

# Representations having vectors fixed by a Levi subgroup associated to a real form

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For any semisimple real Lie algebra  $\mathfrak{g}_{\mathbb{R}}$ , we classify the representations of  $\mathfrak{g}_{\mathbb{R}}$  that have at least one nonzero vector on which the centralizer of a Cartan subspace, also known as the centralizer of a maximal split torus, acts trivially. In the process, we revisit the notion of  $\mathfrak{g}$ -standard Young tableaux, introduced by Lakshmibai and studied by Littelmann, that provides a combinatorial model for the characters of the irreducible representations of any classical semisimple Lie algebra  $\mathfrak{g}$ . We construct a new version of these objects, which differs from the old one for  $\mathfrak{g} = \mathfrak{so}(2r)$  and seems, in some sense, simpler and more natural.

## 1 Introduction

### 1.1 Background and motivation

The present work is motivated by the following geometrical result, proved by the author earlier. Recall that, for a semisimple real Lie group  $G_{\mathbb{R}}$ , the *restricted Weyl group* is the group  $W := N_{G_{\mathbb{R}}}(\mathfrak{a})/Z_{G_{\mathbb{R}}}(\mathfrak{a})$ , where  $\mathfrak{a}$  is the Cartan subspace (or maximal split torus) of  $G_{\mathbb{R}}$ ; the *longest element* of  $W$  is the unique element that maps all positive restricted roots to negative restricted roots. Also let  $L$  denote the centralizer  $Z_{G_{\mathbb{R}}}(\mathfrak{a})$ .

**Theorem 1.1** ([30]). *Let  $G_{\mathbb{R}}$  be a semisimple real Lie group,  $\rho$  a representation of  $G_{\mathbb{R}}$  on a finite-dimensional real vector space  $V$ . Assume that  $\rho$  satisfies the following algebraic condition:*

(\*) *the longest element  $w_0$  of the restricted Weyl group  $W$  of  $G_{\mathbb{R}}$  acts (via  $\rho$ ) nontrivially on the subspace  $V^L$  of vectors of  $V$  that are fixed by all elements of  $L$ .*

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Then the representation  $\rho$  has the following geometric property:

(\*\*) The affine group  $G_{\mathbb{R}} \ltimes_{\rho} V$  contains a Zariski-dense subgroup  $\Gamma$  that is free (of rank at least 2) and acts properly discontinuously on the affine space corresponding to  $V$ .

Moreover, it is conjectured that the converse is true, i.e. that every representation with the property (\*\*) satisfies the condition (\*); the author has partially proved it [29].

Representations with property (\*\*) are called *non-Milnor*, as they provide counterexamples to a conjecture by Milnor [25]. A weaker version of this conjecture, due to Auslander [5], remains open to this day, and spurs a large body of work that includes, besides the author's two papers cited above, [2, 3, 4, 9, 10, 11, 13, 24, 31] and many others. For a concise statement of the two conjectures and a brief overview of work on it, see the introduction to [30]. For a more detailed exposition, see the surveys [1] or [8].

This theorem naturally raises the problem of explicitly classifying the representations that satisfy (\*). In an earlier work [19], we did this in the special case where  $G_{\mathbb{R}}$  is split: then the Cartan subspace (or maximal split torus)  $\mathfrak{a}$  is also a Cartan subalgebra (or maximal torus) of  $G_{\mathbb{R}}$ , hence its centralizer  $L$  reduces to the Lie group  $A$  generated by  $\mathfrak{a}$ , and  $V^L = V^A$  is just the weight space in  $V$  corresponding to the weight 0. So in [19], we classified the representations where  $w_0$  acts nontrivially on this zero-weight space.

In the general case, a natural first step towards this problem consists in classifying the representations for which at least the subspace  $V^L$  itself is nontrivial. This latter classification is the goal of the present paper.

## 1.2 Basic notations

We now introduce some notations necessary to formulate the main theorem, and used throughout the paper.

1. Let  $\mathfrak{g}$  be a semisimple complex Lie algebra,  $\mathfrak{g}_{\mathbb{R}}$  some real form of  $\mathfrak{g} = (\mathfrak{g}_{\mathbb{R}})^{\mathbb{C}}$ .
2. We choose in  $\mathfrak{g}_{\mathbb{R}}$  a Cartan subspace  $\mathfrak{a}_{\mathbb{R}}$  (an abelian subalgebra of  $\mathfrak{g}_{\mathbb{R}}$  whose elements are diagonalizable over  $\mathbb{R}$  and maximal for these properties); we set  $\mathfrak{a} := (\mathfrak{a}_{\mathbb{R}})^{\mathbb{C}}$ .
3. We choose in  $\mathfrak{g}$  a Cartan subalgebra  $\mathfrak{h}$  (an abelian subalgebra of  $\mathfrak{g}$  whose elements are diagonalizable and which is maximal for these properties) that contains  $\mathfrak{a}$ .
4. We denote by  $\mathfrak{l}(\mathfrak{g}_{\mathbb{R}})$ , or simply  $\mathfrak{l}$  when clear from context, the centralizer of  $\mathfrak{a}$  in  $\mathfrak{g}$ .
5. Let  $\Delta$  be the set of roots of  $\mathfrak{g}$  in  $\mathfrak{h}^*$ . We shall identify  $\mathfrak{h}^*$  with  $\mathfrak{h}$  via the Killing form. We call  $\mathfrak{h}_{(\mathbb{R})}$  the  $\mathbb{R}$ -linear span of  $\Delta$ ; it is given by the formula  $\mathfrak{h}_{(\mathbb{R})} = \mathfrak{a}_{\mathbb{R}} \oplus i\mathfrak{t}_{\mathbb{R}}$ , where  $\mathfrak{t}_{\mathbb{R}}$  is the orthogonal complement of  $\mathfrak{a}_{\mathbb{R}}$  in  $\mathfrak{h} \cap \mathfrak{g}_{\mathbb{R}}$ .
6. We choose on  $\mathfrak{h}_{(\mathbb{R})}$  a lexicographical ordering that “puts  $\mathfrak{a}_{\mathbb{R}}$  first”, i.e. such that every vector whose orthogonal projection onto  $\mathfrak{a}_{\mathbb{R}}$  is positive is itself positive. We call  $\Delta^+$  the set of roots in  $\Delta$  that are positive with respect to this ordering, and we let  $\Pi = \{\alpha_1, \dots, \alpha_r\}$  be the set of simple roots in  $\Delta^+$ . Let  $\varpi_1, \dots, \varpi_r$  be the corresponding fundamental weights.

7. We call  $P$  (resp.  $Q$ ) the weight lattice (resp. root lattice), i.e. the abelian subgroup of  $\mathfrak{h}^*$  spanned by  $\varpi_1, \dots, \varpi_r$  (resp. by  $\Delta$ ). Elements of  $P$  are called *integral weights*.
8. We introduce the dominant Weyl chamber  $\mathfrak{h}^+ := \{X \in \mathfrak{h}_{(\mathbb{R})} \mid \forall \alpha \in \Pi, \alpha(X) \geq 0\}$ .
9. When  $\mathfrak{g}$  is simple, we call  $(e_1, \dots, e_n)$  the vectors called  $(\varepsilon_1, \dots, \varepsilon_n)$  in the appendix to [6]. Throughout the paper, we use the Bourbaki conventions [6] for the numbering of simple roots and their expressions in the coordinates  $e_i$ .
10. For all  $\lambda \in P$ , we always denote  $\lambda_1, \dots, \lambda_n$  its coordinates in this last basis:

$$\lambda =: \lambda_1 e_1 + \dots + \lambda_n e_n. \quad (1.1)$$

11. In the sequel, all representations are supposed to be finite-dimensional and (except for a brief discussion at the beginning of Subsection 1.3) complex. Recall ([15] or [14]) that to every irreducible representation of  $\mathfrak{g}$ , we may associate, in a bijective way, a vector  $\lambda \in P \cap \mathfrak{h}^+$  called its *highest weight*. We denote by  $\rho_\lambda(\mathfrak{g})$  the irreducible representation of  $\mathfrak{g}$  with highest weight  $\lambda$ , and by  $V_\lambda(\mathfrak{g})$  the space on which it acts. When clear from context, we will shorten  $V_\lambda(\mathfrak{g})$  to  $V_\lambda$ .
12. Given a representation  $V$  of  $\mathfrak{g}$ , we denote  $V^\mathfrak{l} := \{v \in V \mid \forall l \in \mathfrak{l}, l \cdot v = 0\}$ .
13. We denote by  $\mathfrak{sp}_{2,n}(\mathbb{C})$ ,  $\mathfrak{sp}_{2,n}(\mathbb{R})$  and  $\mathfrak{sp}_2(p, q)$  some Lie algebras that have rank  $n$  (or  $p+q$ ) and a standard representation of dimension  $2n$  (or  $2p+2q$ ). Some authors (e.g. [6]) denote them respectively by  $\mathfrak{sp}_{2n}(\mathbb{C})$ ,  $\mathfrak{sp}_{2n}(\mathbb{R})$  and  $\mathfrak{sp}(2p, 2q)$ ; while others (e.g. [15]) denote them respectively by  $\mathfrak{sp}_n(\mathbb{C})$ ,  $\mathfrak{sp}_n(\mathbb{R})$  and  $\mathfrak{sp}(p, q)$ .

### 1.3 Statement of main result

Here is the main theorem of this paper. In the process of proving it, we develop some other results that may be interesting on their own: we will describe these in Subsubsection 1.4.1.

This theorem solves the problem raised above, after doing the following reductions:

1. from representations of the Lie group  $G_{\mathbb{R}}$  to representations of its Lie algebra  $\mathfrak{g}_{\mathbb{R}}$ ;
2. from real representations to complex representations;
3. from representations of  $\mathfrak{g}_{\mathbb{R}}$  to representations of its complexification  $\mathfrak{g}$ ;
4. from arbitrary to irreducible representations;
5. from the case where  $\mathfrak{g}_{\mathbb{R}}$  is semisimple to the case where  $\mathfrak{g}_{\mathbb{R}}$  is simple.

These reductions rely on classical results of Lie theory and representation theory, and happen without any surprises. See also [28] for more details about the steps 1 and 3–5.

**Main Theorem.** *Let  $\mathfrak{g}_{\mathbb{R}}$  be a simple real Lie algebra. Then the set*

$$\mathcal{M}_{\mathfrak{l}\text{-inv}}(\mathfrak{g}_{\mathbb{R}}) := \left\{ \lambda \in P \cap \mathfrak{h}^+ \mid V_\lambda^{\mathfrak{l}(\mathfrak{g}_{\mathbb{R}})} \neq 0 \right\}$$

*is equal to the set  $\mathcal{M}_{\text{Table}}(\mathfrak{g}_{\mathbb{R}})$ , defined as follows:*

- (i)  $\mathcal{M}_{\text{Table}}$  is listed in Table 1 when  $\mathfrak{g} = (\mathfrak{g}_{\mathbb{R}})^{\mathbb{C}}$  is a classical simple Lie algebra;
- (ii)  $\mathcal{M}_{\text{Table}} = \{0\}$  when  $\mathfrak{g}_{\mathbb{R}}$  is the compact real form of an exceptional simple Lie algebra;
- (iii)  $\mathcal{M}_{\text{Table}} = Q \cap \mathfrak{h}^+$  when  $\mathfrak{g}_{\mathbb{R}}$  is any noncompact real form of an exceptional simple Lie algebra;
- (iv)  $\mathcal{M}_{\text{Table}} = Q \cap \mathfrak{h}^+$  if  $\mathfrak{g}$  is not simple.

Table 1: Conditions for  $V_{\lambda}^{\mathfrak{l}} \neq 0$  for real forms of classical simple Lie algebras. Here we decompose  $\lambda = \sum_{i=1}^n \lambda_i e_i$ ; when  $\mathfrak{g}$  is of type  $B_r$ ,  $C_r$  or  $D_r$ , we adopt the additional convention that  $\lambda_i = 0$  for all  $i > r$ .

$\mathfrak{g}$	$\mathfrak{g}_{\mathbb{R}}$	Parameter range	$\lambda \in \mathcal{M}_{\text{Table}}$ iff...
$A_r$ $r \geq 1$	$\mathfrak{su}(p, r+1-p)$	$0 \leq p < \frac{r+1}{2}$	$\lambda \in Q \cap \mathfrak{h}^+$ and $\lambda_{r+1-2p} \geq 0 \geq \lambda_{2p+1}$
	$\mathfrak{su}(p, p)$	$p = \frac{r+1}{2}$ ( $r$ odd)	$\lambda \in Q \cap \mathfrak{h}^+$
	$\mathfrak{sl}_{r+1}(\mathbb{R})$		$\lambda \in Q \cap \mathfrak{h}^+$
	$\mathfrak{sl}_m(\mathbb{H})$	$m = \frac{r+1}{2}$ ( $r$ odd)	$\lambda \in Q \cap \mathfrak{h}^+$ and $\sum_{i=2}^{m+1} \lambda_i \geq 0 \geq \sum_{i=m}^{2m-1} \lambda_i$
$B_r$ $r \geq 1$	$\mathfrak{so}(p, 2r+1-p)$	$0 \leq p \leq r$	$\begin{cases} \lambda \in Q \cap \mathfrak{h}^+, \lambda_{2p+1} = 0 \text{ and} \\ \text{if } \sum_{i=1}^r \lambda_i \equiv 1 \pmod{2}, \text{ then } \lambda_{2r-2p+1} > 0 \end{cases}$
$C_r$ $r \geq 1$	$\mathfrak{sp}_2.(1, 1)$		$\lambda \in Q \cap \mathfrak{h}^+$ and $\lambda_2 \in 2\mathbb{Z}$
	$\mathfrak{sp}_2.(p, r-p)$	$\begin{cases} 0 \leq p \leq \frac{r}{2} \\ (p, r) \neq (1, 2) \end{cases}$	$\lambda \in Q \cap \mathfrak{h}^+$ and $\lambda_{4p+1} = 0$
	$\mathfrak{sp}_{2,r}(\mathbb{R})$		$\lambda \in Q \cap \mathfrak{h}^+$
$D_r$ $r \geq 3$	$\mathfrak{so}(p, 2r-p)$	$0 \leq p \leq r$	$\lambda \in Q \cap \mathfrak{h}^+$ and $\lambda_{2p+1} = 0$
	$\mathfrak{so}^*(6)$		$\lambda \in Q \cap \mathfrak{h}^+$ and $ \lambda_3  \leq \lambda_1 - \lambda_2$
	$\mathfrak{so}^*(2r)$	$r \geq 4$	$\lambda \in Q \cap \mathfrak{h}^+$

## 1.4 Strategy of the proof

We start by treating some trivial cases in Subsection 2.1. In Subsection 2.2, we make some preliminary remarks about a class of subalgebras that includes  $\mathfrak{l}$ : the Levi subalgebras.

The first key idea, suggested to the author by E. B. Vinberg and presented in Subsection 2.3, is the following. By using the properties of the so-called Cartan product, we prove (in two lines) that the set  $\mathcal{M}_{\mathfrak{l}\text{-inv}}$  is closed under addition. For any given value of  $\mathfrak{g}_{\mathbb{R}}$ , this reduces the problem to a finite number of computations.

Now in order to compute the dimension of  $V_{\lambda}^{\mathfrak{l}}$  given  $\lambda$  and  $\mathfrak{l}$ , we need branching rules. For exceptional Lie algebras, of which there are a finite number, we only need (thanks to the previous paragraph) an algorithmic implementation of them, which is readily available. This case is treated in Section 3.

For classical Lie algebras, we need a conceptual description of branching rules, which does exist in the special case of restrictions to Levi subalgebras. Note that, with similar methods, we could probably generalize the Main Theorem to all Levi subalgebras  $\mathfrak{l}$  of  $\mathfrak{g}$ .

The easier case, treated in Section 4, is when  $\mathfrak{g} = (\mathfrak{g}_{\mathbb{R}})^{\mathbb{C}}$  is of type  $A_r$  (i.e.  $\mathfrak{g} = \mathfrak{sl}_{r+1}(\mathbb{C})$ ). Then there is a classical theory (that we recall in Subsection 4.1) that identifies the character of each representation  $V_{\lambda}(\mathfrak{g})$  with the set of semistandard Young tableaux of shape  $\lambda$ , and gives a branching rule for Levi subalgebras  $\mathfrak{l} \subset \mathfrak{g}$  in terms of an easily-described subset of these tableaux. This reduces the proof of the main theorem for real forms of  $A_r$  to a problem of combinatorics, that we solve in Subsections 4.2 and 4.3.

#### 1.4.1 Doubled Young tableaux

In order to treat the remaining classical algebras, we need a similar description of the characters of representations of  $\mathfrak{g}$ , when  $\mathfrak{g}$  is of type  $B_r$ ,  $C_r$  or  $D_r$ . Various such descriptions exist.

1. Lakshmibai, Musili and Seshadri introduced objects that they also called “standard Young tableaux” ([17, Definition 10.2]), which they define in an abstract way, in terms of the Bruhat order and of the action of the Weyl group. They then establish that the set of standard Young tableaux of a given shape is in bijection with the character of the corresponding representation. In fact they proved a much more general statement [17, Theorem 15.2], involving arbitrary Schubert varieties; we are only interested in particular case  $\phi = w_0$  of their theorem (corresponding to the full flag variety).
2. Littelmann introduced the so-called *path model*  $B_{\pi^+}$  [21], which is a certain set of piecewise affine paths in the Cartan subalgebra  $\mathfrak{h}$ , parametrized by the (arbitrary) choice of a starting path  $\pi^+$  that connects 0 to some integral dominant weight  $\lambda$ , and lies entirely within the dominant Weyl chamber  $\mathfrak{h}^+$ . Every such path model is then also in bijection with the character of the representation  $V_{\lambda}(\mathfrak{g})$ .

The path model provides a generalization of the Lakshmibai–Musili–Seshadri tableaux, insofar as these tableaux can be reinterpreted as a description of the path model  $B_{\pi_{0,L}^+(\lambda)}$  obtained from the starting path

$$\pi_{0,L}^+(\lambda) = x_1\varpi_1 * \cdots * x_r\varpi_r, \quad (1.2)$$

defined (see Definition 5.1) as the concatenation of  $r$  linear segments, each of them equal to (a suitable translation of) the linear path going from 0 to  $x_i\varpi_i$ , where  $x_1, \dots, x_r$  denote the coordinates of  $\lambda$  in the basis  $\varpi_1, \dots, \varpi_r$ .

3. We also need a translation of the abstract defining condition of a “standard Young tableau” (as given in [17]) into a purely combinatorial condition. (The key point that needs most of all to be translated is the notion of an “admissible pair”.)

In this paper, for the reader’s convenience, we provide a full such translation with a complete proof (see [16] for an announcement of a similar result, and [20] for another

similar result with a sketch of a proof). In the  $D_r$  case, however, we use a slight modification of Lakshmibai, Musili and Seshadri's construction, which leads to a simpler notion of standard Young tableau — at least in our special case of the full flag variety (we do not know whether this simpler notion still makes sense in the more general setting of arbitrary Schubert varieties).

This proof, and this modification (comprising our Section 5), may be of interest independently of the remainder of the paper. We call the resulting objects  *$\mathfrak{g}$ -standard doubled Young tableaux*: see Definition 5.23. (We chose this terminology because every column is “split into two” in some sense.) Thus we reduce the proof of the main theorem for  $\mathfrak{g}$  of types  $B_r$ ,  $C_r$  and  $D_r$  to a problem of combinatorics, which we solve in Section 6.

More precisely, our modification consists in considering a path model  $B_{\pi_0^+(\lambda)}$  with a slightly different starting path: namely, the path

$$\pi_0^+(\lambda) := (\lambda_2 - \lambda_1)c_1 * \cdots * (\lambda_r - \lambda_{r-1})c_{r-1} * |\lambda_r|c_r^{\text{sgn}(\lambda_r)}, \quad (1.3)$$

with  $c_1, \dots, c_{r-1}, c_r^\pm$  defined in (5.14).

When  $\mathfrak{g}$  is of type  $B_r$  or  $C_r$ , or when  $\mathfrak{g}$  is of type  $D_r$  but we have  $x_{r-1} = x_r = 0$ , then the two starting paths  $\pi_{0,L}^+(\lambda)$  and  $\pi_0^+(\lambda)$  actually coincide; and indeed, in this case our “ $\mathfrak{g}$ -standard doubled Young tableaux” coincide (up to some cosmetic differences) with the “ $G$ -standard Young tableaux” defined in the appendix of [20].

However when  $\mathfrak{g}$  is of type  $D_r$  (and  $x_{r-1} > 0$  or  $x_r > 0$ ), the two starting paths differ, and so do the two notions of standard Young tableaux. Our notion is simpler: namely the complicated condition (3) from [20, A.3] is replaced by the much simpler condition (H2) from our Definition 5.23. The key result that makes this simplification possible is Proposition 5.18, that allows us to circumvent the problem of the non-transitivity of the “Bruhat order” on weights (see the remark preceding Definition 5.8.)

In the future, the author has some hope of extending this improved construction to exceptional Lie algebras, by finding, for arbitrary  $\mathfrak{g}$  and  $\lambda$ , a starting path  $\pi_0^+(\lambda)$  that satisfies some sort of generalization of Proposition 5.18.

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## 2 Basic remarks

We start by establishing some basic properties of the set  $\mathcal{M}_{\text{I-inv}}$  of integral weights  $\lambda$  such that  $V_\lambda^{\text{I}} \neq 0$ , which will allow us to prove the Main Theorem in some easy cases and lay the groundwork for the proof in the remaining cases.

## 2.1 Trivial cases

We start with some trivial remarks.

**Proposition 2.1.** *Let  $\lambda \in P \cap \mathfrak{h}^+$  be a dominant integral weight of  $\mathfrak{g}$ .*

- (i) *If  $V_\lambda^\mathfrak{l} \neq 0$ , then necessarily  $\lambda \in Q$ .*
- (ii) *If  $\mathfrak{g}_\mathbb{R}$  is split, quasi-split (i.e.  $\mathfrak{l}$  is abelian), or complex, then, conversely,  $V_\lambda^\mathfrak{l} \neq 0$  for all  $\lambda \in Q \cap \mathfrak{h}^+$ .*
- (iii) *If  $\mathfrak{g}_\mathbb{R}$  is compact, then  $V_\lambda^\mathfrak{l} \neq 0$  if and only if  $\lambda = 0$ .*

This settles in particular point (iv) of the Main Theorem.

*Proof.* (i) We always have  $\mathfrak{a} \subset \mathfrak{h} \subset \mathfrak{l}$ , hence

$$V_\lambda^\mathfrak{l} \subset V_\lambda^\mathfrak{h} = V_\lambda^0,$$

where  $V_\lambda^0$  denotes the zero-weight space of  $V_\lambda$ . The latter is nontrivial if and only if 0 is a weight of  $V_\lambda$ , which, by the well-known characterization of the set of weights of a representation (see e.g. [14, Theorem 10.1]), occurs if and only if  $\lambda \in Q$ .

- (ii) In all three cases, one can check that  $\mathfrak{l} = \mathfrak{h}$ , so that the converse also holds.
- (iii) If  $\mathfrak{g}_\mathbb{R}$  is compact, then obviously  $\mathfrak{a} = 0$ ,  $\mathfrak{l} = \mathfrak{g}$ , and  $V_\lambda^\mathfrak{l} \neq 0$  if and only if  $V_\lambda$  is the trivial representation, i.e. if and only if  $\lambda = 0$ .  $\square$

For the remainder of the paper, we assume that both the real Lie algebra  $\mathfrak{g}_\mathbb{R}$  and its complexification  $\mathfrak{g}$  are simple.

## 2.2 Levi subalgebras

The goal of this subsection is to describe  $\mathfrak{l}$  in purely complex terms, so that we will (almost) not need to care about  $\mathfrak{g}_\mathbb{R}$  any more.

**Definition 2.2.** Let  $\Theta \subset \Pi$ . We define the *Levi subalgebra of type  $\Theta$*  in  $\mathfrak{g}$  to be

$$\mathfrak{l}(\Theta) := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta \cap \langle \Theta \rangle} \mathfrak{g}^\alpha,$$

where  $\langle \Theta \rangle$  denotes the linear span of  $\Theta$ . In other terms,  $\mathfrak{l}(\Theta)$  is a reductive Lie algebra whose Cartan subalgebra coincides with  $\mathfrak{h}$ , and whose root system is the subsystem of  $\Delta$  generated by  $\Theta$ . Now the following result is straightforward (and well-known).

**Proposition 2.3.** *For every real form  $\mathfrak{g}_\mathbb{R}$  of  $\mathfrak{g}$ , define  $\Theta(\mathfrak{g}_\mathbb{R}) := \Pi \cap \mathfrak{a}^\perp$ . Then the subalgebra  $\mathfrak{l}$  corresponding to  $\mathfrak{g}_\mathbb{R}$  is a Levi subalgebra of  $\mathfrak{g}$ , of type  $\Theta(\mathfrak{g}_\mathbb{R})$ :*

$$\mathfrak{l}(\mathfrak{g}_\mathbb{R}) = \mathfrak{l}(\Theta(\mathfrak{g}_\mathbb{R})).$$

The only information about  $\mathfrak{g}_\mathbb{R}$  that will matter to us is the data of the set  $\Theta(\mathfrak{g}_\mathbb{R})$ . It can be read off the Satake diagram of  $\mathfrak{g}_\mathbb{R}$  (see [26], Chapter 5, §4, 3° for a definition): it is precisely the set of blackened nodes (called  $\Pi_0$  in [26]). A table of the Satake diagrams of all the simple real Lie algebras is given in [26], Reference Chapter, Table 9.

### 2.3 Additivity property

We finish this section by proving Proposition 2.5, which will greatly simplify our task for proving that  $\mathcal{M}_{\mathfrak{l}\text{-inv}} \supset \mathcal{M}_{\text{Table}}$ . Indeed it will now suffice to check that  $\mathcal{M}_{\mathfrak{l}\text{-inv}}$  contains a basis of the monoid  $\mathcal{M}_{\text{Table}}$ , which, for any given group, is only a finite computation.

Let  $G$  be a simply-connected complex Lie group with Lie algebra  $\mathfrak{g}$  and  $N$  a maximal unipotent subgroup of  $G$ . Let  $\mathbb{C}[G/N]$  denote the space of regular functions on the variety  $G/N$ . Pointwise multiplication of functions is  $G$ -equivariant and makes  $\mathbb{C}[G/N]$  into a  $\mathbb{C}$ -algebra without zero divisors (because the variety  $G/N$  is irreducible).

**Theorem 2.4** ([27, (3.20)–(3.21)]). *Each finite-dimensional irreducible representation  $V_\lambda$  of  $G$  (or equivalently of its Lie algebra  $\mathfrak{g}$ ) occurs exactly once as a direct summand of the representation  $\mathbb{C}[G/N]$ . The  $\mathbb{C}$ -algebra  $\mathbb{C}[G/N]$  is graded by the highest weight  $\lambda$ , in the sense that the product of a vector in  $V_\lambda$  by a vector in  $V_\mu$  lies in  $V_{\lambda+\mu}$ .*

For given  $\lambda$  and  $\mu$ , we call *Cartan product* the induced bilinear map  $\odot : V_\lambda \times V_\mu \rightarrow V_{\lambda+\mu}$ . Given  $u \in V_\lambda$  and  $v \in V_\mu$ , this defines  $u \odot v \in V_{\lambda+\mu}$  as the projection of  $u \otimes v \in V_\lambda \otimes V_\mu = V_{\lambda+\mu} \oplus \dots$ . Since  $\mathbb{C}[G/N]$  has no zero divisor,  $u \odot v \neq 0$  whenever  $u \neq 0$  and  $v \neq 0$ . We deduce the following.

**Proposition 2.5.** *The set  $\mathcal{M}_{\mathfrak{l}\text{-inv}}$  is a submonoid of the additive monoid  $Q \cap \mathfrak{h}^+$ .*

*Proof.* Let  $\lambda_1$  and  $\lambda_2$  be two elements of this set. Choose nonzero vectors  $u_i \in V_{\lambda_i}^{\mathfrak{l}}$  for  $i = 1, 2$ . Then the vector  $u_1 \odot u_2$  is in  $V_{\lambda_1+\lambda_2}$ , is invariant by  $\mathfrak{l}$ , and is still nonzero.  $\square$

## 3 Exceptional Lie algebras

We are now ready to prove the Main Theorem for all exceptional simple real Lie algebras. Proposition 2.1 (iii) takes care of their compact real forms. Proposition 2.1 (ii) takes care of their split real forms, and of the quasi-split real form  $E_{III}$ .

For the remaining noncompact real forms of exceptional simple Lie algebras, Proposition 2.5 together with Proposition 2.1 (i) show that it suffices to verify that  $V_\lambda^{\mathfrak{l}} \neq 0$  for all  $\lambda \in Q \cap \mathfrak{h}^+$  that are primitive (i.e. not expressible as the sum of two nonzero elements of  $Q \cap \mathfrak{h}^+$ ), of which there are finitely many. We have done this verification using branching rules implemented in the software LiE [32].

For example,  $\mathfrak{g} = E_6$  has two real forms to treat:  $E_{III}$  and  $E_{IV}$ . One of the primitive elements of  $Q \cap \mathfrak{h}^+$  is  $\lambda = 3\varpi_1$  (or  $-2e_6 - 2e_7 + 2e_8$  in the Bourbaki basis). For this element, the software computes  $\dim V_{3\varpi_1}^{\mathfrak{l}(E_{III})} = 2$  and  $\dim V_{3\varpi_1}^{\mathfrak{l}(E_{IV})} = 1$ .

## 4 Type $A_r$

We now prove the Main Theorem for  $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ , which has rank  $r = n - 1$ ; for the duration of this section,  $n$  is some integer greater than or equal to 2.

We will start, in Subsection 4.1, by establishing some notations and terminology about Young tableaux. We will then treat the case  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p, q)$  in Subsection 4.2, and the case



$\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_m(\mathbb{H})$  in Subsection 4.3. (The case  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_n(\mathbb{R})$  follows from Proposition 2.1.(ii).) We will put the pieces together in the brief subsection 4.4.

#### 4.1 Young tableaux: notations and definitions

We start by establishing some basic conventions (Definition 4.1) and notations (Definition 4.2) about Young tableaux and diagrams. They will also serve us in the Section 5.

**Definition 4.1.** Let  $n \geq 0$ . A *Young diagram of order  $n$*  is a top- and left-aligned Young diagram with at most  $n$  rows. The *shape* of a Young diagram  $\mathcal{P}$  is the  $n$ -tuple  $(\#_1\mathcal{P}, \dots, \#_n\mathcal{P})$ , where  $\#_i\mathcal{P}$  stands for the length of the  $i$ -th row of  $\mathcal{P}$  (see also the next definition); we will often identify the diagram with this tuple.

Let  $\mathcal{P}, \mathcal{Q}$  be two Young diagrams. We say that  $\mathcal{Q}$  is *contained* in  $\mathcal{P}$ , denoted by  $\mathcal{Q} \subset \mathcal{P}$ , if we have  $\#_i\mathcal{Q} \leq \#_i\mathcal{P}$  for all  $i$ . If this is the case, we define the *skew diagram*  $\mathcal{P}/\mathcal{Q}$  to be the diagram comprising all the boxes that are in  $\mathcal{P}$  but not in  $\mathcal{Q}$ .

Fix some ordered set  $\mathcal{A}$ . A *Young tableau on the alphabet  $\mathcal{A}$* , denoted for example by  $\mathcal{T}$ , is a Young diagram  $\mathcal{P}$  in which each box is filled with an element of  $\mathcal{A}$ ; we then say that  $\mathcal{T}$  is an  $\mathcal{A}$ -*filling* of  $\mathcal{P}$ . We define similarly a *skew tableau on the alphabet  $\mathcal{A}$* . For  $n \geq 0$ , a *Young tableau of order  $n$*  is a  $\{1, \dots, n\}$ -filling of a Young diagram of order  $n$ .

We say that a Young tableau, or skew tableau, is *semistandard* if the values written in its boxes form a strictly increasing sequence along each column (from top to bottom), and a nondecreasing sequence along each row (from left to right).

**Definition 4.2.** For any diagram or tableau  $\mathcal{P}$ , we define the following notations. For  $i \in \mathbb{N}$ , let  ${}_i\mathcal{P}$  (resp.  ${}^i\mathcal{P}$ ) denote the  $i$ -th row (resp. column) from the top (resp. left) of  $\mathcal{P}$ . For  $I \subset \mathbb{N}$ , let  ${}_I\mathcal{P}$  (resp.  ${}^I\mathcal{P}$ ) denote the subtableau or subdiagram comprising all the rows (resp. columns) of  $\mathcal{P}$  indexed by  $I$ . Let  $\#\mathcal{P}$  denote the number of boxes in  $\mathcal{P}$ .

For any tableau  $\mathcal{T}$  on the alphabet  $\mathcal{A}$ , we define the following notations. For  $s \in \mathcal{A}$  (resp.  $S \subset \mathcal{A}$ ), we denote by  $\boxed{s}\mathcal{T}$  (resp.  $\boxed{S}\mathcal{T}$ ) the subtableau of  $\mathcal{T}$  comprising only the boxes containing the symbol  $s$  (resp. symbols from  $S$ ). We denote by  $\square\mathcal{T}$  the underlying diagram of  $\mathcal{T}$ , i.e. the diagram obtained by erasing all the symbols from all the boxes.

These notations can of course be combined *ad libitum*. For example,  $\# \boxed{\begin{smallmatrix} [1,x] \\ [p,q] \\ [1,y] \end{smallmatrix}} \mathcal{T}$  stands for the total number of occurrences of symbols lying between  $p$  and  $q$  in the top left  $x$ -by- $y$  rectangle of  $\mathcal{T}$ . We also convene that  ${}^i_j\mathcal{T}$  stands (by slight notation abuse) for the symbol that fills the  $(i, j)$ -th box of  $\mathcal{T}$ .

The following simple (and well-known) trick is a useful point of view for studying semistandard Young tableaux. Define the *thickness* of a skew diagram  $\mathcal{P}/\mathcal{Q}$  as the largest height of one of its columns:  $\text{thickness}(\mathcal{P}/\mathcal{Q}) := \max_j \#^j(\mathcal{P}/\mathcal{Q}) = \max_j (\#^j\mathcal{P} - \#^j\mathcal{Q})$ .

**Proposition 4.3** (Horizontal strip decomposition). *The set of semistandard Young tableaux of order  $n$  with underlying diagram  $\mathcal{P}$  is in bijection with the set of nested sequences of Young diagrams  $\emptyset = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \dots \subset \mathcal{P}_n = \mathcal{P}$  where each skew diagram  $\mathcal{P}_s/\mathcal{P}_{s-1}$  is a horizontal strip, i.e. has thickness at most 1.*

We introduce a few more definitions in order to state the Young tableau character formula and Levi branching rule for  $\mathfrak{sl}_n(\mathbb{C})$ . They are only valid for this section; in Section 5, when  $\mathfrak{g}$  will be of type  $B_r$ ,  $C_r$  or  $D_r$ , we will use Definition 5.24 instead.

**Definition 4.4** (Passing from diagrams and tableaux to weights, in type  $A_r$ ).

- (i) Let  $\mathcal{P} = (\#_1\mathcal{P}, \dots, \#_n\mathcal{P})$  be a Young diagram of order  $n$ . We define its *offset*

$$a(\mathcal{P}) := \frac{1}{n}\#\mathcal{P} = \frac{1}{n}\sum_{i=1}^n \#_i\mathcal{P}, \quad (4.1)$$

and its  $\mathfrak{sl}_n$ -*shape* to be  $\lambda := \sum \lambda_i e_i$  with

$$\forall i = 1, \dots, n, \quad \lambda_i := \#_i\mathcal{P} - a(\mathcal{P}). \quad (4.2)$$

We observe that this  $\lambda$  is always an element of  $P \cap \mathfrak{h}^+$ . Given some  $\lambda \in P \cap \mathfrak{h}^+$ , the *reduced* Young diagram of  $\mathfrak{sl}_n$ -shape  $\lambda$  is the one whose  $n$ -th row has length 0.

- (ii) We define the *total weight*  $\nu(\mathcal{T})$  of a Young tableau or skew tableau  $\mathcal{T}$  as

$$\nu(\mathcal{T}) := \sum_{i,j} \nu \left( \begin{smallmatrix} j \\ i \end{smallmatrix} \mathcal{T} \right), \quad (4.3)$$

where, for all  $s = 1, \dots, n$ , we define  $\nu(s)$  as the orthogonal projection of  $e_s$  onto  $\mathfrak{h}$ :

$$\nu(s) := e_s - \frac{1}{n} \sum_{i=1}^n e_i. \quad (4.4)$$

- (iii) Given a linear form  $\alpha \in \mathfrak{h}^*$ , we say that a Young tableau  $\mathcal{T}$  is:

- $\alpha$ -*dominant* if for all  $j = 0, \dots, N$ , we have  $\alpha \left( \nu \left( \begin{smallmatrix} j+1, N \\ \end{smallmatrix} \mathcal{T} \right) \right) \geq 0$ ;
- $\alpha$ -*codominant* if for all  $j = 0, \dots, N$ , we have  $\alpha \left( \nu \left( \begin{smallmatrix} 1, j \\ \end{smallmatrix} \mathcal{T} \right) \right) \leq 0$ ,

where  $N = \#_1\mathcal{T}$  is the width of  $\mathcal{T}$ . For a subset  $\Theta \subset \Pi$ , we say that  $\mathcal{T}$  is  $\Theta$ -*(co)dominant* if it is  $\alpha$ -(co)dominant for all  $\alpha \in \Theta$ .

In this paper, we will usually consider tableaux of total weight 0, for which these two properties are obviously equivalent. Dominance is the most natural property in general, but we will find it more convenient to use codominance.

We then have the following classical character formula. It is given only for general context; we will not use it directly in the sequel.

**Proposition 4.5** (Character formula with Young tableaux). *Let  $\lambda \in P \cap \mathfrak{h}^+$  be a dominant integral weight of  $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ . Then the character of  $V_\lambda(\mathfrak{g})$  is given by:*

$$\text{char}(V_\lambda) = \sum_{\mathcal{T}} e^{\nu(\mathcal{T})},$$

where  $\mathcal{T}$  runs over all reduced semistandard Young tableaux of order  $n$  and of  $\mathfrak{sl}_n$ -shape  $\lambda$ .

For a proof, see e.g. [12], Proposition 15.15 together with the discussion that follows.

We also have the following (closely related) classical branching rule, on which we will rely in the sequel.

**Proposition 4.6** (Branching rule with Young tableaux). *Let  $\Theta \subset \Pi$  be a set of simple roots of  $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ , and let  $\lambda \in P \cap \mathfrak{h}^+$  be a dominant integral weight. Then we have*

$$V_\lambda(\mathfrak{g})|_{\mathfrak{l}(\Theta)} = \bigoplus_{\mathcal{T}} V_{\nu(\mathcal{T})}(\mathfrak{l}(\Theta)),$$

where  $\mathcal{T}$  runs over all reduced  $\Theta$ -dominant semistandard Young tableaux of order  $n$  and of  $\mathfrak{sl}_n$ -shape  $\lambda$ .

This is stated in this form in [20, Theorem 2.2.(b)], and can be deduced from Littelmann's more general branching rule ([21, Restriction Rule], restated here as Proposition 5.5) by using the  $\mathfrak{sl}_n$ -analog of Proposition 5.25 (that links Littelmann paths with Young tableaux). It was however certainly known before Littelmann, as one of the multiple avatars of the Littlewood-Richardson rule; see [23].

Now of course  $V_\lambda^{\mathfrak{l}}$  is obtained by selecting, in this decomposition, the summands isomorphic to the trivial representation, i.e. such that  $\nu(\mathcal{T}) = 0$ . So we obtain a criterion for the nontriviality of  $V_\lambda^{\mathfrak{l}}$ , namely Corollary 4.9 below. The following definition and remark will allow us to state this criterion in a purely combinatorial way.

**Definition 4.7.** We say that a Young tableau or skew tableau  $\mathcal{T}$  on an alphabet  $\mathcal{A}$  is *balanced* (with respect to  $\mathcal{A}$ ) if each symbol from  $\mathcal{A}$  occurs the same number of times:

$$\forall s \in \mathcal{A}, \quad \# \boxed{s} \mathcal{T} = \frac{1}{\#\mathcal{A}} \# \mathcal{T}. \quad (4.5)$$

Clearly a Young tableau  $\mathcal{T}$  of order  $n$  is then balanced if and only if  $\nu(\mathcal{T}) = 0$ ; and in this case the offset of  $\mathcal{T}$  is integer.

*Remark 4.8.* Every simple root  $\alpha \in \Pi$  is of the form  $\alpha = \alpha_i = e_i - e_{i+1}$ , for some  $i = 1, \dots, n-1$ . Then  $\mathcal{T}$  is  $\alpha_i$ -codominant if and only if, for any  $j$ , there are at least as many symbols  $i+1$  as symbols  $i$  among the first  $j$  columns of  $\mathcal{T}$ :

$$\forall j \geq 0, \quad \# \boxed{i+1}^{[1,j]} \mathcal{T} \geq \# \boxed{i}^{[1,j]} \mathcal{T}. \quad (4.6)$$

**Corollary 4.9.** *Let  $\Theta \subset \Pi$  be a set of simple roots of  $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ . Let  $\lambda \in P \cap \mathfrak{h}^+$  be a dominant integral weight, and let  $\mathcal{P}$  be any Young diagram with  $\mathfrak{sl}_n$ -shape  $\lambda$ . Then  $V_\lambda^{\mathfrak{l}(\Theta)} \neq 0$  if and only if  $\mathcal{P}$  has a  $\Theta$ -codominant balanced semistandard  $\{1, \dots, n\}$ -filling.*

## 4.2 The case $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p, n-p)$

For the duration of this subsection, fix some  $p \leq \frac{n}{2}$  and assume that  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p, n-p)$ .

Let us then describe  $\Theta(\mathfrak{g}_{\mathbb{R}})$ . We introduce, for the whole remaining duration of the paper, the following notation shortcuts (recall that here  $r = n - 1$ ):

$$\Pi_{[x,y]} := \{\alpha_x, \alpha_{x+1}, \dots, \alpha_y\} \subset \Pi = \Pi_{[1,r]}; \quad (4.7)$$

$$\Pi_{\text{odd}} := \{\alpha_i \in \Pi \mid i \text{ is odd}\}, \quad (4.8)$$

with the convention  $\Pi_{[x,x-1]} = \emptyset$  for all  $x$ . Then [26], Reference Chapter, Table 9 gives

$$\Theta(\mathfrak{su}(p, n - p)) = \begin{cases} \Pi_{[p+1, n-p-1]} & \text{if } p < \frac{n}{2}; \\ \emptyset & \text{if } p = \frac{n}{2}. \end{cases}$$

This subsection proves the following result, from which we conclude in Subsection 4.4.

**Proposition 4.10.** *Let  $\mathcal{P}$  be a Young diagram of order  $n$ , and let  $k \in \{1, \dots, n\}$ . Then  $\mathcal{P}$  has a  $\Pi_{[1,k-1]}$ -codominant balanced semistandard  $\{1, \dots, n\}$ -filling if and only if the offset  $a$  of  $\mathcal{P}$  is integer, and satisfies the inequalities*

$$\#_k \mathcal{P} \geq a \geq \#_{n-k+1} \mathcal{P}. \quad (4.9)$$

The proof relies on the following straightforward application of Proposition 4.3.

**Lemma 4.11.** *Let  $\Theta \subset \Pi$ , and suppose that  $k \in \{1, \dots, n - 1\}$  is such that  $\alpha_k = e_k - e_{k+1} \notin \Theta$ . Then a Young diagram  $\mathcal{P}$  admits a  $\Theta$ -codominant, balanced, semistandard  $\{1, \dots, n\}$ -filling if and only if there exists a diagram  $\mathcal{Q} \subset \mathcal{P}$  such that:*

- $\mathcal{Q}$  has a  $(\Theta \cap \Pi_{[1,k-1]})$ -codominant balanced semistandard  $\{1, \dots, k\}$ -filling;
- $\mathcal{P}/\mathcal{Q}$  has a  $(\Theta \cap \Pi_{[k+1,n-1]})$ -codominant balanced semistandard  $\{k+1, \dots, n\}$ -filling;
- the offset of  $\mathcal{Q}$  (as a diagram of order  $k$ ) coincides with the offset of  $\mathcal{P}$ .

In our case,  $\Theta = \Pi_{[1,k-1]}$ . It remains to characterize diagrams  $\mathcal{Q}$  and  $\mathcal{P}/\mathcal{Q}$  with these properties; this is respectively the object of the following two lemmas.

**Lemma 4.12.** *For  $k \geq 0$  and  $a \geq 0$ , define the tableau  $\mathcal{R}_k^a$  that is shaped like a rectangle with  $k$  rows of length  $a$ , with, for each  $s = 1, \dots, k$ , the  $s$ -th row filled with the symbol  $s$ .*

*Let  $k \geq 1$ . Then the only  $\Pi_{[1,k-1]}$ -codominant balanced semistandard Young tableaux of order  $k$  are the rectangles  $\mathcal{R}_k^a$ , for all (integer) offsets  $a \geq 0$ .*

*Proof.* By Corollary 4.9, this is equivalent to the obvious fact that  $V_{\lambda}^{\mathfrak{g}}(\mathfrak{g}) \neq 0$  if and only if  $\lambda = 0$  (for  $\mathfrak{g} = \mathfrak{sl}_k(\mathbb{C})$ ).  $\square$

**Lemma 4.13.** *Let  $m \geq 0$ , and let  $\mathcal{P}/\mathcal{Q}$  be a skew diagram. Then it admits a balanced semistandard  $\{1, \dots, m\}$ -filling if and only if it has thickness at most  $m$  and its number of boxes is divisible by  $m$ .*

*Proof.* The “only if” part is obvious. Conversely, let  $\mathcal{P}/\mathcal{Q}$  be a skew diagram of thickness at most  $m$  and containing  $ma$  boxes, for some integer  $a \geq 0$ . By induction on  $m$  (using Proposition 4.3), it suffices to find a Young diagram  $\mathcal{P}'$  with  $\mathcal{Q} \subset \mathcal{P}' \subset \mathcal{P}$  such that  $\mathcal{P}/\mathcal{P}'$  (resp.  $\mathcal{P}'/\mathcal{Q}$ ) has exactly  $a$  (resp.  $(m-1)a$ ) boxes and thickness at most 1 (resp.  $m$ ).

Denote by  $X$  (resp.  $Y$ ) the set of indices  $j$  such that the height of the  $j$ -th column of the skew diagram  $\mathcal{P}/\mathcal{Q}$  is exactly  $m$  (resp. is nonzero). By the pigeonhole principle  $\#X \leq a \leq \#Y$ . Then the following specifications of column heights define a suitable  $\mathcal{P}'$ :

- for  $j$  in  $X$  or among the largest  $(a - \#X)$  values in  $Y \setminus X$ , we set  $\#^j \mathcal{P}' := \#^j \mathcal{P} - 1$ ;
- for  $j$  among the remaining values in  $Y \setminus X$  or outside of  $Y$ , we set  $\#^j \mathcal{P}' := \#^j \mathcal{P}$ .  $\square$

*Proof of Proposition 4.10.* Let  $\mathcal{P}$  be any Young diagram with  $\mathfrak{sl}_n$ -shape  $\lambda$  and offset  $a$ . The three lemmas show that  $\mathcal{P}$  has a filling as required if and only if  $a$  is integer and:

$$\begin{cases} \text{the rectangular diagram } \square \mathcal{R}_k^a \text{ is contained in } \mathcal{P}; \\ \text{the skew diagram } \mathcal{P}/\square \mathcal{R}_k^a \text{ has thickness at most } n - k. \end{cases} \quad (4.10)$$

One easily checks that the condition (4.10) is equivalent to the inequalities (4.9).  $\square$

### 4.3 The case $\mathfrak{sl}_m(\mathbb{H})$

Fix some  $m \geq 1$ . For the duration of this subsection, assume  $n = 2m$  and  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_m(\mathbb{H})$ .

From [26], Reference Chapter, Table 9, we then get

$$\Theta(\mathfrak{sl}_m(\mathbb{H})) = \Pi_{\text{odd}} = \{e_1 - e_2, e_3 - e_4, \dots, e_{2m-1} - e_{2m}\}.$$

This subsection proves the following result, from which we conclude in Subsection 4.4.

**Proposition 4.14.** *Let  $\mathcal{P}$  be a Young diagram of order  $n$ . Then it admits a  $\Pi_{\text{odd}}$ -codominant balanced semistandard  $\{1, \dots, n\}$ -filling if and only if its offset  $a$  is integer, and it satisfies the two inequalities*

$$\begin{cases} -p_1 + \sum_{i=2}^{m+1} p_i - \sum_{i=m+2}^{2m} p_i \geq 0 \\ -\sum_{i=1}^{m-1} p_i + \sum_{i=m}^{2m-1} p_i - p_{2m} \leq 0, \end{cases} \quad (4.11.m)$$

with the notation shortcut  $p_i := \#^i \mathcal{P}$ .

The proof relies once again on Lemma 4.11, but now uses *all* even indices as “cutting points” and is much more technical. The proof structure, summarized in the schematic given in the final proof (Subsubsection 4.3.3), is centered on two big lemmas:

- Subsubsection 4.3.1 is dedicated to proving Lemma 4.16, which, roughly, gives a condition for a suitable filling of the “bottom” skew tableau, namely  $\boxed{2m-1, 2m} \mathcal{T}$ .
- Subsubsection 4.3.2 is dedicated to proving Lemma 4.18, which, roughly, deduces the result from the induction hypothesis and from this characterization.

#### 4.3.1 Skew tableaux of thickness 2

Note that, without loss of generality, we may replace the alphabet  $\{2m-1, 2m\}$  by  $\{1, 2\}$ .

**Definition 4.15.** For each  $i$ , the *bridge at height  $i$*  in a skew diagram  $\mathcal{P}/\mathcal{Q}$  is the rectangle formed by all columns  $j$  such that  $\#^j \mathcal{Q} = i-1$  and  $\#^j \mathcal{P} = i$ ; and  $b_i$  denotes its length.

**Lemma 4.16.** *Let  $\mathcal{P}$  be a Young diagram of order  $n$ , and let  $\mathcal{Q} \subset \mathcal{P}$ . Then  $\mathcal{P}/\mathcal{Q}$  admits an  $\alpha_1$ -codominant balanced semistandard  $\{1, 2\}$ -filling if and only if  $\#\mathcal{P}/\mathcal{Q}$  is even,  $\mathcal{P}/\mathcal{Q}$  has thickness at most 2 and*

$$\forall i = 1, \dots, n, \quad 2b_i \leq b_1 + \dots + b_n. \quad (4.12)$$

*Proof.* Let  $\mathcal{T}$  be any  $\{1, 2\}$ -filling of the skew-tableau  $\mathcal{P}/\mathcal{Q}$ . First of all, note that this filling is semistandard if and only if no columns of height more than 2 exist, each column of height 2 is filled with the symbols 1 and 2 in that order, and, for every  $i$ , there exists a number  $c_i$  satisfying  $0 \leq c_i \leq b_i$  with the  $i$ -th bridge of  $\mathcal{T}$  having the first  $c_i$  boxes filled with 1 and the last  $b_i - c_i$  boxes filled with 2.

Assume now that  $\mathcal{T}$  is semistandard. One easily checks that a semistandard filling  $\mathcal{T}$  is  $\alpha_1$ -codominant if and only if it satisfies

$$\forall i = 1, \dots, n, \quad 2(c_1 + \dots + c_i) \leq b_1 + \dots + b_{i-1} + c_i. \quad (4.13)$$

and balanced if and only if it satisfies

$$2(c_1 + \dots + c_n) = b_1 + \dots + b_n. \quad (4.14)$$

Also observe that, assuming that the thickness does not exceed 2, we have  $\#\mathcal{P}/\mathcal{Q} \equiv \sum_{i=1}^n b_i \pmod{2}$ . The conclusion now follows from the following lemma.  $\square$

**Lemma 4.17.** *Given a tuple of integers  $(b_1, \dots, b_n)$ , there exists a tuple of integers  $(c_1, \dots, c_n)$  satisfying  $0 \leq c_i \leq b_i$  and conditions (4.13) and (4.14) if and only if the  $b_i$  have even sum and satisfy the system (4.12).*

*Proof.* Suppose first that such a tuple  $(c_1, \dots, c_n)$  exists. Then (4.14) directly implies that  $\sum_{i=1}^n b_i$  is even. Furthermore, by combining (4.13) and (4.14), we may see that the tuple  $(c'_1, \dots, c'_n)$ , defined by  $c'_i := b_i - c_i$ , then satisfies

$$\forall i = 1, \dots, n, \quad 2(c'_i + \dots + c'_n) \leq c'_i + b_{i+1} + \dots + b_n. \quad (4.15)$$

By adding together (4.13) and (4.15), we then obtain, for all  $i = 1, \dots, n$ :

$$2(c_1 + \dots + c_{i-1} + b_i + c'_{i+1} + \dots + c'_n) \leq b_1 + \dots + b_n. \quad (4.16)$$

Since the left-hand side is not less than  $b_i$ , (4.12) follows.

Conversely, suppose that the tuple  $(b_1, \dots, b_n)$  has an even sum, that we shall denote by  $b$ , and satisfies the system (4.12). Then consider the tuple  $(c_1, \dots, c_n)$  defined by:

$$c_i := \begin{cases} 0 & \text{if } \sum_{j=1}^i b_j \leq \frac{1}{2}b; \\ b_i & \text{if } \sum_{j=1}^{i-1} b_j \geq \frac{1}{2}b; \\ \frac{1}{2}b - \sum_{j=i+1}^n b_j & \text{if } \sum_{j=1}^{i-1} b_j < \frac{1}{2}b < \sum_{j=1}^i b_j. \end{cases} \quad (4.17)$$

Clearly the  $c_i$  lie in the appropriate range, and satisfy (4.14). Moreover for all  $i$  such that  $\sum_{j=1}^i b_j \leq \frac{1}{2}b$  or  $\sum_{j=1}^{i-1} b_j \geq \frac{1}{2}b$ , clearly  $(c_1, \dots, c_n)$  satisfies respectively (4.13) or (4.15), with the latter equivalent to the former given (4.14). As for the index  $i$  such that neither holds (if it exists), (4.12) implies (4.16), since all the extra terms on the left-hand side vanish; and (4.13) and (4.15) being equivalent, their sum (4.16) is equivalent to both.  $\square$

### 4.3.2 The induction step

This subsubsection is dedicated to proving the following result, which provides the equivalence (D) in the outline given in the final proof (Subsubsection 4.3.3).

In this whole subsubsection, we assume that  $m \geq 2$  and  $n = 2m$ .

**Lemma 4.18.** *Let  $\mathcal{P}$  be a Young diagram of order  $n = 2m \geq 4$ . Then  $\mathcal{P}$  satisfies the inequalities (4.11.m) and has integer offset (i.e.  $\#\mathcal{P}$  is divisible by  $n$ ) if and only if there exists a Young diagram  $\mathcal{Q}$  of order  $n - 2$  with the following properties:*

(i)  $\mathcal{Q} \subset \mathcal{P}$  with  $\mathcal{P}/\mathcal{Q}$  satisfying Lemma 4.16, and  $\mathcal{Q}$  satisfies the system

$$\begin{cases} -q_1 + \sum_{i=2}^m q_i - \sum_{i=m+1}^{2m-2} q_i \geq 0 \\ -\sum_{i=1}^{m-2} q_i + \sum_{i=m-1}^{2m-3} q_i - q_{2m-2} \leq 0, \end{cases} \quad (4.11.m-1)$$

i.e. (4.11.m) with  $m$  replaced by  $m - 1$  (with the notation shortcut  $q_i := \#_i \mathcal{Q}$ );

(ii)  $\#\mathcal{Q} = \frac{n-2}{n} \#\mathcal{P}$ .

The proof will require some preliminary work. We will prove the “if” part simply by rewriting the conditions on  $\mathcal{P}$  and  $\mathcal{Q}$  as a system of inequalities, and suitably combining some of them; the “only if” part is the hardest. Very roughly, we will find such a  $\mathcal{Q}$  for a few values of  $\mathcal{P}$ , and then use additivity; but we will need to slightly modify condition (ii).

**Definition 4.19.** The *basis* of a commutative monoid is the set of its *primitive* elements, i.e. nonzero elements that are not the sum of any two other nonzero elements.

**Definition 4.20.** For each integer  $x \geq 0$ , let  $\mathcal{M}^{(x)}$  denote the monoid of all the Young diagrams of order  $x$ , with the addition defined by adding the numbers of boxes row-wise:

$$\forall i = 1, \dots, x, \quad \#_i(\mathcal{P} + \mathcal{Q}) := \#_i \mathcal{P} + \#_i \mathcal{Q}. \quad (4.18)$$

When  $x = n$ , we will usually omit the index, i.e.  $\mathcal{M} := \mathcal{M}^{(n)} = \mathcal{M}^{(2m)}$ . Let  $\mathcal{M}_{(4.11)}$  denote the submonoid of  $\mathcal{M}$  determined by the system (4.11.m). For each  $k > 0$ , let

$$\mathcal{M}_{k|\#} := \{\mathcal{P} \in \mathcal{M} \mid \#\mathcal{P} \in k\mathbb{Z}\}. \quad (4.19)$$

For each  $i = 0, \dots, n$ , let  $\mathcal{C}_i \in \mathcal{M}$  be the diagram consisting of a single column of height  $i$ .

**Lemma 4.21.** *The set of pairs  $(\mathcal{P}, \mathcal{Q})$  that satisfy (i) is a submonoid of  $\mathcal{M}^{(n)} \oplus \mathcal{M}^{(n-2)}$ .*

*Proof.* It suffices to show that (i) is equivalent to a system of the form

$$\forall i \in I, \quad \phi_i(\mathcal{P}, \mathcal{Q}) \geq 0, \quad (4.20)$$

where  $(\phi_i)_{i \in I}$  is some family of linear forms, i.e. linear maps from  $\mathcal{M}^{(n)} \oplus \mathcal{M}^{(n-2)}$  to  $\mathbb{R}$ . Let us show this for (4.12) (for the remaining parts it is obvious). It can be rewritten as

$$\forall i = 1, \dots, n-1, \quad \sum_{j=1}^{n-1} (-1)^{\delta_{ij}} b_j \geq 0, \quad (4.21)$$

using the Kronecker delta ( $b_n = 0$  always). The identity  $\#_i \mathcal{P} \geq j \iff \#^j \mathcal{P} \geq i$  gives

$$\forall i = 1, \dots, n-1, \quad b_i = \min(q_{i-1}, p_i) - \max(q_i, p_{i+1}), \quad (4.22)$$

with the convention  $q_0 = +\infty$ ,  $q_{n-1} = -\infty$ . Plugging (4.22) into (4.21), we rewrite it as

$$(-1)^{\delta_{i1}} p_1 + \sum_{j=1}^{n-1} \left( -(-1)^{\delta_{i,j-1}} \max(q_{j-1}, p_j) + (-1)^{\delta_{ij}} \min(q_{j-1}, p_j) \right) - (-1)^{\delta_{i,n-1}} p_n \geq 0$$

for each  $i = 1, \dots, n-1$ . This can once again be rewritten as

$$\forall i = 1, \dots, n-1, \quad \left( \sum_{\substack{j=1 \\ j \neq i, i+1}}^n T_j \right) - \hat{T}_i + \hat{T}_{i+1} \geq 0, \quad (4.23.i)$$

where we set  $T_1 = \hat{T}_1 := p_1$ ,  $T_n := -p_n$ ,  $\hat{T}_n := p_n$  and, for all  $j = 2, \dots, n-1$ ,  $T_j := -|q_{j-1} - p_j|$  and  $\hat{T}_j := q_{j-1} + p_j$ . Now each inequality (4.23.i) is equivalent to a system of  $2^{N_i}$  linear inequalities ( $N_i$  counts the terms involving absolute values).  $\square$

**Lemma 4.22.** *Let  $\mathcal{B}'$  be the basis of the monoid  $\mathcal{M}_{(4.11)} \cap \mathcal{M}_{2|\#}$ . Then for each  $\mathcal{P} \in \mathcal{B}'$ , there exist two Young diagrams  $\mathcal{Q}^+(\mathcal{P})$  and  $\mathcal{Q}^-(\mathcal{P})$  of order  $n-2$  such that both pairs  $(\mathcal{P}, \mathcal{Q}^+)$  and  $(\mathcal{P}, \mathcal{Q}^-)$  satisfy condition (i) from Lemma 4.18 and the identities*

$$\begin{cases} \# \mathcal{Q}^- = 2 \left\lfloor \frac{1}{2} \frac{n-2}{n} \# \mathcal{P} \right\rfloor; \\ \# \mathcal{Q}^+ = 2 \left\lceil \frac{1}{2} \frac{n-2}{n} \# \mathcal{P} \right\rceil. \end{cases} \quad (4.24)$$

*Proof.* Let  $\mathcal{P}$  be an element of  $\mathcal{M}$ , with  $x_1, \dots, x_n$  its coordinates in the basis  $(\mathcal{C}_1, \dots, \mathcal{C}_n)$  introduced in Definition 4.20. Then (4.11.m) can be rewritten in terms of the  $x_i$  as

$$\begin{cases} \sum_{i=1}^{n-1} \min(i-2, n-i) x_i \geq 0; \\ \sum_{i=1}^{n-1} \min(i, n-i-2) x_i \geq 0 \end{cases} \quad (4.25.m)$$

(recall  $n = 2m$ ). It is then easy to see that the basis  $\mathcal{B}$  of the monoid  $\mathcal{M}_{(4.11)}$  is equal to

$$\begin{aligned} \mathcal{B} = & \left\{ \mathcal{C}_i \mid 2 \leq i \leq n-2 \right\} \cup \{ \mathcal{C}_n \} \cup \left\{ \mathcal{C}_i + a \mathcal{C}_1 \mid 0 < a \leq \min(i-2, n-i) \right\} \cup \\ & \cup \left\{ \mathcal{C}_i + a \mathcal{C}_{n-1} \mid 0 < a \leq \min(i, n-i-2) \right\}. \end{aligned} \quad (4.26)$$



Now consider a diagram  $\mathcal{P} \in \mathcal{M}_{(4.11)} \cap \mathcal{M}_{2|\#}$ . Its decomposition as a sum of elements of  $\mathcal{B}$  will then involve an even number of odd-sized diagrams (where by “size” we mean the number of boxes). Denoting by  $\mathcal{B}_{\text{even}}$  (resp.  $\mathcal{B}_{\text{odd}}$ ) the subset of  $\mathcal{B}$  comprising the diagrams of even (resp. odd) size, we obtain that the set  $\mathcal{B}_{\text{even}} \cup (\mathcal{B}_{\text{odd}} + \mathcal{B}_{\text{odd}})$  (where the “+” sign denotes the Minkowski sum) generates the monoid  $\mathcal{M}_{(4.11)} \cap \mathcal{M}_{2|\#}$ .

Clearly all elements of  $\mathcal{B}_{\text{even}}$ , being already primitive in  $\mathcal{M}_{(4.11)}$ , remain primitive in  $\mathcal{M}_{(4.11)} \cap \mathcal{M}_{2|\#}$ . As for elements of  $(\mathcal{B}_{\text{odd}} + \mathcal{B}_{\text{odd}})$ , they have in general the form

$$\mathcal{P} = \mathcal{C}_i + \mathcal{C}_j + a\mathcal{C}_1 + b\mathcal{C}_{n-1}$$

where  $a, b \geq 0$ , and  $i$  and  $j$  satisfy  $2 \leq i \leq j \leq n-2$ . With some work, we check that such a sum is primitive if and only if  $a = b = 0$ ; hence

$$\mathcal{B}' = \mathcal{B}_{\text{even}} \cup \{\mathcal{C}_i + \mathcal{C}_j \mid 2 \leq i \leq j \leq n-2 \text{ with } i, j \text{ odd}\}. \quad (4.27)$$

We now define, for each element  $\mathcal{P} \in \mathcal{B}'$ , two diagrams  $\mathcal{Q}^-(\mathcal{P})$  and  $\mathcal{Q}^+(\mathcal{P})$  as given in Table 2. Checking that, for each  $\mathcal{P}$ , both  $\mathcal{Q}^-(\mathcal{P})$  and  $\mathcal{Q}^+(\mathcal{P})$  satisfy all of the required properties is then somewhat laborious, but straightforward.  $\square$

We are now ready to conclude this subsubsection.

*Proof of Lemma 4.18.* • Assume first that a diagram  $\mathcal{Q}$  satisfying conditions (i) and (ii) exists. Then the fact that  $\#\mathcal{P}/\mathcal{Q}$  is even combined with (ii) implies that  $\#\mathcal{P}$  is divisible by  $n$ . Now consider condition (4.12): we have seen that it is equivalent to the system (4.23). Consider specifically (4.23.1), i.e. the first inequality of that system ( $i = 1$ ). It implies in particular that

$$-p_1 + q_1 + p_2 + \sum_{i=2}^m (-q_i + p_{i+1}) + \sum_{i=m+1}^{2m-2} (q_i - p_{i+1}) - p_{2m} \geq 0,$$

which we may rewrite as

$$-p_1 + \sum_{i=2}^{m+1} p_i - \sum_{i=m+2}^{2m} p_i \geq -q_1 + \sum_{i=2}^m q_i - \sum_{i=m+1}^{2m-2} q_i; \quad (4.28)$$

and the first part of (4.11. $m$ ) becomes a consequence of the first part of (4.11. $m-1$ ). Similarly, the second part of (4.11. $m$ ) follows from (4.11. $m-1$ ) by (4.23. $n-1$ ).

- Conversely, let  $\mathcal{P}$  be a Young diagram satisfying the assumptions, i.e. let  $\mathcal{P}$  be in  $\mathcal{M}_{(4.11)} \cap \mathcal{M}_{n|\#} \subset \mathcal{M}_{(4.11)} \cap \mathcal{M}_{2|\#}$ . Let us decompose it as  $\mathcal{P} = \sum_{l=1}^N \mathcal{P}_l$ , with each  $\mathcal{P}_l$  lying in the basis  $\mathcal{B}'$  of the latter monoid. We then set, for each  $k = 0, \dots, N$ :

$$\mathcal{Q}_k := \mathcal{Q}^-(\mathcal{P}_1) + \dots + \mathcal{Q}^-(\mathcal{P}_k) + \mathcal{Q}^+(\mathcal{P}_{k+1}) + \dots + \mathcal{Q}^+(\mathcal{P}_N), \quad (4.29)$$

where  $\mathcal{Q}^\pm(\mathcal{P}_l)$  are the diagrams constructed in Lemma 4.22. By construction of these diagrams and by Lemma 4.21, it follows that each of the pairs  $(\mathcal{P}, \mathcal{Q}_k)$

Table 2: Table listing the Young diagrams  $\mathcal{Q}^-(\mathcal{P})$  and  $\mathcal{Q}^+(\mathcal{P})$  claimed to exist in Lemma 4.22, for each primitive diagram  $\mathcal{P}$  lying in the monoid  $\mathcal{M}_{(4.11)} \cap \mathcal{M}_{2|\#}$ .

$\mathcal{P}$	Parameter range	Subrange	$\mathcal{Q}^-(\mathcal{P})$	$\mathcal{Q}^+(\mathcal{P})$
$\mathcal{C}_i$	$\begin{cases} 2 \leq i \leq 2m \\ i \text{ even} \end{cases}$	$i < 2m$	as below	$\mathcal{C}_i$
		$i = 2m$	$\mathcal{C}_{i-2}$	
$\mathcal{C}_i + a\mathcal{C}_1$	$\begin{cases} 0 < a \leq \min(i-2, 2m-i) \\ i+a \text{ even} \end{cases}$	$a < 2m-i$	as below	$\mathcal{C}_i + a\mathcal{C}_1$
		$a = 2m-i$	$\mathcal{C}_{i-1} + (a-1)\mathcal{C}_1$	
$a\mathcal{C}_{2m-1} + \mathcal{C}_i$	$\begin{cases} 0 < a \leq \min(i, 2m-2-i) \\ i+a \text{ even} \end{cases}$	$a = i$	$\mathcal{C}_{2m-2} + (a-1)\mathcal{C}_{2m-3} + \mathcal{C}_{i-1}$	
		$a < i$	$a\mathcal{C}_{2m-3} + \mathcal{C}_{i-2}$	as above
$\mathcal{C}_j + \mathcal{C}_i$	$\begin{cases} 2 \leq i < j \leq 2m-2 \\ i, j \text{ odd} \end{cases}$	$i+j < 2m$	as below	$\mathcal{C}_j + \mathcal{C}_i$
		$i+j = 2m$	$\mathcal{C}_{j-1} + \mathcal{C}_{i-1}$	
		$i+j > 2m$	$\mathcal{C}_{j-2} + \mathcal{C}_{i-2}$	as above
$2\mathcal{C}_i$	$\begin{cases} 2 \leq i \leq 2m-2 \\ i \text{ odd} \end{cases}$	$i < m$	as below	$2\mathcal{C}_i$
		$i = m$	$\mathcal{C}_i + \mathcal{C}_{i-2}$	
		$i > m$	$2\mathcal{C}_{i-2}$	as above

satisfies condition (i). On the other hand, also by construction, the numbers  $\#Q_0, \#Q_1, \dots, \#Q_N$  are all even, form a nondecreasing sequence with consecutive terms differing by at most 2, and satisfy  $\#Q_0 \leq \frac{n-2}{n} \#P \leq \#Q_N$ . This implies that for a suitable choice of  $k$ , we have  $\#Q_k = \frac{n-2}{n} \#P$  as required.  $\square$

### 4.3.3 The case $\mathfrak{sl}_m(\mathbb{H})$ : conclusion

It remains to put everything together.

*Proof of Proposition 4.14.* As announced, we proceed by induction on  $m$ .

For  $m = 1$ , we have  $\Theta = \{\alpha_1\} = \Pi$ , and (4.11.m) reduces to the condition  $p_1 = p_2$ . The result is then a particular case of Lemma 4.12 (for  $k = 2$ ).

Assume now that  $m \geq 2$ , and that the result holds for  $m - 1$ . Let  $\mathcal{P}$  be of order  $2m$ . The result for  $\mathcal{P}$  then follows by combining the lemmas above, along the following outline:

$$\begin{array}{c}
 \mathcal{P} \text{ has } \Theta_m\text{-cbsf} \xLeftrightarrow{(A)} \left\{ \begin{array}{l} Q \subset \mathcal{P} \\ \mathcal{P}/Q \text{ has } \Theta_1\text{-cbsf} \end{array} \right\} \xLeftrightarrow{(C)} \left\{ \begin{array}{l} \#Q \in (2m-2)\mathbb{Z} \\ \#P/Q \in 2\mathbb{Z} \\ \forall j, 0 \leq \#^j \mathcal{P} - \#^j Q \leq 2 \\ \forall i, b_i \leq \frac{1}{2} \sum_i b_i \\ \#Q = \frac{m-1}{m} \#P \end{array} \right\} \xLeftrightarrow{(D)} \left\{ \begin{array}{l} \#P \in 2m\mathbb{Z} \\ \mathcal{P} \vdash (4.11.m) \end{array} \right\}
 \end{array}$$

$\exists Q \text{ of ord. } 2m-2, \quad \exists Q \text{ of ord. } 2m-2,$

$\left\{ \begin{array}{l} Q \text{ has } \Theta_{m-1}\text{-cbsf} \\ \#Q \in (2m-2)\mathbb{Z} \end{array} \right\} \xLeftrightarrow{(B)} \left\{ \begin{array}{l} Q \vdash (4.11.m-1) \\ \#P/Q \in 2\mathbb{Z} \end{array} \right\} \rightarrow \text{redundant}$

Here “has  $\Theta_m$ -cbsf” is shorthand for “has a  $\{\alpha_1, \alpha_3, \dots, \alpha_{2m-1}\}$ -codominant balanced semistandard  $\{1, 2, \dots, 2m\}$ -filling”, and the symbol  $\vdash$  is taken to mean “satisfies”. Naturally,  $b_i$  here stands for the length of the  $i$ -th bridge (see Definition 4.15) of the skew-diagram  $\mathcal{P}/Q$ .

The ingredients of the proof are then as follows: (A) is the (obvious) “divide-and-conquer” Lemma 4.11, applied to  $k = 2m - 2$ ; (B) is the induction hypothesis; (C) is Lemma 4.16 (the main result of Subsubsection 4.3.1); and (D) is Lemma 4.18 (the main result of Subsubsection 4.3.2). Also the condition “ $\#Q \in (2m - 2)\mathbb{Z}$ ” is marked as redundant, as it follows from  $\#P/Q \in 2\mathbb{Z}$  together with  $\#Q = \frac{m-1}{m} \#P$ .  $\square$

### 4.4 The case $\mathfrak{g} = A_r$ : conclusion

*Proof of Main Theorem for  $\mathfrak{g}$  of type  $A_{r \geq 1}$ .*

- For  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_n(\mathbb{R})$ , the conclusion follows from Proposition 2.1.(ii) as already noted.
- For  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p, n - p)$  with arbitrary  $p$ , the result follows from Proposition 4.10, applied to  $k = n - 2p$  if  $p < \frac{n}{2}$  or  $k = 1$  otherwise, by Corollary 4.9 and by (4.2).

We need to explain how we pass from the set  $\Theta(\mathfrak{g}_{\mathbb{R}})$  to the set  $\Pi_{[1, k-1]}$ . If  $p = \frac{n}{2}$ , then both sets are empty, hence equal. Otherwise, we have  $\Theta(\mathfrak{g}_{\mathbb{R}}) = \Pi_{[p+1, n-p-1]}$

and  $\Pi_{[1,k-1]} = \Pi_{[1, n-2p-1]}$ . Some element of the Weyl group maps these two sets to each other, so that the corresponding Levi subalgebras, say  $\mathfrak{l}_1, \mathfrak{l}_2$ , are conjugate in  $G_{\mathbb{R}} = \mathrm{SU}(p, n-p)$  and the two spaces  $V_{\lambda}^{\mathfrak{l}_{1,2}}$  have the same dimension.

- For  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_m(\mathbb{H})$ , the result follows from Proposition 4.14 by Corollary 4.9. The inequalities (4.11.m) are simply a homogeneous version of the inequalities appearing in the line  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_m(\mathbb{H})$  of Table 1: we substituted  $\lambda_i = p_i - a$ , and then expanded the  $a$  terms in terms of the  $p_i$ .  $\square$

## 5 Types $B_r, C_r$ and $D_r$ : the setup

The goal of this section is to obtain Corollary 5.28, which is a purely combinatorial characterization of the weights  $\lambda$  such that  $V_{\lambda}^{\mathfrak{l}} \neq 0$  in the case when  $\mathfrak{g}$  is of type  $B_r, C_r$  or  $D_r$ , analogous to Corollary 4.9 from the previous section. It will allow us, in the next section, to actually classify these weights  $\lambda$ .

This criterion relies on so-called “ $\mathfrak{g}$ -standard doubled Young tableaux”, which play in types  $B, C$  and  $D$  the same role as ordinary semistandard Young tableaux in type  $A$ . More generally, these tableaux lead to a combinatorial character formula (Proposition 5.27) in types  $B, C$  and  $D$ , analogous to Proposition 4.5 in type  $A$ , which may be of independent interest. In types  $B$  and  $C$ , this character formula already appears (without proof) in [20, Appendix A.2]. In type  $D$ , a similar formula appears (also without proof) in [20, Appendix A.3], but our formula constitutes a slight improvement, as discussed in the introduction (Subsubsection 1.4.1).

All of this work is based on the Littelmann path model (that gives a character formula for any semisimple Lie algebra  $\mathfrak{g}$ ), whose construction we briefly recall in Subsection 5.1.

In Subsection 5.2, we explain how to describe the path model based on a long starting path in terms of the path models based on its segments, using the Bruhat order. This part is also essentially due to Littelmann.

Starting from this point, we specialize to the case where  $\mathfrak{g}$  is of type  $B, C$  or  $D$ . In Subsection 5.3, we present a characterization of the Bruhat order in terms of Young tableaux, given some (reasonable) assumptions. This simple characterization is the key point that allows us to simplify the definition of the “doubled Young tableaux” in type  $D$ .

In Subsection 5.4, we describe the path model on a “short” starting path of the form  $e_1 + \dots + e_{k-1} \pm e_k$ , in terms of “admissible pairs” (a notion due to Lakshmibai-Seshadri and Littelmann); and we give a combinatorial description of these admissible pairs.

Finally, in Subsection 5.5, we define a  $\mathfrak{g}$ -standard doubled Young tableau, and give the announced character formula and Levi branching rule in terms of these tableaux.

### 5.1 The Littelmann path model

In this subsection, we briefly recall Littelmann’s path technique, that provides a character formula for representations of an arbitrary semisimple Lie algebra  $\mathfrak{g}$  (Proposition 5.3), as well as a generalization of the Littlewood-Richardson rule and its multiple avatars, including a branching rule from  $\mathfrak{g}$  to any Levi subalgebra (Proposition 5.5).

**Definition 5.1** ([21]). Let  $\mathcal{P}$  be the set of continuous piecewise-linear paths in  $\mathfrak{h}_{(\mathbb{R})}$  starting at 0, i.e. maps  $\pi : [0, 1] \rightarrow \mathfrak{h}_{(\mathbb{R})}$  such that  $\pi(0) = 0$ , considered up to reparametrization (by any increasing homeomorphism  $[0, 1] \rightarrow [0, 1]$ ). We denote by  $\mathcal{P}^+$  the subset of  $\mathcal{P}$  formed by paths lying entirely within the Weyl chamber  $\mathfrak{h}^+$ .

For all  $\nu \in \mathfrak{h}_{(\mathbb{R})}$ , we identify  $\nu$  with the linear path

$$\nu : [0, 1] \rightarrow \mathfrak{h}_{(\mathbb{R})}, \quad t \mapsto t\nu;$$

and, given two paths  $\pi, \rho \in \mathcal{P}$ , we define the concatenated path  $\pi * \rho$  by

$$\pi * \rho : [0, 1] \rightarrow \mathfrak{h}_{(\mathbb{R})}, \quad t \mapsto \begin{cases} \pi(2t) & \text{for } t \leq \frac{1}{2}; \\ \pi(1) + \rho(2t - 1) & \text{for } t \geq \frac{1}{2}. \end{cases}$$

For every simple root  $\alpha \in \Pi$ , Littelmann introduces two functions  $e_\alpha$  and  $f_\alpha$  from  $\mathcal{P} \sqcup \{0\}$  to itself; we refer to [21, Section 1] for their definition. Here 0 denotes a special element, that can be considered as the zero of the free  $\mathbb{Z}$ -module generated by  $\mathcal{P}$ ; it is not to be confused with the constant zero path, about which we will never need to talk.

**Definition 5.2** ([21]). For every  $\pi \in \mathcal{P}$ , we define the *path model* corresponding to  $\pi$  as the smallest subset  $B_\pi \subset \mathcal{P}$  containing  $\pi$  and such that  $B_\pi \sqcup \{0\}$  is closed under all the operators  $e_\alpha$  and  $f_\alpha$ , for all simple roots  $\alpha \in \Pi$ .

**Proposition 5.3** ([21], Character Formula). *Let  $\lambda \in P \cap \mathfrak{h}^+$  be a dominant integral weight of  $\mathfrak{g}$ . Then for any starting path  $\pi^+ \in \mathcal{P}^+$  with target  $\pi^+(1) = \lambda$ , we have*

$$\text{char}(V_\lambda) = \sum_{\pi \in B_{\pi^+}} e^{\pi(1)}.$$

For every set  $\Theta \subset \Pi$  of simple roots, let us define the “ $\Theta$ -dominant Weyl chamber”

$$\mathfrak{h}^{\Theta,+} := \{X \in \mathfrak{h}_{(\mathbb{R})} \mid \forall \alpha \in \Theta, \alpha(X) \geq 0\}, \quad (5.1)$$

which is just the dominant Weyl chamber of the reductive algebra  $\mathfrak{l}(\Theta)$ . Then we have:

**Proposition 5.4** ([21], Restriction Rule). *Let  $\lambda$  and  $\pi^+$  be as before, and let  $\Theta \subset \Pi$  be some set of simple roots. Then the subset  $B_{\pi^+}^\Theta$  of  $B_{\pi^+}$  formed by paths lying entirely within  $\mathfrak{h}^{\Theta,+}$  parametrizes the restriction of  $V_\lambda(\mathfrak{g})$  to  $\mathfrak{l}(\Theta)$ :*

$$V_\lambda(\mathfrak{g})|_{\mathfrak{l}(\Theta)} = \bigoplus_{\pi \in B_{\pi^+}^\Theta} V_{\pi(1)}(\mathfrak{l}(\Theta)).$$

As a corollary, this allows us to compute the dimension of  $V_\lambda^{\mathfrak{l}(\Theta)}$ , which is just the multiplicity of the trivial representation of  $\mathfrak{l}(\Theta)$  in that decomposition:

**Corollary 5.5.** *Let  $\lambda$ ,  $\pi^+$  and  $\Theta$  be as before. Then we have*

$$\dim V_\lambda^{\mathfrak{l}(\Theta)} = \# \{ \pi \in B_{\pi^+}^\Theta \mid \pi(1) = 0 \}.$$

## 5.2 The Bruhat order

In this subsection, we give a partial characterization (essentially due to Littelmann) of the path model  $B_{\pi^+}$ , given some fairly natural assumptions on the starting path  $\pi^+$  (Proposition 5.11), satisfied in particular by all starting paths of the form

$$\pi^+ = \nu_1^+ * \cdots * \nu_N^+ \quad (5.2)$$

where each  $\nu_i^+$  is a dominant integral weight; both Littelmann's (1.2) and our (1.3) choice of a starting path for  $\mathfrak{g}$  of types  $B_r$ ,  $C_r$  and  $D_r$  follow this pattern. For such paths  $\pi^+$ , we shall then decompose this result into two subresults.

The first part (Corollary 5.12) is a characterization of the path model  $B_{\pi^+}$  in terms of the path models  $B_{\nu_i^+}$  corresponding to its segments. The second part (Corollary 5.20) will be given only later, in subsection 5.4; it consists of a description of each of these path models  $B_{\nu_i^+}$ , assuming that  $\nu_i^+$  is “small enough”.

The main tool for this characterization is the Bruhat order; let us recall its definition.

**Definition 5.6.** The *Bruhat order*  $\preceq_B$  is the partial order on  $W$  defined as the transitive closure of the relations

$$\{w \preceq s_\alpha w \mid w \in W, \alpha \in \Delta \text{ such that } \ell(w) < \ell(s_\alpha w)\},$$

where  $\ell(w)$  stands for the length of  $w$  as a word on the generators  $\{s_\alpha \mid \alpha \in \Pi\}$ .

The following classical characterization of such pairs  $(\alpha, w)$  is useful to have in mind:

**Lemma 5.7** (see e.g. Proposition 3.2.14.(4) in [7]). *For all  $w \in W$  and  $\alpha \in \Delta^+$ , we have  $\ell(s_\alpha w) > \ell(w)$  if and only if  $\alpha \in w\Delta^+$ , or equivalently if and only if*

$$\forall X \in \mathfrak{h}^+, \quad \alpha(wX) \geq 0. \quad (5.3)$$

We now use the Bruhat order to define the notion of a “Bruhat-nondecreasing” tuple of elements on  $\mathfrak{h}_{(\mathbb{R})}$ . Note however that such a tuple can not, in general, be thought of as a sequence that is nondecreasing for some partial order on  $\mathfrak{h}_{(\mathbb{R})}$ : indeed, the relationship of forming a Bruhat-nondecreasing pair is not transitive (see Example 5.9).

**Definition 5.8.** Let  $\nu_1, \dots, \nu_N \in \mathfrak{h}_{(\mathbb{R})}$  be some weights. We say that the tuple  $(\nu_1, \dots, \nu_N)$  is *Bruhat-nondecreasing* (once again, this is *not* a transitive relation, see Example 5.9) if there exist some elements  $w_i \in W$  such that for every  $i = 1, \dots, N$ , the weight  $\nu_i$  lies in the Weyl chamber  $w_i \mathfrak{h}^+$ , and satisfying  $w_1 \preceq_B \dots \preceq_B w_N$ .

We say that a path  $\pi$  is *Bruhat-nonincreasing* if the segments  $(\nu_N, \dots, \nu_1)$  of its subdivision  $\pi = \nu_1 * \cdots * \nu_N$  into linear segments form a Bruhat-nondecreasing tuple.

(By convention, the order of paths is reversed compared to tuples: see Definition 5.24.(i).)

**Example 5.9.** For  $\mathfrak{g} = \mathfrak{so}_6(\mathbb{C})$ , take  $\nu_1 = \frac{1}{2}(-e_1 - e_2 + e_3)$ ,  $\nu_2 = \frac{1}{2}(e_1 - e_2 + e_3)$  and  $\nu_3 = e_1$ . Then the pair  $(\nu_1, \nu_2)$  is Bruhat-nondecreasing (take for example  $w_1 = w_2 = w$  with  $w : (e_1, e_2, e_3) \mapsto (e_3, -e_2, -e_1)$ ), and so is the pair  $(\nu_2, \nu_3)$  (take for example  $w_2 = w_3 = w'$  with  $w' : (e_1, e_2, e_3) \mapsto (e_1, e_3, e_2)$ ). But the pair  $(\nu_1, \nu_3)$ , and *a fortiori* the triple  $(\nu_1, \nu_2, \nu_3)$ , is *not* Bruhat-nondecreasing (we will be easily able to check this once we obtain Proposition 5.18).

**Definition 5.10.** Let  $\pi$  be a path,  $\pi = \nu_1 * \dots * \nu_N$  its subdivision into linear segments. We define the *multishape* of  $\pi$  to be the path  $\pi^+ = \nu_1^+ * \dots * \nu_N^+$ , where, for each  $i$ ,  $\nu_i^+$  is the unique dominant element of the Weyl orbit of  $\nu_i$ , i.e.  $\{\nu_i^+\} := W\nu_i \cap \mathfrak{h}^+$ .

Here is now the announced result.

**Proposition 5.11.** *Let  $\nu_1^+, \dots, \nu_N^+$  be some dominant weights such that the path  $\pi^+ = \nu_1^+ * \dots * \nu_N^+$  is a locally integral concatenation (see [22, Definition 5.3]). Then a path  $\pi$  lies in  $B_{\pi^+}$  if and only if it is a locally integral concatenation and has multishape  $\pi^+$ .*

*Proof.* By Proposition 5.9 in [22], local integrality is preserved by the root operators. So is (obviously) the multishape. Denoting by  $\hat{B}_{\pi^+}$  the set of locally integral concatenations with multishape  $\pi^+$ , it then follows from Lemma 6.11 in [22] that

$$\hat{B}_{\pi^+} = \bigcup_{\pi \in \hat{B}_{\pi^+} \cap \mathcal{P}^+} B_\pi.$$

Moreover, one can check that the only path with multishape  $\pi^+$  that is a locally integral concatenation and lies entirely within  $\mathfrak{h}^+$  is  $\pi^+$  itself.  $\square$

Here is the announced interpretation of this result as a “divide-and-conquer” strategy.

**Corollary 5.12.** *Let  $\nu_1^+, \dots, \nu_N^+ \in P \cap \mathfrak{h}^+$  be some dominant integral weights, and let  $\pi^+ = \nu_1^+ * \dots * \nu_N^+$ . Then a path  $\pi$  lies in  $B_{\pi^+}$  if and only if it is Bruhat-nonincreasing and of the form  $\pi = \pi_1 * \dots * \pi_N$ , with  $\pi_k \in B_{\nu_k^+}$  for each  $k$ .*

*Proof.* The “only if” part is an immediate consequence of the previous proposition (Proposition 5.11) and of the combination of Lemma 6.12 and Theorem 6.13 from [22].

For the “if” part, we only need to remark that if each of the paths  $\pi_1, \dots, \pi_N$  is a “weakly locally integral concatenation” (i.e. satisfies all of the conditions from Definition 5.3 in [22], except possibly Bruhat-nonincreasingness) and ends at an integral weight, then their concatenation is automatically weakly locally integral.  $\square$

### 5.3 The Young order

For the remainder of the paper, we assume that  $\mathfrak{g}$  is either of type  $B_r$  for some  $r \geq 1$ , or of type  $C_r$  for some  $r \geq 1$ , or of type  $D_r$  for some  $r \geq 3$ . In this setting, we have  $n = r$ , so we drop the notation  $n$ .

We use Littelmann paths whose segments lie (up to occasional  $\frac{1}{2}$  factors) in the set

$$\mathcal{X} := \{-1, 0, 1\}^r \setminus \{0\} \tag{5.4}$$

of vectors with integer coordinates (in the basis  $(e_1, \dots, e_r)$ ) that have  $\|\cdot\|_\infty$ -norm 1. We will encode these vectors as “strongly-standard” columns (i.e. Young tableaux of width 1) on a certain alphabet: this is the object of Definitions 5.13, 5.14 and 5.15.

We then introduce (Definition 5.16) a “Young order” on the set  $\mathcal{C}$  of such columns, with an additional parity condition when  $\mathfrak{g}$  is of type  $D_r$ . This leads us to the central result of this subsection: Proposition 5.18, which says that a sequence of elements of  $\mathcal{X}$  whose Weyl orbits are ordered in some natural way is Bruhat-nondecreasing if and only if it is nondecreasing for the Young order (or Young order with parity). Thus this assumption on the ordering of Weyl orbits gets rid of the nontransitivity issues outlined in Example 5.9.

**Definition 5.13** (The alphabet). We introduce the alphabet  $\mathcal{A}_r := \{1, \dots, r, \bar{r}, \dots, \bar{1}\}$ ; we also set  $\mathcal{A} := \bigcup_{r \in \mathbb{N}} \mathcal{A}_r = \mathbb{N} \cup \bar{\mathbb{N}}$ . We adopt the convention  $\bar{\bar{s}} = s$ , and we define an *absolute value*  $|\cdot| : \mathcal{A} \rightarrow \mathbb{N}$  and a *sign*  $\text{sgn} : \mathcal{A} \rightarrow \{\pm 1\}$  by identifying  $\bar{s}$  with  $-s$ .

We introduce on  $\mathcal{A}_r$  two very similar orders: the total order  $\preceq_{\mathcal{A}}$  given by

$$1 \prec \dots \prec r \prec \bar{r} \prec \dots \prec \bar{1}, \quad (5.5)$$

and the not quite total order  $\preceq'_{\mathcal{A}}$  given by

$$1 \prec' \dots \prec' r-1 \prec' r, \bar{r} \prec' \overline{r-1} \prec' \dots \prec' \bar{1}. \quad (5.6)$$

The order that we shall use will depend on  $\mathfrak{g}$ : more precisely, we set

$$\preceq_{\mathcal{A}}^{\mathfrak{g}} := \begin{cases} \preceq_{\mathcal{A}} & \text{if } \mathfrak{g} \text{ is of type } B_r \text{ or } C_r; \\ \preceq'_{\mathcal{A}} & \text{if } \mathfrak{g} \text{ is of type } D_r. \end{cases}$$

However the total order  $\preceq_{\mathcal{A}}$  will occasionally be useful even in type  $D_r$  (see Remark 5.17).

**Definition 5.14** (Strongly standard columns). A column  $\mathcal{C}$  filled with this alphabet is *strongly standard* if, for each  $s$ , it contains at most one of the symbols  $s$  and  $\bar{s}$ , and the symbols read from top to bottom form a strictly increasing sequence for the order  $\prec_{\mathcal{A}}$  (or equivalently for the order  $\prec'_{\mathcal{A}}$ ). We denote by  $\mathcal{C}$  the set of all strongly standard columns.

**Definition 5.15** (Identification of columns and weights). We define the *weight* of a strongly standard column  $\mathcal{C}$  to be the vector

$$\nu(\mathcal{C}) := \sum_{i=1}^{\#\mathcal{C}} \nu \left( \begin{smallmatrix} \mathcal{C} \\ i \end{smallmatrix} \right), \quad (5.7)$$

with the function  $\nu$  defined on  $\mathcal{A}_r$  by

$$\forall s = 1, \dots, r, \quad \begin{cases} \nu(s) := e_s; \\ \nu(\bar{s}) := -e_s. \end{cases} \quad (5.8)$$

We shall henceforth identify  $\mathcal{C}$  with  $\mathcal{X}$ , using the bijection given by  $\nu$ .

**Definition 5.16** (Order on strongly standard columns). We endow the set  $\mathcal{C}$  (and, using the identification  $\nu$ , also the set  $\mathcal{X}$ ) with an order  $\preceq_Y^{\mathfrak{g}}$ , that once again depends on  $\mathfrak{g}$ :

$$\preceq_Y^{\mathfrak{g}} := \begin{cases} \preceq_Y & \text{if } \mathfrak{g} \text{ is of type } B_r \text{ or } C_r, \\ \preceq'_Y & \text{if } \mathfrak{g} \text{ is of type } D_r; \end{cases}$$

it remains to explain what  $\preceq_Y$  and  $\preceq'_Y$  are.



1. We define the *Young order*  $\preceq_Y$  by saying that  $\mathcal{C} \preceq_Y \mathcal{C}'$  if the two columns set next to each other form a semistandard Young tableau for the order  $\preceq_A$ . Formally:

$$\mathcal{C} \preceq_Y \mathcal{C}' : \iff \begin{cases} \#\mathcal{C} \geq \#\mathcal{C}'; \\ \forall i = 1, \dots, \#\mathcal{C}', \quad {}_i\mathcal{C} \preceq_A {}_i\mathcal{C}'. \end{cases} \quad (5.9)$$

2. We define the *Young order with parity*  $\preceq'_Y$  by saying that  $\mathcal{C} \preceq'_Y \mathcal{C}'$  if and only if the two columns set next to each other form a semistandard Young tableau for the order  $\preceq'_A$ , and this tableau satisfies the following additional condition:

$$\begin{aligned} & \text{if } \exists i_0, k \text{ with } 1 \leq i_0 \leq i_0 + k - 1 \leq \#\mathcal{C}' \text{ such that} \\ & \left\{ \left| {}_{i_0}\mathcal{C} \right|, \dots, \left| {}_{i_0+k-1}\mathcal{C} \right| \right\} = \left\{ \left| {}_{i_0}\mathcal{C}' \right|, \dots, \left| {}_{i_0+k-1}\mathcal{C}' \right| \right\} = \{r - k + 1, \dots, r\}, \quad (5.10) \\ & \text{then } \#[\mathbb{N}]\mathcal{C} \equiv \#[\mathbb{N}]\mathcal{C}' \pmod{2}. \end{aligned}$$

One easily checks that this relation is transitive.

*Remark 5.17.* Note that the case  $k = 1$  of the condition (5.10) tells us that in a tableau whose columns form a  $\preceq'_Y$ -nondecreasing sequence,  $r$  and  $\bar{r}$  can never occur next to each other. So such a tableau will in particular be semistandard, not only for  $\preceq'_A$ , but also for  $\preceq_A$ ; and, for that matter, also for the total order  $\preceq''_A$  in which  $r$  and  $\bar{r}$  are swapped:

$$1 \prec'' \dots \prec'' r - 1 \prec'' \bar{r} \prec'' r \prec'' \overline{r - 1} \prec'' \dots \prec'' \bar{1}. \quad (5.11)$$

Finally, as announced, we explain how the order  $\preceq_{\mathfrak{g}}$  is related to the Bruhat order. The remainder of this subsection is dedicated to proving the following proposition.

**Proposition 5.18.** *Let  $\mathcal{C}_1, \dots, \mathcal{C}_N \in \mathcal{C}$ , and let  $\nu_i := \nu(\mathcal{C}_i)$  be the corresponding weights.*

- (i) *For  $\mathfrak{g}$  of type  $B_r$  or  $C_r$ , the sequence  $(\mathcal{C}_1, \dots, \mathcal{C}_N)$  is Young-nondecreasing if and only if  $(\nu_1, \dots, \nu_N)$  is Bruhat-nondecreasing and  $\|\nu_1\|^2 \geq \dots \geq \|\nu_N\|^2$ .*
- (ii) *For  $\mathfrak{g}$  of type  $D_r$ , the sequence  $(\mathcal{C}_1, \dots, \mathcal{C}_N)$  is Young-nondecreasing with parity if and only if  $(\nu_1, \dots, \nu_N)$  is Bruhat-nondecreasing,  $\|\nu_1\|^2 \geq \dots \geq \|\nu_N\|^2$  and all  $\nu_i$  such that  $\|\nu_i\|^2 = r$  lie in the same  $W$ -orbit.*

In order to prove this proposition, we need some preliminary work. Recall that a partially ordered set  $(X, \preceq)$  can be characterized by its *Hasse diagram*, i.e. the oriented graph where two vertices  $x, y \in X$  are connected by an edge if and only if  $y$  “covers”  $x$ , i.e.  $x \preceq y$  and  $\{z \mid x \preceq z \preceq y\} = \{x, y\}$ . We then have the following description of the Hasse diagram of the order  $\preceq_Y^{\mathfrak{g}}$ , for both  $\mathfrak{g} = B_r$  or  $C_r$  and  $\mathfrak{g} = D_r$ .

**Lemma 5.19.** *Let  $\mathcal{C}, \mathcal{C}'$  be two strongly standard columns. Then the pair  $(\mathcal{C}, \mathcal{C}')$  is an edge of the Hasse diagram for  $\preceq_Y^{\mathfrak{g}}$  if and only if it has one of the following forms:*

- *$\mathcal{C}$  and  $\mathcal{C}'$  have the same height, and  $\mathcal{C}'$  is obtained from  $\mathcal{C}$  by a single substitution  $s \rightarrow t$ , with  $s \prec_A^{\mathfrak{g}} t$  and such that whenever  $s \prec_A^{\mathfrak{g}} x \prec_A^{\mathfrak{g}} t$ , the value  $\bar{x}$  is contained in some box of  $\mathcal{C}$  (and of  $\mathcal{C}'$ ).*

- $\mathcal{C}$  and  $\mathcal{C}'$  have the same height, and  $\mathcal{C}'$  is obtained from  $\mathcal{C}$  by exactly two substitutions  $s \rightarrow t$  and  $\bar{t} \rightarrow \bar{s}$ , where  $(s, t)$  is an edge of the Hasse diagram for the order  $\preceq_{\mathcal{A}}^{\mathfrak{g}}$ .
- $\mathcal{C}'$  is obtained from  $\mathcal{C}$  by deleting the last box, whose contents  $s$  is a maximal element (for the order  $\preceq_{\mathcal{A}}^{\mathfrak{g}}$ ) among the symbols that do not occur in  $\mathcal{C}'$ .

We omit the proof, a somewhat tedious but elementary exercise in combinatorics. Before proving Proposition 5.18, we also need to describe the set  $\mathcal{X}^+ := \mathcal{X} \cap \mathfrak{h}^+$ :

$$\mathcal{X}^+ = \begin{cases} \{c_1, \dots, c_r\} & \text{if } \mathfrak{g} \text{ is of type } B_r \text{ or } C_r; \\ \{c_1, \dots, c_{r-1}, c_r^+, c_r^-\} & \text{if } \mathfrak{g} \text{ is of type } D_r, \end{cases} \quad (5.13)$$

where, for all  $k = 1, \dots, r$ , we set

$$c_k := \sum_{i=1}^k e_i; \quad c_r^+ := c_r; \quad c_r^- := c_{r-1} - e_r, \quad (5.14)$$

Also observe that the restriction of the order  $\preceq_Y^{\mathfrak{g}}$  to this set is given by:

$$\forall v, v' \in \mathcal{X}^+, \quad v \preceq_Y^{\mathfrak{g}} v' \quad : \iff \quad v = v' \text{ or } \|v\|^2 > \|v'\|^2. \quad (5.15)$$

*Proof of Proposition 5.18.* • The “if” part is now equivalent to proving that the application map

$$\pi : W \times \mathcal{X}^+ \rightarrow \mathcal{X}, \quad (w, v) \mapsto wv$$

is order-preserving, where  $W \times \mathcal{X}^+$  is endowed with the product order  $\preceq_B \times \preceq_Y^{\mathfrak{g}}$  and  $\mathcal{X}$  is endowed with the order  $\preceq_Y^{\mathfrak{g}}$ . This is straightforward (using Lemma 5.7).

- Conversely, assume now that the sequence  $(\nu_1, \dots, \nu_N)$  is nondecreasing for the order  $\preceq_Y^{\mathfrak{g}}$ . Then clearly the Young ordering ensures that the heights of the columns  $\|\nu_i\|^2 = \#\mathcal{C}_i$  form a nonincreasing sequence, and (in type  $D_r$ ) the parity condition ensures that all the columns of height  $r$  lie in the same Weyl orbit. It remains to prove that the sequence  $(\nu_1, \dots, \nu_N)$  is Bruhat-nondecreasing.

This can be proved by exhibiting a section  $\xi : \mathcal{X} \rightarrow W$ , i.e. a map such that every vector  $\nu \in \mathcal{X}$  lies in the Weyl chamber  $\xi(\nu)\mathfrak{h}^+$ , which is also order-preserving.

We construct  $\xi$  as follows. Let  $\nu \in \mathcal{X}$ , and let  $\mathcal{C}$  be the corresponding strongly standard column; let  $k = \#\mathcal{C} = \|\nu\|^2$ . We define  $\xi(\nu)$  as the unique element of  $W$  whose action on  $\{\pm e_1, \dots, \pm e_r\}$ , that we identify with  $\mathcal{A}_r$  as usual, satisfies:

- for all  $i \leq k$  (except possibly  $i = r$  if  $\mathfrak{g}$  is of type  $D_r$ ), we have  $\xi(\nu) \cdot i = {}_i\mathcal{C}$ ;
- $|\xi(\nu) \cdot (k+1)| < |\xi(\nu) \cdot (k+2)| < \dots < |\xi(\nu) \cdot r|$ ;
- for all  $i > k$  (except possibly  $i = r$  if  $\mathfrak{g}$  is of type  $D_r$ ), we have  $\text{sgn}(\xi(\nu) \cdot i) = -1$ .

We easily verify that  $\xi$  is indeed a section and (using Lemmas 5.7 and 5.19) that it is order-preserving.  $\square$

## 5.4 Admissible pairs

We now give a reformulation of Proposition 5.11 when applied to “short” starting paths: this is Corollary 5.20 (in its notations, “short” means that  $k$  is small). We then further specify it to starting paths that lie in the set  $\mathcal{X}^+$  introduced (5.13) in the previous section.

**Corollary 5.20.** *Let  $\nu^+ \in P \cap \mathfrak{h}^+$  be a dominant integral weight, and consider the integer  $k := \max_{\alpha \in \Delta} |\langle \nu^+, \alpha^\vee \rangle|$  (recall the notation  $\alpha^\vee := \frac{2\alpha}{\|\alpha\|}$ ). Then:*

- (i) *If  $k = 1$  (i.e.  $\nu^+$  is minuscule), then  $B_{\nu^+}$  is just the  $W$ -orbit of  $\nu^+$ .*
- (ii) *If  $k \leq 2$ , then  $B_{\nu^+}$  is the set of paths  $\pi$  of the form  $\pi = (\frac{1}{2}\nu_1) * (\frac{1}{2}\nu_2)$  with  $\nu_1, \nu_2$  two elements of the  $W$ -orbit of  $\nu^+$  that form an admissible pair, in the sense of [20] (definition given in Remark 3.4, and originally due to [18, Definition 2.4]).*

In order to characterize the path model for these starting paths, it remains to give a combinatorial characterization of admissible pairs (in terms of strongly standard columns). This is done in the following proposition, whose proof is the main goal of this subsection.

**Proposition 5.21.** *Let  $\mathcal{C}$  and  $\mathcal{C}'$  be two strongly standard columns. Let  $0 \leq a_1 < b_1 < \dots < a_k < b_k \leq r$  be integers such that*

$$\left\{ \left| {}_i\mathcal{C} \right| \mid i = 1, \dots, \#\mathcal{C} \right\} = \bigcup_{i=1}^k \{a_i + 1, a_i + 2, \dots, b_i\}; \quad (5.16)$$

*and, for each  $i$ , let  $x_i$  denote the number of symbols in  $\mathcal{C}$  with sign  $+1$  and with absolute value in  $\{a_i + 1, \dots, b_i\}$ . Define similarly  $a'_1 < b'_1 < \dots < a'_{k'} < b'_{k'}$  and  $x'_i$  for  $\mathcal{C}'$ .*

*Then the pair  $(\nu(\mathcal{C}), \nu(\mathcal{C}'))$  is admissible if and only if all of the following hold:*

- (A1)  $\mathcal{C} \succeq_Y^{\mathfrak{g}} \mathcal{C}'$ ;
- (A2)  $\#\mathcal{C} = \#\mathcal{C}'$ ;
- (A3)  $k = k'$  and, for all  $i = 1, \dots, k$ ,  $a_i = a'_i$  and  $b_i = b'_i$ ;
- (A4) for all  $i = 1, \dots, k$ , the integers  $x_i$  and  $x'_i$  satisfy the following condition:

$$\begin{cases} \text{no restriction} & \text{if } b_i = r \text{ and } \mathfrak{g} \text{ is of type } B_r; \\ x_i \equiv x'_i \pmod{2} & \text{if } b_i = r \text{ and } \mathfrak{g} \text{ is of type } D_r; \\ x_i = x'_i & \text{otherwise.} \end{cases}$$

**Remark 5.22.** One easily checks that, in types  $B_r$  and  $D_r$ , any pair of columns of height  $r$  that satisfies (A1) is admissible; whereas in type  $C_r$ , the only admissible pairs of columns of height  $r$  are of the form  $(\mathcal{C}, \mathcal{C})$ .

*Proof.* Unpacking the definition (and taking into account Lemma 5.7), we see that  $\nu(\mathcal{C})$  and  $\nu(\mathcal{C}')$  form an admissible pair if and only if one can pass from  $\mathcal{C}'$  to  $\mathcal{C}$  by a series of steps of the form  $s_\alpha$  for some  $\alpha \in \Pi$ , where:

- An operation of the form  $s_{e_i - e_{i+1}}$  (for  $1 \leq i \leq r-1$ ) is allowed only if both symbols  $i$  and  $\overline{i+1}$  occur somewhere; it then replaces them by  $i+1$  and  $\overline{i}$  respectively.
- The operation  $s_{e_{r-1} + e_r}$  is allowed only if both of the symbols  $r-1$  and  $r$  occur somewhere; it then replaces them by  $\overline{r}$  and  $\overline{r-1}$  respectively.
- The operation  $s_{e_r}$  is allowed only if the symbol  $r$  occurs; it then replaces it by  $\overline{r}$ .
- The operation  $s_{2e_r}$  is never allowed.

Clearly each of these operations satisfies the conditions (A1) through (A4), which are transitive; this proves the “only if” part.

Conversely, let a pair  $(\mathcal{C}, \mathcal{C}')$  satisfy these four conditions. Find a path going from  $\mathcal{C}'$  to  $\mathcal{C}$  in the Hasse diagram of the order  $\preceq_Y^{\mathfrak{g}}$ . Using Lemma 5.19 we can then easily check that each step of this path is an admissible operation (as described above).  $\square$

## 5.5 Doubled Young tableaux

We are now ready to define a  $\mathfrak{g}$ -standard doubled Young tableau, and to show (Proposition 5.25) that these tableaux describe the path model with starting path  $\pi_0^+(\lambda)$  as given in (1.3). This yields the announced character formula (Proposition 5.27). We also give a combinatorial characterization (Corollary 5.28) of representations  $V_\lambda$  satisfying  $V_\lambda^{(\Theta)} \neq 0$  for any Levi subalgebra  $\mathfrak{l}(\Theta)$  (where  $\Theta \subset \Pi$ ), accompanied by a slightly modified version of this result (Corollary 5.30) that exploits the outer automorphism of  $D_r$ , and will save us some work in the next section.

**Definition 5.23.** A  $\mathfrak{g}$ -standard doubled Young tableau is a Young tableau  $\mathcal{T}$  on the alphabet  $\mathcal{A}_r$  with the following properties:

- (H1) All columns of  $\mathcal{T}$  are strongly standard.
- (H2) The sequence formed by the columns of  $\mathcal{T}$  is Young-nondecreasing if  $\mathfrak{g}$  is of type  $B_r$  or  $C_r$ , Young-nondecreasing with parity if  $\mathfrak{g}$  is of type  $D_r$ :

$$\forall j = 2, \dots, \#_1 \mathcal{T}, \quad {}^{j-1} \mathcal{T} \preceq_Y^{\mathfrak{g}} {}^j \mathcal{T}.$$

- (H3) For all  $j$  with  $1 < j \leq \#_1 \mathcal{T}$  and  $\#_1 \mathcal{T} - j$  even, the pair  $({}^j \mathcal{T}, {}^{j-1} \mathcal{T})$  is admissible.

In order to verify (H3), in practice, it suffices to check that every such pair satisfies conditions (A3) (which implies (A2)) and (A4) from Proposition 5.21, since condition (A1) is already covered by (H2).

**Definition 5.24** (Passing from doubled diagrams and tableaux to weights, in types  $B_r$ ,  $C_r$  and  $D_r$ ). Given a  $\mathfrak{g}$ -standard doubled Young tableau  $\mathcal{T}$ , we define:

- (i) the *corresponding path*  $\pi(\mathcal{T}) := \left(\frac{1}{2}\nu({}^N \mathcal{T})\right) * \dots * \left(\frac{1}{2}\nu({}^1 \mathcal{T})\right)$ , where  $N = \#_1 \mathcal{T}$ .

(ii) the *total weight* of  $\mathcal{T}$  as

$$\nu(\mathcal{T}) := \pi(\mathcal{T})(1) = \frac{1}{2} \sum_{j=1}^{\#_1 \mathcal{T}} \nu(j\mathcal{T}) = \frac{1}{2} \sum_{i,j} \nu_i^{(j)}(\mathcal{T}). \quad (5.17)$$

Note that, in comparison with Definition 4.4.(ii), there is an extra factor  $\frac{1}{2}$ : in fact, it is reasonable to think of doubled Young tableaux as having columns “of width  $\frac{1}{2}$ ”.

We say that  $\mathcal{T}$  is *null* if  $\nu(\mathcal{T}) = 0$ .

(iii) the *sign*  $\epsilon$  of  $\mathcal{T}$  as follows:

- If  $\mathfrak{g}$  is of type  $B_r$  or  $C_r$ , we adopt the convention that  $\epsilon$  is always equal to  $+1$ .
  - If  $\mathfrak{g}$  is of type  $D_r$  and  $\#_r \mathcal{T} > 0$ , we take  $\epsilon = (-1)^x$ , where  $x$  is the number of symbols with bars in any column of height  $r$ . The parity condition (5.10) ensures that  $\epsilon$  does not depend on the choice of the column.
  - If  $\mathfrak{g}$  is of type  $D_r$  but  $\#_r \mathcal{T} = 0$ , we adopt the convention that  $\epsilon = 0$ .
- (iv) for  $\alpha \in \Pi$  or  $\Theta \subset \Pi$ , we define  $\alpha$ -(co)dominance and  $\Theta$ -(co)dominance for doubled Young tableaux in the same way as for ordinary Young tableaux (see Definition 4.4.(iii)). Clearly a doubled Young tableau  $\mathcal{T}$  is  $\Theta$ -dominant if and only if the path  $\pi(\mathcal{T})$  lies within the  $\Theta$ -dominant Weyl chamber  $\mathfrak{h}^{\Theta,+}$ , as defined in (5.1).

Finally, we introduce a correspondence  $\Psi$  between the set  $P \cap \mathfrak{h}^+$  of dominant integral weights  $\lambda$  and the set  $\mathcal{M}^{(r)}$  of Young diagrams of height  $r$ . It is given by the formula

$$(\#_1 \Psi(\lambda), \dots, \#_r \Psi(\lambda)) := (2\lambda_1, \dots, 2\lambda_{r-1}, 2|\lambda_r|), \quad (5.18)$$

where, as usual, we decompose  $\lambda = \sum_{i=1}^r \lambda_i e_i$ .

**Proposition 5.25.** *Let  $\lambda \in P \cap \mathfrak{h}^+$  be a dominant integral weight of  $\mathfrak{g}$ . Then the set of paths  $\pi(\mathcal{T})$ , where  $\mathcal{T}$  runs over all  $\mathfrak{g}$ -standard doubled Young tableaux of shape  $\Psi(\lambda)$  with the same sign as  $\lambda_r$ , is equal to the path model  $B_{\pi_0^+(\lambda)}^+$  (with  $\pi_0^+(\lambda)$  defined in (1.3)).*

*Proof.* Let  $\mathcal{T}$  be any doubled Young tableau,  $\pi(\mathcal{T})$  the corresponding path. Our goal is to apply Corollary 5.12.

Note that it is straightforward to check that  $\mathcal{T}$  has shape  $\Psi(\lambda)$  and sign  $\text{sgn}(\lambda_r)$  if and only if  $\pi(\mathcal{T})$  has multishape  $\pi_0^+(\lambda)$ ; and, assuming that this is the case, it follows from Proposition 5.18 that the columns of  $\mathcal{T}$  form a nondecreasing sequence for the order  $\preceq_Y^{\mathfrak{g}}$  if and only if the path  $\pi(\mathcal{T})$  is Bruhat-nonincreasing.

On the other hand, from the integrality of  $\lambda$ , it follows that all the coefficients in the decomposition (1.3) are integer, except possibly  $|\lambda_r|$  which can be half-integer when  $\mathfrak{g}$  is of type  $B_r$  or  $D_r$ . So let us decompose  $\pi_0^+(\lambda)$  into a concatenation that first involves  $\lfloor \lambda_1 \rfloor$  segments that lie in  $\mathcal{X}^+ = \mathcal{X} \cap \mathfrak{h}^+$  (recall (5.13)) and then possibly ends with a segment equal to  $\frac{1}{2}c_r^\pm$ ; and then apply Corollary 5.12 to this decomposition.

We now conclude by Corollary 5.20. Indeed, all weights  $\nu \in \mathcal{X}^+$  satisfy  $\max_{\alpha \in \Delta} |\langle \nu, \alpha^\vee \rangle| \leq 2$ , while the weights  $\frac{1}{2}c_r$  (if  $\mathfrak{g}$  is of type  $B_r$ ) and  $\frac{1}{2}c_r^\pm$  (if  $\mathfrak{g}$  is of type  $D_r$ ) are minuscule.  $\square$

*Remark 5.26.* One can easily see that *every*  $\mathfrak{g}$ -standard doubled Young tableau  $\mathcal{T}$  is in fact of shape  $\Psi(\lambda)$  for some integral weight  $\lambda \in P \cap \mathfrak{h}^+$ . In particular its weight  $\nu(\mathcal{T})$  is then a weight of the representation  $V_\lambda$ , hence  $\lambda - \nu(\mathcal{T})$  lies in the root lattice  $Q$ .

If the tableau  $\mathcal{T}$  is null (which will always be the case in the sequel), this condition reduces to  $\lambda \in Q$ ; and one can check that in this case all the  $\lambda_i$  are integer. Hence all the row lengths of  $\mathcal{T}$  are even; and in condition (H3), the admissible pairs form a partition of all columns of  $\mathcal{T}$ , without any unpaired column.

By combining this with the Littelmann character formula (Proposition 5.3), we obtain the following, purely combinatorial character formula.

**Proposition 5.27** (Character formula with  $B_r$ ,  $C_r$ ,  $D_r$ -standard doubled Young tableaux). *Let  $\lambda \in P \cap \mathfrak{h}^+$  be a dominant integral weight of  $\mathfrak{g}$ . Then*

$$\text{char}(V_\lambda) = \sum_{\mathcal{T}} e^{\nu(\mathcal{T})},$$

where  $\mathcal{T}$  runs over all  $\mathfrak{g}$ -standard doubled Young tableaux of shape  $\Psi(\lambda)$  that have the same sign as  $\lambda_r$ .

We can also combine this with Corollary 5.5 to obtain a purely combinatorial characterization of representations having  $\mathfrak{l}$ -invariant vectors:

**Corollary 5.28.** *Let  $\lambda \in P \cap \mathfrak{h}^+$  and  $\Theta \subset \Pi$ . Then  $V_\lambda^{(\Theta)} \neq 0$  if and only if there exists a doubled Young tableau  $\mathcal{T}$  satisfying the following seven conditions:*

(H1) through (H3) as in Definition 5.23, i.e. the tableau  $\mathcal{T}$  is  $\mathfrak{g}$ -standard.

(H4) The tableau is null:  $\nu(\mathcal{T}) = 0$ .

(H5) The tableau has shape  $\Psi(\lambda) = (2\lambda_1, \dots, 2\lambda_{r-1}, 2|\lambda_r|)$ .

(H6) The tableau has the same sign as  $\lambda_r$  (if  $\mathfrak{g}$  is of type  $D_r$ ).

(H7) The tableau is  $\Theta$ -codominant (or equivalently  $\Theta$ -dominant, since it is null).

We end this section with an additional small simplification: we can in fact get rid of condition (H6), at the expense of replacing  $\Theta$  by a slightly larger set. Indeed, when studying the properties of such tableaux, we will simply make no use of this property. When trying to construct such tableaux, we will avoid having to check this condition by way of the following (obvious) remark. Let  $\sigma$  denote the outer automorphism of  $D_r$ : it acts on  $\mathfrak{h}$  by  $\sigma(e_i) = e_i$  for  $i < r$  and  $\sigma(e_r) = -e_r$ , hence on  $\mathcal{A}_r$  by exchanging  $r$  and  $\bar{r}$ .

*Remark 5.29.* The set of  $D_r$ -standard doubled Young tableaux is invariant by  $\sigma$ . If  $\mathcal{T}$  is such a tableau, then  $\sigma(\mathcal{T})$  has the same shape, but opposite sign compared to  $\mathcal{T}$ ; and  $\sigma(\mathcal{T})$  is null or  $\alpha$ -(co)dominant (for any  $\alpha \in \Pi$ ) if and only if  $\mathcal{T}$  is so.

**Corollary 5.30.** *Let  $\lambda \in P \cap \mathfrak{h}^+$  and  $\Theta \subset \Pi$ . Suppose that there exists a doubled Young tableau  $\mathcal{T}$  satisfying the conditions:*

(H1) through (H5) as in Corollary 5.28.

(H7') The tableau  $\mathcal{T}$  is  $(\Theta \cup \sigma(\Theta))$ -codominant (say that  $\sigma = \text{Id}$  if  $\mathfrak{g}$  is of type  $B_r$  or  $C_r$ ).

Then we have both  $V_\lambda^{l(\Theta)} \neq 0$  and  $V_{\sigma\lambda}^{l(\Theta)} \neq 0$ .

## 6 Types $B_r$ , $C_r$ and $D_r$ : the proof

In this section, we prove the Main Theorem, i.e. the equality  $\mathcal{M}_{\text{t-inv}} = \mathcal{M}_{\text{Table}}$ , for  $\mathfrak{g}$  of types  $B_r$  ( $r \geq 1$ ),  $C_r$  ( $r \geq 1$ ) and  $D_r$  ( $r \geq 3$ ).

### 6.1 The inclusion $\mathcal{M}_{\text{t-inv}} \subset \mathcal{M}_{\text{Table}}$

Let  $\mathfrak{g}_{\mathbb{R}}$  be some real form of  $\mathfrak{g} = B_r, C_r$  or  $D_r$ , and let  $\lambda$  be a dominant integral weight such that the doubled Young diagram  $\Psi(\lambda)$  admits a  $\Theta(\mathfrak{g}_{\mathbb{R}})$ -codominant null  $\mathfrak{g}$ -standard filling. It then suffices to prove that  $\lambda$  satisfies the corresponding condition from Table 1.

From [26], Reference Chapter, Table 9, we obtain the values of the sets  $\Theta(\mathfrak{g}_{\mathbb{R}})$  for all such real forms  $\mathfrak{g}_{\mathbb{R}}$ ; for quicker reference, we have reproduced them here in Table 3.

The bulk of this subsection is devoted to proving a few inequalities (Proposition 6.1 and Corollary 6.2) satisfied by  $\Pi_{[x+1,r]}$ -codominant null  $\mathfrak{g}$ -standard doubled Young tableaux. At the end, we put the pieces together. The other simple roots contained in  $\Theta(\mathfrak{g}_{\mathbb{R}})$  will not matter (except for two low-rank cases, that we treat by using exceptional isomorphisms).

Table 3: Values of  $\Theta(\mathfrak{g}_{\mathbb{R}})$  for all real forms of simple Lie algebras of types  $B_r$ ,  $C_r$  and  $D_r$ . The notations  $\Pi_{[x,y]}$  and  $\Pi_{\text{odd}}$  are defined in (4.7) and (4.8).

$\mathfrak{g}$	$\mathfrak{g}_{\mathbb{R}}$	Parameter range	$\Theta(\mathfrak{g}_{\mathbb{R}})$
$B_r$ $r \geq 1$	$\mathfrak{so}(p, 2r + 1 - p)$	$0 \leq p \leq r$	$\Pi_{[p+1,r]}$
$C_r$ $r \geq 1$	$\mathfrak{sp}_{2 \cdot}(p, r - p)$ $\mathfrak{sp}_{2 \cdot r}(\mathbb{R})$	$0 \leq p \leq \frac{r}{2}$	$\Pi_{\text{odd}} \cup \Pi_{[2p+1,r]}$ $\emptyset$
$D_r$ $r \geq 3$	$\mathfrak{so}(p, 2r - p)$ $\mathfrak{so}(r - 1, r + 1)$ $\mathfrak{so}^*(2r)$	$\begin{cases} 0 \leq p \leq r \\ p \neq r - 1 \end{cases}$	$\Pi_{[p+1,r]}$ $\emptyset$ $\Pi_{\text{odd}} \setminus \{\alpha_r\}$

**Proposition 6.1.** Suppose that  $\mathfrak{g}$  is of type  $B_r$ ,  $C_r$  or  $D_r$ ; let  $\mathcal{T}$  be any  $\mathfrak{g}$ -standard doubled Young tableau. Let  $h := \#^1 \mathcal{T}$  be the height of  $\mathcal{T}$ , and let  $t := \max_{i,j} \left| \begin{smallmatrix} j \\ i \end{smallmatrix} \mathcal{T} \right|$  be the largest number such that either  $t$  or  $\bar{t}$  appears somewhere in  $\mathcal{T}$ ; these numbers satisfy

$$h \leq t. \quad (6.1)$$

Moreover, for every integer  $x$  satisfying  $0 \leq x \leq r$ , we have the following inequalities.

(i) If  $\mathcal{T}$  is  $\Pi_{[x+1,r]}$ -codominant and null, then we have:

$$h \geq 2(t - x). \quad (6.2)$$

(ii) If moreover the (automatically integer) number  $\frac{1}{2}\#\mathcal{T}$  is odd, then necessarily  $\mathfrak{g}$  is of type  $B_r$ ,  $t = r$ , and the inequality (6.2) becomes strict, i.e.

$$h \geq 2(r - x) + 1. \quad (6.3)$$

By rewriting (6.2) as  $t \leq \frac{h}{2} + x$  and combining it with (6.1), we obtain the following.

**Corollary 6.2.** *Under the same assumptions on  $\mathfrak{g}$  and  $x$ , every  $\Pi_{[x+1,r]}$ -codominant, null,  $\mathfrak{g}$ -standard doubled Young tableau  $\mathcal{T}$  has height at most  $2x$ .*

The proof of Proposition 6.1 relies on the following lemma.

**Definition 6.3.** Given a Young tableau  $\mathcal{T}$  and a symbol  $s$ , we define the number

$$\text{mincol}_{\mathcal{T}}(s) := \min \left\{ j \mid \exists i, \quad {}^j_i\mathcal{T} = s \right\}$$

and similarly  $\text{maxcol}_{\mathcal{T}}(s)$ , with the usual conventions  $\min \emptyset = +\infty$  and  $\max \emptyset = -\infty$ .

**Lemma 6.4.** *Let  $\mathfrak{g}$ ,  $x$  and  $\mathcal{T}$  be as in Proposition 6.1.(i). Then the identity*

$$\bigcup_{s' \in \{s, s+1, \overline{s+1}, \bar{s}\}} [\text{mincol}_{\mathcal{T}}(s'), \text{maxcol}_{\mathcal{T}}(s')] \subset [\text{mincol}_{\mathcal{T}}(\bar{s}), \text{maxcol}_{\mathcal{T}}(s)] \quad (6.4.s)$$

holds for every  $s$  such that  $x < s \leq r - \mathbb{1}_D$ , where we define  $\mathbb{1}_D$  as 1 if  $\mathfrak{g} = D_r$  for some  $r$  and 0 otherwise.

The proof of this lemma relies on the following obvious remark.

*Remark 6.5.* Let  $\mathcal{T}$  be a null  $\Theta$ -codominant  $\mathfrak{g}$ -standard doubled Young tableau  $\mathcal{T}$ , and let  $\alpha \in \Theta$ . Then the first (resp. last) column of  $\mathcal{T}$  with a nonzero total  $\alpha$ -height has negative (resp. positive) total  $\alpha$ -height, where we define the  $\alpha$ -height of a column  $\mathcal{C}$  as the number  $\langle \nu(\mathcal{C}), \alpha^\vee \rangle$ .

*Proof of Lemma 6.4.* We shall prove the lower bound  $\text{mincol}_{\mathcal{T}}(\bar{s})$  in (6.4), for all  $s$  within the given range; the upper bound is similar. Assume that  $\mathcal{T}$  contains at least one of the symbols  $s$ ,  $s+1$ ,  $\overline{s+1}$  or  $\bar{s}$  (otherwise the inequalities are vacuously true), and let  $j$  be the index of the first column where one of these four symbols occurs. Let us prove, by descending induction on  $s$ , that the  $j$ -th column of  $\mathcal{T}$  contains the symbol  $\bar{s}$ .

- Let us first prove it for  $s = r - \mathbb{1}_D$ . We distinguish two cases:
  - Assume first that  $\mathfrak{g}$  is of type  $B_r$  or  $C_r$ , so that  $s = r$ , and  $\alpha_r$  is equal to (possibly the double of)  $e_r$ . In particular the symbols  $r+1$  and  $\overline{r+1}$  do not occur anywhere in  $\mathcal{T}$ , and we may ignore them. By Remark 6.5 applied to  $\alpha_r$ , it then follows that the  $j$ -th column of  $\mathcal{T}$  contains  $\bar{r}$ , as required.



- Assume now that  $\mathfrak{g}$  is of type  $D_r$ , so that  $s = r - 1$ . Both  $\alpha_{r-1} = e_{r-1} - e_r$  and  $\alpha_r = e_{r-1} + e_r$  lie in  $\Pi_{[x+1, r]}$ ; by applying Remark 6.5 to these two roots and using strong standardness, we obtain that the  $j$ -th column contains  $\overline{r-1}$  as required.
- For any  $s$  such that  $x < s < r - \mathbb{1}_D$ , we have  $\alpha_s = e_s - e_{s+1}$ ; and we easily deduce the lower bound in (6.4.s) from the lower bound in (6.4.s+1) by Remark 6.5.  $\square$

*Proof of Proposition 6.1.* The inequality (6.1) is immediate by strong standardness.

- (i) Assume now that  $\mathcal{T}$  is  $\Pi_{[x+1, r]}$ -codominant and null. Clearly (6.2) holds if  $t \leq x$ ; so assume that  $t > x$ . We introduce the numbers

$$\forall s = x+1, \dots, t, \quad \begin{cases} j_{\bar{s}} := \mincol_{\mathcal{T}}(\bar{s}); \\ j_s := \maxcol_{\mathcal{T}}(s). \end{cases} \quad (6.5)$$

Since  $\mathcal{T}$  is null, in fact, both  $t$  and  $\bar{t}$  must appear somewhere in  $\mathcal{T}$ . Now we apply Lemma 6.4: it follows respectively from the lower and upper bound in (6.4.x+1), ..., (6.4.t-1) that

$$j_{\overline{x+1}} \leq \dots \leq j_{\overline{t-1}} \leq j_{\bar{t}} < +\infty; \quad (6.6)$$

$$-\infty < j_t \leq j_{t-1} \leq \dots \leq j_{x+1}. \quad (6.7)$$

In particular, for each  $s = x+1, \dots, t$ , the value  $j_s$  (resp.  $j_{\bar{s}}$ ) is finite, i.e. is a valid index of a column of  $\mathcal{T}$  that contains the symbol  $s$  (resp.  $\bar{s}$ ). So let  $i_s$  (resp.  $i_{\bar{s}}$ ) denote the (unique) index such that  ${}^{j_s}\mathcal{T} = s$  (resp.  ${}^{j_{\bar{s}}}\mathcal{T} = \bar{s}$ ), for every such  $s$ .

We will now establish some inequalities concerning the numbers  $i_s$ : either (6.8) or (6.9), depending on the order between  $j_{\bar{t}}$  and  $j_t$ .

- Assume first that  $t \leq r - \mathbb{1}_D$ . Then we have  $j_{\bar{t}} = \mincol_{\mathcal{T}}(\bar{t}) \leq \mincol_{\mathcal{T}}(t) \leq \maxcol_{\mathcal{T}}(t) = j_t$ , so that we can combine (6.6) and (6.7) into

$$j_{\overline{x+1}} \leq \dots \leq j_{\bar{t}} \leq j_t \leq \dots \leq j_{x+1}.$$

By  $\preceq_{\mathcal{A}}$ -semistandardness of  $\mathcal{T}$  (which in type  $D_r$  follows from Remark 5.17), this implies as desired that

$$i_{\overline{x+1}} > \dots > i_{\bar{t}} > i_t > \dots > i_{x+1}. \quad (6.8)$$

- Assume now that  $t > r - \mathbb{1}_D$ , i.e.  $\mathfrak{g}$  is of type  $D_r$  and  $t = r$ . If we still have  $j_{\bar{r}} \leq j_r$ , then the same proof works, and (6.8) still holds. So assume that  $j_r \leq j_{\bar{r}}$ ; we then have (by more fully exploiting (6.4.r-1)):

$$j_{\overline{x+1}} \leq \dots \leq j_{\overline{r-1}} \leq j_r \leq j_{\bar{r}} \leq j_{r-1} \leq \dots \leq j_{x+1}.$$

Taking into account  $\preceq''_{\mathcal{A}}$ -semistandardness (see Remark 5.17), we get similarly

$$i_{\overline{x+1}} > \dots > i_{\overline{r-1}} > i_r > i_{\bar{r}} > i_{r-1} > \dots > i_{x+1}. \quad (6.9)$$

In both cases, the integers  $i_s$  are all distinct; the inequality (6.2) follows.

(ii) Assume now that additionally  $\frac{1}{2}\#\mathcal{T}$  is odd.

- Since  $\mathcal{T}$  is null, we have  $\#\boxed{\mathbb{N}}\mathcal{T} = \#\boxed{\mathbb{N}}\mathcal{T}$ , and (recall Remark 5.26) the width  $\#_1\mathcal{T} = 2\lambda_1$  of  $\mathcal{T}$  is even. Hence

$$\frac{1}{2}\#\mathcal{T} = \#\boxed{\mathbb{N}}\mathcal{T} = \sum_{j=1}^{2\lambda_1} \#\boxed{\mathbb{N}}^j\mathcal{T} = \sum_{j'=1}^{\lambda_1} \left( \#\boxed{\mathbb{N}}^{2j'-1}\mathcal{T} + \#\boxed{\mathbb{N}}^{2j'}\mathcal{T} \right),$$

so this last sum is odd. Thus there exists an index, let us call it  $j'_0$ , such that

$$\#\boxed{\mathbb{N}}^{2j'_0-1}\mathcal{T} \not\equiv \#\boxed{\mathbb{N}}^{2j'_0}\mathcal{T} \pmod{2}. \quad (6.10)$$

This pair of columns satisfies condition (A4) from Proposition 5.21, from which we deduce that  $\mathfrak{g}$  is of type  $B_r$  and  $t = r$ .

- It remains to prove (6.3). If  $x = r$ , then we only need to prove that  $h \geq 1$  i.e. that  $\mathcal{T}$  is nonempty, which follows from (6.10). So assume that  $x < r$ , which also means that  $x < t$ , so that the inequalities (6.6), (6.7) and (6.8) from part (i) still hold. We know that

$$j_{\bar{r}} \leq 2j'_0 - 1 < 2j'_0 \leq j_r.$$

Using semistandardness and (6.10), we deduce

$$\#\boxed{\mathbb{N}}^{j_{\bar{r}}}\mathcal{T} \geq \#\boxed{\mathbb{N}}^{2j'_0-1}\mathcal{T} > \#\boxed{\mathbb{N}}^{2j'_0}\mathcal{T} \geq \#\boxed{\mathbb{N}}^{j_r}\mathcal{T}. \quad (6.11)$$

On the other hand, by definition of  $i_s$  and by column-standardness, we have:

$$i_{\bar{r}} = \#\boxed{\mathbb{N}}^{j_{\bar{r}}}\mathcal{T} + 1 \quad \text{and} \quad i_r = \#\boxed{\mathbb{N}}^{j_r}\mathcal{T}. \quad (6.12)$$

Plugging these identities into (6.11), we obtain

$$i_{\bar{r}} > i_{\bar{r}} - 1 > i_r, \quad (6.13)$$

which lengthens (6.8) by one step and thus improves (6.2) to (6.3).  $\square$

*Proof that  $\mathcal{M}_{\text{t-inv}} \subset \mathcal{M}_{\text{Table}}$  for  $\mathfrak{g}$  of types  $B_{r \geq 1}$ ,  $C_{r \geq 1}$  or  $D_{r \geq 3}$ .* Let  $\lambda \in \mathcal{M}_{\text{t-inv}} \subset P \cap \mathfrak{h}^+$ . Then by Proposition 2.1 (i), we get that  $\lambda \in Q$ ; and Corollary 5.28 tells us that the doubled Young diagram  $\Psi(\lambda)$  admits a  $\Theta(\mathfrak{g}_{\mathbb{R}})$ -codominant null  $\mathfrak{g}$ -standard filling, with  $\Theta(\mathfrak{g}_{\mathbb{R}})$  given in Table 3.

We can then check one by one the conditions of Table 1 (note that  $\lambda_i = 0 \iff \#^1\Psi(\lambda) < i$ ), using Corollary 6.2, Proposition 6.1.(ii) (in type  $B$ ) and exceptional isomorphisms for  $\mathfrak{sp}_2(1, 1)$  and  $\mathfrak{so}^*(6)$ .  $\square$

## 6.2 The inclusion $\mathcal{M}_{\text{Table}} \subset \mathcal{M}_{\text{t-inv}}$

In this subsection, we prove that, conversely, all elements  $\lambda \in \mathcal{M}_{\text{Table}}$  satisfy  $V_\lambda^1 \neq 0$ . We rely for this on Proposition 2.5, that reduces the problem to the basis of the monoid  $\mathcal{M}_{\text{Table}}$  (which, in contrast to the  $A_r$  case, can be easily described). For each  $\lambda$  lying in this basis, thanks to the work done in Section 5, our goal is to construct a doubled Young tableau of shape  $\Psi(\lambda)$  satisfying conditions (H1)–(H5) and (H7') from Corollary 5.30.

We start by presenting (Definition 6.6) nine infinite families of doubled Young tableaux, and checking their properties (Proposition 6.7). All the required doubled Young tableaux will then be picked from this pool, up to a “shift” operation defined in Definition 6.8.

**Definition 6.6.** We introduce the doubled Young tableaux  $\mathcal{T}_K$  and  $\mathcal{T}'_K$  (of shape  $2\mathcal{C}_K$ ),  $\mathcal{T}_{K,L}$  and  $\mathcal{T}'_{K,L}$  (of shape  $2\mathcal{C}_K + 2\mathcal{C}_L$ ),  $\mathcal{S}_{K,K}$  and  $\mathcal{S}'_{K,K}$  (of shape  $4\mathcal{C}_K$ ) for some values of the parameters  $K$  and  $L$ , as given in Figure 4.

**Proposition 6.7.** *Let  $\mathcal{T}$  be any Young tableau from this list, and let  $\mathfrak{g}$  be any of the Lie algebras listed in the second column of Table 5. Then  $\mathcal{T}$  is a  $\mathfrak{g}$ -standard doubled Young tableau, is null, and is  $\alpha$ -codominant for all simple roots  $\alpha$  of  $\mathfrak{g}$  that do not belong to the list given in the third column of Table 5.*

Beware that the value of  $r$  is now allowed to vary, while the tableaux are fixed.

*Proof.* The proof is an extremely laborious, but mostly straightforward case distinction. The reader may start by checking that each column of  $\mathcal{T}$  is filled with symbols with consecutive absolute values, i.e.  $k = 1$  in the notations of Proposition 5.21; which reduces the parity condition (5.10) and conditions (A3) and (A4) to simpler forms.  $\square$

**Definition 6.8.** Given a doubled Young tableau  $\mathcal{T}$  and an integer  $x \geq 0$ , we define the *shifted tableau*  $\boxed{x+}\mathcal{T}$  to be the tableau with the same shape, with every symbol  $s$  replaced by  $s + x$  and every symbol  $\bar{s}$  replaced by  $\overline{s + x}$ .

The following statement is then obvious:

**Lemma 6.9.** *Keeping the same setup, let us also fix some integer  $r$ ; and let  $\mathfrak{g} = B_r$  (resp.  $C_r, D_r$ ) and  $\mathfrak{g}' = B_{r+x}$  (resp.  $C_{r+x}, D_{r+x}$ ). For all  $s$  within the appropriate bounds, we denote by  $\alpha_s$  (resp.  $\alpha'_s$ ) the  $s$ -th simple root of  $\mathfrak{g}$  (resp. of  $\mathfrak{g}'$ ) in the usual Bourbaki ordering.*

*Then  $\boxed{x+}\mathcal{T}$  is  $\mathfrak{g}'$ -standard (resp. null) if and only if  $\mathcal{T}$  is  $\mathfrak{g}$ -standard (resp. null). It is always  $\alpha'_s$ -codominant when  $s < x$ , is  $\alpha'_x$ -codominant as soon as  $\mathcal{T}$  is semistandard for the  $\preceq_A$  order, and, for  $s > x$ , is  $\alpha'_s$ -codominant if and only if  $\mathcal{T}$  is  $\alpha_{s-x}$ -codominant.*

*Proof that  $\mathcal{M}_{\text{Table}} \subset \mathcal{M}_{\text{t-inv}}$  for  $\mathfrak{g}$  of types  $B_{r \geq 1}, C_{r \geq 1}$  or  $D_{r \geq 3}$ .* By Proposition 2.5, it suffices to prove that, for every  $\lambda$  lying in the basis of the monoid  $\mathcal{M}_{\text{Table}}$ , we have  $V_\lambda^1 \neq 0$ . By Corollary 5.30, it suffices to find, for every such  $\lambda$ , a  $(\Theta \cup \sigma\Theta)$ -codominant null  $\mathfrak{g}$ -standard filling of the doubled Young diagram  $\Psi(\lambda)$ .

This can be done for all values of  $\mathfrak{g}_{\mathbb{R}}$ , using the nine infinite families of doubled Young tableaux introduced above and possibly shifting them. (Note however that for  $\mathfrak{sp}_2(1, 1)$

Figure 4: Doubled Young tableaux introduced in Definition 6.6. Here it is understood that each block of the form  $\begin{array}{|c|} \hline x \\ \hline \vdots \\ \hline y \\ \hline \end{array}$  (resp.  $\begin{array}{|c|} \hline \overline{y} \\ \hline \vdots \\ \hline \overline{x} \\ \hline \end{array}$ ) is filled with consecutive symbols in increasing  $\prec$  order, so that it has (possibly 0) height  $y - x + 1$ .

1	$k+2$
$\vdots$	$\vdots$
$\vdots$	$2k+1$
$k+1$	$\overline{k+1}$
$\overline{2k+1}$	$\vdots$
$\vdots$	$\vdots$
$\overline{k+2}$	$\overline{1}$

Fig. 4.1:  $\mathcal{T}_{2k+1}$ ,  
for  $k \geq 0$ .

1	$k+1$
$\vdots$	$\vdots$
$k$	$2k$
$\overline{2k}$	$\overline{k}$
$\vdots$	$\vdots$
$\overline{k+1}$	$\overline{1}$

Fig. 4.2:  $\mathcal{T}_{2k}$ ,  
for  $k \geq 1$ .

1	$k$
$\vdots$	$k+2$
$k-1$	$\vdots$
$k+1$	$2k$
$\overline{2k}$	$\overline{k+1}$
$\vdots$	$\overline{k-1}$
$\overline{k+2}$	$\vdots$
$\overline{k}$	$\overline{1}$

Fig. 4.3:  $\mathcal{T}'_{2k}$ ,  
for  $k \geq 2$ .

1	1	$k+1$	$k+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$k$	$2k$	$2k+1$
$k+1$	$\overline{2k+1}$	$\overline{2k+1}$	$\overline{k+1}$
$2k+1$	$\vdots$	$\overline{k}$	$\vdots$
$\overline{2k}$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\overline{k+2}$	$\overline{k+1}$	$\overline{1}$	$\overline{1}$

Fig. 4.4:  $\mathcal{S}_{2k+1, 2k+1}$ ,  
for  $k \geq 1$ .

1	1	$2k-1$	$2k+1$
$\vdots$	$\vdots$	$\overline{2k+1}$	$\overline{2k}$
$\vdots$	$\vdots$	$\overline{2k}$	$\overline{2k-1}$
$\vdots$	$\vdots$	$\overline{2k-2}$	$\overline{2k-2}$
$2k-2$	$2k-2$	$\vdots$	$\vdots$
$2k-1$	$2k$	$\vdots$	$\vdots$
$2k$	$\overline{2k+1}$	$\vdots$	$\vdots$
$2k+1$	$\overline{2k-1}$	$\overline{1}$	$\overline{1}$

Fig. 4.5:  $\mathcal{S}'_{2k+1, 2k+1}$ ,  
for  $k \geq 2$ .

1	$k-l+1$	$k+2$	$k+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$k+l+1$	$k+l+1$
	$k+1$	$\overline{k+1}$	$\overline{k+1}$
	$k+l+2$	$\vdots$	$\vdots$
	$\vdots$		
$k+1$	$2k+1$		
$\overline{2k+1}$	$\overline{k+l+1}$		
$\vdots$	$\vdots$		
	$\overline{k+2}$		
	$\overline{k-l}$		
	$\vdots$	$\overline{k-l+1}$	$\overline{k-l+1}$
$\overline{k+2}$	$\overline{1}$		

Fig. 4.6:  $\mathcal{T}_{2k+1,2l+1}$ ,  
for  $k \geq l \geq 0$ .

1	$k-l+1$	$k+3$	$k+3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$k+l+1$	$k+l+1$
	$k$	$\overline{k+2}$	$\overline{k+2}$
	$k+1$	$\overline{k+1}$	$\overline{k+1}$
	$k+2$	$\overline{k}$	$\overline{k}$
	$k+l+2$	$\vdots$	$\vdots$
	$\vdots$		
$k+2$	$2k+1$		
$\overline{2k+1}$	$\overline{k+l+1}$		
$\vdots$	$\vdots$		
	$\overline{k+3}$		
	$\overline{k-l}$		
	$\vdots$	$\overline{k-l+1}$	$\overline{k-l+1}$
$\overline{k+3}$	$\overline{1}$		

Fig. 4.7:  $\mathcal{T}'_{2k+1,2l+1}$ ,  
for  $k \geq l > 0$ .

1	$k-l+1$	$k+2$	$k+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$k+l+1$	$k+l+1$
	$k+1$	$\overline{k+1}$	$\overline{k+1}$
	$k+l+2$	$\vdots$	$\vdots$
	$\vdots$		
	$2k$		
$k+1$	$\overline{k+l+1}$		
$\overline{2k}$	$\vdots$		
$\vdots$	$\overline{k+2}$		
	$\overline{k-l}$		
	$\vdots$	$\overline{k-l+1}$	$\overline{k-l+1}$
$\overline{k+2}$	$\overline{1}$		

Fig. 4.9:  $\mathcal{T}_{2k,2l+1}$ ,  
for  $k > l \geq 0$ .

1	$k+1$	$\overline{k+2}$	$\overline{k+2}$
$\vdots$	$\vdots$		
$k$	$\vdots$		
$k+2$	$2k+1$		
$\overline{2k+1}$	$\overline{k}$		
$\vdots$	$\vdots$		
$\overline{k+3}$	$\vdots$		
$\overline{k+1}$	$\overline{1}$		

Fig. 4.8:  $\mathcal{T}'_{2k+1,1}$ ,  
for  $k \geq 1$ .

Table 5: Properties of the tableaux introduced in Figure 4.

Tableau $\mathcal{T}$	$\mathcal{T}$ is $\mathfrak{g}$ -standard for $\mathfrak{g} = \dots$	$\mathcal{T}$ is $\alpha$ -codominant for all $\alpha$ except...
$\mathcal{T}_{2k+1}$	$B_{2k+1}$	$e_{k+1} - e_{k+2}$ or $e_{k+1}$
$\mathcal{T}_{2k}$	$B_r, C_r, D_r$ for $r \geq 2k$	$e_k - e_{k+1}$
$\mathcal{T}'_{2k}$ for $k \neq 1$	$C_r, D_r$ for $r \geq 2k$	$\begin{cases} e_{k-1} - e_k; \\ e_{k+1} - e_{k+2} \end{cases}$
$\mathcal{T}_{2k+1,2l+1}$	$B_r, C_r, D_r$ for $r \geq 2k+1$ , except $D_{2k+1}$ if $l = k$	$\begin{cases} e_{k+1} - e_{k+2} \text{ or } e_{k+1} \text{ or } 2e_{k+1}; \\ e_2 + e_3 \quad \text{if } k = 1, l = 0 \end{cases}$
$\mathcal{T}'_{2k+1,2l+1}$ for $l \neq 0$	$C_r, D_r$ for $r \geq 2k+1$ , except $D_{2k+1}$ if $l = k$	$\begin{cases} e_{k+2} - e_{k+3} \text{ or } 2e_{k+2}; \\ e_4 + e_5 \quad \text{if } k = 2, l = 1 \end{cases}$
$\mathcal{T}'_{2k+1,1}$ for $k \neq 0$	$C_r, D_r$ for $r \geq 2k+1$	$\begin{cases} e_k - e_{k+1}; \\ e_{k+2} - e_{k+3} \text{ or } 2e_{k+2}; \\ e_2 + e_3; e_4 + e_5 \end{cases}$
$\mathcal{T}_{2k,2l+1}$	$B_{2k}$	$e_{k+1} - e_{k+2}$ or $e_{k+1}$
$\mathcal{S}_{2k+1,2k+1}$ for $k \neq 0$	$D_{2k+1}$	$\begin{cases} e_k - e_{k+1}; e_{k+1} - e_{k+2}; \\ e_2 + e_3 \end{cases}$
$\mathcal{S}'_{2k+1,2k+1}$ for $k \neq 0, 1$	$D_{2k+1}$	$\begin{cases} e_{2k-2} - e_{2k-1}; \\ e_{2k} - e_{2k+1} \text{ and } e_{2k} + e_{2k+1} \end{cases}$

 Table 6: Images by  $\Psi$  of the primitive elements of  $\mathcal{M}_{\text{Table}}$  for  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}_2(p, r-p)$  (for  $r \geq 3$  and  $0 \leq p \leq \frac{r}{2}$ ), and their  $(\Theta \cup \sigma\Theta)$ -codominant null  $\mathfrak{g}$ -standard fillings. Recall that  $\sigma = \text{Id}$  and  $\Theta = \Pi_{\text{odd}} \cup \Pi_{[2p+1, r]}$  in this case.

$\Psi(\lambda)$	Parameter range	Subrange	$\mathcal{T}$	Syndrome
$2\mathcal{C}_{2k}$	$\begin{cases} 1 \leq k \leq 2p \\ 2k \leq r \end{cases}$	$k$ even	$\mathcal{T}_{2k}$	$\alpha_k$
		$k > 1$ odd	$\mathcal{T}'_{2k}$	$\alpha_{k-1}, \alpha_{k+1}$
		$k = 1$	$\boxed{1+}\mathcal{T}_2$	$\alpha_2$
$2(\mathcal{C}_{2k+1} + \mathcal{C}_{2l+1})$	$\begin{cases} 0 \leq l \leq k < 2p \\ 2k+1 \leq r \end{cases}$	$k$ odd	$\mathcal{T}_{2k+1,2l+1}$	$\alpha_{k+1}$
		$k$ even, $l > 0$	$\mathcal{T}'_{2k+1,2l+1}$	$\alpha_{k+2}$
		$k > 0$ even, $l = 0$	$\mathcal{T}'_{2k+1,1}$	$\alpha_k, \alpha_{k+2}$
		$k = l = 0$	$\boxed{1+}\mathcal{T}_{1,1}$	$\alpha_2$

and  $\mathfrak{so}^*(6)$ , it is probably easier to take advantage of exceptional isomorphisms). In Table 6, we have given the details for  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}_2(p, r - p)$ .

It is then straightforward to verify that, first of all, the image of the basis of the monoid  $\mathcal{M}_{\text{Table}}$  by the map  $\Psi$  is as listed in the first two columns. (We have, for every  $i$ ,  $2\mathcal{C}_i = \Psi(c_i^{\pm})$ , where  $c_i^{\pm}$  is as defined in (5.14)).

Moreover, for every such diagram  $\Psi(\lambda)$ , we deduce as a particular case of Proposition 6.6, possibly using Lemma 6.9 when a shift is involved, that its filling  $\mathcal{T}$  listed in the fourth column is null and  $\mathfrak{g}$ -standard, has “syndrome” (which we define as the set of simple roots  $\alpha \in \Pi(\mathfrak{g})$  for which  $\mathcal{T}$  is *not* codominant) as listed in the fifth column, and this syndrome is disjoint from  $(\Theta \cup \sigma\Theta)$ .  $\square$

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