

Voxel-Based Full-Field Eigenstrain Reconstruction of Residual Stresses

Fatih Uzun* and Alexander M. Korsunsky*

Inverse eigenstrain (inherent strain) analysis methods are shown to be effective for the reconstruction of residual stresses in plane eigenstrain problems (continuously processed bodies) while conversely residual stress reconstruction in discontinuously processed bodies is extremely challenging and necessitates the use of complex regularizing assumptions. Herein, a new generic inverse eigenstrain method suitable for the reconstruction of residual stresses along with residual elastic strains and displacements in discontinuously processed bodies is introduced. The proposed method uses the superposition of eigenstrain radial basis functions together with a set of limited experimental data for model-free (unconstrained) determination of unknown eigenstrain fields. This approach eliminates the limitations introduced by global basis functions such as polynomials. The novel point of this method is the ability to account for all six components of strain in an isotropic body without using regularizing assumptions. By lifting complex guiding formulation, the fidelity of full-field eigenstrain reconstruction becomes directly related to the quality of experimental data and proper discretisation of the model domain. The FEniCS implementation has been validated using the experimental data of pointwise high-energy synchrotron X-ray diffraction measurements from a bent titanium alloy bar. A hybrid high throughput computing approach is also introduced for effective parallel computing.

1. Introduction

Deformation processing of solid bodies generates recoverable elastic and nonrecoverable inelastic deformations as permanent inelastic strains.^[1] Eigenstrains are irregularly distributed permanent plastic strains that cause the formation of residual stresses. The term eigenstrain introduced by Mura^[2] in 1987 established the relation between eigenstresses, i.e., residual self-equilibrated stresses, and eigenstrains, i.e., their underlying deforma-

tion sources. The clarification of the connection between plastic strains and stresses gives rise to the method of stress state quantification and control by means of eigenstrain fields. Experimental realization of this approach can be achieved by introducing permanent inelastic strains by sample plastic deformation. In the simulation context, eigenstrains can be introduced into numerical models to give rise to residual stresses. This kind of modeling approach represents a major shortcut compared to full process modeling because the so-called direct problem of eigenstrain is linearly elastic and allows a fast iteration-free solution. The determination of eigenstrain fields from experimentally determined residual stresses is a more challenging task known as the inverse problem of eigenstrain.^[1]

Solving the inverse problem has been achieved by mapping inherent strains^[3–5] or eigenstrains.^[1] Both terms are used to define the strains that are responsible for deformations in the case of lack of any external load. The common property of

these methods is the ability to reconstruct residual stresses. The distribution of inherent strains is reconstructed corresponding to the distortions of welded^[6–9] and additively manufactured parts^[10–12] while the quantified elastic response of strain relaxation^[13–15] is used to reconstruct the distribution of eigenstrains. Advances in computation power^[16] let the development of the continuum mechanics-based finite element method in the 1950s. The ease of access to high-performance workstations after the 2000s allowed the application of the inverse problem of eigenstrain to the solution of complex solid mechanics problems such as continuously processed bodies,^[15] welds,^[14,17–19] and shot peened materials.^[20–24] Full-field distribution of residual stresses was predicted by solving the inverse problem of eigenstrain^[25–27] and hybrid reconstruction methods were also developed^[28–30] for this purpose. Although these studies provide a very good match of linear elastic numerical model calculations with experimental data, which can be in the form of displacements or residual stresses, the reliability of eigenstrain evaluation is not clear. These models create an eigenstrain field around the experimental measurement zone^[31–33] that provides a match of calculated residual stress with experimental measurements. However, eigenstrain fields determined for continuously processed welded thick material^[17] show that eigenstrains are distributed in the whole body while permanent plastic strains

F. Uzun, A. M. Korsunsky

MBLEM

Department of Engineering Science

University of Oxford

Oxford OX1 3PJ, UK

E-mail: fatihuzun@me.com, fatih.uzun@eng.ox.ac.uk;

alexander.korsunsky@eng.ox.ac.uk

 The ORCID identification number(s) for the author(s) of this article can be found under <https://doi.org/10.1002/adem.202201502>.

© 2023 The Authors. Advanced Engineering Materials published by Wiley-VCH GmbH. This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

DOI: 10.1002/adem.202201502

are expected to be formed around the source of deformations, e.g., the weld zone and the heat-affected zone (HAZ). The first attempt to identify the principal source of welding residual stress via the determination of full-field distribution of eigenstrains was the development of an iterative solution process.^[34,35] This modeling approach is based on regularization functions and basis functions together to determine the eigenstrain fields that cause the formation of partially known residual stresses. The choice on regularization functions depends on the prediction of the stress fields based on the assumption that eigenstrains are formed in and around the fields where inelastic deformation occurred. Ensuring that this estimation is correct increases the risk of errors introduced by the model designer during the selection of parameters of these functions. The probability of increasing the error related to regularization functions is minimized in the cases where type I (macroscopic) stresses are characterized and intragranular or atomistic details of residual stresses are not concerned (type II and III microscopic stresses) by the implementation of principles of artificial intelligence that determines optimum parameters of regularization functions.^[36–38]

The advances in the full-field eigenstrain reconstruction^[34,35] let further actions to be taken for the reconstruction of full-field distribution of residual stresses in surface-treated arbitrary geometries.^[39] Another iterative approach was presented to reconstruct full-field residual stresses formed due to the shot peening process using numerical experiments.^[40] The idea of combining evolutionary algorithms with eigenstrain theory in a fuzzy finite element model^[36] was also applied to a more complex weld form for full-field reconstruction of residual stresses.^[41] However, all these attempts and previously developed full-field eigenstrain reconstruction methods are based on guiding information about the distribution of eigenstrains that aims to find a match between resulting and experimentally determined deformations or residual stresses.

The direct problem of eigenstrain involves formulating distribution of eigenstrains within a domain and calculating residual stresses using numerical methods. Different approaches have been developed to deal with complex nature of the inverse problem of eigenstrain.^[9,15,42–45] These approaches rely on polynomial basis functions for determining the distribution of eigenstrains in a domain based on the principle of superposition. Although Stone–Weierstrass theorem^[46] states that the uniform approximation of real valued functions could be achieved using polynomials in compact intervals, Runge’s phenomenon^[47] shows that high-order polynomial interpolations have nonconvergence problem. This phenomenon could be coped using various strategies summarized by Boyd.^[48] However, these strategies provide an additional computational burden on the approximation calculations and are not preferred in the formulation of the inverse problem of eigenstrain which is already a computationally intensive process. Accordingly, approximation techniques used for solving the inverse problem of eigenstrain are limited to low-order polynomial basis functions^[49] combined with regularization functions that provides a rough estimation of the distribution of eigenstrains and corresponding residual stresses.

The eigenstrain reconstruction process enhances solutions of the contour method, using the data obtained by optical or contact profilometry techniques, into 3D domain.^[15] However, this approach depends on out-of-plane displacements corresponding

to elastic strain relief before noncontact cutting, which is performed using the electric discharge machining (EDM) technique, from various diagonal, horizontal, and vertical planes. To overcome this limitation and minimize the number of cuts to a single horizontal or vertical cut, iterative methods^[34] and artificial intelligence based methods^[36,50] were developed by defining a locked ratio between long-transversal and longitudinal components of eigenstrain formed in a welded plate. This approach allowed accurate determination of weld distortions as well as welding residual stresses using a single component of experimental data in terms of the out-of-plane displacements. Although formulation of eigenstrain reconstruction process allows the determination of all components of eigenstrains and residual stresses, difficulties on the collection of experimental data for all components of deformation limit the reliability of eigenstrain reconstruction. Previous attempts at solving the inverse problem of eigenstrain other than the study Uzun et al.^[35] do not present any example of the application of more than one component of experimental data. Even though authors^[35] only included the short-transverse component of displacements, which is formed as a result of weld distortion, in addition to EDM cut surface profilometry data, that approach allowed the reliable determination of the long-transverse component of residual stresses. Accordingly, it can be stated that improvements can be achieved in the calculations of eigenstrain reconstruction process by the inclusion of additional components of deformations.

A reliable simulation of discontinuous mechanical processes such as additive manufacturing and welding needs to model thermal effects on elastic–plastic deformation using anisotropic temperature-dependent material properties determined for each specimen accurately. However, defining a material with its microproperties for understanding macroscale behaviors is an impossible challenge, and elastoplastic continuum models based on the assumption of isotropy provide only a rough estimation of deformations. Eigenstrain theory provides a shortcut that eliminates the need for elastoplastic material model formulations^[51] and anisotropic temperature-dependent material properties to determine deformations specific to any material by calculating only the elastic response that is needed to create deformations experimentally measured by diffraction or profilometry techniques. However, current examples of eigenstrain reconstruction are based on assumptions about the distribution of eigenstrain fields that are defined by regularization functions. Jun et al.^[21,52] and Song et al.^[53] previously presented formulations for solving the inverse problem of eigenstrain using “pulse” and “tent” functions as local basis functions but these solutions are based on the assumption that eigenstrain distributions are identical or continuously reflecting the processing conditions of samples and necessitate the use of regularization functions.

In this study, a new formulation of the inverse problem of eigenstrain, which uses radial basis functions as local basis functions for the reconstruction of residual stresses, eliminates the need for regularization functions, and is independent of simplifying assumptions about processing conditions, is introduced. This voxel-based eigenstrain reconstruction method simplifies the formulation, does not depend on the shape of elements used for the discretization of the domain, and can be applied to 0D, 2D, and 3D cases. The proposed method eliminated the need for regularization functions and made it possible to predict the

distribution of eigenstrain fields accurately using radial basis functions that are positioned corresponding to voxels of a grid. This novel approach made the solution of the inverse problem of eigenstrain directly related to the density of the voxel grid and the quality of experimental data. The reliability of the proposed model was validated and analyzed using numerical experiments by solving the inverse problem of eigenstrain in continuously and discontinuously processed domains. Numerical experiments provided information about the model accuracy in terms of the determination of full-field 3D distribution of residual stresses using experimental data from a limited section of the domain.

2. Methodology

Permanent plastic strains are formed in bodies as a result of various loading conditions. These strains do not cause formation of residual stresses in the material after the removal of external effects if their distribution is uniform. On the other hand, in the case of nonuniform distribution of permanent plastic strains, a balanced state of internal forces is formed. The term eigenstrain is used to define these nonuniformly distributed permanent strains that create residual stress fields.

The relation between residual stress and eigenstrain creates a link between theoretical and numerical modeling frameworks and experimental data. This link provides new insights into the field of residual stress evaluation.^[1] The formulation of this linear elastic relation for isotropic materials includes six components of stress, strain, and eigenstrains. The calculation of eigenstrain fields from experimental data necessitates determining eigenstrains up to six components. The proposed voxel-based eigenstrain reconstruction method is designed in a way to satisfy this requirement. This method uses radial basis functions that are centered to voxel points in the mesh grid and distributed in the whole domain.

The determination of eigenstrains up to six components necessitates the solution of a high number of radial basis functions for each component independently. To deal with this computationally challenging task, a hybrid high throughput computing (HTC) approach for the voxel-based eigenstrain reconstruction method is presented to allocate the computational burden of independent solutions of radial basis functions among shared memory processor (SMP) nodes parallelly without scaling limits.

2.1. Voxel-Based Eigenstrain Reconstruction

The direct problem of eigenstrain allows the calculation of residual stresses from nonuniformly distributed inelastic strains in a solid body. This problem can be formulated as a linear elastic problem and solved using any partial differential equation (PDE) solver. The direct problem of eigenstrain has been formulated for isotropic bodies^[15] using components of eigenstrain tensor given in Equation (1). This formulation has been based on the principle of superposition and can be solved as inverse problem of eigenstrain,^[17] but this solution requires regularization functions for the selected components of eigenstrain using presumptions about eigenstrain field^[36] or simplifying assumptions.^[15] The proposed novel formulation based on radial basis functions is eliminating the need for regularization functions and further simplifications.

$$\varepsilon^* = \begin{bmatrix} \varepsilon_{11}^* & \varepsilon_{12}^* & \varepsilon_{13}^* \\ \varepsilon_{21}^* & \varepsilon_{22}^* & \varepsilon_{23}^* \\ \varepsilon_{31}^* & \varepsilon_{32}^* & \varepsilon_{33}^* \end{bmatrix} \quad (1)$$

Anisotropic behavior of metallic materials arises at nanoscale due to their crystal structure, but nonequiaxed morphology and preferred orientation of groups of crystals lead to anisotropic mechanical and thermal properties manifested at the macro- and mesoscale. On the other hand, the symmetry condition of isotropic materials reduces the number of independent components of the elastic tensor to Young's modulus, E , and Poisson's ratio, ν . The number of components of stress, σ_{ij} , total strain, ε_{ij}^t , and eigenstrain, ε_{ij}^* , is also reduced to six from nine due to symmetry. Eventually, Hooke's law defines the consecutive relation between six components of stress and elastic strain, which is defined in terms of total strain and eigenstrain, for isotropic materials in Equation (2).

$$\begin{bmatrix} \varepsilon_{11}^t - \varepsilon_{11}^* \\ \varepsilon_{22}^t - \varepsilon_{22}^* \\ \varepsilon_{33}^t - \varepsilon_{33}^* \\ \varepsilon_{23}^t - \varepsilon_{23}^* \\ \varepsilon_{13}^t - \varepsilon_{13}^* \\ \varepsilon_{12}^t - \varepsilon_{12}^* \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \quad (2)$$

The principle of superposition allows the determination of the distribution of separate components of eigenstrain as a sum of the linear combination of radial basis functions in a 3D domain using Equation (3) – (8).

$$\varepsilon_{11}^*(x, y, z) = \sum_{l=1}^n A_l^{11} F_l(x, y, z) \quad (3)$$

$$\varepsilon_{22}^*(x, y, z) = \sum_{l=1}^n A_l^{22} F_l(x, y, z) \quad (4)$$

$$\varepsilon_{33}^*(x, y, z) = \sum_{l=1}^n A_l^{33} F_l(x, y, z) \quad (5)$$

$$\varepsilon_{23}^*(x, y, z) = \sum_{l=1}^n A_l^{23} F_l(x, y, z) \quad (6)$$

$$\varepsilon_{13}^*(x, y, z) = \sum_{l=1}^n A_l^{13} F_l(x, y, z) \quad (7)$$

$$\varepsilon_{12}^*(x, y, z) = \sum_{l=1}^n A_l^{12} F_l(x, y, z) \quad (8)$$

where n is the total number of voxel points, where the center of radial basis functions is located; A_l^{11} , A_l^{22} , A_l^{33} , A_l^{23} , A_l^{13} , and A_l^{12} are the unknown coefficients of radial basis function, which are defined for normal and shear components of eigenstrain, respectively, corresponding to l^{th} voxel point. The radial basis function given in Equation (9) defines the distribution of independent eigenstrain components around x_l , y_l , and z_l coordinates depending on the radial distance variable, r , that specifies the localization range of the eigenstrain field. This is a modified radial basis

function based on the Euclidian distance between the field point (x, y, z) and the voxel point, the center of radial basis function, in 3D space

$$F_l(x, y, z) = \left(\frac{1}{1 + \sqrt{(x - x_l)^2 + (y - y_l)^2 + (z - z_l)^2}} \right)^r \quad (9)$$

Experimental data used for solving the inverse problem of eigenstrain may be any deformation corresponding to the elastic response of the material. Based on the linear relation between the independent solutions of radial basis functions and experimental data, the inverse problem of eigenstrain can be solved using Equation (10) where $\{u\}$ denotes the vector of experimental data, $[C]$ denotes the model-derived matrix of output data, and $\{A\}$ denotes the unknown coefficients of radial basis functions.

$$\{u\} = [C]\{A\} \quad (10)$$

The vector $\{u\}$, given in Equation (11), is composed of information about all components of experimental data collected from m number of measurement points.

$$\{u\} = \left[u_1^{11}, \dots, u_m^{11}, u_1^{22}, \dots, u_m^{22}, u_1^{33}, \dots, u_m^{33}, u_1^{23}, \dots, u_m^{23}, u_1^{13}, \dots, u_m^{13}, u_1^{12}, \dots, u_m^{12} \right]^T \quad (11)$$

Equation (12) defines the model-derived matrix of output data which is composed of $6m$ number of rows and $6n$ number of columns. Each row represents the output data from separate solutions of the radial basis functions that defines distribution of a single normal component of eigenstrain centered to the corresponding voxel point.

$$[C] = \begin{bmatrix} C_{1,1}^{11} & \dots & C_{1,n}^{11} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ C_{m,1}^{11} & \dots & C_{m,n}^{11} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & C_{1,1}^{22} & \dots & C_{1,n}^{22} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & C_{m,1}^{22} & \dots & C_{m,n}^{22} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & C_{1,1}^{33} & \dots & C_{1,n}^{33} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & C_{m,1}^{33} & \dots & C_{m,n}^{33} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & C_{1,1}^{23} & \dots & C_{1,n}^{23} & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & C_{m,1}^{23} & \dots & C_{m,n}^{23} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & C_{1,1}^{13} & \dots & C_{1,n}^{13} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & C_{m,1}^{13} & \dots & C_{m,n}^{13} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & C_{1,1}^{12} & \dots & C_{1,n}^{12} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & C_{m,1}^{12} & \dots & C_{m,n}^{12} \end{bmatrix} \quad (12)$$

The vector of unknown coefficients corresponding to the separate solutions of radial basis functions is given in Equation (13).

$$\{A\} = \left[A_1^{11}, \dots, A_n^{11}, A_1^{22}, \dots, A_n^{22}, A_1^{33}, \dots, A_n^{33}, A_1^{23}, \dots, A_n^{23}, A_1^{13}, \dots, A_n^{13}, A_1^{12}, \dots, A_n^{12} \right]^T \quad (13)$$

The coefficients in the vector $\{A\}$ are calculated using the least squares fitting method formulated in Equation (14) which minimizes the error between model-derived and experimental datasets.

$$\{A\} = ([C]^T[C])^{-1}[C]^T\{u\} \quad (14)$$

The formulation of the inverse problem of eigenstrain allows calculations of six components of residual stress from

six components of experimental data. However, the experimental data may include a limited number of components of displacement. In such cases, components of missing experimental data and corresponding outputs of solutions of radial basis functions should be ignored. This formulation allows the reconstruction of all components of residual stress using limited number of components of experimental data, but the reliability of calculations depends on the availability and quality of all components of experimental data.

2.2. Scalability

The inverse problem of eigenstrain creates a shortcut to process modeling by solving linear elastic numerical models with small forces to reconstruct residual stresses based on

the principle of superposition. The PDEs obtained by applying Newton's laws to calculate deformations formed in a fixed domain as a result of small forces are solved using numerical methods. Finite element is the preferred method for solving PDEs of linear elasticity models because of its ability to approximate complex shaped domains accurately by high-order polynomial functions or fine mesh distributions. Commercial finite element packages provide graphical user interfaces, computer-aided design (CAD) environments, and meshing tools that allow performing simulations for product design smoothly. However, their license costs and proprietary software structure create limitations on research purposes. In the case of solving solid mechanics problems using the eigenstrain theory, eigenstrains can be imported into finite element model as pseudothermal strains in such packages, but that creates an additional computational burden of thermal elasticity, necessitates the formulation of additional coefficients, and prevents obtaining necessary outputs like distribution of separate components of eigenstrain natively. Another limitation of commercial finite element packages is their limited options for parallelization. They provide effective solutions in their optimized form, but users are not allowed to change the structure of the parallel computation. On the other hand, FEniCS computing platform provides a collection of free and open-source software components that allow the automated solution of PDEs using C++ and Python interfaces. This flexible interface also allows the implementation of eigenstrain components natively as it is formulated in Equation (2) and the use of alternative parallelization options like HTC to perform the independent linear elastic calculations of basis functions in parallel for solving the inverse problem of eigenstrain.

High-performance computing (HPC) allows parallelization of solutions at large-scale domains across a high number of nodes but the scalability of HPC is a challenge for most problems. As Amdahl^[54] stated that scale of problems is limited by the fraction of nonparallelizable computational load and Gustafson's numerical experiments^[55] showed that unless the scale of a problem is increased with the increasing number of resources, HPC has a bottleneck of reaching a steady state of performance improvement related to increasing number of computational units. On the other hand, HTC is used to perform independent one-way solutions whose outputs are not returned to the calculations. This characteristic of HTC makes it a remarkable alternative to HPC for solving the independent and lightweight linear elastic finite element models of radial basis functions of the inverse problem of eigenstrain in parallel despite running them sequentially as illustrated in Figure 1.

3. Experimental Validation of the FEniCS Eigenstrain Model

This study presents the first attempt to solve the direct and inverse problems of eigenstrain using the FEniCS computing platform. The model scripts written for converting the formulation of the eigenstrain model into a finite element code are sole and they need to be validated using experimental data whose reliability in the quantification of residual elastic strains was accepted to be reliable by scientists. Accordingly, the inverse

problem of eigenstrain was solved for the reconstruction of residual elastic strains in a titanium bent bar using residual elastic strains quantified experimentally employing pointwise high-energy synchrotron X-ray diffraction measurements.

Ti64 specimen with dimensions of $50.0 \times 8.5 \times 4.0$ mm was bent using 100 kN load by setting 4 pins as illustrated in Figure 2. The distance between the outer pins is set to be 20 mm while the inner pins are located 10 mm apart from each other. The applied bending moment caused plastic deformations at the outer edges of the specimen along the y -axis. In this study, the section between the bottom pins, which covers 20 mm in length, is assumed to have a bent form similar to the bending over a cylinder, and accordingly, the distribution of plastic strains along the longitudinal x -axis of the specimen is assumed to be continuous within this bound.

Residual elastic strains formed after removing the loads were quantified along the vertical line that is parallel to the y -axis and passes from the origin, as illustrated in Figure 2b, between the top and bottom edges of the specimen with a step size of 0.25 mm. The voxel-based eigenstrain reconstruction of residual elastic strains was accomplished in the section between the bottom pins that is assumed to accommodate continuously distributed plastic deformations. Accordingly, residual elastic strains within this section are assumed to distribute continuously along the longitudinal x -axis and the short-transverse z -axis. The finite element mesh composed of tetrahedral elements was created using a single layer of cells with dimensions of $0.5 \times 0.5 \times 0.5$ mm to form a rectangular domain with dimensions of $20.0 \times 8.5 \times 0.5$ mm.

Line distribution of reconstructed residual elastic strains has an excellent agreement with the residual elastic strains quantified using the high-energy synchrotron X-ray diffraction measurements as illustrated in Figure 2a. The formation of residual elastic strains depends on the availability of the irregular distribution of permanent plastic strains. In the case of bending problem, the irregularity is expected to be formed due to availability of three different regions of permanent plastic strains that are tensile, compressive, and null. Results of distribution of eigenstrains clearly present these three regions and consistent with the boundaries of expected irregularities. Accordingly, the line distribution of residual elastic strains and eigenstrains has good agreement with the expected distributions formed in bars^[13] that are bent into a developable surface^[56] and it can be concluded that the voxel-based FEniCS eigenstrain model for solving the inverse problem of eigenstrain has been formulated and coded correctly.

4. Numerical Experiments

The voxel-based eigenstrain reconstruction method was analyzed using numerical experiments. Solutions to all numerical experiments were performed using FEniCS eigenstrain model in a rectangular domain that was meshed using second-order Lagrange family Continuous Galerkin element by dividing the domain into 2304 rectangular cells that accommodate 13 824 tetrahedral elements and 21 609° of freedom points. The center of the base of the domain is located at the origin that covers a plane that extends from -9.6 to 9.6 mm in x -dimension and

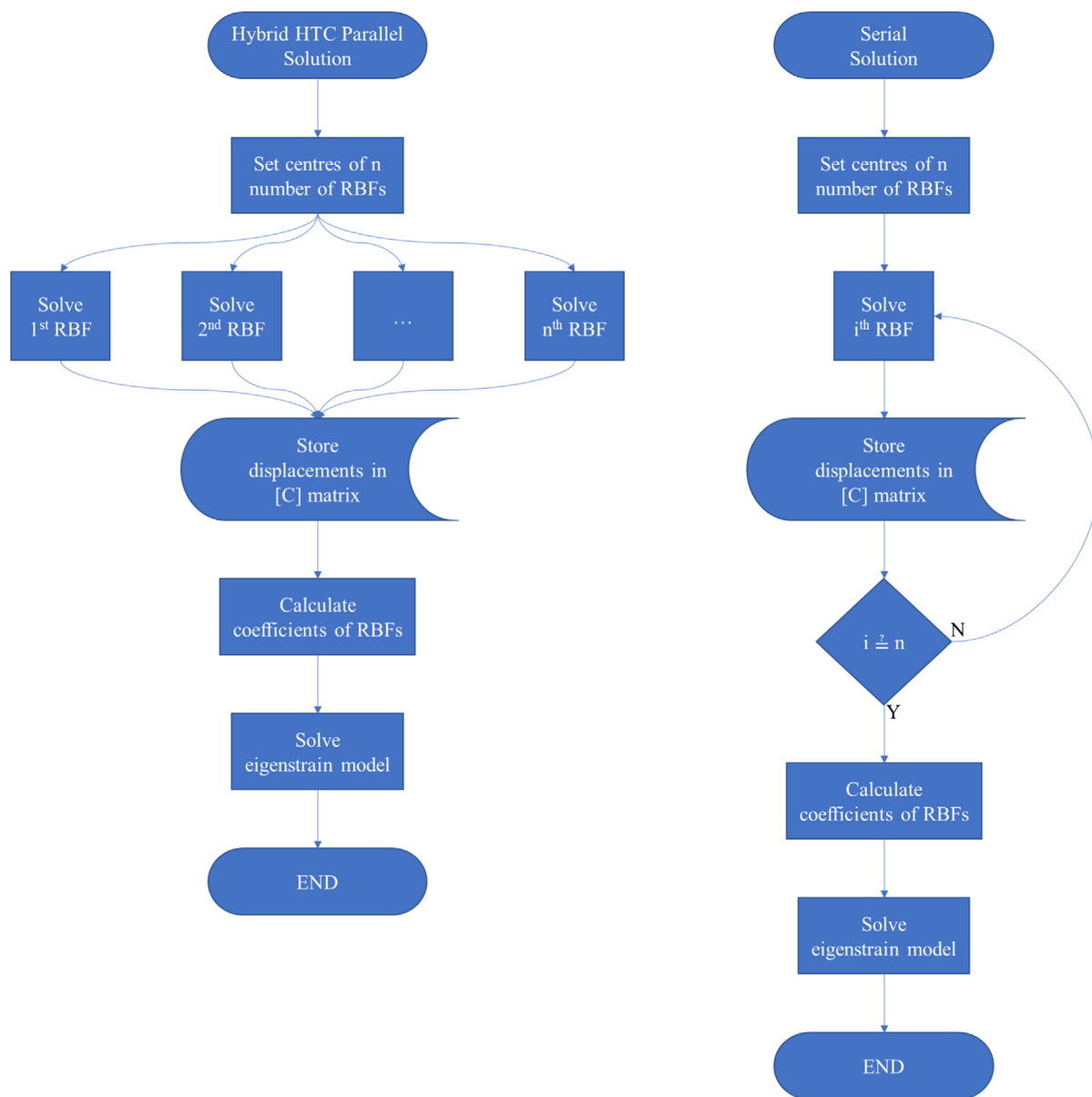


Figure 1. Flowcharts of hybrid HTC parallel and sequential solutions of the inverse problem of eigenstrain.

from -10.0 to 10.0 mm in y -dimension while its height extends from 0.0 to 3.2 mm in z -dimension as illustrated in **Figure 3**. The moduli of elasticity and Poisson's ratio for this body are set as 200 GPa and 0.3 , respectively, in all numerical experiments.

Numerical experiments start by solving the direct problem of eigenstrain using a predetermined eigenstrain field imported to the model domain. Dirichlet boundary conditions that prevent movement of the z -plane were applied as the z -component of the degree of freedom to the point at node $(0.0, 0.0, 0.0)$ in all stages of numerical experiments. This allows the deformation of the body freely as a result of relaxation of planar residual stresses, which were accommodated while the body is fixed to the substrate, to simulate the state of the body after being

removed from the substrate. Three components of displacement from the top surface of the domain, where the z -coordinate is 3.2 mm, were recorded after solving the direct problem of eigenstrain for being used as an input for solving the inverse problem of eigenstrain.

The voxel-based eigenstrain reconstruction process in the presented numerical experiments accomplishes the determination of eigenstrain state that causes already recorded three components of displacements. Then it becomes possible to calculate deformations and residual stresses within the 3D domain. Two different numerical experiments that mimic the conditions of continuously and discontinuously processed bodies were designed to investigate accuracy in solving the inverse problem of eigenstrain. The quality of reconstructions of residual stresses

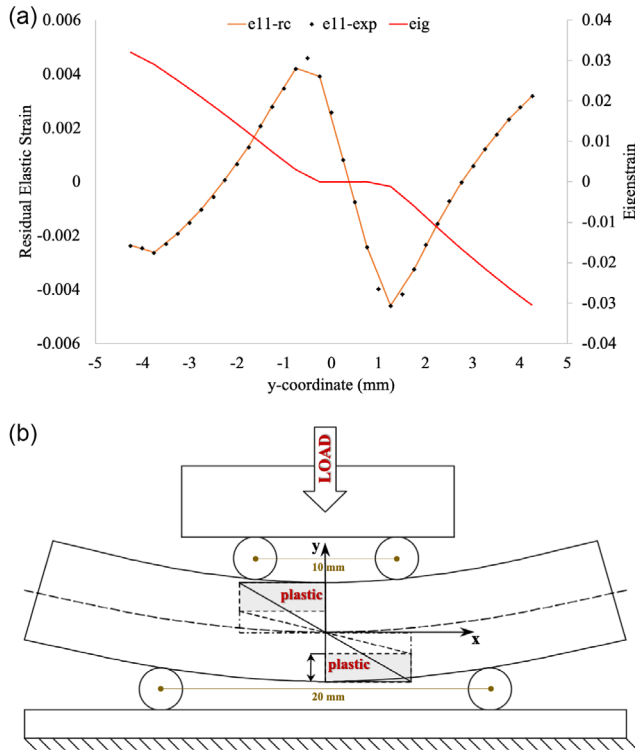


Figure 2. a) The illustration of four-point bending of the Ti64 specimen and b) the distribution of experimentally determined and reconstructed residual elastic strains along with eigenstrains through the long-transverse y-axis.

corresponding to two conditions was analyzed for presenting as 3D surface and transparent in-volume illustration and line plots along the red diagonal paths given in Figure 3. Line plots present a comparison of all components of eigenstrain, residual stress, and displacement that belong to the solutions of direct and inverse problems of eigenstrain and error analysis w.r.t. the domain height that varies along the z-dimension. Finally, the hybrid HTC parallelization performance of the FEniCS eigenstrain model was analyzed.

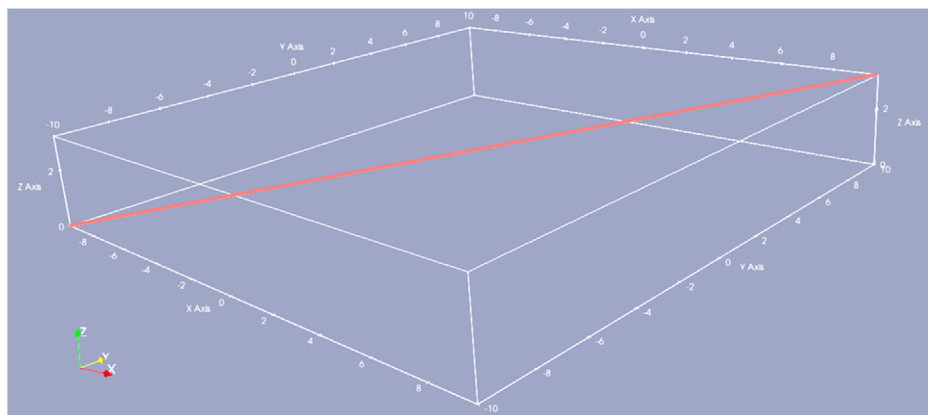


Figure 3. The illustration of the domain with dimensions of 19.2 × 20.0 × 3.2 mm. Red diagonal line represents the path used for line plots.

4.1. Continuously Processed Body

The analysis of voxel-based full-field eigenstrain reconstruction in continuously processed bodies is composed of two stages. The first stage is solving the direct problem of eigenstrain problem using normal components of eigenstrain defined by Equation (15), where d_x and d_y are the size of the domain in x - and y -axes, respectively, and k_{ij} is a constant which was set for 11, 22, and 33 components of eigenstrain as 0.96×10^{-3} , 1.0×10^{-3} , and 0.32×10^{-3} , respectively. The second stage is solving the inverse problem of eigenstrain using the outputs of direct problem of eigenstrain.

Equation (15) defines the distribution of eigenstrain in xy -plane without variation in z -dimension that satisfies the continuous processing condition. The output of the direct problem of eigenstrain, which is used for solving the inverse problem of eigenstrain, is the displacement data with its triaxial x -, y -, and z -components from the top surface of the domain where z -coordinate is 3.2 mm.

$$\epsilon_{ij}^*(x, y) = k_{ij} \exp\left(-\frac{4x}{d_x}\right)^2 \exp\left(-\frac{4y}{d_y}\right)^2 \quad (15)$$

The top surface distribution of magnitude of eigenstrain, residual stress, and displacement in continuously processed body after solving the direct problem of eigenstrain is given in Figure 4. Magnitudes are determined as the sum of absolute values of all components of eigenstrain, residual stress, and displacement. These illustrations show an excellent match of the 3D distribution of reconstructed values with the results of direct solution. Given magnitudes include the sum of shear components as well as normal ones while the input data, displacement, from the calculations of the direct problem correspond only to the normal components. Accordingly, the results present reliability of the model in the case of calculation of six components of deformations and residual stress using only three components of experimental data.

Variations of separate components of eigenstrain, residual stress, and displacement between the results of direct and inverse solutions were analyzed using line plots along the red diagonal path that covers whole thickness. Results given in

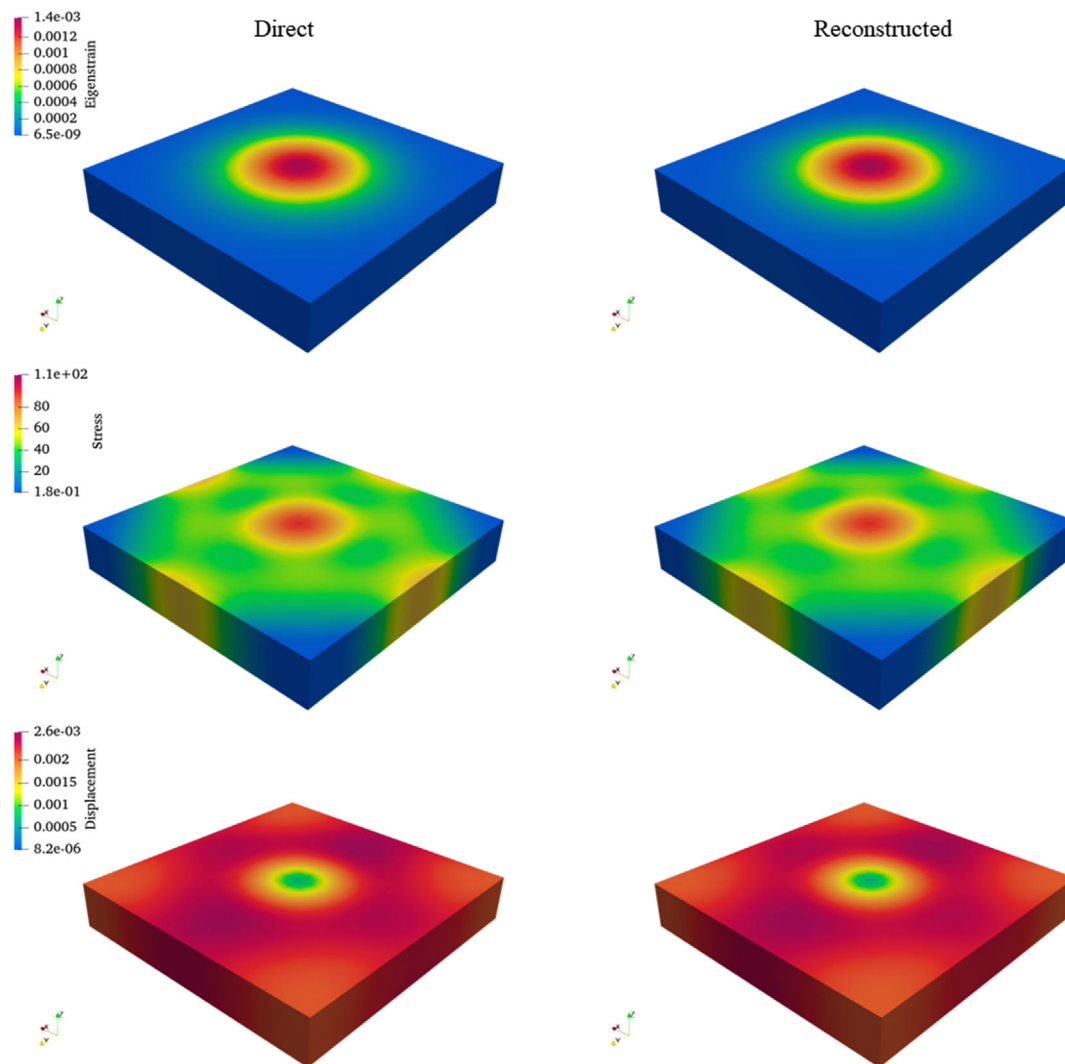


Figure 4. Distribution of eigenstrain, residual stress (MPa), and displacement (mm) after solving the inverse problem of eigenstrain in continuously processed body.

Figure 5 show an excellent agreement between direct solution and reconstruction calculations. The root mean squared error (RMSE) between direct solution and reconstruction calculations up to the corresponding depth was also calculated using magnitudes of eigenstrain, residual stress, and displacement separately. Results show that magnitudes after eigenstrain reconstruction have an excellent agreement with magnitudes of direct solution and RMSE does not have a significant change with increasing depth and remains close to zero at all levels of domain depth starting from the top surface.

It should be mentioned that this model uses radial basis functions that are distributed in the whole domain and the result of the solution of the inverse problem of eigenstrain is based on the weighted sum of the solution of independent radial basis functions without using any guiding information about the distribution of eigenstrains in a continuously processed body. Accordingly, it can be stated that the proposed voxel-based

eigenstrain reconstruction is highly successful in the determination of the exact distribution eigenstrains without regularization functions. This is the most important achievement of the voxel-based eigenstrain reconstruction method when compared to previous attempts on full-field eigenstrain reconstruction mentioned in the introduction section.

4.2. Discontinuously Processed Body

The analysis of voxel-based eigenstrain reconstruction was repeated in a discontinuously processed body by solving the inverse problem of eigenstrain using the outputs of the direct problem of eigenstrain corresponding to the eigenstrain field defined by Equation (16). In this equation, d_z is the size of the domain in z-dimension and k_{ij} was set for 11, 22, and 33 components of eigenstrain as 0.96×10^{-3} , 1.0×10^{-3} , and

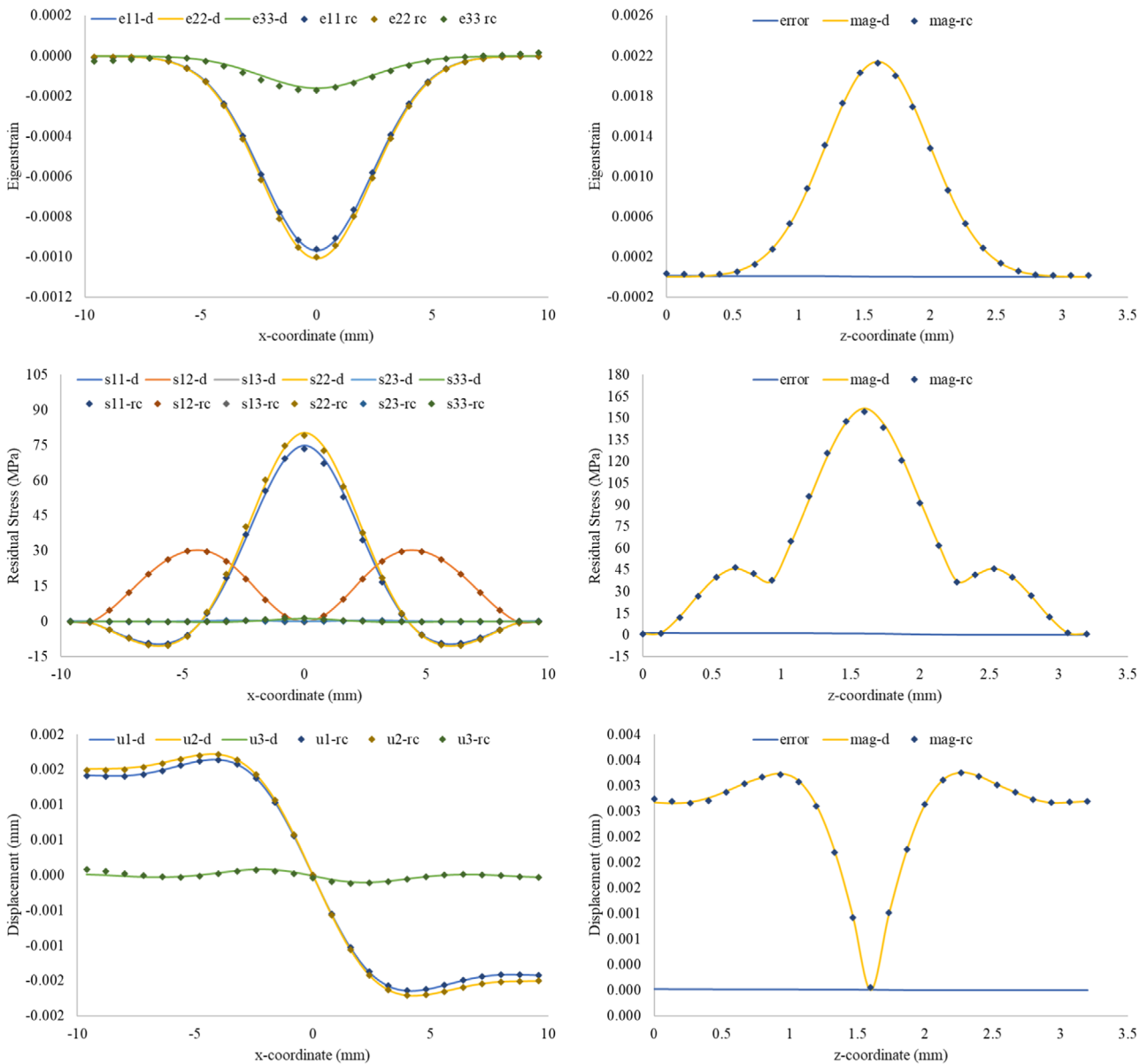


Figure 5. Line plots of separate components of eigenstrain, residual stress, and displacement along the diagonal red path, given in Figure 3, and error analysis after solving the inverse problem of eigenstrain in continuously processed body using the magnitudes of sum of all components.

0.32×10^{-3} , respectively. Equation (16) defines an eigenstrain field that varies in all three coordinate axes and provides an example of a discontinuously processed body. Similar to the first numerical experiment, triaxial displacement data from the top surface of the domain were collected as an output to be used for solving the inverse problem of eigenstrain.

$$\varepsilon_{ij}^*(x, y, z) = k_{ij} \exp\left(-\frac{4x}{d_x}\right)^2 \exp\left(-\frac{4y}{d_y}\right)^2 \sqrt{z/d_z} \quad (16)$$

Representations of surface and volume distributions of the magnitude of eigenstrain, residual stress, and displacement in the discontinuously processed body after solving the direct

problem of eigenstrain and eigenstrain reconstruction are given in Figure 6 and 7. Both surface and transparent in-volume illustrations show a highly accurate match of the 3D distribution of reconstruction calculations with the results of the direct solution. The slight deviation of reconstructed residual stresses from the results of the solution of the direct problem of eigenstrain is hardly visible but the magnitude of reconstructed eigenstrain and displacement has an excellent match with their direct solution counterparts. Residual stresses calculated after eigenstrain reconstruction process and direct solution have highly reliable match in the whole volume as illustrated in Figure 7.

Variation of separate components of eigenstrain, residual stress, and displacement between the results of direct and

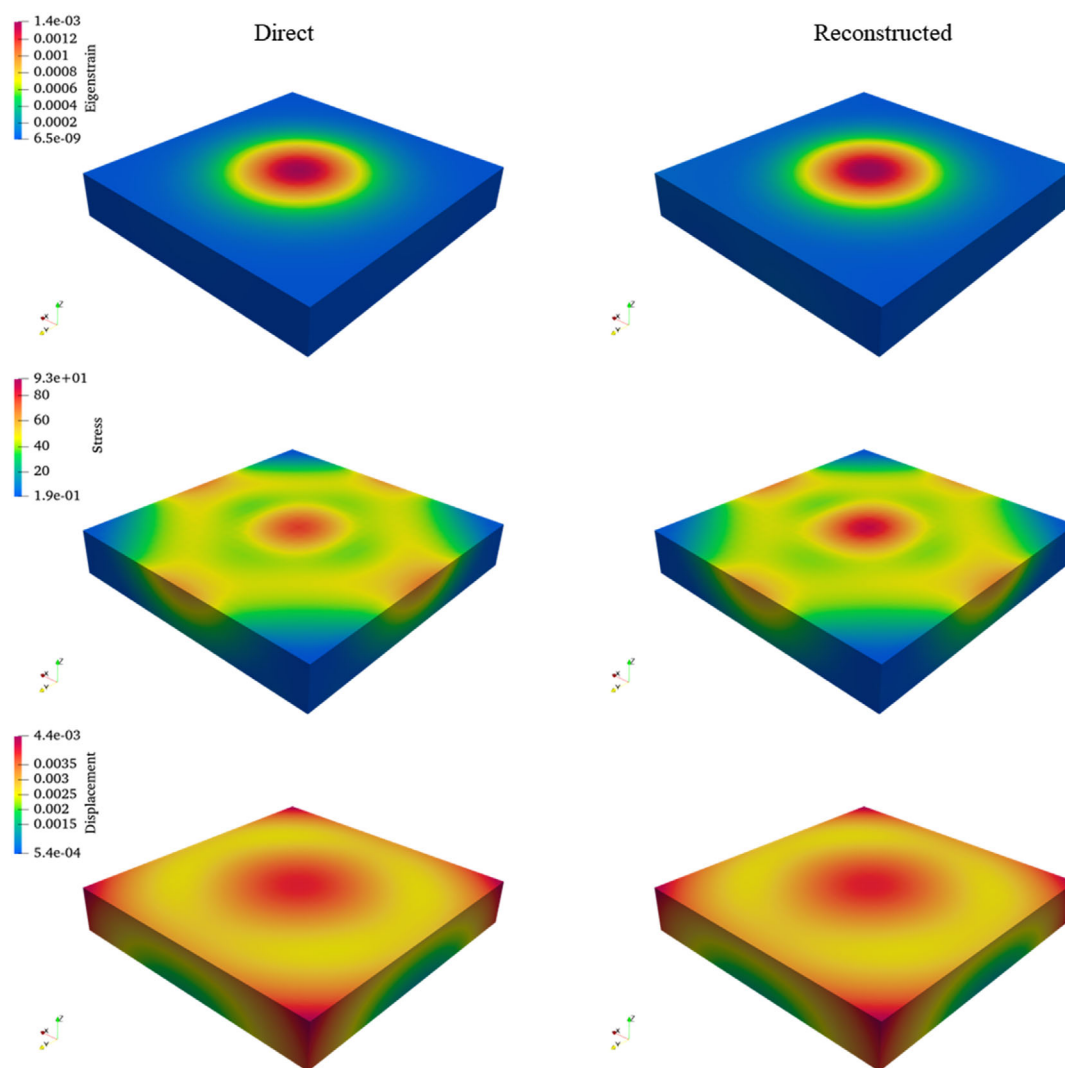


Figure 6. The distribution of eigenstrain, residual stress (MPa), and displacement (mm) after solving the inverse problem of eigenstrain in discontinuously processed body.

inverse solutions was analyzed using line plots along the diagonal red path. Results given in **Figure 8** show an excellent agreement of reconstructed displacements with direct solution calculations while eigenstrain and residual stress distributions have slight deviations. Deviation of z -component of eigenstrain occurs independent of the depth which is an important parameter because experimental data are imported to the top surface of the domain, where the z -coordinate is 3.2 mm, only and calculations get distant from the experimental data with increasing depth. Logical expectation for that is getting a higher error with increasing distance from the surface of experimental data but the result shows that the influence of distance from experimental data is negligible because magnitudes of reconstructed residual stress show that deviations are high at spots where magnitudes are relatively higher. Even though the z -component of eigenstrain has the lowest magnitude, it has a higher deviation when compared to the other components of eigenstrain. The reason for this is expected to be the relatively low number of layer of cells

and elements distributing along the z -dimension that prevents capturing details about the variation of eigenstrains through the thickness. Other components of eigenstrain presumably have the same problem but their high magnitude suppresses deviations related to the cell density through the thickness. In addition, shear components of reconstructed residual stress have an excellent match with direct solution calculations. This is an unexpected result because the inverse problem of eigenstrain is solved using radial basis functions that are defined for normal components of eigenstrains. The expectation in this condition is having a higher error of shear components while having a better match of normal components residual stress, but the result is opposite. This shows the capability of the proposed method on the determination of all components of residual stresses using a limited number of components of experimental data corresponding to measured deformations from a limited region.

The RMSE between direct solution and reconstruction calculations shows that the magnitude of reconstruction displacement

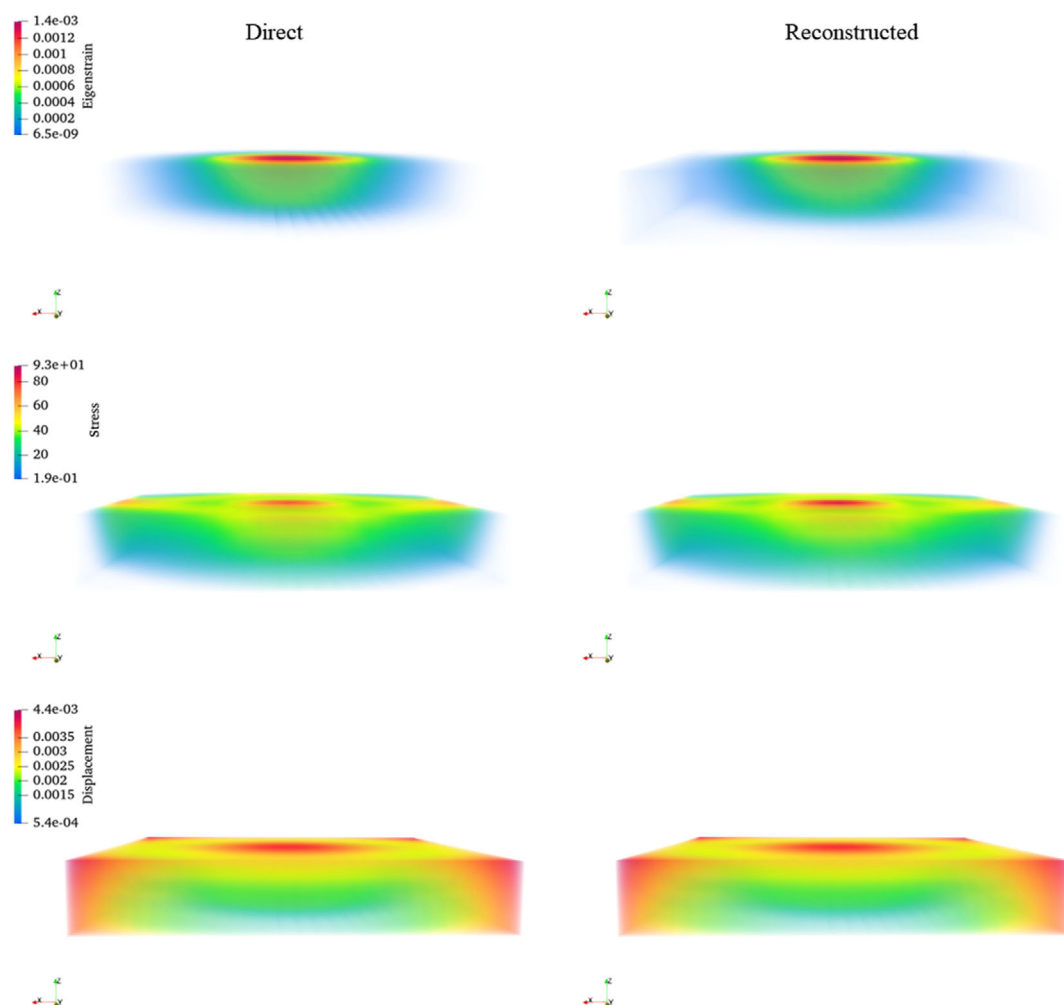


Figure 7. The illustration of 3D distribution of eigenstrain, residual stress (MPa), and displacement (mm) after solving the inverse problem of eigenstrain in discontinuously processed body.

has an excellent agreement with the direct solution in the full-field while eigenstrain and residual stress magnitudes have a slightly increasing trend of RMSE with increasing distance from the source of experimental data. Because errors of eigenstrain reconstruction in discontinuously processed bodies have a complex nature. The PDE solver has a high influence on the accuracy of residual stress calculations and the combined influence of parameters of finite element solver, such as cell density or element properties, and formulation of basis functions should be analyzed carefully for each problem. In this numerical experiment, the source of deviation of residual stresses is predicted to be the high deviation of the z -component of eigenstrain and its relationship with the cell and element density through the thickness and parameters of the finite element solution. Consequently, it can be stated that the voxel-based full-field eigenstrain reconstruction method is highly reliable on solving the inverse problem of eigenstrain in discontinuously processed bodies, but it necessitates the use of higher cell density and denser distribution of radial basis functions that will increase the computation cost and solution time. This study also aims

to show the limits that can be achieved by a single-node SMP unit using a reasonable number of radial basis functions.

Numerical experiments on the analysis of the voxel-based eigenstrain reconstruction method provided an ideal environment free from characteristic errors and uncertainties that eliminated the need to show agreement with residual stress measurements of independent techniques for the validation of the proposed method.^[57,58] The excellent agreement of the results of the continuously processed case shows that the model formulation is valid. On the other hand, errors appear in the discontinuous processed case are related to the complexity of eigenstrain distribution. Improvement in the discontinuously processed case is expected to be achieved by using higher density radial basis functions and corresponding higher mesh density. However, current solutions reached the limits of single node SMP workstation. Accordingly, in order to show that this claim is valid, the solution of the discontinuous case was repeated using a lower radial basis function density and less number of cells. In these solutions, all steps of the voxel-based eigenstrain reconstruction method were repeated using second-order

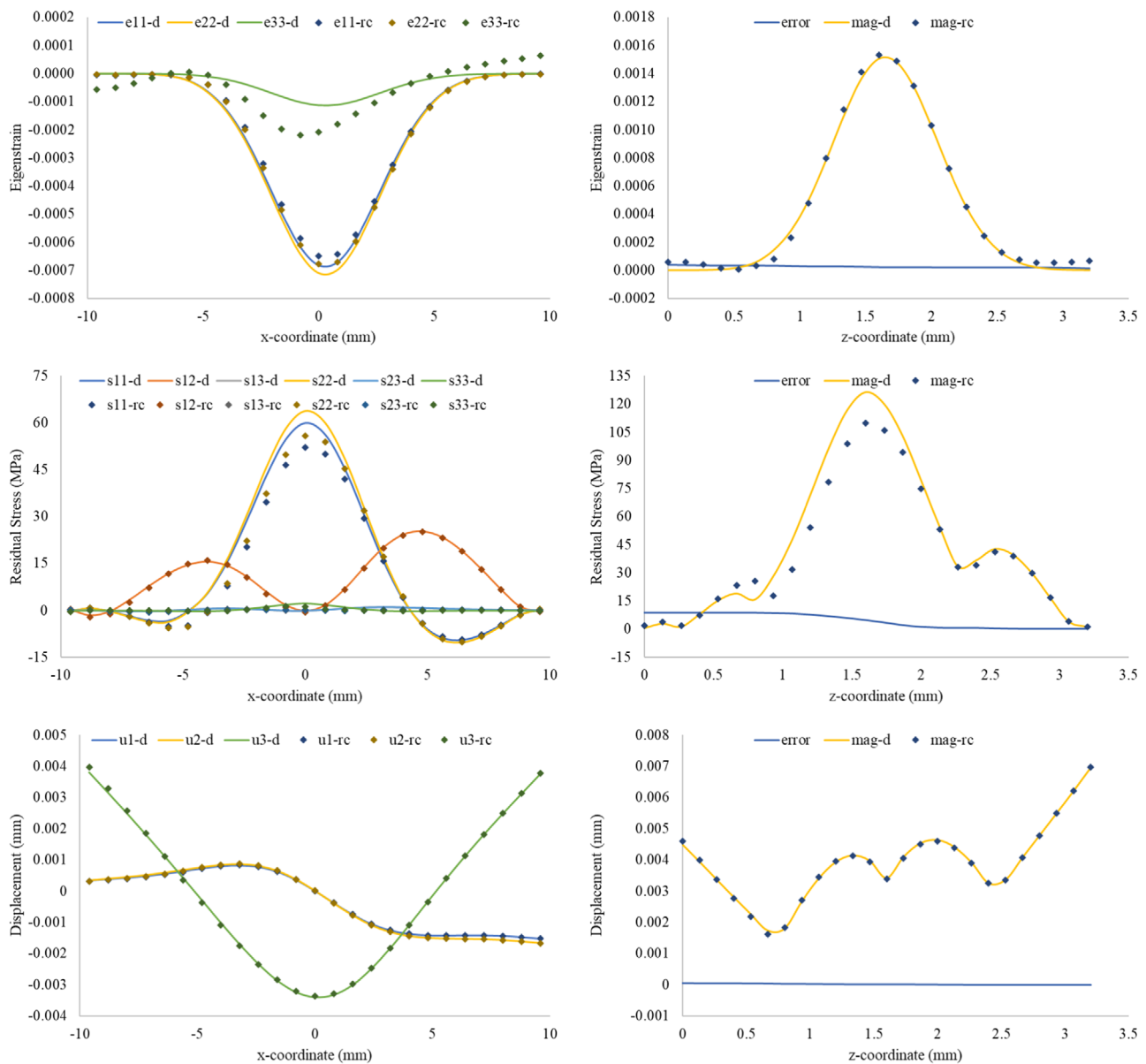


Figure 8. Line plots of separate components of eigenstrain, residual stress, and displacement along the diagonal red path, given in Figure 3, and error analysis after solving the inverse problem of eigenstrain in discontinuously processed body using the magnitudes of sum of all components.

Lagrange elements distributed in $12 \times 12 \times 2$, $18 \times 18 \times 3$, and $24 \times 24 \times 4$ number cells. Line plots given in Figure 9a show the increase of RMSE with increasing depth starting from the top surface where the z-dimension is 3.2 mm. Increasing the number of cells increases the number of tetrahedral elements and provides denser distribution of voxel points that are used for locating voxel points. The results of this test prove the statement on the inverse relationship between the number of basis functions and RMSE. Increasing the number of basis functions reduces the RMSE and increases the quality of the results of the solution of the inverse problem of eigenstrain. In addition, the lack of cell symmetry in the z-direction, when $18 \times 18 \times 3$ number of cells is used, causes an irregular variation of

RMSE with increasing depth, but this result is specific to this problem. Other problems can have different behaviors depending on their domain, eigenstrain distribution, and boundary conditions.

4.3. Hybrid HTC Parallelization

Hybrid HTC parallelization for solving the inverse problem of eigenstrain distributes the shared memory parallelization of independent solutions of radial basis functions across the nodes. The only input needed for independent solutions is the coordinate of voxel points where radial basis functions are centered. The outputs of these solutions are saved in the storage

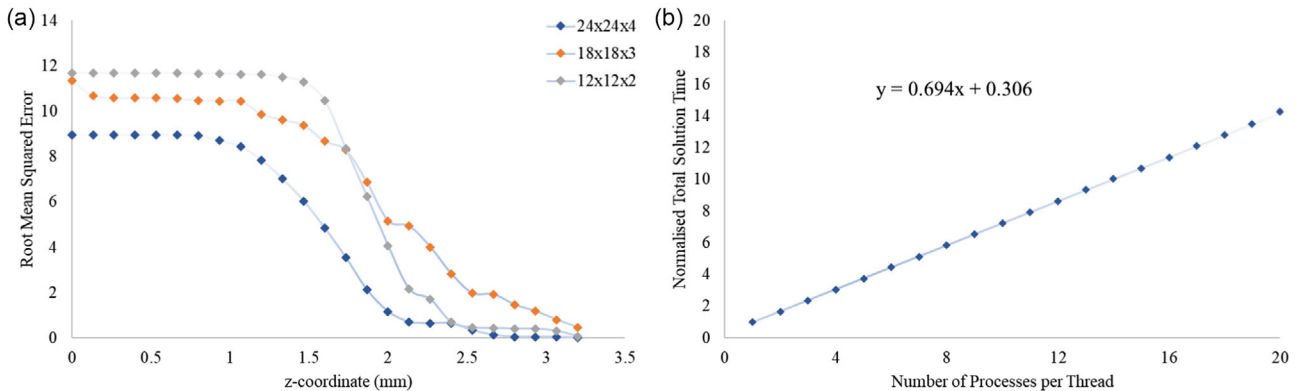


Figure 9. a) Relation between the density of radial basis functions and the variation of RMSE of magnitude of residual stress with increasing depth along the diagonal red path, given in Figure 3 and b) normalized total solution time w.r.t. the number of processes per thread.

system for the reconstruction process without transferring back to the memory. Consequently, the independent solutions of radial basis functions can be effectively parallelized across an unlimited number of nodes without scaling limits using this approach, but it is necessary to determine the optimum number of solutions per node due to the bottlenecks of SMPs.

In order to achieve effective hybrid HTC parallelization at any single SMP, a two-step analysis scheme is demonstrated by a numerical experiment using the FEniCS finite element model parameters. Tests were performed using a workstation with Intel Core i7-8700 CPU 3.2 GHz processor that has 6 physical and 12 logical cores. Prebuilt FEniCS Docker image, which was last modified on January 14, 2020, was used to run FEniCS on the Windows 10 platform for solving linear elastic finite element model of radial basis functions. In order to create a safe environment of calculations that dedicates the use of CPU threads for FEniCS eigenstrain model calculations only, the maximum number of active threads is limited to 5. Tests were repeated 5 times and benchmark results were averaged. The standard error of benchmarks is less than 0.1% of the average in all cases and accordingly, error related to the sampling distribution was ignored.

The main bottleneck of parallelization at an SMP is start-up overhead that causes the initial process to have a higher solution time when compared to subsequent solutions. Different number

of processes were sent to a single thread for observing the relation between bottleneck and the number of sequential processes. Solution times normalized to the solution time of a single process given in Figure 9b show a linear variation between solution time and the number of processes per thread where normalized solution time of a single process is 1. This linear relation shows that subsequent processes have a very similar solution time that is 30.6% faster than the first process. It can be concluded that the solution of the first basis function at a single thread is much slower than the subsequent solutions and accordingly the determination of the optimum number of processes per thread is crucial.

The solution time per process is investigated w.r.t. the number of sequential processes sent to a single thread to determine the optimum number of processes per thread. The plots given in Figure 10a show that solution time reaches a steady state after a certain number of processes. Accordingly, the optimum number of processes for the independent solution of basis functions is determined to be 10 for analyzing the efficiency of shared memory parallelization but it should be noted that the number of independent solutions can be much higher in a real case problem depending on the number of central processing unit (CPU) cores. Results in Figure 10b show that the efficiency of a single-node parallel process is higher than 80% when the number of active threads is 2 and decreases with each additional thread

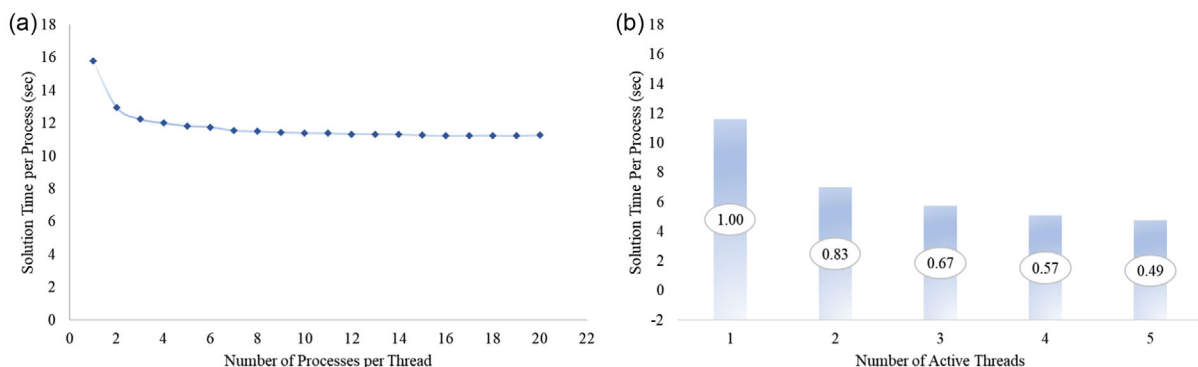


Figure 10. a) The solution time per process w.r.t. the number of processes per thread and b) the number of active threads in the SMP with efficiencies given per process on bars when the number of processes per thread is set to 10.

while solution time per process decreases continuously with a diminishing rate. As solution times are presented in terms of a single process while each thread deals with 10 sequential processes, the solution time per process is less than 5 s when 50 basis functions are solved in parallel using five active threads.

This demonstration does not provide any advice on the optimum number of processes to be sent to a single thread of an SMP for dealing with start-up overhead. Different problems with the different number of cells and elements can necessitate using different number of processes per thread in alternative SMPs. Accordingly, any SMP architecture needs to be tested independently for each problem in order to achieve optimum hybrid HTC parallelization performance.

5. Conclusions

The use of radial basis functions in the voxel grid for the full-field eigenstrain reconstruction of residual stresses eliminated the need for regularization functions, which depend on assumptions about the nonregular distribution of permanent plastic strains, reduced the complexity of the formulation of basis functions and created a new parameter, the resolution of the voxel-based domain, for solving the inverse eigenstrain problem. This new parameter has direct influence on the reliability of calculations of the proposed method. The resolution of the domain can be increased subjected to available computational power in order to reduce the error of reconstruction. In the case of availability of computation power for high-resolution domain, the reliability of reconstruction depends only on the quality of experimental data. This property of the proposed novel approach has tightened the link between modeling and experimentation in a practical way without dealing with computationally heavy formulations.

Results of numerical experiments in continuously and discontinuously processed bodies show that the combination of experimental data with linear elastic finite element models in a remarkable algorithm provides a great achievement in the prediction of the 3D distribution of residual stresses and their permanent plastic strain sources. Excellent reconstruction was achieved in the case of a continuously processed body and highly reliable results were obtained from the reconstruction in the case of a discontinuously processed body. As expected, the error in the case of a discontinuously processed body increases with the increasing distance from the top surface of the domain which is the source of experimental data used for the reconstruction. Although the appearance of a slight increase in error with increasing depth, transparent maps show that it is difficult to distinguish direct and reconstructed solutions while the reconstruction on the surface of experimental data are excellent.

The proposed full-field eigenstrain reconstruction method depends on the density of basis functions located at each voxel in the domain. This nature of the proposed reconstruction method allows for defining the density of basis functions as model resolution. The error analysis on the influence of model resolution showed that the error of reconstruction appearing in discontinuously processed bodies can be minimized by increasing the density of radial basis functions in return for higher computation cost. When sufficient model resolution is achieved, the model is expected to provide highly reliable results

in the case of the availability of error-free experimental data collected from the whole top plane of a rectangular body.

Residual elastic strains measured by nondestructive diffraction techniques or elastic responses corresponding to destructive processes determined by profilometry or microscopy techniques can also be used for full-field eigenstrain reconstruction of residual stresses but this time experimental errors will take place. Accordingly, it can be stated that as the excellence of this reconstruction method has been proven by the numerical experiments, the reliability of reconstruction based on the experimentally measured data will solely depend on the quality of measurements.

The results of numerical experiments show that the error of reconstruction increases with depth because the experimental data are located on the top surface and this influence can be minimized by increasing the model resolution. However, increasing the density of basis functions increases the computational cost. The volumetric distribution of error of reconstruction given in this study applies to problems with the same domain in the case of the availability of impeccable experimental data. As this study is not able to cover all other domains with different geometries and experimental measurement techniques, volumetric error estimation is advised to be done by numerical experiments in the domain geometry of the problem using the maximum achievable model resolution and revised according to the experimental method used for the collection of experimental data.

Acknowledgements

The authors acknowledge the support of EPSRC through grant EP/S005072/1 Strategic Partnership in Computational Science for Advanced Simulation and Modelling of Engineering Systems (ASiMoV).

Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Keywords

discontinuously processed bodies, eigenstrain, full-field reconstruction, high throughput computing, inherent strain, radial basis functions, residual stresses

Received: October 17, 2022
Revised: January 30, 2023
Published online: April 14, 2023

- [1] A. M. Korsunsky, *A Teaching Essay on Residual Stresses and Eigenstrains*, Butterworth-Heinemann, Oxford, UK **2017**.
- [2] T. Mura, *Micromechanics of Defects in Solids*, Springer Netherlands, Dordrecht **1987**.
- [3] T. Fujimoto, *J. Jpn. Weld. Soc.* **1970**, 39, 26.
- [4] Y. Ueda, K. Fukuda, K. Nakanaga, S. Endo, *Trans. Soc. Nav. Archit. Jpn.* **1975**, 1975, 499.

- [5] Y. Ueda, H. Murakawa, N. Ma, *Welding Deformation and Residual Stress Prevention*, Butterworth-Heinemann, United States of America **2012**, p. 35.
- [6] J. M. Lee, H. D. Seo, H. Chung, *J. Mater. Process. Technol.* **2018**, 256, 36.
- [7] J. Wang, X. Yin, H. Murakawa, *J. Mater. Process. Technol.* **2013**, 213, 1447.
- [8] Y. X. Wang, P. Zhang, Z. G. Hou, C. Z. Li, *J. Mech.* **2008**, 24, 301.
- [9] M. R. Hill, D. V. Nelson, *Am. Soc. Mech. Eng. Press. Vessels Pip. Div. PVP* **1995**, 318, 343.
- [10] I. Setien, M. Chiumenti, S. van der Veen, M. San Sebastian, F. Garcandía, A. Echeverría, *Comput. Math. Appl.* **2019**, 78, 2282.
- [11] H. S. Park, H. S. Shin, N. H. Tran, *Int. J. Adv. Manuf. Technol.* **2022**, 121, 6507.
- [12] M. Bugatti, Q. Semeraro, *Addit. Manuf.* **2018**, 23, 329.
- [13] A. M. Korsunsky, *J. Mech. Mater. Struct.* **2006**, 1, 259.
- [14] A. M. Korsunsky, G. M. Regino, D. Nowell, *Int. J. Solids Struct.* **2007**, 44, 4574.
- [15] A. T. DeWald, M. R. Hill, *Exp. Mech.* **2006**, 46, 473.
- [16] M. Turner, R. Clough, H. Martin, J. Topp, *J. Aeronaut. Sci.* **1956**, 23, 805.
- [17] M. E. Kartal, Y. H. Kang, A. M. Korsunsky, A. C. F. F. Cocks, J. P. Bouchard, *Int. J. Solids Struct.* **2016**, 80, 420.
- [18] A. M. Korsunsky, T. Jun, P. Road, in *Miner. Met. Mater. Soc. - 3rd Int. Conf. Process. Mater. Prop. 2008, PMP III, TMS - Minerals Metals and Materials Society* **2009**, Vol. 2, p. 738.
- [19] A. M. Korsunsky, G. M. Regino, D. Nowell, M. Karadge, B. Grant, P. J. Withers, M. Preuss, G. Baxter, *J. Strain Anal. Eng. Des.* **2009**, 44, 159.
- [20] S. Coratella, M. Sticchi, M. B. Toparli, M. E. Fitzpatrick, N. Kashaev, *Surf. Coat. Technol.* **2015**, 273, 39.
- [21] T. S. Jun, A. M. Venter, A. M. Korsunsky, *Exp. Mech.* **2011**, 51, 165.
- [22] E. Salvati, A. J. G. Lunt, S. Ying, T. Sui, H. J. Zhang, C. Heason, G. Baxter, A. M. Korsunsky, *Comput. Methods Appl. Mech. Eng.* **2017**, 320, 335.
- [23] X. Song, W. C. Liu, J. P. Belnoue, J. Dong, G. H. Wu, W. J. Ding, S. A. J. Kimber, T. Buslaps, A. J. G. Lunt, A. M. Korsunsky, *Int. J. Fatigue* **2012**, 42, 284.
- [24] C. Correa, A. Gil-Santos, J. A. Porro, M. Díaz, J. L. Ocaña, *Mater. Des.* **2015**, 79, 106.
- [25] F. Wang, K. Mao, B. Li, *Int. J. Mech. Sci.* **2018**, 140, 68.
- [26] G. H. Farrahi, S. A. Faghidian, D. J. Smith, *Int. J. Press. Vessels Pip.* **2009**, 86, 777.
- [27] H. E. Coules, D. J. Smith, K. Abburi Venkata, C. E. Truman, K. A. Venkata, C. E. Truman, *Int. J. Solids Struct.* **2014**, 51, 1980.
- [28] S. Naskar, B. Banerjee, *Int. J. Mech. Sci.* **2021**, 196, 106295.
- [29] J. R. Chukkan, G. Wu, M. E. Fitzpatrick, S. Jones, J. Kelleher, *Int. J. Mech. Sci.* **2019**, 160, 421.
- [30] A. W. T. Gregg, J. N. Hendriks, C. M. Wensrich, A. Wills, A. S. Tremis, V. Luzin, T. Shinohara, O. Kirstein, M. H. Meylan, E. H. Kisi, *Phys. Rev. Appl.* **2018**, 10, 064034.
- [31] F. Uzun, A. I. Salimon, E. S. Statnik, C. Besnard, J. Chen, T. Moxham, E. Salvati, Z. Wang, A. M. Korsunsky, *Measurement* **2019**, 151, 107193.
- [32] E. S. Statnik, A. I. Salimon, F. Uzun, A. M. Korsunsky, in *Proc. World Congr. Eng. 2019*, Newswood Limited, London **2019**, p. 2.
- [33] T. S. Jun, K. Dragnevski, A. M. Korsunsky, *Mater. Des.* **2010**, 31, S121.
- [34] F. Uzun, A. M. Korsunsky, *Int. J. Mech. Sci.* **2018**, 145, 231.
- [35] F. Uzun, J. Everaerts, L. R. Brandt, M. Kartal, E. Salvati, A. M. Korsunsky, *J. Manuf. Process.* **2018**, 36, 601.
- [36] F. Uzun, A. M. Korsunsky, *Finite Elem. Anal. Des.* **2019**, 155, 43.
- [37] F. Uzun, A. M. Korsunsky, *Mater. Sci. Eng., A* **2019**, 752, 180.
- [38] F. Uzun, C. Papadaki, Z. Wang, A. M. Korsunsky, *Mech. Mater.* **2020**, 143, 103316.
- [39] S. Cai, Y. Zhang, *Int. J. Solids Struct.* **2022**, 236–237, 111372.
- [40] S. Cai, Y. Zhang, *Surf. Coat. Technol.* **2022**, 436, 128237.
- [41] S. Cai, Y. Zhang, J. Wu, F. Wu, *J. Manuf. Process.* **2022**, 77, 451.
- [42] S. A. Faghidian, *Int. J. Solids Struct.* **2014**, 51, 4427.
- [43] A. M. Korsunsky, *J. Strain Anal. Eng. Des.* **2005**, 40, 817.
- [44] J. Everaerts, E. Salvati, F. Uzun, L. Romano Brandt, H. Zhang, A. M. Korsunsky, *Acta Mater.* **2018**, 156, 43.
- [45] H. T. Luckhoo, T.-S. Jun, A. M. Korsunsky, *Procedia Eng.* **2009**, 1, 213.
- [46] M. H. Stone, *Trans. Am. Math. Soc.* **1937**, 41, 375.
- [47] C. Runge, *Z. Math. Phys.* **1901**, 46, 224.
- [48] J. P. Boyd, *Comput. Math. Appl.* **2010**, 60, 3108.
- [49] A. M. Korsunsky, W. J. J. Vorster, S. Y. Zhang, D. Dini, D. Latham, M. Golshan, J. Liu, Y. Kyriakoglou, M. J. Walsh, *Acta Mater.* **2006**, 54, 2101.
- [50] F. Uzun, A. M. Korsunsky, *Mater. Today Proc.* **2020**, 4.
- [51] F. Uzun, A. N. Bilge, *J. Found. Appl. Phys.* **2017**, 4, 1.
- [52] T. Jun, A. M. Korsunsky, *Int. J. Solids Struct.* **2010**, 47, 1678.
- [53] X. Song, A. M. Korsunsky, *J. Appl. Mech. Trans. ASME* **2011**, 78, 031019.
- [54] G. M. Amdahl, in *AFIPS Conf. Proc. - 1967 Spring Jt. Comput. Conf. AFIPS 1967* **1967**, p. 483.
- [55] J. L. Gustafson, *Commun. ACM* **1988**, 31, 532.
- [56] S. Timoshenko, *Theory of Plates and Shells*, McGraw-Hill Book Company, United States of America **1989**.
- [57] M. C. Smith, A. C. Smith, C. Ohms, R. C. Wimpory, *Int. J. Press. Vessels Pip.* **2018**, 164, 3.
- [58] E. Bemporad, M. Brisotto, L. E. Depero, M. Gelfi, A. M. Korsunsky, A. J. G. Lunt, M. Sebastiani, *Thin Solid Films* **2014**, 572, 224.