

Threshold amplitudes for transition to turbulence in a pipe

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Abstract

Although flow in a circular pipe is stable to infinitesimal perturbations, it can be excited to turbulence by finite perturbations whose minimal amplitude shrinks as $R \rightarrow \infty$ ($R = \text{Reynolds number}$). Laboratory experiments have appeared to disagree with one another and with theoretical predictions about the dependence of this minimal amplitude on R , with published results ranging approximately from $R^{-1/4}$ to $R^{-3/2}$. Here it is shown that these dis-

crepancies can be explained by the use of different definitions of amplitude by different authors. An attempt is made to convert the existing results to a uniform definition of amplitude, the nondimensionalized L^2 definition common in the theoretical literature. Although subtleties in the physics raise some questions, agreement appears to be reached on a minimal amplitude that scales as $R^{-3/2\pm 0.3}$.

I. INTRODUCTION

Laminar incompressible flow in an infinite circular pipe is mathematically stable, but in practice, pipe flows invariably undergo transition to turbulence if the Reynolds number R is high. It is generally accepted that an explanation for this phenomenon is that although the laminar state is stable to infinitesimal perturbations of the velocity field, certain small finite amplitude perturbations are enough to excite transition for large R . A natural question is, if $\epsilon = \epsilon(R)$ denotes the minimal amplitude of all perturbations that may excite transition, and if ϵ scales with R according to

$$\epsilon = O(R^\gamma) \tag{1.1}$$

as $R \rightarrow \infty$, then what is the exponent γ ? A value of γ substantially below zero would correspond to a sensitivity of the laminar flow that increases rapidly with R .

We six, coming from diverse backgrounds in applied mathematics, scientific computing, and laboratory experimentation, have all been interested in (1.1), but the exponents γ that our different lines of research have suggested have varied by as much as a factor of six, from $\approx -1/4$ to $-3/2$. In discussions at the ERCOFTAC Workshop on Subcritical Transition in Delft in October, 1999, it became clear that we have been using inconsistent definitions of the amplitude of a velocity perturbation. Without a consistent definition, (1.1) of course has little meaning. The purpose of this note is to attempt to cast our various results in terms of a single definition of amplitude. The definition we shall use is essentially the one employed previously by Chapman [1], Schmid and Henningson [2], Trefethen, et al. [3], and others, and we shall call it the L^2 amplitude. We do not argue that this definition is more or less appropriate physically than any other, merely that it is precise and that it provides a reasonable starting point for discussion.

Specifically, in this note we attempt to convert the experimental results of Draad and Nieuwstadt [4,5] (henceforth DN) and Darbyshire and Mullin [6] (henceforth DM) to L^2 amplitudes. We conclude that in the L^2 framework, DN's published exponent of -1 should

be adjusted to between about -2 and -1 , and DM's published exponent of between -0.4 and -0.2 should be adjusted to between -1.8 and -1.15 . One reason why these adjusted values are expressed as ranges rather than single numbers is that both sets of experiments introduce perturbations by injection from the side of the pipe, and it is not known exactly what perturbations these injections induce in the velocity field within the pipe. We emphasize that these ranges are rough, having nothing like the two-digit precision suggested by a number like -1.15 .

Based on an asymptotic analysis of the Navier–Stokes equations, Chapman [1] has predicted the values $\gamma = -5/4$ for plane Couette flow and $\gamma = -3/2$ for plane Poiseuille flow, and discussed the relationship of these predictions to existing evidence from direct numerical simulation of the Navier–Stokes equations (DNS) [7,8]. (In the plane Poiseuille case one restricts attention to perturbations that avoid the Tollmien–Schlichting instability.) In work not yet written for publication, Chapman has extended the prediction $\gamma = -3/2$ also to pipe flow. Thus if our adjusted exponents for the DN and DM experiments are correct, there would appear to be reasonable agreement between two independent laboratories and a theoretical calculation on an exponent for the pipe in the vicinity

$$\gamma \approx -\frac{3}{2}. \tag{1.2}$$

This result would be consistent with the conjecture of Ref. [3] that γ is strictly less than -1 , an inequality also satisfied by most of the low-dimensional ODE models of the Navier–Stokes equations that have been published in the 1990s [9]. The apparent convergence of various lines of evidence on the estimate (1.2) looks promising, but we urge that it not be taken as definitive or as numerically precise. There are uncertainties at many points on both the experimental and theoretical sides, and no relevant data at all yet from DNS simulations for the pipe. Moreover, as we mention in Section 3, Chapman's asymptotic arguments are based on pipe lengths much longer than those in the DN and DM experiments, and for these pipes of finite lengths, somewhat less sensitivity to perturbations may be expected. We regard (1.2) as a rough working approximation.

II. NONDIMENSIONALIZATION AND L^2 AMPLITUDE

One source of confusion about γ has been the nondimensionalization of the Navier–Stokes equations. The L^2 definition of amplitude is formulated within a particular choice of nondimensional variables, the standard one. We shall review this choice and explain why it can be a point of confusion.

We are concerned with the idealized problem of laminar flow through an infinite circular pipe. The standard nondimensionalization takes the pipe radius as the space scale and the centerline velocity as the velocity scale. Thus, after nondimensionalization, the radius and velocity become

$$\text{radius} = 1, \quad \text{velocity} = 1.$$

These choices imply that the nondimensional time scale is the convective one, i.e., the time it takes the flow to travel downstream a distance of one pipe radius:

$$\text{time to travel one pipe radius} = 1.$$

Now there is also another time scale physically present in the problem, on which the effects of viscosity are felt. In our nondimensionalization this viscous time scale is R , the Reynolds number. Thus we have the following situation: a typical flow perturbation of small amplitude and of spatial extent comparable to the pipe radius is convected down the pipe at speed $O(1)$ for a time $O(R)$ and a distance $O(R)$ before the effects of viscosity become significant.

With these scales agreed upon, we imagine an initial value problem in which at time $t = 0$, the velocity field consists of the laminar solution plus a divergence-free finite perturbation $u(0) = u(x, r, \theta, 0)$. The flow now evolves according to the Navier–Stokes equations, with the result that the initial perturbation develops as a time-dependent divergence-free function $u(t) = u(x, r, \theta, t)$. At any time t , we measure the amplitude of u in an L^2 fashion:

$$\|u(t)\| = \left(\int_{-\infty}^{\infty} \int_0^1 \int_0^{2\pi} u(x, r, \theta, t)^2 d\theta r dr dx \right)^{1/2}. \quad (2.1)$$

Thus $\|u(t)\|$ is the root-mean-square velocity perturbation over the whole pipe.

We now return to the matter of why these formulations may sometimes be confusing. In most laboratory experiments, and certainly in DM and DN, the Reynolds number R is controlled by varying the speed of the flow, not the viscosity. On the other hand in nondimensional units the speed of the flow is always 1, and other velocities are defined as ratios to this one. Thus to convert from laboratory to nondimensional units we must

$$\text{multiply time measured in seconds by } R \tag{2.2}$$

and

$$\text{divide velocity measured in meters/second by } R \tag{2.3}$$

(as well as R -independent scalings by pipe diameter divided by kinematic viscosity). In particular, the nondimensionalized $O(R^\gamma)$ and $O(R^{-3/2})$ formulas of (1.1) and (1.2) would appear as $O(R^{\gamma+1})$ and $O(R^{-1/2})$ in laboratory units. Thus (1.2) can be paraphrased by the statement that if you double the speed of flow of water through an infinitely long pipe, the minimal velocity perturbation needed to excite transition becomes smaller in meters/second by a factor of about $\sqrt{2}$ and smaller relative to the flow speed by a factor of about $2\sqrt{2}$.

III. ASYMPTOTIC ESTIMATES OF CHAPMAN

Chapman's paper [1] estimates γ for channel flows by asymptotic analysis of the Navier-Stokes equations. The exponent $\gamma = -3/2$ is obtained for plane Poiseuille flow, and though this has not yet been written for publication, the same exponent results from an analogous analysis of pipe flow. Chapman uses the L^2 definition of amplitude as described above, except that he assumes a periodic flow perturbation and defines amplitude by an integral over one period. We believe that this does not affect the final result, so that his conclusion can be fairly summarized by (1.2).

We shall say nothing of the arguments of Ref. [1] except to note that they are based on the specific initial condition that appears to be most effective at exciting transition,

a streamwise vortex plus small non-streamwise components. If the analysis is correct, the threshold amplitude for such perturbations to excite transition will scale as $R^{-3/2}$ as $R \rightarrow \infty$. In principle $-3/2$ is thus a proposed upper bound for γ in the sense that there is the possibility that some other initial configuration might be found that would excite transition more effectively.

It must be noted, however, that Chapman's analysis is based on flow structures that evolve on a time scale $O(R)$, during which they move a distance $O(R)$ and stretch a distance $O(R)$. For the mechanisms involved to come fully into play, a pipe would have to have length at least $O(R)$, i.e., $O(R)$ pipe diameters. This exceeds the actual pipe lengths of experiments, which are in the hundreds, not thousands or tens of thousands. Thus an exponent as low as the theoretical value of $-3/2$ should not necessarily be observable in any existing pipe experiment. (Conversely, Chapman also identifies other finite- R effects that act in the opposite direction, effects which make exponents γ estimated from data with $R < 10^4$ more negative than the asymptotic values for $R \rightarrow \infty$.)

IV. PIPE EXPERIMENTS OF DRAAD AND NIEUWSTADT

The DN experiments in the 36m pipe at the Delft University of Technology are described in detail in Ref. [4]. In these experiments, a disturbance is introduced into the laminar flow through a set of slits in the side of the pipe. These are pumped in an oscillatory fashion so that water is injected and extracted sinusoidally at a controllable frequency and amplitude. As a measure of disturbance amplitude, DN take injection velocity nondimensionalized according to (2.3), i.e., divided by the flow velocity in the pipe. Their experiments lead to the following estimates, summarized for example in Figure 6.8 on p. 140 of Ref. [4]:

$$\text{DN as published: } \gamma \approx \begin{cases} -2/3 & \text{for long wavelengths,} \\ -1 & \text{for short wavelengths.} \end{cases} \quad (4.1)$$

“Long” and “short” wavelengths are defined in the usual nondimensional space scale, i.e., relative to the pipe radius. For simplicity, since our interest is in smallest perturbations that

may excite turbulence, from now on we shall consider just the value -1 reported by DN for short wavelengths.

Several matters arise in the attempt to convert the DN result to the amplitude measure (2.1). The most obvious is the fact that since the DN perturbations are periodic in time, they have infinite L^2 -amplitude. For the conversion to (2.1) we must guess how short a finite-length perturbation might have led to approximately the same observations. If a perturbation of length $O(1)$ (i.e., of length independent of R) would suffice, then the exponent $\gamma \approx -1$ can be taken at face value. On the other hand one might also imagine that perturbations of length and time scale $O(R)$ (i.e., laboratory time $O(1)$ as measured in seconds) would be needed. In this case the disturbance amplitudes must be multiplied by $O(R^{1/2})$ for conversion to L^2 amplitude because of the square root in (2.1), so that the estimate $\gamma \approx -1$ should be increased by $1/2$ to $-1/2$.

Thus at this stage of the discussion it would appear that the DN minimal exponent -1 corresponds in the amplitude measure (2.1) to a figure in the range -1 to $-1/2$.

It appears to us that there is also a second adjustment that should be applied to cast these results in terms of the L^2 amplitude (2.1). DN measure disturbance amplitude by velocity in the injection and extraction slits. However, the velocity of the water in the slits is not proportional to the velocity of the perturbation it induces in the pipe. The reason is that as the flow speed in the pipe increases with R , a proportionally greater volume of water is disturbed by the injection and extraction, implying that the pointwise velocity disturbance shrinks.

Penetration scenario. Suppose that for any flow speed R , the injected perturbation penetrates approximately the full width of the pipe. Then because the amount of fluid into which it is injected scales as $O(R)$, the velocity amplitude reduces pointwise by $O(R)$. The exponents -1 to $-1/2$ would need to be decreased by 1, giving the range from -2 to $-3/2$.

Non-penetration scenario. On the other hand, it is not obvious that injected perturbations penetrate the pipe effectively, and the other extreme scenario would seem to be that the injected perturbation affects a region near the pipe wall of width $O(R^{-1})$. In this

case, the perturbation is distributed over a volume $O(R^{-1}) \times O(R) = O(1)$, i.e., a volume independent of R . The pointwise velocity amplitude will accordingly be independent of R in that region. At the same time, the fraction of the pipe filled by the velocity perturbation is now not $O(1)$ but $O(R^{-1})$, implying an L^2 correction factor of $O(R^{-1/2})$. Thus according to the L^2 definition of amplitude, the exponents -1 to $-1/2$ would need to be decreased by $1/2$, giving the range from $-3/2$ to -1 .

Our discussion has raised two physical questions (whose answers may be related). Rather than attempt to resolve them on the basis of meagre evidence, we summarize our current understanding of the DN observations for short wavelengths by the range

$$\text{DN adjusted to } L^2 \text{ amplitudes: } -2 \leq \gamma \leq 1.$$

In principle these are estimated upper bounds for γ , as it is always possible that the perturbations actually injected are not maximally efficient in exciting turbulence.

V. PIPE EXPERIMENTS OF DARBYSHIRE AND MULLIN

The DM experiments, described in Ref. [6], were carried out in a 3.8m pipe at Oxford University. (The equipment subsequently moved with Mullin to the University of Manchester, where a new 17m pipe has recently been built based on the same design.) These experiments differ in many ways from those of DN, the most fundamental one being that water is sucked out of the pipe at fixed speed rather than pushed into it at fixed pressure. Another important difference is that whereas the DN perturbation is periodic, the DM perturbation is injected just once.

The DM paper does not propose a value for γ except to suggest that it seems to be just slightly less than 0. Based on the plots in Ref. [6], a rough estimate would seem to be

$$\text{DM as published: } -0.4 \leq \gamma \leq -0.2. \tag{5.1}$$

However, the disturbance amplitudes reported by DM are not normalized by the velocity in the pipe. Introducing the adjustment (2.3) gives

DM after nondimensionalization: $-1.4 \leq \gamma \leq -1.2$.

Now the more difficult question arises, as in the DN case, of what further adjustment may be needed because of the complex relationship between the flow velocity in the slits and the perturbation velocities induced in the pipe. As with DN, the DM flow perturbations are measured in the slits, not in the pipe. The form of the injections even in the slits is complicated by the geometry of a drive mechanism mounted on a rotating plate. In adjusting what they call the amplitude A , DM increase the injected volume in proportion to A , the injection time approximately in proportion to $A^{1/2}$, and the maximum injection velocity approximately in proportion to $A^{1/2}$. Within the pipe, this will produce a velocity perturbation extending a distance on the order of $A^{1/2}R$.

Penetration scenario. Suppose the perturbation penetrates a distance $O(1)$ into the pipe. Then the volume of the effective disturbance is $O(A^{1/2}R)$ and its pointwise amplitude is $O(A^{1/2}R^{-1})$, giving an L^2 amplitude $O(A^{3/4}R^{-1/2})$.

Non-penetration scenario. Suppose the perturbation penetrates only a distance $O(R^{-1})$ into the pipe. Then the volume of the effective disturbance is $O(A^{1/2})$ and its pointwise amplitude is $O(A^{1/2})$, giving an L^2 amplitude of $O(A^{3/4})$.

We conclude that two further adjustments of the DM results are needed to convert them to L^2 amplitudes. First, because what DM call A becomes $O(A^{3/4})$ in the L^2 measure, the numbers -0.4 and -0.2 should be multiplied by $3/4$, becoming -0.3 and -0.15 . Second and more important, the final numbers obtained should be reduced by between 0 and $1/2$. Putting all these adjustments together gives:

DM adjusted to L^2 amplitudes: $-1.8 \leq \gamma \leq -1.15$.

We emphasize once more that our arguments and measurements are not as precise as these numbers may suggest.

In principle these are again estimated upper bounds for γ , as it is again possible that the perturbations actually injected are not maximally efficient. In particular, it might be

possible to excite turbulence more efficiently by disturbances shaped to have a streamwise length with a different dependence on R and on amplitude.

VI. DISCUSSION

The existing experimental and theoretical literature on threshold exponents for transition in a pipe is based on inconsistent definitions of amplitudes, so the published results are not comparable. Here we have attempted to reformulate some of these results in a matter consistent enough for a meaningful comparison. There are numerous uncertainties in this process, including spatial vs. temporal growth of disturbances, solitary vs. periodic disturbances, form of the velocity field perturbation effectively introduced by injection, non-“optimality” of experimentally injected disturbances from the point of view of exciting transition, the effects of finite pipe length, and, of course, experimental error. For all these reasons, no decisive conclusion can be drawn from our comparison. The tentative conclusion we draw is that the reformulated experimental (Draad and Nieuwstadt; Darbyshire and Mullin) and theoretical (Chapman) results appear to agree upon a critical exponent roughly in the range $\gamma = -3/2 \pm 0.3$, if the centerline velocity is nondimensionalized to 1.

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