Non-Price Competition in College Admissions



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Abstract

This study theoretically analyses college admission mechanisms where students' applications have uncertain outcomes and the number of applications they can make are limited. I construct a model where two colleges engage in costly investment to compete for better students, and students apply to colleges facing admission uncertainty. I examine the equilibrium of this game when students are limited to a single application, and when they are not. I show, echoing previous research, that in some equilibria, the low-ranked college might benefit from an application limit and admission uncertainty by attracting cautious applicants desirable by its rival. I then offer the novel prediction that this generates a competitive effect to the benefit of the high-ranked college. By improving the students the low-ranked college enrols, these frictions mitigate the competition between colleges. Subsequently, the high-ranked college competes for the better students at a more favourable investment, increasing its payoff. I then embed this game to a larger model where colleges choose whether to impose an application limit to students. I find that when the equilibrium is as described, both colleges choose to impose an application limit to students, distorting the stable matching.

Keywords: college admissions, decentralized two-sided matching, prematch investments, frictional matching

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Any limitations and errors are strictly my own.

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1 Introduction

In their seminal work, Gale and Shapley (1962) showed that a stable matching exists in any college admissions problem, and proposed a mechanism which always implements it. Nonetheless, college admission systems in many countries remain puzzlingly far from the stable mechanism they proposed. In the United States, applications are considerably costly, and students often face rejections[1]. In the UK, undergraduate students are barred from applying both to Oxford and Cambridge in the same year[2]. In South Korea, prior to 1994, students were limited to at most two college applications, effectively able to apply to only one selective college as these all held their admission exams on the same day (Avery, Lee, & Roth, 2014).

Two frictions; costly or restricted applications, and admission uncertainties, seemingly prevail across many admission systems. Hence, how these influence college admissions attracted substantial attention from economists. Theoretical studies on different structures that effectuate these frictions agreed on one prediction: when two colleges, one commonly or widely preferred to, i.e. "ranked above", the other, compete for talented applicants, costly applications and uncertainty might make the low-ranked college better off at its rival's expense. These two frictions generate a trade-off between applying to the safer but low-ranked college, or its riskier but higher-ranked rival. Provided students' preferences towards the high-ranked college are not sufficiently strong, cautious applicants who would be desirable by both might prefer applying to the low-ranked college. Were these frictions circumvented, they would be enrolled by the higher-ranked college. By distorting stability, these frictions convey the low-ranked college a competitive benefit. This comes at its rival's expense, who loses some desirable applicants. Hence, low-ranked colleges might disfavour stable admission mechanisms and strategise to distort them. A consensus towards centralized, stable admissions mechanisms can be impeded.

This prediction emerges in various contexts. Avery and Levin (2010) show that when students are unsure about their calibre and can early apply to a single college, committing to enrol upon admission, a *strategic effect* can emerge; whereby the low-ranked college can attract desirable applicants away from its rival, to its detriment. Chen and Kao (2014)

[1] "This high school senior says he is spending $1,700 on college applications", Sarah Berger, *CNBC*, January 2, 2018. Retrieved May 26, 2020, from `https://www.cnbc.com/2018/01/02/high-price-of-college-applications.html`.

[2] See the University of Oxford's "Guide for Applicants". Retrieved May 20, 2020, from `http://www.ox.ac.uk/admissions/undergraduate/applying-to-oxford/guide/ucas-application`.

show that, with noisy screening and limited applications, a *coordination failure* between students might similarly help the lower quality college get the best student; provided colleges' qualities are sufficiently close. Analysing college admissions in South Korea, Avery, Lee, and Roth (2014) echo these results. They show that by forcing students to at most one application, a conflict with its rival's admission date can convey similar benefits to the low-ranked college when students have imperfect knowledge of their ability. In admissions models geared towards different foci, Chade, Lewis, and Smith (2014) and Che and Koh (2016) offer similar predictions.

These studies treat students' preferences towards colleges, or colleges' "rankings", as exogenous; focusing on an admissions game where these are given. However, colleges do not only strategise in their admissions policies to attract talented applicants. They also spend substantial effort towards improving or maintaining their rankings, often critical for students' preferences and application decisions. Indeed, due to their influence on students[3], these rankings have been accused of forging prestige-wars, ramping up education costs, and diverting colleges' attention and resources on improving how they rank[4], sometimes even by illicit means[5].

In this paper, I aim to analyse how two frictions in college admissions – limited applications and admission uncertainty – interact with the competition between colleges, when students' preferences are determined by colleges' actions. I extend the literature on college admissions with frictions by endogenising students' preferences towards colleges. These preferences depend on the costly investment each college offers as they compete for talented applicants.

I show that once this endogeneity is accounted for, the competitive benefits these frictions convey to the low-ranked college need not be at the expense of its rival. Echoing earlier studies, the low-ranked college can indeed benefit from these frictions as they distort stability in its favour, helping it enrol better students. The high-ranked college loses some talented applicants it would enrol as a consequence. This paper aims to show

[3] "College Rankings Really Do Influence Which Schools Students Apply To", Eleanor Barkhorn, *The Atlantic*, January 17, 2014. Retrieved May 26, 2020, from `https://www.theatlantic.com/education/archive/2014/01/college-rankings-really-do-influence-which-schools-students-apply-to/283151/`

[4] "How Dangerous Are College Rankings and the Rat Race For Prestige?", Stephen Joel Trachtenberg, September 28, 2011. Retrieved May 22, 2020, from `https://www.theatlantic.com/business/archive/2011/09/how-dangerous-are-college-rankings-and-the-rat-race-for-prestige/245850/`

[5] "Your Annual Reminder to Ignore the U.S. News & World Report College Rankings", John Tierney, September 10, 2013. Retrieved May 26, 2020, from `https://www.theatlantic.com/education/archive/2013/09/your-annual-reminder-to-ignore-the-em-us-news-world-report-em-college-rankings/279103/`

that this distortion can serve to increase its payoff nonetheless: as the low-ranked college gets better students in equilibrium, the costly competition in investments to attract better students is ameliorated, and the high-ranked college can attract these students at less cost. Hence, these distortions might not be uniquely favoured by the low-ranked college to the detriment of its rival; by mitigating the competition to attract the best applicants, they can benefit both colleges.

To show this, I build two college admission games with two ex-ante identical colleges and a unit mass of students. I assume that there are two dates where colleges can accept applications, and a student can apply to a single college at any given date. These two games differ only in the dates colleges receive applications. In the *Sequential Applications* game, colleges hold admissions on different dates; thus, there is no limit on applications. In the *Simultaneous Applications* game, dates conflict, limiting students to a single application. In addition, colleges' capacities are ex-ante unknown, revealed only at the end of the game. This causes admission uncertainty for students; some might find a college has sufficient capacity to admit them in one state, whereas not in another.

In these sequential move, perfect information games, first, colleges sequentially set the costly investment they will offer their students. They are then followed by students, who choose where to apply at each date. After the game ends, the college-proposing Gale and Shapley (1962) algorithm matches applicants to colleges, treating any student who has not applied to a college as having declared it unacceptable.

Colleges are matched to students to produce human capital, where their costly investments and students' abilities are complements. Colleges' payoffs are a share of the human capital they help produce, less their investment costs. Therefore, they prefer higher ability students holding investment constant. Similarly, students' payoffs are a share of the human capital they help produce; therefore they prefer the college offering the highest investment. As it is commonly preferred over its rival, I call the college with higher investment as the "high-ranked" college, and its rival the "low-ranked" college.

When colleges' investment levels differ, the positive assortative matching is the unique stable matching in both games. This matches the high-ranked college to the best students subject to its capacity, and its rival to the second best. This unique stable matching is always implemented in the Sequential Admissions game; students apply to both colleges, and are subsequently matched to colleges with a stable mechanism. Without limits on applications, uncertain admissions cannot preclude stability.

Here, the competition for students becomes a "winner-take-all" game. The college to bid the highest investment gets all the best students it can enrol, leaving its rival only the second best. This leads to a stark equilibrium prediction: both colleges' equilibrium payoffs equal the maximum payoff the second best students can bring. The reasoning echoes a classical Bertrand competition. In equilibrium, the low-ranked college settles for the second best students instead of competing with its rival for the best. The high-ranked college can deter such competition only by bidding investment suboptimally high; its cost erodes any payoff the best students would yield over that the second best would. It cannot secure higher payoff; then, outcompeting it would dominate settling for the second best for its rival. Though the high-ranked college gets the best students in any equilibrium, its payoff is determined by its rival's outside option: the payoff the second best students can yield.

In the Simultaneous Applications game, students are limited to a single application. Coupled with admission uncertainty in equilibrium, this precludes the stable matching, leading instead to two possible outcomes. When a college outcompetes its rival by a wide margin, it gets all the best applicants the stable outcome affords. Some, even though they are sure admits for the low-ranked college, *gamble* on their preferred; despite a chance of being rejected. Rejected, they are left unmatched, when capacities are insufficient. On the other hand, when colleges are closer alternatives, similar students – fearing rejection – apply to the low-ranked college, as it is a safer option, and close competitor. Subsequently, though their preferred college would admit them should capacities be sufficient, they always attend the low-ranked college; *insuring* themselves against uncertainty. Hence, competition no longer exhibits a winner-take-all structure. The high-ranked college loses preferable students regardless of its capacity, while its rival enrols students better than its admits in some equilibria. Echoing earlier findings, the low-ranked enrols better students to the detriment of the ability of those its rival expects.

When this latter outcome emerges, the low-ranked college's payoff improves relative to the Sequential Applications game. Expecting better students, it invests to maximize its payoff conditional on enrolling them. This costs the high-ranked college strong applicants it would admit in some states. However, it does not imply a lower equilibrium payoff. As the low-ranked college can secure a higher payoff without having to outbid its rival, competing for the better students as aggressive as it would is no longer profitable. The stakes to emerge as the highest bidder falls, decreasing competitive pressure. Though the

high-ranked college gets a worse set of applicants, it does so without having to bid as aggressively as it would in Sequential Applications. Due to this favourable competitive effect, it attracts the better applicants at a higher payoff, facing milder competition for them.

I find that when it is sufficiently unlikely for the high-ranked college to have sufficient capacity to admit gamble applicants, this emerges as the equilibrium in Simultaneous Admissions. The unlikelier that their preferred college will admit them, the more the investment premium students demand to gamble for the high-ranked college. This makes bidding investment to secure all best applicants becomes prohibitively expensive for both colleges. Therefore, colleges emerge as close competitors in equilibrium. As they expect students closer in quality, the low-ranked college gains from the better students it enrols, and the high-ranked college gains from the favourable competitive effect this generates by mitigating competition.

This is the central result of this paper. Previous studies on college admissions with frictions showed that low-ranked colleges might benefit from a distortion of stability due to admission uncertainty and limited applications. I extend these studies by endogenising colleges' rankings and therefore students preferences, building a model where colleges engage in costly competition which shapes students' preferences towards them. This reveals a piece missing to this puzzle. By improving the students the low-ranked college gets at the expense of those its rival does, these frictions can ameliorate the competition for better students. Though it gets worse students in equilibrium, the high-ranked college benefits from this favourable competitive effect; it competes for these better students at less cost. In equilibrium, the low-ranked college benefits from these frictions not at the expense of the high-ranked college, but rather, like it.

Following this result, I construct a larger game, where, instead of having them exogenously determined, colleges set their application dates sequentially before the college admissions game is played. When the equilibrium of the Simultaneous Admissions game is as described above, I show that the Sequential Admissions game is never played in the equilibrium of this larger game. Unlike suggested by previous research, no college favours this game over Simultaneous Admissions; the low-ranked college is better off as it attracts better applicants, and the high-ranked college benefits from the favourable competitive environment this generates. Whichever college can strategise to impose these frictions does so.

Previous literature assumed students' preferences towards colleges, and colleges' rankings, to be exogenous when analysing college admissions. The result this paper finds was therefore largely overlooked. Analysing an environment where colleges can influence students' preferences towards them completes the missing piece of the puzzle. A consensus towards centralized, stable mechanisms might not be impeded only as they are to lower-ranked colleges' detriment. By introducing stauncher competition for the better students, they might deteriorate both high-ranked and low-ranked colleges' payoffs, attracting opposition from both sides.

The rest of the paper is structured as follows: Section 2 presents the model. Section 3 analyses the Sequential Applications game and its equilibria. Section 4 analyses the Simultaneous Applications game and its equilibria. Section 5 builds a larger game where colleges choose their admission dates before the admissions game is played. It then derives its equilibria, using the analyses in Sections 3 and 4. Section 6 offers further discussion around the main results, and concludes. Some results not central to the discussion are omitted in the main text, and reserved instead to Section 7. In many instances, proofs for the results presented in the main text are reserved to Section 8 instead of accompanying the results in the main body.

2 The Model

There are two ex-ante identical colleges A and B, and a unit mass of students with types $a \sim U[0,1]$, where a represents intrinsic ability. Constrained by their capacities κ, ex-ante unknown, colleges enrol students to produce human capital by combining the costly investment they undertake with their students' abilities. Ability and costly investment are complements in human capital production[6]. For a mass Q of students with ability a, human capital produced with investment h_i is $k_i := Q \times k(a, h_i) = Q \times K \times a \times h_i$, $K > 1$.

College i's payoff is a portion of the human capital it produces less the costs of its investment, $\Pi_i(h_i) = \frac{k_i}{K} - h_i^2$. There are no complementarities between students. Therefore, colleges' preferences over sets of students are responsive, and a higher ability student is commonly preferred to a low ability student. I therefore refer to students with higher ability as "better" students, and their peers as "worse" students.

Students retain a share of the human capital they help produce. A student with

[6]The marginal product of investment increases in ability, and vice versa. Where $k(a, h_i)$ is the human capital production function, $\frac{\partial^2 k(a,h_i)}{\partial a \partial h_i} > 0$.

ability a, and matched to College i, gets a payoff $u(a, h_i) := (T - \frac{1}{K}) \times k(a, h_i) = (T - \frac{1}{K}) \times (K \times a) \times h_i$, where $T \leq 1$. A student matched to no college gets a payoff of 0. Therefore, all students prefer the college which offers higher investment, and any college over being unmatched. Hence, I will refer to the college offering higher investment as the "high-ranked" college, and to its rival as the "low-ranked" college.

The surplus any match between a student and college generates cannot be redistributed with side payments; i.e. utility is non-transferable. All agents are assumed to be risk-neutral, and maximize expected payoff.

Students are matched to colleges through a sequential move, symmetric and perfect information college admissions game. This game has the following timing: first, each college is assigned a date, $t_A, t_B \in \{1, 2\}$, where they will receive applications. Then, A chooses its investment $h_A \geq 0$, and pays the sunk cost $c(h_A) = h_A^2$, followed by B. Colleges' actions are commonly observed upon execution. Then, students apply to colleges. Applications are costless, but only a single one can be submitted at each date. Effectively, this bars students from applying to both colleges when $t_A = t_B$. Following students' applications, Nature chooses the – ex-ante unknown – capacities κ, common for both colleges. With probability $p \in (0, 1)$, $\kappa = Q$; otherwise $\kappa = Q'$. $\frac{1}{2} \geq Q' > Q$. This concludes the game, which is followed by the college-proposing Gale and Shapley (1962) algorithm to match applicants to colleges. Any student who has not applied to a college is treated as having declared it unacceptable by the algorithm.

I seperately analyse the equilibria of the college admissions game where $t_A \neq t_B$ and $t_A = t_B$; referring to the former as the *Sequential Applications* game and to the latter as the *Simultaneous Applications* game. Then, I embed this college admissions game to a larger game. In this larger game, instead of being assigned one, colleges choose the dates they will receive applications sequentially before the college admissions game proceeds. I analyse the equilibrium of this game when either A or B move first to set their dates before the admissions game begins.

I assume that agents play their weakly dominant strategies in equilibrium, when available. When two investment strategies yield equal expected equilibrium payoffs, I focus on equilibria where the college plays the strategy it expects to attract better students with. Since I analyse sequential, perfect information games, the solution concept is Subgame Perfect Nash Equilibria (SPNE).

3 Sequential Applications

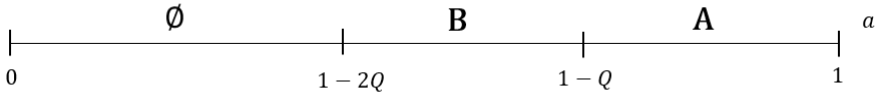
Let $t_A = 1$ and $t_B = 2$, leading to the Sequential Applications game. Students face no restraints in their application decisions. Since applications are costless, an application to a college can, at worst, result in no offer or one to be rejected. Therefore, the weakly dominant strategy for any student is to apply to both colleges. I assume that students follow this strategy in equilibrium.

When students have strict preferences between colleges ($h_A \neq h_B$), the positive assortative matching of students to colleges is the unique stable matching of the college admissions game.

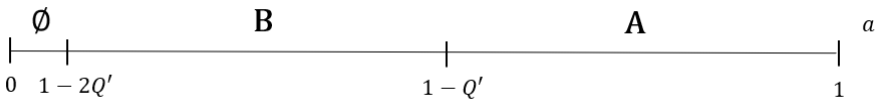
Definition 3.1. *Let $h_A > h_B$. Where colleges have capacities κ, $\mu^s(a, \kappa)$ denotes the positive assortative matching of students to colleges, defined as:*

$$\mu^s(a,\kappa) = \begin{cases} A & a \geq 1-\kappa \\ B & a \in [1-2\kappa, 1-\kappa) \\ \emptyset & a < 1-2\kappa \end{cases}$$

When $h_B > h_A$, the definition holds symmetrically.



(a) $\mu^s(a, Q)$



(b) $\mu^s(a, Q')$

Figure 3.1: The Positive Assortative Matchings

Lemma 3.2. *Let $h_A \neq h_B$. Where colleges have capacities κ, the positive assortative matching $\mu^s(a, \kappa)$ is the unique stable matching in the college admissions game.*

As students apply to both colleges, the college-proposing Gale and Shapley algorithm implements the unique stable matching μ^s irrespective of the realized capacity. Without limited applications, enrolment uncertainty becomes irrelevant for students' behaviour and the resulting matching. On its own, enrolment uncertainty cannot preclude stability.

Lemma 3.3. *Suppose $h_A \neq h_B$. Then, the outcome of the Sequential Admissions game is the unique stable matching, $\mu^s(a, \kappa)$.*

When colleges' investments are not equal, denote the students the high-ranked college enrols when capacities are κ as $\mu^s(H, \kappa)$, and those the low-ranked college enrols as $\mu^s(L, \kappa)$. Denote the sum of these students' abilities as $\Sigma^s_{H,\kappa}$ and $\Sigma^s_{L,\kappa}$ respectively. Then:

$$\Sigma^s_{H,\kappa} := \int_{a \in \mu^s(H,\kappa)} a.da = \int_{1-\kappa}^{1} a.da$$
$$\Sigma^s_{L,\kappa} := \int_{a \in \mu^s(L,\kappa)} a.da = \int_{1-2\kappa}^{1-\kappa} a.da$$

As $\kappa = Q$ with probability p and $\kappa = Q'$ otherwise, the high and low ranked college expect their students' total ability to be Σ^s_H and Σ^s_L respectively, where:

$$\Sigma^s_H := p \times \Sigma^s_{H,Q} + (1-p) \times \Sigma^s_{H,Q'}$$
$$\Sigma^s_L := p \times \Sigma^s_{L,Q} + (1-p) \times \Sigma^s_{L,Q'}$$

Then, the payoffs the high and low ranked college expect in the Sequential Applications game are respectively:

$$\Pi^s_H(h) := h \times \Sigma^s_H - h^2$$
$$\Pi^s_L(h) := h \times \Sigma^s_L - h^2$$

Figure 3.2 illustrates the payoff curves these functions correspond to.

The functions $\Pi^s_i(h)$ attain their global maxima $(\Pi^s_i)^*$ at $h^s_i := \frac{\Sigma^s_i}{2}$ for $i \in \{H, L\}$. Note that h^s_i are, for each college, constant multiples of the average level of ability of the students they expect to be matched with[7]. I will refer to h^s_i as the *conditionally optimal investment* for the respective expected students, $\mu^s(i, \kappa)$.

Armed with these observations, we can now analyse the equilibrium investment levels in the Sequential Admissions game, denoted as h^s_A and $h^s_B(h_A)$.

[7]The expected level of ability for a college expecting to be matched to students $\mu^s(i, \kappa)$ is $\frac{\Sigma^s_i}{p.Q+(1-p).Q'}$. For given (p, Q, Q'), this is a constant multiple of $h^s_i := \frac{\Sigma^s_i}{2}$.

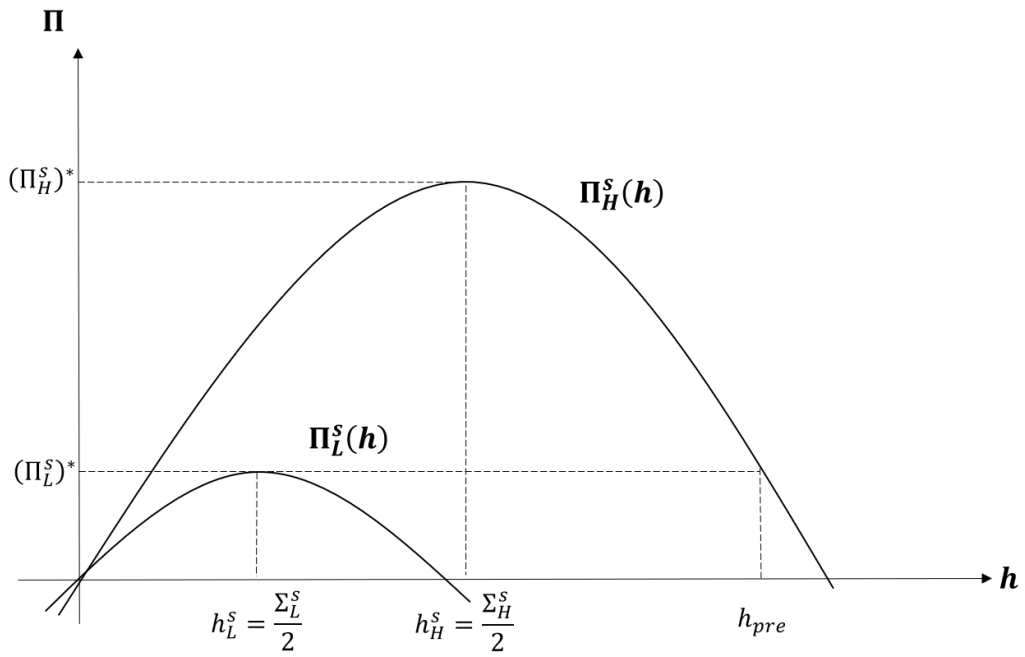


Figure 3.2: Expected Payoffs: Sequential Applications

Lemma 3.4. *$h_B^s(h_A) = h_A$ is a strictly dominated strategy.*

Proof. Students apply to both colleges in the Sequential Admissions game. Those who would be matched to the top ranked college under μ^s receive offers from both colleges under the college-proposing Gale and Shapley algorithm. Left indifferent, they randomize between A and B when accepting offers. As any college who cannot enrol some student in this set has to replace her with a lower ability peer, each college expects a total ability strictly less than Σ_H^s from its cohort. Then, for any given h, $\Pi_T(h) < \Pi_H^s(h)$, where $\Pi_T(h)$ is a college's expected payoff when $h_A = h_B$ in Sequential Admissions. Since $\Pi_H^s(h)$ continuous in h, there exists some $\epsilon > 0$ for all $h \geq 0$ such that $\Pi_H^s(h+\epsilon) > \Pi_T(h)$. Hence, $h_B^s(h_A) = h_A$ is strictly dominated by $h_B^s(h_A) = h_A + \epsilon$ for some $\epsilon > 0$ and all $h_A \geq 0$.

□

Lemma 3.4 establishes that, upon observing h_A, it's never optimal for B to set h_B to leave students indifferent in their choices. Outbidding its rival marginally brings about a negligible cost but discrete increase in the sum of abilities of the students B expects, making a tie strictly dominated. Hence, restricting analysis to the case where students'

preferences are strict ($h_A \neq h_B$), as we have done so far, is sufficient to analyse the equilibrium of the Sequential Applications game.

Claim 3.5. *Let h_{pre} be the lowest level of investment for which $h > h_{pre}$ implies $\Pi_H^s(h) < (\Pi_L^s)^*$. Then, $h_{pre} = \frac{\Sigma_H^s + \sqrt{(\Sigma_H^s)^2 - (\Sigma_L^s)^2}}{2}$.*

Lemma 3.6. *In any SPNE of the Sequential Applications game, $h_A = h_L^s$ and $h_A = h_{pre}$ are the only strategies A plays with positive probability. B's corresponding best responses are $h_B^s(h_L^s) = h_H^s$ and $h_B^*(h_{pre}) = h_L^s$.*

In equilibrium, A is indifferent between two investment levels which lead to equilibrium outcomes with different characteristics. Its first option is to avoid competing with B for the best students, $\mu^s(H, \kappa)$; settling instead only for the second best, $\mu^s(L, \kappa)$. One can see the latter as the outside option for any college; students weakly preferable to them are guaranteed in any outcome. Without worrying about deterring competition from its rival, A can enrol these students at their conditionally optimal investment, h_L^s. B, as its competition for $\mu^s(H, \kappa)$ withdraws, has its cake and eats it. It enrols the best students by outbidding its rival with their respective conditionally optimal investment, h_H^s.

A's alternative is to compete for $\mu^s(H, \kappa)$ instead. To do so, it must bid investment high enough to deter B from outbidding it. This is only possible if B cannot outbid A for a payoff higher than $(\Pi_L^s)^*$; the highest payoff a losing bid can bring. h_{pre} is the bid that achieves this at least detriment to A's expected payoff. Since payoff functions are continuous, any investment below h_{pre} would not deter B; it could still outbid h_A and achieve payoff higher than $(\Pi_L^s)^*$. In turn, as payoff functions are strictly convex, any investment above h_{pre} would yield lower payoff for A. Hence, to attract $\mu^s(H, \kappa)$, A bids $h_A = h_{pre}$.

In equilibrium, A cannot expect to beat its rival's expected payoff; B would outbid it instead of settling for the outside option whenever that were the case. The best it can hope is to match $(\Pi_L^s)^*$, the maximum expected payoff from a losing bid. When $h_A = h_{pre}$, A's equilibrium payoff with the better students equals exactly this. Instead of outbidding A for $\mu^s(H, \kappa)$ at the expense of a lower payoff, B prefers to settle for $\mu^s(L, \kappa)$ at their conditionally optimal investment, yielding the same payoff its rival expects.

These outcomes are identical for the low-ranked college. It always enrols the students $\mu^s(L, \kappa)$ at their conditionally optimal investment h_L^s. This always yields expected equilibrium payoff $(\Pi_L^s)^*$. However, they differ critically for the high-ranked college. This

college always receives the students $\mu^s(H,\kappa)$. When the first-mover, A, does not compete for $\mu^s(H,\kappa)$, B becomes the high-ranked college. It invests at their conditionally optimal investment h^s_H, in equilibrium expecting payoff $(\Pi^s_H)^*$. Both colleges mimic monopolists, setting investment to maximize their expected payoff conditional on the students they get. The high-ranked college gains two-fold: as it gets better students, it is better off under any given h, and it complements their higher ability with a higher investment, further increasing its gains. In equilibrium, all students receive investment proportional to the total ability of the cohort their college expects to enrol.

When the first-mover, A, competes for $\mu^s(H,\kappa)$ instead, it becomes the high-ranked college. Absent side-payments, it must bid for the overdemanded[8] students $\mu^s(H,\kappa)$ with the investment it offers[9]. Setting $h_A = h^s_H$ is not sufficient to deter B from competing for these students; it could outbid h_A and still be better off relative to settling for $\mu^s(L,\kappa)$. For the same reason, any investment below h_{pre} fails to deter B from doing so. When $h_A \geq h_{pre}$ however, B can only compete for $\mu^s(H,\kappa)$ at a cost which drags its payoff below its outside option, hence prefers settling for $\mu^s(L,\kappa)$ instead.

Any investment above h_{pre} leaves A worse off, therefore it bids for $\mu^s(H,\kappa)$ by setting $h_A = h_{pre}$. This erodes any gains to enrolling $\mu^s(H,\kappa)$ over $\mu^s(L,\kappa)$. The costs to the pre-emptive investment A bids erode the benefits from enrolling better students. The students expecting to be matched to the high-ranked college receive an investment premium above the ability of the students their college expects. Their peers attending the low-ranked college receive investment proportional to the ability of the students their college expects. With both strategies, A, the first-mover, expects the same payoff; bounded above by the outside option in the competition.

Proposition 3.7. *Assume colleges break ties between investments in favour of better students. Then, the Subgame Perfect Nash Equilibrium in Sequential Applications is unique. Students apply to both colleges, and colleges' equilibrium investments are* $h^s_A = h_{pre}$, $h^s_B = h^s_L$. *The resulting matching is described by* $\mu^s(a,\kappa)$.

We assumed colleges set the investment which attracts the better students when any two yield equal expected payoff. Thence, $h_A = h_{pre}$ is the strategy A employs in the SPNE

[8] In a slight abuse of their definition, following Roth and Sotomayor (1992), I use the term *overdemanded* for the set of students who are both colleges' first preferences. Here, this set is $\mu^s(H,\kappa)$; both colleges' first preference when capacities are κ.

[9] This is analogous to an all-pay auction with common valuations, where bidders sequentially commit to a bid and the highest bidder wins the highest value object. Colleges' bids are the investments they offer.

of the Sequential Applications game. Along with students' weakly dominant strategies and B's unique best response, Proposition 3.7 then pins the unique SPNE of Sequential Applications. In this equilibrium, both colleges attain the same payoff, $(\Pi_L^s)^*$. A bids preemptively high to get the best students, eroding any profit to be enjoyed from enrolling the best students over their weaker peers. Its payoff is bounded above by the outside option of its rival, B. In turn, B settles for $\mu^s(L, \kappa)$, setting $h_B = h_L^s$ instead of outcompeting A.

Note that the analysis when $t_B = 1$ and $t_A = 2$ is exactly the same. The labels of colleges' admission dates matter only insofar as they indicate a timing conflict or not. How dates are labelled make no difference on students' application behaviour, as they are matched to colleges after the game ends. Colleges, are similarly indifferent as they care only about students' application behaviour. Therefore, we can assume without loss of generality, that $t_A = 1$ and $t_B = 2$ in the Sequential Applications game.

4 Simultaneous Applications

4.1 Application Strategies

Now let $t_A = t_B = 1$, which leads to the Simultaneous Application game. Note that $t_A = t_B = 2$ produces an identical analysis; dates' labels do not make a difference for the game played. Students trivially apply nowhere at $t = 2$, as no college accepts applications at this date. At $t = 1$, application strategies are no longer trivial however. Students must choose between applying to either A or B (or no college).

At $t = 1$, students can choose any application strategy which mixes over applying to A, B, or no college ($\emptyset$); i.e. any strategy in the set $\triangle\{A, B, \emptyset\}$. We can then define $\sigma(a)$ as the mapping of students' application strategies in $t = 1$.

Definition 4.1. *Let $t_A = t_B = 1$. Then, the function $\sigma(a) : [0, 1] \to \triangle\{A, B, \emptyset\}$ maps a student of type $a \in [0, 1]$ to a mixed application strategy in the set $\triangle\{A, B, \emptyset\}$.*

Applicants are matched to colleges by the college-proposing Gale and Shapley algorithm. A student can be admitted to a college only if she applies. She is then admitted if, given where her peers apply and the realized capacity κ, the Gale and Shapley algorithm matches her to this college. As κ varies with known probabilities, the probability a student of type a is admitted to College i when she applies can be easily computed given

her peers' applications.

Definition 4.2. *Let σ_{-a} denote the function mapping any student of type $a' \in [0,1] \setminus \{a\}$ to an application strategy. Then, conditional on applying to i and her peers' application strategies σ_{-a}, $\rho_i(a, \sigma_{-a})$ denotes the probability a student of type a is admitted to College i.*

As students maximize expected utility, a student's expected benefit from an application to College i is simply $\rho_i(a, \sigma_{-a}) \times u(a, h_i)$. As her admission probability depends on her peers' application strategies, σ_{-a}, so does her expected payoff. A student, to apply to a college, must forgo applying to its rival. Then, the expected opportunity cost of an application is simply the expected benefit of the forgone application. In equilibrium, she will follow the strategy with the highest expected benefit, randomizing when indifferent.

Note in passing that a student's application strategy depends on her type a only insofar as it determines the probability she is admitted to either college. As $u(a, h_i)$ is linear in a, holding ρ_A and ρ_B constant, how the expected benefits from applying to A and B compare depend only on h_A and h_B.

In what follows, I again restrict analysis to the case where $h_A > h_B$; as the analysis for $h_B > h_A$ is symmetric. In Sections 4.1.1 and 4.1.2, I analyse students' equilibrium application strategies and the resulting matching for two subgames arising from different investment strategies: (i) $(1-p) \times h_A > h_B$, and (ii) $(1-p) \times h_A < h_B < h_A$, respectively.

In the first subgame, the expected payoff of applying to A for a student admitted only when capacities are sufficiently large ($\kappa = Q'$), is higher than the expected payoff of applying to B; despite she is a sure admit in the latter. Consequently, applicants strong enough to be sure admits in B, but face rejection in A when capacities are insufficient, apply to A nonetheless in this subgame. The difference in the investments of the high and low-ranked colleges justifies this *gamble*.

In the second subgame, this ceases to be the case. For students who are sure admits in B, but fear a rejection from A when capacities are insufficient, differences in investments no longer justify the risk applying to A entails; though it is still preferred over B. They instead apply to its safer, close competitor B, *insuring* themselves against the variability of capacities. This difference in students' behaviour gives rise to two distinct matchings in equilibrium, therefore distinct expected payoffs for the high and low-ranked colleges.

I then show that, analogously to what Lemma 3.4 establishes for the Sequential Ap-

plications game, either one of these two subgames are played in the equilibrium of the Simultaneous Applications game; leaving some students who a college might admit indifferent can never be an equilibrium strategy.

4.1.1 Application Strategies: $(1-p) \times h_A > h_B$

Let $(1-p) \times h_A > h_B$. As students are limited to a single application in Simultaneous Admissions, some might face a trade-off between applying to the high-ranked college or the low-ranked college. While the high-ranked college offers higher payoff conditional on admission, the low-ranked college might be a safer choice. As the high-ranked college is preferred by all students, and subsequently more competitive, admission to the low-ranked college might be likelier.

Students with types $a \geq (1-Q)$ face no such dilemma. Since $\kappa \geq Q$, no college can fill its capacity with students preferable to them in any equilibrium. Therefore, regardless of their peers' application strategies, these students are sure admits at both colleges, and always apply to the high-ranked college. They face no enrolment uncertainty, making limited applications irrelevant for their equilibrium matches.

For students with types $(1-Q) > a \geq (1-Q')$, a tougher decision awaits. The high-ranked college, A, rejects them when $\kappa = Q$, in favour of their better peers. However, when $\kappa = Q'$, they are admitted to A regardless of where their peers apply, as no college can fill its capacity with students favourable to them. Then, these students are admitted to A with probability $(1-p)$. Facing this probability, when $(1-p) \times h_A > h_B$, regardless of the probability B admits them, applying to A yields higher expected payoff than to B. Consequently, these students *gamble* for the high-ranked college in this game; applying there in spite of a higher likelihood of admission to the low-ranked college[10].

Given this application behaviour by their stronger peers, students with types $(1-Q') > a \geq (1-2Q')$ can never be admitted to the high-ranked college A. The high-ranked college fills its capacity with better students in any equilibrium. However, they face a positive admission probability to the low-ranked college B. Students with $a \geq (1-Q')$ apply to A; therefore, when $\kappa = Q'$, the best students B can enrol are those with types $(1-Q') > a \geq (1-2Q')$. Consequently, their unique equilibrium strategy is applying to B.

[10] Any such student is guaranteed admission to B when she unilaterally deviates and applies to B; as all her better peers apply to A.

The rest of the students are never admitted to any college; similar to Sequential Applications, they are always left unmatched. Indifferent, any application strategy mixing over the set $\triangle\{A, B, \emptyset\}$, constitutes an equilibrium strategy for them.

These equilibrium strategies give rise to a unique matching of students to colleges in equilibrium. Proposition 4.3 summarizes these.

Proposition 4.3. *Let $(1-p) \times h_A > h_B$, and $\hat{\sigma}^{hi}(a)$ denote the equilibrium application strategy of a student with type a when investments are such. In any SPNE of the Simultaneous Admissions game,*

$$\hat{\sigma}^{hi}(a) = \begin{cases} A & a \in [1-Q', 1] \\ B & a \in [1-2Q', 1-Q') \end{cases}$$

For students with types $a < (1-2Q')$, any $\hat{\sigma}^{hi}(a) \in \triangle\{A, B, \emptyset\}$ constitutes an equilibrium strategy.

The equilibrium strategies $\hat{\sigma}^{hi}(a)$ lead to a unique matching. Denote this matching by $\mu^{hi}(a, \kappa)$. Then:

$$\mu^{hi}(a, Q) = \begin{cases} A & a \geq 1-Q \\ \emptyset & a \in [1-Q', 1-Q) \\ B & a \in [1-Q-Q', 1-Q') \\ \emptyset & a < 1-Q-Q' \end{cases}$$

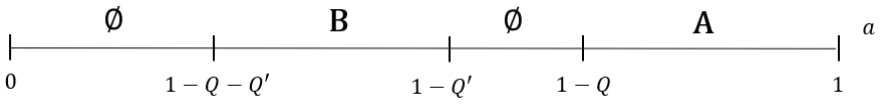
$$\mu^{hi}(a, Q') = \begin{cases} A & a \geq 1-Q' \\ B & a \in [1-2Q', 1-Q-Q') \\ \emptyset & a < 1-2Q' \end{cases}$$

Figure 4.1 illustrates μ^{hi}. $\mu^{hi}(a, Q')$ replicates the stable matching $\mu^s(a, Q')$. When capacities are sufficiently large, the students who gamble for the high-ranked college see their application pay off. Just as they would be in the stable matching, they are matched to their first preference. The second best students who apply to the low-ranked college are similarly matched there. The resulting matching is stable.

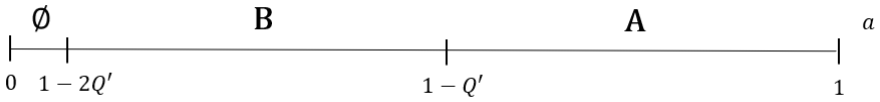
When $\kappa = Q$, students who gambled for the high-ranked college are left unmatched. The high-ranked college lacks sufficient capacity to admit them, and without an application, they cannot be admitted to the low-ranked college. This causes justified envy, and

a subsequent departure from stability. B enrols students strictly worse than those left unmatched subsequent to gambling for A. Similarly, these students, now rejected by A, would prefer B over not being matched to any college.

Figure 4.1 also makes this evident; when $\kappa = Q$, some students are left unmatched while their worse peers attend B. Lemma 4.4 formalises this observation.



(a) $\mu^{hi}(a, Q)$



(b) $\mu^{hi}(a, Q')$

Figure 4.1: Matchings in Simultaneous Adm.: $(1-p) \times h_A > h_B$

Lemma 4.4. *$\mu^{hi}(a, Q')$ is pairwise stable. $\mu^{hi}(a, Q)$ is not stable; it causes justified envy.*

Note that when this subgame is played, the high-ranked college enrols those students it would in the Sequential Applications game regardless of capacity. The students $\mu^s(H, Q')$, whom it would enrol with positive probability in Sequential Applications are its applicants. Therefore, conditional on its investment level, its expected payoff is also the same as it is in Sequential Applications.

This comes at the expense of the low-ranked college. This college would enrol some students in $\mu^s(H, Q')$ when $\kappa = Q$; since some would be rejected by the high-ranked college who lacked sufficient capacity to admit them all. In this subgame, this ceases to be the case. As these students gamble for the high-ranked college, the low-ranked college cannot admit them even when they are rejected. The low-ranked college's applicants are the students it would admit in Sequential Applications when $\kappa = Q'$, $\mu^s(L, Q')$. Enrolling students strictly worse than it would in Sequential Applications when $\kappa = Q$, its expected payoff is also lower conditional on its investment.

4.1.2 Application Strategies: $(1-p) \times h_A < h_B$

Now let $(1-p) \times h_A < h_B < h_A$. The decision facing students with types $a \geq (1-Q)$ is undisturbed. They are guaranteed admission to any college upon application, hence apply

to the high-ranked college A. However, for students who would be admitted to A only when $\kappa = Q'$, but are sure admits in B, the decision changes. The investment premium A offers is now not sufficient to motivate these students to gamble for the high-ranked college.

Students with types $(1-2Q) \leq a < (1-Q)$ are always admitted by B. As students with types $a \geq (1-Q)$ apply to A, B can never saturate its capacity with students it prefers over any in this group. They are admitted to A with probability at most $(1-p)$, as when $\kappa = Q$, A fills its capacity with students it favours over them. Therefore, the expected payoff from an application to B exceeds that from A for these students. Consequently, they apply to B; the safer, close competitor. They *insure* themselves against enrolment uncertainty; applying to the low-ranked college even though they would be admitted to the high-ranked college with a positive probability.

These two groups of students then form those admitted to some college regardless of capacities. The best students among the remaining, students with types $(1-Q-Q') \leq a < (1-Q')$ face equal admission chances at either college. Any college will admit them if and only if $\kappa = Q'$. Subsequently, applying to A is a clear equilibrium strategy for them. Consequently, no student weaker than them faces any admission probability to A. Remaining students who face a positive admission probability to B, with types $(1-2Q') \leq a < (1-Q-Q')$, apply there. The rest of the students are never admitted to any college, and as in Section 4.1.1, randomize over the set $\triangle\{A, B, \emptyset\}$.

These equilibrium strategies give rise to a unique matching of students to colleges in equilibrium. Proposition 4.5 summarizes these.

Proposition 4.5. *Let $h_A > h_B > (1-p) \times h_A$, and $\hat{\sigma}^{lo}(a)$ denote the equilibrium application strategy of a student with type a when investments are such. In any SPNE of the Simultaneous Applications game,*

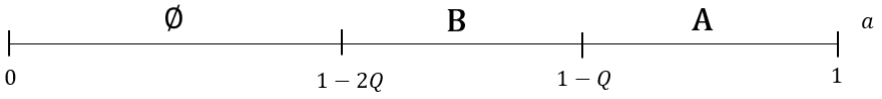
$$\hat{\sigma}^{lo}(a) = \begin{cases} A & a \in [1-Q, 1] \\ B & a \in [1-2Q, 1-Q) \\ A & a \in [1-Q-Q', 1-2Q) \\ B & a \in [1-2Q', 1-Q-Q') \end{cases}$$

For students with types $a < (1-2Q')$, any $\hat{\sigma}^{lo}(a) \in \triangle\{A, B, \emptyset\}$ constitutes an equilibrium strategy.

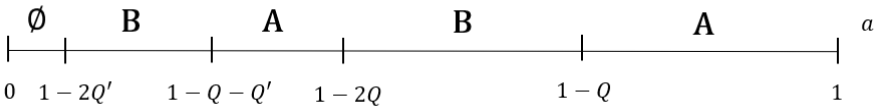
The equilibrium strategies $\hat{\sigma}^{lo}(a)$ lead to a unique matching. Denote this matching by $\mu^{lo}(a,\kappa)$. Then,

$$\mu^{lo}(a,Q) = \begin{cases} A & a \geq 1-Q \\ B & a \in [1-2Q, 1-Q) \\ \emptyset & a < 1-2Q \end{cases}$$

$$\mu^{lo}(a,Q') = \begin{cases} A & a \geq 1-Q \\ B & a \in [1-2Q, 1-Q) \\ A & a \in [1-Q-Q', 1-2Q) \\ B & a \in [1-2Q', 1-Q-Q') \\ \emptyset & a < 1-2Q' \end{cases}$$



(a) $\mu^{lo}(a,Q)$



(b) $\mu^{lo}(a,Q')$

Figure 4.2: Matchings in Simultaneous Adm.: $(1-p) \times h_A < h_B < h_A$

Figure 4.2 illustrates μ^{lo}. Note that the reduced discrepancy between investment levels does not preclude an unstable matching and justified envy, but alters their form. Students who, when facing the same probabilities, gambled for the high-ranked college in the earlier subgame analysed, now settle for the safer option. Having insured themselves, they are never left unmatched. When $\kappa = Q$, the equilibrium matching mirrors the stable matching $\mu^s(a,Q)$.

However, when $\kappa = Q'$, they still regret the application strategy they followed. The high-ranked college has the sufficient capacity to admit them, but cannot despite not having better applicants, as it lacks their applications. Subsequently, worse students, who applied to A as they would not be admitted by either college when capacities were insufficient, attend the high-ranked college, while their stronger peers are enrolled by the low-ranked college.

Figure 4.2 makes this evident; when $\kappa = Q'$, some students attend the low-ranked college, even though their weaker peers attend the top ranked college. Lemma 4.6 formalizes this observation.

Lemma 4.6. *$\mu^{lo}(a, Q)$ is pairwise stable. $\mu^{lo}(a, Q)$ is not stable, inducing justified envy.*

Note that when this subgame is played, the high-ranked college cannot attract all students it would enrol in Sequential Applications. For attracting the very best students, admitted regardless of κ, even marginally outbidding its rival is sufficient. However, the investment premium it offers over its rival must be sufficiently large, $(1-p) \times h_A > h_B$, to attract the students it would enrol only when $\kappa = Q'$. As these students are good enough to be sure admits at the low-ranked college, they require a high enough investment premium to forsake applying there.

$\mu^{lo}(A, Q')$ defines the applicants the high-ranked college gets when $h_A > h_B > (1-p) \times h_A$, referred to as $\mu^{lo}(H, Q')$ in the remainder of the text. $\mu^{s}(H, Q')$ is preferred over $\mu^{lo}(H, Q')$; the high-ranked college enrols worse students than it would in Sequential Applications. Consequently, it expects lower payoff conditional on its investment level.

As a consequence, the low-ranked college expects to enrol better students. Attracting some students its rival would admit when $\kappa = Q'$ as insurance applicants, it locks them in; enrolling them even when its rival would in Sequential Applications. When $h_A > h_B > (1-p) \times h_A$, the set $\mu^{lo}(B, Q')$ are its applicants, denoted in the remainder of this text as $\mu^{lo}(L, Q')$. As $\mu^{lo}(L, Q')$ is preferred over $\mu^{s}(L, Q')$, in this subgame, the low-ranked college expects higher payoff conditional on its investment. Definition 4.7 outlines the notation I introduced here to denote the sets of applicants to each college.

Definition 4.7. *In the subgame played when $(1-p) \times h_A > h_B$, the set of applicants to the high and low-ranked college are respectively $\mu^{lo}(H, Q') := \mu^{lo}(A, Q')$ and $\mu^{s}(L, Q') := \mu^{lo}(B, Q')$.*

4.2 Expected Payoffs in Simultaneous Applications

Unlike in Sequential Applications, the difference between the investments of the high and low-ranked college influence the students these colleges are matched to in Simultaneous Applications.

In Section 4.1, we analysed two subgames where the high and low-ranked colleges are matched to different sets of students. We deduced that in the first subgame, induced by

$(1-p) \times h_A > h_B$, the high-ranked college attracts all applicants it would by virtue of offering a substantial investment premium above its rival. It expects the same students it would in equilibrium. This deteriorates the students the low-ranked college expects, as some students it would enrol in Sequential Applications gamble for the high-ranked college.

When $h_A > h_B > (1-p) \times h_A$ and colleges are closer competitors instead, allocations shift. As they fear being rejected when $\kappa = Q$, some students the high-ranked college would admit insure themselves by applying to the low-ranked college; the close competitor. This deteriorates the students the high-ranked college expects. At its expense, the low-ranked college expects better students. It captures some desirable students who would be enrolled by its rival when $\kappa = Q'$ as applicants by virtue of offering a safer option.

Below, I investigate the total ability of the students in these sets, and formalize the discussion above. These will in turn determine the payoffs colleges expect in each of these subgames conditional on their investment levels.

In the subgame played when $(1-p) \times h_A > h_B$, the high and low-ranked colleges get the applicants $\mu^s(H, Q')$ and $\mu^s(L, Q')$ respectively. The high-ranked college enrols the students $\mu^s(H, Q)$ and $\mu^s(H, Q')$, when $\kappa = Q$ and $\kappa = Q'$ respectively. Then, the expected total ability for the high-ranked college in equilibrium is Σ_H^s, previously defined.

The low-ranked college enrols the students $\mu^s(L, Q')$ when $\kappa = Q'$, but those with types $a \in [1-Q-Q', 1-Q')$ when $\kappa = Q'$. Then, the expected total ability for the low-ranked college in equilibrium is lower than that in Sequential Applications, and can be defined as:

$$\Sigma_L^{hi} := p \times \int_{1-Q-Q'}^{1-Q'} a.da + (1-p) \times \int_{1-2Q'}^{1-Q'} a.da$$

where $\Sigma_L^{hi} < \Sigma_L^s$ for all $p \in (0,1)$.

In the subgame played when $h_A > h_B > (1-p) \times h_A$, the high and low-ranked college get the applicants $\mu^{lo}(H, Q')$ and $\mu^{lo}(L, Q')$ respectively. The high-ranked college enrols the students $\mu^s(H, Q)$ when $\kappa = Q$. When $\kappa = Q'$, it enrols students with types $a \in [1-Q-Q', 1-2Q) \cup [1-Q, 1]$. Then, the expected total ability for the high-ranked college in equilibrium is lower than that in Sequential Applications, and can be defined

as:

$$\Sigma_H^{lo} := p \times \int_{1-Q}^{1} a.da + (1-p) \times \left(\int_{1-Q-Q'}^{1-2Q} a.da + \int_{1-Q}^{1} a.da \right)$$

where $\Sigma_H^{lo} < \Sigma_H^s$ for all $p \in (0,1)$.

The low-ranked college enrols the students $\mu^s(L,Q)$ when $\kappa = Q$. When $\kappa = Q'$, it enrols students with types $a \in [1-2Q', 1-Q-Q') \cup [1-2Q, 1-Q)$. Then, the expected total ability for the low-ranked college in equilibrium is higher than that in Sequential Applications, and can be defined as:

$$\Sigma_L^{lo} := p \times \int_{1-2Q}^{1-Q} a.da + (1-p) \times \left(\int_{1-2Q}^{1-Q} a.da + \int_{1-2Q'}^{1-Q-Q'} a.da \right)$$

where $\Sigma_L^{lo} > \Sigma_L^s$ for all $p \in (0,1)$.

Departing from these, Lemma 4.8 derives colleges' expected payoffs under the subgames studied in Section 4.1, illustrated by Figure 4.3. The notation is again analogous to that used in Section 3.

Lemma 4.8. *Let $h_A > h_B$ without loss of generality. $\Pi_i^j(h)$ is the payoff function defined as $\Pi_i^j(h) := (\Sigma_i^j \times h) - h^2$. Then, the expected payoff for the high-ranked college, A, under Simultaneous Applications is:*

$$\Pi_A(h_A, h_B) = \begin{cases} \Pi_H^s(h_A) & (1-p) \times h_A > h_B \\ \Pi_H^{lo}(h_A) & h_A > h_B > (1-p) \times h_A \end{cases}$$

For all $p \in (0,1)$ and $h_A > 0$, $\Pi_H^{lo}(h_A) < \Pi_H^s(h_A)$. For the low-ranked college, B, the expected payoff is:

$$\Pi_B(h_B, h_A) = \begin{cases} \Pi_L^{hi}(h_B) & (1-p) \times h_A > h_B \\ \Pi_L^{lo}(h_B) & h_A > h_B > (1-p) \times h_A \end{cases}$$

For all $p \in (0,1)$ and $h_B > 0$, $\Pi_L^{lo}(h_B) > \Pi_L^s(h_B) > \Pi_L^{hi}(h_B)$.

Any departure from stability can only leave the high-ranked college with an inferior subset of students. It cannot hope to allocatively benefit from Simultaneous Applications, at best it preserves its students in Sequential Applications.

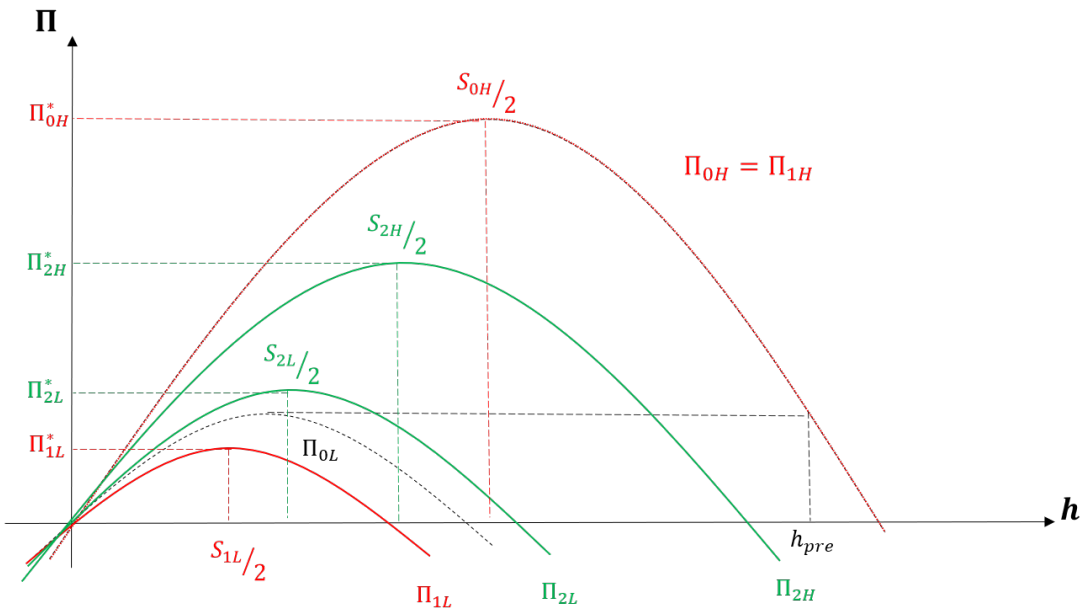


Figure 4.3: Expected Payoffs: Simultaneous Applications

On the other hand, depending on its competition, Simultaneous Applications can be a blessing or a curse for the low-ranked college. If it's strongly outcompeted by its rival, it loses the best students it could hope to attain in any equilibrium. Its expected payoff conditional on h, decreases vis-à-vis Sequential Applications. However, when its competitor is close in quality, limited applications help it lock its best students in even when its competitor would admit them, by offering them a safer option. It is then strictly better off in expectation relative to Sequential Applications.

Observe that the allocation of students to colleges is a zero-sum game: the low-ranked college is better off in this latter outcome only at the expense of its rival, who loses valuable applicants, replaced instead by worse ones. Similarly, the high-ranked college can retain all students who could be its first preference only by locking them in; avoiding its competitor from admitting them even when it rejects.

Competition for students no longer exhibits a "winner-take-all" structure, as in Sequential Applications. Colleges can now compete in different intensities. Outbidding a rival by a large margin secures all the best students' applications, leaving one's rival only with the residual applicants: those who are no college's first preference in any state. In contrast, when colleges are close competitors, the high-ranked college cannot attract all such students as applicants. Fearing rejection, some apply to the inferior college instead, insuring themselves. The resulting matching fails to sort students monotonically; when

$\kappa = Q'$, some students are matched to the worse college while their weaker peers to the best. This reduces the discrepancy between the students the higher and lower ranked colleges expect to be matched, and their expected payoffs conditional on their investments.

4.3 Investment Strategies

So far we only focused on the cases $(1-p) \times h_A > h_B$ and $h_A > h_B > (1-p) \times h_A$, but avoided $(1-p) \times h_A = h_B$ or $h_A = h_B$; sidestepping any possible indifference on students' part. Lemma 4.9 – Lemma 3.4's analogue – assures that analysing the former is sufficient to study the equilibrium under Simultaneous Applications.

When investment strategies leave some students whom colleges would admit with positive probability indifferent, they randomise between colleges. Were capacities sufficient to admit these students, a college would need to replace any such student who applied to its rival by a weaker peer. Hence, instead of a tie with its rival, the second-mover, B, always prefers marginally incrementing its bid at no cost, to attract all students who would be left indifferent.

Lemma 4.9. *In the Simultaneous Applications game, the strategies* $h_B(h_A) = (1-p) \times h_A$, $h_B(h_A) = h_A$, *and* $h_B(h_A) = \frac{h_A}{1-p}$ *are strictly dominated.*

Lemma 4.9 establishes that either of the subgames (or, their symmetries, where $h_B > h_A$ instead) will be played in the equilibrium of Simultaneous Applications. Then, we can proceed to analyse this equilibrium.

4.3.1 College B's Best Response Function: $h_B(h_A)$

Given h_A, College B's optimal investment can be found by breaking its decision to two questions: (i) which applicants are optimal to attract, and (ii) among those that attract these applicants, which level of investment is optimal. (ii) is clear once (i) is answered. As revealed in Section 4.1 and made more explicit in Lemma 4.8, h_A imposes different thresholds B must outbid to attract different sets of applicants. Figure 4.4 illustrates the thresholds and applicants B gets given how h_B relates to h_A for clarity.

The better the applicants, the higher the threshold. If the threshold h_A imposes to attract the desired applicants is below their conditionally optimal investment[11], B's best

[11] The term *conditionally optimal investment* and its notation carry over naturally from Section 3. The conditionally optimal investment for the applicants $\mu^j(i, Q')$ is $h_i^j := \arg\max_h \Pi_i^j(h) = \frac{\Sigma_i^j}{2}$.

response is to set the conditionally optimal investment. If not, since every expected payoff function is strictly concave, the best B can do is to marginally outbid this threshold.

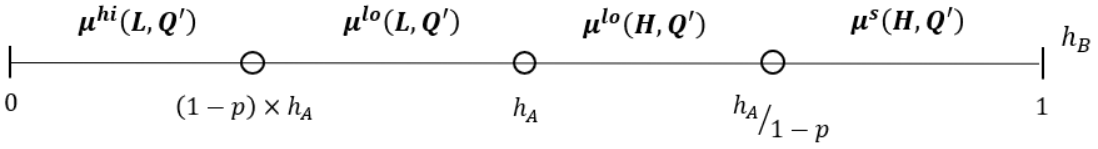


Figure 4.4: Simultaneous Admissions: Applicants B Gets Given h_A

Given h_A and the thresholds it imposes, conducting this reasoning for any possible set of applicants gives B its maximized expected payoff conditional on any set of applicants. The answer to (i) then, is simply whichever set of applicants maximizes B's expected payoff.

Observe that h_A imposes no threshold for the weakest set of applicants, $\mu^{hi}(L, Q')$. These serve as the "outside option"[12]. Excluding them, the threshold h_A imposes for any set of applicants is tighter the greater h_A. As A invests more aggressively, the opportunity cost of applying to B – the expected payoff from a forgone application to A – increases for any student admissible to A with positive probability. The higher h_A is, the higher the h_B required to justify forgoing an application to A in favour of B.

Note also that the thresholds for different sets of applicants increase at different rates with h_A. Students' expected payoffs, hence application strategies depend on how their relative admission probabilities $\left(\frac{\rho_A}{\rho_B}\right)$ compare to the relative investments colleges offer $\left(\frac{h_B}{h_A}\right)$[13]. The better the applicants, the higher their relative admission probability to A. The investment premium they demand is proportional to this probability, therefore increases at a higher rate for better applicants.

The maximum payoff B can attain with a given set of applicants given h_A can then be traced using the *constrained conditional payoff functions* I define below. Figure 4.5 illustrates these functions.

Definition 4.10. *Let* $\tilde{\Pi}^s_H$, $\tilde{\Pi}^{lo}_H$, $\tilde{\Pi}^{lo}_L$, *and* $\tilde{\Pi}^{hi}_L$ *be the constrained conditional payoff func-*

[12] This term is used similar to the way it was in Section 3. In Simultaneous Applications, any college is guaranteed applicants weakly better than $\mu^{hi}(L, Q')$.

[13] $\rho_A \times u(a, h_A) > \rho_B \times u(a, h_B) \iff \left(\frac{\rho_A}{\rho_B}\right) > \left(\frac{h_B}{h_A}\right)$

tions, defined as:

$$\tilde{\Pi}_H^s := \Pi_H^s\left(\max\left\{h_H^s, \frac{h_A}{1-p}\right\}\right)$$
$$\tilde{\Pi}_H^{lo}(h_A) := \Pi_H^{lo}\left(\max\left\{h_H^{lo}, h_A\right\}\right)$$
$$\tilde{\Pi}_L^{lo}(h_A) := \Pi_L^{lo}\left(\max\left\{h_L^{lo}, (1-p)\times h_A\right\}\right)$$
$$\tilde{\Pi}_L^{hi} := \Pi_{1L}^*$$

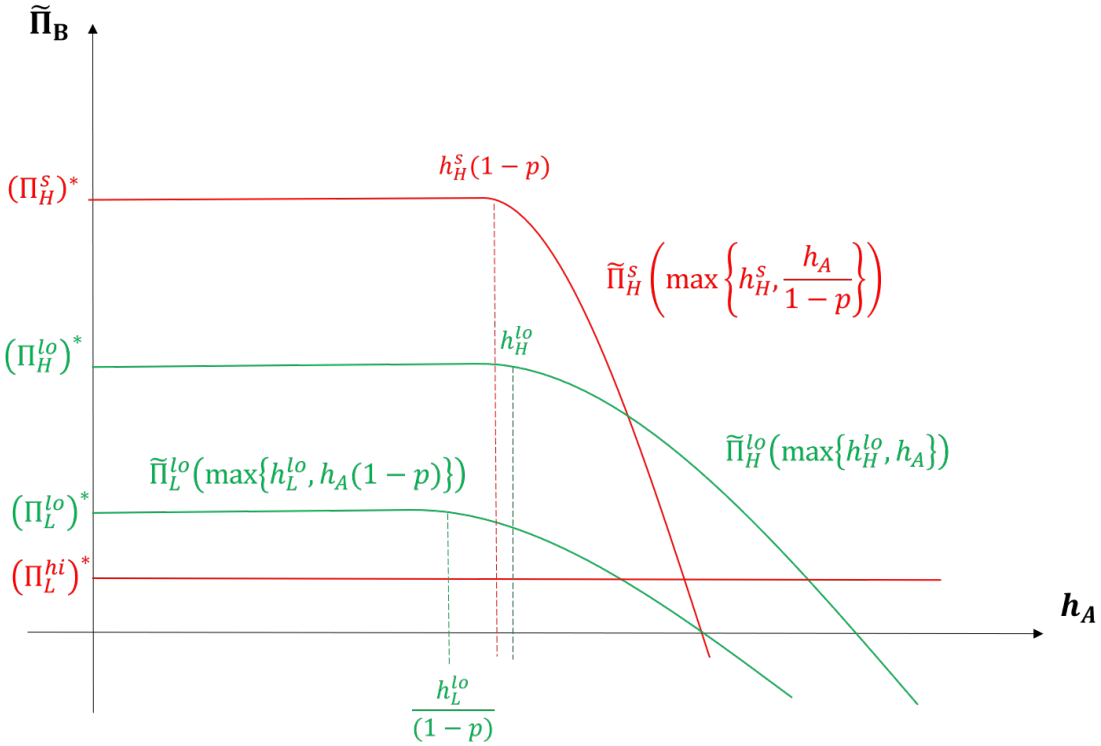


Figure 4.5: The Constrained Conditional Payoff Functions

When the threshold h_A imposes for the applicants $\mu^j(i, Q')$ does not bind, i.e. setting $h_B = h_i^j$ outbids the threshold, B can attain the maximum payoff these applicants allow for: $(\Pi_i^j)^*$. Whenever this is the case, the respective constrained conditional payoff function takes the value $(\Pi_i^j)^*$, and indicates B's maximized payoff conditional on attracting these applicants, given h_A.

When the constraint h_A imposes binds, B cannot attract $\mu^j(i, Q')$ at their conditionally optimal investment $h_B = h_i^j$. Since $\Pi_i^j(h)$ is strictly concave, it instead wants to bid investment as low as possible, should it bid higher than h_i^j. The optimal bid attracting

these applicants is only marginally above the imposed constraint[14].

Since payoff functions are continuous, B's payoff at this optimal bid is then only marginally less than the value the respective payoff function attains at the imposed threshold. When h_A is such, the respective constrained conditional payoff function attains this latter value. So, when the threshold for some applicants binds given h_A, the constrained conditional function indicates only the *supremum* for B's payoff conditional on these applicants. The maximum payoff B can attain by attracting these applicants is infinitesimally close, but never equal to this supremum at this h_A. Lemma 4.11 formalizes this discussion.

Lemma 4.11. *Given h_A, the supremum of B's expected payoff conditional on attracting the applicants $\mu_j(i, Q')$ is given by $\tilde{\Pi}_i^j(h_A)$. When the threshold h_A imposes for $\mu_j(i, Q')$ is less than h_i^j, B's maximum expected payoff equals this supremum. Otherwise, this payoff is infinitesimally less than the supremum.*

As h_A increases, competing for any set of applicants becomes (weakly) less profitable for B. As the thresholds for attracting different applicants increase, since payoff functions are concave, B's payoff conditional on attracting them can at best stay constant. This is reflected in Figure 4.5 as well, which shows that constrained conditional payoff functions are monotone decreasing in h_A.

Though the discussion so far makes this evident, it does not yet clarify which applicants B favours to compete for given h_A. Lemma 4.12 helps clarify how B's preferences between two sets of applicants change with h_A. It establishes that the constrained conditional payoff functions are single-crossing; given any two, below a single critical investment h_A – their pairwise crossing point – the function corresponding to the better students is higher, and above it, lower.

Lemma 4.12. *The constrained conditional payoff functions are pairwise single-crossing.*

Then, keeping Lemma 4.11 in mind, B's preferences between two sets of applicants depend on how h_A compares to the pairwise crossing point of their respective constrained conditional payoff functions. When h_A is less than this crossing point, B prefers to compete for the better set of applicants. When h_A exceeds this point, it instead prefers attracting the worse set of applicants.

[14]Recall from Lemma 4.9 that a tie with the threshold leaves some applicants in this set indifferent. It is never an optimal strategy for B.

Lemma 4.11 also helps us discern B's preferences when h_A equals the pairwise crossing point. Recall our analysis in Section 3. There, the threshold for the students $\mu^s(H,\kappa)$ was h_A, and for $\mu^s(L,\kappa)$, the outside option, there was none. The pairwise crossing point for these two applicants would be h_{pre}; when $h_A < h_{pre}$, B could profitably outbid A for $\mu^s(H,\kappa)$; but not otherwise. When h_A equalled the crossing point, $h_A = h_{pre}$, B would prefer the students $\mu^s(L,\kappa)$. A would attain the same payoff settling for the worse students brings, but outbidding A would yield strictly less payoff than than this.

A similar reasoning holds here. At the pairwise crossing point, the value of the payoff function corresponding to both sets of students are equal. Therefore, the threshold for the better students must be binding[15]. Then, to attract the better students, B must bid marginally above this threshold. But since this threshold is the pairwise crossing point for the corresponding sets of applicants, at bids above this level, the worse applicants are preferable to the better. Then, B prefers the worse applicants whenever h_A is weakly greater than the pairwise crossing point. Corollary 4.14 formalizes this observation.

Definition 4.13. *Define the pairwise crossing points of the constrained conditional payoff functions as follows:*

$$\tilde{\Pi}^s_H(h_{14}) = \tilde{\Pi}^{lo}_H(h_{13}) = \tilde{\Pi}^{lo}_L(h_{12}) = \Pi^*_{1L} = \tilde{\Pi}^{hi}_L(h)$$
$$\tilde{\Pi}^s_H(h_{24}) = \tilde{\Pi}^{lo}_L(h_{24}) \;\&\; \tilde{\Pi}^{lo}_H(h_{23}) = \tilde{\Pi}^{lo}_L(h_{23})$$
$$\tilde{\Pi}^s_H(h_{34}) = \tilde{\Pi}^{lo}_H(h_{34})$$

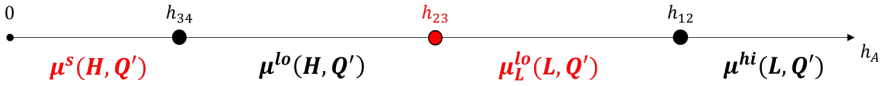
Corollary 4.14. *Take two sets of applicants $\mu_j(i,Q')$ and $\mu_{j'}(i',Q')$; where the former are better, $\Sigma^j_i > \Sigma^{j'}_{i'}$. Where h is the pairwise crossing point of their respective constrained conditional payoff functions; i.e. $\tilde{\Pi}^j_i(h) = \tilde{\Pi}^{j'}_{i'}(h)$, B prefers the better, $\mu_j(i,Q')$, when $h_A < h$, and the worse, $\mu_{j'}(i',Q')$, otherwise.*

Corollary 4.14 can be conveniently illustrated to facilitate understanding and further analysis. Suppose we revealed:

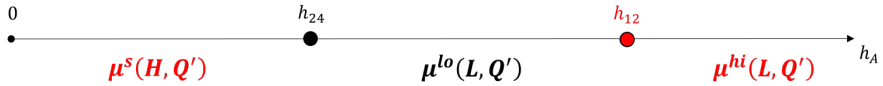
$$h_{34} < h_{24} \;\&\; h_{34} < h_{14} \tag{4.1}$$
$$h_{23} < h_{13} \tag{4.2}$$

[15]Otherwise, they could be attracted at their conditionally optimal investment, which would always yield larger payoff than their worse peers could bring.

Equation 4.1 shows that as h_A increases, the first set of applicants to dominate $\mu^s(H, Q')$ are $\mu^{lo}(H, Q')$. h_A needs to climb higher for B to prefer $\mu^{lo}(L, Q')$ or $\mu^{hi}(L, Q')$ over $\mu^s(H, Q')$. Equation 4.2 similarly shows that $\mu^{lo}(H, Q')$ is dominated first by $\mu^{lo}(L, Q')$. Figure 4.6a illustrates how B's preferences over sets of applicants evolve as h_A increases.



(a) Example 1



(b) Example 2

Figure 4.6: B's Preferred Applicants Given h_A

Suppose instead:

$$h_{24} \leq h_{34} \;\&\; h_{24} < h_{14} \tag{4.3}$$

Equation 4.3 now implies that $\mu^s(H, Q')$ is dominated first by $\mu^{lo}(L, Q')$. By Lemma 4.12, this implies that $\mu^{lo}(H, Q')$ is dominated by $\mu^{lo}(L, Q')$ as well at this point[16]. Hence, $\mu^{lo}(H, Q')$ is never B's preferred set. These preferences are illustrated in Figure 4.6b.

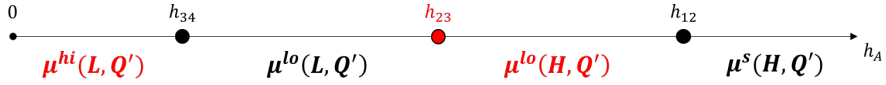
Given the pairwise crossing points, the illustrations in Figure 4.6 make mapping $h_B(h_A)$ an easy task. If the crossing points were as described by Equation 4.3, Figure 4.6b would reveal the optimal set of applicants for B for any given h_A. Given these optimal applicants, if the threshold h_A imposes binds, B's best response is to marginally outbid h_A. Otherwise, B simply sets the conditionally optimal investment to attract these applicants.

4.3.2 Optimal Investment for A Given $h_B(h_A)$

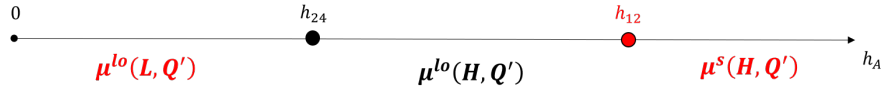
As h_A increases, the threshold it imposes to attract any set of applicants (excluding $\mu^{hi}(L, Q')$) increases; the better the applicants, the faster the rate of increase. Hence, when it confronts a larger h_A, B finds it increasingly prohibitive to compete for the better applicants. When setting h_A, alongside its marginal cost and benefit from complementing its applicants abilities, A must also consider this competitive effect; the greater h_A is, the better the applicants A retains.

[16]Suppose this were not the case. Then, $h_{23} > h_{24}$. But that implies $\tilde{\Pi}^{lo}_H(h_{24}) > \tilde{\Pi}^{lo}_L(h_{24}) = \tilde{\Pi}^{s}_H(h_{24})$, contradicting our first statement.

In equilibrium, A gets the best students who do not apply to B; the complements of B's applicants. Hence, where Figure 4.6 describes B's preferences on which applicants to attract, its complement, Figure 4.7, shows the applicants A would get given any h_A.



(a) Example 1



(b) Example 2

Figure 4.7: A's Applicants Given h_A

B's best response function, $h_B(h_A)$, determines which applicants A can attract with any h_A then. Corollary 4.15 simply establishes that A's equilibrium investment is the one which maximizes its payoff, subject to the applicants it attracts, given $h_B(h_A)$.

Corollary 4.15. *Given the pairwise crossing points and $h_B(h_A)$, the following denote the sets of invesments which match A to distinct cohorts:*

$$S_H^s := \left\{h : h > \frac{h_B(h)}{1-p}\right\}$$
$$S_H^{lo} := \left\{h : \frac{h_B(h)}{1-p} > h > h_B(h)\right\}$$
$$S_H^{lo} := \{h : h_B(h) > h > (1-p) \times h_B(h)\}$$
$$S_H^{hi} := \{h : (1-p) \times h_B(h) > h\}$$

Where h_A^* denotes A's investment strategy in the SPNE of the Simultaneous Applications game, $h_A^* = \arg\max_h \Pi_A(h)$ where $\Pi_A(h) = \Pi_i^j(h)$ for $h \in S_i^j$.

4.4 Equilibrium in Simultaneous Applications

As this analysis reveals, pinning down equilibria in Simultaneous Applications is less straightforward than in Sequential Applications. It's difficult to characterise how equilibria change as the parameters Q, Q', and p shift. Nonetheless, analysing the equilibrium for sufficiently large p; i.e. for $\kappa = Q'$ sufficiently unlikely, unveils a striking result: when excess capacities – $\kappa = Q'$ – are sufficiently unlikely, the equilibrium outcome in Simulta-

neous Applications is always determined[17]. In this equilibrium, both colleges have strictly greater payoff relative to Sequential Applications, despite one being allocatively worse off.

Proposition 4.16 establishes that when it is sufficiently unlikely that $\kappa = Q'$, in equilibrium, colleges align as *close competitors*. The difference between their investment levels does not suffice for the high-ranked college to retain all students it could admit as applicants. It loses some such students to its rival as insurance applicants. Subsequently, it enrols worse students than it would in the stable outcome to the benefit of its rival, who enrols better ones.

As p grows and $\kappa = Q'$ becomes less likely, it becomes increasingly risky to gamble for the high-ranked college. The probability that it will have sufficient capacity to enrol gamble applicants decreases. This increases the investment premium these students demand to make a gamble application. Then, competing for all best applicants, including these students becomes prohibitively costly for both colleges when p is sufficiently large. For B, given h_A, the premium B must bid to attract gamble applicants grows exceedingly large. Likewise, B can challenge A as the low-ranked college for these gamble applicants by setting $h_B > (1-p) \times h_A$. For this to become prohibitively expensive, h_A must be exceedingly large.

Then, for sufficiently large p, no college finds it profitable to compete for all best applicants. Instead, they compete as close competitors; the high-ranked college concedes some strong students whom it would admit to the low-ranked college as insurance applicants.

Proposition 4.16. *Assume colleges break ties between investments in favour of better students. Then, there exists some $\bar{p} \in (0,1)$ such that for $p > \bar{p}$, only one outcome can be supported in any SPNE of Simultaneous Applications. In any such SPNE, colleges set equilibrium investments $(h_A^*, h_B^*) = (h_{23}, h_L^{lo})$, students' application strategies are described by $\hat{\sigma}^{lo}(a)$, and the resulting matching is $\mu^{lo}(a, \kappa)$. Both colleges' equilibrium expected payoffs are $(\Pi_L^{lo})^*$.*

Note that this not only increases the payoff of the low-ranked college, but of both colleges. Previous literature pointed out that when colleges are close competitors, limited applications and enrolment uncertainty can benefit the low-ranked college by attracting strong, but cautious applicants. This model replicates this finding. When colleges are close competitors, the low-ranked college locks insurance applicants in, even when they

[17] Keeping our initial assumption that colleges break ties in favour of better students.

would be preferable by its rival. Proposition 4.16 shows that in equilibrium, colleges align as close competitors, and the low-ranked college enrols better students than it would in the stable matching at the conditionally optimal investment. Hence, as established by previous literature, this increase in its students abilities translates to an increase in its equilibrium payoff.

Previous literature established that this would cause the high-ranked college enrol weaker students in equilibrium. Holding colleges' investments, i.e. students' preferences, constant, the allocation of students to colleges is a zero-sum game. As some students the high-ranked college would enrol with positive probability apply to its rival, it replaces them with strictly worse students. However, when the endogeneity of students' preferences is taken into account, we reveal that this does not imply a lower equilibrium payoff for the high-ranked college. As previous models in the literature assumed exogenous preferences for students, this result was overlooked.

In Sequential Applications, competition takes a winner-take-all structure. The outcompeted college settles for a strictly worse cohort, therefore, a staunch competition for the better students ensues. The high-ranked college can only secure them by bidding investment high enough to erode any payoff it could expect above its rival's outside option. At any investment level which would bring higher payoff, it would be outcompeted by its rival. Consequently, its equilibrium payoff is determined by the outside option of its rival.

The frictions in Simultaneous Applications impose ameliorate the competition between colleges by increasing the outside option of the outcompeted college, improving the students the low-ranked college is matched to. Thus, they generate a favourable competitive effect. As the payoff to being the outcompeted college increases, colleges no longer compete as willingly to become the high-ranked college and attract the better set of students. The students the high-ranked college is matched to deteriorates in favour of its rival, but it can compete for them at a bid which leaves it better off relative to Sequential Applications without fearing being outbid. It gets worse students, but with a less aggressive competitor, can compete for them with a more favourable bid.

In both equilibria discussed, the payoff of the high-ranked college depends on its rival's payoff. Its bid must ensure that its rival's payoff from settling for its outside option is higher than from than outbidding it. By redistributing the students, or prizes, in this game, the frictions in Simultaneous Applications increase this outside option. Though this comes at the expense of the prize the high-ranked college gets, it is better off nonetheless

due to the favourable competitive effect this increase in the outside option has.

This favourable competitive effect only arises when the endogeneity of students' preferences is incorporated to the model. Without costly competition for students, the allocation of students to colleges is a zero-sum game, a college gets better students only at the expense of its rival. Once this costly competition is accounted for however, by conceding some strong applicants to the low-ranked college, the high-ranked college can also be better off, as this ameliorates the competition for students.

5 Equilibrium Under Endogenous Application Dates

We derived the unique SPNE in Sequential Applications, and the unique outcome of any SPNE in Simultaneous Applications, when p is large enough. We assumed that agents play their weakly dominant strategies where available. Where multiple equilibria were possible – as two investment strategies brought equal equilibrium payoff – we assumed that colleges broke ties between investment strategies by choosing the one attracting the best set of students.

In Sequential Applications, these assumptions single out an equilibrium where both colleges' payoffs are pinned down by the outside option of the low-ranked college. The high-ranked college cannot attract the better students at a higher payoff; similarly to a Bertrand competition, it would then be outbid by its rival.

In Simultaneous Applications, when p is large enough, competing for all applicants the high-ranked college would enrol in the stable matching becomes prohibitively costly. Applicants who would only be admitted when $\kappa = Q'$ demand an exorbitant investment premium to apply to this college. Therefore, in equilibrium, the high-ranked college loses such favourable students as insurance applicants to its rival. By virtue of this, the low-ranked college expects higher payoff. In turn, this creates a favourable competitive effect. Facing a less aggressive competitor, the high-ranked college can compete for the better students at higher payoff. The ability of the students it expects deteriorates, but its payoff increases nonetheless due to the ameliorated competitive environment.

These games differ uniquely in when colleges accept applications, where these dates are exogenous. Now, I build a larger game where this is endogenous, referring to it as the "extended college admissions game". In this game, colleges sequentially choose their admission dates; $t_A, t_B \in \{1, 2\}$. When $t_A = t_B$, the Simultaneous Applications game

follows. When $t_A \neq t_B$ instead, the Sequential Applications game follows. While these two subgames start by h_A being set first, I analyse the equilibrium of the extended college admissions game both when t_A and t_B are set first.

Now let p be large enough so that Proposition 4.16 holds. Then, recalling Proposition 3.7, in Simultaneous Applications, both colleges achieve higher equilibrium payoffs than in Sequential Applications. Consequently, regardless of which college sets its date last, it conflicts its admission date with its rival, setting its admission date equal to what the first-mover set. Subsequently, the Simultaneous Applications subgame is played in the SPNE, regardless of which college moves first to set its date.

Proposition 5.1. *Assume ties between investments are broken in favour of better students. For $p \in (0,1)$ such that Proposition 4.16 holds, in any SPNE of the extended college admissions game, colleges set equilibrium admission dates such that $t_A^* \neq t_B^*$, and the Simultaneous Admissions subgame is played.*

When p is sufficiently large, making $\kappa = Q'$ increasingly unlikely, both colleges' payoffs increase in the Simultaneous Applications game relative to Sequential Applications. The frictions turn to a competitive weapon for both colleges, as they mitigate the competition for the better students. The low-ranked college gains due to the better students it enrols, and the high-ranked college from the competitive environment this ameliorates.

Hence, when p is such, regardless of the college determining the subgame to be played in the extended college admissions game, the Simultaneous Admissions subgame is played. The equilibrium exhibits justified envy, and the high-ranked college enrols worse students than it would in the stable matching. However, both colleges are better off relative to the stable college admissions mechanism, due to the competitive effect these frictions generate.

6 Conclusion

Gale and Shapley (1962) showed that a stable matching exists in any college admissions game, and proposed a simple mechanism that always implements it. Yet, many college admission systems are still puzzlingly far from this mechanism, exhibiting two key frictions: costly, or restricted applications, and admission uncertainties. Previous studies converged on the prediction that these frictions might benefit a low-ranked college by distorting stability in its favour. If students' preferences between this college and its

high-ranked rival are not sufficiently strong, the low-ranked college can attract cautious applicants its rival would admit. This comes at the expense of the high-ranked college, which is deprived of these desirable applicants.

These studies treat students' preferences towards colleges – and colleges' rankings – as exogenous. However, colleges spend substantial effort to improve their rankings and affect students' preferences by various means. These efforts are not limited to strategically setting their admission policies, which previous studies focus on.

With this motivation, I paper attempt to extend this literature by endogenising students' preferences for colleges. I analyse a model where colleges compete for better applicants by bidding costly investments. Confirming earlier studies, I find that when colleges are close competitors, the low-ranked college can benefit from a single application limit and uncertainty. These frictions distort the stable matching in its favour, helping it attract cautious applicants desirable by its rival. In similar accord, I find that this deteriorates the quality of the students the high-ranked college expects.

However, endogenising the costly competition for better students affords the novel insight that this can nonetheless improve the high-ranked college's payoff as well, contradicting what previous studies implied. This owes to a favourable competitive effect: as the students the low-ranked college gets improve, it becomes less willing to compete for the better set of students. Consequently, though the high-ranked college gets worse students, facing mitigated competition, it can compete for them at a higher payoff without fearing being outbid by its rival. Thus, these frictions can become a competitive weapon for both colleges: the low-ranked college benefits as the students it enrols improve, and the high-ranked college benefits as the competition for better students is mitigated.

I find that this outcome characterises the equilibrium of Simultaneous Applications when it is sufficiently unlikely that the high-ranked college will have enough capacity to admit all best applicants. In this equilibrium, both colleges are made better off relative to Sequential Applications, which always implements the stable matching.

I then construct a larger game where colleges endogenously set their admission policies before the admissions game is played. When the equilibrium in Simultaneous Applications is as described, both colleges choose to conflict their admissions date with that of their rival, thereby imposing a single application limit to students.

This paper highlights that admission uncertainties and application limits can become a competitive weapon for both high-ranked and low-ranked colleges in their competition to

attract better students. They might help low-ranked colleges attract desirable applicants away from their high-ranked rivals, by virtue of being safer options. In turn, high-ranked colleges can also benefit as this serves to mitigate the competition for the best applicants, allowing to compete for them at a higher payoff. Therefore, I find that a consensus towards centralized, stable mechanisms might be impeded not only because they leave the low-ranked college worse off; but also because the high-ranked college suffers as a result of the staunch competition this introduces.

Shortcomings and Avenues for Future Research

The model this paper analyses imposes simplifying assumptions which aid analysis. Foremost, it assumes that there are no side payments between students and colleges; i.e. that utility is non-transferable. Therefore, colleges compete for students by investing sub-optimally high towards human capital production. As costs to investment are borne solitarily by colleges, students' preferences are monotone in this investment. However, colleges might also be found to compete for talented students by other means which could be modelled in the form of side-payments: by levying, or waiving tuition, for instance.

Empirical studies and evidence are indicative of such direct transfers having an increasingly limited role in higher education markets. In the United Kingdom, tuition fees universities can charge are capped[18]. Hoxby (2009) documents that especially the most selective colleges in the United States rely significantly on future payments by students as donations to finance their investments. She also highlights that, especially in such colleges, the human capital investment students receive substantially exceeds the tuition they pay, a trend which has become more pronounced over time. Donations to colleges might be reasonably expected to depend on the earnings of their alumni in the labour market. These in turn, are linked to the human capital investment they receive whilst in college. Where these trends become increasingly relevant, modelling college markets as such promises to bring yet more informative results.

This analysis also departs from earlier studies in the way admission uncertainty is modelled. The literature has favoured adopting the more elegant and realistic approach of modelling a noisy screening of students' abilities. In such models, either colleges imperfectly screen students' abilities, or students have imperfect information of their types,

[18] "How much does university really cost students – and their parents?", Debbie Andalo, *The Guardian*, July 2, 2014. Retrieved May 27, 2020, from `https://www.theguardian.com/education/2014/jul/02/how-much-does-university-cost-students-parents`

revealed only once they apply. Subsequently, students face admission uncertainty; their types might be revealed to be very discrepant than their signal, or colleges might err in their evaluations of the student.

The novelty of this model is endogenising colleges' costly competition for strong applicants. In this framework, modelling uncertainty as such markedly complicates the analysis. First, in these models, students' relative admission probabilities vary continuously with their types. This would lead to application strategies changing continuously with relative investments in the Simultaneous Applications game we analyse; substanially complicating equilibrium analysis. Further, Chade, Lewis, and Smith (2014) show that such models can be prone to multiple equilibria in application strategies; with characteristics depending on college qualities, capacities, and signal structures. This risks complicating analysis further and making progress harder.

Modelling admission uncertainty as a potential discrete increase in capacities retains the key strategic effect of uncertainty on students' application behaviour that the previous literature captures, and builds on. This model successfully captures such central strategic behaviour on students' part, such as non-monotonicity in application strategies[19] and *gamble* and *insurance* applications from students are reminiscent of earlier studies' findings. The simple uncertainty structure it imposes pins student behaviour down, freeing us from dealing with a multiplicity of equilibrium strategies. By imposing a discrete nature to the uncertainty students face, it avoids the complications that could arise from continuously varying admission probabilities. At the cost of decreased elegance, it affords a more general analysis and economises on analytical complications.

This study also hopes to inspire future research. It contributes to existing literature on college admissions with frictions by analysing them as matching markets where one side commits to a costly investment, which affects the payoffs of both sides of the match. This affords a new insight into how decentralized admissions mechanisms might affect competition between colleges. Further insight into college markets and admissions processes can be gained by expanding this model further, and analysing college markets where students can also invest prior to matchings.

Several studies have examined matching markets with ex-ante investments. Felli

[19]Echoing Chade, Lewis, and Smith (2014), this refers to students with lower ability forming more aggressive application portfolios than their stronger peers. In this model, this emerges in the subgame of the Simultaneous Applications game where $h_A > h_B > (1-p) \times h_A$; some weaker applicants apply to the high-ranked college A while their stronger peers apply to its rival, B.

and Roberts (2016) analyse a market where heterogenous buyers and sellers make ex-ante investments, and are matched by Bertrand competition; in a one-to-one context. Burdett and Coles (2001) study a similar market with overlapping generations and non-transferable utilities. Mailath, Postlewaite, and Samuelson (2013) study how the absence of personalized pricing can cause inefficiencies in a marriage market with ex ante investments. Integrating elements from these models towards a richer examination of college admission mechanisms is a promising avenue for future research.

Previous literature that this paper builds on offered compelling insights into the strategic interactions between colleges. These were subsequently backed with empirical evidence. Avery and Levin (2010) refer to convincing anecdotal and empirical evidence to illustrate the relevance of the strategic interactions predicted by their model. Avery, Lee, and Roth (2014) depart from substantial empirical evidence to formulate their model, and relate how the strategic interactions their model suggests can relate to this evidence; with some predictions of the model proving more robust than others.

This model offers novel insights on the strategic interaction and competition between colleges; suggesting that frictions in admissions mechanisms might favour colleges by mitigating staunch competition for better students. This prediction awaits empirical testing, which would also inform models better suited to understand how the strategic interactions between colleges shape competition.

7 Omitted Results

Proposition 7.1. *There exists some* $\bar{p} \in (0,1)$ *such that for all* $p > \bar{p}$, $h_{34} < h_H^{lo}$, $h_{23} < \frac{h_L^{lo}}{1-p}$, *and* $h \geq h_{12}$ *implies* $\Pi_H^s(h) < (\Pi_L^{lo})^*$.

Proof. Recall that $h_H^{lo} = \frac{\Sigma_H^{lo}}{2}$, and $h_L^{lo} = \frac{\Sigma_L^{lo}}{2}$ are the corresponding conditionally optimal investment levels.

I show that for large enough p, each individual claim holds:

(i) Consider h_{34}, outlined in Equation 8.4. Note that Σ_H^s and Σ_H^{lo} are continuous and bounded functions of $p \in (0,1)$. Therefore, $\lim_{p\to 1}(1-p) \times \frac{\Sigma_H^s - \Sigma_H^{lo}(1-p)}{p(2-p)} = 0$. As $\frac{\Sigma_H^{lo}}{2}$ is bounded below by $\Sigma_{H,Q}^s > 0$, by continuity there is some $\bar{p_1} \in (0,1)$ such that for all $p > \bar{p_1}$, $(1-p) \times \frac{\Sigma_H^s - \Sigma_H^{lo}(1-p)}{p(2-p)} < \frac{\Sigma_H^{lo}}{2}$ and $h_{34} = \frac{(1-p)}{2}\left(\Sigma_H^s + \sqrt{(\Sigma_H^s)^2 - (\Sigma_H^{lo})^2}\right)$. Again, as Σ_H^s and Σ_H^{lo} are continuous and bounded by positive real numbers for $p \in (0,1)$, there is some $\bar{p_2} \in (0,1)$ such that $p > \bar{p_2}$ implies $h_{34} < \frac{\Sigma_H^{lo}}{2}$. This proves the first claim; for $p > \max\{\bar{p_1}, \bar{p_2}\}$, $h_{34} < \frac{\Sigma_H^{lo}}{2} = h_H^{lo}$.

(ii) Now consider h_{23}, outlined in Equation 8.6. The inequality $\frac{\frac{\Sigma_H^s}{1-p} - (1-p).\Sigma_L^{lo}}{\frac{1}{(1-p)^2} - (1-p)^2} \geq \frac{\Sigma_L^{lo}}{2(1-p)}$ can be written equivalently as $\frac{\Sigma_H^s.(1-p) - (1-p)^3.\Sigma_L^{lo}}{1-(1-p)^4} \geq \frac{\Sigma_L^{lo}}{2(1-p)}$. Due to similar continuity and boundedness assumptions, as $p \to 1$, the LHS tends to 0, and the RHS to ∞. Similar to (i), there exists some $\bar{p_3} < 1$ such that $p > \bar{p_3}$ implies $\frac{\frac{\Sigma_H^s}{1-p} - (1-p).\Sigma_L^{lo}}{\frac{1}{(1-p)^2} - (1-p)^2} < \frac{\Sigma_L^{lo}}{2(1-p)}$ then. Subsequently, for $p > \bar{p_3}$, $h_{24} = \frac{(1-p)}{2}.\left(\Sigma_H^s + \sqrt{(\Sigma_H^s)^2 - (\Sigma_L^{lo})^2}\right)$. But then, again similarly to (i), for some $\bar{p_4} < 1$, $p > \bar{p_4}$ implies that $\frac{(1-p)}{2}.\left(\Sigma_H^s + \sqrt{(\Sigma_H^s)^2 - (\Sigma_L^{lo})^2}\right) < \frac{\Sigma_L^{lo}}{2(1-p)} = \frac{h_L^{lo}}{1-p}$. This proves our second claim.

(iii) $\Pi_H^s(h) \geq (\Pi_L^{lo})^* \iff h \in \left[\frac{1}{2}\left(\Sigma_H^s - \sqrt{(\Sigma_H^s)^2 - (\Sigma_L^{lo})^2}\right), \frac{1}{2}\left(\Sigma_H^s + \sqrt{(\Sigma_H^s)^2 - (\Sigma_L^{lo})^2}\right)\right]$. Note from Equation 8.3 that, since Σ_L^{lo} and Σ_L^{hi} are bounded for $p \in (0,1)$, $\lim_{p\to 1} h_{12} = \infty$, but $\lim_{p\to 1}\frac{1}{2}\left(\Sigma_H^s + \sqrt{(\Sigma_H^s)^2 - (\Sigma_L^{lo})^2}\right)$ is bounded above. Since h_{12} is continuous for $p < 1$, there exists some $\bar{p_5} < 1$ such that $p > \bar{p_5}$ implies $\Pi_H^s(h) < (\Pi_L^{lo})^*$ when $h \geq h_{12}$.

For $\bar{p} := \max\{\bar{p_1}, \bar{p_2}, \bar{p_3}, \bar{p_4}, \bar{p_5}\}$, subcases (i), (ii), and (iii) hold when $p > \bar{p}$; proving Proposition 7.1.

□

Corollary 7.2. *When Proposition 7.1 holds, $h_{34} < h_{24}$, $h_{34} < h_{14}$, and $h_{23} < h_{13}$.*

Proof. $h_{34} < h_H^{lo}$ implies that $\tilde{\Pi}_H^{lo}(h_{34}) = (\Pi_H^{lo})^* = \tilde{\Pi}_H^s(h_{34})$. Since for all $h \geq 0$, $\tilde{\Pi}_L^{lo}(h) < (\Pi_H^{lo})^*$, $\tilde{\Pi}_L^{lo}(h_{24}) = \tilde{\Pi}_H^s(h_{24}) < (\Pi_H^{lo})^*$. As $\tilde{\Pi}_H^s(h)$ is monotone non-increasing, this implies that $h_{34} < h_{24}$. $\tilde{\Pi}_H^s(h_{14}) < (\Pi_H^{lo})^*$ similarly; implying $h_{34} < h_{14}$.

$h_{23} < \frac{h_L^{lo}}{1-p}$implies that $\tilde{\Pi}_L^{lo}(h_{23}) = (\Pi_L^{lo})^* = \tilde{\Pi}_H^{lo}(h_{23})$. As $\tilde{\Pi}_H^{lo}(h_{13}) = (\Pi_L^{hi})^*$, $h_{23} < h_{13}$ similarly. □

8 Omitted Proofs

Section 3

Lemma 3.2: Proof. That $\mu^s(a, \kappa)$ is individually rational is clear. Since A strictly prefers any student it's matched to over any student it's not under any κ, no pairwise block can involve A. Any pairwise block involving B must involve a student with type $a \geq 1 - \kappa$. However these students are matched to A, which they strictly prefer over B. Hence, $\mu^s(a, \kappa)$ is stable.

Now consider any other stable matching $\mu(a, \kappa) \neq \mu^s(a, \kappa)$. Since all students are acceptable and preferences are responsive, no matching where some college has unsaturated capacity can be stable, so restrict attention to matchings where colleges' capacities are saturated. $\mu(a, \kappa)$'s stability implies $\mu(A, \kappa) = \mu^s(A, \kappa)$; otherwise A would pairwise block μ with any student of type a such that $\mu(a, \kappa) \neq A$ but $\mu^s(a, \kappa) = A$.

But then, $\mu(B, \kappa) = \mu^s(B, \kappa)$ also; since students matched to A are unattainable for B, and any student with type a such that $\mu^s(a, \kappa) = B$ is strictly preferable over any with type $\mu^s(a, \kappa) = \emptyset$. $\mu = \mu^s$ therefore; yielding a contradiction. □

Lemma 3.3: Proof. The college-proposing Gale and Shapley algorithm is stable. In the Sequential Admissions game, students follow their weakly dominant strategy and apply to both colleges, meaning the Gale and Shapley algorithm will not be impeded towards implementing a stable matching. Lemma 3.2 established that the unique stable matching with respect to the described preferences for colleges and students is the positive assortative matching, μ^s. Therefore the outcome of the algorithm is μ^s. □

Claim 3.5: Proof. Note that $\Pi_H^s(h)$ is continuously differentiable in h. $\bar{h} > \frac{\Sigma_H^s}{2}$, as $\Pi_H^s(\frac{\Sigma_H^s}{2}) = (\Pi_H^s)^* > (\Pi_L^s)^*$. For all $h > \frac{\Sigma_H^s}{2}$, $\Pi_H^s{}'(h) < 0$ and $\Pi_H^s(h_{pre}) = (\Pi_L^s)^*$.

Therefore, for all $h \in \left[\frac{\Sigma_H^s}{2}, h_{pre}\right]$, $\Pi_H^s(h) > (\Pi_L^s)^*$ and for all $h > h_{pre}$, $\Pi_H^s(h) < (\Pi_L^s)^*$; proving the claim. □

Lemma 3.6: Proof. Due to Claim 3.5, for all $h < h_{pre}$, there exists some $h' > h$ such that $\Pi_H^s(h') > (\Pi_L^s)^*$. Since ties are never optimal by Lemma 3.4, for all $h_A < h_{pre}$, B's best response is to outbid A; $h_B^s(h_A) > h_A$, where $\Pi_H^s(h_B^s(h_A)) > (\Pi_L^s)^*$. Then, for all $h_A < h_{pre}$, $\Pi_A(h_A) = \Pi_L^s(h_A)$ where Π_A is A's expected payoff. Hence, any $h_A < h_{pre}$ is strictly dominated by $h_A = \frac{\Sigma_L^s}{2}$,which yields expected payoff $\Pi_A\left(\frac{\Sigma_L^s}{2}\right) = (\Pi_L^s)^*$. B's best response to this strategy is $h_B^s\left(\frac{\Sigma_L^s}{2}\right) = \arg\max_{h|h>\frac{\Sigma_L^s}{2}} \Pi_H^s(h) = \frac{\Sigma_H^s}{2}$.

On the other hand, for $h > h_{pre}$, $\Pi_H^s(h) < (\Pi_L^s)^*$; again due to Claim 3.5. Using Lemma 3.4 again, this implies that B's best response to $h_A \geq h_{pre}$ is $h_B^s(h_A) = \frac{\Sigma_L^s}{2} < h_{pre}$, yielding $\Pi_B = (\Pi_L^s)^*$. Then, for $h_A \geq h_{pre}$, $\Pi_A(h_A) = \Pi_H^s(h_A)$, and since ${\Pi_H^s}'(h) < 0$ for all $h \geq h_{pre}$, any $h_A > h_{pre}$ is strictly dominated by $h_A = h_{pre}$, which yields expected equilibrium payoff $\Pi_A = (\Pi_L^s)^*$.

Since $h_A = \frac{\Sigma_L^s}{2}$ and $h_A = h_{pre}$ yield the same equilibrium payoff $(\Pi_L^s)^*$ for A, it is indifferent between the two. In any SPNE, only these strategies can be played with positive probability, as they strictly dominate the rest. □

Proposition 3.7. By Lemma 3.6, only $h_A = \frac{\Sigma_L^s}{2}$ and $h_A = h_{pre}$ are played with positive probability in any SPNE, and yield the same equilibrium payoff. $h_B^s(\frac{\Sigma_L^s}{2}) = \frac{\Sigma_H^s}{2} > \frac{\Sigma_L^s}{2}$. Then, $h_A = \frac{\Sigma_L^s}{2}$ matches A to the students $\mu^s(L,\kappa)$. $h_B^*(h_{pre}) = \frac{\Sigma_L^s}{2} < h_{pre}$, so A is matched to $\mu^s(H,\kappa)$. Assuming ties between investments are broken in favour of the better students, as $\mu^s(H,\kappa)$ are preferable to $\mu^s(L,\kappa)$, $h_A = h_{pre}$ is the unique equilibrium strategy to survive. $h_A = h_{pre}$ is then the unique equilibrium strategy A employs in the SPNE. □

Section 4

Proposition 4.3: Proof. I use the Iterated Elimination of Strictly Dominated Strategies (IESDS).

Consider students with types $a \geq 1-Q$. As no college can form a cohort of size $\kappa = Q$ with students of types $a' > a$, for any profile of strategies σ_{-a}, $\rho_A(a,\sigma_{-a}) = \rho_B(a,\sigma_{-a}) = 1$. Hence, $h_A > h_B \implies \frac{h_A}{h_B} > \frac{\rho_B(a,\sigma_{-a})}{\rho_A(a,\sigma_{-a})}$. Then, $\hat{\sigma}^{hi}(a) = A$.

Now consider $a \in [1-Q', 1-Q)$. Since no college can form a cohort of size $\kappa = Q'$ from students of types $a' > a$, $\rho_A(a,\sigma_{-a}) \geq (1-p)$ for any profile of strategies σ_{-a}. Therefore,

for any $\rho_B(a, \sigma_{-a}) \leq 1$, $(1-p) \times h_A > h_B \implies \frac{h_A}{h_B} > \frac{\rho_B(a,\sigma_{-a})}{\rho_A(a,\sigma_{-a})}$. Then, $\hat{\sigma}^{hi}(a) = A$.

The unique strategy surviving IESDS is $\hat{\sigma}^{hi}(a) = A$ for students with types $a \geq 1-Q'$. Then, by IESDS, $\rho_A(a, \hat{\sigma}^{hi}_{-a}) = 0$ for $a < 1-Q'$. However, excluding those who applied to A, the students with $1-2Q' \leq a < 1-Q'$ are the students with the highest ability of mass Q'. Then, $1-2Q' \leq a < 1-Q' \implies \rho_B(a, \hat{\sigma}^{hi}) > 0$ by IESDS. Therefore, for students with $a \geq \delta^{Q'}_{0h}$, $\hat{\sigma}^{hi}(a) = B$.

By IESDS, students with $a < \delta^{Q'}_{0l}$ can never be admitted to any college. They are then indifferent between any application strategy. These equilibrium strategies straightforwardly pin μ^{hi} down for $\kappa \in \{Q, Q'\}$. □

Lemma 4.4: Proof. Consider $h_A > h_B$, without loss of generality. By Lemma 3.2, μ^s is the unique stable matching. $\mu^{hi}(a, Q') = \mu^s(a, Q')$, hence it's stable. Since $\mu^{hi}(a, Q) \neq \mu^s(a, Q)$, $\mu^{hi}(a, Q)$ is not stable. Consider a student with type $a \in [1-Q', 1-Q)$. For this student, $\mu^{hi}(a, Q) = \emptyset$. B prefers her to any student it's matched to since for all $a' \in \mu^{hi}(B, Q)$, $a > a'$. She prefers being matched to B rather than $\emptyset$. Then, $\mu^{hi}(a, Q)$ causes justified envy. □

Proposition 4.5: Proof. The proof is again by IESDS.

As established in Proposition 4.3's proof, $a \geq 1-Q$ implies $\hat{\sigma}^{lo}(a) = A$ when $h_A > h_B$. Then, by IESDS, for $a \in [1-2Q, 1-Q)$, $\rho_A(a, \hat{\sigma}_{-a}) \leq (1-p)$. However, also by IESDS, B cannot from a cohort of size Q from students with $a' > a$ for any $a \in [1-2Q, 1-Q)$. Therefore, $a \in [1-2Q, 1-Q) \implies \rho_B(a, \hat{\sigma}^{lo}_{-a}) = 1$ by IESDS. $(1-p) \times h_A < h_B \implies \frac{h_A}{h_B} < \frac{\rho_B(a,\hat{\sigma}_{-a})}{\rho_A(a,\hat{\sigma}_{-a})}$ then, implying $\hat{\sigma}^{lo}(a) = B$.

Now consider students with $a \in [1-Q-Q'1-2Q)$. Given $\hat{\sigma}^{lo}(a)$ for $a \geq 1-2Q$, these students are not admissible at either college when $\kappa = Q$. However, when $\kappa = Q'$, they are the best students with mass $(Q'-Q)$ excluding those who fill colleges' quotas when $\kappa = Q$, hence are admitted by either college. Then, $a \in [1-Q-Q'1-2Q) \implies \rho_A(a, \hat{\sigma}^{lo}_{-a}) = \rho_B(a, \hat{\sigma}^{lo}_{-a}) = (1-p)$ by IESDS. This implies that $\frac{h_A}{h_B} > \frac{\rho_B(a,\hat{\sigma}^{lo}_{-a})}{\rho_A(a,\hat{\sigma}^{lo}_{-a})}$ for all $a \in [1-Q-Q', 1-2Q)$. Then, $\hat{\sigma}^{lo}(a) = A$.

Finally, consider $a \in [1-2Q'1-Q-Q')$. Given the strategies of their peers analysed so far, $\rho_A(a, \hat{\sigma}^{lo}_{-a}) = 0$ evidently, but $\rho_B(a, \hat{\sigma}^{lo}_{-a}) = (1-p)$. Then, $\hat{\sigma}^{lo}(a) = B$, again by IESDS.

As in Proposition 4.3, for any $a < 1-2Q'$, $\rho_A(a, \hat{\sigma}^{lo}_{-a}) = \rho_B(a, \hat{\sigma}^{lo}_{-a}) = 0$ by IESDS; implying any $\hat{\sigma}^{lo} \in \triangle\{A, B, \emptyset\}$ constitutes an equilibrium strategy. These students are

never matched to any college.

These equilibrium strategies straightforwardly pin the equilibrium matching down as μ^{lo}. □

Lemma 4.6: Proof. Consider $h_A > h_B$, without loss of generality. By Lemma 3.2, $\mu^{lo}(a, Q') \neq \mu^s(a, Q')$ implies that $\mu^{lo}(a, Q')$ is not stable. As $\mu^{lo}(a, Q) = \mu^s(a, Q)$, $\mu^{lo}(a, Q)$ is pairwise stable. Consider a student with type $a \in [1 - 2Q, 1 - Q)$. $\mu^{lo}(a, Q') = B$; but there exists some $a' < a$ such that $\mu^{lo}(a', Q') = A$. A prefers a over a'. As any student also prefers A over B, $\mu^{lo}(a, Q')$ causes justified envy. □

Lemma 4.8: Proof. Suppose $(1-p) \times h_A > h_B$. The applicants to A and B are $\mu^s(H, Q')$ and $\mu^s(L, Q')$ respectively. The total ability these colleges expect to enrol are then respectively Σ^s_H and Σ^{hi}_L. For all $p \in (0, 1)$, $\Sigma^{hi}_L < \Sigma^s_H$. Subsequently, conditional on their investment levels, their expected payoffs are $\Pi_A(h_A) = (\Sigma^s_H \times h_A) - (h_A)^2 = \Pi^s_H(h_A)$, and $\Pi_B(h_B) = (\Sigma^{hi}_L \times h_B) - (h_B)^2 < \Pi^s_L(h_B)$ respectively, for all $h_B > 0$ and $p \in (0, 1)$.

Now let $h_A > h_B > (1 - p) \times h_A$. The applicants to A and B are $\mu^{lo}(H, Q')$ and $\mu^{lo}(L, Q')$ respectively; and the total ability of those they expect to enrol are $\Sigma^{lo}_H < \Sigma^s_H$ and $\Sigma^{lo}_L > \Sigma^s_L$, for all $p \in (0, 1)$. Subsequently, conditional on their investment levels, their expected payoffs are $\Pi_A(h_A) = (\Sigma^{lo}_H \times h_A) - (h_A)^2 < \Pi^s_H(h_A)$, and $\Pi_B(h_B) = (\Sigma^{lo}_L \times h_B) - (h_B)^2 > \Pi^s_L(h_B)$ respectively, for all $h_A, h_B > 0$ and $p \in (0, 1)$. □

Lemma 4.9: Proof. Only the first subcase is proved since the rest follow exactly the same reasoning. Suppose $h_B = (1 - p) \times h_A$. Then, there exists some $\epsilon > 0$ such that students with types $a \in (1 - Q - \epsilon, 1 - Q)$ would randomize between A and B in their applications as $\rho_A(a, \hat{\sigma}_{-a}) = (1 - p)$ and $\rho_B(a, \hat{\sigma}_{-a}) = 1$. Since any such student who applies to A has to be replaced by an inferior student, $\Pi_B((1-p) \times h_A) < \Pi^{lo}_L(h_B)$, where $\Pi_B((1-p) \times h_A)$ is B's expected payoff given $h_B = (1 - p) \times h_A$. Since payoff functions are continuous, there exists some $\delta > 0$, where δ is arbitrarily small, such that $\Pi^{lo}_L((1 - p) \times h_A + \delta) > \Pi_B((1-p) \times h_A)$. Hence, $h_B = (1-p) \times h_A + \delta$ strictly dominates $h_B = (1-p) \times h_A$. □

Lemma 4.11: Proof. For a given set of applicants, suppose the threshold h_A imposes does not bind. Then, the applicants can be attracted at their conditionally optimal payoff, yielding the maximum of their respective payoff curve. When h_A is as such, the respective conditionally constrained payoff function attains this value.

When the threshold binds, the maximum payoff B can attain conditional on enrolling the applicants is marginally less than the value of the respective payoff function at the threshold. The respective conditionally constrained payoff function attains this value; hence it is the supremum of the payoff B can achieve conditional on attracting these applicants. B can never achieve this supremum, but can get infinitesimally close to it. □

Proposition 4.12: Proof. I will prove each case. Take $\tilde{\Pi}_L^{hi} = (\Pi_L^{hi})^*$ and any other constrained conditional payoff function $\tilde{\Pi}$. Since $\lim_{h_A \to \infty} \tilde{\Pi} = -\infty$, $\tilde{\Pi}$ is weakly monotone, and $\tilde{\Pi}(0) > \tilde{\Pi}_L^{hi}$, it crosses $\tilde{\Pi}_L^{hi}$ exactly once. This leaves us with the pairwise combinations of $\left\{\tilde{\Pi}_H^s, \tilde{\Pi}_H^{lo}, \tilde{\Pi}_L^{hi}\right\}$.

1. $\tilde{\Pi}_H^s$ and $\tilde{\Pi}_H^{lo}$

   (i) Take the "parent" payoff functions $\Pi_H^s(\frac{h_A}{1-p})$ and $\Pi_H^{lo}(h_A)$, which, to the right of their peak, characterize these constrained conditional payoff functions. They cross at $h_A = 0$ and $h_A = h_x := (1-p)\frac{\Sigma_H^s - \Sigma_H^{lo}(1-p)}{p(2-p)}$. $\Pi_H^s(\frac{h_A}{1-p})$ peaks at $\hat{h} := (1-p)\frac{\Sigma_H^s}{2}$. Albeit tedious, it's straightforward to verify that for all $p \in (0,1)$, $h_x > \hat{h}$. The parent curves cross after the point where the original curve Π_H^s peaks.

   (ii) Suppose they cross before the parent curve with the lower peak, $\Pi_H^{lo}(h_A)$, peaks; $h_x < \frac{\Sigma_H^{lo}}{2}$. Then, by the Intermediate Value Theorem (IVT) (since $\tilde{\Pi}_H^s$ is monotone decreasing at the crossing point) $\tilde{\Pi}_H^s$ must have crossed $\tilde{\Pi}_H^{lo}$'s peak earlier. This must be the only place they cross since $\tilde{\Pi}_H^s$ is monotone decreasing at this point and the parent curves do not cross again (no crossing after the peak of the lower one). Then, $\tilde{\Pi}_H^s$ and $\tilde{\Pi}_H^{lo}$ cross only at $h_{34} = \frac{(1-p)}{2}(\Sigma_H^s + \sqrt{(\Sigma_H^s)^2 - (\Sigma_H^{lo})^2})$ then – where $\Pi_H^s(\frac{h_A}{1-p})$ crosses $\tilde{\Pi}_H^{lo}$'s peak.

   (iii) Now suppose $h_x \geq \frac{\Sigma_H^{lo}}{2}$. Then, I argue that this is the unique point $\tilde{\Pi}_H^s$ and $\tilde{\Pi}_H^{lo}$ cross. It's easy to see that these curves indeed cross at h_x. Suppose $\tilde{\Pi}_H^s$ crossed $\tilde{\Pi}_H^{lo}$ where the latter was horizontal too (the only possibility left for another cross). At this point, the latter has derivative zero, and the former has negative derivative. Notice however that $\tilde{\Pi}_H^s$ always has second derivative less than the second derivative of $\tilde{\Pi}_H^{lo}$. Had they crossed here, the latter cross we found could not have happened; yielding a contradiction. Hence, $h_{34} = h_x = (1-p)\frac{\Sigma_H^s - \Sigma_H^{lo}(1-p)}{p(2-p)}$.

2. $\tilde{\Pi}^s_H$ and $\tilde{\Pi}^{lo}_L$

   (i) Again, take the parent functions $\Pi^s_H(\frac{h_A}{1-p})$ and $\Pi^{lo}_L((1-p)\times h_A)$. The non-zero crossing of these functions is $h_x := \frac{\frac{\Sigma^s_H}{1-p} - (1-p)\Sigma^{lo}_L}{\frac{1}{(1-p)^2} - (1-p)^2}$. Again, tediously but straightforwardly verifiable is that $h_x > \frac{\Sigma^s_H(1-p)}{2}$ for all $p > 0$; so the parent functions cross after the peak of the higher one.

   (ii) Suppose $h_x < \frac{\Sigma^{lo}_L}{2(1-p)}$; the crossing is before $\Pi^{lo}_L((1-p)\times h_A)$'s peak. Then, by IVT, $\tilde{\Pi}^s_H$ must have crossed $\tilde{\Pi}^{lo}_L$'s peak earlier. By an argument similar to the first case, this must be the only point they cross. Hence, $\tilde{\Pi}^s_H$ and $\tilde{\Pi}^{lo}_L$ cross only at this point where $\Pi^s_H(\frac{h_A}{1-p})$ crosses $\tilde{\Pi}^{lo}_L$'s peak. $h_{24} = \frac{(1-p)}{2}.(\Sigma^s_H + \sqrt{(\Sigma^s_H)^2 - (\Sigma^{lo}_L)^2})$.

   (iii) Now suppose $h_x \geq \frac{\Sigma^{lo}_L}{2(1-p)}$. Then, h_x is the unique point $\tilde{\Pi}^s_H$ and $\tilde{\Pi}^{lo}_L$ cross; the argument is similar to the one constructed for the first case. $h_{24} = h_x = \frac{\frac{\Sigma^s_H}{1-p} - (1-p)\Sigma^{lo}_L}{\frac{1}{(1-p)^2} - (1-p)^2}$.

3. $\tilde{\Pi}^{lo}_H$ and $\tilde{\Pi}^{lo}_L$

   (i) The parent functions are $\Pi^{lo}_H(h_A)$ and $\Pi^{lo}_L((1-p)\times h_A)$. The non-zero point where they cross is $h_x := \frac{\Sigma^{lo}_H - (1-p)\Sigma^{lo}_L}{p(2-p)}$. That for all $p > 0$ $h_x > \frac{\Sigma^{lo}_H}{2}$ is easily verifiable.

   (ii) Suppose $h_x < \frac{\Sigma^{lo}_L}{2(1-p)}$; the crossing is before $\Pi^{lo}_L((1-p)\times h_A)$'s peak. With an argument similar to those made in the previous two subcases, $h_{23} = \frac{1}{2}(\Sigma^{lo}_H + \sqrt{(\Sigma^{lo}_H)^2 - (\Sigma^{lo}_L)^2})$.

   (iii) Suppose instead $h_x \geq \frac{\Sigma^{lo}_L}{2(1-p)}$. Then, again with similar arguments, $h_x = h_{23}$; the unique point where these curves cross.

The crossing points we found can be summarized as follows.

$$h_{14} = \frac{1-p}{2} \times (\Sigma_H^s + \sqrt{(\Sigma_H^s)^2 - (\Sigma_L^{hi})^2}) \tag{8.1}$$

$$h_{13} = \frac{(\Sigma_H^{lo} + \sqrt{(\Sigma_H^{lo})^2 - (\Sigma_L^{hi})^2})}{2} \tag{8.2}$$

$$h_{12} = \frac{(\Sigma_L^{lo} + \sqrt{(\Sigma_L^{lo})^2 - (\Sigma_L^{hi})^2})}{2(1-p)} \tag{8.3}$$

$$h_{34} = \begin{cases} (1-p) \times \frac{\Sigma_H^s - \Sigma_H^{lo}(1-p)}{p(2-p)} & (1-p) \times \frac{\Sigma_H^s - \Sigma_H^{lo}(1-p)}{p(2-p)} \geq \frac{\Sigma_H^{lo}}{2} \\ \frac{(1-p)}{2}(\Sigma_H^s + \sqrt{(\Sigma_H^s)^2 - (\Sigma_H^{lo})^2}) & o/w \end{cases} \tag{8.4}$$

$$h_{24} = \begin{cases} \frac{\frac{\Sigma_H^s}{1-p} - (1-p)\Sigma_L^{lo}}{\frac{1}{(1-p)^2} - (1-p)^2} & \frac{\frac{\Sigma_H^s}{1-p} - (1-p)\Sigma_L^{lo}}{\frac{1}{(1-p)^2} - (1-p)^2} \geq \frac{\Sigma_L^{lo}}{2(1-p)} \\ \frac{(1-p)}{2} \times (\Sigma_H^s + \sqrt{(\Sigma_H^s)^2 - (\Sigma_L^{lo})^2}) & o/w \end{cases} \tag{8.5}$$

$$h_{23} = \begin{cases} \frac{\Sigma_H^{lo} - (1-p)\Sigma_L^{lo}}{p(2-p)} & \frac{\Sigma_H^{lo} - (1-p)\Sigma_L^{lo}}{p(2-p)} \geq \frac{\Sigma_L^{lo}}{2(1-p)} \\ \frac{(\Sigma_H^{lo} + \sqrt{(\Sigma_H^{lo})^2 - (\Sigma_L^{lo})^2})}{2} & o/w \end{cases} \tag{8.6}$$

□

Corollary 4.14: Proof. When $h_A < h$, $\tilde{\Pi}_i^j(h_A) > \tilde{\Pi}_{i'}^{j'}(h_A)$; outbidding h_A to compete for $\mu_j(i, Q')$ yields higher payoff than settling for $\mu_{j'}(i', Q')$, due to Lemma 4.11. Hence, $\mu_j(i, Q')$ is preferred over $\mu_{j'}(i', Q')$.

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Now let $h_A = h$. The threshold must be binding for the applicants $\mu_j(i, Q')$. Otherwise, they could be attracted at their constrained conditional payoff, and enrolling $\mu_{j'}(i', Q')$ would yield a strictly lower payoff. Then, attracting $\mu_j(i, Q')$ brings payoff less than $\tilde{\Pi}_i^j(h_A)$; and is equal to $\tilde{\Pi}_i^j(h_A + \epsilon)$ for some $\epsilon > 0$, due to Lemma 4.11. If the threshold for $\mu_{j'}(i', Q')$ is not binding, they are preferable since $\tilde{\Pi}_{i'}^{j'}(h_A) > \tilde{\Pi}_i^j(h_A + \epsilon)$. Otherwise, for all $\epsilon > 0$, $\tilde{\Pi}_{i'}^{j'}(h_A + \epsilon) > \tilde{\Pi}_i^j(h_A + \epsilon)$ and $\mu_{j'}(i', Q')$ is still preferable over $\mu_j(i, Q')$. □

Corollary 4.15: Proof. Note that an investment $h_A \in S_i^j$ attracts the applicants $\mu^j(i, Q')$. Therefore, the expected payoff is given by $\Pi_i^j(h_A)$. A chooses the investment strategy which maximizes its payoff subject to this constraint. □

Proposition 4.16: Proof. Suppose that for $p > \bar{p}$, Proposition 7.1 and Corollary 7.2 hold. Corollary 7.2 asserts that B's preferences between applicants are as in Figure 4.6a and the applicants A can attract given h_A are then given by Figure 4.7a.

By Proposition 7.1, $\Pi_H^{lo}(h_{23}) = (\Pi_L^{lo})^*$. Since when $h_A = h_{23}$, B prefers settling for the applicants $\mu^{lo}(L, Q')$ rather than competing for $\mu^{lo}(H, Q')$, $\Pi_A(h_{23}) = (\Pi_L^{lo})^*$.

Setting $h \in (h_{23}, h_{12})$ cannot be optimal, as $\Pi_A(h) = \Pi_H^{lo}(h) < (\Pi_L^{lo})^*$ for such h. The last part of Proposition 7.1 asserts that setting $h \geq h_{12}$ cannot be optimal either, for $h \geq h_{12}$ implies $\Pi_A = \Pi_H^s(h) < (\Pi_L^{lo})^*$. For $h < h_{23}$, A can at most get an equilibrium payoff of $(\Pi_L^{lo})^*$ since it will get students weakly inferior to $\mu^{lo}(L, Q')$. As colleges break ties in favour of better students, A's equilibrium investment strategy h_A^* is then $h_A^* = h_{23}$.

By Proposition 7.1, $h_{23} < \frac{h_L^{lo}}{1-p}$. Then, $h_B^*(h_{23}) = \frac{\Sigma_L^{lo}}{2}$; B sets the conditionally optimal investment h_L^{lo} for the candidates $\mu^{lo}(L, Q')$.

As $h_{23} > \frac{\Sigma_L^{lo}}{2} > (1-p) \times h_{23}$ by Proposition 7.1, $\hat{\sigma}^{lo}$ describes students application behaviour. A and B get the applicants Σ_{2H} and Σ_{2L} respectively, in equilibrium, expecting payoff $(\Pi_L^{lo})^*$.

□

Section 5

Proposition 5.1. Let p be large enough so that Proposition 4.16 holds. Then, in Simultaneous Applications, both colleges equilibrium payoffs are $(\Pi_L^{lo})^*$. In Sequential Applications, due to Proposition 3.7, both achieve an equilibrium payoff of $(\Pi_L^s)^*$. Lemma 4.8 establishes that for all $p \in (0, 1)$ and $h > 0$, $\Pi_L^{lo}(h) > \Pi_L^s(h)$. Hence, $(\Pi_L^{lo})^* > (\Pi_L^s)^*$.

Then, in the extended college admissions game, both colleges attain higher payoffs in the SPNE of the Simultaneous Applications subgame. Consequently, regardless of which college moves second to set its date, in equilibrium it conflicts its admission date with the first-mover, leading to the Simultaneous Applications subgame being played. □

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