

MODIFIED TURAEV-VIRO INVARIANTS FROM QUANTUM $\mathfrak{sl}(2|1)$

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ABSTRACT. The category of finite dimensional modules over the quantum superalgebra $U_q(\mathfrak{sl}(2|1))$ is not semi-simple and the quantum dimension of a generic $U_q(\mathfrak{sl}(2|1))$ -module vanishes. This vanishing happens for any value of q (even when q is not a root of unity). These properties make it difficult to create a fusion or modular category. Loosely speaking, the standard way to obtain such a category from a quantum group is: 1) specialize q to a root of unity; this forces some modules to have zero quantum dimension, 2) quotient by morphisms of modules with zero quantum dimension, 3) show the resulting category is finite and semi-simple. In this paper we show an analogous construction works in the context of $U_q(\mathfrak{sl}(2|1))$ by replacing the vanishing quantum dimension with a modified quantum dimension. In particular, we specialize q to a root of unity, quotient by morphisms of modules with zero modified quantum dimension and show the resulting category is generically finite semi-simple. Moreover, we show the categories of this paper are relative G -spherical categories. As a consequence we obtain invariants of 3-manifold with additional structures.

1. INTRODUCTION

The numerical $6j$ -symbols associated with the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ were first introduced in theoretical physics by Eugene Wigner in 1940 and Giulio (Yoel) Racah in 1942. These symbols have been generalized to quantum $6j$ -symbols coming from tensor categories. If the category is fusion and spherical then the quantum $6j$ -symbols lead to Turaev-Viro invariants of 3-manifold (see [2, 3, 23, 28, 27]). The prototype of such a topological invariant arises from a particular category of modules over the quantum algebra $U_q(\mathfrak{sl}_2(\mathbb{C}))$.

Let us describe the T-V construction for this example. Without modification the category of finite dimensional modules over $U_q(\mathfrak{sl}_2(\mathbb{C}))$ is not fusion. If q is generic then there are an infinite number of non-isomorphic simple modules. When q is a root of unity then the quantum dimension of some of these modules becomes zero. Loosely speaking, by taking quotient with respect to such modules one obtains a category with a finite number of simple modules. More precisely, taking the quotient

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of the category of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ -modules by negligible morphisms one obtains the desired spherical and fusion category \mathcal{S} . Here a morphism $f : V \rightarrow W$ is negligible if for all morphisms $g : W \rightarrow V$ we have

$$Tr_q(f \circ g) = 0$$

where Tr_q is the quantum trace. If V is a simple module whose quantum dimension $qdim(V) = Tr_q(\text{Id}_V)$ is zero then any morphism to or from V is negligible; such a module is called negligible. The simple modules which are not negligible are said to be in the alcove.

The Turaev-Viro invariant is defined as a certain state sum computed on an arbitrary triangulation of a 3-manifold. The state sum on a triangulation \mathcal{T} of a closed 3-manifold M is defined, roughly speaking, as follows: Consider states of \mathcal{T} which are maps from the edges of \mathcal{T} to a finite index set I corresponding to isomorphism classes of simple objects in the category \mathcal{S} . Given a state $\varphi : \{\text{edges of } \mathcal{T}\} \rightarrow I$ one associates with each tetrahedron T of \mathcal{T} a particular quantum $6j$ -symbol denoted by $|T|_\varphi$. The state sum is defined by taking the product of these symbols over all tetrahedra of \mathcal{T} and summing up the resulting products (with certain weights) over all I colorings of \mathcal{T} :

$$(1) \quad TV(\mathcal{T}) = \mathcal{D}^{-2v} \sum_{\varphi \text{ state}} \left(\prod_{e \in \{\text{edges of } \mathcal{T}\}} qdim(\varphi(e)) \right) \left(\prod_{T \in \{\text{tetrahedron of } \mathcal{T}\}} |T|_\varphi \right)$$

where $qdim(\varphi(e))$ is the quantum dimension of the simple module associated to $\varphi(e)$ and $\mathcal{D}^2 = \sum_{i \in I} qdim(i)^2$.

The main point of this construction is that the state sum $TV(\mathcal{T})$ is independent of the choice of triangulation. This can be verified in two steps. First, the quantum $6j$ -symbols satisfy the symmetries of the tetrahedron. Second, any two triangulations of a closed 3-manifold can be transformed into one another by a finite sequence of the so called Pachner moves and an ambient isotopy (see [26]). Thus, it is enough to check that the state sum is invariant under the Pachner moves. For the category \mathcal{S} , these moves correspond to well-known algebraic identities which the quantum $6j$ -symbols satisfy.

Obstructions to applying this construction to a general pivotal tensor category \mathcal{C} include:

- (1) zero quantum dimensions,
- (2) non-semi-simplicity of \mathcal{C} ,
- (3) infinitely many isomorphism classes of simple objects of \mathcal{C} .

Kashaev [20] and later Baseilhac and Benedetti [4] and Geer, Kashaev and Turaev [8], considered 3-manifold invariants arising from a category with such obstructions, namely the category of modules over the Borel subalgebra of quantum $\mathfrak{sl}(2)$ at a root of unity. Geer, Patureau-Mirand, and Turaev [15] gave an alternate general approach to dealing with these obstructions and defined a secondary Turaev-Viro invariant of oriented 3-manifolds M . This is accomplished by three main modifications of the T-V invariant.

First, to address obstruction (1) they replace the vanishing $qdim$ and $|T|_\varphi$ in Equation (1) with corresponding non-zero modified quantum dimension and modified $6j$ -symbol, see [14, 15].

The second modification is dealing with the last two obstructions. If the usual state sum described above is applied to a category with infinitely many isomorphism classes of simple objects, this sum is of course infinite. With this in mind, the authors of [15] required that the pivotal tensor category \mathcal{C} had additional structure, including a G -grading on \mathcal{C} where G is an abelian group. To overcome the infinite sum problem, a finite number of modules are selected using a cohomology class in $H^1(M, G)$. This step also addresses obstruction (2) by requiring that generically graded pieces of the category are semi-simple.

The final modification is to introduce a link L in the manifold M . If one applies the changes described above, in the first two steps the invariant can still be zero or not well defined (see [15]). In particular, if the sum of the modified dimensions over a graded piece is zero, i.e. the analogous quantity associated to \mathcal{D}^2 in Equation (1) is zero. To construct a non-zero invariant, one can consider triangulations \mathcal{T} of M that realize the isotopy class of L as a so called Hamiltonian link in \mathcal{T} (see [4]). The Hamiltonian link is used to modify the weights in the state sum.

Let us be more precise. The construction of the modified TV-invariant works in the context of a relative G -spherical category (see Section 2 for details): Let G be an abelian group with a small symmetric subset $\mathcal{X} \subset G$. Let $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ be a G -graded pivotal \mathbb{k} -category such that

- (1) \mathcal{C}_g is finitely semi-simple for each $g \in G \setminus \mathcal{X}$,
- (2) there exists a t-ambi pair $(\mathbf{A}, \mathbf{d} : \mathbf{A} \rightarrow \mathbb{k}^\times)$ where each object of \mathbf{A} is isomorphic to a unique element of $\bigcup_{g \in G \setminus \mathcal{X}} \mathcal{I}_g$ and \mathcal{I}_g is a set of representatives of the isomorphism classes of simple objects of \mathcal{C}_g ,
- (3) there exists a map $\mathbf{b} : \mathbf{A} \rightarrow \mathbb{k}$ satisfying the condition in Definition 2.3.1.

In [15] it is shown that a relative G -spherical category (with basic data) gives rise to modified quantum $6j$ -symbols. In this context, these $6j$ -symbols are not numbers but rather tensors on certain multiplicity spaces.

Let M be a closed orientable 3-manifold and L a link in M . Following [4], we consider H -triangulation $(\mathcal{T}, \mathcal{L})$ of (M, L) : \mathcal{T} is a quasi-regular triangulation of M , \mathcal{L} is a set of unoriented edges of \mathcal{T} such that every vertex of \mathcal{T} belongs to exactly two edges of \mathcal{L} and the union of the edges in \mathcal{L} is the link L . Let $\Phi : \{\text{edges}\} \rightarrow G$ be a G -valued 1-cocycle on \mathcal{T} which takes values in $G \setminus \mathcal{X}$. A *state* of the G -coloring Φ is a map φ assigning to every oriented edge e of \mathcal{T} an element $\varphi(e)$ of $\mathcal{I}_{\Phi(e)}$ such that $\varphi(-e) = \varphi(e)^*$ for all e . The set of all states of Φ is denoted $\text{St}(\Phi)$.

Given a state φ and a tetrahedron T of \mathcal{T} , we can associate a modified $6j$ -symbol $|T|_\varphi$, for details see [15]. Any face of \mathcal{T} belongs to exactly two tetrahedra of \mathcal{T} , and the associated multiplicity modules are dual to each other. The tensor product of the $6j$ -symbols $|T|_\varphi$ associated to all tetrahedra T of \mathcal{T} can be contracted using this duality. We denote by cntr the tensor product of all these contractions. Let \mathcal{T}_1 be the set of unoriented edges \mathcal{T} and let \mathcal{T}_3 the set of tetrahedra of \mathcal{T} . Set

$$TV(\mathcal{T}, \mathcal{L}, \Phi) = \sum_{\varphi \in \text{St}(\Phi)} \prod_{e \in \mathcal{T}_1 \setminus \mathcal{L}} \mathbf{d}(\varphi(e)) \prod_{e \in \mathcal{L}} \mathbf{b}(\varphi(e)) \text{cntr} \left(\bigotimes_{T \in \mathcal{T}_3} |T|_\varphi \right) \in \mathbb{k}.$$

Theorem 1.0.1. *$TV(\mathcal{T}, \mathcal{L}, \Phi)$ depends only on the isotopy class of L in M and the cohomology class $[\Phi] \in H^1(M, G)$. It does not depend on the choice of the H -triangulation of (M, L) and on the choice of Φ in its cohomology class.*

In this paper we show that the category \mathscr{D} of finite dimensional modules over the quantum universal enveloping algebra of the Lie superalgebra $U_q(\mathfrak{sl}(2|1))$ leads to a relative G -spherical category and so gives rise to a modified TV-invariant. Our approach is based on a generalization of the usual technique: we take a quotient by negligible morphisms corresponding to the *modified trace*. As we will discuss, the obtained invariants should have different properties than the usual TV-invariants.

The category \mathscr{D} has the obstructions (1)–(3) listed above. When q is generic, the simple $U_q(\mathfrak{sl}(2|1))$ -modules are indexed by pairs (n, α) where $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$. Even when q is generic most of these simple modules have vanishing quantum dimensions. Therefore, taking the quotient of \mathscr{D} by the negligible morphisms corresponding to the quantum trace would trivialize most of the category. Alternatively, the ideal \mathcal{I} generated by the four dimensional module $V(0, \alpha)$ has a non-zero modified trace and corresponding non-zero modified quantum dimensions. When $q = e^{2\pi i/l}$ is a root of unity some of these modified quantum dimensions become zero. In particular, the modified quantum dimensions of modules on the boundary of the “alcove” vanish. The idea behind this paper is to take the quotient of \mathscr{D} by these modules to obtain a relative G -spherical category.

This construction has a novel feature: The alcove has an infinite number of non-isomorphic simple modules $V(n, \tilde{\alpha})$ where $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$, $0 \leq n \leq l' - 2$ and $l' = \frac{2l}{3+(-1)^l}$. Let $V(n, \alpha)$ be the simple module over $U_q(\mathfrak{sl}(2|1))$ when q is generic. By setting $q = e^{2\pi i/l}$ the module $V(n, \alpha)$ specializes to $V(n, \alpha + l\mathbb{Z})$ for $\alpha \in \mathbb{C}$. Modified $6j$ -symbols corresponding to these modules exist for both q generic and $q = e^{2\pi i/l}$. An interesting problem is to see if the $6j$ -symbols coming from generic q will specialize to the $6j$ -symbols constructed in this paper for roots of unity of **any** order. If this is true one could see if there exist an invariant defined for generic q which, when specialized to $q = e^{2\pi i/l}$ for any l , recovers the modified TV-invariants corresponding to the relative G -spherical categories defined in this paper. Results in this direction have been obtained for the usual Reshetikhin-Turaev quantum invariants, see [18, 19, 24, 25] and references within.

What is different in our context is that the modified invariants arising from the work of this paper require manifolds with additional structures, including the cohomology class discussed above. This makes the definition of the invariants more technical but can also add new information. For example, the modified RT-type invariants of [7, 6] recover the multivariable Alexander polynomial and Reidemeister torsion, which allows the reproduction of the classification of lens spaces. Also, the invariants of [7, 6] give rise to TQFTs and mapping class group representations with the notable property that the action of a Dehn twist has infinite order. This is in strong contrast with the usual quantum mapping class group representations where all Dehn twists have finite order. We expect similar properties for the invariants coming from this paper. Combining such properties with the ideas discussed in the previous paragraph could lead to appealing applications.

We should also mention that 3-manifold invariants arising from a quantization of $\mathfrak{sl}(2|1)$ have already been constructed by Ha in [17]. Ha’s construction uses a “unrolled” version of quantum $\mathfrak{sl}(2|1)$. He also uses a modified trace on a different ideal which consists of projective modules, most of which only exist when q is a root of unity. In comparison, in this paper the quotient of \mathscr{D} we take contains all the projective modules of \mathscr{D} . Also, for $0 \leq n \leq l' - 2$ and $\alpha \in \mathbb{C}$ the ideal

Ha works with does not contain a module which is the specialization of the simple $U_q(\mathfrak{sl}(2|1))$ -module $V(n, \alpha)$ discussed above.

2. CATEGORICAL PRELIMINARIES

As mentioned above our main theorem will be that $U_q(\mathfrak{sl}(2|1))$ gives rises to a relative G -spherical category. With this in mind, in this section we will recall the general definition of such a category, for more details see [12, 15].

2.1. \mathbb{k} -categories. Let \mathbb{k} be a field. A *tensor \mathbb{k} -category* is a tensor category \mathcal{C} such that its hom-sets are \mathbb{k} -modules, the composition and tensor product of morphisms is \mathbb{k} -bilinear, and the canonical \mathbb{k} -algebra map $\mathbb{k} \rightarrow \text{End}_{\mathcal{C}}(\mathbb{I}), k \mapsto k \text{Id}_{\mathbb{I}}$ is an isomorphism (where \mathbb{I} is the unit object). In the sequel, we will use the term *\mathbb{k} -category* referring to a *tensor \mathbb{k} -category*. We will need the notion of pivotal categories. A tensor category is *pivotal* if it has dual objects and duality morphisms

$$\overrightarrow{\text{coev}}_V: \mathbb{I} \rightarrow V \otimes V^*, \quad \overrightarrow{\text{ev}}_V: V^* \otimes V \rightarrow \mathbb{I}, \quad \overleftarrow{\text{coev}}_V: \mathbb{I} \rightarrow V^* \otimes V, \quad \overleftarrow{\text{ev}}_V: V \otimes V^* \rightarrow \mathbb{I}$$

which satisfy compatibility conditions (see for example [2, 10]). We will recall the definitions below.

Definition 2.1.1. (Left duality)

Let \mathcal{C} be a strict tensor category. We say that \mathcal{C} has a left duality if for each object V of \mathcal{C} there is an object V^* of \mathcal{C} and morphisms

$$(2) \quad \overrightarrow{\text{coev}}_V: \mathbb{I} \rightarrow V \otimes V^* \quad \text{and} \quad \overrightarrow{\text{ev}}_V: V^* \otimes V \rightarrow \mathbb{I}$$

such that

$$(\text{Id}_V \otimes \overrightarrow{\text{ev}}_V)(\overrightarrow{\text{coev}}_V \otimes \text{Id}_V) = \text{Id}_V \quad \text{and} \quad (\overrightarrow{\text{ev}}_V \otimes \text{Id}_{V^*})(\text{Id}_{V^*} \otimes \overrightarrow{\text{coev}}_V) = \text{Id}_{V^*}.$$

Remark 2.1.2. A left duality on a category \mathcal{C} determines for every morphism $f: V \rightarrow W$ in \mathcal{C} the dual morphism $f^*: W^* \rightarrow V^*$ defined as

$$f^* = (\overrightarrow{\text{ev}}_W \otimes \text{Id}_{V^*})(\text{Id}_{W^*} \otimes f \otimes \text{Id}_{V^*})(\text{Id}_{W^*} \otimes \overrightarrow{\text{coev}}_V).$$

Moreover, for any objects V, W of \mathcal{C} , it determines an isomorphism

$$\gamma_{V,W}: W^* \otimes V^* \rightarrow (V \otimes W)^*$$

defined by the formula

$$\gamma_{V,W} = (\overrightarrow{\text{ev}}_W \otimes \text{Id}_{(V \otimes W)^*})(\text{Id}_{W^*} \otimes \overrightarrow{\text{ev}}_V \otimes \text{Id}_W \otimes \text{Id}_{(V \otimes W)^*})(\text{Id}_{W^*} \otimes \text{Id}_{V^*} \otimes \overrightarrow{\text{coev}}_{V \otimes W}).$$

Definition 2.1.3. (Right duality)

A strict tensor category \mathcal{C} has a right duality if for each object V of \mathcal{C} there is an object V^\bullet of \mathcal{C} and morphisms

$$(3) \quad \overleftarrow{\text{coev}}_V: \mathbb{I} \rightarrow V^\bullet \otimes V \quad \text{and} \quad \overleftarrow{\text{ev}}_V: V \otimes V^\bullet \rightarrow \mathbb{I}$$

such that

$$(\text{Id}_{V^\bullet} \otimes \overleftarrow{\text{ev}}_V)(\overleftarrow{\text{coev}}_V \otimes \text{Id}_{V^\bullet}) = \text{Id}_{V^\bullet} \quad \text{and} \quad (\overleftarrow{\text{ev}}_V \otimes \text{Id}_V)(\text{Id}_V \otimes \overleftarrow{\text{coev}}_V) = \text{Id}_V.$$

Remark 2.1.4. Suppose that \mathcal{C} is a category that has a right duality. Then, for every morphism $f: V \rightarrow W$ in \mathcal{C} , it determines a dual morphism $f^\bullet: W^\bullet \rightarrow V^\bullet$ by

$$f^\bullet = (\text{Id}_{V^\bullet} \otimes \overleftarrow{\text{ev}}_W)(\text{Id}_{V^\bullet} \otimes f \otimes \text{Id}_{W^\bullet})(\overleftarrow{\text{coev}}_V \otimes \text{Id}_{W^\bullet}).$$

Moreover, for any objects $V, W \in \mathcal{C}$, it gives an isomorphism

$$\gamma'_{V,W} : W^\bullet \otimes V^\bullet \rightarrow (V \otimes W)^\bullet$$

defined as follows

$$\gamma'_{V,W} = (\text{Id}_{(V \otimes W)^\bullet} \otimes \overleftarrow{\text{ev}}_V)(\text{Id}_{(V \otimes W)^\bullet} \otimes \text{Id}_V \otimes \overleftarrow{\text{ev}}_W \otimes \text{Id}_{V^\bullet})(\overleftarrow{\text{coev}}_{V \otimes W} \otimes \text{Id}_{W^\bullet} \otimes \text{Id}_{V^\bullet}).$$

Definition 2.1.5. (*Pivotal category*)

A pivotal category is a tensor category with a left duality $\{\overrightarrow{\text{coev}}_V, \overrightarrow{\text{ev}}_V\}_V$ and a right duality $\{\overleftarrow{\text{coev}}_V, \overleftarrow{\text{ev}}_V\}_V$ which are compatible in the following manner:

$$V^* = V^\bullet; \quad f^* = f^\bullet; \quad \gamma_{V,W} = \gamma'_{V,W}$$

for all V, W, f as above.

Every pivotal category gives a natural tensor isomorphism

$$(4) \quad \phi = \{\phi_V = (\overleftarrow{\text{ev}}_V \otimes \text{Id}_{V^{**}})(\text{Id}_V \otimes \overrightarrow{\text{coev}}_{V^*}) : V \rightarrow V^{**}\}_{V \in \mathcal{C}}.$$

An object V of \mathcal{C} is *simple* if $\text{End}_{\mathcal{C}}(V) = \mathbb{k} \text{Id}_V$. Let V and W be objects in \mathcal{C} and let $\alpha : V \rightarrow W$ and $\beta : W \rightarrow V$ be morphisms. The triple (V, α, β) (or just the object V) is a *retract* of W if $\beta\alpha = \text{Id}_V$. An object W is a *direct sum* of the finite family $\{V_i\}_i$ of objects of \mathcal{C} if there exist retracts (V_i, α_i, β_i) of W with $\beta_i\alpha_j = 0$ for $i \neq j$ and $\text{Id}_W = \sum_i \alpha_i\beta_i$. An object which is a direct sum of simple objects is called *semi-simple*.

2.2. Colored ribbon graph invariants. Let \mathcal{C} be a pivotal \mathbb{k} -category. A ribbon graph is formed from several oriented framed edges colored by objects of \mathcal{C} and several coupons colored with morphisms of \mathcal{C} ([27] Chapter I- Section 1.6, [21]). We say a \mathcal{C} -colored ribbon graph in \mathbb{R}^2 (resp. $S^2 = \mathbb{R}^2 \cup \{\infty\}$) is *planar* (resp. *spherical*). Let F be the usual Reshetikhin-Turaev functor from the category of \mathcal{C} -colored planar ribbon graphs to \mathcal{C} (see [27]).

Let $T \subset S^2$ be a closed \mathcal{C} -colored ribbon spherical graph. Let e be an edge of T colored with a simple object V of \mathcal{C} . Cutting T at a point of e , we obtain a \mathcal{C} -colored ribbon graph T_V in $\mathbb{R} \times [0, 1]$ where $F(T_V) \in \text{End}(V) = \mathbb{k} \text{Id}_V$. We call T_V a *cutting presentation* of T and let $\langle T_V \rangle \in \mathbb{k}$ denote the isotopy invariant of T_V defined from the equality $F(T_V) = \langle T_V \rangle \text{Id}_V$.

Let A be a class of simple objects of \mathcal{C} and $d : A \rightarrow \mathbb{k}^\times$ be a mapping such that $d(V) = d(V^*)$ and $d(V) = d(V')$ if V is isomorphic to V' . We say (A, d) is *t-ambi pair* if for any closed \mathcal{C} -colored trivalent ribbon spherical graph T with any two cutting presentations T_V and $T_{V'}$, $V, V' \in A$ the following equation holds:

$$d(V)\langle T_V \rangle = d(V')\langle T_{V'} \rangle.$$

2.3. G -graded and generically G -semi-simple categories. Let G be a group. A pivotal \mathbb{k} -category is *G -graded* if for each $g \in G$ we have a non-empty full subcategory \mathcal{C}_g of \mathcal{C} such that

- (1) $\mathbb{I} \in \mathcal{C}_e$, (where e is the identity element of G)
- (2) $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$,
(the direct sum refers to the objects of the category: $Ob(\mathcal{C}) = \bigoplus_{g \in G} Ob(\mathcal{C}_g)$)
- (3) if $V \in \mathcal{C}_g$, then $V^* \in \mathcal{C}_{g^{-1}}$,
- (4) if $V \in \mathcal{C}_g$, $V' \in \mathcal{C}_{g'}$ then $V \otimes V' \in \mathcal{C}_{gg'}$,
- (5) if $V \in \mathcal{C}_g$, $V' \in \mathcal{C}_{g'}$ and $\text{Hom}_{\mathcal{C}}(V, V') \neq 0$, then $g = g'$.

For a subset $\mathcal{X} \subset G$ we say:

- (1) \mathcal{X} is *symmetric* if $\mathcal{X}^{-1} = \mathcal{X}$,
- (2) \mathcal{X} is *small* in G if the group G can not be covered by a finite number of translated copies of \mathcal{X} , in other words, for any $g_1, \dots, g_n \in G$, we have $\bigcup_{i=1}^n (g_i \mathcal{X}) \neq G$.
- (1) A \mathbb{k} -category \mathcal{C} is *semi-simple* if all its objects are semi-simple.
- (2) A \mathbb{k} -category \mathcal{C} is *finitely semi-simple* if it is semi-simple and has finitely many isomorphism classes of simple objects.
- (3) A G -graded category \mathcal{C} is a *generically finitely G -semi-simple* category if there exists a small symmetric subset $\mathcal{X} \subset G$ such that for each $g \in G \setminus \mathcal{X}$, \mathcal{C}_g is finitely semi-simple. By a *generic simple* object we mean a simple object of \mathcal{C}_g for some $g \in G \setminus \mathcal{X}$.

The notion of generically G -semi-simple categories appears in [15, 13] through the following generalization of fusion categories (in particular, fusion categories satisfy the following definition when G is the trivial group, $\mathcal{X} = \emptyset$ and $\mathbf{d} = \mathbf{b} = \text{qdim}_{\mathcal{C}}$ is the quantum dimension):

Definition 2.3.1 (Relative G -spherical category). *Let \mathcal{C} be a generically finitely G -semi-simple pivotal \mathbb{k} -category with small symmetric subset $\mathcal{X} \subset G$ and let \mathbf{A} be a class of generic simple objects of \mathcal{C} . We say that \mathcal{C} is $(\mathcal{X}, \mathbf{d})$ -relative G -spherical if*

- (1) *there exists a map $\mathbf{d} : \mathbf{A} \rightarrow \mathbb{k}^\times$ such that (\mathbf{A}, \mathbf{d}) is a t-ambi pair,*
- (2) *there exists a map $\mathbf{b} : \mathbf{A} \rightarrow \mathbb{k}$ such that $\mathbf{b}(V) = \mathbf{b}(V^*)$, $\mathbf{b}(V) = \mathbf{b}(V')$ for any isomorphic objects $V, V' \in \mathbf{A}$, and for any $g_1, g_2, g_1 g_2 \in G \setminus \mathcal{X}$ and $V \in G_{g_1 g_2}$ we have*

$$\mathbf{b}(V) = \sum_{V_1 \in \text{irr}(\mathcal{C}_{g_1}), V_2 \in \text{irr}(\mathcal{C}_{g_2})} \mathbf{b}(V_1) \mathbf{b}(V_2) \dim_{\mathbb{k}}(\text{Hom}_{\mathcal{C}}(V, V_1 \otimes V_2))$$

where $\text{irr}(\mathcal{C}_{g_i})$ denotes a set of representatives of isomorphism classes of simple objects of \mathcal{C}_{g_i} .

If \mathcal{C} is a category with the above data, for brevity we say \mathcal{C} is a *relative G -spherical category*. In [15], to construct a 3-manifold invariant from a relative G -spherical category \mathcal{C} the authors assume that \mathcal{C} satisfies a technical requirement to have a *basic data*. The following lemma (proved in [12]) says that in most situations \mathcal{X} can be enlarged so that \mathcal{C} has basic data. This lemma implies that we can assume the categories we considered in this paper have basic data.

Lemma 2.3.2. *If no object of \mathbf{A} is isomorphic to its dual, then \mathcal{C} contains a basic data. In particular, a basic data exists if \mathcal{X} contains the set $\{g \in G : g = g^{-1}\}$.*

2.4. Traces on ideals in pivotal categories. In this subsection we recall some facts about right traces in a pivotal \mathbb{k} -category \mathcal{C} , for more details see [16, 10]. In this paper we will use right traces to show that a t-ambi pair exists.

By a *right ideal* of \mathcal{C} we mean a full subcategory \mathcal{I} of \mathcal{C} such that:

- (1) If $V \in \mathcal{I}$ and $W \in \mathcal{C}$, then $V \otimes W \in \mathcal{I}$.
- (2) If $V \in \mathcal{I}$ and if $W \in \mathcal{C}$ is a retract of V , then $W \in \mathcal{I}$.

A *right trace* on a right ideal \mathcal{I} is a family of linear functions

$$\{\mathbf{t}_V : \text{End}_{\mathcal{C}}(V) \rightarrow \mathbb{k}\}_{V \in \mathcal{I}}$$

such that:

- (1) If $U, V \in \mathcal{I}$ then for any morphisms $f : V \rightarrow U$ and $g : U \rightarrow V$ in \mathcal{C} we have

$$\mathbf{t}_V(gf) = \mathbf{t}_U(fg).$$

- (2) If $U \in \mathcal{I}$ and $W \in \mathcal{C}$ then for any $f \in \text{End}_{\mathcal{C}}(U \otimes W)$ we have

$$\mathbf{t}_{U \otimes W}(f) = \mathbf{t}_U \left((\text{Id}_U \otimes \overleftarrow{\text{ev}}_W)(f \otimes \text{Id}_{W^*})(\text{Id}_U \otimes \overrightarrow{\text{coev}}_W) \right)$$

Next we recall how to construct a right trace. Given an object V of \mathcal{C} we define the ideal generated by V as

$$\mathcal{I}_V = \{W \in \mathcal{C} \mid W \text{ is a retract of } V \otimes X \text{ for some object } X\}.$$

In [16] the notion of a right ambidextrous simple object is developed (see Sections 4.2 and 4.4 of [16]). Theorem 10 of [16] implies:

Theorem 2.4.1 ([16]). *If V is a right ambidextrous simple object then there exists a non-zero right trace $\{\mathbf{t}_V\}$ on the ideal \mathcal{I}_V ; this trace is unique up to multiplication by a non-zero scalar.*

Now we will recall a way to show a simple object is right ambidextrous. Let us assume \mathcal{C} is an additive pivotal \mathbb{k} -category. Let V be a simple object in \mathcal{C} . We fix a direct sum decomposition of $V \otimes V^*$ into indecomposable objects W_i indexed by a set I :

$$(5) \quad V \otimes V^* = \bigoplus_{k \in I} W_k.$$

Let $i_k : W_k \rightarrow V \otimes V^*$ and $p_k : V \otimes V^* \rightarrow W_k$ be the morphisms corresponding to this decomposition. In particular, $\sum_{k \in I} i_k p_k = \text{Id}_{V \otimes V^*}$ and $p_k i_k = \text{Id}_{W_k}$, for all $k \in I$. From Lemma 3.1.1 of [10] we have the following lemma:

Lemma 2.4.2 ([10]). *There exist unique $j, j' \in I$ such that*

- (1) $\text{Hom}_{\mathcal{C}}(\mathbb{I}, W_j)$ is non-zero and is spanned by $p_j \overrightarrow{\text{coev}}_V$ and
- (2) $\text{Hom}_{\mathcal{C}}(W_{j'}, \mathbb{I})$ is non-zero and is spanned by $\overleftarrow{\text{ev}}_V i_{j'}$.

Theorem 3.1.3. of [10] gives the following theorem.

Theorem 2.4.3 ([10]). *The simple object V is right ambidextrous if and only if $j = j'$.*

Finally, let us explain how to produce a \mathbf{t} -ambi pair from a right trace. Let \mathbf{t} be a right trace on a right ideal \mathcal{I} of \mathcal{C} . Let \mathbf{d} be the modified dimension associated with \mathbf{t} defined by $\mathbf{d}(V) = \mathbf{t}_V(\text{Id}_V)$ for $V \in \mathcal{I}$. Set

$$\mathcal{B} = \{V \in \mathcal{I} \cap \mathcal{I}^* \mid V \text{ is simple and } \mathbf{d}(V) = \mathbf{d}(V^*)\}$$

where $\mathcal{I}^* = \{V \in \mathcal{C} \mid V^* \in \mathcal{I}\}$. The following theorem is Corollary 7 in [16].

Theorem 2.4.4 ([16]). *The pair $(\mathcal{B}, \mathbf{d})$ is a \mathbf{t} -ambi pair.*

3. QUANTUM $\mathfrak{sl}(2|1)$ AT ROOTS OF UNITY

3.1. Notation. Fix a positive integer $l \geq 3$ and let $q = e^{\frac{2\pi\sqrt{-1}}{l}}$ be a l^{th} -root of unity. Set

$$l' = \begin{cases} l & \text{if } l \text{ is odd} \\ l/2 & \text{if } l \text{ is even} \end{cases}.$$

We use two quotients of the complex numbers: \mathbb{C}/\mathbb{Z} and $\mathbb{C}/l\mathbb{Z}$. We will use greek letters to denote elements of \mathbb{C} . We will denote the corresponding elements in \mathbb{C}/\mathbb{Z} and $\mathbb{C}/l\mathbb{Z}$ by adding bars and tildes, respectively. In this paper, both \mathbb{C}/\mathbb{Z} and $\mathbb{C}/l\mathbb{Z}$ are considered with the abelian group structures induced from the additive group structure of \mathbb{C} .

For $\alpha \in \mathbb{C}$ let $\tilde{\alpha}$ be the element of $\mathbb{C}/l\mathbb{Z}$ such that α is in the equivalence class $\tilde{\alpha}$. In other words, α maps to $\tilde{\alpha}$ under the canonical map $\mathbb{C} \rightarrow \mathbb{C}/l\mathbb{Z}$. Similarly, for $\alpha \in \mathbb{C}$ or $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ let $\bar{\alpha} \in \mathbb{C}/\mathbb{Z}$ be such that α or $\tilde{\alpha}$ is mapped to $\bar{\alpha}$ under the map $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$ or $\mathbb{C}/l\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}$, respectively. For x in \mathbb{C} or $\mathbb{C}/l\mathbb{Z}$ set $\{x\} = q^x - q^{-x}$ and $[x] = \frac{\{x\}}{\{1\}}$.

3.2. Superspaces. In the sequel, we will use the notation: $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. A *superspace* is a \mathbb{Z}_2 -graded vector space $V = V_{|0|} \oplus V_{|1|}$ over \mathbb{C} . We denote the parity of a homogeneous element $x \in V$ by $|x| \in \mathbb{Z}_2$. We say x is even (odd) if $x \in V_{|0|}$ (resp. $x \in V_{|1|}$). If V and W are \mathbb{Z}_2 -graded vector spaces then the space of linear maps $\text{Hom}_{\mathbb{C}}(V, W)$ has a natural \mathbb{Z}_2 -grading given by $f \in \text{Hom}_{\mathbb{C}}(V, W)_{|j|}$ if $f(V_{|i|}) \subset W_{|i|+|j|}$ for $|i|, |j| \in \mathbb{Z}_2$. Throughout, all modules over a \mathbb{Z}_2 -graded ring will be \mathbb{Z}_2 -graded modules.

3.3. The superalgebra $U_q(\mathfrak{sl}(2|1))$. Let $A = (a_{ij})$ be the 2×2 matrix given by $a_{11} = 2$, $a_{12} = a_{21} = -1$ and $a_{22} = 0$. Let $U_q(\mathfrak{sl}(2|1))$ be the \mathbb{C} -superalgebra generated by the elements K_i, K_i^{-1}, E_i and F_i , $i = 1, 2$, subject to the relations:

$$\begin{aligned} K_i^{\pm 1} K_j &= K_j K_i^{\pm 1}, & K_i^{-1} K_j^{\pm 1} &= K_j^{\pm 1} K_i^{-1}, & K_i F_j K_i^{-1} &= q^{-a_{ij}} F_j, \\ K_i E_j K_i^{-1} &= q^{a_{ij}} E_j, & [E_i, F_j] &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \end{aligned}$$

(6)

$$E_1^2 E_2 - (q + q^{-1}) E_1 E_2 E_1 + E_2 E_1^2 = 0, \quad F_1^2 F_2 - (q + q^{-1}) F_1 F_2 F_1 + F_2 F_1^2 = 0$$

where $[,]$ is the super-commutator given by $[x, y] = xy - (-1)^{|x||y|} yx$. All generators are even except for E_2 and F_2 which are odd. The algebra $U_q(\mathfrak{sl}(2|1))$ is a Hopf superalgebra where the coproduct, counit and antipode are defined by

$$\begin{aligned} \Delta(E_i) &= E_i \otimes 1 + K_i^{-1} \otimes E_i, & \varepsilon(E_i) &= 0 & S(E_i) &= -K_i E_i \\ \Delta(F_i) &= F_i \otimes K_i + 1 \otimes F_i, & \varepsilon(F_i) &= 0 & S(F_i) &= -F_i K_i^{-1} \\ \Delta(K_i) &= K_i \otimes K_i & \varepsilon(K_i) &= 1, & S(K_i) &= K_i^{-1}, \\ \Delta(K_i^{-1}) &= K_i^{-1} \otimes K_i^{-1} & \varepsilon(K_i^{-1}) &= 1, & S(K_i^{-1}) &= K_i. \end{aligned}$$

This is a Hopf object in the symmetric monoidal category of super vector spaces.

3.4. Representations of $U_q(\mathfrak{sl}(2|1))$. Let V be a $U_q(\mathfrak{sl}(2|1))$ -module.

An element $v \in V$ is called a *weight vector* if $\exists \lambda_1, \lambda_2 \in \mathbb{C}$ such that:

$$K_i v = \lambda_i v, \forall i \in \{1, 2\}.$$

A *weight vector* is a *highest weight vector* if the following conditions hold:

$$E_i v = 0, \forall i \in \{1, 2\}.$$

A *weight vector* is a *lowest weight vector* if it satisfies the relations:

$$F_i v = 0, \forall i \in \{1, 2\}.$$

A $U_q(\mathfrak{sl}(2|1))$ -module V is a *weight module* if V is finite dimensional and both K_1 and K_2 act diagonally on V with respect to some homogeneous basis.

A module V over $U_q(\mathfrak{sl}(2|1))$ is called *highest weight module*, if it is generated by a highest weight vector (over the quantum group $U_q(\mathfrak{sl}(2|1))$).

Definition 3.4.1. *We define the trivial representation of $U_q(\mathfrak{sl}(2|1))$ to be the one dimensional complex vector space \mathbb{C} , generated by an even vector, with the following action of the quantum group:*

$$E_i v = 0, \quad F_i v = 0, \quad K_i v = v, \quad \forall v \in \mathbb{C}, \forall i \in \{1, 2\}.$$

Remark 3.4.2. *We notice that \mathbb{C} is the one dimensional module corresponding to the counit ε of the quantum group $U_q(\mathfrak{sl}(2|1))$.*

Let \mathcal{D} be the tensor category of finite dimensional \mathbb{Z}_2 -graded weight $U_q(\mathfrak{sl}(2|1))$ -modules, whose unit object is the trivial module $\mathbb{I} = \mathbb{C}$, more specifically $\mathbb{I}_{|0|} = \mathbb{C}$ and $\mathbb{I}_{|1|} = 0$. It is easy to see that this category is a \mathbb{C} -category.

A direct calculation shows that $S^2(x) = K_2^{-2} x K_2^2$ for all $x \in U_q(\mathfrak{sl}(2|1))$. Thus, the square of the antipode is equal to the conjugation by a group-like element and so \mathcal{D} is a pivotal category (see [5, Proposition 2.9]). In particular, for any object V in \mathcal{D} , the dual object and the duality morphisms are defined as follows: $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and

$$(7) \quad \begin{aligned} \overrightarrow{\text{coev}}_V: \mathbb{C} &\rightarrow V \otimes V^* \text{ is given by } 1 \mapsto \sum v_j \otimes v_j^*, \\ \overrightarrow{\text{ev}}_V: V^* \otimes V &\rightarrow \mathbb{C} \text{ is given by } f \otimes w \mapsto f(w), \\ \overleftarrow{\text{coev}}_V: \mathbb{C} &\rightarrow V^* \otimes V \text{ is given by } 1 \mapsto \sum (-1)^{|v_j|} v_j^* \otimes K_2^2 v_j, \\ \overleftarrow{\text{ev}}_V: V \otimes V^* &\rightarrow \mathbb{C} \text{ is given by } v \otimes f \mapsto (-1)^{|f||v|} f(K_2^{-2} v), \end{aligned}$$

for homogeneous v and f ,

where $\{v_j\}$ is a homogeneous basis of V and $\{v_j^*\}$ is the dual basis of V^* . These morphisms satisfy the compatibility conditions of a pivotal category.

The simple $U_q(\mathfrak{sl}(2|1))$ -modules have been studied in [1]. Here we will consider what they call the *typical type A representations*: let $\omega \in \{\pm 1\}$, $0 \leq n \leq l' - 1$ and $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ then there exists a highest weight module $V(\omega, n, \tilde{\alpha})$ with highest weight vector v which is even homogeneous such that

$$E_i v = 0, \quad K_1 v = \omega q^n v \quad \text{and} \quad K_2 v = q^{\tilde{\alpha}} v.$$

In [1] it is shown that under certain conditions $V(\omega, n, \tilde{\alpha})$ is a simple module of dimension $4(n+1)$. Let us now give these conditions. To simplify notation if $\omega = 1$ we set $V(n, \tilde{\alpha}) = V(1, n, \tilde{\alpha})$.

Proposition 3.4.3 ([1, page 873]). *If $[\tilde{\alpha}] \cdot [\tilde{\alpha} + n + 1] \neq 0$ then $V(n, \tilde{\alpha})$ is simple.*

Remark 3.4.4. *We use slightly different notation than [1]. Our module $V(n, \tilde{\alpha})$ corresponds to the module from [1, page 873] with the following parameters:*

$$\omega = 1, \quad N = n + 1, \quad \lambda_1 = q^n, \quad \mu_1 = n = N - 1, \quad \lambda_2 = q^\alpha, \quad \mu_2 = \alpha.$$

Since

$$[x] = 0 \Leftrightarrow \frac{q^x - q^{-x}}{q - q^{-1}} = 0 \Leftrightarrow q^x - q^{-x} = 0 \Leftrightarrow q^{2x} = 1 \Leftrightarrow x \in \frac{l}{2}\mathbb{Z}$$

then the above proposition implies that $V(n, \tilde{\alpha})$ is simple if $\tilde{\alpha} \in (\frac{l}{2}\mathbb{Z})/l\mathbb{Z} \cup ((\frac{l}{2}\mathbb{Z}) - (n+1))/l\mathbb{Z}$. In particular, if $\tilde{\alpha} \notin \{\bar{0}, \bar{\frac{l}{2}}\} \subseteq \mathbb{C}/\mathbb{Z}$ then $V(n, \tilde{\alpha})$ is simple.

Theorem 3.4.5 ([1]). *Let $n \in \{0, \dots, l' - 1\}$ and $\tilde{\alpha} \notin (\frac{l}{2}\mathbb{Z})/l\mathbb{Z} \cup ((\frac{l}{2}\mathbb{Z}) - (n+1))/l\mathbb{Z}$. Then $V(n, \tilde{\alpha})$ has a homogeneous basis $\{w_{\rho, \sigma, p} | p \in \{0, \dots, n\}; \rho, \sigma \in \{0, 1\}\}$ and the action of $U_q(\mathfrak{sl}(2|1))$ is given by:*

$$(8) \quad K_1 \cdot w_{\rho, \sigma, p} = q^{\rho - \sigma + n - 2p} w_{\rho, \sigma, p}, \quad K_2 \cdot w_{\rho, \sigma, p} = q^{\tilde{\alpha} + \sigma + p} w_{\rho, \sigma, p},$$

$$(9)$$

$$F_1 \cdot w_{\rho, \sigma, p} = q^{\sigma - \rho} w_{\rho, \sigma, p+1} - \rho(1 - \sigma) q^{-\rho} w_{\rho-1, \sigma+1, p}, \quad F_2 \cdot w_{\rho, \sigma, p} = (1 - \rho) w_{\rho+1, \sigma, p},$$

$$(10) \quad E_1 \cdot w_{\rho, \sigma, p} = -\sigma(1 - \rho) q^{n-2p+1} w_{\rho+1, \sigma-1, p} + [p][n-p+1] w_{\rho, \sigma, p-1},$$

$$(11) \quad E_2 \cdot w_{\rho, \sigma, p} = \rho[\tilde{\alpha} + p + \sigma] w_{\rho-1, \sigma, p} + \sigma(-1)^\rho q^{-\tilde{\alpha}-p} w_{\rho, \sigma-1, p+1}.$$

Here the super grading of this basis is given by $|w_{\rho, \sigma, p}| = \rho + \sigma \in \mathbb{Z}/2\mathbb{Z}$.

Since $w_{1,1,n}$ is a lowest weight vector of $V(n, \tilde{\alpha})$ with weight $(-n, \tilde{\alpha} + \tilde{n} + \tilde{1})$ then we have

$$V(n, \tilde{\alpha})^* = V(n, -\tilde{\alpha} - \tilde{n} - \tilde{1}).$$

We will use the modules of the form $V(0, \tilde{\alpha})$ extensively. With this in mind we highlight the structure of such modules. If $\tilde{\alpha} \notin (\frac{l}{2}\mathbb{Z})/l\mathbb{Z} \cup ((\frac{l}{2}\mathbb{Z}) - (n+1))/l\mathbb{Z}$ then $V(0, \tilde{\alpha})$ is a 4-dimensional module with the following action:

	$w_{0,0}$ $\rho = 0, \sigma = 0$	$w_{1,0}$ $\rho = 0, \sigma = 0$	$w_{0,1}$ $\rho = 0, \sigma = 0$	$w_{1,1}$ $\rho = 0, \sigma = 0$
K_1	$w_{0,0}$	$q w_{1,0}$	$q^{-1} w_{0,1}$	$w_{1,1}$
K_2	$q^{\tilde{\alpha}} w_{0,0}$	$q^{\tilde{\alpha}} w_{1,0}$	$q^{\tilde{\alpha}+1} w_{0,1}$	$q^{\tilde{\alpha}+1} w_{1,1}$
E_1	0	0	$w_{1,0}$	0
E_2	0	$[\tilde{\alpha}] w_{0,0}$	0	$[\tilde{\alpha} + 1] w_{0,1}$
F_1	0	$w_{0,1}$	0	0
F_2	$w_{1,0}$	0	$w_{1,1}$	0

Remark 3.4.6. *In this remark and the following lemma let \mathcal{C} is an additive pivotal \mathbb{k} -category (\mathcal{C} will be a subcategory of \mathcal{D}). Let $n \in \mathbb{N}$ and $V, V_i \in \text{Ob}(\mathcal{C})$ for $i \in \{1, \dots, n\}$. Then, the following property holds (in the category \mathcal{C}):*

$$\text{Hom}_{\mathcal{C}}(V, V_1 \oplus \dots \oplus V_n) = \text{Hom}_{\mathcal{C}}(V, V_1) \oplus \dots \oplus \text{Hom}_{\mathcal{C}}(V, V_n).$$

Proof. For $i \in \{1, \dots, n\}$, let us consider the following inclusions and projections:

$$\iota_i \in \text{Hom}_{\mathcal{C}}(V_i, V_1 \oplus \dots \oplus V_n)$$

$$p_i \in \text{Hom}_{\mathcal{C}}(V_1 \oplus \dots \oplus V_n, V_i)$$

We notice that the above relations hold:

$$(12) \quad p_i \circ \iota_i = \text{Id}_{V_i}$$

$$(13) \quad \text{Id}_{V_1 \oplus \dots \oplus V_n} = \sum_{i=1}^n \iota_i \circ p_i.$$

Let us consider the following functions:

$$\mathcal{F} : \text{Hom}_{\mathcal{C}}(V, V_1 \oplus \dots \oplus V_n) \rightarrow \text{Hom}_{\mathcal{C}}(V, V_1) \oplus \dots \oplus \text{Hom}_{\mathcal{C}}(V, V_n)$$

$$\mathcal{F}(f) := (p_1 \circ f, \dots, p_n \circ f).$$

$$\begin{aligned}\mathcal{G} : \text{Hom}_{\mathcal{C}}(V, V_1) \oplus \dots \oplus \text{Hom}_{\mathcal{C}}(V, V_n) &\rightarrow \text{Hom}_{\mathcal{C}}(V, V_1 \oplus \dots \oplus V_n) \\ \mathcal{G}((g_1, \dots, g_n)) &:= \iota_1 \circ g_1 + \dots + \iota_n \circ g_n.\end{aligned}$$

From 12, 13, it follows that the functions \mathcal{F} and \mathcal{G} are inverse one to the other. \square

We will use the following lemma in the proof of the Decomposition Lemma below.

Lemma 3.4.7. *Let \mathcal{C} be an additive pivotal \mathbb{k} -category. Let V be an object in \mathcal{D} . Suppose V_1, \dots, V_n are simple submodules of V such that V_i is not isomorphic to V_j for all $i \neq j$ and*

$$\dim(V_1) + \dots + \dim(V_n) = \dim(V).$$

Then $V = V_1 \oplus \dots \oplus V_n$.

Proof. Consider the statement $P(k)$: if $i_1, \dots, i_k, j \in \{1, \dots, n\}$ are all pairwise different then

$$V_j \cap (V_{i_1} + \dots + V_{i_k}) = \{0\}.$$

If $P(k)$ were true for $k \in \{1, \dots, n-1\}$ then $V_1 \oplus \dots \oplus V_n$ would be a submodule of V . The hypothesis on the dimensions would then imply

$$V = V_1 \oplus \dots \oplus V_n.$$

Thus, it suffices to prove $P(k)$ holds for $k \in \{1, \dots, n-1\}$. We will do this by induction.

First, we will show $P(1)$ holds. Let $i, j \in \{1, \dots, n\}$ be such that $V_i \cap V_j \neq \{0\}$ and $i \neq j$. Therefore, there is a non-zero vector in $V_i \cap V_j$ which generates a submodule W of $V_i \cap V_j$. In particular, W is a submodule of both V_i and V_j . Since these modules are simple we have W is equal to both V_i and V_j . Thus, $V_i = V_j$ which is a contradiction.

Next assuming $P(k)$ is true we will show $P(k+1)$ holds. Let i_1, \dots, i_{k+1} and j be pairwise distinct elements of $\{1, \dots, n\}$. Suppose by contradiction that $V_j \cap (V_{i_1} + \dots + V_{i_{k+1}}) \neq \{0\}$ and let v be a non-zero vector in this intersection. Let $\langle v \rangle$ be the non-zero module generated by v . Since $v \in V_j$ then $\langle v \rangle$ is a submodule of V_j . But V_j is simple so $\langle v \rangle = V_j$. Similarly, $v \in V_{i_1} + \dots + V_{i_{k+1}}$ implying $\langle v \rangle$ is in this sum and we conclude

$$V_j \subseteq V_{i_1} + \dots + V_{i_{k+1}}.$$

From the induction step for $P(k)$, we deduce that $V_{i_1} + \dots + V_{i_{k+1}} = V_{i_1} \oplus \dots \oplus V_{i_{k+1}}$. Combining the last two observations, we have V_j is a submodule of $V_{i_1} \oplus \dots \oplus V_{i_{k+1}}$. This implies $\text{Hom}_{\mathcal{C}}(V_j, V_{i_1} \oplus \dots \oplus V_{i_{k+1}}) \neq \{0\}$ since the inclusion morphism is in this space.

On the other hand, since the simple modules V_j and V_{i_s} are non-isomorphic for $s \in \{1, \dots, k+1\}$ then $\text{Hom}_{\mathcal{C}}(V_j, V_{i_s}) = \{0\}$. This implies

$$\text{Hom}_{\mathcal{C}}(V_j, V_{i_1} \oplus \dots \oplus V_{i_{k+1}}) = \text{Hom}_{\mathcal{C}}(V_j, V_{i_1}) \oplus \dots \oplus \text{Hom}_{\mathcal{C}}(V_j, V_{i_{k+1}}) = \{0\}.$$

But above we showed this homomorphism space was non-zero so we have a contradiction. Thus, the induction step is complete. \square

Lemma 3.4.8. (*Decomposition Lemma*) *Let $\tilde{\alpha}, \tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ with $\bar{\alpha}, \bar{\beta} \notin \{\bar{0}, \bar{\frac{l}{2}}\}$ and $\bar{\alpha} + \bar{\beta} \notin \{\bar{0}, \bar{\frac{l}{2}}\}$. Then for any $n \in \{0, \dots, l' - 2\}$ we have*

$$V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta}) = V(n, \tilde{\alpha} + \tilde{\beta}) \oplus V(n+1, \tilde{\alpha} + \tilde{\beta}) \oplus V(n-1, \tilde{\alpha} + \tilde{\beta} + 1) \oplus V(n, \tilde{\alpha} + \tilde{\beta} + 1)$$

where we use the convention that $V(-1, \tilde{\alpha} + \tilde{\beta} + 1) = 0$ in the case when $n = 0$.

Proof. We will prove the case when $n \neq 0$ (the case $n = 0$ will be analogous). Since $\tilde{\alpha}, \tilde{\beta} \notin \{0, \frac{1}{2}\}$, it means that $V(0, \tilde{\alpha})$ and $V(n, \tilde{\beta})$ have the structure described in Theorem 3.4.5: let $\{w_{\rho, \sigma}^{(0, \tilde{\alpha})}\}$ and $\{w_{\rho, \sigma, p}^{(n, \tilde{\beta})}\}$ be the corresponding bases of $V(0, \tilde{\alpha})$ and $V(n, \tilde{\beta})$, respectively.

We will prove that there are four highest weight vectors:

$$v_{(n, \tilde{\alpha} + \tilde{\beta})}, v_{(n+1, \tilde{\alpha} + \tilde{\beta})}, v_{(n-1, \tilde{\alpha} + \tilde{\beta} + 1)}, v_{(n, \tilde{\alpha} + \tilde{\beta} + 1)} \in V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta})$$

where the weight of $v_{(i, \tilde{\gamma})}$ is $(q^i, q^{\tilde{\gamma}})$. First, clearly $v_{(n, \tilde{\alpha} + \tilde{\beta})} := w_{0,0}^{(0, \tilde{\alpha})} \otimes w_{0,0,0}^{(n, \tilde{\beta})}$ is a highest weight vector of weight $(n, \tilde{\alpha} + \tilde{\beta})$. Second, we want to find a highest weight vector $v_2 = v_{(n+1, \tilde{\alpha} + \tilde{\beta})}$ with weight $(q^{n+1}, q^{\tilde{\alpha} + \tilde{\beta}})$. We'll search for v_2 as a combination of the form:

$$v_2 = w_{0,0}^{(0, \tilde{\alpha})} \otimes w_{1,0,0}^{(n, \tilde{\beta})} + c \cdot w_{1,0}^{(0, \tilde{\alpha})} \otimes w_{0,0,0}^{(n, \tilde{\beta})}.$$

To find c we check that E_1 and E_2 act by zero. For any c we have $E_1(v_2) = 0$. On the other hand, $E_2(v_2) = 0$ implies $c = -q^{-\tilde{\alpha}} \cdot \frac{[\tilde{\beta}]}{[\tilde{\alpha}]}$. So,

$$v_2 = w_{0,0}^{(0, \tilde{\alpha})} \otimes w_{1,0,0}^{(n, \tilde{\beta})} - q^{-\tilde{\alpha}} \cdot \frac{[\tilde{\beta}]}{[\tilde{\alpha}]} \cdot w_{1,0}^{(0, \tilde{\alpha})} \otimes w_{0,0,0}^{(n, \tilde{\beta})}$$

is a highest weight vector.

Third, we want a highest weight vector $v_3 = v_{(n-1, \tilde{\alpha} + \tilde{\beta} + 1)}$ of the form

$$c_1 w_{0,0}^{(0, \tilde{\alpha})} \otimes w_{0,1,0}^{(n, \tilde{\beta})} + c_2 w_{0,0}^{(0, \tilde{\alpha})} \otimes w_{1,0,1}^{(n, \tilde{\beta})} + c_3 w_{1,0}^{(0, \tilde{\alpha})} \otimes w_{0,0,1}^{(n, \tilde{\beta})} + c_4 w_{0,1}^{(0, \tilde{\alpha})} \otimes w_{0,0,0}^{(n, \tilde{\beta})}.$$

After checking the conditions which come from the action of E_1 and E_2 , and setting $c_2 = 1$ we obtain:

$$\begin{aligned} v_3 = & q^{-(n+1)} [1][n] \cdot w_{0,0}^{(0, \tilde{\alpha})} \otimes w_{0,1,0}^{(n, \tilde{\beta})} + w_{0,0}^{(0, \tilde{\alpha})} \otimes w_{1,0,1}^{(n, \tilde{\beta})} \\ & - \frac{1}{[\tilde{\alpha}]} (q^{-(\tilde{\alpha} + \tilde{\beta} + n + 1)} [1][n] + q^{-\tilde{\alpha}} [\tilde{\beta} + 1]) \cdot w_{1,0}^{(0, \tilde{\alpha})} \otimes w_{0,0,1}^{(n, \tilde{\beta})} \\ & - \frac{q^{-1} [1][n]}{[\tilde{\alpha}]} (q^{-(\tilde{\alpha} + \tilde{\beta} + n + 1)}) [1][n] + q^{-\tilde{\alpha}} [\tilde{\beta} + 1]) \cdot w_{0,1}^{(0, \tilde{\alpha})} \otimes w_{0,0,0}^{(n, \tilde{\beta})}. \end{aligned}$$

Similarly we obtain:

$$\begin{aligned} v_4 = v_{(n, \tilde{\alpha} + \tilde{\beta} + 1)} = & -q^{-\tilde{\alpha}} \frac{[\tilde{\alpha}]}{[\tilde{\beta} + 1]} w_{0,0}^{(0, \tilde{\alpha})} \otimes w_{1,1,0}^{(n, \tilde{\beta})} \\ & + q^{-\tilde{\alpha} - 1} \frac{[\tilde{\beta}]}{[\tilde{\alpha} + 1]} (q^{-n} + q^{-1 - \tilde{\beta}} \frac{[1][n]}{[\tilde{\beta} + 1]}) w_{1,1}^{(0, \tilde{\alpha})} \otimes w_{0,0,0}^{(n, \tilde{\beta})} \\ & + (q^{-n} + q^{-1 - \tilde{\beta}} \frac{[1][n]}{[\tilde{\beta} + 1]}) w_{0,1}^{(0, \tilde{\alpha})} \otimes w_{1,0,0}^{(n, \tilde{\beta})} \\ & + w_{1,0}^{(0, \tilde{\alpha})} \otimes w_{0,1,0}^{(n, \tilde{\beta})} - \frac{q^{-\tilde{\beta}}}{[\tilde{\beta} + 1]} w_{1,0}^{(0, \tilde{\alpha})} \otimes w_{1,0,1}^{(n, \tilde{\beta})}. \end{aligned}$$

Consider the submodule $W_{(i, \tilde{\gamma})}$ of $V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta})$ generated by one of the highest weight vectors $v_{(i, \tilde{\gamma})}$ constructed above. As mentioned above the classification of $U_q(\mathfrak{sl}(2|1))$ -highest weight modules is given in [1]. From this classification, since

$W_{(i,\tilde{\gamma})}$ is a highest weight module of weight $(q^i, q^{\tilde{\gamma}})$, with $\tilde{\gamma} = \bar{\alpha} + \bar{\beta} \notin \{\bar{0}, \frac{\bar{1}}{2}\}$ it follows that $W_{(i,\tilde{\gamma})}$ is isomorphic to $V(i, \tilde{\gamma})$ and is a simple of dimension $4(i+1)$.

Thus, we have

$$\begin{aligned} & \dim(W_{(n,\bar{\alpha}+\bar{\beta})}) + \dim(W_{(n+1,\bar{\alpha}+\bar{\beta})}) + \dim(W_{(n-1,\bar{\alpha}+\bar{\beta}+1)}) + \dim(W_{(n,\bar{\alpha}+\bar{\beta}+1)}) \\ &= 4((n+1) + (n+2) + n + (n+1)) = 4(4n+4) = 16(n+1). \end{aligned}$$

But $\dim(V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta})) = 4 \cdot 4(n+1) = 16(n+1)$. So, the four submodules satisfy the conditions of Lemma 3.4.7, which means that their direct sum is isomorphic to $V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta})$. \square

From the previous result, we obtain that, with some weight restrictions, the decomposition of the tensor product of two typical modules depends just on the total weight-sum, and it is independent of the two separate components. More precisely:

Corollary 3.4.9. *Consider $\tilde{\alpha}, \tilde{\beta} \in \mathbb{C}/l\mathbb{Z}, n \in \{0, \dots, l' - 1\}$ such that $\bar{\alpha}, \bar{\beta}, \bar{\alpha} + \bar{\beta} \notin \{\bar{0}, \frac{\bar{1}}{2}\}$. Then*

$$V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta}) \simeq V(0, \widetilde{\bar{\alpha} + \epsilon}) \otimes V(n, \widetilde{\bar{\beta} - \epsilon})$$

for any $\epsilon \in \mathbb{C}$ such that $\overline{\bar{\alpha} + \epsilon}, \overline{\bar{\beta} - \epsilon} \notin \{\bar{0}, \frac{\bar{1}}{2}\}$.

For $g \in \mathbb{C}/\mathbb{Z}$, let \mathcal{D}_g be the full subcategory of \mathcal{D} whose objects are all $U_q(\mathfrak{sl}(2|1))$ -modules V such that the central element K_2^l acts as multiplication by q^{lg} . In particular, for $0 \leq n \leq l' - 1$ and $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ we have $V(n, \tilde{\alpha}) \in \mathcal{D}_{\tilde{\alpha}}$. This gives a \mathbb{C}/\mathbb{Z} -grading on the category \mathcal{D} and we write $\mathcal{D} = \bigoplus_{g \in \mathbb{C}/\mathbb{Z}} \mathcal{D}_g$.

3.5. The subcategory \mathcal{C} of \mathcal{D} . Now we want to construct a subcategory \mathcal{C} of \mathcal{D} that will eventually (after taking a quotient) lead to our invariants for 3-manifolds.

Definition 3.5.1. *Set $\mathcal{Y} = (\frac{1}{4}\mathbb{Z})/\mathbb{Z}$. Let \mathcal{C} be the full sub-category of \mathcal{D} containing the trivial module and all retracts of a module of the form*

$$(14) \quad V(0, \tilde{\alpha}_1) \otimes V(0, \tilde{\alpha}_2) \otimes \dots \otimes V(0, \tilde{\alpha}_n)$$

where $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\alpha}_1, \dots, \bar{\alpha}_n \in (\mathbb{C}/\mathbb{Z}) \setminus \mathcal{Y}$.

Lemma 3.5.2. *The category \mathcal{C} is a \mathbb{C}/\mathbb{Z} -graded pivotal \mathbb{C} -category, where the grading and pivotal structure are induced from those of \mathcal{D} .*

Proof. Let W_1 and W_2 be in \mathcal{C} . From the definition, for $j = 1, 2$, there exist $\tilde{\alpha}_{j,1}, \dots, \tilde{\alpha}_{j,n_j} \in \mathbb{C}/l\mathbb{Z}$, with $\bar{\alpha}_{j,1}, \dots, \bar{\alpha}_{j,n_j} \in (\mathbb{C}/\mathbb{Z}) \setminus \mathcal{Y}$ such that W_j be is retract of $V_j := V(0, \tilde{\alpha}_{j,1}) \otimes \dots \otimes V(0, \tilde{\alpha}_{j,n_j})$. Let $p_j : V_j \rightarrow W_j$ and $q_j : W_j \rightarrow V_j$ be the morphisms of this retract. Then $V_1 \otimes V_2$ is of the form of the module in Equation (14) with all $\bar{\alpha}_{j,n} \notin \mathcal{Y}$. It follows that $W_1 \otimes W_2$ is an object of \mathcal{C} since it is a retract of $V_1 \otimes V_2$ with maps $p_1 \otimes p_2$ and $q_1 \otimes q_2$. Therefore, \mathcal{C} is a tensor category. Moreover, \mathcal{C} is \mathbb{C} -category since it is a full sub-category of the \mathbb{C} -category \mathcal{D} .

Finally, we will check that \mathcal{C} is closed under duality. Let $W \in \mathcal{C}$. Then W is a retract of some $V := V(0, \tilde{\alpha}_1) \otimes \dots \otimes V(0, \tilde{\alpha}_n)$ such that $\bar{\alpha}_1, \dots, \bar{\alpha}_n \in (\mathbb{C}/\mathbb{Z}) \setminus \mathcal{Y}$. Then W^* is a retract of V^* and we have that:

$$\begin{aligned} V^* &\cong (V(0, \tilde{\alpha}_1) \otimes \dots \otimes V(0, \tilde{\alpha}_n))^* \cong V(0, \tilde{\alpha}_n)^* \otimes \dots \otimes V(0, \tilde{\alpha}_1)^* \\ &\cong V(0, -\tilde{\alpha}_n - \tilde{1}) \otimes \dots \otimes V(0, -\tilde{\alpha}_1 - \tilde{1}). \end{aligned}$$

But $-\bar{\alpha}_n - \bar{1}, \dots, -\bar{\alpha}_1 - \bar{1} \in (\mathbb{C}/\mathbb{Z}) \setminus \mathcal{Y}$ so we have $W^* \in \mathcal{C}$. Thus, since \mathcal{C} is a full subcategory of \mathcal{D} then the duality morphisms of \mathcal{D} give a pivotal structure in \mathcal{C} . Finally, the \mathbb{C}/\mathbb{Z} -grading on \mathcal{D} induces a \mathbb{C}/\mathbb{Z} -grading on \mathcal{C} . \square

The Decomposition Lemma 3.4.8 says we can decompose the tensor product $V(0, \tilde{\alpha}) \otimes V(0, \tilde{\beta})$ into simple modules if $\bar{\alpha} + \bar{\beta} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. Given a module as in Equation (14), the following lemma says we can always find a pair $\tilde{\alpha}_i, \tilde{\alpha}_j$ with this property. This fact is one of the motivations for the choice of the set \mathcal{Y} .

Lemma 3.5.3. *For any $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n \in \mathbb{C}/\mathbb{Z}$ such that $\bar{\alpha}_1, \dots, \bar{\alpha}_n \in (\mathbb{C}/\mathbb{Z}) \setminus \mathcal{Y}$ and*

$$\bar{\alpha}_1 + \dots + \bar{\alpha}_n \notin \{\bar{0}, \frac{\bar{l}}{2}\}$$

there exist $i, j \in \{1, \dots, n\}$ such that $i \neq j$ and $\bar{\alpha}_i + \bar{\alpha}_j \notin \{\bar{0}, \frac{\bar{l}}{2}\}$.

Proof. If $n = 2$, we have just two numbers and from the hypothesis they have the desired sum.

Let us consider the case $n \geq 3$ and let suppose by contradiction that

$$(15) \quad \overline{\alpha_i + \alpha_j} \in \{\bar{0}, \frac{\bar{l}}{2}\},$$

for all $i, j \in \{1, \dots, n\}$ with $i \neq j$. Up to a reordering, we can suppose that there exists $m \in \{2, \dots, n\}$ such that:

- $\overline{\alpha_1 + \alpha_i} = \bar{0}, \forall i \in \{2, \dots, m\}$
- $\overline{\alpha_1 + \alpha_j} = \frac{\bar{l}}{2}, \forall j \in \{m+1, \dots, n\}$.

This implies the following:

- $\bar{\alpha}_i = -\bar{\alpha}_1, \forall i \in \{2, \dots, m\}$
- $\bar{\alpha}_i = \frac{\bar{l}}{2} - \bar{\alpha}_1, \forall j \in \{m+1, \dots, n\}$.

Now we have three cases.

Case 1. If $m \geq 3$, then $\bar{\alpha}_2 = \bar{\alpha}_3 = -\bar{\alpha}_1$ which implies

$$\bar{\alpha}_2 + \bar{\alpha}_3 = -2\bar{\alpha}_1 \notin \{\bar{0}, \frac{\bar{l}}{2}\}, \quad \text{since } \bar{\alpha}_1 \notin \{\frac{\bar{l}}{2}, \frac{\bar{l}}{4}\}$$

which is a contradiction with our supposition.

Case 2. If $n - m \geq 2$, then $\bar{\alpha}_{m+1} = \bar{\alpha}_{m+2} = \frac{\bar{l}}{2} - \bar{\alpha}_1$ which implies

$$\bar{\alpha}_{m+1} + \bar{\alpha}_{m+2} = -2\bar{\alpha}_1.$$

Here as above this leads to a contradiction.

Case 3. If we are not in the first two cases and $n \neq 2$ then it means $n = 3$ and $m = 2$. In this case we have

- $\bar{\alpha}_2 = -\bar{\alpha}_1$
- $\bar{\alpha}_3 = \frac{\bar{l}}{2} - \bar{\alpha}_1$.

The relations above lead to:

$$\bar{\alpha}_2 + \bar{\alpha}_3 = \frac{\bar{l}}{2} - 2\bar{\alpha}_1.$$

But from the initial supposition, we have that $\bar{\alpha}_2 + \bar{\alpha}_3 \in \{\bar{0}, \frac{\bar{l}}{2}\}$.

If $\bar{\alpha}_2 + \bar{\alpha}_3 = \bar{0}$, it implies that $\frac{\bar{l}}{2} - 2\bar{\alpha}_1 = \bar{0}$, so $\bar{\alpha}_1 = \frac{\bar{l}}{4}$ which contradicts that $\bar{\alpha}_1 \notin \mathcal{Y} = \frac{1}{4}\mathbb{Z}/\mathbb{Z}$.

If $\bar{\alpha}_2 + \bar{\alpha}_3 = \frac{\bar{l}}{2}$, then $\frac{\bar{l}}{2} - 2\bar{\alpha}_1 = \frac{\bar{l}}{2}$, and it means $\bar{\alpha}_1 \in \{\bar{0}, \frac{\bar{l}}{2}\}$ which is impossible since $\bar{\alpha}_1 \notin \mathcal{Y}$.

Thus all cases lead to contradictions and so the lemma follows. \square

The next part is devoted to an argument that will lead to the fact that the tensor product of simple modules in the alcove is commutative. The proof uses the braiding of the “un-rolled” quantum $U_q^H(\mathfrak{sl}(2|1))$, studied by Ha in [17]. In his paper he works with odd ordered roots of unity but as we observe his proof also works for even roots of unity (at least for the existence of a braiding, it may not extend to the twist).

Let $U_q^H = U_q^H(\mathfrak{sl}(2|1))$ be the superalgebra generated by the elements K_i, K_i^{-1}, h_i, E_i and F_i , $i = 1, 2$, subject to the relations in (6) and

$$[h_i, E_j] = a_{ij}E_j, \quad [h_i, F_j] = -a_{ij}F_j, \quad [h_i, h_j] = 0, \quad [h_i, K_j] = 0$$

for $i, j = 1, 2$. All generators are even except E_2 and F_2 which are odd. This superalgebra is a Hopf superalgebra where the coproduct, counit and antipode of K_i, K_i^{-1}, E_i and F_i are given in Subsection 3.3 and

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad \epsilon(h_i) = 0, \quad S(h_i) = -h_i$$

for $i, j = 1, 2$.

For a U_q^H -module V let $q^{h_i} : V \rightarrow V$ be the operator defined by $q^{h_i}(v) = q^{\lambda_i}v$ where v is a weight vector with respect to h_i of weight λ_i . The superalgebra ideal I generated by $E_1^{l'}$ and $F_1^{l'}$ is a Hopf superalgebra ideal (i.e. an ideal in the kernel of the counit, a coalgebra coideal and stable under the antipode). Let \mathcal{D}^H be the category of finite dimensional U_q^H/I -modules with even morphisms such that $q^{h_i} = K_i$ as operators for $i = 1, 2$. Since U_q^H/I is a Hopf superalgebra then \mathcal{D}^H is a tensor category. Moreover, the maps given in Equation (7) define a pivotal structure on \mathcal{D}^H . There is a forgetful functor from \mathcal{D}^H to \mathcal{D} which forgets the action of h_1 and h_2 . Given two objects V, W of \mathcal{D}^H let $\mathcal{K} : V \otimes W \rightarrow V \otimes W$ be the operator defined by

$$\mathcal{K}(v \otimes w) = q^{-\lambda_1\mu_2 - \lambda_2\mu_1 - 2\lambda_2\mu_2} v \otimes w$$

where $h_i v = \lambda_i v$ and $h_i w = \mu_i w$ for $i = 1, 2$. Consider the truncated R -matrix:

$$(16) \quad \check{R} = \sum_{k=0}^{l'-1} \frac{\{1\}^k}{(k)_q!} E_1^k \otimes F_1^k \sum_{s=0}^1 \frac{(-\{1\})^s}{(s)_q!} E_3^s \otimes F_3^s \sum_{t=0}^1 \frac{(-\{1\})^t}{(t)_q!} E_2^t \otimes F_2^t$$

where $E_3 = E_1 E_2 - q^{-1} E_2 E_1$, $F_3 = F_2 F_1 - q F_1 F_2$, $(n)_q = \frac{1-q^n}{1-q}$ and $(n)_q! = (1)_q \cdots (n)_q$.

Theorem 3.5.4. *The family $\{c_{V,W} : V \otimes W \rightarrow W \otimes V\}_{V,W \in \mathcal{D}^H}$ defined by*

$$c_{V,W}(v \otimes w) = \tau(\check{R}\mathcal{K}(v \otimes w))$$

is a braiding on \mathcal{D}^H where τ is the super flip map $\tau(v \otimes w) = (-1)^{|w||v|} w \otimes v$.

Proof. The proof is essentially given by Ha in [17]. As mentioned above, in Theorem 3.6 of [17], Ha proves the theorem for odd order roots of unity. Ha’s proof works for even order roots of unity as well. In particular, before Proposition 3.5 of [17] Ha uses the PBW basis of $U_q(\mathfrak{sl}(2|1))$ to define an algebra $\mathcal{U}^<$. In our case, when defining this algebra one should take powers of E_1 and F_1 from 0 to $l' - 1$

not $l-1$. Then use this algebra to define the projection $p : U_q(\mathfrak{sl}(2|1)) \rightarrow \mathcal{U}^<$ and the element

$$\mathcal{R}^< = p \otimes p(\mathcal{R}_q)$$

where \mathcal{R}_q is the R -matrix defined in [22, 29] for $U_q(\mathfrak{sl}(2|1))$ when q is generic. With these modifications the proofs of Proposition 3.5 and Theorem 3.6 in [17] hold word for word for both the even and odd case. Note that at the end of the proof of Theorem 3.6 in [17] Ha says, “The element $\mathcal{R}^<$ has no pole when q is a root of unity of the order l .” This is true in our case because we defined $\mathcal{R}^<$ using p which only allows powers of E_1 or F_1 smaller than $l'-1$ which is analogous to the definition of \tilde{R} above. \square

For $(n, \alpha) \in \mathbb{N} \times \mathbb{C}$ with $0 \leq n \leq l'-1$ and $\bar{\alpha} \notin \{\bar{0}, \frac{\bar{1}}{2}\}$, one can check directly that there is a $U_q^H(\mathfrak{sl}(2|1))$ -module $V^H(n, \alpha)$ with basis $\{w_{\rho, \sigma, p}^\alpha | p \in \{0, \dots, n\}; \rho, \sigma \in \{0, 1\}\}$ whose action is given by

$$(17) \quad h_1 \cdot w_{\rho, \sigma, p}^\alpha = (\rho - \sigma + n - 2p)w_{\rho, \sigma, p}^\alpha, \quad h_2 \cdot w_{\rho, \sigma, p}^\alpha = (\alpha + \sigma + p)w_{\rho, \sigma, p}^\alpha,$$

and Equations (8), (9), (10), and (11) with $\tilde{\alpha}$ replaced with α . Moreover, by definition the operators q^{h_i} and K_i are equal on $V^H(n, \alpha)$.

Lemma 3.5.5. *For $n \in \{0, \dots, l'-1\}$ and $\alpha \in \mathbb{C}$ with $\bar{\alpha} \notin \{\bar{0}, \frac{\bar{1}}{2}\}$, the actions of $E_1^{l'}$ and $F_1^{l'}$ are zero on $V(n, \tilde{\alpha})$ and $V^H(n, \alpha)$.*

Proof. We will prove the theorem for $V(n, \tilde{\alpha})$ the proof for $V^H(n, \alpha)$ is essentially identical. Let us prove the action of $F_1^{l'}$ is zero on $V(n, \tilde{\alpha})$, the proof that $E_1^{l'}$ acts as zero is similar and left to the reader.

It is enough to prove that $F_1^{l'} w_{\rho, \sigma, p} = 0$ where $w_{\rho, \sigma, p}$ is any of the basis vectors given in Theorem 3.4.5. Equation (9) gives the action of F_1 on $V(n, \tilde{\alpha})$. In particular, if $\rho \neq 1$ and $\sigma \neq 0$ then $F_1 w_{\rho, \sigma, p} = q^{\sigma - \rho} w_{\rho, \sigma, p+1}$. Therefore, in this case,

$$F_1^{l'} w_{\rho, \sigma, p} = q^{l'(\sigma - \rho)} w_{\rho, \sigma, p+l'} = 0$$

since $w_{\rho, \sigma, i} = 0$ if $i \geq l'$.

Now a direct calculation implies:

$$F_1^k w_{1, 0, p} = q^{-k} w_{1, 0, p+k} - q^{k-2} \left(\sum_{i=0}^{k-1} q^{-2i} \right) w_{0, 1, p+k-1}.$$

When $k = l'$ we see each of these terms is zero, since $w_{\rho, \sigma, i} = 0$ if $i \geq l'$ and $\sum_{i=0}^{l'-1} q^{-2i} = \frac{1-q^{-2l'}}{1-q^{-2}} = 0$. \square

Corollary 3.5.6. *For $n \in \{0, \dots, l'-1\}$ and $\alpha \in \mathbb{C}$ with $\bar{\alpha} \notin \{\bar{0}, \frac{\bar{1}}{2}\}$, the $U_q^H(\mathfrak{sl}(2|1))$ -module $V^H(n, \alpha)$ is an object in \mathcal{D}^H .*

Lemma 3.5.7. *(Commutativity Lemma) Let $n, n' \in \mathbb{N}$ be such that $0 \leq n, n' \leq l'-1$ and $\tilde{\alpha}, \tilde{\alpha}' \in \mathbb{C}/l\mathbb{Z}$ such that $\tilde{\alpha}, \tilde{\alpha}' \notin \{\bar{0}, \frac{\bar{1}}{2}\}$. Let $\{w_{\rho, \sigma, p}\}$ and $\{w'_{\rho', \sigma', p'}\}$ be the bases given in Theorem 3.4.5 for $V(n, \tilde{\alpha})$ and $V(n', \tilde{\alpha}')$, respectively. Choose $\alpha, \alpha' \in \mathbb{C}$ such that $[\alpha] = \tilde{\alpha}$ and $[\alpha'] = \tilde{\alpha}'$ in $\mathbb{C}/l\mathbb{Z}$. Then there exists an isomorphism*

$$\psi_{\alpha, \alpha'} : V(n, \tilde{\alpha}) \otimes V(n', \tilde{\alpha}') \rightarrow V(n', \tilde{\alpha}') \otimes V(n, \tilde{\alpha})$$

such that

$$\psi_{\alpha, \alpha'}(w_{0, 0, 0} \otimes w'_{\rho', \sigma', p'}) = q^{-n(\alpha' + \sigma' + p') - \alpha(\rho' - \sigma' + n' - 2p') - 2\alpha(\alpha' + \sigma' + p')} w'_{\rho', \sigma', p'} \otimes w_{0, 0, 0}$$

and

(18)

$$\psi_{\alpha,\alpha'}(w_{\rho,\sigma,p} \otimes w'_{0,0,0}) = q^{-(\rho-\sigma+n-2p)\alpha' - (\alpha+\sigma+p)n' - 2\alpha'(\alpha+\sigma+p)} w'_{0,0,0} \otimes w_{\rho,\sigma,p} + \sum_{i \in I} c_i x_i \otimes y_i.$$

where each x_i is a basis element in $\{w'_{\rho',\sigma',p'}\}$ not equal to $w'_{0,0,0}$ and the coefficients $c_i \in \mathbb{C}$ for all $i \in I$.

Proof. Recall the forgetful functor from \mathcal{D}^H to \mathcal{D} . Lemma 3.5.5 and Corollary 3.5.6 imply that $V^H(n, \alpha)$ maps to $V(n, \tilde{\alpha})$ under this functor. Similarly, $V^H(n', \alpha')$ maps to $V(n', \tilde{\alpha}')$. Now the braiding $c_{V^H(n, \alpha), V^H(n', \alpha')}$ of Theorem 3.5.4 under the forgetful functor gives the desired isomorphism $\psi_{\alpha,\alpha'}$ in \mathcal{D} .

We have

$$\psi_{\alpha,\alpha'}(w_{\rho,\sigma,p} \otimes w'_{\rho',\sigma',p'}) = \tau(\check{R}\mathcal{K}(w_{\rho,\sigma,p}^\alpha \otimes w'_{\rho',\sigma',p'}^{\alpha'}))$$

where

$$\mathcal{K}(w_{0,0,0}^\alpha \otimes w'_{\rho',\sigma',p'}^{\alpha'}) = q^{-n_1(\alpha'+\sigma'+p') - \alpha(\rho'-\sigma'+n'-2p') - 2\alpha(\alpha'+\sigma'+p')} w_{0,0,0}^\alpha \otimes w'_{\rho',\sigma',p'}^{\alpha'}$$

Since $E_1 w_{0,0,0}^\alpha = E_2 w_{0,0,0}^\alpha = 0$ it follows that $\check{R}(w_{0,0,0}^\alpha \otimes w'_{\rho',\sigma',p'}^{\alpha'}) = w_{0,0,0}^\alpha \otimes w'_{\rho',\sigma',p'}^{\alpha'}$ and the first formula in the lemma holds.

To prove the second formula, recall from Equation (16) that

$$\check{R} = 1 \otimes 1 + \sum_i d_i a_i \otimes b_i$$

where each b_i is of the form $F_1^k F_3^s F_2^t$ where at least one of the indices k, s or t is non-zero. Therefore, from the defining relations of Theorem 3.4.5 we have $b_i w'_{0,0,0}^{\alpha'}$ is a linear combination of basis vectors $w'_{\rho',\sigma',p'}^{\alpha'}$ where ρ', σ', p' are not all zero (since the action of either F_1 or F_2 on any basis vector increases at least one of the indices of the vector, see Equation (9)). Combining the above we have

$$\begin{aligned} \check{R}(w_{\rho,\sigma,p}^\alpha \otimes w'_{0,0,0}^{\alpha'}) &= w_{\rho,\sigma,p}^\alpha \otimes w'_{0,0,0}^{\alpha'} + \sum_i d_i (a_i \otimes b_i) (w_{\rho,\sigma,p}^\alpha \otimes w'_{0,0,0}^{\alpha'}) \\ &= w_{\rho,\sigma,p}^\alpha \otimes w'_{0,0,0}^{\alpha'} + \sum_j d'_j y_j \otimes x_j \end{aligned}$$

where each x_i is a basis element in $\{w'_{\rho',\sigma',p'}^{\alpha'}\}$ not equal to $w'_{0,0,0}^{\alpha'}$. Thus, since \mathcal{K} acts diagonally on the basis, we just need to compute $\mathcal{K}(w_{\rho,\sigma,p}^\alpha \otimes w'_{0,0,0}^{\alpha'})$. This can be done as above to obtain Equation (18). \square

Remark 3.5.8. Clearly, the isomorphism $\psi_{\alpha,\alpha'}$ in Lemma 3.5.7 depends on the choice of α and α' .

Lemma 3.5.9. Consider $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n \in \mathbb{C}/l\mathbb{Z}$ with $\tilde{\alpha}_i \notin \mathcal{Y}, i \in \{1, \dots, n\}$. From the Lemma 3.5.3, there exists i, j such that $i < j$ and $\tilde{\alpha}_i + \tilde{\alpha}_j \notin \{\bar{0}, \frac{l}{2}\}$. Then, for any $\epsilon \in \mathbb{C}/l\mathbb{Z}$ such that: $\tilde{\alpha}_j - \bar{\epsilon} \notin \{\bar{0}, \frac{l}{2}\}$ and $\tilde{\alpha}_i + \bar{\epsilon} \notin \{\bar{0}, \frac{l}{2}\}$ we can modify the weights without changing the tensor product in the following way:

$$\begin{aligned} V(0, \tilde{\alpha}_1) \otimes \dots \otimes V(0, \tilde{\alpha}_i) \otimes \dots \otimes V(0, \tilde{\alpha}_j) \otimes \dots \otimes V(0, \tilde{\alpha}_n) &\simeq \\ V(0, \tilde{\alpha}_1) \otimes \dots \otimes V(0, \tilde{\alpha}_i + \epsilon) \otimes \dots \otimes V(0, \tilde{\alpha}_j - \epsilon) \otimes \dots \otimes V(0, \tilde{\alpha}_n). \end{aligned}$$

Proof. From the choice of ϵ , Lemma 3.4.9 implies

$$V(0, \tilde{\alpha}_i) \otimes V(0, \tilde{\alpha}_j) \simeq V(0, \tilde{\alpha}_i + \epsilon) \otimes V(0, \tilde{\alpha}_j - \epsilon).$$

Combining this isomorphism with Lemma 3.5.7 we have the following isomorphisms:

$$\begin{aligned} & V(0, \tilde{\alpha}_1) \otimes \dots \otimes V(0, \tilde{\alpha}_i) \otimes \dots \otimes V(0, \tilde{\alpha}_j) \otimes \dots \otimes V(0, \tilde{\alpha}_n) \\ & \simeq V(0, \tilde{\alpha}_1) \otimes \dots \otimes \hat{V}(0, \tilde{\alpha}_i) \otimes \dots \otimes \hat{V}(0, \tilde{\alpha}_j) \otimes \dots \otimes V(0, \tilde{\alpha}_n) \otimes V(0, \tilde{\alpha}_i) \otimes V(0, \tilde{\alpha}_j) \\ & \simeq V(0, \tilde{\alpha}_1) \otimes \dots \otimes \hat{V}(0, \tilde{\alpha}_i) \otimes \dots \otimes \hat{V}(0, \tilde{\alpha}_j) \otimes \dots \otimes V(0, \tilde{\alpha}_n) \otimes V(0, \tilde{\alpha}_i + \epsilon) \otimes V(0, \tilde{\alpha}_j - \epsilon) \\ & \simeq V(0, \tilde{\alpha}_1) \otimes \dots \otimes V(0, \tilde{\alpha}_i + \epsilon) \otimes \dots \otimes V(0, \tilde{\alpha}_j - \epsilon) \otimes \dots \otimes V(0, \tilde{\alpha}_n). \end{aligned}$$

This concludes the proof. \square

4. THE RIGHT TRACE AND ITS MODIFIED DIMENSION

4.1. The existence of the right trace. In Subsection 2.4 we recalled several results about right traces. Here we apply these results to construct a right trace on the ideal generated by $V(0, \tilde{\alpha})$ for $\tilde{\alpha} \notin \{\bar{0}, \frac{\bar{1}}{2}\}$.

We've seen in the Decomposition Lemma 3.4.8 that for $\tilde{\alpha}, \tilde{\beta} \in \mathbb{C}/\mathbb{Z}$ such that $\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha} + \tilde{\beta} \notin \{\bar{0}, \frac{\bar{1}}{2}\}$ we have the following decomposition:

$$V(0, \tilde{\alpha}) \otimes V(0, \tilde{\beta}) = V(0, \tilde{\alpha} + \tilde{\beta}) \oplus V(0, \tilde{\alpha} + \tilde{\beta} + 1) \oplus V(1, \tilde{\alpha} + \tilde{\beta}).$$

In the case

$$V(0, \tilde{\alpha}) \otimes V(0, \tilde{\alpha})^* = V(0, \tilde{\alpha}) \otimes V(0, -\tilde{\alpha} - 1)$$

the decomposition is no longer semi-simple, and the two 4-dimensional modules corresponding to $V(0, -1)$ and $V(0, 0)$ merge into an indecomposable non-simple 8-dimensional module which we will denote by $V_1(\tilde{\alpha})$. More precisely we have the following result:

Proposition 4.1.1. *Let $\tilde{\alpha} \in \mathbb{C}/\mathbb{Z}$ be such that $\tilde{\alpha} \notin \{\bar{0}, \frac{\bar{1}}{2}\}$. We have the following decomposition:*

$$(19) \quad V(0, \tilde{\alpha}) \otimes V(0, \tilde{\alpha})^* = V_1(\tilde{\alpha}) \oplus V_2(\tilde{\alpha})$$

where $V_2(\tilde{\alpha})$ is an 8-dimensional simple module and $V_1(\tilde{\alpha})$ is an indecomposable module such that $\text{Hom}_{\mathcal{E}}(\mathbb{C}, V_1(\tilde{\alpha}))$ and $\text{Hom}_{\mathcal{E}}(V_1(\tilde{\alpha}), \mathbb{C})$ are both non-zero, where \mathbb{C} is the trivial module (3.4.1).

Proof. Recall $V(0, \tilde{\alpha})^*$ is isomorphic to $V(0, -\tilde{\alpha} - 1)$. Let $\{w_{\rho, \sigma}^{\tilde{\alpha}}\}$ and $\{w_{\rho, \sigma}^{-\tilde{\alpha}-1}\}$ be the bases of $V(0, \tilde{\alpha})$ and $V(0, -\tilde{\alpha} - 1)$ given in Theorem 3.4.5. Consider the vectors of $V(0, \tilde{\alpha}) \otimes V(0, -\tilde{\alpha} - 1)$:

$$v_7 = q^{-\tilde{\alpha}-1}[\tilde{\alpha}]w_{1,1}^{\tilde{\alpha}} \otimes w_{0,0}^{-\tilde{\alpha}-1} + q^{\tilde{\alpha}}[\tilde{\alpha} + 1]w_{0,0}^{\tilde{\alpha}} \otimes w_{1,1}^{-\tilde{\alpha}-1}$$

and

$$u_0 = [\alpha]w_{0,0}^{\tilde{\alpha}} \otimes w_{1,0}^{-\tilde{\alpha}-1} + q^{-\tilde{\alpha}}[\alpha + 1]w_{1,0}^{\tilde{\alpha}} \otimes w_{0,0}^{-\tilde{\alpha}-1}.$$

Let $V_1(\tilde{\alpha})$ and $V_2(\tilde{\alpha})$ be the modules generated by v_7 and u_0 , respectively. The action of these modules is given in Tables 1 and 2 where $\{v_i\}$ and $\{u_i\}$ are bases for the corresponding modules.

We will show that module $V_1(\tilde{\alpha})$ is indecomposable. Suppose W_1 and W_2 are modules such $V_1(\tilde{\alpha}) = W_1 \oplus W_2$. Since $\{v_i\}$ is a basis of $V_1(\tilde{\alpha})$ there exists

$$v = c_0v_0 + c_1v_1 + c_2v_2 + \dots + c_7v_7$$

such that $c_7 \neq 0$ and $v \in W_1$ or $v \in W_2$. Without loss of generality assume $v \in W_1$. From Table 1 we have $F_2 E_2 E_1 E_2 v$ is a non-zero multiple of v_1 . So $v_1 \in W_1$. Then Table 1 implies that

$$\{v_1, c^{-1} E_2 v_1, F_1 v_1, F_2 F_1 v_1\} = \{v_0, v_1, v_2, v_3\} \subset W_1.$$

Similarly, $E_2 F_2 F_1 F_2 v$ is a non-zero multiple of v_5 so $v_5 \in W_1$ and

$$\{v_5, E_1 v_5, -c^{-1} F_2 v_5\} = \{v_4, v_5, v_6\} \subset W_1.$$

Since W_1 is a submodule we have

$$v_7 = c_7^{-1}(v - c_0 v_0 - c_1 v_1 - c_2 v_2 - \dots - c_6 v_6) \in W_1$$

Thus, $W_1 = V_1(\tilde{\alpha})$ and we have shown that $V_1(\tilde{\alpha})$ is indecomposable.

Next we will show that $V_2(\tilde{\alpha})$ is simple. Suppose U is a non-zero submodule of $V_2(\tilde{\alpha})$. Notice that the generator u_0 of $V_2(\tilde{\alpha})$ is a highest weight vector. The idea is to push any non-zero vector of U to a multiple of u_0 . So let u be a non-zero vector of U .

Write u in terms of the basis $\{u_i\}$:

$$u = c_0 u_0 + c_1 u_1 + c_2 u_2 + \dots + c_7 u_7.$$

If there exists an element x in $U_q(\mathfrak{sl}(2|1))$ such that xu is a non-zero multiple of u_0 then since u_0 is a generator of $V_2(\tilde{\alpha})$ we would have $U \cong V_2(\tilde{\alpha})$. We will show this is true for all possible non-zero coefficients of u .

- (1) If $c_4 \neq 0$ then from the $U_q(\mathfrak{sl}(2|1))$ -action given in the above table we have $E_1 E_2 E_1 E_2 u$ is a non-zero multiple of u_0 .
- (2) If $c_4 = 0$ and $c_7 \neq 0$ then $E_2 E_1 E_2 u$ is a non-zero multiple of u_0 .
- (3) If $c_4 = c_7 = 0$ and $c_3 \neq 0$ then $E_1 E_2 E_1 u$ is a non-zero multiple of u_0 .
- (4) If $c_4 = c_7 = c_3 = 0$ and $c_6 \neq 0$ then $E_1 E_2 u$ is a non-zero multiple of u_0 .
- (5) If $c_4 = c_7 = c_3 = c_6 = 0$ and $c_2 \neq 0$ then $E_2 E_1 u$ is a non-zero multiple of u_0 .
- (6) If $c_4 = c_7 = c_3 = c_6 = c_2 = 0$ and $c_5 \neq 0$ then $E_2 u$ is a non-zero multiple of u_0 .

TABLE 1. Action on $V_1(\tilde{\alpha})$, where $c = q^{-\tilde{\alpha}}(q^{-1}[\tilde{\alpha}] - [\tilde{\alpha} + 1])$.

$V_1(\tilde{\alpha})$	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7
E_1	0	0	v_1	0	0	v_4	0	0
E_2	0	$c \cdot v_0$	0	0	$[\tilde{\alpha}][\tilde{\alpha} + 1]v_3$	0	v_5	$-[\tilde{\alpha}][\tilde{\alpha} + 1]v_2$
F_1	0	v_2	0	0	v_5	0	0	0
F_2	v_1	0	v_3	0	0	$-c \cdot v_6$	0	v_4

TABLE 2. Action on $V_2(\tilde{\alpha})$, where $c = q^{-\tilde{\alpha}}(q^{-1}[\tilde{\alpha}] - [\tilde{\alpha} + 1])$.

$V_2(\tilde{\alpha})$	u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7
E_1	0	u_0	u_5	u_6	u_7	0	$(q + q^{-1})v_5$	0
E_2	0	0	0	0	$-c \cdot u_3$	$c \cdot u_0$	$c \cdot u_1$	$c \cdot u_2$
F_1	u_1	0	u_3	0	0	u_6	0	u_4
F_2	u_5	u_2	0	u_4	0	0	u_7	0

- (7) Finally, if $c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0$ and $c_1 \neq 0$ then $E_1 u$ is a non-zero multiple of u_0 .
- (8) Finally, if $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0$ then $c_0 \neq 0$ and u is a non-zero multiple of u_0 .

Thus, $U \cong V_2(\tilde{\alpha})$ and $V_2(\tilde{\alpha})$ is simple.

Next, we consider the head and socle of $V_1(\tilde{\alpha})$. We have

$$v_3 = q^{2(-\tilde{\alpha}-1)} w_{1,1}^{\tilde{\alpha}} \otimes w_{0,0}^{-\tilde{\alpha}-1} - q^{-\tilde{\alpha}-1} w_{0,1}^{\tilde{\alpha}} \otimes w_{1,0}^{-\tilde{\alpha}-1} + q^{-\tilde{\alpha}} w_{1,0}^{\tilde{\alpha}} \otimes w_{0,1}^{-\tilde{\alpha}-1} + w_{0,0}^{\tilde{\alpha}} \otimes w_{1,1}^{-\tilde{\alpha}-1}$$

which generates the trivial module in $V_1(\tilde{\alpha})$ (as defined in 3.4.1). Thus, $\text{Hom}_{\mathcal{C}}(\mathbb{C}, V_1(\tilde{\alpha}))$ is non-zero. Also, from Table 1 we can see the map

$$V_1(\tilde{\alpha}) \rightarrow \mathbb{C} \text{ given by } c_0 v_0 + c_1 v_1 + \dots + c_7 v_7 \mapsto c_7$$

is a $U_q(\mathfrak{sl}(2|1))$ -module morphism. Thus, $\text{Hom}_{\mathcal{C}}(V_1(\tilde{\alpha}), \mathbb{C})$ is non-zero.

Finally, we prove that Equation (19) holds. Since the dimension of $V(0, \tilde{\alpha}) \otimes V(0, \tilde{\alpha})^*$ is equal to the sum of the dimensions of $V_1(\tilde{\alpha})$ and $V_2(\tilde{\alpha})$, it suffices to show that $V_1(\tilde{\alpha}) \cap V_2(\tilde{\alpha}) = \{0\}$. Suppose this is not true. Then there exists a non-zero $v \in V_1(\tilde{\alpha}) \cap V_2(\tilde{\alpha})$. Since $V_2(\tilde{\alpha})$ is simple then $V_2(\tilde{\alpha})$ is isomorphic to the module $\langle v \rangle$ generated by v . But since $v \in V_1(\tilde{\alpha})$ then $V_2(\tilde{\alpha}) \cong \langle v \rangle \subset V_1(\tilde{\alpha})$. Since $V_1(\tilde{\alpha})$ and $V_2(\tilde{\alpha})$ have the same dimension this implies that $V_2(\tilde{\alpha}) \cong V_1(\tilde{\alpha})$ which is a contradiction because $V_1(\tilde{\alpha})$ contains the trivial module as a submodule and $V_2(\tilde{\alpha})$ is simple. Thus, we have the decomposition. \square

Corollary 4.1.2. *Let $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ be such that $\bar{\alpha} \notin \{\bar{0}, \bar{\frac{1}{2}}\}$. Then $V(0, \tilde{\alpha})$ is a right ambidextrous object in the category \mathcal{C} .*

Proof. Equation (19) gives a decomposition of $V(0, \tilde{\alpha}) \otimes V(0, \tilde{\alpha})^*$ into indecomposable as in Equation (5) where $W_1 = V_1(\tilde{\alpha})$ and $W_2 = V_2(\tilde{\alpha})$. Since $W_2 = V_2(\tilde{\alpha})$ is an 8-dimensional simple module then $\text{Hom}_{\mathcal{C}}(\mathbb{C}, W_2) = \text{Hom}_{\mathcal{C}}(W_2, \mathbb{C}) = 0$. From Lemma 2.4.2 there are unique $j, j' \in \{0, 1\}$ such that $\text{Hom}_{\mathcal{C}}(\mathbb{I}, W_j)$ and $\text{Hom}_{\mathcal{C}}(W_{j'}, \mathbb{I})$ are non-zero. Thus, $j = j' = 1$ and Theorem 2.4.3 implies $V(0, \tilde{\alpha})$ is right ambidextrous. \square

4.2. The modified trace. From Theorem 10 of [10] (for a statement see Theorem 2.4.1 above) we have that the right ambidextrous object $V(0, \tilde{\alpha})$ gives a unique right trace:

Theorem 4.2.1 ([10]). *Let $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\alpha} \notin \{\bar{0}, \bar{\frac{1}{2}}\}$. There exists a non-zero right trace $\{\text{tr}_V\}$ on the ideal $\mathcal{I}_{V(0, \tilde{\alpha})}$ which is unique up to multiplication by a non-zero scalar.*

The following lemma shows that the ideal generated by $V(0, \tilde{\alpha})$ contains all objects of \mathcal{C} except the trivial module \mathbb{C} and thus the right trace is defined on all these objects. It follows that this ideal is independent of $\tilde{\alpha}$ and we will denote it by \mathcal{I} .

Lemma 4.2.2. *For any $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\alpha} \notin \{\bar{0}, \bar{\frac{1}{2}}\}$ we have $\mathcal{I}_{V(0, \tilde{\alpha})} = \mathcal{C} \setminus \{\mathbb{C}\}$.*

Proof. First, we will show $\mathcal{I}_{V(0, \tilde{\alpha})} \subseteq \mathcal{C} \setminus \{\mathbb{C}\}$. By definition this ideal is contained in \mathcal{C} so we only need to show $\mathbb{C} \notin \mathcal{I}_{V(0, \tilde{\alpha})}$. Suppose on the contrary that $\mathcal{I}_{V(0, \tilde{\alpha})} = \mathcal{C}$. From Lemma 2.4.2 and Theorem 2.4.3 it follows that the trivial module \mathbb{C} is right ambidextrous. By Theorem 2.4.1 there is a unique right trace on $\mathcal{I}_{\mathbb{C}} = \mathcal{C}$. It is easy to see this trace is equal to the usual quantum trace in \mathcal{C} and its associated

modified dimension is the usual quantum dimension qdim . Since $\mathcal{I}_{V(0,\tilde{\alpha})} = \mathcal{C} = \mathcal{I}_{\mathbb{C}}$ then from the proof of Lemma 4.2.2 in [9] we have $\text{qdim}(V(0,\tilde{\alpha})) \neq 0$ (note [9] requires a braiding but it is easy to see the cited proof works in our pivotal context). But this is a contradiction so we have the desired inclusion.

Now, we will show the converse inclusion. First, notice that if $\tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ satisfies $\tilde{\beta}, \tilde{\alpha} + \tilde{\beta} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$ then Lemma 3.4.8 implies that $V(0, \tilde{\alpha} + \tilde{\beta})$ is a retract of $V(0, \tilde{\alpha}) \otimes V(0, \tilde{\beta})$. Therefore, $V(0, \tilde{\alpha} + \tilde{\beta}) \in \mathcal{I}_{V(0,\tilde{\alpha})}$. Now if $\tilde{\mu} \in \mathbb{C}/l\mathbb{Z}$ such that $\tilde{\mu} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$ then there exists $\tilde{\beta}, \tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}$ such that $\tilde{\beta}, \tilde{\gamma} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$, $\tilde{\mu} = \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma}$ and $\tilde{\alpha} + \tilde{\beta} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. Lemma 3.4.8 implies $V(0, \tilde{\mu})$ is a retract of $V(0, \tilde{\alpha} + \tilde{\beta}) \otimes V(0, \tilde{\gamma})$. We have proved that if $\tilde{\mu} \in \mathbb{C}/l\mathbb{Z}$ with $\tilde{\mu} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$ then $V(0, \tilde{\mu}) \in \mathcal{I}_{V(0,\tilde{\alpha})}$.

Now let $V \in \mathcal{C} \setminus \{\mathbb{C}\}$. By definition of \mathcal{C} there exists $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n \in \mathbb{C}/l\mathbb{Z}$ with $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n \notin \mathcal{Y}$ such that V is a retract of $V(0, \tilde{\alpha}_1) \otimes V(0, \tilde{\alpha}_2) \otimes \dots \otimes V(0, \tilde{\alpha}_n)$. Since $V(0, \tilde{\alpha}_1)$ is in the ideal $\mathcal{I}_{V(0,\tilde{\alpha})}$ then $V \in \mathcal{I}_{V(0,\tilde{\alpha})}$. \square

4.3. Computations of modified dimensions. In the proof of Theorem 5.2.4 we will show that the ideal \mathcal{I} contains $V(n, \tilde{\alpha})$ such that $0 \leq n \leq l' - 2$ and $\tilde{\alpha} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. In the next lemma we will compute the modified quantum dimension of such modules. Recall that the modified quantum dimension is defined to be $d(V) = \text{t}_V(\text{Id}_V)$ for $V \in \mathcal{I}$.

Lemma 4.3.1. *If $V(n, \tilde{\alpha}) \in \mathcal{I}$ for $0 \leq n \leq l' - 1$ and $\tilde{\alpha} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$ then the right trace $\{\text{t}_V\}_{V \in \mathcal{I}}$ can be normalized so*

$$(20) \quad d(V(n, \tilde{\alpha})) = \frac{\{n+1\}}{\{1\}\{\tilde{\alpha}\}\{\tilde{\alpha} + n + 1\}}$$

where $\{z\} = q^z - q^{-z}$.

Proof. In the proof of Lemma 4.2.2 we showed that $V(0, \frac{\bar{l}}{3}) \in \mathcal{I}$. The trace is unique up to a global scalar and we choose a normalization so that

$$d(V(0, \frac{\bar{l}}{3})) = \text{t}_{V(0, \frac{\bar{l}}{3})}(\text{Id}_{V(0, \frac{\bar{l}}{3})}) = \frac{1}{\{\frac{\bar{l}}{3}\}\{\frac{\bar{l}}{3} + 1\}}.$$

Let $V(n, \tilde{\alpha}) \in \mathcal{I}$ for $0 \leq n \leq l' - 1$ and $\tilde{\alpha} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. Fix $\alpha \in \mathbb{C}$ such that $\tilde{\alpha} = \alpha$ modulo $l\mathbb{Z}$. For $U, W \in \mathcal{C}$ and $f \in \text{End}_{\mathcal{C}}(U \otimes W)$ define

$$\text{ptr}^W(f) = \left((\text{Id}_U \otimes \overleftarrow{\text{ev}}_W)(f \otimes \text{Id}_{W^*})(\text{Id}_U \otimes \overrightarrow{\text{coev}}_W) \right).$$

Recall the isomorphisms $\psi_{\alpha, \frac{1}{3}}$ and $\psi_{\frac{1}{3}, \alpha}$ of Lemma 3.5.7.

Let $S'_{\alpha, \frac{1}{3}}$ and $S'_{\frac{1}{3}, \alpha}$ be the complex numbers defined by the following equations:

$$S'_{\frac{1}{3}, \alpha} \text{Id}_{V(n, \tilde{\alpha})} = \text{ptr}^{V(0, \frac{\bar{l}}{3})} \left(\psi_{\frac{1}{3}, \alpha} \psi_{\alpha, \frac{1}{3}} \right), \quad S'_{\alpha, \frac{1}{3}} \text{Id}_{V(0, \frac{\bar{l}}{3})} = \text{ptr}^{V(n, \tilde{\alpha})} \left(\psi_{\alpha, \frac{1}{3}} \psi_{\frac{1}{3}, \alpha} \right).$$

Now from the properties of the modified trace we have

$$\begin{aligned}
d(V(n, \tilde{\alpha}))S'_{\frac{1}{3}, \alpha} &= \mathbf{t}_{V(n, \tilde{\alpha})} \left(\text{ptr}^{V(0, \frac{1}{3})} \left(\psi_{\frac{1}{3}, \alpha} \psi_{\alpha, \frac{1}{3}} \right) \right) \\
&= \mathbf{t}_{V(n, \tilde{\alpha}) \otimes V(0, \frac{1}{3})} \left(\psi_{\frac{1}{3}, \alpha} \psi_{\alpha, \frac{1}{3}} \right) \\
&= \mathbf{t}_{V(0, \frac{1}{3}) \otimes V(n, \tilde{\alpha})} \left(\psi_{\alpha, \frac{1}{3}} \psi_{\frac{1}{3}, \alpha} \right) \\
&= \mathbf{t}_{V(0, \frac{1}{3})} \left(\text{ptr}^{V(n, \tilde{\alpha})} \left(\psi_{\alpha, \frac{1}{3}} \psi_{\frac{1}{3}, \alpha} \right) \right) \\
&= d \left(V(0, \frac{1}{3}) \right) S'_{\alpha, \frac{1}{3}}.
\end{aligned}$$

Now, if $S'_{\frac{1}{3}, \alpha} \neq 0$ (which we will show below) then

$$(21) \quad d(V(n, \tilde{\alpha})) = \frac{S'_{\alpha, \frac{1}{3}}}{\left(\left\{ \frac{1}{3} \right\} \left\{ \frac{1}{3} + 1 \right\} \right) S'_{\frac{1}{3}, \alpha}}.$$

Thus, to find a formula for $d(V(n, \tilde{\alpha}))$ it suffices to compute $S'_{\frac{1}{3}, \alpha}$ and $S'_{\alpha, \frac{1}{3}}$.

We now compute $S'_{\frac{1}{3}, \alpha}$. Let $\{w_{\rho, \sigma, p}\}_{\rho, \sigma \in \{0, 1\}, p \in \{0, \dots, n-1\}}$ and $\{w'_{\rho', \sigma', 0}\}_{\rho', \sigma' \in \{0, 1\}}$ be the weight bases of $V(n, \tilde{\alpha})$ and $V(0, \frac{1}{3})$, respectively. Any endomorphism of $V(n, \tilde{\alpha})$ maps the highest weight vector $w_{0, 0, 0}$ of $V(n, \tilde{\alpha})$ to a multiple of itself. Since $V(n, \tilde{\alpha})$ is simple it is enough to compute this coefficient, in other words

$$(22) \quad \text{ptr}^{V(0, \frac{1}{3})} \left(\psi_{\frac{1}{3}, \alpha} \psi_{\alpha, \frac{1}{3}} \right) (w_{0, 0, 0}) = S'_{\frac{1}{3}, \alpha} w_{0, 0, 0}.$$

Now $\psi_{\frac{1}{3}, \alpha}$ and $\psi_{\alpha, \frac{1}{3}}$ are determined by the action of the R -matrix $\check{R}\mathcal{K}$ on the $U_q^H(\mathfrak{sl}(2|1))$ -modules $V^H(n, \alpha)$ and $V^H(0, \frac{1}{3})$. Since we are taking a partial trace, only the diagonal quantities of this action contribute when writing on the weight vector basis $\{w_{\rho, \sigma, 0}^{\frac{1}{3}}\}_{\rho, \sigma \in \{0, 1\}}$ of $V^H(0, \frac{1}{3})$ given above. So it is enough to know the values of $\psi_{\alpha, \frac{1}{3}}(w_{0, 0, 0} \otimes w'_{\rho', \sigma', 0})$ and $\psi_{\frac{1}{3}, \alpha}(w'_{\rho', \sigma', 0} \otimes w_{0, 0, 0})$ which are computed in Lemma 3.5.7. Note, the terms $c_i x_i \otimes y_i$ in Equation (18) are not diagonal and so do not contribute. Thus, evaluating the left side of Equation (22) we have

$$\begin{aligned}
S'_{\frac{1}{3}, \alpha} &= \sum_{\rho', \sigma' = 0}^1 q^{(-2n-4\alpha)(\frac{1}{3}+\sigma')-2\alpha(\rho'-\sigma')} (-1)^{\rho'+\sigma'} w'^*_{\rho', \sigma', 0} (K_2^{-2} w'_{\rho', \sigma', 0}) \\
&= \sum_{\rho', \sigma' = 0}^1 q^{(-2n-4\alpha)(\frac{1}{3}+\sigma')-2\alpha(\rho'-\sigma')-2(\frac{1}{3}+\sigma')} (-1)^{\rho'+\sigma'} \\
&= q^{(-\frac{2}{3}-1)(2\alpha+n+1)} (q^{2\alpha+n+1} - q^{-n-1} - q^{n+1} + q^{-2\alpha-n-1}) \\
&= q^{-(\frac{2}{3}+1)(2\alpha+n+1)} \{\alpha\} \{\alpha + n + 1\}.
\end{aligned}$$

Similarly,

$$S'_{\alpha, \frac{1}{3}} = q^{-(2\alpha+n+1)(\frac{2}{3}+1)} \frac{\{n+1\}}{\{1\}} \{1/3\} \{4/3\}.$$

Finally, since $\{\tilde{x}\} = \{x\}$ for any $x \in \mathbb{C}$ then Equation (21) implies the result. \square

5. THE RELATIVE \mathbb{C}/\mathbb{Z} -SPHERICAL CATEGORY

5.1. Purification of \mathcal{C} . The category \mathcal{C} that we've constructed still needs to be modified in order to obtain a relative G -spherical category. One of the main problems is that there are an infinite number of non-isomorphic simple objects in each graded piece of \mathcal{C} . In order to obtain a finite number of objects in each grading, we will “purify” the category using the modified trace. This will have the effect of removing all modules outside the alcove. This generalizes the well known purification of a category discussed in Chapter XI of [27].

Let $V, W \in \mathcal{I} = \mathcal{C} \setminus \{\mathbb{C}\}$. A morphism $f \in \text{Hom}_{\mathcal{C}}(V, W)$ is called *negligible* with respect to the right trace t if

$$\text{t}_W(f \circ g) = \text{t}_V(g \circ f) = 0$$

for all $g \in \text{Hom}_{\mathcal{C}}(W, V)$. Denote $\text{Negl}(V, W)$ the set of negligible morphisms from V to W . The set $\text{Negl}(V, W)$ is actually a vector subspace of $\text{Hom}_{\mathcal{C}}(W, V)$. Thus, we can take the quotient and obtain a \mathbb{C} -vector space $\text{Hom}_{\mathcal{C}}(V, W)/\text{Negl}(V, W)$. We set $\text{Negl}(V, \mathbb{C}) = \text{Negl}(\mathbb{C}, V) = 0$ for any $V \in \mathcal{C}$.

We describe a purification process of \mathcal{C} which will produce a category \mathcal{C}^N where all negligible morphisms are zero. We define a new pivotal \mathbb{C} -category \mathcal{C}^N whose objects are the same as in \mathcal{C} . The set of morphisms between two objects V and W of \mathcal{C}^N is

$$\text{Hom}_{\mathcal{C}^N}(V, W) = \text{Hom}_{\mathcal{C}}(V, W)/\text{Negl}(V, W).$$

The composition, tensor product, pivotal structure and grading in \mathcal{C}^N are induced from \mathcal{C} :

Lemma 5.1.1. *The category \mathcal{C}^N is a pivotal \mathbb{C} -category with a \mathbb{C}/\mathbb{Z} -grading induced from the grading of \mathcal{C} .*

Proof. First, we will show that \mathcal{C}^N is a pivotal \mathbb{C} -category. There is an obvious functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}^N$ which is the identity on objects and maps a morphism to its class modulo negligible morphisms:

- (1) $\mathcal{F}(A) = A, \forall A \in \text{Ob}(\mathcal{C})$,
- (2) $\mathcal{F}(f) = [f] \in \text{Hom}_{\mathcal{C}^N}(A, B), \forall f \in \text{Hom}_{\mathcal{C}}(A, B)$.

Using this functor we can induce a tensor \mathbb{C} -linear structure on \mathcal{C}^N from that of \mathcal{C} . We also define the dual structure on \mathcal{C}^N as the one coming from \mathcal{C} , via the functor \mathcal{F} . Since the duality morphisms in \mathcal{C} satisfy the compatibility conditions for a pivotal structure, then the corresponding image dualities under \mathcal{F} will also satisfy these compatibility conditions in \mathcal{C}^N .

Recall the definition of a G -graded category given in Subsection 2.3. From Lemma 3.5.2, we know that \mathcal{C} is \mathbb{C}/\mathbb{Z} -graded. For any $g \in \mathbb{C}/\mathbb{Z}$, define $\mathcal{C}_g^N := \mathcal{F}(\mathcal{C}_g)$. It is easy to see this gives a \mathbb{C}/\mathbb{Z} -grading on \mathcal{C}^N . \square

We will use the same notation for the object $V(n, \tilde{\alpha})$ of \mathcal{C} and the corresponding object in \mathcal{C}^N .

Lemma 5.1.2. *If $W \in \mathcal{C}$ such that W is simple and $\text{d}(W) = 0$ then every morphism to or from W is negligible.*

Proof. It suffices to prove that $\text{t}_W(h) = 0$ for any $h \in \text{End}_{\mathcal{C}}(W)$. This will imply that if $V \in \mathcal{C}$ and $f \in \text{Hom}_{\mathcal{C}}(V, W)$ then

$$\text{t}_W(f \circ g) = 0$$

for any $g \in \text{End}_{\mathcal{C}}(W, V)$. Thus, f is negligible. A similar statement holds for $f \in \text{Hom}_{\mathcal{C}}(W, V)$.

To prove the first statement, let $h \in \text{End}_{\mathcal{C}}(W)$. Since W is simple, $\text{End}_{\mathcal{C}}(W) = \mathbb{C} \text{Id}_W$ and we will define the scalar $\langle h \rangle \in \mathbb{C}$ as the solution to the equation $h = \langle h \rangle \text{Id}_W$. But $d(W) = 0$, in other words $\text{t}_W(\text{Id}_W) = 0$. Thus,

$$\text{t}_W(h) = \text{t}_W(\langle h \rangle \text{Id}_W) = \langle h \rangle \text{t}_W(\text{Id}_W) = 0.$$

□

Lemma 5.1.3. *Let V and W be objects in \mathcal{C} such that W is simple and $d(W) = 0$. Then V, W are also objects in \mathcal{C}^N with the property that the direct sum $V \oplus W$ is isomorphic to V in \mathcal{C}^N , in other words $V \oplus W \simeq_{\mathcal{C}^N} V$.*

Proof. Let $i_1 : V \rightarrow V \oplus W$ and $pr_1 : V \oplus W \rightarrow V$ be the injection and projection morphisms with $pr_1 \circ i_1 = \text{Id}_V$ in \mathcal{C} . This gives the relation $pr_1 \circ i_1 = \text{Id}_V$ in \mathcal{C}^N . We want to show $i_1 \circ pr_1 = \text{Id}_{V \oplus W}$ in \mathcal{C}^N . To do this consider the other inclusion and projection morphisms $i_2 : W \rightarrow V \oplus W$ and $pr_2 : V \oplus W \rightarrow W$ in \mathcal{C} . Then by definition $\text{Id}_{V \oplus W} = i_1 \circ pr_1 + i_2 \circ pr_2$ in \mathcal{C} . But from Lemma 5.1.2 we have $i_2 \circ pr_2$ is negligible. Thus, in \mathcal{C}^N we have $\text{Id}_{V \oplus W} = i_1 \circ pr_1$ and so i_1 is the inverse of pr_1 . □

Corollary 5.1.4. *Let $\tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}$ be such that $\tilde{\gamma} \notin \{\bar{0}, \bar{\frac{1}{2}}\}$. If $V(l' - 1, \tilde{\gamma}) \in \mathcal{I}$ then for any $V \in \mathcal{C}^N$ we have $V \oplus V(l' - 1, \tilde{\gamma}) \simeq_{\mathcal{C}^N} V$. In particular, $V(l' - 1, \tilde{\gamma}) \simeq_{\mathcal{C}^N} \{0\}$.*

Proof. From Lemma 4.3.1, we have that $d(V(l' - 1, \tilde{\gamma})) = 0$. Applying the previous lemma we conclude the isomorphism. □

5.2. Generically finitely semi-simple.

Lemma 5.2.1. *Let V be a simple object in \mathcal{C} . As an object of \mathcal{C}^N , V is either simple or $V \simeq_{\mathcal{C}^N} \{0\}$.*

Proof. Since V is simple we have that $\text{End}_{\mathcal{C}}(V) \simeq \mathbb{C} \cdot \text{Id}_V$ is the 1-dimensional vector space. By definition

$$\text{End}_{\mathcal{C}^N}(V) = \text{End}_{\mathcal{C}}(V) / \text{Negl}(V, V) = (\mathbb{C} \cdot \text{Id}_V) / \text{Negl}(V, V).$$

Thus, $\text{End}_{\mathcal{C}^N}(V)$ is either 0 or 1-dimensional corresponding to the two cases of the lemma. □

Lemma 5.2.2. *Let $U \in \mathcal{C}$ be such that $U = (\mathcal{C} \oplus_{j \in J} S_j) \oplus W$ where J is a finite indexing set and S_j is simple for all $j \in J$. Let $V \in \mathcal{C}$ be a retract of U with maps $i : V \rightarrow U$ and $p : U \rightarrow V$. Then the following statements are true:*

1) *There exist $J' \subseteq J$ and $W' \subseteq W$ such that:*

$$\text{Im}(i) = (\oplus_{j \in J'} S_j) \oplus W'.$$

Moreover, if $i' : V \rightarrow \text{Im}(i)$ is the function i but with range $\text{Im}(i)$ then i' is an isomorphism with inverse $p' := p|_{\text{Im}(i)}$.

2) *W' is a retract of W .*

Proof. 1) Denote by $p_j : U \rightarrow S_j$ and $p_W : U \rightarrow W$ the projections onto direct summands of U .

Consider $J' := \{j \in J | p_j \circ i \neq 0\}$ and $W' = \text{Im}(p_W \circ i)$. Since for any $j \in J$, S_j is simple then it is generated by any non-zero element. Using this and the fact that $p_j \circ i : V \rightarrow S_j$ is a non-zero morphism for all $j \in J'$, we obtain that this morphism

is surjective. We conclude that $Im(i) = \mathcal{C}(\oplus_{j \in J'} S_j) \oplus W'$. So i' is surjective and injective. Moreover, $p' \circ i' = p \circ i = Id_V$.

2) We prove the second statement in two steps.

Step 1. We will show that $(\oplus_{j \in J'} S_j) \oplus W'$ is a retract of $(\oplus_{j \in J} S_j) \oplus W$. Consider $\iota : (\oplus_{j \in J'} S_j) \oplus W' \rightarrow (\oplus_{j \in J} S_j) \oplus W$ the natural inclusion, of each component of the direct sum in the left to the corresponding one on the right hand side. From the first part of the proof we have

$$p \circ \iota = p'.$$

Define $\pi : (\oplus_{j \in J} S_j) \oplus W \rightarrow (\oplus_{j \in J'} S_j) \oplus W'$ by $\pi := i' \circ p$. Then since i' and p' are inverses of each other we have:

$$\pi \circ \iota = (i' \circ p) \circ \iota = i' \circ (p \circ \iota) = i' \circ p' = Id_{(\oplus_{j \in J'} S_j) \oplus W'}.$$

This concludes the Step 1.

Second 2. Consider $\iota_{W'} : W' \rightarrow (\oplus_{j \in J'} S_j) \oplus W'$ and $\pi_{W'} : (\oplus_{j \in J'} S_j) \oplus W' \rightarrow W'$ the injection and projection with respect to the direct summand of W' . Similarly, consider the injection $\iota_W : W \rightarrow (\oplus_{j \in J} S_j) \oplus W$ and projection $\pi_W : (\oplus_{j \in J} S_j) \oplus W \rightarrow W$.

Define $\iota' : W' \rightarrow W$ and $\pi' : W \rightarrow W'$ as:

$$\iota' := \pi_W \circ \iota \circ \iota_{W'} \quad \text{and} \quad \pi' := \pi_{W'} \circ \pi \circ \iota_W.$$

By definition we have:

$$\pi' \circ \iota' = \pi_{W'} \circ \pi \circ (\iota_W \circ \pi_W) \circ \iota \circ \iota_{W'}.$$

Since $Im(\iota \circ \iota_{W'}) \subseteq 0 \oplus W \subseteq (\oplus_{j \in J} S_j) \oplus W$, this means that

$$(\iota_W \circ \pi_W) \circ \iota \circ \iota_{W'} = \iota \circ \iota_{W'}.$$

So, we obtain:

$$\pi' \circ \iota' = \pi_{W'} \circ \pi \circ \iota \circ \iota_{W'}.$$

Using the conclusion of the first step ($\pi \circ \iota = Id$) we have:

$$\pi' \circ \iota' = \pi_{W'} \circ \iota_{W'} = Id_{W'}.$$

This finishes the proof of the second part. \square

Lemma 5.2.3. *Let $V, W \in \mathcal{C}$ be such that W is a retract of V in \mathcal{C} . If V is isomorphic to the zero module in \mathcal{C}^N (i.e. $V \simeq_{\mathcal{C}^N} \{0\}$) then so is W :*

$$W \simeq_{\mathcal{C}^N} \{0\}.$$

Proof. Let $i : W \rightarrow V$ and $\pi : V \rightarrow W$ be the retract in \mathcal{C} . Let $[i] : W \rightarrow V$ and $[\pi] : V \rightarrow W$ be their images in \mathcal{C}^N . Since $V \simeq_{\mathcal{C}^N} \{0\}$, this means that the images are zero maps in \mathcal{C}^N :

$$[0]_V : V \rightarrow \{0\} \quad \text{and} \quad [0]^V : \{0\} \rightarrow V$$

and they are inverses of each other in \mathcal{C}^N . In particular, we have:

$$(23) \quad [0]^V \circ [0]_V =_{\mathcal{C}^N} [Id_V].$$

Consider the zero maps:

$$[0]_W : W \rightarrow \{0\} \quad \text{and} \quad [0]^W : \{0\} \rightarrow W.$$

We have $[0]_W \circ [0]^W =_{\mathcal{C}^N} [0] =_{\mathcal{C}^N} [Id_{\{0\}}]$. For the other composition, notice that

$$[0]_W =_{\mathcal{C}^N} [0]_V \circ [i] \quad \text{and} \quad [0]^W =_{\mathcal{C}^N} [\pi] \circ [0]^V$$

so we have:

$$\begin{aligned} [0]^W \circ [0]_W &=_{\mathcal{C}^N} ([\pi] \circ [0]^V) \circ ([0]_V \circ [i]) =_{\mathcal{C}^N} [\pi] \circ ([0]^V \circ [0]_V) \circ [i] \\ &=_{\mathcal{C}^N} \pi \circ [\text{Id}_V] \circ i =_{\mathcal{C}^N} \pi \circ i =_{\mathcal{C}^N} [\text{Id}_W] \end{aligned}$$

where the third equality comes from Equation (23). Thus, we have shown $[0]_W$ and $[0]^W$ are inverses of each other. \square

Theorem 5.2.4. *Let $g \in G, g \notin \{\bar{0}, \frac{\bar{1}}{2}\}$.*

1) *The category \mathcal{C}_g^N is semi-simple.*

2) *The set of isomorphism classes of simple objects in $\bigcup_{g \in G \setminus \{\bar{0}, \frac{\bar{1}}{2}\}} \mathcal{C}_g^N$ is represented by*

$$\left\{ V(n, \tilde{\gamma}) \mid 0 \leq n \leq l' - 2, \tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}, \tilde{\gamma} \notin \left\{ \bar{0}, \frac{\bar{1}}{2} \right\} \right\}.$$

Proof. Proof of part 1). To prove the first statement we begin by showing that the elementary tensor products of $V(0, \tilde{\alpha})$ which arrive in grading g are semi-simple in \mathcal{C}_g^N . To do this we first work in \mathcal{C} and use induction on the number of terms in the tensor product. Then we show that such a tensor product is semi-simple in \mathcal{C}_g^N .

Let $P(n)$ be the following statement:

If $h \in G$ and $\tilde{\alpha}_i \in \mathbb{C}/l\mathbb{Z}$ are such that $h \notin \{\bar{0}, \frac{\bar{1}}{2}\}$, $\tilde{\alpha}_i \notin \mathcal{Y}$ for $i = 1, \dots, n$ and $\tilde{\alpha}_1 + \tilde{\alpha}_2 + \dots + \tilde{\alpha}_n = h$ then, as an object in \mathcal{C} , the tensor product

$$V(0, \tilde{\alpha}_1) \otimes V(0, \tilde{\alpha}_2) \otimes \dots \otimes V(0, \tilde{\alpha}_n)$$

can be written as a direct sum of modules of the following form:

- (a) $V(m, \tilde{\beta})$ where $m \leq \min\{n-1, l'-2\}$ and $\tilde{\beta} = h$,
- (b) $V(l'-1, \tilde{\delta}) \otimes W$ where $\tilde{\delta} \notin \{\bar{0}, \frac{\bar{1}}{2}\}$ and W is an object of \mathcal{C} .

Moreover, this decomposition contains at least one module of the form $V(\min\{n-1, l'-2\}, \tilde{\beta})$ for some $\tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ with $\tilde{\beta} = h$ and $W = \{0\}$ if $n < l' - 1$.

We prove this statement by induction.

The case $n = 2$. Let $h \in G \setminus \{\bar{0}, \frac{\bar{1}}{2}\}$ and $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathbb{C}/l\mathbb{Z}$ be such that $\tilde{\alpha}_1 + \tilde{\alpha}_2 = h$. It follows from Lemma 3.4.8 that:

$$V(0, \tilde{\alpha}_1) \otimes V(0, \tilde{\alpha}_2) =_{\mathcal{C}} V(0, \tilde{\alpha}_1 + \tilde{\alpha}_2) \oplus V(1, \tilde{\alpha}_1 + \tilde{\alpha}_2) \oplus V(0, \tilde{\alpha}_1 + \tilde{\alpha}_2 + 1).$$

As we can see, all the modules have the right form $V(n, \tilde{\alpha})$ with $n \leq 1$ and there is one $V(1, \tilde{\alpha}_1 + \tilde{\alpha}_2)$ which occurs.

Next we assume $P(n)$ is true and show $P(n+1)$ holds. To do this we need to consider two cases $n \geq l' - 1$ and $n < l' - 1$.

Case 1: $n \geq l' - 1$. Let $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{n+1}$ be as in the statement of $P(n+1)$. From Lemma 3.5.3, there exist $i, j \in \{1, \dots, n+1\}$ such that $i < j$ and $\tilde{\alpha}_i + \tilde{\alpha}_j \notin \{\bar{0}, \frac{\bar{1}}{2}\}$. Choose $\epsilon \in \mathbb{C}/l\mathbb{Z}$ such that:

- (1) $\tilde{\alpha}_1 + \tilde{\alpha}_2 + \dots + \tilde{\alpha}_j + \dots + \tilde{\alpha}_{n+1} + \epsilon \neq \bar{0}, \frac{\bar{1}}{2}$,
- (2) $\tilde{\alpha}_j - \epsilon \neq \bar{0}, \frac{\bar{1}}{2}$,
- (3) $\tilde{\alpha}_i + \epsilon \neq \bar{0}, \frac{\bar{1}}{2}$.

Using Lemma 3.5.9 and the Commutativity Lemma 3.5.7, we obtain that:

$$\begin{aligned} V(0, \tilde{\alpha}_1) \otimes \dots \otimes V(0, \tilde{\alpha}_i) \otimes \dots \otimes V(0, \tilde{\alpha}_j) \otimes \dots \otimes V(0, \tilde{\alpha}_{n+1}) &\simeq_{\mathcal{C}} \\ V(0, \tilde{\alpha}_1) \otimes \dots \otimes V(0, \tilde{\alpha}_i + \epsilon) \otimes \dots \otimes V(0, \tilde{\alpha}_j - \epsilon) \otimes \dots \otimes V(0, \tilde{\alpha}_{n+1}) &\simeq_{\mathcal{C}} \\ V(0, \tilde{\alpha}_1) \otimes \dots \otimes \widehat{V(0, \tilde{\alpha}_i)} \otimes \dots \otimes \widehat{V(0, \tilde{\alpha}_j)} \otimes \dots \otimes V(0, \tilde{\alpha}_{n+1}) \otimes V(0, \tilde{\alpha}_i + \epsilon) \otimes V(0, \tilde{\alpha}_j - \epsilon). \end{aligned}$$

This shows that it suffices to prove that the statement $P(n+1)$ holds for weights of the form

$$(\tilde{\alpha}_1, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \tilde{\alpha}_{n+1}, \tilde{\alpha}_i + \epsilon, \tilde{\alpha}_j - \epsilon).$$

By the choice of ϵ we have $\tilde{\alpha}_1, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \tilde{\alpha}_{n+1}, \tilde{\alpha}_i + \epsilon$ has the total grading different than $\bar{0}, \frac{\bar{l}}{2}$, so it satisfies the property in the statement of $P(n)$. Therefore, by the induction hypothesis there exist:

$m_1, \dots, m_k \in \{0, \dots, l' - 3\}, \tilde{\beta}_1, \dots, \tilde{\beta}_k, \tilde{\gamma}_1, \dots, \tilde{\gamma}_s, \tilde{\delta}_1, \dots, \tilde{\delta}_p \in \mathbb{C}/l\mathbb{Z}$, and $W_1, \dots, W_p \in \mathcal{C}$ such that

$$\begin{aligned} V(0, \tilde{\alpha}_1) \otimes \dots \otimes \hat{V}(0, \tilde{\alpha}_i) \otimes \dots \otimes \hat{V}(0, \tilde{\alpha}_j) \otimes \dots \otimes V(0, \tilde{\alpha}_{n+1}) \otimes V(0, \tilde{\alpha}_i + \epsilon) \\ \cong_{\mathcal{C}} (\oplus_u V(m_u, \tilde{\beta}_u)) \oplus (\oplus_t V(l' - 2, \tilde{\gamma}_t)) \oplus (\oplus_k (V(l' - 1, \tilde{\delta}_k) \otimes W_k)) \end{aligned}$$

Taking the tensor product with $V(0, \tilde{\alpha}_j - \epsilon)$ we obtain:

$$\begin{aligned} V(0, \tilde{\alpha}_1) \otimes \dots \otimes \hat{V}(0, \tilde{\alpha}_i) \otimes \dots \otimes \hat{V}(0, \tilde{\alpha}_j) \otimes \dots \otimes V(0, \tilde{\alpha}_{n+1}) \otimes V(0, \tilde{\alpha}_i + \epsilon) \otimes V(0, \tilde{\alpha}_j - \epsilon) \\ \cong_{\mathcal{C}} (\oplus_u (V(m_u, \tilde{\beta}_u) \otimes V(0, \tilde{\alpha}_j - \epsilon))) \oplus (\oplus_t (V(l' - 2, \tilde{\gamma}_t) \otimes V(0, \tilde{\alpha}_j - \epsilon))) \\ \oplus (\oplus_k ((V(l' - 1, \tilde{\delta}_k) \otimes W_k) \otimes V(0, \tilde{\alpha}_j - \epsilon))). \end{aligned}$$

Since the tensor product preserves the grading we have

$$\bar{\beta}_u = \bar{\gamma}_t = \bar{\alpha}_1 + \dots + \hat{\alpha}_j + \bar{\alpha}_{n+1} + \bar{\epsilon}$$

for all $u \in \{1, \dots, k\}$ and $t \in \{1, \dots, s\}$. It follows that

$$\bar{\beta}_u + \bar{\alpha}_j - \bar{\epsilon} = \bar{\alpha}_1 + \dots + \bar{\alpha}_{n+1} \notin \{\bar{0}, \frac{\bar{l}}{2}\}.$$

Similarly, $\bar{\gamma}_t + \bar{\alpha}_j - \bar{\epsilon} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. Lemma 3.4.8 implies that the final expression in the previous tensor decomposition is isomorphic to

$$\begin{aligned} \oplus_u (V(m_u, \tilde{\beta}_u + \tilde{\alpha}_j - \epsilon) \oplus V(m_u + 1, \tilde{\beta}_u + \tilde{\alpha}_j - \epsilon) \oplus \\ \oplus V(m_u - 1, \tilde{\beta}_u + \tilde{\alpha}_j - \epsilon + 1) \oplus V(m_u, \tilde{\beta}_u + \tilde{\alpha}_j - \epsilon + 1)) \oplus \\ \oplus \oplus_t (V(l' - 2, \tilde{\gamma}_t + \tilde{\alpha}_j - \epsilon) \oplus (V(l' - 1, \tilde{\gamma}_t + \tilde{\alpha}_j - \epsilon) \otimes \mathbb{I}) \oplus \\ \oplus V(l' - 3, \tilde{\gamma}_t + \tilde{\alpha}_j - \epsilon + 1) \oplus V(l' - 2, \tilde{\gamma}_t + \tilde{\alpha}_j - \epsilon + 1)) \oplus \\ \oplus \oplus_k (V(l' - 1, \tilde{\delta}_k) \otimes W'_k) \end{aligned}$$

where $W'_k = W_k \otimes V(0, \tilde{\alpha}_j - \epsilon)$. We notice that from the induction hypothesis $\bar{\delta}_k \notin \{\bar{0}, \frac{\bar{l}}{2}\}$ and from the previous relation $\bar{\gamma}_t + \bar{\alpha}_j - \bar{\epsilon} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$, so all the second components that occur are not in $\{\bar{0}, \frac{\bar{l}}{2}\}$. Also, notice that the decomposition contains $V(l' - 2, \tilde{\gamma}_t + \tilde{\alpha}_j - \epsilon + 1)$ as a summand. Thus, we proved the step $P(n+1)$ in this case.

Case 2: $n < l' - 1$. The proof of the previous case also works here except that things are slightly simpler in this case because no module of the form $V(l' - 1, \tilde{\gamma}) \otimes W$

appears in the large tensor product. We highlight the differences: the first part of the proof is the same. Then the induction hypothesis implies there exist:

$$m_1, \dots, m_k \in \{0, \dots, n-1\}, \text{ and } \tilde{\beta}_1, \dots, \tilde{\beta}_k \in \mathbb{C}/l\mathbb{Z}$$

such that

$$V(0, \tilde{\alpha}_1) \otimes \dots \otimes \hat{V}(0, \tilde{\alpha}_i) \otimes \dots \otimes \hat{V}(0, \tilde{\alpha}_j) \otimes \dots \otimes V(0, \tilde{\alpha}_{n+1}) \otimes V(0, \tilde{\alpha}_i + \epsilon) \cong \oplus_u V(m_u, \tilde{\beta}_u)$$

where at least one $m_i = n-1$. As above take the tensor product with $V(0, \tilde{\alpha}_j - \epsilon)$ then the Decomposition Lemma 3.4.8 implies that

$$(24) \quad V(m_u, \tilde{\beta}_u) \otimes V(0, \tilde{\alpha}_j - \epsilon)$$

decomposes into a direct sum of modules of the form $V(m, \tilde{\beta})$ where $m \leq m_u + 1 \leq n < l' - 1$ and $\tilde{\beta} = \tilde{\beta}_u + \tilde{\alpha}_j - \epsilon \notin \{\bar{0}, \frac{l}{2}\}$. Also notice that when $m_u = m_i = n-1$, the tensor product in Equation (24) has a summand of the form $V(m_i + 1, \tilde{\beta}) = V(n, \tilde{\beta})$. Thus, we have proved that the statement for $P(n+1)$ holds.

Now we will show that \mathcal{C}_g^N is **semi-simple**

Let $V \in \mathcal{C}_g^N$. Then, from the definition, V is a \mathcal{C} -retract of a module

$$(25) \quad V(0, \tilde{\alpha}_1) \otimes V(0, \tilde{\alpha}_2) \otimes \dots \otimes V(0, \tilde{\alpha}_n)$$

where $\tilde{\alpha}_i \notin \mathcal{Y}$ and $\tilde{\alpha}_1 + \tilde{\alpha}_2 + \dots + \tilde{\alpha}_n = g$. From the first part, we know that there exist

$$m_1, \dots, m_k \in \{0, \dots, l' - 2\}, \tilde{\beta}_1, \dots, \tilde{\beta}_k, \tilde{\delta}_1, \dots, \tilde{\delta}_p \in \mathbb{C}/l\mathbb{Z}, \text{ and } W_1, \dots, W_p \in \mathcal{C}$$

such that

$$V(0, \tilde{\alpha}_1) \otimes \dots \otimes V(0, \tilde{\alpha}_n) \cong_{\mathcal{C}} \left(\bigoplus_u V(m_u, \tilde{\beta}_u) \right) \oplus \left(\bigoplus_t (V(l' - 1, \tilde{\delta}_t) \otimes W_t) \right)$$

where $\tilde{\beta}_u, \tilde{\delta}_t \notin \{\bar{0}, \frac{l}{2}\}$. Applying Lemma 5.2.2 to the right side of previous equation there exist a subset $J' \subset \{1, \dots, k\}$ and a retract W' of $\bigoplus_t (V(l' - 1, \tilde{\delta}_t) \otimes W_t)$ such that

$$V \simeq_{\mathcal{C}} (\bigoplus_{u \in J'} V(m_u, \tilde{\beta}_u)) \oplus W'.$$

Now, for all $t \in \{1, \dots, p\}$, Corollary 5.1.4 implies

$$V(l' - 1, \tilde{\delta}_t) \simeq_{\mathcal{C}^N} \{0\}.$$

This shows that

$$\bigoplus_t (V(l' - 1, \tilde{\delta}_t) \otimes W_t) \simeq_{\mathcal{C}^N} \{0\}.$$

Using 5.2.3, we obtain that:

$$W' \simeq_{\mathcal{C}^N} \{0\}$$

Thus we conclude that in \mathcal{C}^N :

$$(26) \quad V \simeq \bigoplus_{u \in J'} V(m_u, \tilde{\beta}_u).$$

Since the modules in the previous decomposition are simples in \mathcal{C} , then Lemma 5.2.1 implies they are also simple in \mathcal{C}^N and we obtain that V is semi-simple in \mathcal{C}^N .

Proof of part 2). Now we will prove the second part of the theorem. Let $V \in \mathcal{C}_g^N$ be a simple object (i.e. $\text{End}_{\mathcal{C}^N}(V) = \mathbb{C}\text{Id}_V$), with $g \in G, g \notin \{\bar{0}, \frac{l}{2}\}$. By definition, V is obtained from a \mathcal{C} -retract of tensor products of modules of the form $V(0, \alpha)$ and from Equation (26) we have

$$V \simeq_{\mathcal{C}^N} \bigoplus_{u \in J'} V(m_u, \tilde{\beta}_u)$$

where each $V(m_u, \tilde{\beta}_u)$ is simple in both \mathcal{C} and \mathcal{C}^N . If the cardinality of J' was strictly greater than one then $\dim(\text{End}_{\mathcal{C}^N}(V)) \geq 2$ which is a contradiction. So

$$V \simeq_{\mathcal{C}^N} V(m_u, \tilde{\beta}_u)$$

for some $0 \leq m_u \leq l' - 2$ and $\tilde{\beta}_i = g$. This shows that any simple object that occurs in \mathcal{C}_g^N is of the desired form.

For the other inclusion, let $0 \leq s \leq l' - 2, \tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}, \tilde{\gamma} = g \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. We will show that $V(s, \tilde{\gamma})$ is in \mathcal{C}_g^N .

In the first part of this proof we showed the statements $P(n)$ hold. The last parts of these statements imply that for all $0 \leq m \leq l' - 2$ and $h \in \mathbb{C}/\mathbb{Z} \setminus \{\bar{0}, \frac{\bar{l}}{2}\}$ there exists $\tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ such that $\tilde{\beta} = h$ and $V(m, \tilde{\beta})$ is a simple object in \mathcal{C}_h^N . We use this as follows.

Choose $\bar{\beta} \in \mathbb{C}/\mathbb{Z}$ such that $\bar{\beta}, \bar{\gamma} - \bar{\beta} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. There exists a lift $\tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ of $\bar{\beta}$ so that $V(s, \tilde{\beta})$ is $\mathcal{C}_{\tilde{\beta}}^N$ as discussed above. Set $\tilde{\epsilon} = \tilde{\gamma} - \tilde{\beta}$ then $\tilde{\epsilon}$ and $\tilde{\beta} + \tilde{\epsilon} = \tilde{\gamma}$ are not in $\{\bar{0}, \frac{\bar{l}}{2}\}$.

In \mathcal{C} , by definition, $V(s, \tilde{\beta})$ is a retract of a module:

$$V(0, \tilde{\alpha}_1) \otimes V(0, \tilde{\alpha}_2) \otimes \dots \otimes V(0, \tilde{\alpha}_n).$$

Taking the tensor product of this module with $V(0, \tilde{\epsilon})$, we get that $V(s, \tilde{\beta}) \otimes V(0, \tilde{\epsilon})$ is a \mathcal{C} -retract of

$$(27) \quad V(0, \tilde{\alpha}_1) \otimes V(0, \tilde{\alpha}_2) \otimes \dots \otimes V(0, \tilde{\alpha}_n) \otimes V(0, \tilde{\epsilon}).$$

Since $\tilde{\beta}, \tilde{\epsilon}, \tilde{\beta} + \tilde{\epsilon} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$ we have

$$V(s, \tilde{\beta}) \otimes V(0, \tilde{\epsilon}) \simeq_{\mathcal{C}} V(s, \tilde{\gamma}) \oplus V(s+1, \tilde{\gamma}) \oplus (1 - \tilde{\delta}_{s,0})V(s-1, \tilde{\gamma}+1) \oplus V(s, \tilde{\gamma}+1)$$

and we see that $V(s, \tilde{\gamma})$ is a \mathcal{C} -retract of $V(s, \tilde{\beta}) \otimes V(0, \tilde{\epsilon})$. Using properties of \mathcal{C} -retracts (if A is a retract of B and B is a retract of C , then A is a retract of C) and the previous two decompositions, we have $V(s, \tilde{\gamma})$ is a \mathcal{C} -retract of the module in Equation (27). This concludes the proof. \square

A set of simple objects A is said to be *represented* by a set of simple objects R_A if any element of A is isomorphic to a unique element of R_A . Lemma 5.1.1 and Theorem 5.2.4 imply the following corollary.

Corollary 5.2.5. *The category \mathcal{C}^N is a generically finitely \mathbb{C}/\mathbb{Z} -semi-simple pivotal \mathbb{C} -category with small symmetric subset $\mathcal{X} = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. The class of generic simple objects A of \mathcal{C}^N is represented by*

$$(28) \quad R_A = \{V(n, \tilde{\gamma}) \mid 0 \leq n \leq l' - 2, \tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}, \tilde{\gamma} \notin \mathcal{X}\}.$$

5.3. Trace. Here we will show that the right trace t on \mathcal{I} induces a trace in \mathcal{C}^N .

Lemma 5.3.1. *The full subcategory $\mathcal{I}^N := \mathcal{C}^N \setminus \{\mathbb{C}\}$ is a right ideal in \mathcal{C}^N .*

Proof. We need to show that \mathcal{I}^N satisfies the two conditions to be a right ideal (see Subsection 2.4). The first condition is true from the definitions of \mathcal{C} and \mathcal{C}^N . For the second condition we need to check that the trivial object \mathbb{C} is not a retract of an object in \mathcal{C}^N . On the contrary, suppose there exist an object W in \mathcal{C}^N and morphisms $f \in \text{Hom}_{\mathcal{C}^N}(\mathbb{C}, W)$ and $g \in \text{Hom}_{\mathcal{C}^N}(W, \mathbb{C})$ such that $gf = \text{Id}_{\mathbb{C}}$. But by definition $\text{Hom}_{\mathcal{C}^N}(\mathbb{C}, W) = \text{Hom}_{\mathcal{C}}(\mathbb{C}, W)$ and $\text{Hom}_{\mathcal{C}^N}(W, \mathbb{C}) = \text{Hom}_{\mathcal{C}}(W, \mathbb{C})$ so f

and g give a \mathcal{C} -retract of the trivial module which would imply that $\mathcal{I} = \mathcal{C}$ which is a contradiction to Lemma 4.2.2. Thus, $\mathbb{C} \notin \mathcal{C}^N$. \square

Lemma 5.3.2. *For $V \in \mathcal{I}^N$ the assignment $\mathbf{t}_V^N : \text{End}_{\mathcal{C}^N}(V) \rightarrow \mathbb{C}$ given by $[f] \mapsto \mathbf{t}_V(f)$ is a well defined linear function. Moreover, the family $\{\mathbf{t}_V^N\}_{V \in \mathcal{I}^N}$ is a right trace on \mathcal{I}^N .*

Proof. We need to show \mathbf{t}_V^N does not depend on the representative of $[f]$ in

$$\text{End}_{\mathcal{C}^N}(V) = \text{Hom}_{\mathcal{C}^N}(V, V) = \text{Hom}_{\mathcal{C}}(V, V) / \text{Negl}(V, V).$$

Suppose $[f] = [g]$ then $f = g + h$ for some $h \in \text{Negl}(V, V)$. Then $\mathbf{t}_V(f) = \mathbf{t}_V(g + h) = \mathbf{t}_V(g)$, implying $\mathbf{t}_V^N([f]) = \mathbf{t}_V^N([g])$.

In order to prove that \mathbf{t}_V^N is a right trace on \mathcal{I}^N , we have to prove that this satisfies the conditions 1) and 2) from the definition.

1) Let $U, V \in \mathcal{I}^N$ and $[f] \in \text{Hom}_{\mathcal{C}^N}(V, U)$, $[g] \in \text{Hom}_{\mathcal{C}^N}(U, V)$. Let $f \in \text{Hom}_{\mathcal{C}}(V, U)$, $g \in \text{Hom}_{\mathcal{C}}(U, V)$ be such that the class of f and g in \mathcal{C}^N are $[f]$ and $[g]$ respectively. Then

$$\mathbf{t}_V^N([g][f]) = \mathbf{t}_V^N([gf]) = \mathbf{t}_V(gf) = \mathbf{t}_U(fg) = \mathbf{t}_U^N([fg]) = \mathbf{t}_U^N([f][g]).$$

2) Consider $U \in \mathcal{I}^N$ and $W \in \mathcal{C}$ and $f \in \text{End}_{\mathcal{C}}^N(U \otimes W)$. Let $f \in \text{End}_{\mathcal{C}}(U \otimes W)$ be such that the class of f in \mathcal{C} is $[f]$. Then we obtain:

$$\begin{aligned} \mathbf{t}_{U \otimes W}^N([f]) &= \mathbf{t}_{U \otimes W}(f) = \mathbf{t}_U \left((\text{Id}_U \otimes \overleftarrow{\text{ev}}_W)(f \otimes \text{Id}_{W^*})(\text{Id}_U \otimes \overrightarrow{\text{coev}}_W) \right) = \\ &= \mathbf{t}_U^N \left(([\text{Id}_U] \otimes [\overleftarrow{\text{ev}}_W])([f] \otimes [\text{Id}_{W^*}])([\text{Id}_U] \otimes [\overrightarrow{\text{coev}}_W]) \right) \end{aligned}$$

The previous two equalities conclude the statement. \square

5.4. T-ambi pair. Let $\mathbf{d} : R_A \rightarrow \mathbb{C}$ be the function given in Equation (20), in other words:

$$(29) \quad \mathbf{d}(V(n, \tilde{\alpha})) = \frac{\{n+1\}}{\{1\}\{\tilde{\alpha}\}\{\tilde{\alpha}+n+1\}}$$

for $V(n, \tilde{\alpha}) \in R_A$. Extend this function to A by requiring $\mathbf{d}(V) = \mathbf{d}(V(n, \tilde{\alpha}))$ if V is isomorphic to $V(n, \tilde{\alpha})$.

Lemma 5.4.1. *The pair (A, \mathbf{d}) is a t-ambi pair in \mathcal{C}^N .*

Proof. Consider \mathbf{d}^N the modified dimension on \mathcal{I}^N coming from the right trace \mathbf{t}^N on \mathcal{C}^N .

Let $B := \{V \in \mathcal{I}^N \cap (\mathcal{I}^N)^* \mid V \text{ is simple, } \mathbf{d}^N(V) = \mathbf{d}^N(V^*)\}$. From Theorem 2.4.4, it follows that (B, \mathbf{d}^N) is a t-ambi pair. We notice that $(\mathcal{I}^N)^* = \mathcal{I}^N$.

We will prove that $A \subseteq B$ and that \mathbf{d}^N is determined by Equation (29). Let $V \in A$. By definition, there exist $0 \leq n \leq l' - 2$ and $\tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}$ with $\tilde{\gamma} \notin \mathcal{X}$ such that $V \simeq V(n, \tilde{\gamma})$. We have

$$\mathbf{d}^N(V(n, \tilde{\gamma})) = \mathbf{t}^N([\text{Id}_V(n, \tilde{\gamma})]) = \mathbf{t}(\text{Id}_V(n, \tilde{\gamma})) = \mathbf{d}(V(n, \tilde{\gamma}))$$

and so $\mathbf{d}^N(V(n, \tilde{\gamma}))$ is given by the formula in Equation (29).

Since $V(n, \tilde{\alpha})^* = V(n, -\tilde{\alpha} - \tilde{n} - \tilde{1})$, Equation (29) implies

$$\mathbf{d}(V(n, \tilde{\alpha})) = \mathbf{d}((V(n, \tilde{\alpha}))^*).$$

We conclude that

$$\mathbf{d}^N(V(n, \tilde{\alpha})) = \mathbf{d}^N((V(n, \tilde{\alpha}))^*)$$

for any $V \in \mathbf{A}$. This shows that $\mathbf{A} \subseteq \mathbf{B}$. Thus, since $(\mathbf{B}, \mathbf{d}^N)$ is a t-ambi pair, it is easy to check that (\mathbf{A}, \mathbf{d}) is a t-ambi pair. \square

5.5. The \mathbf{b} map. Here we show \mathcal{C}^N has a map \mathbf{b} as in the definition of a relative G -spherical category. To do this we need the following technical lemmas.

Definition 5.5.1. (*Isotypic components*) Let $V \in \mathcal{C}$ (or \mathcal{C}^N) be a semi-simple object such that V has the following decomposition:

$$V = \bigoplus_{i \in I} S_i, \text{ with } S_i \in \mathcal{C} \text{ (or } \mathcal{C}^N \text{) simple objects.}$$

Let $j \in I$ and consider the following indexing set: $\mathcal{I}_{S_j} := \{i \in I \mid S_i \simeq S_j\}$. The isotypic component of S_j in V is defined as:

$$\bigoplus_{k \in \mathcal{I}_{S_j}} S_k.$$

Lemma 5.5.2. Let $L, R \in \mathcal{C}^N$ with $L \in \mathcal{C}_g^N$, $R \in \mathcal{C}_h^N$ with $g, h \notin \{\bar{0}, \bar{1}, \bar{2}\}$. Suppose $L =_{\mathcal{C}^N} L_1 \oplus L_2$ and $R =_{\mathcal{C}^N} R_1 \oplus R_2$ are such that

$$L \simeq_{\mathcal{C}^N} R \text{ and } L_1 \simeq_{\mathcal{C}^N} R_1.$$

Then

$$L_2 \simeq_{\mathcal{C}^N} R_2.$$

Proof. Using Theorem 5.2.4, we have that both L and R are semi-simple in \mathcal{C}^N . More precisely, there exists $N \in \mathbb{N}$, and $S_1, \dots, S_N \in R_{\mathbf{A}}$ all pairwise different such that:

$$L = \eta_1 S_1 \oplus \dots \oplus \eta_N S_N$$

$$R = \eta'_1 S_1 \oplus \dots \oplus \eta'_N S_N \oplus J$$

where $\eta_i, \eta'_i \in \mathbb{N}$, are the multiplicities of the simple object S_i and J is a direct sum of elements of $R_{\mathbf{A}}$ which are all different than $S_i, i \in \{1, \dots, N\}$. As an observation, from the computation of \mathbf{d} we have:

$$\mathbf{d}(V) \neq 0 \quad \text{for all} \quad V \in R_{\mathbf{A}},$$

in particular $\mathbf{d}(S_i) \neq 0$, for all $i \in \{1, \dots, N\}$.

We have that:

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(L, R) &= \text{Hom}_{\mathcal{C}}(\eta_1 S_1 \oplus \dots \oplus \eta_N S_N, \eta'_1 S_1 \oplus \dots \oplus \eta'_N S_N \oplus J) \\ &= \bigoplus_{i,j} (\text{Hom}_{\mathcal{C}}(\eta_i S_i, \eta'_j S_j)) \oplus \bigoplus_i (\text{Hom}_{\mathcal{C}}(\eta_i S_i, J)). \end{aligned}$$

We notice that $\text{Hom}(S_i, S_j) = 0$ for $i \neq j$ and $\text{Hom}(S_i, J) = 0$ since J has no S_i -isotypic components so:

$$\text{Hom}_{\mathcal{C}}(L, R) = \bigoplus_i (\text{Hom}_{\mathcal{C}}(\eta_i S_i, \eta'_i S_i)).$$

Now we will study the negligible morphisms from this space. By definition $\text{Negl}(L, R) \subseteq \text{Hom}_{\mathcal{C}}(L, R)$ as a vector subspace. From the last two relations we obtain that:

$$\text{Negl}(L, R) = \bigoplus_i (\text{Negl}(\eta_i S_i, \eta'_i S_i)).$$

We will prove that actually we have no negligible morphisms between isotypic components of S_i .

Suppose that there exists $f \in \text{Negl}(\eta_i S_i, \eta'_i S_i)$ which is non-zero. For $k \in \{1, \dots, N\}$, denote by $\iota_k : S_i \rightarrow \eta_i S_i$ and $\pi_k : \eta'_i S_i \rightarrow S_i$ the inclusion and projection of the k^{th} component.

Since f is non-zero, then there exists $k, l \in \{1, \dots, N\}$ such that $\pi_l \circ f \circ \iota_k \neq 0$. Also, since S_i is simple in \mathcal{C} :

$$\pi_l \circ f \circ \iota_k = \pi_l \circ f \circ \iota_k(1) \text{Id}_{S_i}.$$

At the level of the modified trace we have:

$$\mathbf{t}_{S_i}(\pi_l \circ f \circ \iota_k) = (\pi_l \circ f \circ \iota_k(1)) \mathbf{t}_{S_i}(\text{Id}_{S_i}) = (\pi_l \circ f \circ \iota_k(1)) \mathbf{d}(S_i) \neq 0.$$

From the properties of \mathbf{t} , we have:

$$\mathbf{t}_{S_i}(\pi_l \circ f \circ \iota_k) = \mathbf{t}_{\eta'_i S_i}(f \circ \iota_k \circ \pi_l) = 0$$

(since f is negligible).

The last two equalities lead to a contradiction. We conclude that $\text{Negl}(L, R) = \{0\}$ and so:

$$\text{Hom}_{\mathcal{C}}(L, R) = \text{Hom}_{\mathcal{C}^N}(L, R).$$

Now let $[\phi] \in \text{Hom}_{\mathcal{C}^N}(L, R)$ be an isomorphism. Consider $\phi \in \text{Hom}_{\mathcal{C}}(L, R)$ that gives $[\phi]$ in \mathcal{C}^N . From the previous considerations, $\phi : L \rightarrow R$ is an isomorphism in \mathcal{C} .

Using this, we obtain that $J = \{0\}$ (it is not possible to have more isotypic components in R than in L). Also, since we are in a category of representations which are semi-simple and morphisms between representations, we obtain that

$$\eta_i = \eta'_i, \forall i \in \{1, \dots, N\}.$$

So now, both R and L are semi-simple modules in \mathcal{C} with the same isotypic decomposition.

Now both L_1 and R_1 are direct summands in L and R . It means that each of them has a semi-simple decomposition with modules from the set S_i . But L_1 and R_1 are isomorphic in \mathcal{C}^N . Using the same argument as in the first part with L and R , we obtain that L_1 and R_1 are isomorphic in \mathcal{C} . It means that they have the same isotypic decompositions with the same multiplicities. Let us compose ϕ to the right with an automorphism of L that makes a permutation on the isotypic components such that the ones corresponding to L_1 are sent onto the ones corresponding to $\phi^{-1}(R_1)$ respectively. This means that we obtain an isomorphism

$$\tilde{\phi} : L \rightarrow R$$

such that

$$\tilde{\phi}(L_1) = R_1.$$

We conclude that

$$\tilde{\phi}|_{L_2} : L_2 \rightarrow R_2$$

is an isomorphism in \mathcal{C} and also in \mathcal{C}^N . □

Lemma 5.5.3. *For all $\tilde{\alpha}, \tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ and $n \in \mathbb{N}$ such that $\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha} + \tilde{\beta} \notin \{\bar{0}, \bar{\frac{l}{2}}\}$ and $n \leq l' - 2$ and given $m \leq n$ we have*

$$V(m, \tilde{\alpha}) \otimes V(n, \tilde{\beta}) \simeq_{\mathcal{C}^N} V(0, \tilde{\alpha}) \otimes \left(V(n+m, \tilde{\beta}) \oplus V(n+m-2, \tilde{\beta}+1) \oplus \dots \oplus V(n-m, \tilde{\beta}+m) \right)$$

where we set $V(k, \tilde{\beta}) = 0$ if $k \geq l' - 1$.

Proof. We will show the statement by induction on m . The case $m = 0$ is true from Lemma 3.4.8. Next, we will check the case $m = 1$.

Let $\tilde{\alpha}, \tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ and $n \in \mathbb{N}$ be such that $\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha} + \tilde{\beta} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$ and $1 \leq n \leq l' - 2$. Choose $\tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}$ such that $\tilde{\gamma}, \tilde{\alpha} - \tilde{\gamma}, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. From Lemma 3.4.8 we have

$$\begin{aligned} V(0, \tilde{\alpha} - \tilde{\gamma}) \otimes V(n, \tilde{\beta}) &\simeq_{\mathcal{C}^N} V(n, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta}) \oplus V(n, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta} + 1) \\ &\oplus (1 - \delta_{l'-2, n}) V(n+1, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta}) \oplus V(n-1, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta} + 1) \end{aligned}$$

in \mathcal{C}^N . Take the tensor product of both sides of this equation with $V(0, \tilde{\gamma})$. Then, decomposing the left side by grouping the first two simple modules together, we have

$$\begin{aligned} (V(0, \tilde{\gamma}) \otimes V(0, \tilde{\alpha} - \tilde{\gamma})) \otimes V(n, \tilde{\beta}) &\simeq_{\mathcal{C}^N} (V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta})) \\ &\oplus (V(0, \tilde{\alpha} + 1) \otimes V(n, \tilde{\beta})) \oplus (V(1, \tilde{\alpha}) \otimes V(n, \tilde{\beta})) \end{aligned}$$

On the other hand, the left side is

$$\begin{aligned} (V(0, \tilde{\gamma}) \otimes V(n, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta})) \oplus (V(0, \tilde{\gamma}) \otimes V(n, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta} + 1)) \\ \oplus (1 - \delta_{l'-2, n}) (V(0, \tilde{\gamma}) \otimes V(n+1, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta})) \\ \oplus (V(0, \tilde{\gamma}) \otimes V(n-1, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta} + 1)) \end{aligned}$$

Now, using Corollary 3.4.9, we see the first two tensor products in the last two expressions are the same direct sum of simple modules so they are isomorphic. Thus, Lemma 5.5.2 implies

$$\begin{aligned} V(1, \tilde{\alpha}) \otimes V(n, \tilde{\beta}) &\simeq_{\mathcal{C}^N} (1 - \delta_{l'-2, n}) (V(0, \tilde{\gamma}) \otimes V(n+1, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta})) \\ &\oplus (V(0, \tilde{\gamma}) \otimes V(n-1, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta} + 1)). \end{aligned}$$

Using Corollary 3.4.9, we see the right hand side of this equation is isomorphic to

$$(1 - \delta_{l'-2, n}) (V(0, \tilde{\alpha}) \otimes V(n+1, \tilde{\beta})) \oplus (V(0, \tilde{\alpha}) \otimes V(n-1, \tilde{\beta} + 1)).$$

Thus, we have proved the lemma for the case $m = 1$.

Now, assuming the statement is true for $k \leq m$, we will show the statement holds for $m + 1$. Let $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ and n be as above. Let us denote:

$$E_{m, n}^{\tilde{\beta}} := (V(n+m, \tilde{\beta}) \oplus V(n+m-2, \tilde{\beta}+1) \oplus \cdots \oplus V(n-m, \tilde{\beta}+m)).$$

From the induction hypothesis, we have

$$(30) \quad V(0, \tilde{\gamma}) \otimes V(m, \tilde{\alpha} - \tilde{\gamma}) \otimes V(n, \tilde{\beta}) \simeq_{\mathcal{C}^N} V(0, \tilde{\gamma}) \otimes V(0, \tilde{\alpha} - \tilde{\gamma}) \otimes E_{m, n}^{\tilde{\beta}}.$$

Using Lemma 3.4.8 to decompose the first tensor product, we have the left hand side of Equation (30) is isomorphic to

$$\begin{aligned} (V(m, \tilde{\alpha}) \otimes V(n, \tilde{\beta})) \oplus (V(m, \tilde{\alpha} + 1) \otimes V(n, \tilde{\beta})) \oplus \\ \oplus (V(m-1, \tilde{\alpha} + 1) \otimes V(n, \tilde{\beta})) \oplus (V(m+1, \tilde{\alpha}) \otimes V(n, \tilde{\beta})) \end{aligned}$$

Similarly, the right hand side of Equation (30) is isomorphic to

$$\left(V(0, \tilde{\alpha}) \otimes E_{m,n}^{\tilde{\beta}} \right) \oplus \left(V(0, \tilde{\alpha} + 1) \otimes E_{m,n}^{\tilde{\beta}} \right) \oplus \left(V(1, \tilde{\alpha}) \otimes E_{m,n}^{\tilde{\beta}} \right)$$

From the induction hypothesis, the first two terms of the last two expressions are isomorphic, thus from Lemma 5.5.2 we obtain

$$(31) \quad \left(V(m-1, \tilde{\alpha} + 1) \otimes V(n, \tilde{\beta}) \right) \oplus \left(V(m+1, \tilde{\alpha}) \otimes V(n, \tilde{\beta}) \right) \simeq_{\mathcal{C}^N} V(1, \tilde{\alpha}) \otimes E_{m,n}^{\tilde{\beta}}.$$

Next, we decompose the right side of this equation. By induction we know that for any $1 \leq n' \leq l' - 2$ and $k \leq m$ we have

$$V(1, \tilde{\alpha}) \otimes V(n', \tilde{\beta} + k) \simeq_{\mathcal{C}^N} V(0, \tilde{\alpha}) \otimes \left(V(n' + 1, \tilde{\beta} + k) \oplus V(n' - 1, \tilde{\beta} + k + 1) \right).$$

Applying this to each term of the sum $E_{m,n}^{\tilde{\beta}}$ we obtain:

$$V(1, \tilde{\alpha}) \otimes E_{m,n}^{\tilde{\beta}} \simeq_{\mathcal{C}^N} V(0, \tilde{\alpha}) \otimes \left(E_{m,n+1}^{\tilde{\beta}} \oplus E_{m,n-1}^{\tilde{\beta}+1} \right)$$

From the definition $E_{m,n}^{\tilde{\beta}}$ we have

$$\begin{aligned} V(0, \tilde{\alpha}) \otimes \left(E_{m,n+1}^{\tilde{\beta}} \oplus E_{m,n-1}^{\tilde{\beta}+1} \right) &= \\ &= V(0, \tilde{\alpha}) \otimes \left(E_{m,n+1}^{\tilde{\beta}} \oplus \left(E_{m-1,n}^{\tilde{\beta}+1} \oplus V(n-1-m, \tilde{\beta} + m + 1) \right) \right) \\ &= V(0, \tilde{\alpha}) \otimes \left(\left(E_{m,n+1}^{\tilde{\beta}} \oplus V(n-1-m, \tilde{\beta} + m + 1) \right) \oplus E_{m-1,n}^{\tilde{\beta}+1} \right) \\ &= V(0, \tilde{\alpha}) \otimes \left(E_{m+1,n}^{\tilde{\beta}} \oplus E_{m-1,n}^{\tilde{\beta}+1} \right) \\ &\simeq_{\mathcal{C}^N} \left(V(0, \tilde{\alpha}) \otimes E_{m+1,n}^{\tilde{\beta}} \right) \oplus \left(V(m-1, \tilde{\alpha} + 1) \otimes V(n, \tilde{\beta}) \right) \end{aligned}$$

where the isomorphism comes from the induction hypothesis. Combining the last two equations, and using Lemma 5.5.2 we see that Equation (31) implies

$$V(m+1, \tilde{\alpha}) \otimes V(n, \tilde{\beta}) \simeq_{\mathcal{C}^N} V(0, \tilde{\alpha}) \otimes E_{m+1,n}^{\tilde{\beta}}$$

which proves the statement for $m+1$ and concludes the induction step. \square

Now we use this lemma to show \mathcal{C}^N has a \mathbf{b} map. In [15] it is shown how to construct a \mathbf{b} map from a character. Our \mathbf{b} map will be defined on the representative class of simple objects $R_{\mathbf{A}}$ and extended to \mathbf{A} by setting $\mathbf{b}(W) = \mathbf{b}(V)$ if $W \simeq V$ for $W \in \mathbf{A}$ and $V \in R_{\mathbf{A}}$.

Here a *character* is a map $\chi : R_{\mathbf{A}} \rightarrow \mathbb{C}$ satisfying

- (1) $\chi(V^*) = \chi(V)$ for all $V \in R_{\mathbf{A}}$,
- (2) if $V(m, \tilde{\alpha}), V(n, \tilde{\beta}) \in R_{\mathbf{A}}$ such that $\tilde{\alpha} + \tilde{\beta} \notin \mathcal{X}$ then

$$\chi(m, \tilde{\alpha})\chi(n, \tilde{\beta}) = \sum_{k, \tilde{\gamma}} \dim \left(\text{Hom}_{\mathcal{C}^N} (V(k, \tilde{\gamma}), V(m, \tilde{\alpha}) \otimes V(n, \tilde{\beta})) \right) \chi(k, \tilde{\gamma})$$

here for simplicity we denoted $\chi(V(m, \tilde{\alpha})) = \chi(m, \tilde{\alpha})$,

- (3) for any $g \in G \setminus \mathcal{X}$, the element $\mathcal{D}_g = \sum_{V \in \mathcal{C}_g^N \cap R_{\mathbf{A}}} \chi(V)^2$ of \mathbb{C} is non-zero.

If χ is a character, then Lemma 23 of [15] implies the map $G \setminus \mathcal{X} \rightarrow \mathbb{C}$, $g \mapsto \mathcal{D}_g$ is a constant function with value \mathcal{D} . Moreover, the map $\mathbf{b} = \frac{1}{\mathcal{D}}\chi$ satisfies the properties listed in Definition 2.3.1. We will now show that a character exists.

Let $V(m, \tilde{\alpha})$ be in R_A . Consider the formal character $\chi(m, \tilde{\alpha}) = \sum_{k, \tilde{\gamma}} c_{k, \tilde{\gamma}} e^k e^{\tilde{\gamma}}$ of $V(m, \tilde{\alpha})$ in \mathcal{C} . Here e^k and $e^{\tilde{\gamma}}$ are both formal variables for each $k \in \mathbb{Z}$ and $\tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}$ and $c_{k, \tilde{\gamma}}$ is the dimension of the $(k, \tilde{\gamma})$ weight space determined by the action of (K_1, K_2) .

The variables e^k and $e^{\tilde{\gamma}}$ of a character $\chi(m, \tilde{\alpha})$ can be specialized to q^k and 1, respectively to obtain a complex number which we denote by $\chi_q(m, \tilde{\alpha}) \in \mathbb{C}$. We will show χ_q is a character in \mathcal{C}^N .

Using the basis in Theorem 3.4.5 we see

$$\chi_q(m, \tilde{\alpha}) = (2 + q + q^{-1})(q^m + q^{m-2} + \cdots + q^{-m}) = (2 + q + q^{-1})[m + 1].$$

In particular, $\chi_q(l' - 1, \tilde{\alpha}) = 0$. Also, $V(m, \tilde{\alpha})^* = V(m, -\tilde{\alpha} - \tilde{m} - \tilde{1})$ which implies $\chi_q(V^*) = \chi_q(V)$ for $V \in R_A$.

Next, we show property (2) holds for χ_q . We first do this for the case $m = 0$ and $0 \leq n \leq l' - 2$. Let $V(0, \tilde{\alpha}), V(n, \tilde{\beta}) \in R_A$ be such that $\tilde{\alpha} + \tilde{\beta} \notin \mathcal{X}$. From Lemma 3.4.8 we have

$$(32) \quad V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta}) \simeq_{\mathcal{C}} V(n, \tilde{\alpha} + \tilde{\beta}) \oplus V(n + 1, \tilde{\alpha} + \tilde{\beta}) \\ \oplus (1 - \delta_{0,n})V(n - 1, \tilde{\alpha} + \tilde{\beta} + 1) \oplus V(n, \tilde{\alpha} + \tilde{\beta} + 1)$$

in \mathcal{C} . This implies

$$\chi(0, \tilde{\alpha})\chi(n, \tilde{\beta}) = \chi(n, \tilde{\alpha} + \tilde{\beta}) + \chi(n + 1, \tilde{\alpha} + \tilde{\beta}) + (1 - \delta_{0,n})\chi(n - 1, \tilde{\alpha} + \tilde{\beta} + 1) + \chi(n, \tilde{\alpha} + \tilde{\beta} + 1).$$

By specializing the variables of this equation, we have:

$$(33) \quad \chi_q(0, \tilde{\alpha})\chi_q(n, \tilde{\beta}) = \chi_q(n, \tilde{\alpha} + \tilde{\beta}) + \chi_q(n + 1, \tilde{\alpha} + \tilde{\beta}) \\ + (1 - \delta_{0,n})\chi_q(n - 1, \tilde{\alpha} + \tilde{\beta} + 1) + \chi_q(n, \tilde{\alpha} + \tilde{\beta} + 1).$$

Translating Equation (32) to \mathcal{C}^N we have:

$$V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta}) \simeq_{\mathcal{C}^N} V(n, \tilde{\alpha} + \tilde{\beta}) \oplus (1 - \delta_{l'-2,n})V(n + 1, \tilde{\alpha} + \tilde{\beta}) \\ \oplus (1 - \delta_{0,n})V(n - 1, \tilde{\alpha} + \tilde{\beta} + 1) \oplus V(n, \tilde{\alpha} + \tilde{\beta} + 1).$$

Since $\chi_q(l' - 1, \tilde{\alpha} + \tilde{\beta}) = 0$, the last equation implies that we can rewrite Equation (33) as

$$\chi_q(0, \tilde{\alpha})\chi_q(n, \tilde{\beta}) = \sum_{k, \tilde{\gamma}} \dim \left(\text{Hom}_{\mathcal{C}^N}(V(k, \tilde{\gamma}), V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta})) \right) \chi_q(k, \tilde{\gamma})$$

here all but possibly four homomorphism spaces are zero. This implies that if $W \simeq_{\mathcal{C}^N} \bigoplus_{i=1}^s \left(V(0, \tilde{\alpha}) \otimes V(n_i, \tilde{\beta}_i) \right)$ where $\tilde{\alpha}, \tilde{\beta}_i, \tilde{\alpha} + \tilde{\beta}_i \notin \mathcal{X}$ and $0 \leq n_i \leq l' - 2$ for all $i \in \{1, \dots, s\}$ then

$$(34) \quad \sum_{i=1}^s \chi_q(0, \tilde{\alpha})\chi_q(n_i, \tilde{\beta}_i) = \sum_{k, \tilde{\gamma}} \dim(\text{Hom}_{\mathcal{C}^N}(V(k, \tilde{\gamma}), W)) \chi_q(k, \tilde{\gamma}).$$

Next we consider the general case. Let $V(m, \tilde{\alpha}), V(n, \tilde{\beta}) \in R_A$ be such that $\tilde{\alpha} + \tilde{\beta} \notin \mathcal{X}$. A direct computation shows that

$$\chi_q(m, \tilde{\alpha})\chi_q(n, \tilde{\beta}) = (2 + q + q^{-1})^2 [m + 1][n + 1].$$

where it can be shown that

$$[m + 1][n + 1] = [n + 1 + m] + [n + 1 + m - 2] + \dots + [n + 1 - m].$$

This implies,

$$\begin{aligned} & \chi_q(m, \tilde{\alpha}) \chi_q(n, \tilde{\beta}) \\ &= \chi_q(0, \tilde{\alpha}) \left(\chi_q(n+m, \tilde{\beta}) + \chi_q(n+m-2, \tilde{\beta}+1) + \dots + \chi_q(n-m, \tilde{\beta}+m) \right). \end{aligned}$$

But from Lemma 5.5.3 and Equation (34), we have that the right side of the last equation is equal to

$$\sum_{k, \tilde{\gamma}} \dim \left(\text{Hom}_{\mathcal{C}^N} (V(k, \tilde{\gamma}), V(m, \tilde{\alpha}) \otimes V(n, \tilde{\beta})) \right) \chi_q(k, \tilde{\gamma})$$

and we have shown that property (2) holds.

Finally, we show that χ_q satisfies the last property of a character. Fix $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ such that $\tilde{\alpha} = g \notin \mathcal{X}$. Then, for any $k \in \{0, \dots, l'-1\}$, we have

$$\sum_{m=0}^{l'-1} \chi_q(m, \tilde{\alpha} + k)^2 = c^2 \sum_{m=0}^{l'-1} [m+1]^2 = \frac{c^2}{(q - q^{-1})^2} \sum_{m=0}^{l'-1} (q^{m+1} - q^{-m-1})^2$$

where $c = 2 + q + q^{-1}$. Let us compute the sum in this expression:

$$\begin{aligned} \sum_{m=0}^{l'-1} (q^{m+1} - q^{-m-1})^2 &= \sum_{m=0}^{l'-1} (q^{2m+2} + q^{-2m-2} - 2) \\ &= -2l' + q^2 \sum_{m=0}^{l'-1} q^{2m} + q^{-2} \sum_{m=0}^{l'-1} q^{-2m} \\ &= -2l' + q^2 \frac{q^{2l'} - 1}{q^2 - 1} + q^{-2} \frac{q^{-2l'} - 1}{q^{-2} - 1} = -2l'. \end{aligned}$$

Thus, we have

$$\mathcal{D}_g = \sum_{V \in \mathcal{C}_g^N \cap R_A} \chi_q(V)^2 = \sum_{m, k=0}^{l'-1} \chi_q(m, \tilde{\alpha} + k)^2 = \sum_{k=0}^{l'-1} \frac{-2l' c^2}{(q - q^{-1})^2} = \frac{-2(l')^2 c^2}{(q - q^{-1})^2}$$

which is non-zero.

In summary, χ_q is a character and, as explained above, it leads to a map $\mathbf{b} = \frac{1}{D} \chi$ which satisfies the properties listed in Definition 2.3.1.

5.6. Main theorem. Here we summarize the results of this paper in the following theorem.

Theorem 5.6.1. *Let $G = \mathbb{C}/\mathbb{Z}$ and $\mathcal{X} = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Let \mathbf{A} be the set of generic simple objects of \mathcal{C}^N given in Equation (28). Let $\mathbf{d} : \mathbf{A} \rightarrow \mathbb{C}^\times$ be the function defined in Equation (29). Let $\mathbf{b} : \mathbf{A} \rightarrow \mathbb{k}$ be the function constructed in Subsection 5.5. With this data \mathcal{C}^N is a relative G -spherical category with basic data and leads to the modified TV-invariant described in Theorem 1.0.1.*

Proof. The proof follows directly from Corollary 5.2.5 and Lemmas 5.4.1 and 2.3.2. \square

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