

# Topology and Edge Modes in Quantum Critical Chains

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We show that topology can protect exponentially localized, zero energy edge modes at critical points between one-dimensional symmetry-protected topological phases. This is possible even without gapped degrees of freedom in the bulk—in contrast to recent work on edge modes in gapless chains. We present an intuitive picture for the existence of these edge modes in the case of noninteracting spinless fermions with time-reversal symmetry (BDI class of the tenfold way). The stability of this phenomenon relies on a topological invariant defined in terms of a complex function, counting its zeros and poles inside the unit circle. This invariant can prevent two models described by the *same* conformal field theory (CFT) from being smoothly connected. A full classification of critical phases in the noninteracting BDI class is obtained: Each phase is labeled by the central charge of the CFT,  $c \in \frac{1}{2}\mathbb{N}$ , and the topological invariant,  $\omega \in \mathbb{Z}$ . Moreover,  $c$  is determined by the difference in the number of edge modes between the phases neighboring the transition. Numerical simulations show that the topological edge modes of critical chains can be stable in the presence of interactions and disorder.

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**Introduction.**—Topology is fundamental to characterizing quantum phases of matter in the absence of local order parameters [1]. In one spatial dimension, such zero-temperature phases have topological invariants generically protecting edge modes, i.e., zero energy excitations localized near boundaries. These phases are usually referred to as topological insulators or superconductors when noninteracting [2] and as symmetry-protected topological phases when interacting [3,4]. The topological invariants of one-dimensional systems require the presence of a symmetry and have been classified for phases with a gap above the ground state [5–11].

Conventional wisdom says topological edge modes *require* a bulk gap. Recently, there has been work on gapless phases hosting edge modes [12–24], but, when their localization is exponential, it is attributed to gapped degrees of freedom (meaning there are exponentially decaying correlation functions).

We indicate that this picture is at odds with the critical points between topological superconductors in the BDI class (noninteracting spinless fermions with time-reversal symmetry) [25]. Curiously, a 2001 work by Motrunich, Damle, and Huse with different aims implied that some of these transitions host topological edge modes [26]. This phenomenon and its consequences have not been explored since. It is of particular importance given the recent interest in the interplay between topology and criticality, since—as we will argue—the bulk has *no* gapped degrees of freedom.

After reviewing the gapped phases of the BDI class, we present an example of a critical chain with edge modes.

Subsequently, we identify a topological invariant in terms of a complex function—distinct from the transfer matrix approach [27,28] of Ref. [26]—which we prove counts the edge modes (Theorem 1). Similar concepts appear in the literature on *gapped* topological phases [29–31]. The idea of classifying such critical phases is then explored. We find that, in the BDI class, chains with the *same* conformal field theory (CFT) can be smoothly connected *if and only if* the topological invariant coincides (Theorem 2). Moreover, the CFT is determined by the change in topological invariant upon crossing the transition (Theorem 3). Finally, we numerically demonstrate that topological edge modes at criticality can survive disorder and interactions.

**Example.**—We illustrate how a critical phase—without gapped degrees of freedom—can have localized edge modes. First, we decompose every fermionic site  $c_n$ ,  $c_n^\dagger$  into two Majorana modes:  $\gamma_n = c_n^\dagger + c_n$  and  $\tilde{\gamma}_n = i(c_n^\dagger - c_n)$ . The former is real ( $T\gamma_n T = \gamma_n$ , where  $T$  is complex conjugation in the occupation basis) and the latter imaginary ( $T\tilde{\gamma}_n T = -\tilde{\gamma}_n$ ). These Hermitian operators anticommute and square to unity.

We define the  $\alpha$ -chain [29,32,33]:

$$H_\alpha = \frac{i}{2} \sum_n \tilde{\gamma}_n \gamma_{n+\alpha} \quad (\alpha \in \mathbb{Z}). \quad (1)$$

These gapped chains are illustrated in Fig. 1. For  $\alpha = 1$ , it is the Kitaev chain with Majorana edge mode  $\gamma_1$  [34].  $H_\alpha$  has  $|\alpha|$  Majorana zero modes per edge and can be thought of as a stack of Kitaev chains. The edge modes survive quadratic,  $T$ -preserving perturbations due to chirality: If *real* modes

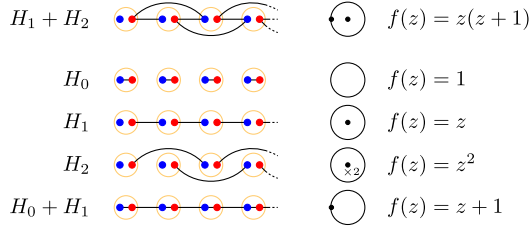


FIG. 1. Representation of the critical Hamiltonian  $H_1 + H_2$  with its edge mode [each fermionic site is decomposed into Majorana modes:  $\gamma$  (blue) and  $\tilde{\gamma}$  (red); a bond signifies a term in the Hamiltonian]. Also shown are the gapped Hamiltonians  $H_\alpha$  ( $\alpha = 0$  is trivial, and  $\alpha = 1$  is the Kitaev chain).  $H_0 + H_1$  is the standard critical Majorana chain. The associated complex function  $f(z)$  [Eq. (3)] and zeros in the complex plane are shown.

prefer to couple to their *left*, some remain decoupled at the left edge. This can be quantified by a topological winding number [35] counting edge modes, meaning each  $H_\alpha$  represents a distinct phase of matter. The  $2|\alpha|$  zero modes imply a  $2^{|\alpha|}$ -fold degeneracy, with a finite-size gap  $\sim e^{-L/\xi}$  when the modes are exponentially localized.

Consider now a critical point between the phases defined by fixed point Hamiltonians  $H_1$  and  $H_2$ , namely,  $H_1 + H_2$ . Despite it being critical, Fig. 1 shows a localized Majorana edge mode. Nevertheless, there is no local operator  $\mathcal{O}$  with  $\langle \mathcal{O}_n \mathcal{O}_m \rangle \sim e^{-|n-m|/\xi}$  [36]. Indeed, for periodic boundary conditions, shifting  $\gamma_n \rightarrow \gamma_{n-1}$  (which one *cannot* smoothly implement in a local and  $T$ -preserving way) maps  $H_1 + H_2$  to  $H_0 + H_1$ , the well-studied critical Majorana chain described by a CFT with central charge  $c = \frac{1}{2}$  [37].

In the next section, we demonstrate the edge mode's stability: A topological invariant protects it. Similar to the winding number for gapped phases, it quantifies the couplings' chirality. In short, we associate to every chain a complex function  $f(z)$  (illustrated in Fig. 1) whose number of zeros (minus poles) in the unit disk counts the edge modes.

**Topology and edge modes in the BDI class.**—Consider the full BDI class: chains of noninteracting spinless fermions with time-reversal symmetry  $T$  (defined above). The aforementioned  $\{H_\alpha\}_{\alpha \in \mathbb{Z}}$  form a basis for arbitrary translation-invariant Hamiltonians in this class:

$$H_{\text{BDI}} = \frac{i}{2} \sum_{\alpha=-\infty}^{+\infty} t_\alpha \left( \sum_{n \in \text{sites}} \tilde{\gamma}_n \gamma_{n+\alpha} \right) = \sum_{\alpha} t_\alpha H_\alpha. \quad (2)$$

We take  $t_\alpha$  to be nonzero for only a finite number of  $\alpha$  (i.e.,  $H$  is finite range). Time-reversal symmetry forbids terms of the form  $i\gamma_n \gamma_m$ , and Hermiticity requires  $t_\alpha \in \mathbb{R}$ . We mainly work in this translation-invariant setting, but the effects of unit cells and disorder are addressed.

$H_{\text{BDI}}$  is determined by the list of numbers  $t_\alpha$  or, equivalently, by its Fourier transform  $f(k) := \sum_{\alpha} t_\alpha e^{ik\alpha}$ . It is efficiently diagonalized: If  $f(k) = \varepsilon_k e^{i\varphi_k}$  (with  $\varepsilon_k, \varphi_k \in \mathbb{R}$ ), then a Bogoliubov rotation over the angle  $\varphi_k$  diagonalizes  $H_{\text{BDI}}$ , with single-particle spectrum  $\varepsilon_k$  [38].

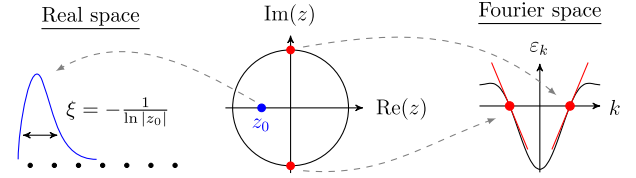


FIG. 2. The middle figure shows the zeros of  $f(z)$ . The zero  $z_0$  within the disk (blue) corresponds to an edge mode (per edge) with localization length  $\xi = -1/\ln|z_0|$ . Each zero on the unit circle (red) implies a massless Majorana field in the low-energy limit ( $c = \frac{1}{2}$ ).

In this language, the invariant for gapped phases is simply the winding number of  $f(k)$  around the origin: Since  $\varepsilon_k$  is nonzero, the phase  $e^{i\varphi_k}$  is a well-defined function from  $S^1$  to  $S^1$ . This fails when the system is gapless but can be repaired using complex analysis. First, interpret the function  $f(k)$  as living on the unit circle in the complex plane—abusing notation, write  $f(z = e^{ik})$ —with the *unique* analytic continuation

$$f(z) = \sum_{\alpha=-\infty}^{\infty} t_\alpha z^\alpha. \quad (3)$$

Now,  $f$  is a function  $\mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$  with a pole at the origin when  $t_\alpha \neq 0$  for some  $\alpha < 0$ . If it has no zero *on* the unit circle (i.e., the system is gapped), then Cauchy's argument principle says that the winding number defined above equals the number of zeros ( $N_z$ , including the degree) minus the order of the pole ( $N_p$ ) within the unit disk. If at least one zero lies on the unit circle, the aforementioned winding number breaks down—the quantity  $N_z - N_p$ , however, remains well defined. Perturbing  $H_{\text{BDI}}$  smoothly moves the zeros of  $f(z)$  around, and changing the support of  $t_\alpha$  produces or destroys zero-pole pairs at the origin or infinity. Hence, by continuity,  $N_z - N_p$  cannot change without affecting the number of zeros on the unit circle. This would change the bulk physics: Every (nondegenerate) zero  $e^{ik_0}$  of  $f(z)$  implies that  $\varepsilon_k \sim k - k_0$ , contributing a massless Majorana fermion (with central charge  $c = \frac{1}{2}$ ) to the CFT (see Fig. 2).

Hence,  $\omega := N_z - N_p$  (*strictly within* the unit disk) defines a topological invariant, for both gapped and gapless chains. We now show its physical significance: If  $\omega > 0$ , it counts the Majorana zero modes which are exponentially localized on the boundary. Moreover, the localization lengths are given by the zeros of  $f(z)$ . Figure 2 illustrates this, with the precise statement being as follows.

**Theorem 1.** If the topological invariant  $\omega > 0$ , then (1) each boundary has  $\omega$  Majorana zero modes, (2) the modes have localization length  $\xi_i = -1/\ln(|z_i|)$ , where  $\{z_i\}$  are the  $\omega$  largest zeros of  $f(z)$  within the unit disk, and (3) the modes on the left (right) are real (imaginary).

If  $\omega < -2c$  (where  $c = \frac{1}{2}$  is half the number of zeros on the unit circle), the left (right) boundary has  $|\omega + 2c|$  imaginary (real) Majorana modes with localization length

$\xi_i = 1/\ln(|z_i|)$ , with  $\{z_i\}$  the  $|\omega + 2c|$  smallest zeros outside the unit disk.

For any other value of  $\omega$ , no localized edge modes exist.

Before we outline the proof, note that in the gapped case  $c$  is zero, with  $|\omega|$  correctly counting edge modes. At criticality,  $2c$  counts the zeros on the unit circle, and, if these are nondegenerate, the bulk is a CFT with central charge  $c$ . However, if  $f(z)$  has a zero  $e^{ik_0}$  with multiplicity  $m$ , then  $\varepsilon_k \sim (k - k_0)^m$ , implying a dynamical critical exponent  $z_{\text{dyn}} = m$ .

If  $\omega > 0$ , we construct for each  $z_i$  (defined above) a real edge mode  $\gamma_{\text{left}}^{(i)} = \sum_{n \geq 1} b_n^{(i)} \gamma_n$  by requiring  $[\gamma_{\text{left}}^{(i)}, H] = 0$ . This gives constraints  $\sum_{a \geq 1} b_a^{(i)} t_{a-n} = 0$  for  $n \geq 1$ , leading to standard solvable recurrence relations, with the function  $f(z)$  appearing as the characteristic polynomial. If  $N_p = 0$ , the solution is simply  $b_n^{(i)} \propto z_i^n$  (hence,  $|b_n^{(i)}| \sim e^{-n/\xi_i}$ ), while  $N_p > 0$  modifies  $b_n^{(i)}$  without affecting its asymptotic form. Details are treated in Supplemental Material [38].

The case  $\omega < -2c$  follows by noting that inverting left and right effectively implements  $f(z) \leftrightarrow f(1/z)$ , and one can show that this changes the topological invariant  $\omega \leftrightarrow -\omega - 2c$ . This completes the proof. Note that the exponential localization implies that the commutator—and hence the energy gap of the edge mode—is exponentially small for finite systems.

We can now appreciate the right-hand side of Fig. 1, showing for each Hamiltonian the function  $f(z)$  and its zeros. The two critical Hamiltonians indeed have a zero on the unit circle, and the edge modes are counted by the zeros strictly within the unit disk. Hence, the edge mode of  $H_1 + H_2$  is stable: The zero will stay within the unit disk for small perturbations.

*Classifying critical phases.*—In this section, we use the above framework to answer two related questions: (i) What is the classification of the critical phases within the BDI class, and (ii) given two gapped phases, what is the universality class of the critical point between them?

We define two Hamiltonians to be in the same phase if and only if they are connected by a path of local Hamiltonians (within the symmetry class) along which the low-energy description of the bulk changes smoothly. (This is different from the notion of Furuya and Oshikawa [39], where CFTs are in the same phase if a renormalization group flow connects them.) Hamiltonians described by CFTs with different central charges are automatically in distinct phases by the  $c$ -theorem [40]. If, on the other hand, both Hamiltonians have the same CFT description, one might expect them to be in the same phase. However, we have shown that  $H_1 + H_2$  and  $H_0 + H_1$  have different topological invariants (respectively,  $\omega = 1$  and  $\omega = 0$ ) yet the same CFT description. Hence, they cannot be connected within the BDI class, as illustrated in Fig. 3.

This is illustrative of the general case. We have seen that any translation-invariant Hamiltonian  $H_{\text{BDI}}$  can be identified

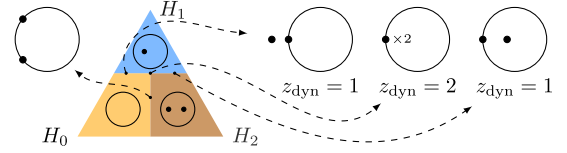


FIG. 3. Phase diagram illustrating how critical points with the same CFT description but different topological invariants cannot be connected: Interpolating  $H_0 + H_1$  and  $H_1 + H_2$  induces a point where the dynamical critical exponent  $z_{\text{dyn}}$  changes discontinuously.

with a complex function  $f(z)$ , with its set of zeros and poles, as in Fig. 2. Conversely, the zeros and poles uniquely identify  $f(z)$  and hence the Hamiltonian. More precisely, let  $\{z_i\}$  be the list of (distinct) zeros of Eq. (3) with a corresponding list of multiplicities  $\{m_i\}$ , and let  $N_p$  be the order of the pole at the origin. By the fundamental theorem of algebra, this uniquely identifies the meromorphic function  $f(z) = \pm(1/z^{N_p}) \prod_i (z - z_i)^{m_i}$  up to a positive multiplicative scalar. This correspondence between a BDI Hamiltonian and a picture of zeros and poles reduces the classification to an exercise in geometric insight.

Let us focus on the physically interesting case where the bulk is described by a CFT—i.e., the zeros on the unit circle have multiplicity one. There is only one rule restricting the movement of the zeros on the unit circle: Since  $t_\alpha$  is real, the zeros of  $f(z)$  are real or come in complex-conjugate pairs. This means we *cannot* move a zero off the real axis. However, we *can* bring  $f(z)$  to a canonical form where the zeros are equidistantly distributed on the unit circle with mirror symmetry about the real axis. There are only two such patterns, given by the solutions of  $z^{2c} = \pm 1$ . Thus, we can always tune to the canonical form  $f(z) = \pm(z^{2c} \pm 1)z^\omega$ . Hence, for a given nonzero  $c \in \frac{1}{2}\mathbb{N}$  and  $\omega \in \mathbb{Z}$ , there are  $\mathbb{Z}_2 \times \mathbb{Z}_2$  translation-invariant phases, labeled by the two signs. These signs are protected by translation symmetry: The first encodes the spatially modulating sign of correlations, the second whether  $\varepsilon_k = |f(e^{ik})|$  vanishes at time-reversal invariant momenta.

Hence, allowing for paths with unit cells, two systems can be smoothly connected *if* they have identical  $c$  and  $\omega$  [38]. To confirm that this is a *necessary* condition, we extend  $\omega$  to systems with an  $N$ -site unit cell,  $H = (i/2) \sum \tilde{\gamma}_n^T T_\alpha \gamma_{n+\alpha}$ , where  $T_\alpha \in \mathbb{R}^{N \times N}$ . Defining  $f(z) = \det(\sum_\alpha T_\alpha z^\alpha)$ , then analogous to before, one can show that  $|f(e^{ik})|$  is the *product* of the energy bands  $\varepsilon_k^{n=1, \dots, N}$  [38]. Thus,  $\omega = N_z - N_p$  cannot change without a bulk transition.

For gapped phases,  $\omega$  is known to be additive under stacking [35]. Our extension to critical systems still satisfies this property (as does  $c$  [37]). Moreover, the classification straightforwardly generalizes to stackings, with a small caveat. For example, a Kitaev chain stacked onto  $H_0 + H_1$  has the same invariants as  $H_1 + H_2$  ( $c = \frac{1}{2}$  and  $\omega = 1$ ). One might expect these to be in the same



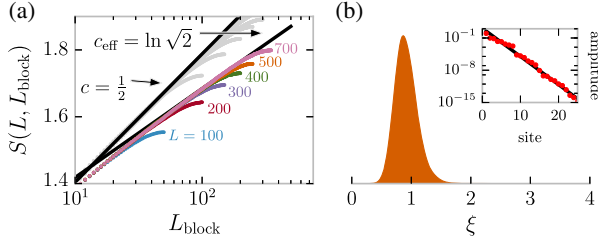


FIG. 4. Phase transition at strong disorder: (a) Entanglement scaling (averaging  $10^5$  states) suggests an infinite randomness fixed point with  $c_{\text{eff}} = \ln \sqrt{2}$  (black lines guide the eye; gray is the clean case), and (b) the distribution of edge mode localization length over disorder realizations (inset: edge mode for one realization).

phase, but the latter has no gapped degrees of freedom. This is resolved by adding a decoupled trivial chain—after which one can indeed connect them. Details are in Supplemental Material [38].

We thus obtain a semigroup classification (“semi” because  $c$  cannot decrease under stacking).

**Theorem 2.** The phases in the BDI class described in the bulk by a CFT and obtained by deforming a translation-invariant Hamiltonian (or a stacking thereof) with an arbitrary unit cell are classified by the semigroup  $\mathbb{N} \times \mathbb{Z}$ . They are labeled by the central charge  $c \in \frac{1}{2}\mathbb{N}$  and the topological invariant  $\omega \in \mathbb{Z}$ .

Translation invariance gives an extra  $\mathbb{Z}_2$  invariant when  $c = 0$  and an extra  $\mathbb{Z}_2 \times \mathbb{Z}_2$  invariant when  $c \neq 0$ .

The second question, regarding the phase transition between two topological phases, is straightforwardly addressed. By continuity, the difference in the winding number between two gapped phases is the number of zeros that must cross the unit circle at the transition, i.e.,

**Theorem 3.** A phase transition between two gapped phases with winding numbers  $\omega_1$  and  $\omega_2$  obeys  $c \geq (|\omega_1 - \omega_2|/2)$ .

As before,  $c$  should be understood as counting half the number of zeros on the unit circle. If these zeros are nondegenerate, the bulk is a CFT with central charge  $c$ . Generically,  $c$  equals  $(|\omega_1 - \omega_2|/2)$ , but one can fine-tune the transition with zeros bouncing off the unit circle [41]. This theorem proves a special case of a recent conjecture concerning all transitions between one-dimensional symmetry protected topological phases [32].

In the remainder, we demonstrate that topological edge modes in critical chains can survive disorder and interactions.

**Disorder.**—We consider  $H = (i/2) \sum_{\alpha=0}^3 \sum_n t_{\alpha}^{(n)} \tilde{\gamma}_n \gamma_{n+\alpha}$ . The clean model  $t_1^{(n)} = t_2^{(n)} = 1$  and  $t_0^{(n)} = t_3^{(n)} = a$ , where  $-1 < a < \frac{1}{3}$ , is critical with  $c = \frac{1}{2}$  and  $\omega = 1$ . This reduces to  $H_1 + H_2$  when  $a = 0$ . We now introduce strong disorder:  $t_1^{(n)}, t_2^{(n)} (t_0^{(n)}, t_3^{(n)})$  are drawn independently from the flat distribution on  $[0, 1]$  ( $[-0.5, 0]$ ).

We confirm the system flows to the infinite randomness fixed point with effective central charge  $c_{\text{eff}} = \ln \sqrt{2}$

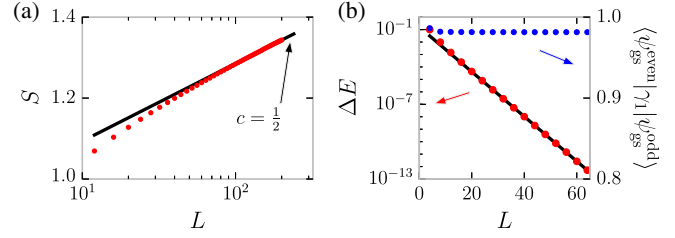


FIG. 5. Finite-size scaling for the interacting Hamiltonian (4) with open boundaries and  $U = 0.3$ : (a) The bulk is described by the  $c = \frac{1}{2}$  Majorana CFT (black line guides the eye), and (b) energy splitting between fermionic parity sectors is exponentially small,  $\Delta E \sim e^{-L/\xi}$  with  $\xi \approx 2.42$ . The two ground states are related by an edge mode.

[26,42,43]. We diagonalize periodic systems of size  $L$ , calculating the entanglement entropy  $S(L, L_{\text{block}})$  of a region of length  $L_{\text{block}}$ . The average is predicted to obey the asymptotic scaling  $S \sim (c_{\text{eff}}/3) \ln L_{\text{block}}$  (for  $1 \ll L_{\text{block}} \ll (L/2)$ ), as shown in Fig. 4(a).

In the presence of open boundary conditions, we observe one Majorana edge mode per boundary. These are exponentially localized, with Fig. 4(b) showing the distribution of localization lengths over different disorder realizations. The inset gives a generic example, plotting  $|b_n|$  where the edge mode is  $\gamma_{\text{left}} = \sum_{n=1}^L b_n \gamma_n$ .

**Interactions.**—Interactions can have interesting effects, with the gapped classification reducing from  $\mathbb{Z}$  to  $\mathbb{Z}_8$  [8,9,44]. Here we simply show that an interacting critical point between the Kitaev chain and  $H_2$ ,

$$H = H_1 + H_2 + U \sum_{n=1}^L \gamma_n \gamma_{n+1} \gamma_{n+2} \gamma_{n+3} + (\gamma \leftrightarrow \tilde{\gamma}), \quad (4)$$

has localized edge modes. The critical point will not shift when  $U \neq 0$ , since (4) is self-dual under  $\gamma_n \rightarrow \gamma_{3-n}$  and  $\tilde{\gamma}_n \rightarrow \tilde{\gamma}_{-n}$ .

We use the density matrix renormalization group method [45] to perform finite-size scaling with open boundaries for  $U = 0.3$ . (Convergence was reached for system sizes shown with bond dimension  $\chi = 60$ .) In Fig. 5(a), we confirm that the system remains critical by using the CFT prediction [46] for the entanglement entropy of a bipartition into two equal halves of length  $(L/2)$ , namely,  $S \sim (c/6) \ln L$ .

Figure 5(b) shows the ground state degeneracy with open boundary conditions. These states differ only near the edge, since  $\langle \psi_{\text{gs}}^{\text{even}} | \gamma_1 | \psi_{\text{gs}}^{\text{odd}} \rangle$  is finite as  $L \rightarrow \infty$ .

**Conclusion.**—We have shown that any two gapped phases in the BDI class with winding numbers  $\omega_1 > \omega_2 > 0$  are separated by a critical point with  $\omega_2$  topological edge modes (and central charge  $(\omega_1 - \omega_2/2)$ ). We have characterized such phases within this class in terms of the zeros and poles of an associated complex function.

Unlike gapped phases, these critical chains do not have the usual bulk-boundary correspondence (i.e., edge modes

implying degenerate Schmidt values when bipartitioning a periodic system [47]), as suggested by Fig. 1. Hence, how the topology is reflected in the entanglement is an open question. The study of possible string orders using Toeplitz determinants constitutes a forthcoming work.

A natural question is how to extend this to the other one-dimensional symmetry classes. We note that our analysis also applies to AIII (identified with a subset of BDI; see, e.g., Ref. [32]) leading to an  $\mathbb{N} \times \mathbb{Z}$  classification with integer central charge. We similarly expect (but do not argue) an  $\mathbb{N} \times \mathbb{Z}$  classification for CII and a *single* critical phase [48] for the other nontrivial classes. Furthermore, the classification of interacting critical chains should be found. Similar to the gapped case, a tensor-network approach may prove insightful, perhaps using the multiscale entanglement renormalization ansatz [49] or infinite-dimensional matrix product states [50]. The generalization to higher dimensions is an open-ended frontier.

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