about the spectral sequence of a double complex. Of course $\mathcal{C}(\mathcal{G}(L,\bar{C}(X)))$ is not a double complex, but the relevant remarks still apply [4, p.331, ¶ 2-3]. F_p/F_{p-1} may be identified with $\sum_{\sigma^p \in L} \otimes^{p+2}\bar{C}(X)$ and the homology with respect to the product differential is the product homology $\bar{H}_*(X^{(p+2)})$. The differential d_1 may be identified with the boundary operator

$$\partial : \mathcal{C}_r(L;\mathcal{G}'') \to \mathcal{C}_{r-1}(L;\mathcal{G}'').$$

<u>Definition</u> 5.6. Let $a \in E^1(X,L,\Lambda)$. By induction, we suppose defined "$b \in E^r(X,L,\Lambda)$ is represented by a."

Suppose $d_r(b) = 0$ then <u>the class of b in</u> $E^{r+1}(X,L,\Lambda)$ <u>is represented by</u> a. We say that a <u>persists</u> to E^r if there exist $b \in E^r$ such that b is represented by a. If b is represented by a we write $b = \{a\}$. We say that a persists non-trivially if b is non-zero.

A better idea of the content of Theorem 5.5 can perhaps be obtained by rephrasing it at greater length. Let $E^r_{p,q} = E^r_{p,q}(X,L,\Lambda)$. Theorem 5.5 says that $E^1_{p,q}$ consists of several copies of $\bar{H}_q(X^{(p+2)})$, indexed by the elements of L_p.

<u>Example</u> 5.7. Let L be an ordered simplicial complex. As usual we can regard $\otimes^{p+2} \bar{H}_*(X)$ as a subgroup of $\bar{H}_*(X^{(p+2)})$. Suppose that a is a class of the form $u_0 \otimes \cdots \otimes u_{p+1}$ belonging to a copy of $\bar{H}_q(X^{(p+2)})$ corresponding to some p-simplex σ. The class $d_1(a)$ is a set of classes b_j belonging to the various copies of $\bar{H}_q(X^{(p+1)})$. The classes b_i are zero except for those which belong to copies of $\bar{H}_q(X^{(p+1)})$ indexed by faces of σ.

<u>Special Example</u> 5.8. Let $L = \Delta^n$, the standard simplicial complex of the standard simplex. The number of p-simplices of Δ^n is the binomial coefficient (p+1, n-p). Hence $E^1_{p,q}$ consists of (p+1, n-p) copies of $\bar{H}_q(X^{(p+2)})$ and, ordering these in some way, we can write an element of

$E^1_{p,q}$ as a (p+1, n-p)-tuple $(v_1,\dots,v_j)$ $(j=(p+1,n-p))$ of classes $v_k \in \bar{H}_q(X^{(p+2)})$. In particular, $E^1_{n,q} = \bar{H}_q(X^{(n+2)})$ and $E^1_{-1,q} = \bar{H}_q(X)$. Suppose that the class $a \in \bar{H}_q(X^{(n+2)})$ is of the form $u_0 \otimes \cdots \otimes u_{n+1}$, then $d_1(a)$ can be written as the set $(v_0,\dots,v_n)$ where $v_k = (-1)^k u_0 \otimes \cdots \otimes u_k u_{k+1} \otimes \cdots \otimes u_{n+1}$, the Pontrjagin product being denoted by juxtaposition.

Suppose further $d_i(a)$ is defined and vanishes for $i \leq n$.

The <u>stack operation</u> $\langle u_0,\dots,u_{n+1}\rangle$ is defined as the set

$$\{ z \in \bar{H}_{q+n}(X) \mid \{z\} = d_{n+1} \{u_0 \otimes \cdots \otimes u_{n+1}\}\}.$$

The notation $\langle\ ,\dots,\ \rangle$ reflects the analogy with the Massey cohomology operations.

<u>Special example</u> 5.9. Let L be ∇^n, then $E^1_{p,q} = \bar{H}_q(X^{(p+2)})$.

For a of the form $u_0 \otimes \cdots \otimes u_{n+1}$ we have $d_1(a) =$

$= \sum (-1)^k u_0 \otimes \cdots \otimes u_k u_{k+1} \otimes \cdots \otimes u_{n+1}$. If a persists to E^{n+1}, the <u>bar operation</u> $]u_0,\dots,u_{n+1}[$ is defined as the set $\{ z \in \bar{H}_{q+n}(X) \mid \{z\} = d_{n+1}\{u_0 \otimes \cdots \otimes u_{n+1}\}\}$.

The adjective "bar" reflects the fact that in the associative case this spectral sequence coincides with part of one which is associated with the bar construction [3].

<u>Comparison of the bar and stack operations</u>. In general, the invariants derived from Δ^n and ∇^n are not the same, although they are related by means of the map j described in Theorem 3.16.

<u>Example</u> 5.10. Let $n=1$, $\Lambda = Z_3$, $X = K(Z,2)$. From [2], we conclude that $H_*(Z,2;\ Z_3)$ can be written as a tensor product $\bigotimes_i T_3(u_i,\ 2\cdot 3^i)$ where each factor $T_3(u_i,\ 2\cdot 3^i)$ is a truncated polynomial algebra over Z_3 of height 2 on a single generator u_i in dimension $2\cdot 3^i$.

Let $u = u_0 \in \bar{H}_2(X)$. Let $a \in \bar{H}_8(X^{(3)})$ be the class $u \otimes u \otimes u^2 + u^2 \otimes u \otimes u$ considered as a class in $E^1(X, \Delta^1, Z_3)$. Let b denote the same class considered as belonging to $E^1(X, \nabla^1, Z_3)$, i.e. $b = j_*(a)$. Then $d_1(a) \neq 0$ while $d_1(b) = 0$. As described in the examples, we can write

$$d_1(a) = (u^2 \otimes u^2 + u^3 \otimes u,\ -u \otimes u^3 - u^2 \otimes u^2)$$

which is non-zero since $u^2 \otimes u^2$ is non-zero in $\bar{H}_8(X \# X)$. However

$$d_1(b) = (u^2 \otimes u^2 + u^3 \otimes u - u \otimes u^3 - u^2 \otimes u^2) = 0$$

since $u^3 = 0$.

6. Naturality of the invariants

Consider the concept of an A_n-map as given in Part II. It is is relevant to the present study since

Theorem 6.1. <u>Let L be ∇^{n-2} or an ordered simplicial complex of dimension n-2. Let</u> $f : (Y; \{h_i\}) \longrightarrow (X; \{g_i\})$ <u>be an A_n-map. Then f induces a map of spectral sequences</u> $\{f^i_*\} : \{E^i(Y,L,\Lambda), d_i\} \rightarrow \{E^i(X,L,\Lambda), d_i\}$.

Proof. The singular theory C being fixed, let $\mathcal{C}(L; X)$ denote $\mathcal{C}(\mathcal{G}(L, C(X)))$. By [4, p. 321] it is sufficient to show that an A_n-map f induces $\bar{f} : \mathcal{C}(L; Y) \rightarrow \mathcal{C}(L; X)$ which is a map of filtered DG Λ-modules. Denote the A(n)-structures on $\bar{C}(X)$ by g_i and those on $\bar{C}(Y)$ by h_i. We define maps of Λ-modules $G_i : [\otimes^i \bar{C}(Y)]_q \rightarrow \bar{C}(X)_{q+i-1}$ such that

(6.2) $G_1 = f_\#$

(6.3) $$\sum \operatorname{sgn} \bar{\varepsilon}(s,k,a)\, G_r(1 \otimes \cdots \otimes h_s \otimes \cdots \otimes 1)(a) + \sum_{\mu} \operatorname{sgn} \bar{\varepsilon}(\mu, a)\, g_r(G_{p_1} \otimes \cdots \otimes G_{p_r})(a) = 0$$

where the second summation extends over all partitions $\mu = (p_1, \ldots, p_r)$ of i and $\bar{\varepsilon}(\mu, a) = \sum_k (p_k+1)(q_k + i + \sum_j^{q_k} \dim u_j + 1) + \sum_{k \neq j} (p_k+1)(p_j+1) + 1$ with the conventions $g_1 = d_X$, $h_1 = d_Y$ and $a = u_1 \otimes \cdots \otimes u_i$.

Theorem 6.4. <u>Let $G_i : I_i \times K_i \times Y^i \to X$, $i = 1,\dots,n$ be as in the definition of A_n-map, then Λ-homomorphisms G_i satisfying (6.2, 6.3) are given by</u>

(6.5) $\quad G_i(u_1 \otimes \cdots \otimes u_i) \;=\; G_{i\#}(\iota \times \kappa \times u_1 \times \cdots \times u_i)$

(where ι is the standard chain on I_i and κ the standard chain on K_i).

Proof. The method of analysis used in Theorem 2.9 applies equally well here to show that (6.5) yields the signs of (6.3).

We now construct the map $\bar{f}$. Let L be an ordered simplicial complex and $\sigma = (v_0,\dots,v_p)$ a p-simplex. Let $\rho = (p_0,\dots,p_m)$ be a partition of p+2. Let $q_j = \sum_{k=0}^{j-1} p_k$ for $j = 1,\dots,m$. Let σ_ρ denote the (m-1)-face of with vertices $(v_{q_1-1},\dots,v_{q_{m-1}-1})$. This establishes a one-to-one correspondence between the partitions of p+2 and faces of σ.

For $a = u_0 \otimes \cdots \otimes u_{p+1} \in G_Y(\sigma)$, define

(6.6) $\quad \bar{f}(a) \,\big|\, G_X(\sigma_\rho) =$

$$\operatorname{sgn} \varepsilon(\rho, a)\, (G_{p_0} \otimes \cdots \otimes G_{p_m})\,(a)$$

where $\varepsilon(\rho, a) = \sum_k (p_k+1)\left(p + \sum_j^{q_k} (\dim u_j + 1) + 1\right) + \sum_{k \neq j} (p_k+1)(p_j+1) + 1$

with

In particular, note that

(6.7) $\quad \bar{f}(a) \,\big|\, G_X(\sigma) = f_\#(u_0) \otimes \cdots \otimes f_\#(u_{p+1})$.

Clearly $\bar{f}$ respects the filtration.

Lemma 6.8. If L is an ordered simplicial complex, then $\partial\bar{f} = \bar{f}\partial$.

Proof. Let $\mathfrak{p} = (p_o, \ldots, p_m)$ be an arbitrary partition of p+2 and $\sigma_{\mathfrak{p}}$ the corresponding face of σ. We wish to compare $\partial\bar{f}(a) \mid G_X(\sigma_{\mathfrak{p}})$ with $\bar{f}\partial(a) \mid G_X(\sigma_{\mathfrak{p}})$. Now $\bar{f}\partial(a) \mid G_X(\sigma_{\mathfrak{p}})$ can be written as a sum of terms

(6.9) $\quad (G_{p_o} \otimes \cdots \otimes G_{p_{j-1}} \otimes G_t \otimes G_{p_{j+1}} \otimes \cdots \otimes G_{p_m})$ (b)

where $b = u_o \otimes \cdots \otimes h_q(u_k \otimes \cdots \otimes u_{k+q-1}) \otimes \cdots \otimes u_{p+1}$ with $q_j < k \leq q_{j+1}$ and $q+t = p_j+1$, each term having coefficient

(6.10) $\quad \operatorname{sgn}\,(\varepsilon(\mathfrak{p}', b) + \varepsilon(q,k,a))$

where $\mathfrak{p}' = (p_o, \ldots, p_{j-1}, t, p_{j+1}, \ldots, p_m)$.

For fixed j, we compare the summand of terms (6.9) with the corresponding terms of $\partial\bar{f}(a) \mid G_X(\sigma_{\mathfrak{p}})$. These can be written

(6. 11) $\quad (G_{p_o} \otimes \cdots \otimes G_{p_{j-1}} \otimes g_s \otimes G_{p_{j+1}} \otimes \cdots \otimes G_{p_m})$ (c)

where $c = u_o \otimes \cdots \otimes [(G_{r_1} \otimes \cdots \otimes G_{r_s})(u_{q_j+1} \otimes \cdots \otimes u_{q_{j+1}})] \otimes \cdots$
$\cdots \otimes u_{p+1}$

and $(r_1, \ldots, r_s)$ is any partition of p_j, each term of the summand having coefficient

(6.12) $\quad \operatorname{sgn}\,(\varepsilon(s,j,c') + \varepsilon(\bar{\mathfrak{p}}, a))$

where $c' = (G_{p_o} \otimes \cdots G_{p_{j-1}} \otimes G_{r_1} \otimes \cdots \otimes G_{r_s} \otimes G_{p_{j+1}} \otimes \cdots \otimes G_{p_m})(a)$

and $\bar{\beta} = (p_0, \ldots, p_{j-1}, r_1, \ldots, r_s, p_{j+1}, \ldots, p_m)$.

Now (6.10) is congruent mod 2 (cf. 3.10 - .12) to

(6.13) $\quad \bar{\varepsilon}(q, k-q_j, u_{q_j+1} \otimes \cdots \otimes u_{q_{j+1}})$ + some constant α

while (6.12) is congruent mod 2 to

(6.14) $\quad \bar{\varepsilon}((r_1, \ldots, r_s), u_{q_j+1} \otimes \cdots \otimes u_{q_{j+1}})$ + the same $\alpha + 1$.

Since the terms (6.9) are summed over all t,q such that $t+q = p_j+1$ and $q_j < k \leq q_{j+1}$ and the terms (6.11) are summed over all partitions of p_j, for arbitrary j we can apply (6.3) to obtain $(\partial\bar{f}(a) - \bar{f}\partial(a)) \mid G_X(\sigma_{\beta}) = 0$. Hence $\partial\bar{f} = \bar{f}\partial$ and the proof of Lemma 6.8 is complete.

To prove Theorem 6.1 when $L = \nabla^{n-2}$, we use the map $j: \mathcal{C}(\Delta^{n-2}; \;) \to \mathcal{C}(\nabla^{n-2}; \;)$. With $\bar{f}$ as defined by (6.6) we see that if $j(a) = j(b)$ for any $a, b \in \mathcal{C}(\Delta^{n-2}; \;)$, then $j\bar{f}(a) = j\bar{f}(b)$; hence we can define $\bar{\bar{f}}: \mathcal{C}(\nabla^{n-2}; Y) \to \mathcal{C}(\nabla^{n-2}; X)$ by the relation $\bar{\bar{f}}j = j\bar{f}$. Since $\partial j = j\partial$, we have $\partial\bar{\bar{f}}j = \partial j\bar{f} = j\partial\bar{f} = j\bar{f}\partial = \bar{\bar{f}}j\partial = \bar{\bar{f}}\partial j$. Since j is onto, we conclude that $\bar{\bar{f}}\partial = \partial\bar{\bar{f}}$. This completes the proof of Theorem 6.1.

The significance of Theorem 6.1 is that the invariants we have been discussing, the bar operations and stack operations in particular, are natural with respect to A_n-maps. That is

Corollary 6.15. Let $f: Y \to X$ be an A_n-map of $(Y;\{h_i\})$ into $(X;\{g_i\})$. Let $u_j \in H_*(X)$ for $j = 1,\dots,i \leq n$, then if $]u_0,\dots,u_i[$ is defined so is $]f_*(u_0),\dots,f_*(u_i)[$ and the latter contains $f_*]u_0,\dots,u_i[$.

This is merely an interpretation of Theorem 6.1 in terms of the operations, using the fact that $f_*^1(u_0 \otimes \cdots \otimes u_i) = f_*(u_0) \otimes \cdots \otimes f\ (u_i)$, which is implied by (6.7).

7. Identification of the term E^{∞}

In computing with spectral sequences it is helpful to have information about the term E^{∞}. For certain Λ-stacks we are able to identify this term.

Theorem 7.1. Let $(X; \{h_i\})$ be an arcwise connected A_n-space and $XP(n)$ the corresponding X-projective n-space (constructed in Part II), then

$$H_{\partial}(\mathcal{C}(\nabla^{n-2}; X)) \approx \bar{H}_*(XP(n)).$$

Proof. It is sufficient to define a chain equivalence $f : \mathcal{C}_r(\nabla^{n-2}; X) \to \bar{C}_{r+2}(XP(n))$. We are free to choose a suitable singular theory C. For our purposes, the appropriate choice is the singular theory based on acyclic models which are arbitrary finite products of the complexes K_i and the complexes ${}^j\Delta$. [We must specify the boundary of a singular chain $u : {}^p\Delta \longrightarrow X$. We have already specified the faces of ${}^p\Delta$ by "imbeddings" $\theta_\alpha : {}^{p-q-1}\Delta \times K_q \longrightarrow {}^p\Delta$. We define

$$(7.2)\quad du = \sum_{\alpha} \omega(\alpha)\ u \circ \theta_\alpha$$

where, for $\alpha = (k+1, \ldots, k+q-1)$, we have $\omega(\alpha) = \operatorname{sgn} k(q-1) + qp$. The necessary relation $dd = 0$ follows from the identifications II.2.4 and II.2.5, utilizing the usual change of sign in passing from $K_r \times K_s$ to $K_s \times K_r$.]

Let $u_i \in \bar{C}(X)$ for $i = 0,\ldots,p-1$.

Define

(7.3) $f(u_0 \otimes \cdots \otimes u_{p-1}) = \sigma^p \otimes u_0 \otimes \cdots \otimes u_{p-1}$

where σ^p denotes the standard chain on ${}^p\Delta$, i.e. the identity map $\sigma^p : {}^p\Delta \longrightarrow {}^p\Delta$. The right hand side of (7.3) is regarded as a chain on $XP(n)$ by including ${}^p\Delta \times X^p$ in $XP(p) \subset XP(n)$ as attached by $\gamma(p)$.

<u>Lemma</u> 7.4. The function f is a chain map.

Proof. We denote the boundary in $\mathcal{E}(\nabla^{n-2};X)$ by ∂, that in $\bar{C}(XP(n))$ by d. We must show $df = f\partial$.

Since $XP(p)$ is represented as ${}^p\Delta \times X^p \smile_{\gamma(p)} XP(p-1)$, for any chain of $\bar{C}(XP(n))$ of the form $\sigma^p \times u_0 \times \cdots \times u_{p-1}$, $u_j \in \bar{C}(X)$, we can write

$$d(\sigma^p \times u_0 \times \cdots \times u_{p-1}) = \gamma(p)_{\#}(d\sigma^p \times u_0 \times \cdots \times u_{p-1}) + (-1)^p \sigma^p \times d(u_0 \times \cdots \times u_{p-1})$$

where $\gamma(p)_{\#}(d\sigma^p \times u_0 \times \cdots \times u_{p-1}) =$

(7.5) $$\sum_{\alpha} \omega(\alpha)\,\gamma(p)_{\#}(\theta_{\alpha\#}(\sigma^{p-q-1} \times \kappa_q) \times u_0 \times \cdots \times u_{p-1})$$

$$= \sum_{\alpha \neq (0),(p)} \omega(\alpha)(-1)^{qb_k} \sigma^{p-q-1} \times u_0 \times \cdots \times h_{q\#}(\kappa_q \times u_k \times \cdots \times u_{k+q-1}) \times \cdots \times u_{p-1}$$

where $\alpha = (k+1,\ldots,k+q-1)$ and $b_k = \sum_{j=0}^{k-1} \dim u_j$.
$0 \le k \le p-q$

with $\mathcal{K}_q$ the standard chain on K_q.

Notice that when $\alpha = (0)$ or (p),

$$\gamma(p)_{\#}\ (\ \theta_{\alpha\#}(\sigma^{p-1}) \times u_0 \times \cdots \times u_{p-1}) = 0$$

since in this case the map $\gamma(p) \mid (\theta_\alpha \times 1)({}^{p-1}\Delta \times X^p)$ is projection onto ${}^{p-1}\Delta \otimes X^{p-1}$, omitting the first or last factor of X^p and we are factoring out $C_0(X)$, i.e. using $\bar{C}(X) = C(X)/C_0(X)$.

If we adopt the convention $K_1 =$ a point, $h_{1\#} = d$, we can write

(7.6) $d(\sigma^p \times u_0 \times \cdots \times u_{p-1}) =$

$$\sum \operatorname{sgn} \varepsilon(q,k,a)\ \sigma^{p-q-1} \times u_0 \times \cdots$$

$$\cdots \times h_{q\#}(\mathcal{K}_q \times u_k \times \cdots \times u_{k+q-1}) \times \cdots \times u_{p-1}.$$

We are now ready to compare df and $f\partial$. We have

$df(u_0 \otimes \cdots \otimes u_{p-1}) =$

(7.7) $\sum \operatorname{sgn} \varepsilon(q,k,a)\ \sigma^{p-q-1} \times u_0 \times \cdots$

$$\cdots \times h_{q\#}(\mathcal{K}_q \times u_k \times \cdots \times u_{k+q-1}) \times \cdots \times u_{p-1}$$

while $f\partial(u_0 \otimes \cdots \otimes u_{p-1}) =$

(7.8) $\sum \operatorname{sgn} \varepsilon(q,k,a)\ \sigma^{p-q-1} \times u_0 \times \cdots \times h_q(u_k \otimes \cdots$

$$\cdots \otimes u_{k+q-1}) \times \cdots \times u_{p-1}.$$

By the definition of h_q in (IV.2.10), we see that $df = f\partial$.

We now prove Theorem 7.1 by induction on n. Since f is a chain map we have a map of exact sequences

$$\begin{array}{ccccccc}
\cdots \to & H_q(\mathcal{C}^1) & \to & H_q(\mathcal{C}) & \to & H_q(\mathcal{C}, \mathcal{C}^1) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H_{q+2}(XP(n-1)) & \to & H_{q+2}(XP(n)) & \to & H_{q+2}(XP(n), XP(n-1)) & \to \cdots
\end{array}$$

where $\mathcal{C}^1 = \mathcal{C}(\nabla^{n-3}; X)$, $\mathcal{C} = \mathcal{C}(\nabla^{n-2}; X)$.

Now $\mathcal{C}^1$ is a direct summand of $\mathcal{C}$, i.e. $\mathcal{C}^1 = \sum_{i<n-2} \mathcal{C}_{i,j}$, hence $(\mathcal{C}, \mathcal{C}^1)$ is chain isomorphic to $\sum_j \mathcal{C}_{n-2,j}$, which is just $\otimes^n \bar{C}(X)$. On the other hand, by excision $H_*(XP(n), XP(n-1)) \approx \bar{H}_*(XP(n)/XP(n-1))$ and $H_*(XP(n)) \approx \bar{H}_*(XP(n))$ in positive dimensions.

<u>Lemma</u> 7.9. $XP(n)/XP(n-1)$ has the homotopy type of $S^n(X^{(n)})$.

Proof. Since the attachment

$\bar{\gamma}(n) : {}^n\Delta \times X^n, \partial\, {}^n\Delta \times X^n \cup {}^n\Delta \times X^{[n]} \to XP(n), XP(n-1)$

is a relative homeomorphism and since we can construct a relative homeomorphism $\varphi : {}^n\Delta, \partial\, {}^n\Delta \to S^n, *$, it follows that $\bar{\gamma}(n)(\varphi^{-1} \times 1)$ defines a map of $S^n(X^{(n)})$ onto $XP(n)/XP(n-1)$ which is a homotopy equivalence.

Theorem 7.1 now follows by using the chain equivalence of $\otimes^n \bar{C}(X)$ with $\bar{C}(X^{(n)})$.

The map f induces

$H_{q+n-2}(\mathcal{C}, \mathcal{C}') \approx \bar{H}_q(X^{(n)}) \to \bar{H}_{q+n}(S^n(X^{(n)})) \approx \bar{H}_{q+n}(XP(n), XP(n-1))$

which is just the suspension isomorphism. Hence by induction $f_* : H_r(\mathcal{C}) \to \bar{H}_{r+2}(XP(n))$ is an isomorphism for all r.

(To start the induction, we note that

$H_r(\mathcal{E}(\nabla^{-1};X)) = H_{r+1}(\bar{C}(X)) \approx \bar{H}_{r+2}(SX)$ is induced by f_*.)

As usual, E^∞ can be interpreted as the graded group associated with $H_\partial(\mathcal{E})$ with the induced filtration. We can filter $\bar{C}(XP(n))$ by the subcomplexes $\bar{C}(XP(p))$; the map f is then filtration preserving. Thus we obtain

Theorem 7.10. $E^\infty(X, \nabla^{n-2}, \Lambda)$ <u>is the graded group associated with</u> $\bar{H}_*(XP(n))$ <u>as filtered by the imbeddings</u> $XP(p) \subset XP(n)$, $p \leq n$.

<u>Remark</u> 7.11. In the particular case $L = \nabla^\infty = (-1,0,1,\ldots)$, it should be possible to prove Theorem 7.1 by reworking Cartan's theory of constructions so as to apply to $A(\infty)$-algebras instead of strictly associative algebras [3,N3].

8. Non-triviality and applications

We use Theorem 7.10 to exhibit spaces X_r such that the spectral sequence $\{E^i(X_r, \nabla^\infty, \Lambda), d_i\}$ has non-trivial differential d_r; in fact there exists $u \in H_*(X_r)$ such that $\rceil u,\dots,u\lceil$, the bar operation of $(r+1)$-factors, is non-trivial.

As in the last section, we can identify $\mathcal{E}_r(\nabla^{n-1};\bar{R})$ with a submodule of $\mathcal{E}_r(\nabla^n;\bar{R})$. In particular $\mathcal{E}_r(\nabla^{n-1};\bar{R})$ is isomorphic with $\mathcal{E}_r(\nabla^n;\bar{R})$ for $r+2 < (n+2)(2+\max\{q \mid \bar{R}_q = 0\})$.

Let $X_r = \Omega\, CP(r)$. By Proposition III.1.11, $\Omega\, CP(r)$ has the homotopy type of $S^1 \times \Omega S^{2r+1}$. In dimensions less than $2r+1$, $H_i(\Omega\, CP(r))$ is non-trivial only for $i = 0, 1$ and $2r$. Let $\Lambda = Z$ and u be a generator of $H_1(\Omega\, CP(r)) \approx Z$ and y a generator of $H_{2r}(\Omega\, CP(r)) \approx Z$.

Theorem 8.1. <u>Let</u> X_r be <u>given the</u> A_∞ <u>-structure corresponding to loop addition. Then the</u> $(r+1)$-<u>fold bar operation</u> $\rceil u,\dots,u\lceil$ <u>is defined and</u> $y = \rceil u,\dots,u\lceil$.

<u>Proof</u>: The $(r+1)$-fold bar operation is defined as a differential in the spectral sequence $\{E^i(X_r, \nabla^{r-1}, Z), d_i\}$. Since $z = u \otimes \cdots \otimes u \in E^1_{r-1,r+1}$, the evaluation of the bar operation involves only $\mathcal{E}_s(\nabla^{r-1}; \bar{H}_*(X_r))$ for $s \leq 2r+1$, which can be identified with $\mathcal{E}_s(\nabla^\infty; \bar{H}_*(X_r))$ since $2r+3 < (r+2)2$.

In order to compute E^{∞}, we must identify $X_r P(\infty)$. A relevant result of [21] is

Theorem 8.2. <u>If</u> $X = \Omega Y$ <u>then</u> Y <u>has the homotopy type of</u> $XP(\infty)$ (constructed with respect to the loop structure).

It follows that $CP(r)$ has the homotopy type of $X_r P(\infty)$. Since $H_i(CP(r)) = 0$ for $i > 2r$, it follows that $E_i^{\infty} = 0$ for $i + 2 > 2r$; in particular $E^{\infty}_{r-1,r+1} = 0$ and $E^{\infty}_{-1,2r} = 0$. The class z belongs to $E^1_{r-1,r+1}$. Thus $d_i\{z\}$ if defined is represented by a class in $E^i_{r-i-1,r+i} = \bar{H}_{r+i}(X^{(r-i+1)})$, which is zero unless $i = r$. Hence $d_i\{z\} = 0$ for $i < r$. Since $E^{\infty}_{r-1,r+1} = 0$, if $d_r\{z\} = 0$ then $d_s\{w\} = \{z\}$ for some class $w \in E^1_{r+s-1,r-s+2} = \bar{H}_{r-s+2}(X^{(r+s+1)})$. Since $\bar{H}_j(X^{(r+s+1)}) = 0$ for $j < r+s+1$, we must have $d_r\{z\} \neq 0$.

$E^1_{-1,2r}$ is generated by y, hence $d_r\{z\} = m\{y\}$ for some non-zero integer m. Now $d_i\{y\} = 0$ for all i since $E^1_{i,j} = 0$ for $i < -1$. Because $E^{\infty}_{-1,2r} = 0$, there must exist an i and a class $w \in E^1_{i-1,2r-i+1}$ such that $d_i\{w\} = \{y\}$. However, the only value of i for which $E^1_{i-1,2r-i+1}$ is non-trivial is $i + 1 = 2r-i+1$, i.e. $i = r$. Since $E^1_{r-1,r+1}$ is generated by z, we conclude $d_r\{z\} = \{y\}$. That is, $]u,\dots,u[= y$.

<u>Remark</u> 8.3. William Browder has mentioned to the author that the stack operations can be computed in $\Omega CP(r)$ by means

of the Adams-Hilton chain algebra methods [1]. The corresponding result is obtained: $\langle u,\ldots,u\rangle = y$. The same approach can be used to compute the bar operations; the computations coincide since in the latter case no use is made of summing chains which remain separate in the former.

<u>Application</u> 8.4. The inclusion $w : S^1 \to \Omega\, CP(r)$ defined by $w(s)(t) = (t,s) \in S(S^1) = S^2 \subset CP(r)$ is an A_r-map but not an A_{r+1}-map.

Proof. It is possible to identify $CP(r)$ with $S^1P(r)$ and thus see that w is an A_r-map by applying Proposition II.3.10. However, there is a more direct method available. By induction, suppose w is an A_{i-1}-map and that $G_j : I_j \times K^j \times (S^1)^j \to \Omega\, CP(r)$ are corresponding homotopies, $2 \leq j \leq i$. This permits us to define $\bar{G}_i = G_i \,|\, \partial\,(I_i \times K_i) \times (S^1)^i$ by formulas (II.3.2.b,c,d). The map $\bar{G}_i$ can be extended as the constant map on $I_i \times K_i \times e \times \cdots \cdots \times e$. The maps G_j for $j < i$ enable us to extend to $I_i \times K_i \times (S^1)^{[i]}$ by using the unitary condition on A_n-spaces (II.1.2.c). Hence the obstructions to extending to all of $I_i \times K_i \times (S^1)^i$ can be regarded as classes in the groups

(8.5) $H^q(\partial\,(I_i \times K_i) \ast (S^1)^{(i)};\ \pi_q(\Omega\, CP(r)))$.

Since $I_i \times K_i$ is homeomorphic to I^{i-1} and $(S^1)^{(i)}$ to S^i, the group (8.5) is isomorphic to $H^q(S^{2i-2}; \pi_q(\Omega\, CP(r)))$. This group is trivial unless $\pi_{2i-2}(\Omega\, CP(r))$ is non-trivial which happens only for $2i-2 \geqslant 2r$; hence w is an A_r-map.

To prove that w is not an A_{r+1}-map, let a denote a generator of $H_1(S^1)$. The map w is adjoint to the imbedding $S^2 \subset CP(r)$ and S^2 carries the generating class of $H_2(CP(r))$, so that $w_*(a) = u$. Since $H_j(S^1) = 0$ for $j > 1$, we have $0 =]a, \ldots, a[$ for any number of j factors > 1, while for $(r+1)$ factors, $]u,\ldots,u[= y$. If w were an A_{r+1}-map, we would have the contradiction $y = w_*(\,]a,\ldots,a[\,) = 0$.

By a precisely similar argument we can show

<u>Application</u> 8.6. Let $g : \Omega\, CP(r) \longrightarrow S^1 \times \Omega^{2r+1}$ be a homotopy equivalence and $p : S^1 \times \Omega\, S^{2r+1} \longrightarrow \Omega\, S^{2r+1}$ the projection onto the second factor, then $pg : \Omega\, CP(r) \longrightarrow \Omega\, S^{2r+1}$ is not an A_{r+1}-map.

Combining Application 8.4 with Corollary III.1.16, we see that the homotopy equivalence $(q + \Omega\pi) : S^1 \times \Omega S^7 \longrightarrow \Omega\, CP(3)$ is an A_2-map but not an A_4-map. We are unable to determine if $(q + \Omega\pi)$ is an A_3-map, although both $\Omega\pi$ and q are, the latter because all the obstructions lie in trivial groups.

Now it is possible to show that $(q + \Omega\pi)$ is an A_3-map if and only if loop addition on $\Omega CP(3)$ is an A_3-map. Hence we could look at possible axial maps $SX \times XP(2) \longrightarrow XP_{\smile}(3) \subset CP(3)$ for $X = \Omega\, CP(3)$. In particular we know $S\,\Omega\, CP(3)$ has the homotopy type of a bunch of spheres. Thus we return to a geometric problem : does there exist a decomposition of $(\Omega\, CP(3))P(2)$ into fairly elementary subspaces?

The combination of geometric and algebraic techniques of this thesis increase our knowledge of the multiplicative structure of $\Omega CP(3)$; they also suggest at least one direction for further research.

BIBLIOGRAPHY

1. J.F.ADAMS and P.J.HILTON. On the chain algebra of a loop space. Comment. Math.Helv.30(1956) 305-329

2. H. CARTAN. Proc. NAS. 40 (1954) 467-471, 704-707.

3. Séminaire H. CARTAN. ENS. 1954-55.

4. H. CARTAN and S.EILENBERG. <u>Homological Algebra.</u> Princeton 1956.

5. S. EILENBERG and S. MACLANE. Acyclic Models. AMJ. 75 (1953) 189-199.

6. S.EILENBERG and N.STEENROD. <u>Foundations of Algebraic Topology</u>. Princeton 1952.

7. P.J.HILTON. Homotopy Theory and Duality (mimeographed). Cornell University. 1959.

8. - On divisors and multiples of continuous maps. Fund. Math. 43(1956) 358-386.

9. F. HIRZEBRUCH. Über die quaternionalen projektiven Räume. S.Ber.math. - naturw. Kl.Bayer. Akad. Wiss. München (1953) 301-312.

10. I.M.JAMES. Reduced product spaces. Ann.Math. 62(1955) 170-197;

11. - and E. THOMAS. Which Lie groups are homotopy abelian? Proc.NAS. 45(1959) 737-740.

12. - and J.H.C.WHITEHEAD. The homotopy theory of sphere bundles over spheres (I). Proc.LMS. (3)4 (1954) 196-218.

13. T. KUDO and S. ARAKI. Topology of H_n-spaces and H-squaring operations. Mem.Fac.Sci.,Kyusyu Univ. Ser.A.10 (1956) 85-120.

14. W.S.MASSEY. Some higher order cohomology operations. Symposium Internacional de Topologia Algebraica 1958

15. J. MILNOR. Construction of Universal Bundles, II. Ann. Math.63(1956) 430-436.

16. .. - The construction FK (mimeographed). Princeton. 1956.

17. - On spaces having the homotopy type of CW-complexes. Trans.AMS 90 (1959) 272-280.

18. J.C.MOORE. The double suspension and p-primary components of the homotopy groups of spheres. Boletin de la Soc. Mat. Mexicana.1 (1956) 28-37.

19. J.-P. SERRE. Homologie singulière des espaces fibrés. Ann.Math.54 (1951) 425-504.

20. J. STASHEFF. On homotopy abelian H-spaces. (to appear)

21. - Thesis. Princeton University.

22. M. SUGAWARA. On a condition that a space is an H-space. Math. J. Okayama Univ. 6 (1957) 109-129.

23. - H-spaces and spaces of loops. Math. J. Okayama Univ. 5 (1955) 5-11.