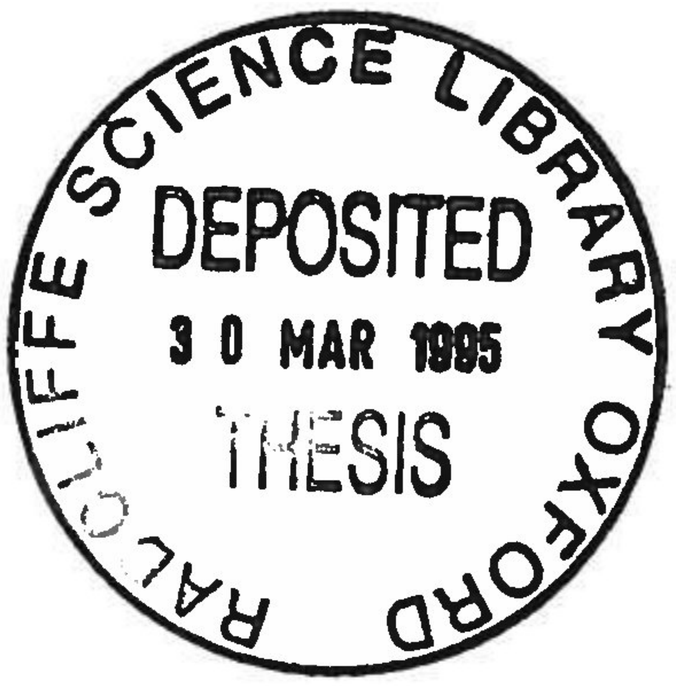
Dynamics of Nonabelian Dirac Monopoles

Jacqueline Faridani
Christ Church

Department of Physics

Thesis submitted for the degree of Doctor of Philosophy
in the University of Oxford

Michaelmas Term 1994



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Abstract

The topological definition of nonabelian magnetic charge is formulated using loop-space techniques. The dynamics of both classical and quantum monopoles are explored, and the equations of motion derived. A new U-symmetry, having its origin in the topological definition of the charge is found to exist. A comparison of scattering processes for identical systems of charges *vs* monopoles is made with the hope of discovering a situation where the predictions of the conventional source formalism and the monopole formalism differ. Both classical and quantum processes are explored; it is shown that in the few simple cases considered, there are no observable experimental consequences in the classical case. Next, quantum scattering is examined and the propagators for the various fields in the theory calculated, by fixing the $\tilde{U}$-gauge. Ghosts are introduced and shown to decouple from the theory. The simplest Feynman diagram is calculated for monopole-monopole scattering, and the scattering amplitude found to be of the same form as the corresponding quark-quark diagram. For the case of quantum monopoles the Feynman path-integral approach is used.

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To my parents

Acknowledgements

I am grateful to many individuals without whose help this thesis would never have been completed. First and foremost, I would like to thank my supervisor Dr. Hong-Mo Chan for his guidance, criticism, support and strong encouragement throughout the past four years. I am particularly grateful to him for the very stimulating discussions, which re-awakened in me, my love for the subject.

I would also like to thank Dr. Tsou Sheung Tsun, for introducing me to Lie groups and for her general help with the mathematics. I am indebted to Dr. S. Sarkar for introducing me to my supervisor, and to Dr. J. E. Paton, for supporting me during the difficult period at the beginning of my time in Oxford.

I would like to thank my friends Magda Lola and Farid Khorrami, for their assistance with technical matters and for their support, and Graham Ward for interesting and valuable discussions on the subject of monopoles.

I acknowledge the support received from The Committee of Vice-Chancellors and Principals of the United Kingdom, The Soudavar Fund, The Mihran and Azniv Essefian Fund and The Gulbenkian Foundation.

My stay in Oxford would not have been the same without my friends: In particular, I would like to thank Cristina White and Andronike Makres for their friendship and encouragement.

A special thanks to my friend Janet Zaroukian, with whom I had my first discussions in physics, and who prompted me to take my childlike enthusiasm seriously.

Finally, I am grateful to my parents and my sister Katrin, for their love, support and encouragement.

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Introduction

It is known that some gauge theories, namely those with nonsimply-connected gauge groups, admit certain topological charges which are conserved and quantised due to the topological nature of the charge [11, 19, 29]. In particular, these charges appear in electromagnetism and pure Yang-Mills theories. In the case of electromagnetism, they are the "abelian" Dirac monopoles first studied by Dirac in 1931 and shown to be the magnetic analogue of the electric charge, and therefore a source of magnetic flux. In the nonabelian case however, there is no direct way of defining the monopole as "a source of magnetic flux" [15]. In fact, the concept of flux itself, cannot be generalised to nonabelian theories in a useful way. The nonabelian monopole is therefore defined to be the topological charge mentioned above.

The interactions of Dirac monopoles with the field in the abelian theory of electromagnetism are dual to those of electric charges as a consequence of duality, but in the more complicated nonabelian theories, due to lack of dual symmetry, the interactions of these topological charges are not simply dual to those of nonabelian charges, and must be investigated. It is the aim of this thesis to study the dynamics of these nonabelian topological charges in both the classical and quantum cases and compare them to those of colour charges, with the object of determining whether such particles exist in nature.

Before proceeding further, it must be noted that in this thesis, we use small letters to denote the abelian potential a_μ and field $f_{\mu\nu}$ and capital letters to

denote the corresponding nonabelian quantities A_μ and $F_{\mu\nu}$. In order to study the dynamics of nonabelian monopoles (mainly in theories with gauge group $SU(N)$), we will find it useful and instructive to refer back to the abelian case, where much more is known about the interactions and properties of monopoles, since they are dual to those of charges.

We begin Chapter 1 by examining gauge theories in some detail, in particular, the concept of parallel-transport and the question of choosing suitable variables for the description of gauge theories in the presence of topological charges [9, 21, 22, 25]. We study the patching of the gauge potential A_μ in the presence of these topologically defined charges and examine the difficulties of working with the potential A_μ as variables.

In Chapter 2, a new set of variables $F_\mu[\xi \mid s]$ which do not present these problems are introduced; these are basically just the logarithmic derivatives of the phase factor $\Phi[\xi]$ [21]. Like $\Phi[\xi]$ they are "loop variables" and are thus complicated to work with, for they depend on the whole loop, rather than a spacetime point x^μ. As variables for describing gauge theories, they are therefore highly redundant, and must be constrained. It is further shown, by using an "extended Poincaré lemma" that these variables are adequate for describing gauge theories, in that given the variable $F_\mu[\xi \mid s]$ subject to certain constraints, it is always possible to define a gauge potential A_μ in terms of the $F_\mu[\xi \mid s]$ such that the two variables describe the same physics [4, 6].

Chapter 3 concentrates on the dynamics of the monopole and the free action is determined for both the classical and quantum cases. The constraints necessary for removing the redundancy of the $F_\mu[\xi \mid s]$ are determined and shown to be closely linked to the presence of monopoles and the definition of the topological charge. These constraints are then imposed on the action by using Lagrange multipliers, one for each constraint. The equations of motion are derived by varying this "auxiliary action" with respect to the now unconstrained variables $F_\mu[\xi \mid s]$, the Lagrange multipliers and the monopole worldline/wavefunction.

It is further shown that the quantum action possesses a symmetry, in addition to its manifest U-symmetry, which is present due to the choice of patch-independent variables. This new $\tilde{U}$-symmetry has its origins in the topological definition of the charge and possesses a chirality opposite to that of the original symmetry [2, 3, 5, 7].

The last two chapters concentrate on attempting to calculate a few simple scattering processes for classical/quantum monopoles, and comparing them to the analogous processes with sources [24]. The monopole-monopole scattering in the classical case is examined first, in Chapter 4. In all the cases considered, it is found that there is no difference between the behaviour of the system of monopoles and sources that could potentially be detectable. However, the comparison is of little practical use, since classical laws do not adequately describe the dynamics of particles in nature and the laws of quantum mechanics should be applied.

This is done in Chapter 5, where the monopole-monopole scattering via the exchange of a single so-called L-particle is examined. The propagators in the theory are calculated, and in doing so, it is found that the $\tilde{U}$-gauge needs to be fixed. The $\tilde{U}$-symmetry of the theory is used to accomplish this. The Feynman diagram for this simple monopole-monopole process is calculated and the result compared to the analogous quark-quark scattering amplitude. The thesis concludes with a brief discussion of calculations underway, and extensions of the Feynman diagram calculations to more complicated processes.

Chapter 1

Useful Gauge Theory Concepts

1.1 Parallel Phase Transport

The concept of monopole is deeply rooted in the structure of gauge theories. In order to understand this concept, and for the later development of this thesis, it is necessary to briefly re-examine gauge theories, with this particular purpose in mind.

In particle physics, gauge theories describe the interactions and dynamics of elementary particles. These are theories which are symmetric under a set of transformations known as gauge transformations. The gauge transformations form a group, the gauge group of the theory.

In the simplest gauge theory, electromagnetism, with the gauge group $U(1)$, the charges of the theory (i.e. electrons) are described by a wavefunction $\psi(x)$. This wavefunction is in general complex-valued and therefore has associated with it, a magnitude and a phase:

$$\psi(x) = |\psi(x)| \exp i\alpha. \tag{1.1}$$

Due to the gauge invariance of the theory, the phase α is completely arbitrary and can be redefined at every spacetime point x without changing the physics. It is therefore not a straightforward matter to compare two phases at two different

spacetime points and the concept of parallelism of phases has to be made clear. To achieve this, a connection $a_\mu(x),\ \mu = 0, 1, 2, 3$, is defined. Two phases are then considered to be parallel at x^μ and $x^\mu + dx^\mu$ if they differ by an amount $e a_\mu(x)\ dx^\mu$. Suppose now that we perform an x-dependent "local" gauge transformation on $\psi(x)$, reassigning the phase at every spacetime point x:

$$\psi(x) \rightarrow \psi'(x) = \exp ie\,\alpha(x)\,\psi(x). \tag{1.2}$$

Under this gauge transformation, the phase of ψ at the neighbouring point $x^\mu + dx^\mu$ will be given by:

$$e\,\alpha(x + dx^\mu) = e\,\alpha(x) + e\,\partial_\mu\,\alpha(x)\,dx^\mu. \tag{1.3}$$

Hence, to ensure that parallel phases remain parallel after performing the gauge transformation, the a_μ must transform as [8]:

$$a_\mu(x) \rightarrow a'_\mu(x) = a_\mu(x) + \partial_\mu\,\alpha(x). \tag{1.4}$$

The procedure outlined above, describes infinitesimal parallel-transport. To generalise this to finite parallel-transport along a path Γ, one can apply repeated infinitesimal parallel-transports. Thus, in parallel-transporting a wavefunction from P to Q along Γ, the total change in phase is given by:

$$e \int_{P\,\Gamma}^{Q} a_\mu(x)\,dx^\mu. \tag{1.5}$$

It can be seen that parallel-transport along different paths Γ_1 and Γ_2 (Fig. 1.1), will not yield the same result in general. The difference:

$$e \int_{P\,\Gamma_2}^{Q} a_\mu(x)\,dx^\mu - e \int_{P\,\Gamma_1}^{Q} a_\mu(x)\,dx^\mu = e \oint_{\Gamma_2-\Gamma_1} a_\mu(x)\,dx^\mu, \tag{1.6}$$

depends on the function $a_\mu(x)$. If $a_\mu(x)$ is curl-free everywhere in the region bounded by $\Gamma_2 - \Gamma_1$, the integral around the closed path $\Gamma_2 - \Gamma_1$ would give zero and hence, parallel-transport would be path-independent. However, if a_μ is not curl-free, the integral would yield:

$$e \oint_{\Gamma_2-\Gamma_1} a_\mu(x)\,dx^\mu = -e \iint_{\Sigma} f_{\mu\nu}\,d\sigma^{\mu\nu}, \tag{1.7}$$

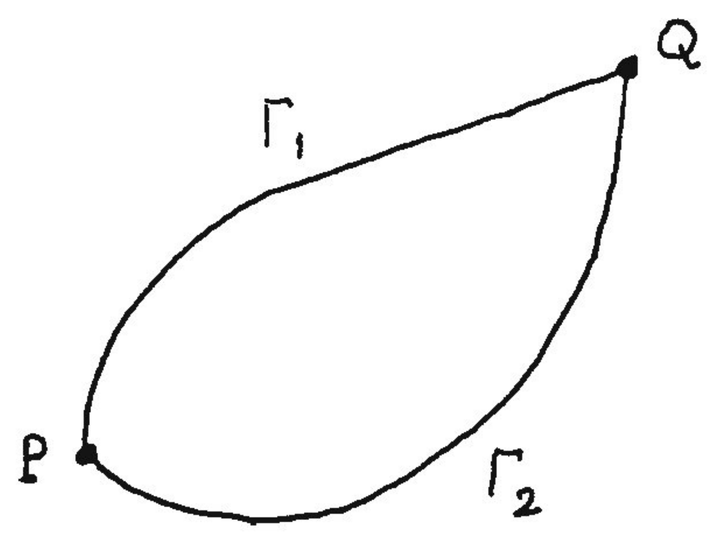


Figure 1.1: Parallel-transport along different paths

where:

$$f_{\mu\nu} = \partial_\nu a_\mu - \partial_\mu a_\nu, \tag{1.8}$$

is the electromagnetic field tensor. Thus the quantity $f_{\mu\nu} d\sigma^{\mu\nu} = f_{\mu\nu}\, dx^\mu\, dx^\nu$ represents the change in phase in going around an infinitesimal rectangle of sides dx^μ and dx^ν, as shown in Fig. 1.2. $f_{\mu\nu}(x)$, being the difference of two phases taken at the same spacetime point, is gauge invariant.

Historically, the functions $a_\mu(x)$ were first introduced in classical electromagnetism as a convenient calculational tool. They were related to $f_{\mu\nu}$ as above, but had no physical significance. This was due to the fact that the particles were not described by complex wavefunctions with a phase but by world-lines. The only interaction between the field and the particle was through the Lorentz equation, which involves $f_{\mu\nu}$ only and not a_μ. In quantum electrodynamics however, the functions a_μ take on a new role. Indeed they can be shown to be physically

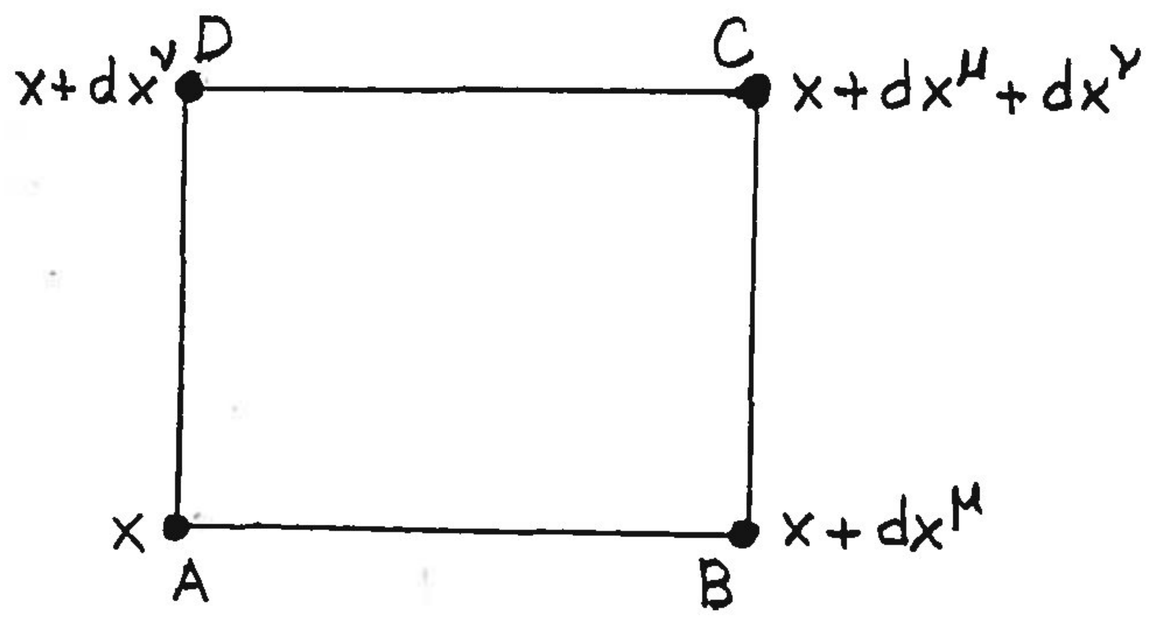


Figure 1.2: Parallel-transport around an infinitesimal rectangle

significant quantities, with observable physical consequences.

This was first pointed out in a thought experiment by Aharonov and Böhm in 1959, and experimentally confirmed by Chambers in 1960. The set-up (Fig. 1.3) consists of a source at which fires electrons at the two slits. The electrons are then allowed to pass on either side of a magnetic solenoid, and subsequently form a diffraction pattern on the screen. The magnetic field of the solenoid is zero everywhere outside the core and non-zero only inside. Therefore $f_{\mu\nu}$ is identically zero everywhere in the region R traversed by the electrons. In classical mechanics this ensures that there will be no interaction between the electrons and the electromagnetic field and hence, the electrons will not sense the presence of the solenoid. This is a situation however, where the predictions of classical and quantum electrodynamics differ significantly. In quantum electrodynamics, although $f_{\mu\nu}$ is identically zero everywhere in R, a_μ is non-zero in this region. The phase of the electron wavefunction at the screen relative to that at the source,

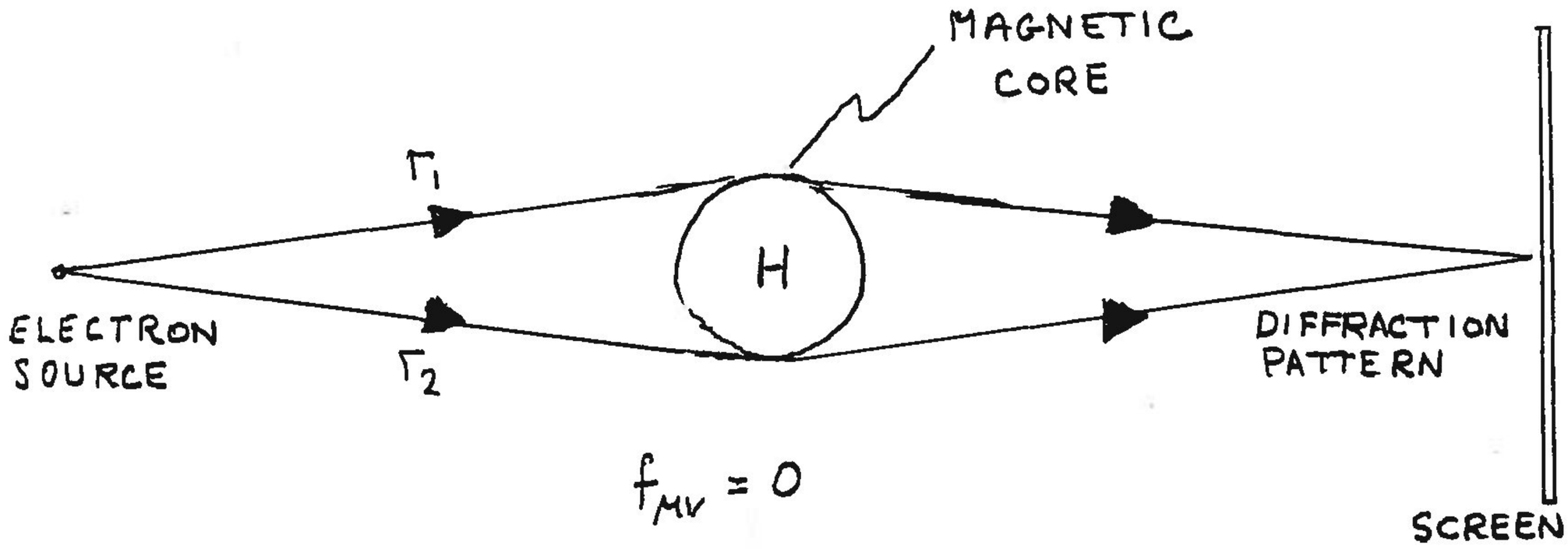


Figure 1.3: Set-up for Böhm-Aharonov experiment

will therefore shift by an amount:

$$e \int_{P\ \Gamma_i}^{Q} a_\mu(x)\, dx^\mu, \tag{1.9}$$

which depends on the path Γ_i traversed by the electrons. The difference in phase between the two paths Γ_1 and Γ_2 through slits 1 and 2 is:

$$\oint_{\Gamma_2-\Gamma_1} a_\mu(x)\, dx^\mu, \tag{1.10}$$

at the screen, and hence, the position of the maximum shifts away from the centre. The quantity:

$$\oint_{\Gamma_2-\Gamma_1} a_\mu(x)\, dx^\mu = -e \iint_\Sigma f_{\mu\nu}(x) d\sigma^{\mu\nu}, \tag{1.11}$$

is just the magnetic flux through the surface bounded by Γ_1 and Γ_2, due to the presence of the solenoid. The effect of the magnetic solenoid is therefore felt by the electrons because of a non-zero $a_\mu(x)$, where none would be expected classically.

In quantum electrodynamics, we see therefore that the gauge potential acquires a new physical significance which it did not possess classically.

Yang-Mills theory is a generalisation of abelian electromagnetism to non-abelian gauge groups. In what follows, we consider only $SU(N)$ theories, which are sufficient for the purposes of this thesis. Here, the state of the particle is once again completely specified by its wavefunction $\psi(x)$, which has N components $\psi_i(x),\ i = 1, \ldots, N$. The Lagrangian possesses an internal symmetry and is invariant under nonabelian gauge transformations. Under such a transformation, the wavefunction of the charged particle undergoes a change in "phase" given by a change in its orientation in internal space:

$$\psi(x) \to \psi'(x) = S(x)\psi(x), \tag{1.12}$$

where S is a unitary matrix with unit determinant (i.e. $S \in SU(N)$) in the fundamental representation. The requirement of gauge invariance thus states that the theory must remain invariant under an arbitrary local $SU(N)$ gauge transformation. To define the parallelism of phases, a set of matrix-valued functions $A_\mu(x)$ must be introduced. Two phases at points x^μ and $x^\mu + dx^\mu$ are then defined to be parallel if they differ by $gA_\mu(x)\,dx^\mu$ where g is the "charge" of the particle. Such a change in phase in ψ is obtainable by applying an infinitesimal gauge transformation:

$$S = \exp igA_\mu(x)\,dx^\mu = 1 + igA_\mu(x)\,dx^\mu, \tag{1.13}$$

to ψ, where $S \in SU(N)$. The gauge potential A_μ therefore, has to be traceless and hermitian, i.e. an element of the gauge algebra $su(N)$. Under an arbitrary local $SU(N)$ transformation S, the functions $A_\mu(x)$ transform as [8]:

$$A_\mu(x) \to A'_\mu(x) = S(x)A_\mu(x)S^{-1}(x) - \frac{i}{g}\,\partial_\mu S(x)\,S^{-1}(x), \tag{1.14}$$

in order that parallel phases at x^μ remain parallel at $x^\mu + dx^\mu$. The field tensor $F_{\mu\nu}$ is obtained (as in abelian theory) by parallel-transporting $\psi(x)$ around an

infinitesimal rectangle (see Fig. 1.2) and represents the total change in ψ, in travelling around the circuit shown in the figure. In Yang-Mills theory, the field tensor $F_{\mu\nu}$ and the potential A_μ are related by:

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) + ig[A_\mu(x), A_\nu(x)], \tag{1.15}$$

where the last term has its origins in the non-commutativity of the algebra elements. The field tensor $F_{\mu\nu}$ is no longer an invariant quantity under $SU(N)$, but transforms covariantly according to:

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = S(x)F_{\mu\nu}(x)S^{-1}(x). \tag{1.16}$$

Parallel-transport over finite distances (say from P to Q along Γ) can once again be achieved by repeated infinitesimal parallel-transports. Once again however, the non-abelian nature of the gauge group introduces complications. Infinitesimal parallel-transport is given by applying $\exp igA_\mu(x)\,dx^\mu$ to ψ, yielding:

$$\psi'(x) = \exp igA_\mu(x)\,dx^\mu\,\psi(x). \tag{1.17}$$

Dividing the interval PQ into N equal segments of length dx and parallel-transporting along each segment, one obtains a product of exponentials:

$$\psi'(x) = \prod_{i=1}^{N} \exp igA_\mu(x_i)dx_i^\mu\,\psi(x). \tag{1.18}$$

This product cannot be written as an exponential of a sum, since if $[A, B] \neq 0$, then:

$$\exp A.\exp B \neq \exp(A+B), \tag{1.19}$$

in general. Thus, the factors need to be "path-ordered", i.e.:

$$\psi'(x) = P\exp ig\int_{P\ \Gamma}^{Q} A_\mu(x)\,dx^\mu\ \ \psi(x), \tag{1.20}$$

where P denotes the path-ordering of the factors $\exp igA_\mu(x)\,dx^\mu$ from P to Q along the curve Γ. The factor:

$$\Phi_{QP}(\Gamma) = P\exp ig\int_{P\ \Gamma}^{Q} A_\mu(x)\,dx^\mu, \tag{1.21}$$

is called the phase factor from P to Q along the open path Γ [29]. Another complication in dealing with nonabelian gauge groups is that the concept of flux is no longer well-defined. In electromagnetism, Stokes' theorem relates the total change in phase due to parallel-transport along a closed curve to the integral over any closed surface bounded by that curve:

$$e \oint_\Gamma a_\mu(x)\, dx^\mu = -e \iint_\Sigma f_{\mu\nu}(x)\, d\sigma^{\mu\nu}, \tag{1.22}$$

where the right-hand side is just the magnetic flux out of the volume bounded by Σ. In Yang-Mills theory, although Stokes' theorem still holds:

$$e \oint A_\mu(x)\, dx^\mu = -e \iint_\Sigma (\partial_\nu A_\mu(x) - \partial_\mu A_\nu(x))\, d\sigma^{\mu\nu}, \tag{1.23}$$

the quantity on the right is not related to $F_{\mu\nu}$, or to the physically interesting concept of parallel-transport in any obvious way. This prevents us from generalising the concept of flux to nonabelian theories. There have been many attempts in this direction, but none have been very useful. In fact, there seems to be no useful nonabelian analogue to the concept of flux.

Under an $SU(N)$ gauge transformation, the phase factor transforms as:

$$P \exp ig \int_P^Q A_\mu(x)\, dx^\mu \to S(Q)\ \ P \exp ig \int_P^Q A_\mu(x)\, dx^\mu\ \ S^{-1}(P), \tag{1.24}$$

showing that it is affected by the gauge transformation only at the endpoints. One important particular case is the phase factor over closed loops:

$$\Phi(C) = P \exp ig \oint A_\mu(x)\, dx^\mu, \tag{1.25}$$

which we will often use. As in the previous case, gauge transformations affect this phase factor only at the basepoint. This is quite easy to handle and for our purposes, these phase factors can be considered to be gauge invariant.

1.2 Determination of Gauge Groups

Gauge groups in physical theories are usually specified through infinitesimal gauge transformations, which are formed by Lie-algebra elements. The elements of the

gauge group are then generated by applying successive infinitesimal gauge transformations. This procedure however, does not completely specify the global structure of the gauge group, since non-isomorphic gauge groups can have the same Lie algebra structure. For studying monopole theory, one needs to understand this global structure and it will be seen that it plays a very important role in determining whether or not monopoles exist in the theory, and if so, in determining their charges. However, in order to determine the global properties of the gauge group, one must examine the theory in some detail to decide what the gauge group of the theory is; that is to say, which of the group elements are to be regarded as physically distinct.

To begin, consider the case of electromagnetism, where the gauge transformation is:

$$\psi \to \psi' = \exp ie\alpha\, \psi, \tag{1.26}$$

where e is the electric charge, and α the gauge parameter. However, note that for $\alpha' = \alpha + \frac{2\pi n}{e}$, we have:

$$\exp ie\alpha' = \exp ie\alpha, \tag{1.27}$$

implying that the two group elements $\exp ie\alpha$ and $\exp ie\alpha'$ have the same effect on the wavefunction ψ, and therefore have to be regarded as the same group element. The gauge group of the theory is therefore, not one obtained by allowing α to vary between $-\infty$ and $+\infty$, but by restricting it to be in the range $[0, \frac{2\pi}{e}]$, with the endpoints identified. Consequently, the gauge group of the theory is not the group of the real numbers $\mathbf{R}$, but the compact group $U(1)$, the group of the circle.

To apply the same reasoning in the more general case, consider the theory of N charged particles with charges $e_1, \ldots, e_N$ moving in an electromagnetic field. Under a gauge transformation, the wavefunction ψ_r of the r-th particle will transform as follows:

$$\psi_r \to \psi_r' = \exp ie_r\alpha\, \psi_r. \tag{1.28}$$

The question of which values of α denote physically distinct gauge group elements now depends on whether or not the values of $\{e_r \mid r = 1, \ldots, N\}$ are rational with respect to one another. If so, then there is a smallest charge e, such that:

$$e_r = n_r e, \qquad n_r \in \mathbf{Z}, \tag{1.29}$$

In this case, all those values of α differing by $\frac{2\pi n}{e}$ will give the same $\{\psi_r\}$ and will therefore be physically indistinguishable. The gauge group will thus be the compact group $U(1)$. This is not the case however, if one of the pairs e_r/e_s is irrational. This forces us to consider all the values for $\alpha \in (-\infty, +\infty)$ as physically distinct, showing that the gauge group in this case is the non-compact group $\mathbf{R}$. From the considerations above, one is led to conclude that compact electrodynamics is synonymous with charge quantisation.

The same type of argument must be applied in the more general case of nonabelian theories. Once again, the particle content of the theory is crucial in determining the global structure of the gauge group. Consider pure Yang-Mills theory: the theory with colour gluons and no quarks. The gauge potential transforms as a connection under $SU(3)$ gauge transformations:

$$A_\mu \to A'_\mu = S A_\mu S^{-1} + \frac{i}{g} S^{-1} \partial_\mu S, \tag{1.30}$$

where $S \in SU(3)$ is a unitary 3×3 matrix with unit determinant. The question now is whether all such matrices are to be regarded as distinct elements of the gauge group.

To answer this question, we note that the matrices:

$$\zeta_r = \exp i \frac{2\pi r}{3}, \qquad r = 0, 1, 2, \tag{1.31}$$

form a cyclic subgroup Z_3 of $SU(3)$ (being unitary with unit determinant). They, being scalar multiples of the identity, commute with $su(3)$ matrices and hence with A_μ. Thus, given a gauge transformation $S \in SU(3)$, the effect of the gauge transformations $S\zeta_r, \quad r = 1, 2$, on A_μ would be indistinguishable from that of S.

Since A_μ is the only field in pure Yang-Mills theory, the gauge transformations $S\zeta_r$ must be regarded as the same gauge transformation, forcing the gauge group to be not the full $SU(3)$, but $SU(3)/Z_3$, which is obtained from $SU(3)$ by identifying triplets of elements $S\zeta_r$.

More generally, for pure gauge theories with gauge algebra $su(N)$, the gauge group can be seen to be $SU(N)/Z_N$, where Z_N is the cyclic group of order N. This is the group obtained from $SU(N)$ by identifying the N gauge transformations given by:

$$S_r = S \exp i\frac{2\pi r}{N}, \qquad r = 0, 1, 2, \ldots, N-1, \tag{1.32}$$

where $S \in SU(N)$ is a gauge transformation. In the case of $su(2)$, the group $SU(2)/Z_2$ is isomorphic to $SO(3)$, the group of special orthogonal transformations.

If other fields are included in the theory, the effect of the gauge transformation on the fields would have to be considered. In the case of quantum chromodynamics containing both quarks and gluons, the gauge group is the full group $SU(3)$, since although the transformations $S_r = S\zeta_r, \quad r = 0, 1, 2$, have an identical effect on A_μ, they act upon the quark wavefunctions differently:

$$S_r\psi_q = S \exp i\frac{2\pi r}{3}\psi_q \neq S\psi_q, \quad r = 1, 2, \tag{1.33}$$

and must therefore be considered as distinct.

Another important case is that of electroweak theory, whose gauge group is usually denoted by $SU(2) \times U(1)$. A closer look at the particle content of the theory however, shows that the gauge group of the theory is not $SU(2) \times U(1)$, but $[SU(2) \times U(1)]/Z_2$, which is isomorphic to the group $U(2)$ of 2×2 unitary matrices. This is obtained by identifying pairs of elements (f, y) and $(\tilde{f}f, \tilde{y}y)$, where $\tilde{f} = \exp 2\pi i T_3 \in SU(2)_f$ and $\tilde{y} = \exp 2\pi i Y \in U(1)_Y$, and we have:

$$(\tilde{f}, \tilde{y})\psi = \exp(2\pi i(T_3 + Y))\psi = \psi. \tag{1.34}$$

For the "standard model", a similar analysis shows that the gauge group is a group obtained from $SU(3)_c \times SU(2)_f \times U(1)_Y$ by identifying the sextuplet of elements:

$$\begin{array}{ccccc} (c,f,y) & = & (cc_1,f,yy_1) & = & (cc_2,f,yy_2) \\ = (c,f\tilde{f},y\hat{y}) & = & (cc_1,f\tilde{f},y\hat{y}y_1) & = & (cc_2,f\tilde{f},y\hat{y}y_2), \end{array} \tag{1.35}$$

for $c \in SU(3)_c$, $f \in SU(2)_f$, $y \in U(1)_Y$, where:

$$T = T_3 - Q, \tag{1.36}$$

$$c_r = \exp \frac{i2\pi r}{\sqrt{3}} \lambda_8, \quad r = 1,2, \tag{1.37}$$

$$y_r = \exp i4\pi r Y, \quad r = 1,2, \tag{1.38}$$

$$\hat{y} = \exp i6\pi Y, \tag{1.39}$$

with $\tilde{f}$ as defined above. This gauge group can symbolically be denoted as $[SU(3)_c \times SU(2)_f \times U(1)_Y]/Z_6$ [8].

The analysis above will be seen to be essential to the study of monopoles. As previously mentioned, it will be shown that the existence of monopoles depends crucially on the global properties of the gauge group, in particular on its topological structure. Since the groups mentioned above (e.g. $SU(N)$ *vs* $SU(N)/Z_N$) have very different topological structures, it can be seen that the precise identification of the gauge group is of supreme importance.

1.3 Abelian Theory

One of the interesting properties of Maxwell's equations in the absence of sources, is their symmetry properties under the duality operation:

$$f_{\mu\nu} \rightarrow^* f_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f^{\rho\sigma}. \tag{1.40}$$

The Maxwell equations (in vacuum) are given by:

$$\partial_\nu f^{\mu\nu} = 0, \tag{1.41}$$

$$\partial^*_\omega f^{\mu\nu} = 0, \tag{1.42}$$

and are easily seen to be symmetric under the * operation. By introducing electric charges (henceforth referred to as sources), the first equation becomes:

$$\partial_\nu f^{\mu\nu} = j^\mu, \tag{1.43}$$

where j^μ is the 4-current. The introduction of an electric charge then, destroys the symmetry.

If however, the existence of magnetic charge (i.e. monopoles) is assumed, the Maxwell equations become:

$$\partial_\nu f^{\mu\nu} = j^\mu, \tag{1.44}$$

$$\partial^*_\omega f^{\mu\nu} = \tilde{j}^\mu, \tag{1.45}$$

where $\tilde{j}^\mu$ is the magnetic 4-current, and symmetry under the * operation (with $j^\mu \leftrightarrow \tilde{j}^\mu$) is restored.

In terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, the equations read:

$$\begin{aligned} \nabla.\mathbf{E} &= 4\pi\rho_e, & \nabla\times\mathbf{B} &= \tfrac{1}{c}\dot{\mathbf{E}} + \tfrac{4\pi}{c}\mathbf{J_e}, \\ \nabla.\mathbf{B} &= 4\pi\rho_m, & \nabla\times\mathbf{E} &= -\tfrac{1}{c}\dot{\mathbf{B}} + \tfrac{4\pi}{c}\mathbf{J_m}. \end{aligned} \tag{1.46}$$

These equations are invariant under a transformation defined by:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}'\cos\xi + \mathbf{B}'\sin\xi, & \rho_e &= \rho'_e\cos\xi + \rho'_m\sin\xi, \\ \mathbf{B} &= -\mathbf{E}'\sin\xi + \mathbf{B}'\cos\xi, & \rho_m &= -\rho'_e\sin\xi + \rho'_m\cos\xi, \\ \mathbf{J_e} &= \mathbf{J'_e}\cos\xi + \mathbf{J'_m}\sin\xi, & \mathbf{J_m} &= -\mathbf{J'_e}\sin\xi + \mathbf{J'_m}\cos\xi. \end{aligned} \tag{1.47}$$

The invariance of the Maxwell equations under this transformation indicates that it is a matter of convention to speak of a particle possessing electric and no magnetic charge, since ξ could always be chosen such that [18]:

$$\rho_m = 0, \qquad\qquad \mathbf{J_m} = 0, \tag{1.48}$$

and hence, the original Maxwell equations would be recovered. The only meaningful question therefore, is whether all particles possess the same ratio of magnetic

to electric charge. One could equally well choose ξ such that $\rho_e = 0$, $\mathbf{J_e} = 0$, and thus the charge would be purely magnetic.

The concept of "duality" is the main property of abelian theory which distinguishes it from the nonabelian theories. The choice of $\rho_m = \mathbf{J}_m = 0$ is purely a matter of convention. It will be seen that the absence of duality in nonabelian theories forces us to define the magnetic charge in a different manner, by utilising the concept of topological obstruction. This will be the topic of the next section.

1.4 Monopole as Topological Obstruction

In gauge theories, as remarked previously, monopoles appear naturally as topological obstructions in the gauge group. To show this, it will be helpful to consider a family of parametrised closed loops $\{C_t\}$, defined by [8]:

$$C_t = \{\xi_t^\mu(s) : t = 0 \to 2\pi,\ s = 0 \to 2\pi\}, \tag{1.49}$$

enveloping a sphere Σ as shown in Fig. 1.5. At $t = 0$, C_t is just the point P_0. As t increases, C_t expands and loops over the surface Σ, until it shrinks back to the point P_0 for $t = 2\pi$. The parameter s labels the points along the loop. The phase factor $\Phi(C_t)$, which is an element of the gauge group G, could be calculated for every value of t. At $t = 0$ and 2π, $\Phi(C_t)$ is thus just the identity element, and for the intermediate values of t, it traces a continuous curve Γ in the gauge group, beginning and ending at the identity. Now suppose the following question is asked: what happens if the surface Σ is continuously shrunk to the point P_0? Does the curve Γ in the gauge group also shrink to a point? If not, then there must be something inside the surface Σ, preventing the contraction. This must be a topological property of the gauge field, and is therefore called a *topological obstruction*. The existence of a topological obstruction depends on the global properties of the gauge group. The question reduces to whether or not the curve Γ in the gauge group could always be shrunk to a point by continuously

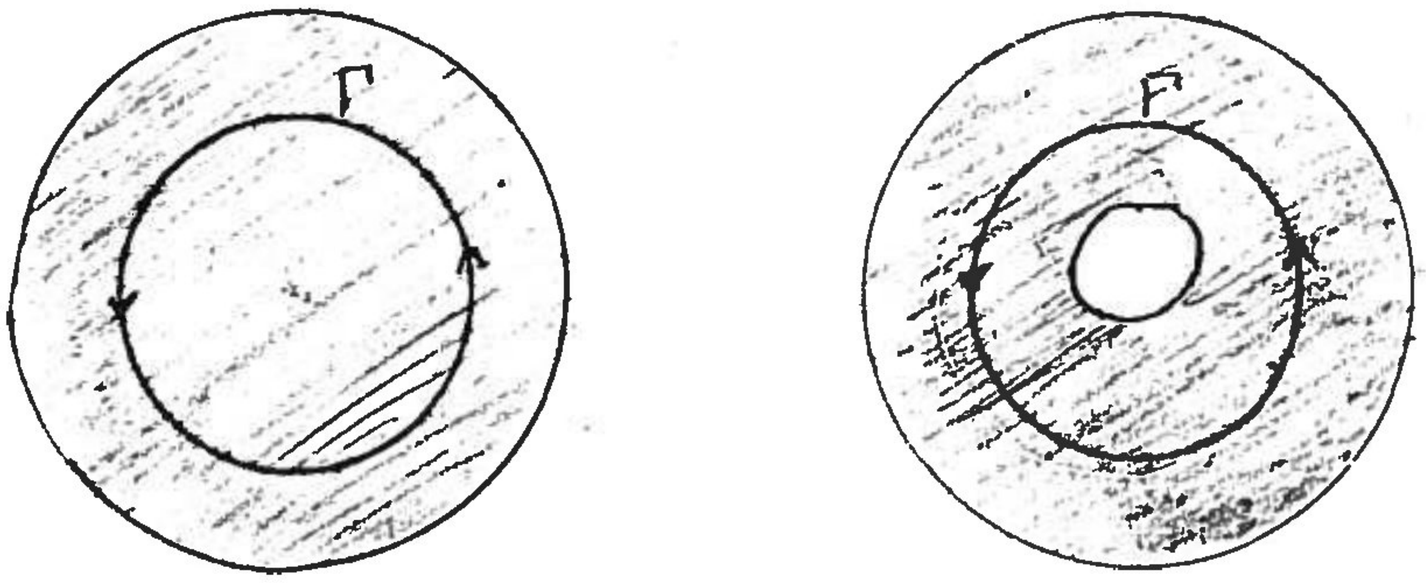


Figure 1.4: An example of a curve in G which cannot be deformed to a point

shrinking Σ to P_0: if the gauge group is simply-connected, this will always be the case. However, for some gauge groups G, i.e. those that are multiply-connected, there would be some curves Γ in G which could never be continuously deformed to a point (see Fig. 1.4). In the latter case, the theory will be shown to yield monopoles.

Once again, the example of electromagnetism can be used to illustrate the essential points. In non-compact electrodynamics where the gauge group is the real line, all closed curves in the gauge group can be continuously deformed to a point, whereas in the case of non-compact electrodynamics with the gauge group $U(1)$, there are some closed curves for which this cannot be done (namely those that wind around the group an integer number of times) and will thus correspond to topological obstructions. Thus, these obstructions exist only in compact electrodynamics or, as argued earlier, only if the charge is quantised.

To show that the latter case corresponds to a magnetic monopole, one must

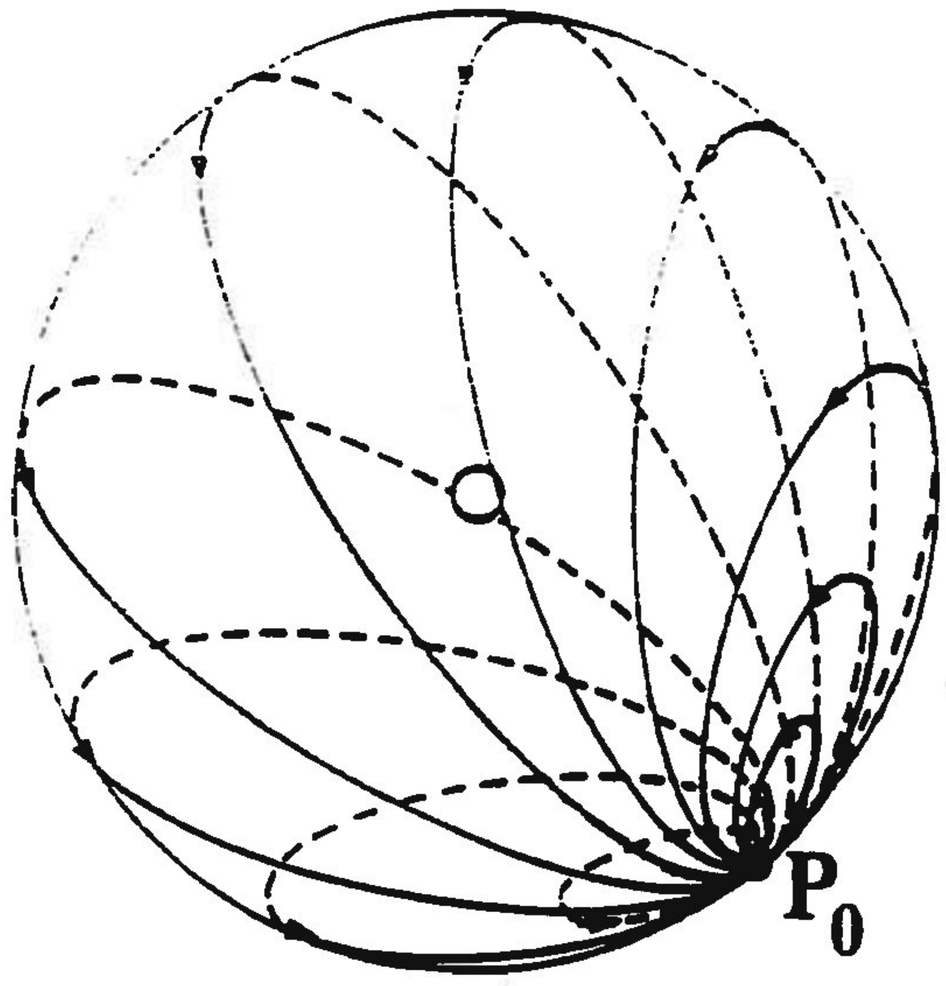


Figure 1.5: Family of loops enveloping a surface Σ

show that the topological obstruction is a source of magnetic flux, so that it can be regarded as a magnetic charge. To see this, consider a closed surface containing a topological obstruction. Now, consider the line integral in the exponent of the phase factor; i.e.:

$$\alpha_t = e \oint_{C_t} a_\mu(x)\, dx^\mu. \tag{1.50}$$

This quantity is the angle giving the position of the point $\Phi(C_t)$ on the curve Γ in the gauge group. Using Stokes' theorem, we obtain:

$$\alpha_t = -e \iint_{\Sigma_t} f_{\mu\nu}(x) d\sigma^{\mu\nu}, \tag{1.51}$$

where Σ_t is the surface swept out by $\{C_{t'}\}$ for $t' = 0 \to t$. As $t \to 2\pi$, the loop C_t shrinks back to the point P_0, and Σ_t becomes the whole surface Σ, as depicted in Fig. 1.5. Thus, for $t = 2\pi$, α_t represents the total magnetic flux leaving the surface Σ_t.

We now consider two possibilities: the curve Γ in the gauge group $U(1)$ might

or might not wind around the group $U(1)$. Consider the case of a Γ that does not. The quantity α_t is zero at $t = 0$. At $t = 2\pi$, the curve of integration shrinks to a point, and once again $\alpha_t = 0$. The total magnetic flux in this case is zero, and thus there are no monopoles inside Σ. In the case of a closed Γ however, although $\alpha_{t=0}$ is once again zero, α_t at $t = 2\pi$ returns to the origin only after winding around the gauge group once. Therefore, at $t = 2\pi$, $\alpha_t = 2\pi$. This means that the total magnetic flux emerging from Σ is $2\pi/e$ in this case. This is exactly the value one would get if there was a magnetic monopole of charge $\tilde{e} = 1/(2e)$ inside Σ. Thus, we see that in the case of abelian theory, the two descriptions of magnetic monopole as source of magnetic flux and as topological obstruction coincide.

This $U(1)$-magnetic charge can take on an infinite number of values, labelled by an integer n. To see this, we must recall that Γ is a closed curve in the gauge group. It must therefore wind around the group an integral number of times. By defining this "winding number" n as positive (negative) if Γ winds around the circle in the counter-clockwise (clockwise) direction, one can easily see that the magnetic flux emerging from Σ is given by $2n\pi/e$, and that the magnetic charge has a value of:

$$\tilde{e} = \frac{n}{2e}. \tag{1.52}$$

These are the only values that the charge can take. This is the famous *Dirac Quantisation Condition*, first derived by Dirac in 1931 [15].

Another property of the $U(1)$-magnetic charge is that it is a conserved quantity in both classical and quantum dynamics. This property is a consequence of the topological definition of magnetic charge, and the property that the winding number n cannot be changed by continuous deformations of the closed curve Γ.

Let us consider the classical case first: the evolution in time of the system is governed by continuous changes in its dynamical variables. For electromagnetism, $a_\mu(x)$ can be taken to be the dynamical variable, and by changing its value,

$\Phi(C_t)$ and hence the curve Γ in G will also vary. These variations however, being continuous in classical mechanics, cannot possibly change the winding number n of Γ around G, and so the value of magnetic charge $\tilde{e}$ cannot change.

In quantum theory, the time evolution of the system is given by the transition matrix, where the amplitude for the transition from an initial state $|\,i\rangle$ to a final state $|\,f\rangle$ is given in terms of the Feynman integral:

$$\langle f\,|\,i\rangle = \int_i^f \delta a\, \exp\{-i\mathcal{A}[a]\}, \tag{1.53}$$

where A is the action and is a functional of the gauge potential $a = \{a_\mu(x)\}$. The integration over a_μ is however, only over continuous paths in the function space, and therefore, since there is no continuous $a_\mu(x)$ linking two states with different winding numbers n, the space over which the Feynman integral is evaluated has zero measure, and hence the transition amplitude is zero. The magnetic charge in quantum theory is therefore conserved.

The conservation law derived above being a consequence of the topology of the gauge group, does not rely on the specific form of the action, and hence is of a more fundamental nature than the conservation laws derived from Nöether's theorem which depend on the details of the dynamics.

In the case of nonabelian theory, it would be interesting to see if similar ideas about the monopole could be applied. One might think that an obvious way of defining a monopole in these theories would be through the concept of flux as in the abelian case. Therefore, we might begin by defining a generalisation of flux to nonabelian theories:

$$flux = -\iint_\Sigma F_{\mu\nu}\, d\sigma^{\mu\nu}, \tag{1.54}$$

and specify that the presence of a monopole in the region bounded by Σ, is characterised by the fact that the flux over Σ is non- zero. Using Stokes' theorem, this integral can be written as an integral of the ordinary divergence $\partial^*_\nu F^{\mu\nu}$. However, since for nonabelian fields in the absence of monopoles:

$$D^*_\nu F^{\mu\nu} = \partial^*_\nu F^{\mu\nu} - ig\,[A_\nu, {}^*F^{\mu\nu}] = 0, \tag{1.55}$$

and hence $\partial^*_\nu F^{\mu\nu} = ig[A_\nu, {}^*F^{\mu\nu}] \neq 0$ in general, the "flux" does not necessarily vanish even in the absence of monopoles and it cannot therefore be used to characterise a monopole. In fact, it seems impossible to generalise the concept of flux to nonabelian theories even though numerous such attempts have been made.

In the case of nonabelian theories then, the flux concept of monopole seems to be inadequate. One must therefore turn to another way of achieving the goal of defining a monopole charge.

At this point, it is necessary to clarify some terminology: by "charge" we mean a $U(1)$ charge which can either be electric or magnetic, i.e. a source of electric or magnetic flux, depending on whether we are discussing the field $f_{\mu\nu}$ or its dual ${}^*f_{\mu\nu}$. The word "source" is synonymous with "electric charge" in electromagnetism, and "monopole" and "magnetic charge" are also interchangeable terms. More generally, in nonabelian theories, "source" refers to the right-hand side of:

$$D_\nu F^{\mu\nu}(x) = j^\mu(x), \tag{1.56}$$

whereas in this context, "monopole" refers to the topological obstruction defined above. In the case of abelian theory, as seen above, a monopole of $f_{\mu\nu}$ can be considered as a source of ${}^*f_{\mu\nu}$ and vice versa, due to duality (see §3.2). As already discussed, there is an alternative method of defining a magnetic charge as a topological obstruction (without explicitly using the concept of flux), which can be adopted for nonabelian theories as well. We notice that in the abelian theory, the answer to the question of whether or not monopoles are admitted in the theory, depends on the global properties of the gauge group of the theory. If the gauge group is multiply-connected, monopoles are admitted and if not, they are not. The existence of monopoles in a theory in other words, depends on whether closed curves Γ in the gauge group G can all be continuously shrunk to a point or not. Those curves which cannot be so eliminated correspond to a topological

charge. The values that this charge can take are labelled by the distinct classes of closed curves Γ in the gauge group which cannot be deformed into one another [8]. These belong to the first homotopy class π_1 of the gauge group G. By repeating the arguments above, one can show that the topological charge corresponding to one homotopy class cannot evolve in time to one corresponding to another class, and is therefore conserved. Also, the topological charge is by definition discrete and hence quantised. We thus generalise the concept of "magnetic monopole" to nonabelian theories and call these topological charges labelled by the homotopy groups, nonabelian monopoles.

Let us examine the values of this charge in a few cases; namely theories with the gauge algebra $su(N)$. Consider the simplest case with the gauge algebra $su(2)$. The group $SU(2)$ has the topology of a 3-dimensional hypersphere S^3, thus it is simply-connected and will admit no monopoles. On the other hand, $SO(3)$ (which is the gauge group of pure Yang-Mills theory, and is obtained from $SU(2)$ by identifying antipodal points on S^3) is not simply-connected, since any curve connecting two antipodal points A and B is a closed curve but cannot be shrunk to zero by continuous deformations. We thus get monopoles in this theory. The values that this charge can take are dictated by the number of distinct classes of homotopic curves. Take any pair of antipodal points A, B, and A', B', as in Fig. 1.6. It is obvious that the closed curve AB can be continuously deformed to the closed curve $A'B'$, so these two belong to the same homotopy class. Also, by joining two curves AB and BA, one obtains a closed curve which winds around $SO(3)$ twice, thus yielding a curve that can be continuously shrunk to a point (i.e. that belongs to the homotopy class of a point). Thus a monopole is its own antimonopole, since the combined charge is zero, and hence the monopoles in the theory can only take charges labelled by the elements of Z_2 (integers modulo 2), or equivalently by a sign $\pm$, where $+$ corresponds to the vacuum and $-$ to a monopole.

In the case of theories with gauge algebra $su(N)$, one can easily show that the

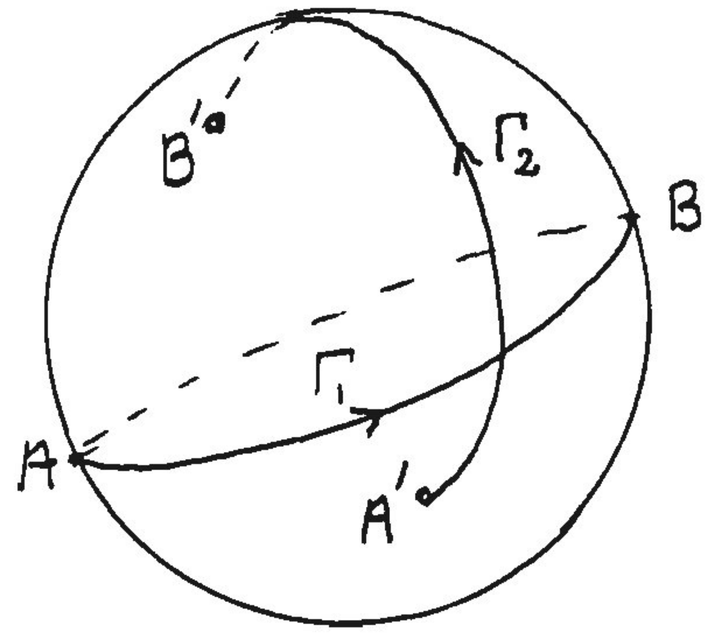


Figure 1.6: The group $SU(2)$

charge must take values in Z_N in the case of pure gauge theories with gauge group $SU(N)/Z_N$, but that theories with gauge group $SU(N)$ being simply-connected, admit no monopoles [8].

It must be noted that in $SU(N)/Z_N$ theory, a generalised Dirac quantisation condition holds:

$$g\tilde{g} = \frac{n}{2N}, \qquad n \in \mathbf{Z}, \tag{1.57}$$

where g is the gauge field self-coupling [8].

As another example, let's take the standard electroweak theory with gauge group $U(2)$, or $[SU(2) \times U(1)]/Z_2$. The group $SU(2) \times U(1)$ itself consists of pairs of elements from $SU(2)$ and $U(1)$ respectively, and it can symbolically be represented as points inside the rectangle (see Fig. 1.7). The opposite sides of the rectangle are to be identified. Taking thus a closed curve PQ winding around the group n-times, one can always continuously deform it to a curve which winds

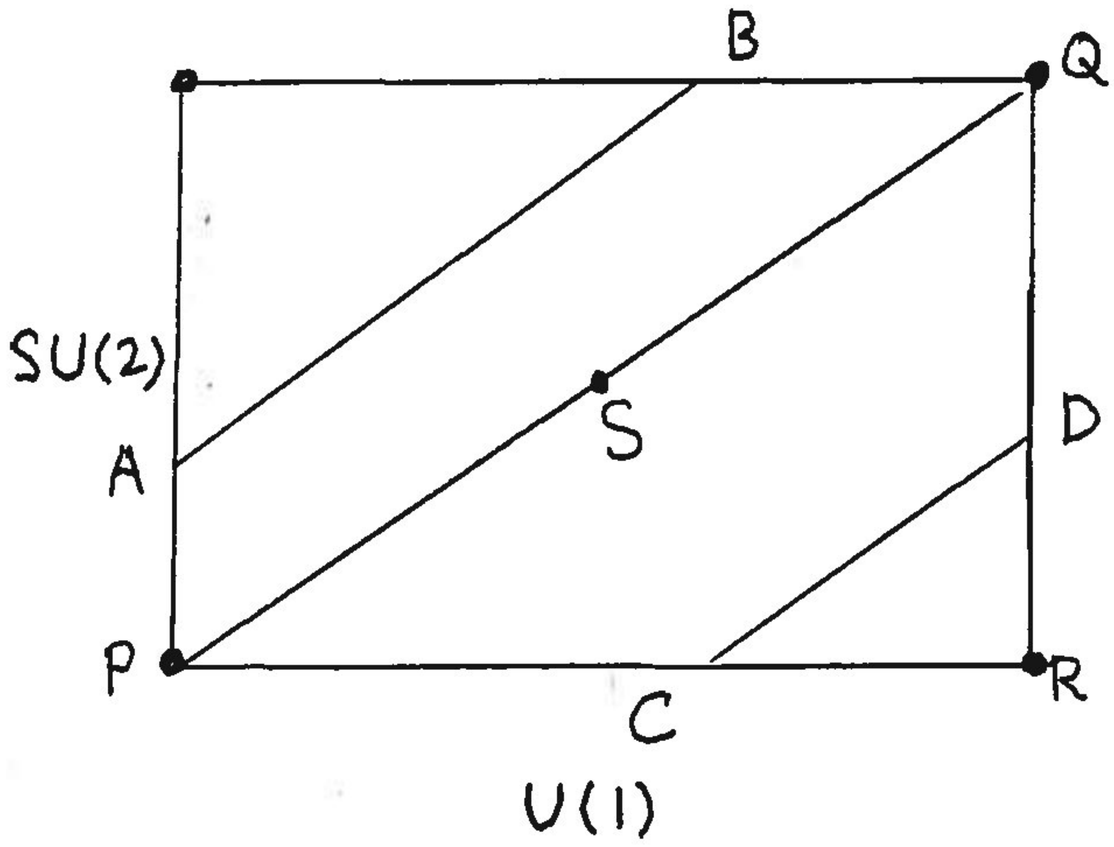


Figure 1.7: The group $SU(2) \times U(1)$

n-times around $U(1)$ but not around $SU(2)$. Thus, a theory with gauge group $SU(2) \times U(1)$ contains monopoles with charges taking integral values, and no others. In the case of electroweak theory however, where the point S representing the pair $(\tilde{f}, \tilde{y})$ in $SU(2) \times U(1)$ is to be regarded as identical to $P = (1, 1)$, PS is already a closed curve in $U(2)$. But PS now winds only halfway around each of $SU(2)$ and $U(1)$, and thus has an $SO(3)$-charge of -1 and a $U(1)$-charge $1/2$ of that represented by PQ or PR. Generalising this to a curve that winds around $[SU(2) \times U(1)]/Z_2$, n times, we see that electroweak theory admits monopoles of $SO(3)$-charge $(-1)^n$ and $U(1)$-charge $\frac{n}{4g'}$ [8].

For the standard model with gauge group $[SU(3) \times SU(2) \times U(1)]/Z_6$ it can be seen that the monopoles in this theory would have $SU(3)/Z_3$-charge $\exp \frac{i2\pi n}{3}$, $SU(3)$-charge $(-1)^n$ and $U(1)$-charge $\frac{n}{6g'}$ [8]. Thus, the question of determining the monopole charge in a theory has been reduced to a mathematical problem. The next task is therefore to study how these monopoles interact with one another

and with other fields in the theory. We thus turn to a study of the dynamics of these objects. In doing so, it is advisable to bear in mind the abelian monopole, since in this case, one has a preconception of the behaviour of magnetic monopoles in the presence of the electromagnetic field. We need to know whether the interactions of these abstractly defined topological charges can be studied in the nonabelian case.

1.5 Monopole Potentials and Patching

Consider now, the abelian theory, in which a magnetic monopole is defined as the source of magnetic flux out of a volume:

$$\Phi_{mag} = e \iint_{\Sigma} f_{\mu\nu}\, d\sigma^{\mu\nu} = 4\pi\tilde{e}, \tag{1.58}$$

where Σ is the surface enveloping the volume and $\tilde{e}$ is the magnetic charge inside Σ. Due to Stokes' theorem, Φ_{mag} could be written as:

$$\Phi_{mag} = e \oint_{\Gamma} a_{\mu}\, dx^{\mu} = 4\pi\tilde{e}. \tag{1.59}$$

The magnetic field due to a single magnetic monopole at the origin is given by:

$$\mathbf{B} = \frac{\tilde{e}\mathbf{r}}{r^3}, \tag{1.60}$$

where $\tilde{e}$ is the magnetic charge of the particle. This field can be derived from the potential:

$$a_x = \frac{\tilde{e}y}{r(r+z)}, \qquad a_y = -\frac{\tilde{e}x}{r(r+z)}, \qquad a_z = 0, \qquad a_t = 0, \tag{1.61}$$

which is singular all along the negative z-axis [8].

As mentioned previously, the potential due to a stationary monopole at the origin is singular. This property is not peculiar to stationary monopoles, and is in fact, a general property of a_μ in the presence of a monopole. To see this, suppose that a_μ is non-singular everywhere on the surface of a sphere surrounding the

monopole. Consider the northern hemisphere N. Applying Stokes' theorem to the expression for the flux leaving N:

$$-\iint_N f_{\mu\nu}\, d\sigma^{\mu\nu} = \oint a_\mu\, dx^\mu, \tag{1.62}$$

where the line integral is taken along the equator in the clockwise direction when viewed from the north pole. The total flux, given by adding (1.62) and the corresponding expression for the flux leaving the southern hemisphere, is already zero, indicating that there is no magnetic monopole inside the sphere. This is clearly a contradiction, and therefore the original assumption of a non-singular a_μ everywhere on the surface Σ of the sphere S must be wrong. Therefore, there must be a point on Σ, where a_μ is singular. By increasing the radius of the sphere S, and applying the same argument, we see that there must be a whole line of singularities starting at the monopole position and extending to ∞. This is the *Dirac string*.

In the case of a stationary, static monopole at the origin, the potential as given in (1.61) is singular along the negative z-axis. However, the field $\mathbf{B} = \frac{\tilde{e}\mathbf{r}}{r^3}$ can equally well be derived from the potential:

$$a'_x = -\frac{\tilde{e}y}{r(r-z)}, \qquad a'_y = \frac{\tilde{e}x}{r(r-z)}, \qquad a'_z = 0, \qquad a'_t = 0, \tag{1.63}$$

which is singular all along the positive z-axis, showing that the negative z-axis has no physical significance. This is to be expected, since the field $\mathbf{B}$ is spherically symmetric. Indeed, the two potentials a_μ and a'_μ are related by a gauge transformation:

$$a'_\mu - a_\mu = \partial_\mu \alpha, \qquad \alpha = 2\tilde{e}\arctan\left(\frac{y}{x}\right) = 2\tilde{e}\phi. \tag{1.64}$$

The Dirac string is therefore seen to be not a physical singularity at all, but due to our particular choice of gauge.

One can give a fully non-singular representation of a_μ on the surface of the sphere S, provided that more than one coordinate patch is used. For example,

in the northern hemisphere N, the potential could be described by (1.61) and in the southern hemisphere S by (1.63). The two patches:

$$\begin{aligned} (N): &\qquad 0 \le \theta < \pi, \qquad 0 \le \phi \le 2\pi, \\ (S): &\qquad 0 < \theta \le \pi, \qquad 0 \le \phi \le 2\pi, \end{aligned} \tag{1.65}$$

overlap in a band along the equator. The potential a_μ is now chosen to be:

$$\begin{aligned} a_i^{(N)}(x) &= a_i(x), \quad a_0^{(N)}(x) = 0, \\ a_i^{(S)}(x) &= a_i'(x), \quad a_0^{(S)}(x) = 0, \end{aligned} \tag{1.66}$$

where $a_\mu^{(N)}$ and $a_\mu^{(S)}$ are related by (1.64) in the overlap region. We thus get a non-singular representation of the monopole potential at the expense of using more than one coordinate patch. The calculation of the line integral of a_μ now gives the correct result, since according to (1.64):

$$\oint_{equator} \left(a_\mu^{(N)} - a_\mu^{(S)}\right) dx^\mu = \oint_{equator} \partial_\mu \alpha \; dx^\mu = 4\pi\tilde{e}, \tag{1.67}$$

which is the value of the total flux emerging from the magnetic charge $\tilde{e}$. This result is quite general and applies equally to moving monopoles in any gauge patched in the manner (1.64).

The extension of the patching process to nonabelian theories is straightforward. In this case also, the gauge potential $A_\mu(x)$ needs to be patched and the patching requires the introduction of a separate gauge potential in each patch A, B while in the overlap region, the potentials must be related by a patching function S_{AB} as follows:

$$A_\mu^{(B)}(x) = S_{BA}(x) A_\mu^{(A)}(x) S_{BA}^{-1}(x) - \frac{i}{g} S_{BA}(x)\, \partial_\mu\, S_{BA}^{-1}(x). \tag{1.68}$$

In the case of a patched gauge potential, all quantities defined so far need to be re-examined, since the potential is no longer specified by one function throughout spacetime. This is particularly true in the overlap region where $A_\mu^{(A)}$ and $A_\mu^{(B)}$ are related by a patching transformation. For local quantities such as $F_{\mu\nu}(x)$, this

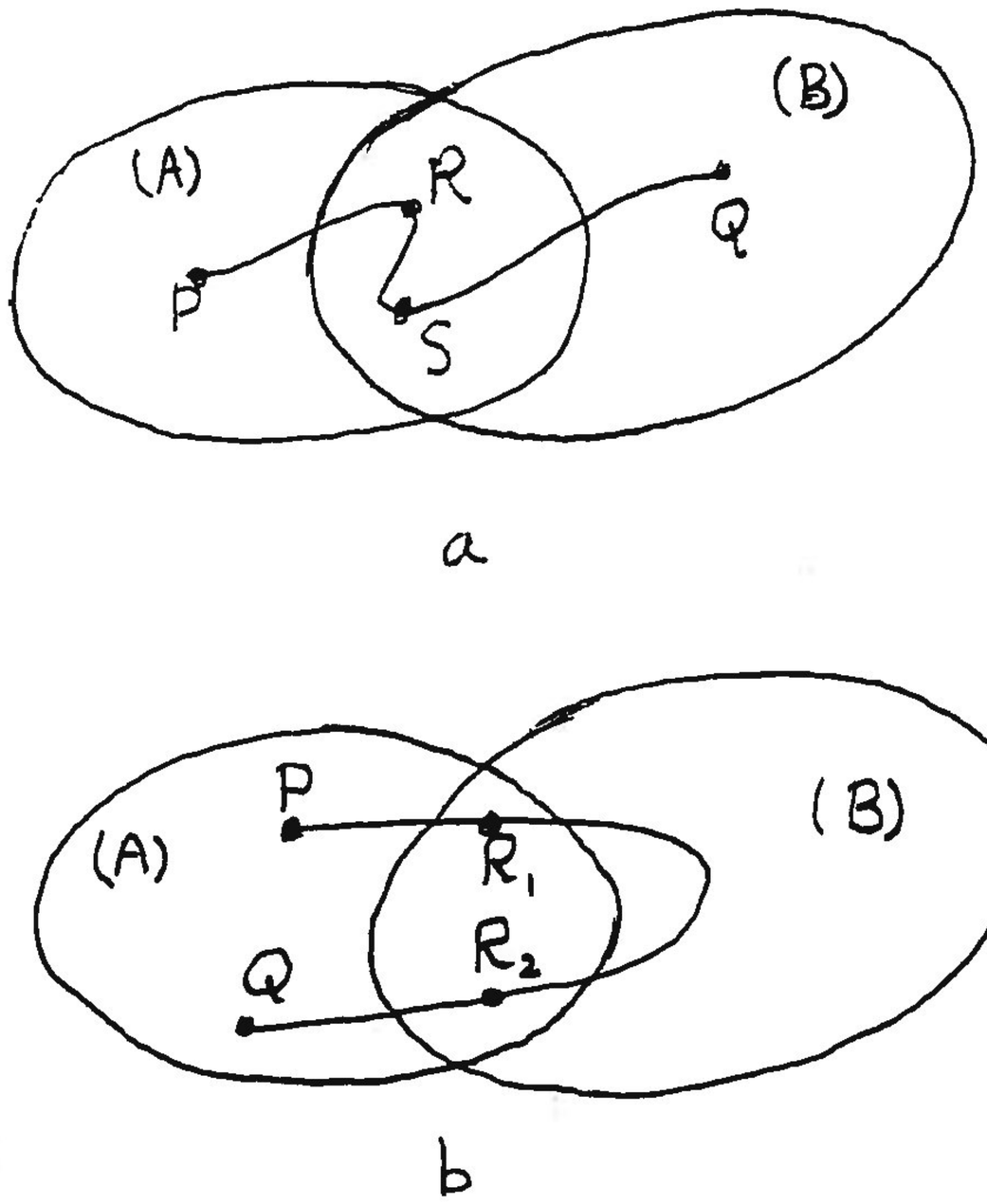


Figure 1.8: Parallel-transport in different patches

means that as long as x lies entirely within one patch, then its value is determined by the value of $A_\mu(x)$ in that patch. For x in the overlap region, $F_{\mu\nu}(x)$ can be defined either using $A_\mu^{(A)}$ or $A_\mu^{(B)}$, and:

$$F_{\mu\nu}^{(B)}(x) = S_{BA}(x) F_{\mu\nu}^{(A)}(x) S_{BA}^{-1}(x). \tag{1.69}$$

In particular, for abelian theory, $f_{\mu\nu}^{(A)} = f_{\mu\nu}^{(B)}$ and there is no need for patching the field tensor.

Parallel-transporting entirely within one patch also presents no problems (see Fig. 1.8a). For example, for a path Γ lying entirely in (A), say PR:

$$\Phi_{RP}(\Gamma) = P \exp ig \int_P^R A_\mu^{(A)}(x) dx^\mu. \tag{1.70}$$

For the path PQ however, which starts at P exclusively in (A) and ends at Q exclusively in (B), after passing through the overlap region, one needs to be careful. Let's select a point R in $(A) \cap (B)$. Then in addition to the equation

above, we have:

$$\Phi_{QP}(\Gamma) = \Phi_{QR}(\Gamma) S_{BA}(R) \Phi_{RP}(\Gamma). \tag{1.71}$$

If the path winds in and out of the overlap region $(A) \cap (B)$ as in Fig. 1.8b, then one can cut up the path into pieces, such that the beginning of each piece lies in one patch exclusively and its endpoint in the overlap region. For the path shown in the figure for example:

$$\Phi_{QP}(\Gamma) = \Phi_{QR_2}(\Gamma) S_{AB}(R_2) \Phi_{R_2R_1} S_{BA}(R_1) \Phi_{R_1P}(\Gamma), \tag{1.72}$$

where Φ_{R_1P} is defined in terms of $A_\mu^{(A)}$, $\Phi_{R_2R_1}$ in terms of $A_\mu^{(B)}$ and Φ_{QR_2} in terms of $A_\mu^{(A)}$. Also, $S_{AB}^{-1} = S_{BA}$ as expected. We therefore have a unique way of parallel-transporting for a general path traversing (A) and (B) or any number of additional patches.

It is instructive to examine what happens to the gauge potential in the presence of a monopole, for nonabelian gauge fields. Consider the simplest example of $SO(3)$, where we have already defined the monopole charge (see §1.4). We anticipate that the potential is patched in this case also. Consider the surface Σ enclosing a monopole. We introduce a north (N) and a south (S) patch as before and an A_μ for each patch. Let us choose the point P_0 on the equator which is to be identified with C_t, $t = t_0$, without loss of generality. The phase factor $\Phi^{(N)}(C_t)$ is therefore to be evaluated using $A_\mu^{(N)}$, and $\Phi^{(S)}(C_t)$ using $A_\mu^{(S)}$. These phase factors, being 2×2 unitary matrices, can be considered to be either elements of $SU(2)$ or $SO(3)$, where in the case of the latter, matrices of opposite sign are to be identified. Let us choose to regard the phase factors as elements of $SU(2)$. Thus, at $t = 0$, $\Phi^{(N)}(C_0)$ is the unit matrix and as t increases, $\Phi^{(N)}(C_t)$ traces a continuous curve in $SU(2)$ until at $t = t_0$, one makes a patching transformation, transforming $\Phi^{(N)}(C_t)$ to $\Phi^{(S)}(C_t)$. Beyond $t = t_0$, $\Phi^{(S)}(C_t)$ again traces a continuous curve until $t = 2\pi$, where $\Phi^{(S)}(C_t)$ becomes the identity. Hence, the only value of t at which $\Phi(C_t)$ need not be continuous is $t = t_0$. Recall that to have a monopole inside Σ, the curve Γ traced by $\Phi(C_t)$ for $t = 0 \to 2\pi$, should

wind only halfway around $SU(2)$, and for this we need:

$$P \exp ig \oint_{equator} A_\mu^{(N)}(x)\, dx^\mu = -P \exp ig \oint_{equator} A_\mu^{(S)}(x)\, dx^\mu, \tag{1.73}$$

where the line-integrals on both sides are taken in the same direction. This condition for an $SO(3)$ monopole with charge $-$ is seen to be stated not in the gauge group $SO(3)$ but in its covering group $SU(2)$. This is in analogy with the abelian case where the existence of a $U(1)$ monopole was stated in terms of the phase which is an element of $\mathbf{R}$ and not in terms of the exponential phase factor. This is found to be a useful tactic in other situations, as will be seen later.

Chapter 2

Loop-Space

2.1 Suitable Variables

The discussion in §1.1 shows that the electromagnetic field tensor $f_{\mu\nu}$ is not adequate for the description of physical phenomena, since there are physical effects (as in the Böhm-Aharonov experiment) which cannot be fully described by $f_{\mu\nu}$ alone. On the other hand, the gauge potential a_μ, although adequate for the description of physical phenomena, is nevertheless superfluous: after all, the gauge freedom of the theory means that two potentials related by a gauge transformation must describe the same physical situation, and thus cannot be regarded as distinct. There is a further redundancy which arises due to the periodicity of the wavefunction $\psi(x)$: $\psi(x)$ does not change if its phase is changed by a multiple of 2π. Hence, the phase difference itself cannot be a suitable variable, since the addition of $2\pi n$ (n an integer) to the phase does not alter the wavefunction and the physics (i.e. the diffraction pattern on the screen).

A natural question to ask at this stage is what is a "suitable" variable for describing electromagnetic phenomena, and more generally those of nonabelian Yang-Mills physics. Taking the above into consideration, leads one to consider

the phase factors:

$$\Phi(C) = \exp ig \oint_C a_\mu(x)\, dx^\mu, \tag{2.1}$$

as the physical variables. They have the required properties: (*i*) they are gauge invariant; thus different $\Phi(C)$'s correspond to distinct physical situations (i.e. to distinct diffraction patterns on the screen), (*ii*) they do not suffer from any of the redundancy problems mentioned above. As pointed out by Wu and Yang for example [29], the phase factors are both observable and adequate, and are exactly what Böhm-Aharonov-type experiments measure. They can thus be adopted as suitable variables for the description of physical phenomena.

A similar situation exists in Yang-Mills theories although here, $F_{\mu\nu}$ is already inadequate at the classical level, since two gauge inequivalent potentials can yield the same $F_{\mu\nu}$ [14]. Also, a nonabelian analogue of the Böhm-Aharonov experiment prevents us from using $F_{\mu\nu}$ as variables. The gauge potential A_μ on the other hand, is once again superfluous due to gauge freedom. Therefore, as in the abelian case, the variables which adequately describe physical phenomena, without overdescribing the theory are the path-ordered phase factors:

$$\Phi(C) = P \exp ig \oint_C A_\mu\, dx^\mu, \tag{2.2}$$

where the integration is over the closed loop C. As already explained, the necessity for path-ordering arises due to the nonabelian nature of the theory. $\Phi(C)$ is not gauge invariant in this case, but its gauge transformation is x-independent and easily handled if C is restricted to curves starting from a fixed basepoint P. This restriction will be justified in §2.3.

The phase factor $\Phi(C)$ is just a global version of the field tensor $F_{\mu\nu}$, in the sense that $\Phi(C)$ describes parallel-transport over finite distances, whereas $F_{\mu\nu}$ specifies infinitesimal parallel-transport. In situations where parallel-transport over finite loops could be built up from repeated infinitesimal parallel-transports, the two descriptions in terms of $F_{\mu\nu}$ and $\Phi(C)$ would be equivalent, and the variables $\Phi(C)$ would not have any advantages over $F_{\mu\nu}$. However, in certain

situations, this is not the case. One example is the Böhm-Aharonov experiment in abelian theory, where the field tensor $f_{\mu\nu}$ is zero everywhere except inside the solenoid, so one might conclude that a_μ could be set equal to zero everywhere, since according to Stokes' theorem:

$$0 = \iint_\Sigma f_{\mu\nu} d\sigma^{\mu\nu} = \oint a_\mu \, dx^\mu. \tag{2.3}$$

However, in order to correctly justify the choice of $a_\mu = 0$ everywhere, the region under consideration must be simply-connected, so that every closed curve C in the region could be shrunk to a point. However, there are some curves C (namely, those that wind around the solenoid) that could not be deformed in this manner, since the presence of the solenoid creates a hole where $f_{\mu\nu} \neq 0$, so that the line integral $\oint a_\mu \, dx^\mu$ need not vanish for these curves C. In such cases, the two descriptions in terms of $f_{\mu\nu}$ and Φ are not equivalent, and as argued above, the $\Phi(C)$ form a superior set of variables. Note also, that if $f_{\mu\nu} \neq 0$ inside a region Σ, then it is not possible to perform a gauge transformation such that $a_\mu = 0$ everywhere in Σ. We thus see that the local description in terms of $f_{\mu\nu}$ does not suffice and the global properties of the spatial region under consideration are crucial. Thus, one needs the phase factor $\Phi(C)$ and not just the local variable $f_{\mu\nu}$.

In the general case of nonabelian theories, as already seen in Chapter 1, the gauge potential A_μ and the field tensor $F_{\mu\nu}$ are not the most convenient variables for discussing gauge theories. It was shown that the phase factor $\Phi(C)$, on the other hand, is a physically measurable quantity which neither overdescribes nor underdescribes the theory. It is therefore, superior to both A_μ and $F_{\mu\nu}$ as a dynamical variable. However, there is the added complication of the dependence of Φ on the loop C rather than the spacetime point x. The latter represents a 4-fold infinity of variables (in four dimensions); to specify a loop C however, one needs an ∞-fold infinity of variables, since the value of the loop has to be specified for a continuum of points. Therefore, as variables $\{\Phi(C)\}$ are seen to be highly redun-

dant, since the dynamical variables $\{A_\mu(x)\}$ are already adequate for describing the gauge theory. We see that in order to use $\Phi(C)$ as variables, this redundancy would have to be removed by imposing constraints which have to be infinite in number. Another problem is how to represent the loops C. Here, one is faced with an option: using parametrised loops (maps of the unit circle to spacetime) $\widehat{\xi} = \{\xi^\mu(s) : s = 0 \to 2\pi\}$ where $\xi(0) = \xi(2\pi) = P_0$ is the "basepoint"; or using unparametrised loops, which are defined to be the equivalence classes of curves under reparametrisation. In choosing the latter however, one is faced with several problems which will be discussed in due course. The loop variables $\Phi(C)$ are then just functionals $\Phi[\xi]$ of the loop ξ, and are parametrisation independent. Thus, by choosing to work with parametrised loops we have increased the redundancy, which would subsequently have to be removed.

The loop-space formulation of gauge theories in terms of loop variables $\Phi(C)$ suffers from this problem and is therefore not a very rewarding approach in terms of being practical: as is well known, the usual formulation in terms of the gauge potential A_μ is already adequate. However, it has the advantage of being explicitly gauge invariant (apart from the trivial gauge transformation at the endpoints, mentioned earlier), and is thus useful in situations where one is dealing with patched potentials, which are very awkward to handle in the conventional approach. One such case is in the theory of monopoles, where the monopole potential is patched as explained earlier. We will develop the loop-space formulation with this particular application in mind.

2.2 Loop-space Kinematics

To begin, we need a way of introducing coordinates for points in loop-space. For loops passing through a fixed reference point $P_0 = \{\xi_0^\mu\}$ in spacetime X we define $\xi^\mu(s)$ as above. The $\{\xi^\mu(s)\}$ represent the coordinates in X of the points on the loop. The space of all such ξ^μ is the parametrised loop-space ΩX, where the

"points" are parametrised loops. Thus the loop variable $\Phi(C)$ can be written as functionals of ξ:

$$\Phi[\xi] = P_s \exp ig \int_0^{2\pi} ds\, A_\mu(\xi(s)) \frac{d\xi^\mu(s)}{ds}, \tag{2.4}$$

where the ordering is with s increasing from right to left and the derivative $\frac{d\xi^\mu(s)}{ds}$ is taken from below. This expression for Φ is invariant under a change in parametrisation $s' = f(s)$ as required which is, as discussed earlier, a further source of redundancy. However, the alternative of choosing unparametrised loops by labelling them by the equivalence classes of $\xi^\mu(s)$ under reparametrisation, presents its own problems: there is no easy way of defining derivatives in this quotient space of equivalence classes. Second derivatives with respect to the loop for example, will not in general be symmetric [4]. ΩX on the other hand, is just a function space, with familiar rules for differentiation and integration. Loop differentiation for example, can be defined as ordinary functional differentiation. For any functional $\Psi[\xi]$ of ξ in ΩX:

$$\delta_\alpha(s)\Psi[\xi] = \lim_{\Delta\to 0} \frac{1}{\Delta}\{\Psi[\xi'] - \Psi[\xi]\}, \tag{2.5}$$

where $\xi'^\mu(s) = \xi^\mu(s) + \Delta\, \delta^\mu_\alpha\, \delta(s-s')$ and $\delta_\alpha(s) = \frac{\delta}{\delta\xi^\alpha(s)}$. The function $\delta(s-s')$ must be viewed as the zero-width limit of a smooth function in the case of ambiguities. This definition results in the second derivatives being symmetric.

We can now define a logarithmic derivative which is an element of the Lie-algebra as follows:

$$F_\mu[\xi \mid s] = \frac{i}{g}\Phi^{-1}[\xi]\delta_\mu(s)\Phi[\xi], \tag{2.6}$$

which will turn out to be of importance in future discussions. In order to write the expression above in terms of loop variables, one must refer to Fig. 2.1. We first vary the curve ξ at s to yield the curve $\xi(s) + \delta\xi(s)$. Note that the variation is around the point $\xi(s)$ only. Due to path-ordering, the exponential factors in $\Phi^{-1}[\xi]$ with parameter values $s \leq s_+$ cancel those originating in the term $\delta_\mu(s)\Phi[\xi]$.

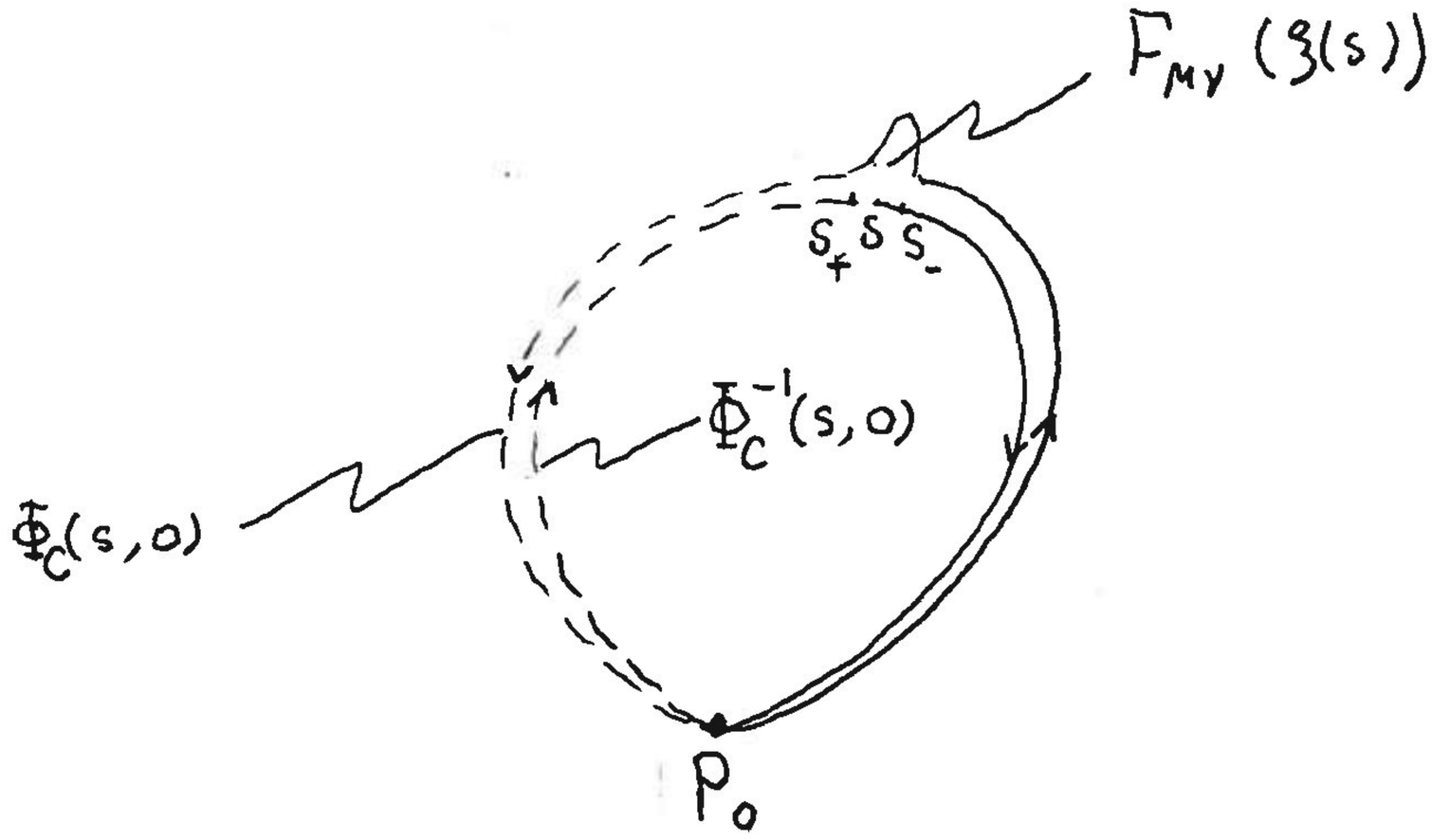


Figure 2.1: Path-ordering in F_μ

Thus, in terms of local variables[4]:

$$F_\mu[\xi \mid s] = \Phi_\xi^{-1}(s,0) F_{\mu\nu}(\xi(s)) \Phi_\xi(s,0)\, \dot{\xi}^\nu(s), \tag{2.7}$$

where $F_{\mu\nu}$ is the usual field tensor, and where we have introduced the notation $\dot{\xi}^\nu(s) = \frac{d\xi^\nu(s)}{ds}$. Here:

$$\Phi_\xi(s_2, s_1) = P_s \exp ig \int_{s_1}^{s_2} ds\; A_\mu(\xi(s))\, \dot{\xi}^\mu(s), \tag{2.8}$$

is the parallel-transport in X from $\xi(s_1)$ to $\xi(s_2)$, along the loop C. One sees that $F_\mu[\xi \mid s]$ depends only on the "early" part of the loop, i.e. on $\xi^\mu(s')$ only for $s' \leq s$ and not for $s' > s$. This is what is meant by the argument $[\xi \mid s]$. Furthermore, we have:

$$F_\mu[\xi \mid s]\dot{\xi}^\mu(s) = 0, \tag{2.9}$$

using the antisymmetry of $F_{\mu\nu}$. This statement reflects that $\Phi[\xi]$ is invariant under reparametrisations of the loop ξ, since reparametrisation just means shifting points along the loop, and reparametrisation invariance leads to a vanishing

derivative along the loop. $F_\mu[\xi\,|\,s]$, being defined in terms of the phase factor $\Phi[\xi]$, has the advantage of being gauge invariant under ordinary gauge transformations and hence patch-independent. We note that $F_\mu[\xi\,|\,s]$ can be viewed as a connection in loop-space ΩX, since it represents the change in phase of Φ as one moves from the "point" ξ to the "point" $\xi + \delta\xi$, in the same way that A_μ represents the change in the phase of ψ in going from x^μ to $x^\mu + dx^\mu$ in spacetime X. Similarly, one can construct the curvature tensor and holonomy in ΩX in analogy to the field tensor $F_{\mu\nu}$ and the holonomy $\Phi(C)$ in X:

$$G_{\mu\nu}[\xi; s, s'] = \delta_\nu(s')F_\mu[\xi\,|\,s] - \delta_\mu(s)F_\nu[\xi\,|\,s'] + ig[F_\mu[\xi\,|\,s], F_\nu[\xi\,|\,s']]. \qquad (2.10)$$

The quantity $igG_{\mu\nu}[\xi; s, s']\,\delta\xi^\mu(s)\,\delta\xi^\nu(s')$ represents the total change in phase as a point moves around an infinitesimal closed circuit, obtained by starting from the point ξ and varying it at s in the μ-direction, then at s' in the ν-direction, then back again at s in the μ-direction and then at s' in the ν-direction in that order. For $s = s'$ this surface will be closed in X and will envelop an infinitesimal volume (see Fig. 2.2), whereas for $s \neq s'$, no such volume will be enclosed. Thus, we expect that the value of $G_{\mu\nu}[\xi; s, s']$ will be proportional to $\delta(s - s')$, and at $s = s'$ its value will depend on what is inside the enclosed volume. $G_{\mu\nu}[\xi; s, s']$ could also be expressed in terms of local field variables [4]. The result:

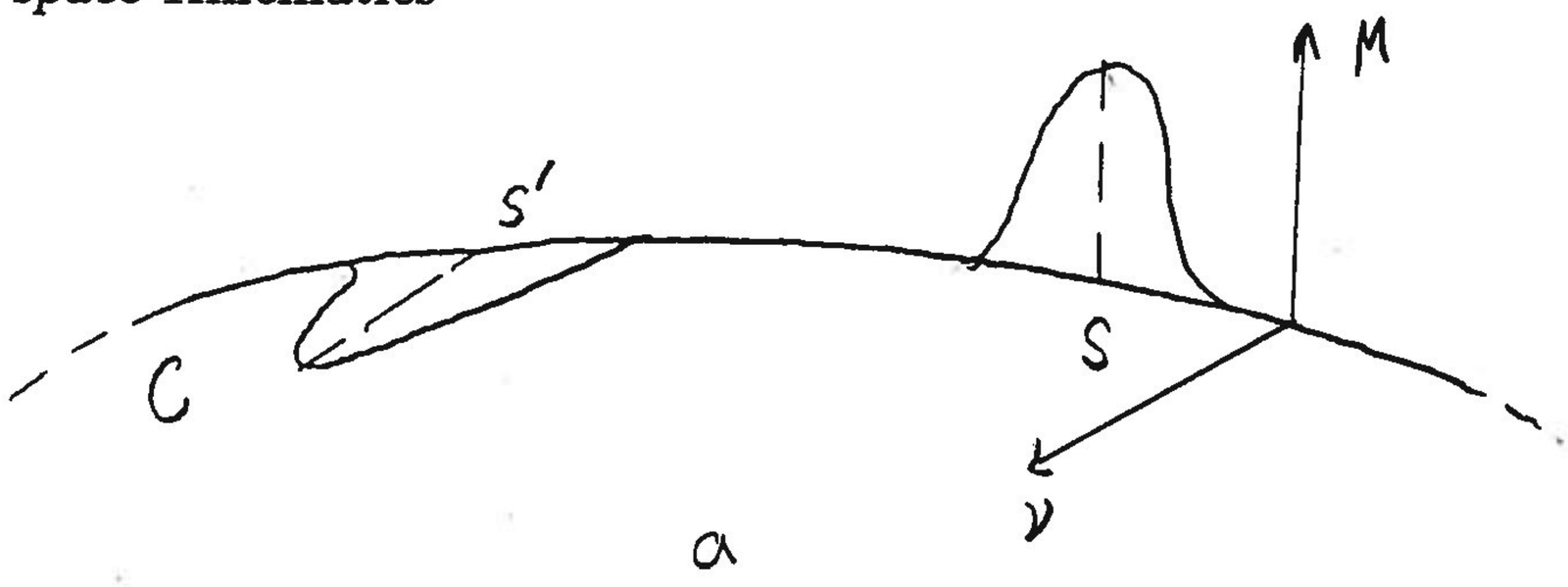
$$G_{\mu\nu}[\xi; s, s'] = \Phi_\xi^{-1}(s, 0)\epsilon_{\mu\nu\rho\sigma}D^*_\alpha F^{\rho\alpha}(\xi(s))\Phi_\xi(s, 0)\dot{\xi}^\sigma(s)\delta(s - s'), \qquad (2.11)$$

is indeed proportional to $\delta(s - s')$ as anticipated.

In order to define the holonomy in ΩX, we need to first define a "closed curve" in loop-space:

$$\Sigma : \{\xi^\mu_t(s) :\ s = 0 \to 2\pi,\ t = 0 \to 2\pi\}, \qquad (2.12)$$

with $\xi^\mu_t(0) = \xi^\mu_t(2\pi) = \xi^\mu_0$, for $t = 0 \to 2\pi$ and $\xi^\mu_0(s) = \xi^\mu_{2\pi}(s) = \xi^\mu_0$, for $s = 0 \to 2\pi$. Thus, for fixed t, $\xi^\mu_t(s)$ for $s = 0 \to 2\pi$ represents a closed loop C_t in X, passing through ξ_0. For $t = 0$ and 2π, C_t consists of the point ξ_0 and as t varies, C_t traces out a closed loop in ΩX beginning and ending at the "zero"



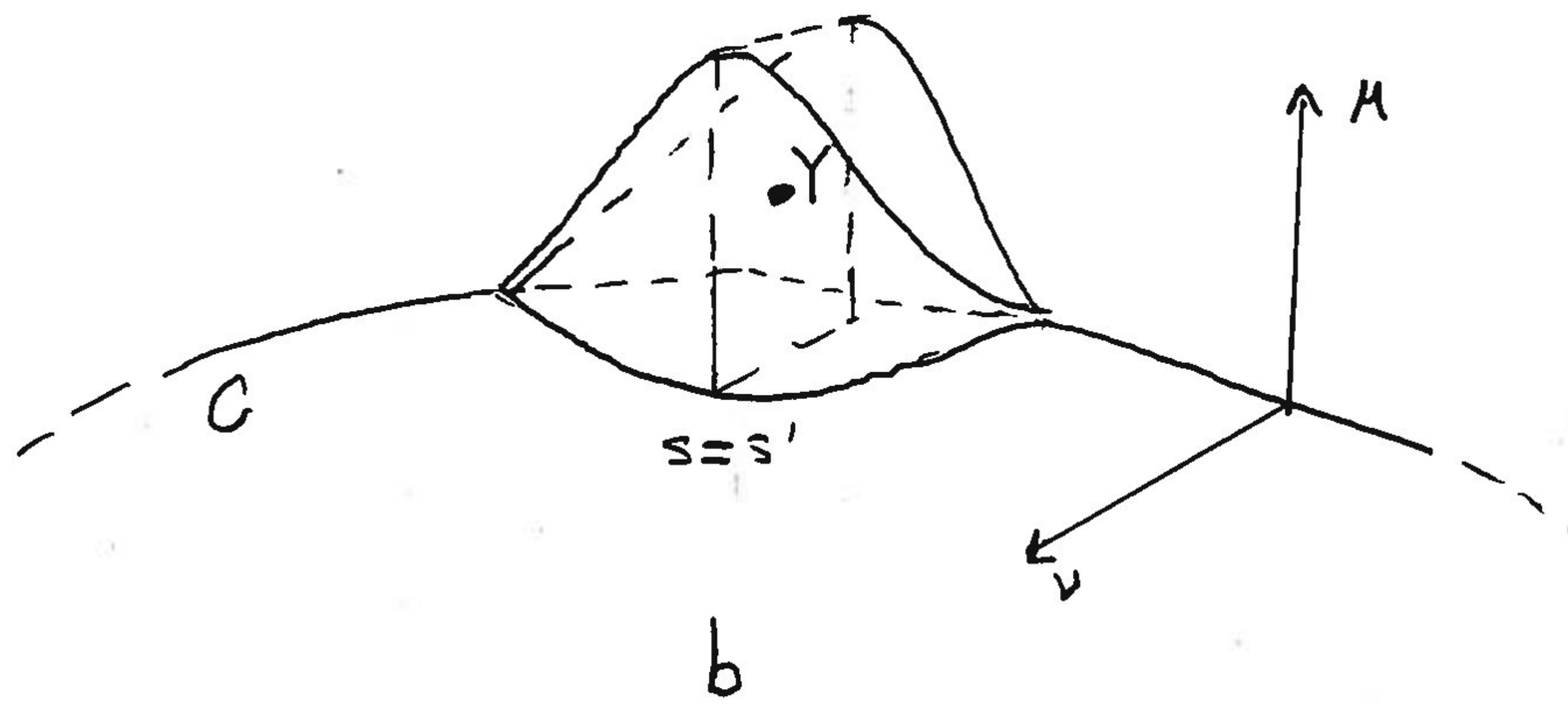


Figure 2.2: Spacetime representation of infinitesimal closed loops in loop-space

loop ξ_0. In spacetime X, Σ envelops a volume swept out by the loops C_t. We can thus construct the holonomy or phase factor as:

$$\Theta[\Sigma] = P_t \exp ig \int_0^{2\pi} dt \int_0^{2\pi} ds\, F_\mu[\xi_t \,|\, s] \frac{\partial \xi_t^\mu(s)}{\partial t}. \tag{2.13}$$

This is in complete analogy to the holonomy in X with the difference that the connection $F_\mu[\xi \,|\, s]$ in ΩX has many more components than $A_\mu(x)$, labelled by the continuous index s as well as μ, both of which have to be summed over. Once again, P_t denotes ordering in t increasing from right to left. As with $G_{\mu\nu}$, the value of $\Theta[\Sigma]$ depends on what is contained inside the volume enveloped by Σ.

To see how $G_{\mu\nu}$ and $\Theta[\Sigma]$ depend on what is inside this volume, let us turn to the abelian theory first. In this case, the covariant derivative D_μ is the same as the ordinary derivative ∂_μ and we have:

$$G_{\mu\nu}[\xi; s, s'] = \epsilon_{\mu\nu\rho\sigma} \partial_\alpha^* f^{\rho\alpha}(\xi(s)) \dot{\xi}^\sigma(s) \delta(s - s'). \tag{2.14}$$

At any point $\xi(s)$ along the curve, where $f_{\mu\nu}(\xi(s))$ is a gauge field, we see that $\partial^*_\alpha f^{\rho\alpha}(\xi(s)) = 0$, since $f_{\mu\nu}$ must satisfy the Bianchi identity. Hence $G_{\mu\nu} = 0$. However, if the worldline of a monopole intersects the curve C at $\xi(s)$ then $a_\mu(\xi(s))$ does not exist, and indeed, as was shown before:

$$\partial^*_\alpha f^{0\alpha}(x) = -4\pi\tilde{e}\int d\tau\ \frac{dY^0(\tau)}{d\tau}\,\delta(x - Y(\tau)). \tag{2.15}$$

From Lorentz covariance it follows that:

$$\partial^*_\nu f^{\mu\nu}(x) = -4\pi\tilde{e}\int d\tau\ \dot{Y}^\mu(\tau)\,\delta(x - Y(\tau)), \tag{2.16}$$

where $\dot{Y}^\mu(\tau) = \frac{dY^\mu(\tau)}{d\tau}$ and the right-hand side is the magnetic current carried by the monopole with charge $\tilde{e}$. Therefore, for the abelian case, monopoles can be regarded as sources of loop-space curvature.

In the case of nonabelian theory, it is still true that wherever $F_{\mu\nu}$ is a gauge field derivable from some potential A_μ, then $G_{\mu\nu} = 0$. Also, at the point where C intersects the worldline of a monopole, A_μ does not exist, and the Bianchi identity is not valid. However, since $F_{\mu\nu}$ and D_μ are now both patched quantities at the monopole position, it is not clear what the value of $G_{\mu\nu}$ would be on the monopole worldline. Let us examine the problem in more detail by considering the holonomy $\Theta[\Sigma]$ over a finite-sized loop in ΩX. Such a parametrised loop represents a closed, parametrised two-dimensional surface Σ in X. We take this surface to enclose a monopole.

Consider the simple case of pure $su(2)$ Yang-Mills theory, where the gauge group is $SO(3)$. Since $F_\mu[\xi \mid s]$ is the logarithmic derivative of $\Phi[\xi]$ we may write:

$$\exp ig\,dt\int_0^{2\pi} ds\,F_\mu[\xi_t \mid s]\frac{\partial\xi_t^\mu(s)}{\partial t} \simeq \Phi^{-1}[\xi_{t+dt}]\,\Phi[\xi]. \tag{2.17}$$

$\Theta[\Sigma]$ is then an ordered product in t of such factors, where t varies over the range $t = 0 \to 2\pi$. We are free to regard Θ and Φ either as elements of $SO(3)$ or $SU(2)$, the covering group of $SO(3)$. Based on our earlier experience, we choose to regard them as elements of $SU(2)$. If Σ encloses a monopole, then $\Phi[\xi_t]$ will trace out for

$t = 0 \to 2\pi$, a curve in $SU(2)$ winding halfway around the group. Hence, $\Theta[\Sigma]$ which measures the total change, is just $(\Phi^{(S)}[\xi_t])^{-1}\,(\Phi^{(N)}[\xi_t])$ evaluated at $t = t_0$, and therefore has a value of $-I$. However, if Σ does not enclose a monopole, then $\Phi[\xi_t]$ traces out a closed curve in $SU(2)$, resulting in Θ having a value of I. The value of $\Theta[\Sigma]$ in an $SO(3)$ theory can thus be denoted by:

$$\Theta[\Sigma] = \zeta_\Sigma\, I, \tag{2.18}$$

where $\zeta_\Sigma = \pm$ is the monopole charge enclosed in ordinary space by the surface Σ, which corresponds to the loop in loop-space ΩX parametrised by:

$$\{\zeta_t : t = 0 \to 2\pi\}. \tag{2.19}$$

This formula thus provides an explicit expression for the monopole charge which was initially defined as a homotopy class. To compare the situation once again with abelian theory, we know that in that case, the abelian monopole charge is given by $\frac{1}{2\pi}$ times the total change in the phase factor $\Phi(C_t)$ for $t = 0 \to 2\pi$, and is labelled by an integer n, which is the winding number. The phase of $\Phi(C_t)$ may be regarded as an element of the real line $\mathbf{R}$, which covers the gauge group $U(1)$, whose image in $U(1)$ via exponentiation is $\Phi(C_t)$. Therefore, already in the abelian theory, we see that the monopole charge may be interpreted as the total change for $t = 0 \to 2\pi$, of an element in the covering group, which corresponds to $\Phi(C_t)$ in the gauge group. The result above, is just a generalisation where the gauge group $U(1)$ is replaced by $SO(3)$ and the covering group $\mathbf{R}$ by $SU(2)$. With other gauge groups, similar results can be derived. For the case of all pure $su(N)$ Yang-Mills theories with gauge groups $SU(N)/Z_N$, the monopole charge can take N values given by:

$$\zeta = \exp i2\pi r/N, \qquad\qquad r = 0, 1, 2, \ldots, N-1. \tag{2.20}$$

These results could also be extended to gauge groups $U(N)$.

We now have a topological definition of the monopole charge in (2.18). This equation however, is "global" in character in both spacetime and the gauge group.

We would like to find a "local" version of this equation, which would specify the change in phase in going around an infinitesimal surface. The curvature tensor $G_{\mu\nu}[\xi; s, s']$ is therefore the quantity of interest: $G_{\mu\nu}[\xi; s, s']\,\delta\xi^{\mu}(s)\,\delta\xi^{\nu}(s')$, is the change in phase of $\Phi[\xi]$ as one goes around a surface with sides $\delta\xi^{\mu}(s)$ and $\delta\xi^{\nu}(s')$, and $G_{\mu\nu}[\xi; s, s']$ is, roughly speaking, the logarithm of $\Theta[\Sigma]$ for an infinitesimal surface. For $s \neq s'$, $G_{\mu\nu}[\xi; s, s']$ corresponds to a surface Σ enclosing zero volume, so that $\Theta[\Sigma] = I$, whose logarithm is zero. We obtain a value of zero even when $s = s'$ but $\xi(s)$ does not sit on the monopole worldline, since in that case, although the surface Σ encloses a non-zero volume, the volume contains no monopole and $\Theta[\Sigma] = I$ as a result. However, when $\xi(s)$ for $s = s'$ coincides with a monopole, $G_{\mu\nu}[\xi; s, s']$ corresponds to a surface Σ enclosing a monopole charge, and its value is equal to the logarithm of the monopole charge at $\xi(s)$. Putting all these facts together, we get the following expression for the curvature:

$$G_{\mu\nu}[\xi; s, s'] = 4\pi\tilde{g}\int d\tau\, \kappa[\xi \,|\, s]\, \epsilon_{\mu\nu\rho\sigma}\dot{\xi}^{\rho}(s)\dot{Y}^{\sigma}(\tau)\,\delta^4(\xi(s) - Y(\tau))\,\delta(s - s'), \quad (2.21)$$

with $\exp i\pi\kappa[\xi \,|\, s] = \zeta$ where ζ is the charge carried by the monopole moving along the worldline $Y(\tau)$. $\tilde{g}$ is given by the generalised Dirac quantisation condition (1.57). This equation must be handled with care, since given a monopole charge $\zeta = -$ in $SO(3)$ for example, the only conclusion we can draw is that $\kappa = 0$ is not a solution. However, there is still an ambiguity involved, originating from the logarithmic nature of the definition of $\kappa[\xi \,|\, s]$. It will be shown in Chapter 3 that this ambiguity leads to the so-called $\tilde{U}$-symmetry, which is also found to exist and will be studied in detail in the quantum case.

2.3 Extended Poincaré Lemma

Consider for now, the abelian theory. Although in the classical case, $f_{\mu\nu}$ are the variables describing the theory, these are not independent but must be expressed in terms of some potential a_{μ}. If we think of a_{μ} as the independent variables of

the theory, then the $f_{\mu\nu}$ are redundant, since they have six components compared to four for a_μ. The extra degrees of freedom are therefore unnecessary, and must be constrained. The Bianchi identity provides the constraints which are necessary and sufficient for the removal of this redundancy. Poincaré's lemma states that the two statements (1.8) and (1.42) are equivalent for antisymmetric $f_{\mu\nu}$. Another way of expressing this is that a necessary and sufficient condition for $f_{\mu\nu}$ to be a gauge field is that $\partial^*_\mu f^{\mu\nu}(x) = 0$ for all x. The physical interpretation of this is that there are no monopoles at x, hence showing that in the absence of monopoles, the redundancy of $f_{\mu\nu}$ as field variables is removed.

The statements above are "local" in x, by which we mean that if a monopole moving along a worldline $Y^\mu(\tau)$ exists, then the Bianchi identity must be replaced by (2.16). This means that $a_\mu(x)$ no longer exists on $Y^\mu(\tau)$, but since the δ-function vanishes for all $x \neq Y(\tau)$, $f_{\mu\nu}(x)$ is still expressible in terms of some gauge potential $a_\mu(x)$ at all points not on the monopole worldline.

Let us now examine nonabelian theory, where we try to determine the constraints needed to be imposed on the loop variables so as to remove their redundancy. The variables we choose to work with are the $F_\mu[\xi \,|\, s]$, which are closely related to $F_{\mu\nu}(x)$ through (2.6), but (like $f_{\mu\nu}$ and unlike the gauge covariant $F_{\mu\nu}$) can be considered to be gauge invariant as discussed in §2.2. We approach the problem by once again comparing the situation to the abelian case. In the latter, the conditions required for removing the redundancy of $f_{\mu\nu}(x)$ are:

(a) $f_{\mu\nu}(x)$ is a local quantity depending on the point x only,

(b) $f_{\mu\nu}(x)$ is antisymmetric under $\mu \leftrightarrow \nu$,

(c) $f_{\mu\nu}(x)$ has no monopoles, so that $\partial^*_\mu f^{\mu\nu}(x) = 0$.

In the general nonabelian case, we look for three conditions analogous to the above, having the same physical content; thus hoping to remove the redundancy in $F_\mu[\xi \,|\, s]$.

The analogous statement to (b) as already seen, is that $F_\mu[\xi \,|\, s]$ be transverse to the loop $\xi_\mu(s)$, i.e. $F_\mu[\xi \,|\, s]\dot{\xi}^\mu(s) = 0$. Statement ($a$) must be replaced by

$\frac{\delta}{\delta\xi^\mu(s')}F_\mu[\xi\,|\,s]=0$ for $s'>s$, which is an alternative way of expressing the "path-ordering" property of $F_\mu[\xi\,|\,s]$: that it depends only on the early part of the loop. In the case of abelian theory, this reduces to (a). The generalisations of (a) and (b) are thus straightforward, as long as we realise that $F_\mu[\xi\,|\,s]$ must satisfy the transversality and path-ordering requirements. Statement (c) however, needs to be considered and a generalised version must be formulated. As far as we know, this statement cannot be generalised in a "local" form to nonabelian theories. However, the absence of a monopole at x can be specified easily if one chooses to work in loop-space and with $F_\mu[\xi\,|\,s]$ as variables. One can then choose between (2.18) with $\zeta_\Sigma=1$ for any infinitesimal Σ surrounding x, or (2.21) with $\kappa[\xi\,|\,s]=0$ for all ξ and s satisfying $\xi(s)=x$. To remove the redundancy completely, we must ensure that there are no monopoles anywhere except at specified positions. This will be the case if and only if (2.18) is satisfied for all surfaces Σ or, (2.21) for all points s on all parametrised loops ξ. This can indeed be accomplished by using a version of the so-called *Extended Poincarè Lemma*, proven in [8], stating that given an $F_\mu[\xi\,|\,s]$,

(A) depending on $\xi(s')$ only for $s'\leq s$,

(B) having only transverse components to the loop parametrised by $\xi(s)$,

(C) satisfying (2.18) for all surfaces Σ, or equivalently (2.21) for all ξ and s,

there always exists a gauge potential $A_\mu(x)$, unique up to gauge transformations and related to $F_\mu[\xi\,|\,s]$ via (2.6) and (2.4) for every ξ and s. This enables us to replace the $A_\mu(x)$ by $F_\mu[\xi\,|\,s]$ as variables, while ensuring that we are still dealing with the same gauge theory.

The particular form of the lemma, which is well-suited to the purposes of this thesis, ensures that given a set of variables $F_\mu[\xi\,|\,s]$ satisfying A, B and C, an $A_\mu(x)$ can be constructed. Hence the $A_\mu(x)$ can be replaced by $F_\mu[\xi\,|\,s]$ and the latter can be used to describe the same theory.

In the case where there are monopoles around, additional complications arise

[8]. However, the lemma remains valid even in this case.

As mentioned earlier, the variables $F_\mu[\xi \mid s]$ have the advantage of being gauge invariant apart from an x-independent gauge rotation at P_0, and are thus preferable to $A_\mu(x)$ in the presence of monopoles. We also note that the claim made in §2.1, that one needs only consider loops passing through a fixed reference point P_0, is substantiated, since given A_μ, one can construct the phase factor $\Phi[\xi]$ for any ξ in spacetime.

Chapter 3

Equations of Motion for Monopoles

3.1 The Constrained Action Principle

The interesting fact about monopoles is that by virtue only of their definition as a topological obstruction, they already possess an intrinsic interaction with the gauge field [7, 30]. To see this, consider a monopole at the position x. The existence of the monopole implies a certain topological configuration of the potential A_μ surrounding the monopole. As the monopole moves, the field must keep the same topological configuration around the monopole in order to ensure the conservation of monopole charge. Therefore, the field variables must change according to the position x to maintain this topological configuration. Thus, we have a coupling between A_μ and the particle coordinate given by the worldline $Y^\mu(\tau)$, (where τ is the proper time) or in other words, an interaction between the monopole and the field.

Consider first the familiar case of abelian theory. Let us see if we can apply this technique in order to "extract" the equations of motion without introducing an interaction term into the action: the free action of the field-plus-particle system

is given by:

$$\mathcal{A}^0 = \mathcal{A}^0_F + \mathcal{A}^0_M, \tag{3.1}$$

where $\mathcal{A}^0_F$ and $\mathcal{A}^0_M$ are the free actions for field and matter respectively, and are given by:

$$\mathcal{A}^0_F = -\frac{1}{16\pi}\int d^4x\, f_{\mu\nu}(x) f^{\mu\nu}(x), \tag{3.2}$$

with:

$$\mathcal{A}^0_M = -m\int d\tau, \tag{3.3}$$

for the classical monopole and:

$$\mathcal{A}^0_M = \int d^4x\, \bar{\psi}(x)(i\partial^\mu\gamma_\mu - m)\psi(x), \tag{3.4}$$

for the quantum monopole.

Let us first examine the classical case, where the particle is described by a worldline $Y(\tau)$, depending on the proper time τ. The topological constraint imposing a magnetic charge on the particle described by $Y^\mu(\tau)$ according to (2.16) is equivalent to:

$$\partial^*_\omega f^{\mu\nu}(x) = -4\pi\tilde{e}\int d\tau\, \dot{Y}^\mu(\tau)\,\delta^4(x - Y(\tau)), \tag{3.5}$$

which has to be satisfied for every μ and x. Thus, in incorporating this constraint into the action, we must introduce a Lagrange multiplier for each value of μ and x. The auxiliary action thus obtained is:

$$\mathcal{A}' = \mathcal{A}^0 + \int dx\, \lambda_\mu(x)\left\{\partial^*_\omega f^{\mu\nu}(x) + 4\pi\tilde{g}\int d\tau \dot{Y}^\mu(\tau)\,\delta^4(x - Y(\tau))\right\}, \tag{3.6}$$

where we have used $\tilde{g}$ for $\tilde{e}$. The equations of motion are now derived by extremising with respect to unconstrained variations in a_μ and Y^μ.

In the presence of a monopole, a_μ has to be patched, making the above variational problem not so easy to handle. We propose instead, to use the unpatched variables $f_{\mu\nu}(x)$. These seem at first sight to be redundant, having six independent components instead of the four of $a_\mu(x)$. In other words, given a set

of values for $f_{\mu\nu}$, there may not in general exist a potential a_μ such that (1.8) holds. However, in this particular case, the constraint (3.5) implies that for all x not on $Y(\tau)$, the Bianchi identity is satisfied, which guarantees via Poincaré's lemma that a_μ exists at all points not lying on the monopole worldline, and the redundancy in the variables $f_{\mu\nu}$ is thus removed. We can therefore replace the variables a_μ by $f_{\mu\nu}$, avoiding the problems arising due to the patched a_μ in the presence of a monopole. Varying the action (3.6) with respect to $f_{\mu\nu}$ and Y^μ, we get for the Euler-Lagrange equations:

$$f^{\mu\nu} = 4\pi \left\{ \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\partial_\sigma \lambda_\rho(x) - \partial_\rho \lambda_\sigma(x)) \right\}, \tag{3.7}$$

and:

$$m\ddot{Y}_\mu(\tau) = -4\pi\tilde{g} \left\{ \partial_\nu \lambda_\mu(Y(\tau)) - \partial_\mu \lambda_\nu(Y(\tau)) \right\} \dot{Y}^\nu(\tau), \tag{3.8}$$

as the equations of motion.

Contracting the first of these with $\epsilon_{\alpha\beta\mu\nu}$, we see that ${}^*f^{\mu\nu}$ is a gauge field with:

$$\bar{a}_\mu(x) = 4\pi\lambda_\mu(x) \tag{3.9}$$

as the gauge potential. Using this in (3.8), we get:

$$m\ddot{Y}_\mu(\tau) = -\tilde{g}\, {}^*f_{\mu\nu}(Y(\tau))\dot{Y}^\nu(\tau). \tag{3.10}$$

These equations are exactly the same as the "conventional" equations for electric charges in an electromagnetic field with e and $f_{\mu\nu}$ replaced by $\tilde{g}$ and ${}^*f_{\mu\nu}$. These "conventional" equations however, are derived by using an explicit interaction term. The equations for the electric charge are derived from the action:

$$\mathcal{A} = -\frac{1}{16\pi} \int d^4x\, f_{\mu\nu}(x) f^{\mu\nu}(x) - m \int d\tau + e \int a_\mu(Y(\tau)) dY^\mu, \tag{3.11}$$

where e is the charge of the particle and the last term in this action is the interaction term, which is put in "by hand" to fit experimental results. The Euler-Lagrange equations are the Maxwell equation and the Lorentz equation

respectively:

$$\partial_\nu f^{\mu\nu}(x) = -4\pi e \int d\tau\, \dot{Y}^\mu(\tau)\, \delta^4(x - Y(\tau)), \tag{3.12}$$

$$m\ddot{Y}_\mu(\tau) = -e f_{\mu\nu}(Y(\tau))\dot{Y}^\nu(\tau). \tag{3.13}$$

These equations couple the field $a_\mu(x)$ to the particle coordinate $Y^\mu(\tau)$, and the interaction is therefore incorporated in this manner. In addition, there is the Bianchi identity for $f_{\mu\nu}$:

$$\partial^*_\nu f^{\mu\nu}(x) = 0, \tag{3.14}$$

which states that the field $f_{\mu\nu}$ is a gauge field, derivable from a gauge potential a_μ. Note that:

$$\partial_\nu f^{\mu\nu}(x) = 0, \tag{3.15}$$

everywhere except on the charge worldline $Y^\mu(\tau)$. We know that:

$$f_{\mu\nu}(x) = -{}^*({}^*f_{\mu\nu}(x)), \tag{3.16}$$

so that (1.41) can be interpreted as the Bianchi identity for ${}^*f_{\mu\nu}(x)$. Therefore, by the Poincaré lemma, there must exist a potential $\bar{a}_\mu(x)$ such that:

$${}^*f_{\mu\nu}(x) = \partial_\nu \bar{a}_\mu(x) - \partial_\mu \bar{a}_\nu(x), \tag{3.17}$$

everywhere except on the charge worldline $Y_\mu(\tau)$, implying that ${}^*f_{\mu\nu}$ is also a gauge field, except on the worldline. It was seen before that a monopole of $f_{\mu\nu}$ is a magnetic charge, which is a source of magnetic flux. Since the * operation just interchanges electric and magnetic fields, a monopole of $f_{\mu\nu}$ can also be regarded as a source of ${}^*f_{\mu\nu}$. This argument could be inverted and applied to ${}^*f_{\mu\nu}$, which is also a gauge field, to deduce that a monopole of ${}^*f_{\mu\nu}$ is equivalent to a source of $f_{\mu\nu}$; i.e. an electric charge such as that in (3.12). We can then choose to regard the electric charge e as the monopole of ${}^*f_{\mu\nu}$ and examine the equations of motion, derived by considering the free action:

$$\mathcal{A}_F = -\frac{1}{16\pi}\int d^4x\, f_{\mu\nu}(x) f^{\mu\nu}(x) - m\int d\tau, \tag{3.18}$$

subject to the topological constraint defining the monopole of ${}^*f_{\mu\nu}$. This procedure yields the Maxwell and Lorentz equations as the Euler-Lagrange equations, and thus, in the case of electromagnetism, we see that the topologically-induced interaction has the same effect as the standard electromagnetic interaction term which has its foundations in experiment.

Next, let us consider a quantum particle described by a wavefunction $\psi(x)$. We can use the same arguments to conclude that there will be an intrinsic interaction between the monopole and the field due to the dual nature of the charge as a monopole of the dual field. Let us derive the equations of motion for the magnetic case, and assume that the magnetic monopole (of $f_{\mu\nu}$) is a Dirac particle and has spin 1/2. In this case, the matter part of the action is given by (3.4), and the constraint is:

$$\partial^*_\mu f^{\mu\nu}(x) = -4\pi\tilde{g}\,\bar{\psi}(x)\gamma^\mu\psi(x), \tag{3.19}$$

where the classical current has been replaced by the quantum one. Let us derive the equations of motion for the magnetic case. Once again, variation with respect to $f_{\mu\nu}$ and ψ of the action:

$$\mathcal{A}' = \mathcal{A} + \int d^4x\,\lambda_\mu(x)\left\{\partial_\nu\,{}^*f^{\mu\nu}(x) + 4\pi\tilde{g}\,\bar{\psi}(x)\gamma^\mu\psi(x)\right\}, \tag{3.20}$$

would result in coupled equations in ψ and a_μ.

However, we notice a problem with this local form of the constraint. For $\mu = 0$, the right-hand side of (3.20) represents the magnetic charge density and must therefore yield $4\pi\tilde{g}$ times an integer, when integrated over any closed surface enclosing a volume. This is clearly not the case for arbitrary ψ. We therefore believe that $\psi(x)$ should be a second-quantised field in order for the theory to be fully consistent. Then, the right-hand side of (3.19) becomes a number operator for the number of monopoles at x and will therefore yield a quantised result when integrated over any closed surface. For the purposes of this thesis, we accept the constraint (3.19) at face value, hoping to explore second-quantisation at a later stage.

Returning to the derivation of the equations of motion, we see that varying the action $\mathcal{A}'$ presents us with the problem of a patched a_μ in the presence of a monopole, and we would therefore like to use $f_{\mu\nu}$ as variables. This may seem unwise, since the Böhm-Aharonov experiment demonstrated that the electromagnetic field tensor $f_{\mu\nu}$ is not enough for a description of electromagnetic phenomena. However, the potential needed for describing the interactions of a *magnetic* charge with the electromagnetic field is not a_μ but $\bar{a}_\mu$, which is the potential of the dual field ${}^*f_{\mu\nu}$.

Let us proceed to finding the equations of motion, given the action (3.20). The Euler-Lagrange equation obtained by extremising with respect to $f_{\mu\nu}$ remains the same as (3.7), while varying with respect to $\bar{\psi}(x)$ yields:

$$(i\partial_\mu\gamma^\mu - m)\,\psi(x) = -\tilde{g}\,\bar{a}_\mu(x)\gamma^\mu\psi(x), \tag{3.21}$$

where $\bar{a}_\mu$ is defined as in (3.9). We notice that in the quantum case, the Lagrange multiplier $\bar{a}_\mu$ can no longer be eliminated from the equations of motion and is coupled to $\psi(x)$ in exactly the same manner as the coupling of a_μ to the charge wavefunction in the Dirac equation.

Here, once again, we see that had we gone through the same process, starting with the action for a charge considered as a monopole of the dual field ${}^*f_{\mu\nu}$, we would have arrived at the "dual" equations, obtained by replacing $\tilde{g}$ by e, ${}^*f_{\mu\nu}$ by $f_{\mu\nu}$ and $\bar{a}_\mu$ by a_μ:

$$\begin{aligned} \partial_\nu f^{\mu\nu}(x) &= -4\pi e\,\bar{\psi}(x)\gamma^\mu\psi(x), \\ (i\partial_\mu\gamma^\mu - m)\psi(x) &= -e\,a_\mu(x)\gamma^\mu\psi(x), \end{aligned} \tag{3.22}$$

together with equation (1.42). We note that these are exactly the same as the standard equations of motion of a Dirac particle carrying electric charge e moving in an electromagnetic field. These equations have therefore now been derived using the topological constraint rather than the usual approach of using the minimal coupling hypothesis.

Note that in the quantum equations, there is a local gauge invariance under a new gauge transformation given by:

$$\begin{aligned} \psi &\rightarrow \exp i\tilde{g}\alpha\psi, \\ \bar{a}_\mu &\rightarrow \bar{a}_\mu + \partial_\mu \alpha. \end{aligned} \tag{3.23}$$

Note that ψ, being a monopole wavefunction, is invariant under $U(1)$-gauge transformations and the new symmetry is in addition to the original gauge transformation for the gauge potential a_μ. The origins of this new symmetry lie in the degeneracy of the solution of the variational problem, which is formulated in terms of $U(1)$-gauge invariant quantities $f_{\mu\nu}$ and ψ. The symmetry of the system has now doubled to $U(1) \times \tilde{U}(1)$ where the "tilde" signifies opposite parity. [1] We call this phenomenon the *chiral doubling* of the gauge symmetry and will discuss it further in the coming chapters.

3.2 Duality, Classical Equations for Monopoles

We have already discussed two special properties of the abelian theory which set it apart from nonabelian theory: (a) at any particular point x where the field $f_{\mu\nu}$ is source-free, $\partial_\nu f^{\mu\nu}(x) = 0$, which can be interpreted as the Bianchi identity for ${}^*f^{\mu\nu}$, implying that the latter is also a gauge field derivable from a potential $\bar{a}_\mu(x)$, (see (3.17)); (b) a monopole of $f_{\mu\nu}$ which is defined as a topological obstruction of $f_{\mu\nu}$, is a source of ${}^*f_{\mu\nu}$ and vice versa. We thus see that there is a symmetry between the field and its dual, as well as between a source and its monopole. This is what is referred to as *duality*, or *dual symmetry* in abelian theories, and was discussed in §1.3.

Yang-Mills theories however, do not possess any of these two properties. Although the Bianchi identity holds for $F_{\mu\nu}$ at any x where there are no sources, due

[1] Due to the presence of the $\epsilon_{\mu\nu\rho\sigma}$ factor in (3.7), the gauge potential $\bar{a}_\mu$ transforms like an axial vector under parity transformations, unlike the vector $U(1)$-gauge potential a_μ. Thus, the new $\tilde{U}(1)$-transformation has opposite parity to the original $U(1)$-gauge transformation.

to the absence of a Poincaré lemma, there is no reason why ${}^*F_{\mu\nu}$ should also be a gauge field derivable from a potential. After all, the Poincaré lemma does not apply to Lie-algebra-valued forms and covariant curls. As already discussed in §1.4, Gauss' theorem cannot be used either to deduce that a monopole charge of $F_{\mu\nu}$ is equivalent to a source of ${}^*F_{\mu\nu}$. Thus, in general, there is no dual symmetry in nonabelian theories.[2]

This lack of duality has deep consequences when considering interactions. In abelian theory, duality allows us to derive the interactions in two ways: by viewing the charge (either electric or magnetic) as either a source or a monopole of the dual theory. The procedure was discussed in §3.1, and involves adding an interaction term to the action as in (3.11), or regarding the source of the field as a monopole of the dual field, and thus imposing a topological constraint on the variables in the action. Duality thus ensures the same result in the two approaches. Nonabelian theory however, does not offer this freedom. Sources and monopoles are different objects, and even their charges take on different values (e.g. $SO(3)$). The interactions could still be introduced as in abelian theory by the two methods mentioned above, but these are no longer interchangeable since a monopole of the field can no longer be regarded as a source of the dual field. In fact, the two methods give results that are in general very dissimilar. The equations so derived for monopoles will thus in principle be new equations previously unknown.

Let us examine how one would go about finding the equations of motion for monopoles in the classical case. We seek to incorporate the interaction in the form of a topological constraint. We begin by writing down the free action for field and particle:

$$\mathcal{A} = \mathcal{A}_F^0 + \mathcal{A}_M^0, \tag{3.24}$$

[2]Recent work by Seiberg [23], Witten [27] and others indicates that there may exist a more genralised interpretation of electromagnetic duality in nonabelian theories.

where $\mathcal{A}_F^0$ is the standard Yang-Mills action:

$$\mathcal{A}_F = -\frac{1}{16\pi}\int d^4x\, Tr\,[F_{\mu\nu}F^{\mu\nu}], \tag{3.25}$$

and $\mathcal{A}_M^0$ the usual action (3.3) for a particle of mass m. The equations of motion are to be obtained by varying $\mathcal{A}^0$ with respect to the variables $A_\mu(x)$ and $Y^\mu(\tau)$, subject to the constraint that the particle at $Y^\mu(\tau)$ should carry a monopole of $F_{\mu\nu}(x)$.

The procedure however, is not so easy to apply. Recall that in abelian theory, the method was successful for two reasons: (1) the topological definition of the magnetic charge which was a global one, was reduced to a differential form, local in x; (2) the complications due to patching prompted us to replace the gauge potential as variables by the field tensor $f_{\mu\nu}$, which is patch-independent. Neither of these properties apply in the nonabelian case: the failure of the concept of flux in nonabelian theories prevents us from finding a "local" form for the constraint, and the patching problem is not avoided by using $F_{\mu\nu}$ as variables, since these are not gauge invariant quantities, but covariant ones, and hence patch-dependent.

These problems prevent us from effectively working in spacetime in a local fashion. However, a strategy which presents itself, is to work in loop-space. As discussed in §2.2, the topological definition of the charge can be stated explicitly in terms of loop variables $F_\mu[\xi\,|\,s]$ as in (2.18) and (2.21) (for the classical monopole), which have the additional property of being gauge invariant, and hence free from difficulties associated with patching. In addition, the extended Poincaré lemma allows us to interchange the variables $A_\mu(x)$ with $F_\mu[\xi\,|\,s]$ satisfying conditions $(A), (B)$ and (C) in §2.3, without changing any of the physics. The important point to note, is that these conditions which are necessary for the removal of this redundancy, are precisely the ones that are implied in imposing the topological constraint defining the monopole charge. This is in analogy with the abelian case, where it was the constraint defining the monopole charge that removed the redundancy of $f_{\mu\nu}$.

We now adopt this strategy, and work entirely with loop variables $F_\mu[\xi \mid s]$. To begin, we write the free action $\mathcal{A}_F^0$ for the field in terms of these variables [6].

$$\mathcal{A}_F^0 = -\frac{1}{4\pi\bar{N}} \int \delta\xi \int_0^{2\pi} ds\, Tr\left[F_\mu[\xi \mid s]\, F^\mu[\xi \mid s]\right] \dot{\xi}^{-2}(s), \tag{3.26}$$

where $\bar{N}$ is the infinite numerical factor:

$$\bar{N} = \int_0^{2\pi} ds \int \prod_{s' \neq s} d^4\xi(s'). \tag{3.27}$$

The free action $\mathcal{A}_M^0$ for the particle must be added to $\mathcal{A}_F^0$ to give us the total free action $\mathcal{A}^0$. The equations of motion are now derived by varying $\mathcal{A}^0$ with respect to $F_\mu[\xi \mid s]$ and $Y^\mu(\tau)$, under the constraint that $Y^\mu(\tau)$ carries a monopole charge. The constraint can be stated in two forms, the integral (2.18) and the differential:

$$G_{\mu\nu}[\xi \mid s] = J_{\mu\nu}[\xi \mid s], \tag{3.28}$$

either of which can be used to derive the equations of motion [3, 7]. Here, we choose the differential version, and introduce Lagrange multipliers $L^{\mu\nu}[\xi \mid s]$ and $b[\xi \mid s]$ for the topological and transversality constraints respectively. Thus, we have for the full action:

$$\mathcal{A} = \mathcal{A}^0 + \int \delta\xi\, ds\, Tr\left\{L^{\mu\nu}\left(G_{\mu\nu} - J_{\mu\nu}\right) + b\, F_\mu \dot{\xi}^\mu\right\}, \tag{3.29}$$

where the arguments for the various quantities have been suppressed. The explicit form of the current $J_{\mu\nu}$ for the classical monopole is based on the definition of the monopole charge ζ as the holonomy in loop-space (2.21):

$$J_{\mu\nu}[\xi \mid s] = 4\pi\tilde{g} \int d\tau\, \kappa[\xi \mid s] \epsilon_{\mu\nu\rho\sigma} \dot{\xi}^\rho(s) \dot{Y}^\sigma(\tau)\, \delta^4(\xi(s) - Y(\tau)), \tag{3.30}$$

with $\kappa[\xi \mid s]$ satisfying:

$$\exp i\pi\kappa = \zeta, \tag{3.31}$$

where ζ is the monopole charge. Note that we need one Lagrange multiplier for each constraint: hence the summation over μ, ν, the loop ξ and the parameter s.

This action must now be varied with respect to the dynamical variables of the theory. We get as the equations of motion:

$$(4\pi\bar{N}\dot{\xi}^2(s))^{-1}F_\mu[\xi\,|\,s] = -\mathcal{D}^\nu(s)L_{\mu\nu}[\xi\,|\,s] - b[\xi\,|\,s]\dot{\xi}_\mu(s), \tag{3.32}$$

$$m\ddot{Y}^\mu(\tau) = -8\pi\tilde{g}\int \delta\xi\, ds\, \epsilon_{\mu\nu\rho\sigma}\,\delta_\lambda(s)\,Tr\left[L^{\lambda\rho}\,\kappa\right]\,\dot{Y}^\nu(\tau)\,\dot{\xi}^\sigma(s)\,\delta^4(\xi(s)-x), \tag{3.33}$$

where $\mathcal{D}^\nu(s) = \delta^\nu - ig\,[F^\nu[\xi\,|\,s],\ ]$ is the covariant derivative in loop-space, with $F^\nu[\xi\,|\,s]$ as connection, and $\delta_\lambda(s) = \frac{\delta}{\delta\xi^\lambda(s)}$. We also have the topological constraint (3.28).

The constraint equations (2.9) and (3.28) can now be used to eliminate the Lagrange multipliers $L_{\mu\nu}[\xi\,|\,s]$ and $b[\xi\,|\,s]$ from these equations: to see this, let us apply the covariant derivative $\mathcal{D}_\mu(s)$ to the first equation and antisymmetrise: [3]

$$(2\pi\bar{N}\dot{\xi}^2(s))^{-1}\,\delta_\mu(s)F^\mu[\xi\,|\,s] = -ig\,[G^{\mu\nu}[\xi\,|\,s], L_{\mu\nu}[\xi\,|\,s]] - 2\mathcal{D}_\mu(s)b[\xi\,|\,s]\dot{\xi}^\mu(s). \tag{3.34}$$

However, the term involving $b[\xi\,|\,s]$ can be written as:

$$\mathcal{D}_\mu(s)\,b[\xi\,|\,s]\dot{\xi}^\mu(s) = (\delta_\mu(s)b[\xi\,|\,s] - ig\,[F_\mu[\xi\,|\,s], b[\xi\,|\,s]])\,\dot{\xi}^\mu(s) \tag{3.35}$$

which by applying the constraint equation (2.9) leads us to;

$$\mathcal{D}_\mu(s)b[\xi\,|\,s]\,\dot{\xi}^\mu(s) = \frac{\delta}{\delta\xi^\mu(s)}b\,[\xi\,|\,s]\,\dot{\xi}^\mu(s) = \frac{d}{ds}b[\xi\,|\,s]. \tag{3.36}$$

The right-hand side of this equation however, is zero, since b is a loop quantity and therefore reparametrisation-invariant. The simplified form of the equation of motion for $F_\mu[\xi\,|\,s]$ is therefore the Polyakov equation:

$$\frac{\delta}{\delta\xi^\mu(s)}F^\mu[\xi\,|\,s] = 0, \tag{3.37}$$

which is valid at all points, except at the monopole position where $G_{\mu\nu}[\xi\,|\,s] \neq 0$.

The Polyakov equation is by (2.7) equivalent to the Yang-Mills equation, with no source:

$$D_\nu F^{\mu\nu}(x) = 0. \tag{3.38}$$

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[3] In deriving this equation, the following have been used: $\mathcal{D}_\mu F^\mu = \delta_\mu F^\mu - ig\,[F_\mu, F^\mu] = \delta_\mu F^\mu$, and: $\mathcal{D}_\mu\mathcal{D}_\nu\,L^{\mu\nu} = \frac{1}{2}[\mathcal{D}_\mu, \mathcal{D}_\nu]L^{\mu\nu} = \frac{ig}{2}[G_{\mu\nu}, L^{\mu\nu}]$.

To eliminate the Lagrange multiplier $L_{\mu\nu}[\xi \mid s]$ from (3.33), we note that:

$$\delta_\mu(s) Tr\{L^{\mu\nu}\kappa\} = Tr\{(\mathcal{D}_\mu(s)L^{\mu\nu})\,\kappa\} + Tr\{L^{\mu\nu}\mathcal{D}_\mu\kappa\}. \tag{3.39}$$

Using the Bianchi identity for $G_{\mu\nu}[\xi \mid s]$ in (2.21):

$$\epsilon^{\mu\nu\rho\sigma}\mathcal{D}_\rho(s)G_{\mu\nu}[\xi \mid s] = 0, \tag{3.40}$$

taking (3.39) into consideration and remembering that κ is a loop quantity with $\frac{d}{ds}\kappa[\xi \mid s] = 0$, we have:

$$\mathcal{D}_\mu(s)\kappa[\xi \mid s] = 0. \tag{3.41}$$

Applying this to (3.39) and using (3.32), we eliminate $L_{\mu\nu}[\xi \mid s]$ from (3.33), and obtain:

$$m\ddot{Y}_\mu(\tau) = \frac{2\tilde{g}}{N}\int \delta\xi\, ds\, \epsilon_{\mu\nu\rho\sigma} Tr\{\kappa F^\nu\}\dot{\xi}^\rho(s)\dot{Y}^\sigma(\tau)\dot{\xi}^{-2}(s)\,\delta^4(\xi(s) - x), \tag{3.42}$$

which is analogous to the Lorentz equation (3.13).

In this section, we have succeeded in deriving the equations of motion for a classical monopole. In fact, equations (2.21), (3.32) and (3.33), form the complete set of equations of motion for such a particle. These equations are analogous to the Wong equations for a classical isospin charge [28], but due to the loop nature of the equations, this is not immediately obvious. We therefore proceed to cast these equations in "local" form.

Strictly speaking, the process of finding the "local" form of the equations of motion for classical nonabelian monopoles is not valid at the monopole position since at that point, local quantities describing the field like $F_{\mu\nu}(x)$ and $A_\mu(x)$ do not really exist. However, we ignore this difficulty for the moment, and proceed to obtain the local equations.

Let us first examine the constraint equation:

$$G_{\mu\nu}[\xi \mid s] = 4\pi\tilde{g}\int d\tau\, \epsilon_{\mu\nu\rho\sigma}\,\kappa[\xi \mid s]\dot{\xi}^\rho(s)\dot{Y}^\sigma(\tau)\,\delta^4(\xi(s) - Y(\tau)), \tag{3.43}$$

and write it in terms of local variables. Substituting (2.11) in the above, we obtain:

$$D^*_\nu F^{\mu\nu}(x) = -4\pi\tilde{g}\int d\tau\ \left[\Phi_\xi(s,0)\kappa[\xi\,|\,s]\Phi^{-1}_\xi(s,0)\right]_{\xi(s)=x}\dot{Y}^\mu(\tau)\,\delta^4(x-Y(\tau)). \tag{3.44}$$

This can be written as:

$$D^*_\nu F^{\mu\nu}(x) = -4\pi\tilde{g}\int d\tau\ K(\tau)\dot{Y}^\mu(\tau)\,\delta^4(x-Y(\tau)), \tag{3.45}$$

where $K(\tau) = \left[\Phi_\xi(s,0)\kappa[\xi\,|\,s]\Phi^{-1}_\xi(s,0)\right]_{\xi(s)=Y(\tau)}$. The latter quantity is a function of τ only, since it depends on the loop only through $x = \xi(s)$, and is evaluated at $Y(\tau)$, by virtue of the δ-function. Next, we examine the "Lorentz" equation (3.42). Substituting for $F_\mu[\xi\,|\,s]$ the expression in (2.7), averaging over all directions of the tangent vector $\dot{\xi}^\mu(s)$, integrating over ξ, and recalling the definition of the normalisation factor $\bar{N}$, we obtain [7]:

$$m\ddot{Y}_\mu(\tau) = -\tilde{g}\,Tr\left[K(\tau)\,{}^*F_{\mu\nu}(Y(\tau))\right]\dot{Y}^\nu(\tau), \tag{3.46}$$

which is the exact dual of the Wong equation (see Chapter 4) for the classical limit of a Yang-Mills source.

There is one more local equation. Note that:

$$\begin{aligned} D_\mu D_\nu\,{}^*F^{\mu\nu} &= \frac{1}{2}[D_\mu, D_\nu]\,{}^*F^{\mu\nu} \\ &= \frac{ig}{2}\epsilon^{\mu\nu\rho\sigma}[F_{\mu\nu}, F_{\rho\sigma}] \\ &= \frac{ig}{2}[{}^*F^{\rho\sigma}, F_{\rho\sigma}] = \frac{ig}{2}[F^{\mu\nu}, {}^*F_{\mu\nu}], \end{aligned} \tag{3.47}$$

and thus, $D_\mu D^*_\nu F^{\mu\nu} = 0$. Hence, by (3.45):

$$\int d\tau\ D_\mu[K(\tau)\dot{Y}^\mu(\tau)]\,\delta^4(x-Y(\tau)) = 0. \tag{3.48}$$

It is easy to show that this can be simplified to:

$$\int d\tau\ \left[-K(\tau)\frac{d}{d\tau}\delta^4(x-Y(\tau)) - ig[A_\mu, K(\tau)]\dot{Y}^\mu(\tau)\,\delta^4(x-Y(\tau))\right], \tag{3.49}$$

and therefore, integrating by parts:

$$\dot{K}(\tau) - ig[A_\mu(Y(\tau)), K(\tau)]\dot{Y}^\mu(\tau) = 0, \tag{3.50}$$

for all τ. Note the presence of the coupling constant g, due to the covariant derivative.

We see that of the four equations governing the motion of a monopole in a gauge field, equations (3.45) and (3.46) are very similar to Wong's equations, and can be obtained from them by making the replacements:

$$\begin{aligned} F_{\mu\nu}(x) &\to {}^*F_{\mu\nu}(x) \\ I(\tau) &\to K(\tau) \\ g &\to \tilde{g}. \end{aligned} \tag{3.51}$$

Equation (3.38) is just the Bianchi identity for the dual Yang-Mills field ${}^*F^{\mu\nu}(x)$. It might be tempting to say therefore, that these equations are the dual transforms of the Wong equations. However, there is one important difference: the covariant derivative D_μ has hidden in it, the gauge potential A_μ, in both the Wong and the monopole equations. This difference manifests itself in (3.50), where g appears instead of $\tilde{g}$. For duality to hold, the latter set of equations should not contain A_μ in the covariant derivative, but rather, a dual potential $\tilde{A}_\mu$ which must be related to ${}^*F^{\mu\nu}$ in the same way as A_μ is related to $F_{\mu\nu}$, i.e.:

$${}^*F_{\mu\nu}(x) = \partial_\nu \tilde{A}_\mu(x) - \partial_\mu \tilde{A}_\nu(x) + i\tilde{g}[\tilde{A}_\mu(x), \tilde{A}_\nu(x)]. \tag{3.52}$$

However, as already explained, such a potential $\tilde{A}_\mu(x)$ does not exist in general. This is precisely what is meant by the lack of duality in nonabelian theories. The presence of the covariant derivative D_μ in these equations is the source of the great difference between the dynamics dictated by the two sets of equations. We shall return to this point later, when we discuss the interactions of monopoles and compare them to those of sources.

3.3 Wave Equations for Monopoles

Now that we have developed the tools for attacking the problem of the dynamics of the classical nonabelian monopole, let us next deal with the nonabelian quantum monopole, which is taken to be a Dirac particle in analogy to the electric and isospin charges. It is therefore described by a wavefunction ψ with internal "isospin" degrees of freedom. The action $\mathcal{A}^0_M$ for the monopole is taken to be (3.4), and that for the field, $\mathcal{A}^0_F$ due to difficulties with patching. Once again, the equations of motion are to be derived by varying the action $\mathcal{A}^0$ with respect to the dynamical variables $F_\mu[\xi\,|\,s]$ and ψ, subject to the constraint defining the monopole charge.

The explicit form of the constraint must now be determined. We seek to replace the classical current (2.21) by a quantum one, as we did in abelian theory. The question we now wish to address is: what is the classical monopole current? One might think that it is represented by the right-hand side of (3.45), because of the similarity of this equation to the Wong equation (see Chapter 4), but this is not correct: the right-hand side of (3.45) is covariantly conserved, as is easily shown:

$$D_\mu D^*_\nu F^{\mu\nu}(x) = -4\pi\tilde{g}\int d\tau\, D_\mu K(\tau)\,\dot{Y}^\mu(\tau)\,\delta^4(x - Y(\tau)) = 0. \tag{3.53}$$

We see a problem in interpreting $D^*_\nu F^{\mu\nu}$ as the monopole current $\tilde{j}^\mu$. Note that unlike Wong's equations (where the current j^μ is covariantly conserved, due to the fact that the field A_μ also carries "colour"), the field in the monopole theory is not "magnetically charged". The magnetic charge is carried solely by the particle at $Y(\tau)$. Hence, the current carried by $Y(\tau)$ should by itself be a conserved quantity and thus, there is no need for the covariant divergence being zero. In other words, the monopole current should have a vanishing *ordinary* divergence:

$$\partial_\mu \tilde{j}^\mu(x) = 0. \tag{3.54}$$

The current on the right-hand side of (3.45) cannot therefore be a monopole

current, and must be modified. We do this by replacing $K(\tau)$ by another quantity $\mathcal{K}(\tau)$:

$$\tilde{j}_\mu(x) = -\tilde{g}\int d\tau\, \mathcal{K}(\tau)\dot{Y}_\mu(\tau)\,\delta^4(x - Y(\tau)), \tag{3.55}$$

where:

$$\mathcal{K}(\tau) = \Omega_\xi(s,0)\kappa[\xi \mid s]\Omega_\xi^{-1}(s,0)\mid_{\xi(s)=Y(\tau)}, \tag{3.56}$$

with:

$$\Omega_\xi(s,0) = \omega(\xi(s_+))\Phi_\xi(s_+,0). \tag{3.57}$$

The meaning of this equation can be made clear by recalling that in the definition of $K(\tau)$, the factor $\Phi_\xi(s_-,0)$ is a parallel-transport from the loop reference point P_0 to the point $\xi(s_-)$, which means that the phase of $K(\tau)$ in internal symmetry space is measured in the local frame at $\xi(s_-)$. The factor $\Omega_\xi(s,0)$ is also a parallel-transport from the base-point P_0 to $\xi(s_+)$, where the argument s_+ means that the value of $\xi(s)$ has to be taken as the limit from above, i.e. $\xi(s_+) = \xi(s_+ + \epsilon), \epsilon \to 0^+$. $\omega(x)$ is a local rotation matrix, allowing for the possibility that the phase of the charge may be measured in a local frame different from the one in which the field is measured. The difference with $\Phi_\xi(s_-,0)$, is that by replacing the argument s_- by s_+, the phase factor $\Phi_\xi(s_+,0)$ is now affected by the loop derivative $\frac{\delta}{\delta\xi^\mu(s)}$, which is not the case for $\Phi_\xi(s_-,0)$. Indeed, we have:

$$\Omega_\xi^{-1}(s,0)\frac{\delta}{\delta\xi^\mu(s)}\Omega_\xi(s,0) = -igF_\mu[\xi \mid s], \tag{3.58}$$

since the logarithmic derivative of $\Omega_\xi(s,0)$ is represented by the solid curve in Fig. 2.1, and is therefore the same as $-igF_\mu[\xi \mid s]$ in (2.6), defined as the logarithmic derivative of $\Phi[\xi]$. (3.55) thus ensures that the monopole current has a vanishing ordinary divergence, as required.

We now proceed to replacing the classical monopole current (3.55) by its quantum analogue:

$$\tilde{j}_\mu(x) = -\tilde{g}\,[\bar{\psi}(x)\gamma_\mu T^i\psi(x)]\tau_i, \tag{3.59}$$

where $\psi(x)$ now belongs to a representation of the gauge group and T^i is a matrix representing the generators τ_i in this representation. The current is therefore an element of the gauge algebra as required.

The constraint we therefore wish to impose for deriving the quantum equations of motion is:

$$G_{\mu\nu}[\xi \mid s] = 4\pi\tilde{g}\,\epsilon_{\mu\nu\rho\sigma}\dot{\xi}^{\rho}(s)[\bar{\psi}(\xi(s))\gamma^{\sigma}T^{i}\psi(\xi(s))]\,\Omega_{\xi}^{-1}(s,0)\tau_i\Omega_{\xi}(s,0). \tag{3.60}$$

The action for the system is still (3.29), with the difference that $J_{\mu\nu}[\xi \mid s]$ is now given by the right-hand side of (3.60). The $F_{\mu}[\xi \mid s]$ variation yields (3.32) as before, whereas variation with respect to $\psi(x)$ leads to:

$$(i\partial_{\mu}\gamma^{\mu} - m)\,\psi(x) = -\tilde{g}\,\tilde{A}_{\mu}(x)\gamma^{\mu}\,\psi(x), \tag{3.61}$$

where:

$$\tilde{A}_{\mu}(x) = 4\pi\int \delta\xi\, ds\, \epsilon_{\mu\nu\rho\sigma}\,\Omega_{\xi}(s,0)L^{\rho\sigma}[\xi \mid s]\Omega_{\xi}^{-1}(s,0)\dot{\xi}^{\nu}(s)\,\delta^{4}(x-\xi(s)). \tag{3.62}$$

Together with (3.32) and (3.61), (3.60) completes the set of equations of motion for a nonabelian Dirac monopole moving in a Yang-Mills field. We have succeeded in deriving equations of motion which bear a strong resemblance to the ordinary Yang-Mills equations. This is especially true of (3.61), which is formally the dual of the ordinary Yang-Mills equation for a "colour" source moving in a gauge field represented by the potential A_{μ}. Note the emergence of the new quantity $\tilde{A}_{\mu}(x)$ coupling to the monopole wavefunction ψ in the same manner as the usual potential $A_{\mu}(x)$ does to the wavefunction of the source charge. The Lagrange multiplier $L_{\mu\nu}$ is explicitly present in the equations of motion (as is the Lagrange multiplier b) and cannot be removed in this case using the equations of motion. We shall call $\tilde{A}_{\mu}(x)$ the *dual potential*, although the term is, strictly speaking, not applicable: this dual potential is not, after all, related to ${}^{*}F_{\mu\nu}$ by:

$${}^{*}F_{\mu\nu}(x) = \partial_{\nu}\tilde{A}_{\mu}(x) - \partial_{\mu}\tilde{A}_{\nu}(x) + i\tilde{g}\,[\tilde{A}_{\mu}(x), \tilde{A}_{\nu}(x)], \tag{3.63}$$

which as we know, might not possess solutions. We shall show that $\tilde{A}_\mu(x)$ has a more intricate relationship with the field ${}^*F_{\mu\nu}$, but nevertheless, it still performs the role of a potential for parallel-transport for the phase of the monopole wavefunction.

3.4 Chiral Doubling of Symmetry

We saw in the case of abelian theory, that there was a "chiral doubling" of the symmetry from $U(1)$ to $U(1) \times U(1)$, where the second $U(1)$ carries a parity opposite to that of the first, and will therefore be denoted by $\tilde{U}(1)$. For abelian theory, this is not unexpected, since the theory is dual symmetric. There is a U-invariance associated with the phase of sources, so there must be a $\tilde{U}$-invariance associated with the phase of monopoles. That this is true in abelian theory however, does not imply that it would be true for nonabelian theory, since in the case of the latter, there is no dual symmetry. But as will be shown, there is nevertheless a $\tilde{U}$-invariance associated with the monopole phase. First, we define a $\tilde{U}$-transformation as a local rotation in phase of the monopole wavefunction $\psi(x)$:

$$\psi(x) \to \left[1 + i\tilde{g}\tilde{\Lambda}(x)\right] \psi(x), \tag{3.64}$$

where $\tilde{\Lambda}(x)$ is an arbitrary, infinitesimal, algebra-valued function. This same function $\psi(x)$ is however, invariant under an ordinary Yang-Mills U-transformation, since the monopole does not possess colour. The variables $F_\mu[\xi \,|\, s]$ are also invariant under a simultaneous U-$\tilde{U}$ transformation (except for an x-independent U-rotation at the reference point P_0, which will be ignored as before). To ensure that the quantity:

$$D^*_\nu F^{\mu\nu}(x) = -4\pi\tilde{g}\left[\bar{\psi}(x)\omega(x)\gamma^\mu T_i \omega^{-1}(x)\psi(x)\right]\tau^i, \tag{3.65}$$

has the correct transformation properties under U and $\tilde{U}$, the following should hold:

$$\begin{aligned} b[\xi \mid s] &\to b[\xi \mid s], \\ \omega(x) &\to \left[1 + i\tilde{g}\tilde{\Lambda}(x)\right] \omega(x) \left[1 - i\tilde{g}\tilde{\Lambda}(x)\right], \\ \Omega_\xi(s,0) &\to \left[1 + i\tilde{g}\tilde{\Lambda}(\xi(s_+))\right] \Omega_\xi(s,0). \end{aligned} \tag{3.66}$$

The transformation properties of $L_{\mu\nu}[\xi \mid s]$ and $\tilde{A}_\mu(x)$ have yet to be determined.

U-transformations could be taken to leave $L_{\mu\nu}$ and hence $\tilde{A}_\mu$ invariant. Under $\tilde{U}$-transformations, the $L_{\mu\nu}$ transformation must be chosen in such a way as to leave the action $\mathcal{A}$ invariant: if $\tilde{A}_\mu$ transforms as a potential under $\tilde{U}$, i.e.:

$$\tilde{\Delta}\tilde{A}_\mu(x) = \partial_\mu \tilde{\Lambda}(x) + i\tilde{g}\,[\tilde{\Lambda}(x), \tilde{A}_\mu(x)], \tag{3.67}$$

then the change due to $\tilde{\Delta}\tilde{A}_\mu(x)$ in:

$$4\pi \int \delta\xi\, ds\, Tr\{L^{\mu\nu}[\xi \mid s] J_{\mu\nu}[\xi \mid s]\} = \tilde{g} \int d^4x\, \bar{\psi}(x) \tilde{A}_\mu(x) \gamma^\mu \psi(x), \tag{3.68}$$

will exactly cancel with the change in the particle free action (3.4) from the transformation of $\psi(x)$, and the action will remain invariant provided that the term proportional to $Tr\{L^{\mu\nu}[\xi \mid s] G_{\mu\nu}[\xi \mid s]\}$ remains invariant under $\tilde{\Delta} L_{\mu\nu}[\xi \mid s]$. The transformation:

$$\tilde{\Delta} L_{\mu\nu}[\xi \mid s] = \epsilon_{\mu\nu\rho\sigma} \mathcal{D}^\rho(s) \tilde{\Lambda}^\sigma[\xi \mid s] \tag{3.69}$$

satisfies the above requirements. The term in $Tr\{L^{\mu\nu}[\xi \mid s] G_{\mu\nu}[\xi \mid s]\}$ changes by:

$$\int \delta\xi\, ds\, Tr\{\tilde{\Delta} L^{\mu\nu} G_{\mu\nu}\} = -\int \delta\xi\, ds\, Tr\left\{\epsilon^{\mu\nu\rho\sigma} \tilde{\Lambda}_\sigma \mathcal{D}_\rho(s) G_{\mu\nu}\right\}, \tag{3.70}$$

which is zero by the Bianchi identity for $G_{\mu\nu}$.

It can also be seen that using (3.69) and (3.66) in (3.70), the following holds:

$$\tilde{\Delta}\tilde{A}_\mu(x) = i\tilde{g}[\tilde{\Lambda}(x), \tilde{A}_\mu(x)] + \tilde{\Delta}_L \tilde{A}_\mu(x), \tag{3.71}$$

where:

$$\tilde{\Delta}_L \tilde{A}_\mu(x) = -8\pi \int \delta\xi\, ds\, \Omega_\xi(s,0) \mathcal{D}_\mu(s) \tilde{\Lambda}_\nu[\xi \mid s] \Omega_\xi^{-1}(s,0) \dot{\xi}^\nu(s)\, \delta^4(\xi(s) - x). \tag{3.72}$$

Using (3.58) we have:

$$\Omega_\xi(s,0)\{\mathcal{D}_\mu(s)\tilde{\Lambda}_\nu[\xi\,|\,s]\}\Omega_\xi^{-1}(s,0) = \delta_\mu(s)\left\{\Omega_\xi(s,0)\tilde{\Lambda}_\nu[\xi\,|\,s]\Omega_\xi^{-1}(s,0)\right\}, \quad (3.73)$$

which when substituted into (3.71) gives (3.67), with:

$$\tilde{\Lambda}(x) = -8\pi\int\delta\xi\,ds\,\Omega_\xi(s,0)\tilde{\Lambda}_\mu[\xi\,|\,s]\Omega_\xi^{-1}(s,0)\dot{\xi}^\mu(s)\,\delta^4(\xi(s)-x), \quad (3.74)$$

so that $\tilde{\Lambda}(x)$ is a sort of average $\tilde{\Lambda}_\mu[\xi\,|\,s]$ over all directions, in the same manner as $\tilde{A}_\mu$ which is a directional average of $L_{\mu\nu}$.

We have therefore succeeded in showing that the action $\mathcal{A}$ is $\tilde{U}$-invariant in addition to its manifest U-invariance. The existence of $\tilde{U}$-symmetry is, in a sense unexpected, since the theory is not dual symmetric as already explained. The source of this extra symmetry can be traced back to the definition of the monopole charge as a homotopy class. Recall that in formulating the dynamics, the "charge" acquired a phase: in the classical case by assigning to it the quantity $\mathcal{K}(\tau)$, and in the quantum case, by the representation of the gauge group G to which $\psi(x)$ belongs. The original definition in terms of the homotopy classes did not involve a phase however. This phase must therefore necessarily be unphysical and the theory must be invariant under local phase rotations of the monopole charge. This is the physical origin of $\tilde{U}$-invariance. To explain the vector-valued gauge parameter $\Lambda_\mu[\xi\,|\,s]$, one must realise that the assignment of phase to the monopole charge was through (3.43) or (3.60), which depend on 3 spacetime indices, namely μ and ν of the curvature $G_{\mu\nu}$ and ρ of the tangent vector $\dot{\xi}^\rho$. Together, these were taken to specify the orientation in spacetime of the elemental 3-volume enclosed by the surface in Fig 1.5. This is equivalent to taking a normal to the elemental volume in 4 dimensions, which will then be represented by the index σ in these equations. We therefore need an extra index on the parameter $\tilde{\Lambda}_\mu[\xi\,|\,s]$, since in redefining the phase, we need to do so for each direction, as well as for each point in spacetime X. This is precisely the reason for $L_{\mu\nu}$ carrying two indices and being a tensor. $L_{\mu\nu}$ plays the role of a "connection" or "gauge potential"

to specify parallel phase transport of monopoles as mentioned in §3.3. This is, once again, analogous to the situation in abelian theory where $\lambda_\mu(x)$ was the connection. The important difference is that since the phase now depends on a direction, the connection has to carry an extra index to specify what is meant by parallel phase transport in neighbouring directions, as well as neighbouring points. This is not the first time that such 2-indexed potentials are encountered: objects with similar properties exist in string theory and supersymmetry[12, 17, 20, 21]. The Freedman-Townsend Model [16, 26] introduced in 1981, has a similar tensor potential with a transformation law given by:

$$\delta\tilde{B}_{\mu\nu} = -\epsilon_{\mu\nu\rho\sigma} D^\rho \Lambda^\sigma, \tag{3.75}$$

which is just the local version of the transformation law (3.69) for $L_{\mu\nu}$.

Chapter 4

Some Considerations of Classical Monopole Dynamics

4.1 Classical Monopole *vs* Wong Equations

In the previous chapters, we derived the classical and quantum equations of motion for nonabelian monopoles. The natural question at this stage, is to ask how these monopoles interact with one another, and what would be the behaviour of a relatively simple system of monopoles, in contrast with a similar system of sources. In this chapter, we examine a few situations where classical monopoles are allowed to interact. In particular, we examine the simple case of two monopoles, and compare their behaviour in various situations with that of nonabelian sources.

To begin, we need to calculate the potential of a single nonabelian classical monopole at the origin. Bearing in mind that the potential has to produce the field:

$$\mathbf{B} = \frac{n}{2g}\frac{\mathbf{r}}{r^3}t_3, \tag{4.1}$$

which is (1.60), with $\tilde{e}$ replaced by $\frac{n}{2g}t_3$, we find that the following is an example

of such a potential, where n is an odd integer:

$$\begin{aligned} A_1^{(N)} &= \tfrac{nt_3}{2g}\tfrac{y}{r(r+z)}, & A_2^{(N)} &= -\tfrac{nt_3}{2g}\tfrac{x}{r(r+z)}, & A_3^{(N)} &= A_0^{(N)} = 0, \\ A_1^{(S)} &= -\tfrac{nt_3}{2g}\tfrac{y}{r(r-z)}, & A_2^{(S)} &= \tfrac{nt_3}{2g}\tfrac{x}{r(r-z)}, & A_3^{(S)} &= A_0^{(S)} = 0, \end{aligned} \tag{4.2}$$

with:

$$t_3 = \frac{1}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.3}$$

$A_\mu^{(N)}$ and $A_\mu^{(S)}$ are thus elements of the $su(2)$ algebra: however, they all commute and the theory is essentially abelian in character.

In order to ensure that the potential above represents a monopole, equation (4.1) must be checked. Indeed it can easily be shown that:

$$\oint_{equator} \left(A_\mu^{(N)} - A_\mu^{(S)}\right) dx^\mu = 2\pi n t_3/g. \tag{4.4}$$

for n odd.

For n even, the potential (4.2) does not correspond to a monopole, since the phase factors $\Phi^{(N)}$ and $\Phi^{(S)}$ have the same sign along the equator and therefore (1.73) would not be satisfied. The absence of a monopole however, means an unpatched potential, which is not the case. This seems to be a contradiction; however, it must be remembered that in the absence of a monopole, all we need is for the potential to be gauge equivalent to an unpatched potential, and indeed, it can be shown that (4.2) is equivalent to the potential:

$$A_i'^{(N)} = A_i'^{(S)} = -\frac{1}{g}\epsilon_{ijk}\frac{x^k}{r^2}, \qquad A_0'^{(N)} = A_0'^{(S)} = 0, \tag{4.5}$$

where the primed and unprimed potentials are related by a gauge transformation:

$$\begin{aligned} A_\mu'^{(N)} &= \xi A_\mu^{(N)}\xi^{-1} - (\tfrac{i}{g})\partial_\mu\xi\ \xi^{-1}, \\ A_\mu'^{(S)} &= \eta A_\mu^{(S)}\eta^{-1} - (\tfrac{i}{g})\partial_\mu\eta\ \eta^{-1}, \end{aligned} \tag{4.6}$$

with:

$$\xi = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\,e^{-i\phi} \\ \sin\frac{\theta}{2}\,e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix}, \qquad \eta = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} \tag{4.7}$$

where θ and ϕ are the polar angles.

The new potential (4.5) can now be continuously deformed to zero, since there is nothing stopping it from doing so. However, for n odd, no gauge transformation of this type can be found, and the potential must necessarily be patched due to the presence of the monopole.

We previously derived the equations of motion for nonabelian monopoles, both classical and quantum, from a constrained action principle. The interaction was imposed by constraints involving Lagrange mulipliers $L_{\mu\nu}$ and b_μ, one for each constraint. But as demonstrated, the Lagrange multipliers can be eliminated from the equations, yielding for the "local" equations:

$$D^*_\nu F^{\mu\nu}(x) = -4\pi\tilde{g}\int d\tau\, K(\tau)\,\dot{Y}^\mu(\tau)\,\delta^4(x - Y(\tau)), \tag{4.8}$$

$$D_\nu F^{\mu\nu}(x) = 0, \tag{4.9}$$

$$m\ddot{Y}^\mu(\tau) = \tilde{g}\,Tr[K(\tau)^*F^{\mu\nu}(Y(\tau))]\,\dot{Y}_\nu(\tau), \tag{4.10}$$

$$\dot{K}(\tau) - ig\,[A_\mu(Y(\tau)), K(\tau)]\,\dot{Y}^\mu(\tau) = 0. \tag{4.11}$$

As already remarked in Chapter 3, these equations (with the exception of (4.11), look like the "dual" of the Wong equations [28], which are the equations of motion for a classical isospin source $I(\tau)$ if we make the following replacements:

$$\begin{aligned} F_{\mu\nu} &\leftrightarrow {}^*F_{\mu\nu}, \\ I(\tau) &\leftrightarrow K(\tau), \\ g &\leftrightarrow g = \tfrac{1}{4g}, \end{aligned} \tag{4.12}$$

except that the covariant derivative is still given by $D_\mu = \partial_\mu - ig[A_\mu, \,]$ rather than the potential $\tilde{A}_\mu$ of the dual field ${}^*F^{\mu\nu}$.

For easy reference, we reproduce here the Wong equations:

$$D_\nu F^{\mu\nu}(x) = -4\pi g\int d\tau\, I(\tau)\,\dot{Y}^\mu(\tau)\,\delta^4(x - Y(\tau)), \tag{4.13}$$

$$D^*_\nu F^{\mu\nu}(x) = 0, \tag{4.14}$$

$$m\ddot{Y}^{\mu}(\tau) = g\,Tr[I(\tau)F^{\mu\nu}(Y(\tau))]\,\dot{Y}_{\nu}(\tau), \tag{4.15}$$

$$\dot{I}(\tau) - ig\,[A_{\mu}(Y(\tau)), I(\tau)]\,\dot{Y}^{\mu}(\tau) = 0. \tag{4.16}$$

It is interesting to note that attempts to derive the Wong equations from an action principle have been unsuccessful. These equations are derived by taking the classical limit of the Yang-Mills equations, and at the present this is the only way of deriving them.

In the case of abelian theory, the monopole equations of motion and the equations of classical electrodynamics exhibit a complete duality under $f_{\mu\nu} \leftrightarrow {}^*f_{\mu\nu}$, $e \leftrightarrow \tilde{g}$. This fact can be interpreted as follows: suppose we consider a system of classical sources. The time evolution of the system is governed by the equations (3.12) and (3.13) for abelian sources and equations (4.13), (4.15) and (4.16) for nonabelian ones. Suppose now that we have a system of classical monopoles, initially in the same configuration as the source system. In this case, the time evolution is determined by equations (3.5) and (3.10) for abelian monopoles and equations (3.45), (3.46) and (3.50) for nonabelian ones. Suppose that now we consider the following question: is there an initial configuration identical for the two systems of monopoles and charges, where there would be a difference in the subsequent time evolution of the two systems due to the differences in the equations of motion for sources and monopoles, and if so, what is this initial configuration?

The answer is quite simple for the abelian case. The equations of motion for monopoles and sources are simply dual, and a system of sources would be subject to a field $f_{\mu\nu}(x)$, which is to be obtained by summing up all the contributions by the sources due to the current. A system of monopoles in the same initial configuration, will contribute to the "dual" field ${}^*f_{\mu\nu}$ via equation (3.5) in exactly the same manner, so that the dependence of $f_{\mu\nu}({}^*f_{\mu\nu})$ on the source (monopole) configuration would be identical. Since the coupling of the field $f_{\mu\nu}$ to the source worldline $Y^{\mu}(\tau)$ (see (3.13)) is identical to the coupling of ${}^*f_{\mu\nu}$ to the monopole worldline $Y^{\mu}(\tau)$ (see (3.10)), the subsequent time evolution of the two systems

would be exactly the same. An observer would therefore not be able to distinguish between the two systems and would be unable to tell them apart as being exclusively sources or exclusively monopoles.

In the case of nonabelian theories however, one is faced with a situation in which the equations of motion are no longer dual. As mentioned, the presence of the covariant derivative containing A_μ destroys the duality. The equations are then non-linear, and the argument above does not hold. It is therefore, in principle possible that one can conceive of a situation in which a system of nonabelian sources and one of nonabelian monopoles, starting from an identical configuration, have different time evolutions, so that an observer would be able to differentiate between the two; and by comparing their behaviour with that predicted by the two sets of equations of motion, determine which is which. The object therefore, is to examine a few cases, with the hope of finding such a situation in a relatively uncomplicated system of say, two interacting particles. We study a few simple cases in the following section.

4.2 Classical Source *vs* Monopole Scattering

This question has been studied for the case of an infinitely heavy source (or monopole) at the origin, and a "test" particle whose effect on the heavy particle will be ignored. Several cases were examined with the identical initial configuration for sources and monopoles. In what follows, sources (being "sources" of $F_{\mu\nu}$) will be referred to as S, and monopoles (being defined topologically and not as a "source" of any field) by M. It must be stated here that not all solutions to the classical equations of motion are stable. In what follows, no attempt has been made to analyse the stability properties of the solutions. The reader is referred to Brandt and Neri [1] for further details.

(1) Case of two stationary static sources S and S' with S infinitely heavy and at the origin: initially, the isospins of the two particles S and S' can be in any

direction. Suppose we choose the initial isospins to be in the 3-rd direction for both particles.[1] Consider the field produced by S. Since S is not interacting with any fields (where we ignore the self-interaction), we get: $\dot{I}(\tau) = 0$, so that $I = I_3\tau_3$ remains a constant, and in the 3-rd direction. The field produced by S will also be in the 3-rd isospin direction and will be given by:

$$\partial_\nu F^{\mu\nu}(x) = -4\pi g \int d\tau\, I(\tau)\, \dot{Y}^\mu(\tau)\, \delta^4(x - Y(\tau)). \tag{4.17}$$

Introducing a second source S' with isospin $I'(0) = I'_3\tau_3$ at $\mathbf{x}$, initially in the 3-rd direction in isospin space, ignoring the field produced by S' and considering only the effect of S on S', we have:

$$\dot{I}'(\tau) = ig\,[A_\mu(Y'(\tau)), I'(\tau)]\,\dot{Y}'^\mu(\tau), \tag{4.18}$$

where the primed quantities refer to S'. Since A_μ and I' both point in the 3-rd direction in isospin space, the direction of I' remains unchanged:

$$\dot{I}'(\tau) = 0. \tag{4.19}$$

The Lorentz equation for S' now reduces to:

$$m\ddot{Y}'^\mu(\tau) = g\, I'_3\, F_3^{\mu\nu}\,(Y'(\tau))\,\dot{Y}'_\nu(\tau), \tag{4.20}$$

which is a Coulomb-type interaction. The field $F_{\mu\nu}$ is the field due to the point-source S at the origin and can be denoted by $F^E_{\mu\nu}$. We thus see that here, the source-source interaction reduces to the abelian case.

(1)′ Let us now consider the parallel situation with monopoles instead of sources. Suppose we have an infinitely heavy monopole M at the origin, with $K(0) = K_3\tau_3$. Once again, using (3.50), the isospin of the monopole does not

[1]Note that due to the gauge invariance of the theory, we are free to rotate the isospin vector at each spacetime point independently, without changing the physics. Thus, we need to make clear how to compare isospin directions at different spacetime points. We do so, by choosing a 3-rd direction for the isospin at the origin and parallel-transporting the isospin to $\mathbf{x}$, and thus defining the 3-rd isospin direction at $\mathbf{x}$.

change, since M is infinitely heavy, and it is assumed to be stationary. The potential and the field of M will also be in the 3-rd isospin direction, and would be given by:

$$\partial^*_\nu F^{\mu\nu}(x) = -4\pi\tilde{g} \int d\tau\, K(\tau)\dot{Y}^\mu(\tau)\,\delta^4(x - Y(\tau)). \tag{4.21}$$

We now introduce a second monopole M', with isospin $K'(0) = K'_3\tau_3$ initially, located at $\mathbf{x}$. The field of M', will be ignored in what follows, as in the source case. Thus, considering the effect of M on M' only, we obtain:

$$\dot{K}'(\tau) = ig[A_\mu(Y'(\tau)), K'(\tau)]\dot{Y}'^\mu(\tau), \tag{4.22}$$

where Y' is the worldline of M' and $A_\mu(Y'(\tau))$, the potential at $Y'(\tau)$, due to the presence of M. Once again, since A_μ and K' point in the same isospin direction, we have:

$$\dot{K}'(\tau) = 0. \tag{4.23}$$

Thus, the analogue of the Lorentz equation (4.10), reduces to:

$$m\ddot{Y}'^\mu(\tau) = \tilde{g}K'^*_3 F^{\mu\nu}_3(Y'(\tau))\dot{Y}'_\nu(\tau), \tag{4.24}$$

which is once again, a Coulomb-type interaction, and the *exact* dual of (4.20), for the S-S case. Also, it must be noted that the field ${}^*F_{\mu\nu}$ in the above equation is produced by the point-monopole M, and is therefore of exactly the same form as the field $F^E_{\mu\nu}$ in (4.20), which is due to the presence of a point-source.

It can easily be seen that the 3-rd isospin direction is not special in any way, and we get a similar result if the initial isospins point in the same (arbitrary) direction.

We now examine the case where the two initial isospins point in two different directions initially, say the 3-rd for the heavy particle and the 1-st for the light. For simplicity, assume that the second isospin is stationary initially.

(2) Once again, the infinitely heavy source S, is at the origin and stationary, with $I(0) = I_3\tau_3$. Since we are ignoring self-interactions, we have once again:

$$\dot{I}(\tau) = 0. \tag{4.25}$$

Thus, once again, from (4.16), we have:

$$[A_\mu(Y(\tau)), I(\tau)]\dot{Y}^\mu(\tau) = 0, \tag{4.26}$$

which holds only if A_μ has components in the 3-rd isospin direction only. Therefore, A_μ points in the 3-rd isospin direction and:

$$A_\mu(x) = A^3_\mu(x)\tau_3, \qquad F_{\mu\nu}(x) = F^3_{\mu\nu}(x)\tau_3, \qquad {}^*F_{\mu\nu}(x) = {}^*F^3_{\mu\nu}(x)\tau_3. \tag{4.27}$$

We now calculate A_μ explicitly. Since the particle is infinitely heavy, we know that the spatial components of the velocity vanish, i.e. $\dot{Y}_\mu(\tau) = (1,0,0,0)$. Thus, in (4.13):

$$\partial_\nu\partial^\nu A^\mu - \partial_\nu\partial^\mu A^\nu = -4\pi g \int d\tau\, I_3(\tau)\tau_3\, \dot{Y}^\mu(\tau)\, \delta^4(x - Y(\tau)). \tag{4.28}$$

Choosing the Lorentz gauge where $\partial_\mu A^\mu = 0$, we have:

$$\begin{aligned} \Box A^\mu(x) &= -4\pi g \int d\tau\, I_3(\tau)\, \tau_3\, \dot{Y}^\mu(\tau)\, \delta^4(x - Y(\tau)) \\ &= -4\pi g \int d\tau\, I_3(\tau)\, \tau_3\, (1,0,0,0)\, \delta^4(x - Y(\tau)). \end{aligned}$$

The solution to this equation is the well-known central potential given by:

$$A^\mu(x) = \left(-\frac{I_3}{|\mathbf{x}|}, 0, 0, 0\right)\, \tau_3, \tag{4.29}$$

which is the same as that of an abelian source of strength gI placed at the origin. $A_\mu(x)$ is time-independent as expected.

We now introduce the second source S', placed at $\mathbf{x}$, with $I'(0) = I'\tau_1$. Once again, ignoring self-interactions, we see that S' is affected only by the field of S. To determine the time-evolution of I', we have using (4.16):

$$\dot{I}'(\tau) - ig[A_\mu(Y'(\tau)), I'(\tau)]\dot{Y}'^\mu(\tau) = 0, \tag{4.30}$$

therefore:

$$\dot{I}'_1\tau_1 + \dot{I}'_2\tau_2 + \dot{I}'_3\tau_3 - ig\ A^3_\mu\,[\tau_3, I'_1\tau_1 + I'_2\tau_2 + I'_3\tau_3]\dot{Y}'^\mu = 0. \tag{4.31}$$

Thus, we get a set of differential equations of first order:

$$\dot{I}'_1(\tau) - gI'_2(\tau)\, A^3_\mu(Y'(\tau))\, \dot{Y}'^\mu(\tau) = 0,$$
$$\dot{I}'_2(\tau) + gI'_1(\tau)\, A^3_\mu(Y'(\tau))\, \dot{Y}'^\mu(\tau) = 0,$$
$$\dot{I}'_3(\tau) = 0,$$

with initial conditions:

$$\begin{aligned} I'_1(0) &= I'_1, & I'_2(0) &= 0, \\ \dot{I}'_1(0) &= 0, & \dot{I}'_2(0) &= -\rho I'_1(0), \\ I'_3(0) &= 0, & \dot{I}'_3(0) &= 0. \end{aligned}$$

where ρ is given by:

$$\rho = g\, A^3_\mu(Y'(0))\dot{Y}'^\mu(0). \tag{4.32}$$

The solution is given by:

$$\begin{aligned} I'_1(\tau) &= I' \cos\rho\tau, \\ I'_2(\tau) &= -I' \sin\rho\tau, \\ I'_3(\tau) &= 0. \end{aligned}$$

The Lorentz equation (4.15), yields for the second particle:

$$m\ddot{Y}'^\mu(\tau) = g\, Tr[I'(\tau)F^{\mu\nu}(Y'(\tau))]\, \dot{Y}'_\nu(\tau), \tag{4.33}$$

where $F_{\mu\nu}$ is given by the potential A_μ above, which refers to the potential produced by a source, and would be denoted by A^E_μ. Thus:

$$\ddot{Y}'^\mu(\tau) = 0. \tag{4.34}$$

The source S' thus remains stationary, and the isospin precesses in the 1-2 plane, tracing a circle.

(2)′ For an infinitely heavy monopole at the origin, with $K(0) = K_3\tau_3$ initially, we have according to (4.2), the patched potential:

$$\begin{aligned} A_1^{(N)} &= \tfrac{t_3}{2g}\tfrac{y}{r(r+z)}, \quad A_2^{(N)} = \tfrac{t_3}{2g}\tfrac{x}{r(r+z)}, \quad A_3^{(N)} = 0, \quad A_0^{(N)} = 0 \\ A_1^{(S)} &= -\tfrac{t_3}{2g}\tfrac{y}{r(r-z)}, \quad A_2^{(S)} = \tfrac{t_3}{2g}\tfrac{x}{r(r-z)}, \quad A_3^{(S)} = 0, \quad A_0^{(S)} = 0, \end{aligned} \tag{4.35}$$

where, as can be seen, the potential and the field $F_{\mu\nu}$ point in the 3-rd direction in isospin space:

$$A_\mu(x) = A_\mu^3(x)\tau_3, \qquad F_{\mu\nu}(x) = F_{\mu\nu}(x)\tau^3, \qquad {}^*F_{\mu\nu}(x) = {}^*F_{\mu\nu}^3(x)\tau_3. \tag{4.36}$$

Relation (4.11) implies:

$$\dot{K}^{(N,S)} = 0, \tag{4.37}$$

where K and other quantities referred to in this section, must be defined in the (N) and (S) patches separately, and are related to one another through a patching transformation in the overlap region. Thus:

$$K(\tau) = K_3\tau_3. \tag{4.38}$$

For M', which is assumed to be initially stationary, we have $K'(0) = K_1'\tau_1$, and:

$$\dot{K}'(\tau) - ig[A_\mu(Y(\tau)), K'(\tau)]\dot{Y}'^\mu(\tau) = 0. \tag{4.39}$$

We now calculate the isospin of M'. Let:

$$K'(\tau) = K_1'(\tau)\tau_1 + K_2'(\tau)\tau_2 + K_3'(\tau)\tau_3. \tag{4.40}$$

Let us now determine the coefficients K_1', K_2' and K_3', by substituting the above into (4.11):

$$\dot{K}_1'\tau_1 + \dot{K}_2'\tau_2 + \dot{K}_3'\tau_3 - ig\,A_\mu^3[\tau_3, K_1'\tau_1 + K_2'\tau_2 + K_3'\tau_3]\dot{Y}'^\mu = 0, \tag{4.41}$$

and therefore, we get the three first-order differential equations below:

$$\begin{aligned} \dot{K}_1'(\tau) - gK_2'(\tau)A_\mu^3(Y'(\tau))\dot{Y}'^\mu(\tau) &= 0, \\ \dot{K}_2'(\tau) + gK_1'(\tau)A_\mu^3(Y'(\tau))\dot{Y}'^\mu(\tau) &= 0, \\ \dot{K}_3'(\tau) &= 0. \end{aligned} \tag{4.42}$$

The initial conditions are analogous to those for Case (2):

$$\begin{aligned} K_1'(0) &= K_1', & K_2'(0) &= 0, \\ \dot{K}_1'(0) &= 0, & \dot{K}_2'(0) &= -\rho K_1'(0), \\ K_3'(0) &= 0, & \dot{K}_3'(0) &= 0. \end{aligned}$$

Thus, we note that:

$$K_3'(\tau) = K_3'(0) = 0. \tag{4.43}$$

We also note that (4.10) for M' yields:

$$m\ddot{Y}_\mu'(\tau) = \tilde{g}\, Tr[{}^*F_{\mu\nu}(Y'(\tau))K'(\tau)]\dot{Y}'^{\nu}(\tau) = 0, \tag{4.44}$$

since $K'(\tau)$ is in the 1-2 plane in isospin space and ${}^*F_{\mu\nu}'$ in the 3-plane, where we have used $Tr[\tau_i\tau_j] = \delta_{ij}$. Thus, M' remains stationary.

The solution to the differential equations above is given by:

$$\begin{aligned} K_1'(\tau) &= K_1' \cos\lambda\tau, \\ K_2'(\tau) &= -K_1' \sin\lambda\tau, \\ K_3'(\tau) &= 0, \end{aligned}$$

where here, $\lambda = g\, A_\mu^{3M}(Y'(0))\dot{Y}'^{\mu}(0)$ is given in terms of the potential (4.2), and is a constant in τ, since M' is stationary. We thus see that the second particle M' is stationary, whereas its isospin precesses along a circle.

Thus, the behaviour of the source and particle in Cases (2) and (2)′ are indistinguishable qualitatively. The only difference is that the two isospins precess with different frequencies given by ρ and λ respectively. This however, is an unobservable effect.

Therefore, so far, we have not been able to find an observable difference between the behaviour of classical sources *vs* charges.

(3)-(3)′ One other case which is a slight generalisation of (2) and (2)′, is the case where the second particle is not stationary, but has a non-zero initial velocity.

In this case, the differential equations for $I(\tau)$ and $K(\tau)$ cannot be integrated as above, and more care must be taken. The solutions for the source are then given by:

$$\begin{aligned} I_1'(\tau) &= I_1' \cos g \int_0^\tau d\tau' A_\mu^E(Y'(\tau'))\dot{Y}'^\mu(\tau'), \\ I_2'(\tau) &= -I_2' \sin g \int_0^\tau d\tau' A_\mu^E(Y'(\tau'))\dot{Y}'^\mu(\tau'), \end{aligned}$$

and for the case of monopoles:

$$\begin{aligned} K_1'(\tau) &= K_1' \cos g \int_0^\tau d\tau' A_\mu^M(Y'(\tau'))\dot{Y}'^\mu(\tau'), \\ K_2'(\tau) &= -K_2' \sin g \int_0^\tau d\tau' A_\mu^M(Y'(\tau'))\dot{Y}'^\mu(\tau'). \end{aligned}$$

Therefore, once again, there are no observable consequences.

It must be realised that the particles in nature do not follow the classical laws and there are in fact very few situations where these laws can be applied with sufficient accuracy. We thus turn our attention to quantum monopoles in the next chapter and explore their dynamics, while continuing the comparison with source interactions.

Chapter 5

Feynman Diagrams for Monopole-Monopole Scattering

5.1 Path-Integral Formulation

In the preceding chapter, we concentrated on the interactions of classical monopoles, comparing them to classical source interactions, with the view of finding a situation where the predictions of the two sets of equations for monopoles and sources would differ. However, as is well-known, there are few situations in particle physics where the classical laws apply with sufficient accuracy. In order to study physical systems, one has to apply the laws of quantum mechanics, and in our case, those of quantum mechanics of monopoles. Now the full quantisation of the present system would be an enormous problem and is way beyond the scope of this thesis. Besides being entirely new, the system involves the double complication of being highly constrained and formulated in terms of loop variables. What we can do is merely to explore the consequences a little, by evaluating the simplest Feynman diagrams.

It is our aim in this chapter to calculate the simplest Feynman diagram for the scattering process involving two monopoles represented by the wavefunction ψ.

After accomplishing this goal, the results of the calculation will be compared with the analogous quark-quark scattering calculation in QCD, in the same manner as in the classical case.

We therefore proceed to study the quantum action, subject to the constraints (2.9) and (3.60), using the Feynman path-integral approach. The procedure applied has two rather unusual features. One is the loop-space formalism, which gives rise to some operations requiring delicate handling. The other is the occurrence of the two-indexed potential $L_{\mu\nu}$, which transforms under $\tilde{U}$-transformations and needs to be gauge-fixed in a special manner. We shall show how these difficulties can be resolved. We begin by writing down the free quantum action:

$$\mathcal{A}_F = \int d^4x\, \bar{\psi}(x)(i\partial_\mu\gamma^\mu - m)\psi(x) + \int \delta\xi\, ds\, a_\xi(s)\, Tr\{F_\mu[\xi \mid s]F^\mu[\xi \mid s]\}, \quad (5.1)$$

where $\psi(x)$ is the wavefunction of the monopole, and where we have introduced the notation $a_\xi(s) = -\frac{1}{4\pi N}\dot{\xi}^{-2}(s)$. In order to write down the path-integral, the constraints $G_{\mu\nu}[\xi \mid s] = J_{\mu\nu}[\xi \mid s]$ and $F_\mu[\xi \mid s]\dot{\xi}^\mu(s) = 0$ have to be incorporated as infinite-dimensional δ-functions:

$$\begin{aligned} FPI \;=\; & \int \delta F \delta\psi\, \delta\bar{\psi} \prod_{[\xi \mid s]} \prod_{\mu\nu} \delta\left(G^{\mu\nu}[\xi \mid s] - J^{\mu\nu}[\xi \mid s]\right) \\ & \times \prod_{[\xi' \mid s']} \delta\left(F_\rho[\xi' \mid s']\dot{\xi}'^\rho(s')\right) \exp i\mathcal{A}_F, \end{aligned} \quad (5.2)$$

where there is one δ-function for each value of μ, ν, ρ and $[\xi \mid s], [\xi' \mid s']$ and:

$$J_{\mu\nu}[\xi \mid s] = 4\pi\tilde{g}\epsilon_{\mu\nu\rho\sigma}\left[\bar{\psi}(\xi(s))\Omega_\xi(s,0)\gamma^\rho T_i \Omega_\xi^{-1}(s,0)\psi(\xi(s))\right]\tau^i\dot{\xi}^\sigma(s), \quad (5.3)$$

is the quantum analogue of the monopole current. [1] This is (3.60), with $G_{\mu\nu} = J_{\mu\nu}$. Note that the third constraint condition represented by (A) in §2.3, is already

[1] Here, in changing variables from A_μ to F_μ, we must remember that the measure in the path-integral must change. The Jacobian factor is rather difficult to compute, since the matrix of transformation is continuous. Another source of difficulty is that the matrix is not "square", since to every possible F_μ correspond a set of A_μ's, related by gauge transformations, and therefore, there is no one-to-one correspondence between the two sets of variables. Efforts are underway to calculate the Jacobian, but the problem will not be considered in this work.

incorporated in the notation $[\xi \mid s]$ for the argument of the loop variables. The infinite product can be written as:

$$\prod_{[\xi \mid s]} \prod_{\mu\nu} \delta[G_{\mu\nu}[\xi \mid s] - J_{\mu\nu}[\xi \mid s]] \prod_{[\xi' \mid s']} \delta[F_\mu[\xi' \mid s']\dot{\xi}'^\mu(s')]$$
$$= \mathcal{N} \int \delta L \exp i \int \delta\xi \, ds \, Tr \{L_{\mu\nu}[\xi \mid s](G^{\mu\nu}[\xi \mid s] - J^{\mu\nu}[\xi \mid s])\}$$
$$\int \delta b \exp i \int \delta\xi \, ds \, Tr \{b[\xi \mid s]F_\mu[\xi \mid s]\}\dot{\xi}^\mu(s),$$

where $L_{\mu\nu}[\xi \mid s]$ and $b[\xi \mid s]$ are the Lagrange multipliers for the topological constraint and the transversality condition respectively and $\mathcal{N}$ is an infinite factor originating in the definition of the δ-function in terms of the integral of an exponential. Substituting this into equation (5.2) yields:

$$\begin{aligned} FPI &= \mathcal{N} \int \delta F \, \delta\psi \, \delta\bar{\psi} \, \delta L \, \delta b \\ &\exp i \left\{ \mathcal{A}_F + \int \delta\xi \, ds \, Tr \left[L_{\mu\nu}(G^{\mu\nu} - J^{\mu\nu}) + bF_\mu\dot{\xi}^\mu \right] \right\}, \end{aligned} \tag{5.4}$$

as the path-integral, with $G_{\mu\nu}$ given by:

$$G_{\mu\nu}[\xi \mid s] = \delta_\nu F_\mu[\xi \mid s] - \delta_\mu F_\nu[\xi \mid s] + ig\,[F_\mu[\xi \mid s], F_\nu[\xi \mid s]]. \tag{5.5}$$

To calculate Feynman diagrams, we first need to obtain the generating functional with external currents from the path-integral (5.4). The generating functional is given by:

$$Z = \int \delta F \delta L \delta\psi \delta\bar{\psi} \exp i\mathcal{A}, \tag{5.6}$$

where $\mathcal{A}$ is the action including terms for "external" sources for each field and Lagrange multiplier introduced:

$$\begin{aligned} \mathcal{A} &= \int \delta\xi \, ds \, a_\xi(s) \, Tr \{F_\mu[\xi \mid s]F^\mu[\xi \mid s]\} + \int d^4x \, \bar{\psi}(x) \, (i\partial_\mu\gamma^\mu - m) \, \psi(x) \\ &+ \int \delta\xi \, ds \, Tr \{b[\xi \mid s]F^\mu[\xi \mid s]\}\dot{\xi}_\mu(s) + \int \delta\xi \, ds \, Tr \{\mathcal{J}_{\mu\nu}[\xi \mid s]L^{\mu\nu}[\xi \mid s]\} \\ &+ \int \delta\xi \, ds \, Tr \{j_\mu[\xi \mid s]F^\mu[\xi \mid s]\} + \int \delta\xi \, ds \, Tr \{j[\xi \mid s]b[\xi \mid s]\} \\ &+ \int \delta\xi \, ds \, Tr \{L_{\mu\nu}[\xi \mid s][G^{\mu\nu}[\xi \mid s] - J^{\mu\nu}[\xi \mid s]]\}. \end{aligned} \tag{5.7}$$

The action $\mathcal{A}$ is thus the sum of three terms:

$$\mathcal{A} = \mathcal{A}^0 + \mathcal{A}_{constraint} + \mathcal{A}_{external}, \tag{5.8}$$

where the terms denote respectively, the free action (3.26), the constraint term action:

$$\int \delta\xi \, ds \, Tr \{bF_\mu\}\dot{\xi}^\mu + \int \delta\xi \, ds \, Tr \{L_{\mu\nu}(G^{\mu\nu} - J^{\mu\nu})\}, \tag{5.9}$$

and the external source action:

$$\mathcal{A}_{external} = \int \delta\xi \, ds \, Tr \{\mathcal{J}_{\mu\nu}L^{\mu\nu}\} + \int \delta\xi \, ds \, Tr \{j_\mu F^\mu\} + \int \delta\xi \, ds \, Tr \{jb\}. \tag{5.10}$$

In simplifying the expression for Z, we follow the usual procedure of retaining terms of $\mathcal{A}$ up to order 2 in the fields, "completing the square" and then integrating these fields out in Z. this will leave us with an expression for Z in terms of the "effective action", containing the external currents only:

$$Z = \exp i\mathcal{A}_{eff}. \tag{5.11}$$

To begin, we consider the terms involving $F_\mu[\xi \,|\, s]$:

$$\mathcal{A}_{F_\mu} = \int \delta\xi \, ds \left\{a_\xi F^i_\mu F^\mu_i + b_i F^i_\mu \dot{\xi}^\mu + j^i_\mu F^\mu_i + L^i_{\mu\nu}\left[\delta^\nu F^\mu_i - \delta^\mu F^\nu_i\right]\right\}, \tag{5.12}$$

where we have made use of:

$$Tr\,(AB) = A_i B^i, \tag{5.13}$$

and where the commutator term of the form $L_{\mu\nu}[ig\,[F^\mu, F^\nu] - J^{\mu\nu}]$ has been dropped, since it is of order 3 in the fields.

Performing integration by parts on the last term in (5.12) and using the antisymmetry property of $L_{\mu\nu}$, we arrive at:

$$\int \delta\xi \, ds \, \{a_\xi F^i_\mu F^\mu_i + P^i_\mu F^\mu_i\}, \tag{5.14}$$

where $P^i_\mu = b^i\dot{\xi}_\mu + j^i_\mu - 2\delta^\nu L^i_{\mu\nu}$. Let us complete the square of the integrand above:

$$\begin{aligned} a_\xi F^i_\mu F^\mu_i + P^i_\mu F^\mu_i &= a_\xi \left[F^i_\mu F^\mu_i + \frac{P^i_\mu F^\mu_i}{a_\xi} \right] \\ &= a_\xi \left[F^i_\mu F^\mu_i + \frac{P^i_\mu F^\mu_i}{a_\xi} + \frac{P^i_\mu P^\mu_i}{4a^2_\xi} \right] - \frac{P^i_\mu P^\mu_i}{4a_\xi} \\ &= a_\xi \left[\left(F^i_\mu + \frac{P^i_\mu}{2a_\xi} \right) \left(F^\mu_i + \frac{P^\mu_i}{2a_\xi} \right) \right] - \frac{P^i_\mu P^\mu_i}{4a_\xi}. \end{aligned} \tag{5.15}$$

We now define:

$$F'^i_\mu = \sqrt{a_\xi} \left[F^i_\mu + \frac{P^i_\mu}{2a_\xi} \right], \tag{5.16}$$

and substitute the result in the generating functional (5.6), where $\delta F' = \sqrt{a_\xi}\delta F$. The part of the generating functional involving F_μ is thus:

$$\begin{aligned} Z_{F_\mu} &= \int \delta F' \, \delta L \, \delta b \, \mathcal{J}_{F'F} \exp i \int \delta\xi \, ds \, \left[a_\xi F'^i_\mu F'^\mu_i \right] \exp i \int \delta\xi \, ds \\ &\quad \left\{ -\frac{1}{4a_\xi} \left[b^i\dot{\xi}_\mu + j^i_\mu - 2\delta^\nu L^i_{\mu\nu} \right] \left[b_i\dot{\xi}^\mu + j^\mu_i - 2\delta_\rho L^{\mu\rho}_i \right] \right\}. \end{aligned} \tag{5.17}$$

The integration over $\delta F'$ can now be performed, where we neglect the Jacobian $\mathcal{J}_{F'F}(\xi; s)$ with the result:

$$\begin{aligned} Z_{F_\mu} &\propto \int \delta L \, \delta b \exp i \int \delta\xi \, ds \\ &\quad \left\{ -\frac{1}{4a_\xi} \left[b^i\dot{\xi}_\mu + j^i_\mu - 2\delta^\nu L^i_{\mu\nu} \right] \left[b_i\dot{\xi}^\mu + j^\mu_i - 2\delta_\rho L^{\mu\rho}_i \right] \right\}, \end{aligned} \tag{5.18}$$

where the proportionality factor will be seen to be of no importance for our future discussions.

The next step would be to take the terms involving $L_{\mu\nu}$, and applying the same technique to complete the square. The relevant terms are:

$$\mathcal{A}_L = -\frac{1}{4a_\xi(s)} \left[-4(\delta^\nu L^i_{\mu\nu}) b_i\dot{\xi}^\mu - 4(\delta^\nu L^i_{\mu\nu}) j^\mu_i + 4(\delta^\nu L^i_{\mu\nu})(\delta_\rho L^{\mu\rho}_i) \right] + \mathcal{J}^i_{\mu\nu} L^{\mu\nu}_i. \tag{5.19}$$

However, in order to write the above as a complete square, we need the inverse of the operator $\delta^\nu\delta_\rho$. This operator however, does not possess an inverse, since:

$$(\delta^\nu\delta_\rho)(\delta^\rho\delta_\pi) = \Box\delta^\nu\delta_\pi, \tag{5.20}$$

and thus, it can be seen that the operator $\delta^\nu\delta_\rho$ is proportional to a projection operator, which is not invertible. This is much the same as in the case of electromagnetism, where the action:

$$\mathcal{A}_{e.m.} = \int d^4x\, f_{\mu\nu}(x) f^{\mu\nu}(x) = -2 \int d^4x\, a_\mu(x)(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)\, a_\nu(x), \qquad (5.21)$$

involves a quadratic term in a_μ and its integrand is proportional to a projection operator, since:

$$(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)(g_{\nu\lambda}\Box - \partial_\nu\partial_\lambda) = \Box(\delta^\mu_\lambda\Box - \partial^\mu\partial_\lambda), \qquad (5.22)$$

and hence the operator $g^{\mu\nu}\Box - \partial^\mu\partial^\nu$ does not possess an inverse. To make the operator in the quadratic term invertible, one thus needs to "fix" a gauge. This has the effect of eliminating the "non-physical" degrees of freedom from the action. The usual choice in the case of electromagnetism is to impose the Lorentz gauge condition $\partial_\mu a^\mu = 0$. Hence, the action reduces to:

$$\mathcal{A}_{e.m.} = -2 \int d^4x\, a_\mu \Box\, a^\mu, \qquad (5.23)$$

where the operator $\Box$ can be inverted with the inverse being proportional to the propagator for a_μ.

A similar operation can be performed for our case, by making use of the $\tilde{U}$-symmetry of the monopole theory. This will be the subject of the next section.

5.2 Gauge-Fixing and Ghosts

In this section, we attempt to fix the $\tilde{U}$-gauge for $L_{\mu\nu}$ and in doing so, remove the non-physical degrees of freedom from $L_{\mu\nu}$ and eliminate the problem with with completing the square mentioned in the previous section. To do this, the following must be demonstrated:

(i) that the addition of the gauge-fixing term to the action will make the operator in the quadratic term in L, invertible;

(*ii*) that the condition thus imposed is indeed a gauge condition: that is, given an arbitrary $L_{\mu\nu}$, it is always possible to define an $L'_{\mu\nu}$ related to $L_{\mu\nu}$ by a $\tilde{U}$-gauge transformation, such that $L'_{\mu\nu}$ satisfies the gauge-fixing condition.

It must be stated that the gauge-fixing of a similar 2-indexed potential has been studied in the context of the non-linear σ-model, by Demarco, Fosco and Trinchero, and Clark, Lee and Love [10, 13]. They calculate the propagators in the theory, obtaining results parallel to ours.

We now proceed to "fix the gauge". Let us start by examining the action:

$$\begin{aligned}\mathcal{A}' &= \int \delta\xi\, ds\, a_\xi\, Tr\,\{F_\mu F^\mu\} + \int d^4x\, \bar{\psi}\,(i\partial_\mu\gamma^\mu - m)\,\psi \\ &\quad + \int \delta\xi\, ds\, Tr\,\{L^{\mu\nu}[G_{\mu\nu} - J_{\mu\nu}]\} + \int \delta\xi\, ds\, Tr\,\{bF_\mu\}\dot{\xi}^\mu, \qquad (5.24)\end{aligned}$$

where:

$$J_{\mu\nu}[\xi \mid s] = 4\pi\tilde{g}\,\epsilon_{\mu\nu\rho\sigma}\left[\bar{\psi}(\xi(s))\Omega_\xi(s,0)\gamma^\rho\dot{\xi}^\sigma(s)\tau_i\Omega_\xi^{-1}(s,0)\psi(\xi(s))\right]\tau^i, \qquad (5.25)$$

which is (5.3).

We write the action in terms of a new set of "hatted" variables:

$$\begin{aligned}Tr\,\{L^{\mu\nu}J_{\mu\nu}\} &= 4\pi\tilde{g}\,\epsilon_{\mu\nu\rho\sigma}\left(\bar{\psi}\Omega_\xi\gamma^\rho\dot{\xi}^\sigma L^{\mu\nu}\Omega_\xi^{-1}\psi\right) \\ &= 4\pi\tilde{g}\,\epsilon_{\mu\nu\rho\sigma}\left(\bar{\psi}\Omega_\xi\Phi^{-1}\Phi L^{\mu\nu}\Phi^{-1}\Phi\Omega_\xi^{-1}\dot{\xi}^\sigma\gamma^\rho\psi\right) \\ &= 4\pi\tilde{g}\,\epsilon_{\mu\nu\rho\sigma}\left(\bar{\psi}\hat{\Omega}_\xi\hat{L}^{\mu\nu}\hat{\Omega}_\xi^{-1}\gamma^\rho\dot{\xi}^\sigma\psi\right), \qquad (5.26)\end{aligned}$$

where $\hat{\Omega}_\xi = \Omega_\xi\Phi^{-1}$ and $\hat{L}_{\mu\nu} = \Phi L_{\mu\nu}\Phi^{-1}$. Similarly:

$$\begin{aligned}Tr\,\{bF_\mu\} &= Tr\left\{\Phi b\Phi^{-1}\Phi F_\mu\Phi^{-1}\right\} = Tr\left\{\hat{b}\hat{F}_\mu\right\}, \\ Tr\,\{L^{\mu\nu}G_{\mu\nu}\} &= Tr\left\{\Phi L^{\mu\nu}\Phi^{-1}\Phi G_{\mu\nu}\Phi^{-1}\right\} \\ &= Tr\left\{\hat{L}^{\mu\nu}\Phi\left[\delta_\nu F_\mu - \delta_\mu F_\nu + ig\,[F_\mu, F_\nu]\right]\Phi^{-1}\right\}. \qquad (5.27)\end{aligned}$$

But since:

$$\begin{aligned}\delta_\nu(\Phi F_\mu\Phi^{-1}) &= (\delta_\nu\Phi)F_\mu\Phi^{-1} + \Phi(\delta_\nu F_\mu)\Phi^{-1} + \Phi F_\mu(\delta_\nu\Phi^{-1}) \\ &= \Phi\delta_\nu F_\mu\Phi^{-1} + ig\,\Phi[F_\mu, F_\nu]\Phi^{-1}, \qquad (5.28)\end{aligned}$$

we can write:

$$Tr\left\{L^{\mu\nu}G_{\mu\nu}\right\} = Tr\left\{\hat{L}_{\mu\nu}\left[\delta^{\nu}\hat{F}^{\mu} - \delta^{\mu}\hat{F}^{\nu} - ig\left[\hat{F}^{\mu},\hat{F}^{\nu}\right]\right]\right\}. \tag{5.29}$$

Note the difference in sign of the term involving the commutator. In terms of these new variables, we have:

$$\begin{aligned}\mathcal{A}' &= \int \delta\xi\, ds\, a_{\xi}\, Tr\left\{\hat{F}_{\mu}\hat{F}^{\mu}\right\} + \int d^4x\, \bar{\psi}\left(i\partial_{\mu}\gamma^{\mu} - m\right)\psi \\ &\quad + \int \delta\xi\, ds\, Tr\left\{\hat{L}_{\mu\nu}\left[\delta^{\nu}\hat{F}^{\mu} - \delta^{\mu}\hat{F}^{\nu} - ig\left[\hat{F}^{\mu},\hat{F}^{\nu}\right] - \hat{J}^{\mu\nu}\right]\right. \\ &\qquad \left. + \left[\hat{b}\hat{F}_{\mu}\right]\dot{\xi}^{\mu}\right\}. \end{aligned} \tag{5.30}$$

Note that strictly speaking, the "hatted" variables no longer depend on $[\xi \,|\, s]$, but on the later part of the loop with $s_{-} \leq s' \leq 2\pi$. This can be seen by observing that $F_{\mu} = -\frac{i}{g}\,\Phi^{-1}\delta_{\mu}\Phi$, and therefore:

$$\hat{F}_{\mu} = \Phi F_{\mu}\Phi^{-1} = -\frac{i}{g}(\delta_{\mu}\Phi)\Phi^{-1}, \tag{5.31}$$

which is a function of the later part of the loop $\xi(s')$ with $s_{-} \leq s' \leq 2\pi$.

The quantity $\hat{\Omega}_{\xi}(s,0)$, is actually independent of $\xi(s)$ and the early part of the loop $[0, s_{+})$, since by (3.57):

$$\begin{aligned}\delta_{\mu}\hat{\Omega}_{\xi}(s,0) &= (\delta_{\mu}(s)\Omega_{\xi}(s,0))\Phi^{-1} + \Omega_{\xi}\delta_{\mu}(s)\Phi^{-1} \\ &= -ig\,\Omega_{\xi}(s,0)F_{\mu}\Phi^{-1} + ig\,\Omega_{\xi}(s,0)F_{\mu}\Phi^{-1} \\ &= 0, \end{aligned} \tag{5.32}$$

and:

$$\begin{aligned}\delta_{\mu}(s')\hat{\Omega}_{\xi}(s,0) &= \delta_{\mu}(s')\left[\omega(\xi(s_{+}))\Phi_{\xi}(s_{+},0)\Phi^{-1}[\xi]\right] \\ &= \omega(\xi(s_{+}))\delta_{\mu}(s')[\Phi_{\xi}(2\pi,s_{+})] \\ &= 0, \end{aligned} \tag{5.33}$$

for $s' < s_{+}$. This property of $\hat{\Omega}_{\xi}$ will be found to be useful in what follows.

Let us recall that the problem in completing the square in the generating functional was due to the presence of the term $\delta^\nu L^i_{\mu\nu}\delta_\rho L^{\mu\rho}_i$ in the action. We therefore attempt to "fix a gauge" in which this term is not present. Let us choose the gauge condition to be:

$$C_\mu(L) = \epsilon_{\mu\nu\rho\sigma}\delta^\nu(s)L^{\rho\sigma}[\xi\,|\,s] = 0. \tag{5.34}$$

In this gauge, the transverse degrees of freedom of L do not propagate and only the longitudinal ones are physical.

We now show that it is always possible to "choose a gauge" in which (5.34) is satisfied. To see this, given any arbitrary $L_{\mu\nu}[\xi\,|\,s]$, let:

$$L'_{\mu\nu}[\xi\,|\,s] = L_{\mu\nu}[\xi\,|\,s] + \tilde{\Delta}L_{\mu\nu}[\xi\,|\,s]. \tag{5.35}$$

Thus, in terms of hatted quantities:

$$\widehat{L}'_{\mu\nu}[\xi\,|\,s] = \widehat{L}_{\mu\nu}[\xi\,|\,s] + \tilde{\Delta}\widehat{L}_{\mu\nu}[\xi\,|\,s], \tag{5.36}$$

since Φ does not transform under a $\tilde{U}$-transformation. We now demand that:

$$\delta_\nu\epsilon^{\mu\nu\rho\sigma}\widehat{L}'_{\rho\sigma} = \delta_\nu\epsilon^{\mu\nu\rho\sigma}\widehat{L}_{\rho\sigma} + \delta_\nu\epsilon^{\mu\nu\rho\sigma}\tilde{\Delta}\widehat{L}_{\rho\sigma} = 0. \tag{5.37}$$

It can be shown that:

$$\epsilon^{\mu\nu\rho\sigma}\delta_\nu\tilde{\Delta}\widehat{L}_{\rho\sigma} = -2\,\delta_\nu\left[\Phi(\mathcal{D}^\mu\Lambda^\nu - \mathcal{D}^\nu\Lambda^\mu)\Phi^{-1}\right]. \tag{5.38}$$

But:

$$\Phi\mathcal{D}^\mu\Lambda^\nu\Phi^{-1} = \delta^\mu(\Phi\Lambda^\nu\Phi^{-1}) = \delta^\mu\widehat{\Lambda}^\nu. \tag{5.39}$$

Substituting in (5.38), we obtain:

$$\epsilon^{\mu\nu\rho\sigma}\delta_\nu\widehat{L}_{\rho\sigma} + \epsilon^{\mu\nu\rho\sigma}\delta_\nu\tilde{\Delta}\widehat{L}_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\delta_\nu\widehat{L}_{\rho\sigma} - 2\,(\delta_\nu\delta^\mu\widehat{\Lambda}^\nu - \delta_\nu\delta^\nu\widehat{\Lambda}^\mu) = C'^\mu(\widehat{L}^\Lambda). \tag{5.40}$$

The left-hand side of this equation is just a function of the loop, and $C'^\mu(\widehat{L}^\Lambda)$ is the gauge-transformed gauge-fixing condition for $\widehat{L}$. In order to show that the

gauge condition can always be satisfied, we need to show that the equation (5.37) always has a solution for $\widehat{\Lambda}_\mu$ for a given $\widehat{L}_{\mu\nu}$. As it stands, this is not obvious. However, we note that we are still free to "choose a gauge" for $\widehat{\Lambda}_\mu$, since under:

$$\widehat{\Lambda}_\nu \to \widehat{\Lambda}'_\nu = \widehat{\Lambda}_\nu + \delta_\nu \widehat{\lambda}, \tag{5.41}$$

$\widetilde{\Delta}\widehat{L}_{\mu\nu}$ does not change:

$$\widetilde{\Delta}\widehat{L}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\delta^\rho(\widehat{\Lambda}^\sigma + \delta^\sigma\widehat{\lambda}) = \epsilon_{\mu\nu\rho\sigma}\delta^\rho\widehat{\Lambda}^\sigma, \tag{5.42}$$

and we can choose $\widehat{\lambda}$ such that $\delta_\nu\widehat{\Lambda}'^\nu = \delta_\nu\widehat{\Lambda}^\nu + \delta_\nu\delta^\nu\widehat{\lambda} = 0$. Thus, (5.40) can always be put in the form:

$$f^\mu = \Box_{\xi(s)}\widehat{\Lambda}^\mu, \tag{5.43}$$

where the box $\Box_{\xi(s)}$ refers to the loop d'Alembertian:

$$\Box_{\xi(s)} = \frac{\delta^2}{\delta\xi_\mu(s)\delta\xi^\mu(s)}. \tag{5.44}$$

(5.43), apart from being a loop equation, is identical to the usual d'Alembertian equation and must therefore have a unique solution for $\widehat{\Lambda}_\mu$, given the appropriate boundary conditions. Thus, (5.34) can always be fulfilled, given an arbitrary $L_{\mu\nu}$.

So far, we have succeeded in showing that it is always possible to choose a gauge for the tensor potential $L_{\mu\nu}$, such that (5.34) is satisfied. Usually, the choice of gauge is incorporated in the action in the form of a constraint. In this particular case, we choose:

$$\mathcal{A}_{gauge\ fixing} = \int \delta\xi\, ds\, \frac{1}{2\alpha(s)}\,\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\,Tr\left\{\delta^\nu\widehat{L}^{\rho\sigma}[\xi\,|\,s]\delta_\alpha\widehat{L}_{\beta\gamma}[\xi\,|\,s]\right\}, \tag{5.45}$$

which is basically the square of the gauge condition (5.37). This is analogous to the suppression factor used in the action in electromagnetism.

At this point, it is necessary to examine the ghosts of the theory that are introduced as a result of gauge-fixing. We note that two sets of ghosts are anticipated, since the gauge-fixing is done in two stages. To begin, we consider the

generating functional without external currents:[2]

$$\begin{aligned} Z &= \int \delta\hat{b}\,\delta\hat{L}\,\delta\hat{F}\,\delta\psi\,\delta\bar{\psi}\,\exp i\int d^4x\,\bar{\psi}\,(i\partial_\mu\gamma^\mu - m)\,\psi \\ &\quad \exp i\int \delta\xi\,ds\,Tr\left\{a_\xi\,\hat{F}_\mu\hat{F}^\mu + \hat{b}\hat{F}_\mu\dot{\xi}^\mu\right. \\ &\quad \left. +\hat{L}_{\mu\nu}\left[\delta^\nu\hat{F}^\mu - \delta^\mu\hat{F}^\nu - ig\left[\hat{F}^\mu,\hat{F}^\nu\right] - \hat{J}^{\mu\nu}\right]\right\}. \quad (5.46) \end{aligned}$$

Under a $\tilde{U}$-transformation, we saw in Chapter 3, that the variation of the Dirac term for ψ cancels that of the $L_{\mu\nu}J^{\mu\nu}$ term and thus, the action remains invariant. We therefore concentrate on the remaining terms:

$$\begin{aligned} Z_{F_\mu} &= \int \delta\hat{b}\,\delta\hat{L}\,\delta\hat{F}\,\exp iC^2\,\exp i\int \delta\xi\,ds \\ &\quad Tr\left[a_\xi\,\hat{F}_\mu\hat{F}^\mu + \hat{L}_{\mu\nu}\left(\delta^\nu\hat{F}^\mu - \delta^\mu\hat{F}^\nu - ig\left[\hat{F}^\mu,\hat{F}^\nu\right]\right) + \hat{b}\hat{F}_\mu\dot{\xi}^\mu\right], \quad (5.47) \end{aligned}$$

where

$$C^2 = \int \delta\xi\,ds\,\frac{1}{2\alpha(s)}\,\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\alpha\beta\gamma}\,Tr\left[\delta^\nu\hat{L}^{\rho\sigma}\,\delta_\alpha\hat{L}_{\beta\gamma}\right], \quad (5.48)$$

is the suppression factor already mentioned. Thus, a Faddeev-Popov determinant Δ_1 is introduced as a result of gauge-fixing:

$$\Delta_1 = \int \delta\hat{\eta}\,\delta\hat{\bar{\eta}}\,\exp i\int \delta\xi\,ds\,Tr\left\{\hat{\bar{\eta}}_\mu[\xi\,|\,s]\left(\frac{\delta C^{\prime\mu}(\hat{L}^\Lambda)}{\delta\hat{\Lambda}^\nu}\right)\hat{\eta}^\nu[\xi\,|\,s]\right\}, \quad (5.49)$$

where $\hat{\eta}$ and $\hat{\bar{\eta}}$ are two independent vector-valued Grassmann variables, depending on the later part of the loop. Here, by $\left(\frac{\delta C^{\prime\mu}(\hat{L}^\Lambda)}{\delta\hat{\Lambda}^\nu}\right)$, we mean the functional derivative of (5.40) with respect to $\hat{\Lambda}^\nu$. Thus:

$$\begin{aligned} Z &= \int \delta\hat{L}\,\delta\hat{b}\,\delta\hat{F}\,\delta\hat{\eta}\,\delta\hat{\bar{\eta}} \\ &\quad \exp i\int \delta\xi\,ds\,Tr\left\{a_\xi\,\hat{F}_\mu\hat{F}^\mu + \hat{L}_{\mu\nu}\left[\delta^\nu\hat{F}^\mu - \delta^\mu\hat{F}^\nu - ig\,[F^\mu,\hat{F}^\nu]\right] + \hat{b}\hat{F}_\mu\dot{\xi}^\mu\right. \\ &\quad \left. +(2\alpha)^{-1}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\alpha\beta\gamma}\,\delta^\nu\hat{L}^{\rho\sigma}\,\delta_\alpha\hat{L}_{\beta\gamma} + 2\hat{\bar{\eta}}_\mu(g^{\mu\nu}\Box_{\xi(s)} - \delta^\mu\delta^\nu)\hat{\eta}_\nu\right\}. \quad (5.50) \end{aligned}$$

We see the appearance of the non-invertible operator $g^{\mu\nu}\Box_{\xi(s)} - \delta^\mu\delta^\nu$ once again. In order to eliminate the term $\hat{\bar{\eta}}_\mu\delta^\mu\delta^\nu\hat{\eta}_\nu$, we must fix the gauge for a second time,

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[2]The change to hatted variables introduces a Jacobian factor, which will be ignored in what follows.

and must therefore find a second gauge-symmetry of the action. This can be done by writing the last term as:

$$2\int \delta\xi\, ds Tr\left[\hat{\bar{\eta}}_\mu(g^{\mu\nu}\delta_\rho\delta^\rho - \delta^\mu\delta^\nu)\hat{\eta}_\nu\right] = -2\int \delta\xi\, ds Tr\left[(\delta_\nu\hat{\bar{\eta}}_\mu - \delta_\mu\hat{\bar{\eta}}_\nu)(\delta^\nu\hat{\eta}^\mu - \delta^\mu\hat{\eta}^\nu)\right]. \tag{5.51}$$

It is now easy to see that we have a second gauge-symmetry of the Faddeev-Popov term, since under:

$$\hat{\eta}^\mu[\xi\,|\,s] \to \hat{\eta}^\mu[\xi\,|\,s] + \delta^\mu\hat{\lambda}[\xi\,|\,s], \tag{5.52}$$

the action remains invariant. We therefore need to gauge-fix the $\hat{\eta}_\mu$. We choose $\hat{\lambda}$ such that:

$$\delta_\mu\hat{\eta}'^\mu = \delta_\mu\hat{\eta}^\mu + \Box_{\xi(s)}\hat{\lambda} = 0, \tag{5.53}$$

which has a solution for $\hat{\lambda}$, given the boundary conditions. The second gauge-fixing term is therefore:

$$D^2 = \int \delta\xi\, ds\, \frac{1}{2\beta} Tr\left[(\delta_\mu\hat{\bar{\eta}}^\mu)(\delta_\nu\hat{\eta}^\nu)\right]. \tag{5.54}$$

Substituting this into Z yields:

$$\begin{aligned} Z \;=\; & \int \delta\hat{L}\,\delta\hat{b}\hat{F}\,\delta\hat{\bar{\eta}}\,\delta\hat{\phi}_i\,\delta\hat{\bar{\phi}}_i\,\Delta_2 \exp i\int \delta\xi\, ds \\ & Tr\left\{a_\xi\,\hat{F}_\mu\hat{F}^\mu + \hat{b}\hat{F}_\mu\dot{\xi}^\mu + 2\hat{\bar{\eta}}_\mu\Box_{\xi(s)}\hat{\eta}^\mu + (4a_\xi)^{-1}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\alpha\beta\gamma}\delta^\nu\hat{L}_{\rho\sigma}\delta_\alpha\hat{L}_{\beta\gamma}\right. \\ & \left.\hat{L}_{\mu\nu}\left[\delta^\nu\hat{F}^\mu - \delta^\mu\hat{F}^\nu - ig\left[\hat{F}^\mu,\hat{F}^\nu\right]\right]\right\}, \end{aligned} \tag{5.55}$$

where we have chosen $\beta = -1/4$, such that the $\delta^\nu\hat{\eta}^\nu$ term cancels the gauge-fixing term for the $\hat{\eta}_\nu$, and where we have included the Faddeev-Popov determinant Δ_2 for the second gauge-fixing condition:

$$\Delta_2 = \int \prod_{i=1.2} \delta\hat{\phi}\,\delta\hat{\bar{\phi}}_i \exp i\int \delta\xi\, ds\, \frac{1}{2\beta} Tr\left\{\hat{\bar{\phi}}[\xi\,|\,s]\Box_{\xi(s)}\hat{\phi}[\xi\,|\,s]\right\}. \tag{5.56}$$

Here, $\hat{\phi}_i$ and $\hat{\bar{\phi}}_i$ for $i = 1, 2$ are four independent commuting fields depending on the later part of the loop.[3]

[3]There is some subtlety involved in counting the number of fields ϕ_i (see [24]), but due to the decoupling of the ghosts from the theory, this subtlety is of no importance for our future discussions.

Thus:

$$\begin{aligned} Z &= \int \delta\hat{L}\,\delta\hat{b}\,\delta\hat{F}\,\delta\hat{\eta}\,\delta\hat{\bar{\eta}}\,\delta\hat{\phi}\,\delta\hat{\bar{\phi}} \\ &\quad \exp i\int \delta\xi\, ds\, Tr\,\Big\{a_\xi\,\hat{F}_\mu\hat{F}^\mu + \hat{b}\hat{F}_\mu\dot{\xi}^\mu + 2\,\hat{\bar{\eta}}_\mu\Box_{\xi(s)}\hat{\eta}^\mu \\ &\quad +\hat{L}_{\mu\nu}\left[\delta^\nu\hat{F}^\mu - \delta^\mu\hat{F}^\nu - ig\,[\hat{F}^\mu,\hat{F}^\nu]\right] - \frac{1}{4}\hat{\bar{\phi}}\,\Box_{\xi(s)}\hat{\phi} \\ &\quad +(4a_\xi)^{-1}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\alpha\beta\gamma}\delta^\nu\hat{L}^{\rho\sigma}\,\delta_\alpha\hat{L}_{\beta\gamma}\Big\}. \end{aligned} \tag{5.57}$$

It is easy to see that the ghosts $\hat{\eta}$, $\hat{\bar{\eta}}$, $\hat{\phi}$ and $\hat{\bar{\phi}}$, can be integrated out and thus decouple from the remaining fields. They therefore disappear from the generating functional, and need not be discussed further. The reason for this decoupling is as follows: although the theory itself is nonabelian in character, the $\tilde{U}$-transformation does not involve a term coupling $\hat{\Lambda}_\mu$ to $\hat{L}_{\mu\nu}$. This is in contrast to the Yang-Mills theory, where the U-gauge transformation:

$$A_\mu \to A_\mu + \partial_\mu\Lambda + ig\,[\Lambda, A_\mu], \tag{5.58}$$

contains the term $[\Lambda, A_\mu]$. In this case, we get the coupling term:

$$\int \delta\eta\,\delta\bar{\eta}\,\exp ig\int d^4x\, f_{abc}\left[\bar{\eta}^a\partial^\mu\eta^b A^c_\mu + \partial^\mu A^c_\mu(\bar{\eta}^a\eta^b)\right]. \tag{5.59}$$

In the following, we thus assume that the ghost fields have been integrated out, and proceed to calculate the propagators of the theory.

5.3 Calculation of Propagators

As already discussed, in order to calculate the propagators for the various fields appearing in the action $\mathcal{A}$ with external sources, the usual procedure is to consider only the terms in Z, which are of order two and lower in the fields and then integrate them out by "completing the square" in the generating functional, such that in the end, the only remaining terms are ones involving external currents. The propagator for a field $H(x)$ is then defined to be:

$$\langle H(x)H(y)\rangle = \frac{1}{i}\frac{\delta}{\delta J(x)}\frac{1}{i}\frac{\delta}{\delta J(y)}\,Z[J]\,|_{J=0}, \tag{5.60}$$

where the notation $\langle AB \rangle$ indicates the time-ordered vacuum expectation value $\langle 0 \,|\, T\,(AB) \,|\, 0 \rangle$, $Z[J]$ the generating functional involving the external currents only, and x and y the arguments of the field H including internal indices, as well as any spinor or Lorentz indices that are necessary in order to specify the field H.

We now apply the above to the case of the quantum monopole. Following the procedure, we recognise four sets of propagators:

$$\langle \psi(x)\bar{\psi}(y) \rangle,\ \langle \widehat{F}^i_\mu[\xi \,|\, s]\widehat{F}^j_\nu[\xi' \,|\, s'] \rangle,\ \langle \widehat{L}^i_{\mu\nu}[\xi \,|\, s]\widehat{L}^j_{\rho\sigma}[\xi' \,|\, s'] \rangle,\ \langle \widehat{b}^i[\xi \,|\, s]\widehat{b}^j[\xi' \,|\, s'] \rangle. \quad (5.61)$$

There are also mixed propagators of the type:

$$\langle \widehat{F}^i_\mu[\xi \,|\, s]\widehat{L}^j_{\rho\sigma}[\xi' \,|\, s'] \rangle, \ldots \quad (5.62)$$

but these are not important for our future calculations of Feynman diagrams, and will not be discussed.

The notation $\widehat{F}^i_\mu[\xi \,|\, s]$ needs some explanation. We define it to be:

$$\begin{aligned} \widehat{F}^i_\mu[\xi \,|\, s] &= (\Phi[\xi]F_\mu[\xi \,|\, s]\Phi^{-1})^i \\ &= (\Phi[\xi]\tau_j\Phi^{-1}[\xi]F^j_\mu[\xi \,|\, s])^i = Tr\,(\Phi[\xi]\tau_j\Phi^{-1}[\xi]\tau_i)F^j_\mu[\xi \,|\, s]. \end{aligned} \quad (5.63)$$

The propagator for the monopole $\psi(x)\bar{\psi}(y)$ is the usual Feynman propagator for the particles of spin 1/2:

$$\langle \psi(x)\bar{\psi}(y) \rangle = i\,S_F(x-y), \quad (5.64)$$

or, Fourier transforming:

$$\langle \psi(x)\bar{\psi}(y) \rangle = \int \frac{d^4k}{(2\pi^4)}\,\widetilde{S}_F(k)\,e^{-ik(x-y)}. \quad (5.65)$$

In what follows, we only consider the "gauge field" terms in the action, since for the purpose of calculating the propagators, terms of order > 2 in the action can be ignored. Thus, the generating functional for the full action including matter can be factored into a "matter" part and a "gauge field" part:

$$Z_{full} = Z_{matter}\,Z_{gauge}, \quad (5.66)$$

and hence the presence of matter in no way affects the calculation of the gauge field propagators and can be safely ignored. To derive the explicit expressions for the field propagators, we begin with the action (5.7) up to second order in the fields, including external sources:

$$\begin{aligned}\mathcal{A} \;=\; \int \delta\xi\, ds\, \Big\{ & a_\xi \widehat{F}^i_\mu \widehat{F}^\mu_i + \widehat{L}^i_{\mu\nu}\left(\delta^\nu \widehat{F}^\mu_i - \delta^\mu \widehat{F}^\nu_i\right) + \widehat{b}_i \widehat{F}^i_\mu \dot{\xi}^\mu(s) + \widehat{L}^i_{\mu\nu}\widehat{\mathcal{J}}^{\mu\nu}_i + \widehat{j}^i_\mu \widehat{F}^\mu_i \\ & + (2\alpha)^{-1}\epsilon_{\mu\nu\rho\sigma}\,\epsilon^{\mu\alpha\beta\gamma}\delta^\nu \widehat{L}^{\rho\sigma}_i\, \delta_\alpha \widehat{L}_{\beta\gamma} + \widehat{j}_i \widehat{b}^i \Big\}. \qquad (5.67)\end{aligned}$$

Substituting (5.67) in (5.6), completing the squares for the fields $\widehat{F}_\mu[\xi \mid s]$, $\widehat{L}_{\mu\nu}[\xi \mid s]$ and $\widehat{b}[\xi \mid s]$ as before and integrating them out, we obtain for the "effective action":

$$\begin{aligned}\mathcal{A}_{eff} \;=\; \int \delta\xi\, ds\, \Big\{ & (4a_\xi)^{-1} j^i_\mu j^\mu_i + \dot{\xi}^{-2}(4a_\xi)^{-1}\Big[-j^i_\mu j^i_\nu \dot{\xi}^\mu \dot{\xi}^\nu + 4a_\xi \Box^{-1}\delta^\rho \mathcal{J}^i_{\mu\rho}\dot{\xi}^\mu j^i_\nu \dot{\xi}^\nu \\ & - 4a^2_\xi j_i j^i + 8a^2_\xi j_i \Box^{-1}\delta^\rho \mathcal{J}^i_{\mu\rho}\dot{\xi}^\mu - 4a^2_\xi \Box^{-1}\delta^\rho \mathcal{J}^i_{\mu\rho}\dot{\xi}^\mu \Box^{-1}\delta^\pi \mathcal{J}^i_{\nu\pi}\dot{\xi}^\nu \\ & - 4a_\xi j_i j^i_\mu \dot{\xi}^\mu - 4a_\xi \delta_\nu \mathcal{J}^{\mu\nu}_i \Box^{-1} j^i_\mu \dot{\xi}^2 - 2a^2_\xi \mathcal{J}^i_{\mu\nu}\Box^{-1}\mathcal{J}^{\mu\nu}_i \dot{\xi}^2\Big]\Big\}, \qquad (5.68)\end{aligned}$$

where we have dropped the "hats" and have chosen $\alpha = 2a_\xi$ such that the gauge fixing term cancels the term $\int \delta\xi\, ds\, \frac{1}{a_\xi}\, \widehat{L}^i_{\mu\rho}\delta^\rho \delta_\pi \widehat{L}^{\mu\pi}_i$ in the action. Substituting thus expression into:

$$Z[j^i, J^i_{\mu\nu}, j^i_\mu] = \exp i\mathcal{A}_{eff}, \qquad (5.69)$$

and differentiating functionally with respect to the currents as in (5.60), yields the propagators:

$$\begin{aligned}\langle F^i_\mu[\xi \mid s] F^{i'}_{\mu'}[\xi' \mid s']\rangle \;=\; & -\frac{i\dot{\xi}'^2(s')}{2a_{\xi'}(s')}\delta^{ii'}\left\{\dot{\xi}'^2(s') g_{\mu\mu'} - \dot{\xi}'_\mu(s')\dot{\xi}'_{\mu'}(s')\right\} \\ & \delta(s-s')\prod_{\bar{s}=0}^{2\pi}\delta^4(\xi(\bar{s}) - \xi'(\bar{s})), \qquad (5.70)\end{aligned}$$

$$\begin{aligned}\langle b^i[\xi \mid s] b^{i'}[\xi' \mid s']\rangle \;=\; & 2i\, a_{\xi'}(s')\dot{\xi}'^{-2}(s')\delta^{ii'}\delta(s-s') \\ & \prod_{\bar{s}=0}^{2\pi}\delta^4(\xi(\bar{s}) - \xi'(\bar{s})), \qquad (5.71)\end{aligned}$$

$$\begin{aligned}\langle L^i_{\mu\nu}[\xi \mid s] L^{i'}_{\mu'\nu'}[\xi' \mid s']\rangle &= \frac{i\, a_{\xi'}(s')}{2}\delta^{ii'}\delta(s-s')\prod_{\bar{s}=0}^{2\pi}\delta^4(\xi(\bar{s})-\xi'(\bar{s})) \\ &\Big\{(g_{\mu\mu'}g_{\nu\nu'}-g_{\mu\nu'}g_{\nu\mu'})\Box^{-1}_{\xi'(s')} \\ &-(g_{\alpha\mu'}g_{\lambda\nu'}-g_{\alpha\nu'}g_{\lambda\mu'})\,(g_{\beta\mu}g_{\pi\nu}-g_{\beta\nu}g_{\pi\mu}) \\ &\dot{\xi}'^{-2}(s')\dot{\xi}'^{\alpha}(s')\dot{\xi}'^{\beta}(s')\delta'^{\lambda}(s')\delta'^{\pi}(s')\Box^{-2}_{\xi'(s')}\Big\}, \quad (5.72)\end{aligned}$$

where we have used the rules for functional differentiation given below for hatted variables:

$$\begin{aligned}\frac{\delta X^i_\mu[\xi \mid s]}{\delta X^\alpha_j[\xi' \mid s']} &= \delta^{ij}\, g_{\mu\alpha}\,\delta(s-s')\prod_{\bar{s}=0}^{2\pi}\delta^4(\xi(\bar{s})-\xi'(\bar{s})), \\ \frac{\delta Y^i_{\mu\nu}[\xi \mid s]}{\delta Y^{\alpha\beta}_j[\xi' \mid s']} &= \frac{1}{2}\delta^{ij}(g_{\mu\alpha}g_{\nu\beta}-g_{\mu\beta}g_{\nu\alpha})\,\delta(s-s')\prod_{\bar{s}=0}^{2\pi}\delta^4(\xi(\bar{s})-\xi'(\bar{s})). \quad (5.73)\end{aligned}$$

5.4 Loop-Space Fourier Transform & Feynman Diagrams

Now that we have expressions for the propagators in spacetime, we would like to proceed to calculating scattering amplitudes. It is common in such cases to work in momentum space rather than spacetime. The conventional approach is to perform a Fourier transform on the various fields appearing in the particular Feynman diagram. The so-called Feynman rules are obtained by applying the procedure outlined in §5.3 and identifying vertex and propagator factors. These rules are usually stated in terms of momentum variables. We propose to calculate the Feynman diagrams in terms of analogous variables: however, we are faced once again with the complication of working with loop variables instead of local ones such as A_μ. The operation of Fourier-transforming has to be made precise for the case of functions depending on a loop. In this section, we will try to define the Fourier transform and cast the propagators calculated above in terms of these new loop-space variables. We start by considering a functional $S[\xi \mid s]$ depending on the loop in the same manner as $F_\mu[\xi \mid s]$. The Fourier transform of

this functional is then defined to be:

$$\widetilde{S}[\pi \,|\, s] = \int \delta\xi \, K_{\pi(s,2\pi]} \, S[\xi \,|\, s] \, \exp i \int_0^{2\pi} dt \, \pi(t) \, \xi(t), \tag{5.74}$$

where:

$$K^{-1}_{\pi(s,2\pi]} = \sqrt{\mathcal{N}} \int \prod_{\bar{s}>s}^{2\pi} d^4\xi(\bar{s}) \, \exp i \int_s^{2\pi} dt \, \pi(t) \, \xi(t). \tag{5.75}$$

Here:

$$\mathcal{N} = \prod_{\bar{s}>s}^{2\pi} (2\pi)^4, \tag{5.76}$$

is an infinite product of factors of 2π. Thus, the integration for $s < \bar{s} \leq 2\pi$ can be performed and the upper limit of the integral can be changed to s, yielding:

$$\widetilde{S}[\pi \,|\, s] = \frac{1}{\sqrt{\mathcal{N}}} \int \prod_{\bar{s}=0}^{s} d^4\xi(s) \, S[\xi \,|\, s] \, \exp i \int_0^{s} dt \, \pi(t) \xi(t). \tag{5.77}$$

The inverse Fourier transform is given by:

$$S[\xi \,|\, s] = \int \delta\pi \, K_{\pi(s,2\pi]} \, \widetilde{S}[\pi \,|\, s] \, \exp -i \int_0^{2\pi} dt \, \pi(t) \, \xi(t), \tag{5.78}$$

where:

$$K^{-1}_{\pi(s,2\pi]} = \sqrt{\mathcal{N}} \int \prod_{\bar{s}>s}^{2\pi} d^4\pi(\bar{s}) \, \exp -i \int_s^{2\pi} dt \, \pi(t) \, \xi(t). \tag{5.79}$$

Once again, by performing the integration for $s < \bar{s} \leq 2\pi$, we obtain a quantity depending on $0 \leq \bar{s} \leq s$ only:

$$S[\xi \,|\, s] = \frac{1}{\sqrt{\mathcal{N}}} \int \prod_{\bar{s}=0}^{s} d^4\pi(\bar{s}) \, \widetilde{S}[\pi \,|\, s] \, \exp -i \int_0^{s} dt \, \pi(t) \, \xi(t). \tag{5.80}$$

Substituting the definition of $\widetilde{S}[\pi \,|\, s]$ into (5.78), we obtain:

$$S[\xi \,|\, s] = \int \prod_{\bar{s}=0}^{s} d^4\xi'(\bar{s}) \, S[\xi' \,|\, s] \prod_{\bar{s}=0}^{s} \delta^4(\xi'(\bar{s}) - \xi(\bar{s})) = S[\xi \,|\, s], \tag{5.81}$$

and hence, the definitions are seen to be consistent.

It must be noted that for the "hatted" variables, which depend on the loop for $s < \bar{s} \leq 2\pi$, analogous expressions for the Fourier transforms can be defined, by taking a different K, which depends on the earlier part of the loop and which

will enable us to integrate out the earlier part of the loop, leaving us with an integral over the later part only.

The above expressions for the Fourier transform can be generalised to the case of loop variables depending on two loops:

$$\begin{aligned}\tilde{S}[\pi,\pi'\,|\,s,s'] &= \int \delta\xi\,\delta\xi'\,K_{\pi\pi'(s;s',2\pi]}\,S[\xi,\xi'\,|\,s,s'] \\ &\qquad \exp -i\int_0^{2\pi} dt\,\pi'(t)\xi'(t)\,\exp i\int_0^{2\pi} dt\,\pi(t)\,\xi(t), \qquad (5.82)\end{aligned}$$

with:

$$K^{-1}_{\pi\pi'} = \mathcal{N}\int \prod_{\bar{s}>s}^{2\pi} d^4\pi(\bar{s}) \prod_{\bar{s}>s'}^{2\pi} d^4\pi'(\bar{s})\,\exp -i\int_s^{2\pi} dt\,\pi(t)\,\xi(t)\,\exp i\int_{s'}^{2\pi} dt\,\pi'(t)\,\xi'(t). \qquad (5.83)$$

Once again, the integrals in the exponentials could be performed for $s < \bar{s} \leq 2\pi$ and $s' < \bar{s} \leq 2\pi$ respectively, yielding:

$$\begin{aligned}\tilde{S}[\pi,\pi'\,|\,s,s'] &= \frac{1}{\mathcal{N}}\int \prod_{\bar{s}=0}^{s} d^4\xi(\bar{s}) \prod_{\bar{s}=0}^{s'} d^4\xi'(\bar{s})\,S[\xi,\xi'\,|\,s,s'] \\ &\qquad \exp i\int_0^{s} dt\,\pi(t)\,\xi(t)\,\exp -i\int_0^{s'} \pi'(t)\,\xi'(t), \quad (5.84)\end{aligned}$$

and:

$$\begin{aligned}S[\xi,\xi'\,|\,s,s'] &= \frac{1}{\mathcal{N}}\int \prod_{\bar{s}=0}^{s} d^4\pi(\bar{s}) \prod_{\bar{s}=0}^{s'} d^4\pi'(\bar{s})\,S[\pi,\pi'\,|\,s,s'] \\ &\qquad \exp -i\int_0^{s} dt\,\pi(t)\,\xi(t)\,\exp i\int_0^{s'} dt\,\pi'(t)\,\xi'(t). \quad (5.85)\end{aligned}$$

Analogous expressions for the "hatted" variables can be found, by using a K which depends on the earlier part of the loop. The Fourier transform could then be written as an integral over the later part of the loop.

We are now ready to calculate the simplest scattering process using the action $\mathcal{A}_{eff}$. These processes must then be compared to the analogous process in QCD, of the same order. In thus comparing the scattering amplitudes, we hope to establish whether or not a system of monopoles would behave differently from a system of sources.

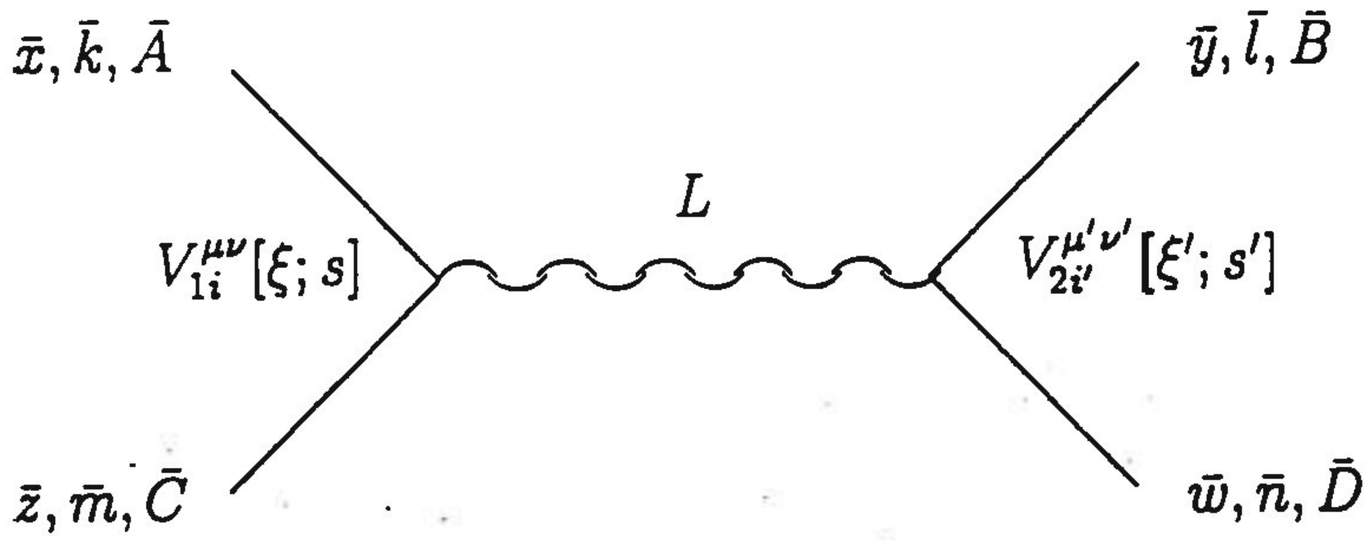


Figure 5.1: Monopole-monopole scattering via the exchange of one L-particle

The simplest case of interest is that of Fig. 5.1, with two monopoles scattering via the exchange of an L-particle, the latter being the mediator of the $L_{\mu\nu}$ field. We now proceed to calculate the scattering amplitude for this process in spacetime and subsequently, cast the result in terms of momentum variables. It would be useful in what follows to use the bra-ket notation of Dirac for purposes of calculating the scattering amplitudes.

For the monopole-monopole process depicted in Fig. 5.1, the scattering amplitude is given by the vacuum expectation value of the time-ordered product:

$$\begin{aligned} A \;=\; \langle 0\,|\,T\psi_{\bar{k}}^{\bar{A}}(\bar{x})\bar{\psi}_{\bar{l}}^{\bar{B}}(\bar{y})\psi_{\bar{m}}^{\bar{C}}(\bar{z})\bar{\psi}_{\bar{n}}^{\bar{D}}(\bar{w}) \int \delta\xi\, ds\, V_{1i}^{\mu\nu}[\xi; s] L_{\mu\nu}^{i}[\xi\,|\,s] \\ \int \delta\xi'\, ds'\, V_{2i'}^{\mu'\nu'} L_{\mu'\nu'}[\xi'; s']\,|\,0\rangle, \quad (5.86) \end{aligned}$$

where ψ and ψ' refer to the incoming and outgoing monopole wavefunctions, and $V_1^{\mu\nu}$ and $V_2^{\mu'\nu'}$ to the two vertex factors coupling to $L_{\mu\nu}$ and $L_{\mu'\nu'}$ respectively. Note that in this section, where applicable, the quantities are the "hatted" ones,

where the "hat" has been omitted for convenience. The vertex factors are given by:

$$\begin{aligned} V_{1i}^{\mu\nu} &= -4\pi\tilde{g}\,\epsilon^{\mu\nu\rho\sigma}\dot{\xi}_\rho(s)\bar{\psi}_k^A(\xi(s))\gamma_\sigma^{AB}\Omega_\xi^{kl}(s,0)T_{lm}^i\Omega_\xi^{-1mn}(s,0)\psi_n^B(\xi(s)), \\ V_{2i'}^{\mu'\nu'} &= -4\pi\tilde{g}\,\epsilon^{\mu'\nu'\rho'\sigma'}\dot{\xi}'_{\rho'}(s')\bar{\psi}_{k'}^{A'}(\xi'(s'))\gamma_{\sigma'}^{A'B'} \\ &\quad\times\Omega_{\xi'}^{k'l'}(s',0)T_{l'm'}^{i'}\Omega_{\xi'}^{-1m'n'}(s',0)\psi_{n'}^{B'}(\xi'(s')), \quad (5.87) \end{aligned}$$

where Greek letters denote spacetime indices, capital Roman letters spinor indices and small Roman letters internal space (isospin) indices. The barred letters refer to external particles.

The amplitude for the figure is therefore:

$$\begin{aligned} A = (-4\pi\tilde{g})^2 \int \delta\xi\delta\xi'\,ds\,ds'\langle 0\,|\,T\psi_{\bar{k}}^{\bar{A}}(\bar{x})\bar{\psi}_{\bar{l}}^{\bar{B}}(\bar{y})\psi_{\bar{m}}^{\bar{C}}(\bar{z})\bar{\psi}_{\bar{n}}^{\bar{D}}(\bar{w})\,\epsilon^{\mu\nu\rho\sigma}L_{\mu\nu}^i[\xi\,|\,s]\dot{\xi}_\rho(s) \\ \times\bar{\psi}_k^A(\xi(s))\gamma_\sigma^{AB}\Omega_\xi^{kl}(s,0)T_{lm}^i\Omega_\xi^{-1mn}(s,0)\psi_n^B(\xi(s))\,\epsilon^{\mu'\nu'\rho'\sigma'}L_{\mu'\nu'}^{i'}[\xi'\,|\,s']\dot{\xi}'_{\rho'}(s') \\ \times\bar{\psi}_{k'}^{A'}(\xi'(s'))\gamma_{\sigma'}^{A'B'}\Omega_{\xi'}^{k'l'}(s',0)T_{l'm'}^{i'}\Omega_{\xi'}^{-1m'n'}(s',0)\psi_{n'}^{B'}(\xi'(s'))\,|\,0\rangle. \quad (5.88) \end{aligned}$$

In order to obtain the time-ordered product for the process, the various fields have to be contracted pairwise, on the basis of the Feynman diagram, using Wick's theorem.

The contraction of a ψ field with a $\bar{\psi}$ field is proportional to the usual Feynman propagator below:

$$\langle 0\,|\,T\psi_i^A(x)\bar{\psi}_{i'}^B(y)\,|\,0\rangle = i\,S_F^{AB}(x-y)\,\delta_{ii'}, \quad (5.89)$$

whereas that of the $L_{\mu\nu}^i$ with $L_{\mu'\nu'}^{i'}$ is proportional to the L-propagator (5.72):

$$\langle 0\,|\,TL_{\mu\nu}^i[\xi\,|\,s]L_{\mu'\nu'}^{i'}[\xi'\,|\,s']\,|\,0\rangle = i\,\Delta_{\mu\nu\mu'\nu'}^{ii'}[\xi,\xi'\,|\,s,s'], \quad (5.90)$$

Δ being the propagator.

Substituting the expressions above for the propagators and making use of Wick's theorem, we obtain for the scattering amplitude:

$$
\begin{aligned}
A \;=\; -\frac{(4\pi\tilde{g})}{2}\int \delta\xi\,\delta\xi'\,ds\,ds'\,a_{\xi'}(s')\,\epsilon^{\mu'\nu'\rho'\sigma'}\dot{\xi}'_{\rho'}(s')\epsilon^{\mu\nu\rho\sigma}\dot{\xi}_{\rho}(s)\dot{\xi}'^{-2}(s')\delta^{ii'}\,\delta(s-s') \\
\Big\{S_F(\bar{x}-\xi(s))\gamma_\sigma\Omega_\xi(s,0)T_i\Omega_\xi^{-1}(s,0)S_F(\xi(s)-\bar{y})\Big\}^{\bar{A}\bar{B}}_{\bar{k}\bar{l}} \\
\prod_{\bar{s}=0}^{2\pi}\delta^4(\xi(\bar{s})-\xi'(\bar{s}))\,\Big[(g_{\mu\mu'}g_{\nu\nu'}-g_{\mu\nu'}g_{\nu\mu'})\,\dot{\xi}'^2(s')\Box^{-1}_{\xi'(s')} \\
-\dot{\xi}'^{\alpha}\dot{\xi}'^{\beta}(g_{\alpha\mu'}g_{\lambda\nu'}-g_{\alpha\nu'}g_{\lambda\mu'})(g_{\beta\mu}g_{\pi\nu}-g_{\beta\nu}g_{\pi\mu})\Box^{-2}_{\xi'(s')}\delta'^{\pi}\delta'^{\lambda}\Big] \\
\Big\{S_F(\bar{z}-\xi'(s'))\gamma_{\sigma'}\Omega_{\xi'}(s',0)T_{i'}\Omega^{-1}_{\xi'}(s',0)S_F(\xi'(s')-\bar{w})\Big\}^{\bar{C}\bar{D}}_{\bar{m}\bar{n}} \qquad (5.91)
\end{aligned}
$$

where $\delta'_\mu = \frac{\delta}{\delta\xi'^{\mu}}$. Note that this amplitude is the sum of two terms. Let us simplify the second term to begin with. Performing the s' integration, we obtain:

$$
\begin{aligned}
A_2 \;=\; -\frac{(4\pi\tilde{g})^2}{2}\int \delta\xi\,\delta\xi'\,ds\,\Big\{S_F(\bar{x}-\xi(s))\Omega_\xi(s,0)\gamma_\sigma T_i\Omega_\xi^{-1}(s,0)S_F(\xi(s)-\bar{y})\Big\}^{\bar{A}\bar{B}}_{\bar{k}\bar{l}} \\
a_{\xi'}(s)\dot{\xi}'^{\alpha}(s)\dot{\xi}'^{\beta}(s)(g_{\alpha\mu}g_{\lambda\nu}-g_{\alpha\nu}g_{\lambda\mu})(g_{\beta\mu'}g_{\pi\nu'}-g_{\beta\nu'}g_{\pi\mu'})\dot{\xi}'^{-2}(s) \\
\Box^{-2}_{\xi'(s)}\delta'^{\pi}(s)\delta'^{\lambda}(s)\epsilon^{\mu'\nu'\rho'\sigma'}\dot{\xi}'_{\rho'}(s)\,\delta^{ii'}\epsilon^{\mu\nu\rho\sigma}\dot{\xi}_{\rho}(s)\prod_{\bar{s}=0}^{2\pi}\delta^4(\xi(\bar{s})-\xi'(\bar{s})) \\
\Big\{S_F(\bar{z}-\xi'(s))\gamma_{\sigma'}\Omega_{\xi'}(s,0)T_{i'}\Omega^{-1}_{\xi'}(s,0)S_F(\xi'(s)-\bar{w})\Big\}^{\bar{C}\bar{D}}_{\bar{m}\bar{n}}\,. \qquad (5.92)
\end{aligned}
$$

The $\Box^{-2}_{\xi'(s)}$ operator acts on the infinite product of the δ-functions. Let us first concentrate on the factors involving Lorentz indices, in particular:

$$
\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\delta'^{\lambda}\delta'^{\pi}(g_{\alpha\mu}g_{\lambda\nu}-g_{\alpha\nu}g_{\lambda\mu})(g_{\beta\mu'}g_{\pi\nu'}-g_{\beta\nu'}g_{\pi\mu'})\,\dot{\xi}_\rho\dot{\xi}'_{\rho'}\dot{\xi}'^{\alpha}\dot{\xi}'^{\beta} \\
= \dot{\xi}_\rho\dot{\xi}'_{\rho'}\dot{\xi}'^{\alpha}\dot{\xi}'^{\beta}\delta'^{\lambda}\delta'^{\pi}(4\epsilon_{\alpha\lambda}{}^{\rho\sigma}\epsilon_{\beta\pi}{}^{\rho'\sigma'}).
\end{aligned} \qquad (5.93)
$$

Interchanging β and ρ', we see that this term vanishes due to the antisymmetry of the Levi-Civita tensor. Thus:

$$
A_2 = 0, \qquad (5.94)
$$

and the amplitude in spacetime coordinates reduces to the first term only:

$$
\begin{aligned}
A \;=\; -\frac{(4\pi\tilde{g})^2}{2}\int \delta\xi\,\delta\xi'\,ds\,\Big\{S_F(\bar{x}-\xi(s))\gamma_\sigma\Omega_\xi(s,0)T^{i}\Omega_\xi^{-1}(s,0)S_F(\xi(s)-\bar{y})\Big\}^{\bar{A}\bar{B}}_{\bar{k}\bar{l}} \\
a_{\xi'}(s)\epsilon^{\mu\nu\rho\sigma}\dot{\xi}_\rho(s)\epsilon^{\mu'\nu'\rho'\sigma'}\dot{\xi}'_{\rho'}(s)(g_{\mu\mu'}g_{\nu\nu'}-g_{\mu\nu'}g_{\nu\mu'})\,\delta^{ii'} \\
\Big\{S_F(\bar{z}-\xi(s))\gamma_{\sigma'}\Omega_{\xi'}(s,0)T^{i'}\Omega^{-1}_{\xi'}(s,0)S_F(\xi(s)-\bar{w})\Big\}^{\bar{C}\bar{D}}_{\bar{m}\bar{n}} \\
\Box^{-1}_{\xi'(s)}\prod_{\bar{s}=0}^{2\pi}\delta^4(\xi(\bar{s})-\xi'(\bar{s})). \qquad (5.95)
\end{aligned}
$$

Contracting the Levi-Civita tensor with $(g_{\mu\mu'}g_{\nu\nu'} - g_{\mu\nu'}g_{\nu\mu'})$ yields:

$$(g_{\mu\mu'}g_{\nu\nu'} - g_{\mu\nu'}g_{\nu\mu'})\,\epsilon^{\mu\nu\rho\sigma}\,\epsilon^{\mu'\nu'\rho'\sigma'} = -2\,(g^{\rho\rho'}g^{\sigma\sigma'} - g^{\rho\sigma'}g^{\sigma\rho'}). \tag{5.96}$$

Thus:

$$\begin{aligned} A \;=\; & (4\pi\tilde{g})^2 \int \delta\xi\,\delta\xi'\,ds\,\left\{S_F(\bar{x}-\xi(s))\gamma_\sigma\Omega_\xi(s,0)T^i\Omega_\xi^{-1}(s,0)S_F(\xi(s)-\bar{y})\right\}^{AB}_{\bar{k}\bar{l}} \\ & a_{\xi'}(s)\dot{\xi}_\rho(s)\,\dot{\xi}'_{\rho'}(s)(g^{\rho\rho'}g^{\sigma\sigma'} - g^{\rho\sigma'}g^{\sigma\rho'})\Box^{-1}_{\xi'(s)}\prod_{\bar{s}=0}^{2\pi}\delta^4(\xi(\bar{s})-\xi'(s)) \\ & \left\{S_F(\bar{z}-\xi(s))\gamma_{\sigma'}\Omega_{\xi'}(s,0)T^i\Omega_{\xi'}^{-1}(s,0)S_F(\xi(s)-\bar{w})\right\}^{\bar{C}\bar{D}}_{\bar{m}\bar{n}}. \end{aligned} \tag{5.97}$$

The next step is to do a Fourier transform and write the amplitude in terms of momentum variables. In the bra-ket notation, we are interested in:

$$\langle p-k\,|\,A\,|\,p'-k'\rangle = \int \delta\pi\,\delta\pi'\,\langle p-k\,|\,\Gamma\,|\,\pi\rangle\,\langle\pi\,|\,\Delta\,|\,\pi'\rangle\langle\pi'\,|\,\Gamma'\,|\,p'-k'\rangle, \tag{5.98}$$

where A is the amplitude in spacetime variables and Γ, Γ' and Δ are, in the spacetime basis, given by:

$$\langle\bar{x}\,\bar{y}\,|\,\Gamma\,|\,\xi\rangle = 4\pi\tilde{g}\dot{\xi}_\rho\left\{S_F(\bar{x}-\xi(s))\gamma_\sigma\Omega_\xi T_i\Omega_\xi^{-1}S_F(\xi(s)-\bar{y})\right\}^{\bar{A}\bar{B}}_{\bar{k}\bar{l}}, \tag{5.99}$$

$$\langle\xi\,|\,\Delta\,|\,\xi'\rangle = (g^{\rho\rho'}g^{\sigma\sigma'} - g^{\rho\sigma'}g^{\sigma\rho'})\Box^{-1}_{\xi(s)}\prod_{\bar{s}=0}^{2\pi}\delta^4(\xi(\bar{s})-\xi'(\bar{s}')), \tag{5.100}$$

$$\langle\xi'\,|\,\Gamma'\,|\,\bar{z}\,\bar{w}\rangle = 4\pi\tilde{g}a_{\xi'}\dot{\xi}'_{\rho'}\left\{S_F(\bar{z}-\xi'(s))\gamma_{\sigma'}\Omega_{\xi'}T_i\Omega_{\xi'}^{-1}S_F(\xi'(s)-\bar{w})\right\}^{\bar{C}\bar{D}}_{\bar{m}\bar{n}}, \tag{5.101}$$

where the indices on the left-hand side above and in (5.97) have been suppressed, and where it must be remembered that Ω_ξ stands for $\widehat{\Omega}_\xi$.

To change the basis from spacetime to momentum space, we proceed as follows:

$$\begin{aligned} \langle\pi\,|\,\Delta\,|\,\pi'\rangle \;=\; & \int \delta\xi\,\delta\xi'\,\langle\pi\,|\,\xi\rangle\,\langle\xi\,|\,\Delta\,|\,\xi'\rangle\,\langle\xi'\,|\,\pi'\rangle \\ =\; & \int \delta\xi\,\delta\xi'\,\left[(g^{\rho\rho'}g^{\sigma\sigma'} - g^{\rho\sigma'}g^{\sigma\rho'})\Box^{-1}_{\xi'(s)}\prod_{\bar{s}=0}^{2\pi}\delta^4(\xi(\bar{s})-\xi'(\bar{s}))\right] \\ & \exp i\int_0^{2\pi} dt\,\pi(t)\,\xi(t)\,\exp -i\int_0^{2\pi} dt\,\pi'(t)\,\xi'(t), \end{aligned} \tag{5.102}$$

where:

$$\langle \pi \,|\, \xi \rangle = \exp i \int_0^{2\pi} dt\, \pi(t)\, \xi(t), \tag{5.103}$$

is the analogue of a "plane wave" with:

$$\langle \xi' \,|\, \pi' \rangle = \exp -i \int_0^{2\pi} dt\, \pi'(t)\, \xi'(t). \tag{5.104}$$

The operator $\Box^{-1}_{\xi'(s)}$ acts on the δ-function, but by an operation similar to integration by parts, it can be made to act on the exponential function and the above expression becomes:

$$\begin{aligned} \langle \pi \,|\, \Delta \,|\, \pi' \rangle \;=\; & \int \delta\xi\, \delta\xi' \,(g^{\rho\rho'} g^{\sigma\sigma'} - g^{\rho\sigma'} g^{\sigma\rho'}) \prod_{\bar{s}=0}^{2\pi} \delta^4(\xi(\bar{s}) - \xi'(\bar{s})) \\ & \left\{ \Box^{-1}_{\xi'(s)} \exp -i \int_0^{2\pi} dt\, \pi'(t)\, \xi'(t) \right\} \exp i \int_0^{2\pi} dt\, \pi(t) \xi(t). \end{aligned} \tag{5.105}$$

$\Box^{-1}_{\xi'(s)}$ acting on $\exp -i \int dt\, \pi'(t) \xi'(t)$ yields:

$$\begin{aligned} \langle \pi \,|\, \Delta \,|\, \pi' \rangle \;=\; & \int \delta\xi\, \delta\xi' \int_{s_-}^{s_+} ds'\, ds'' \frac{\beta_\eta(s' - s)\beta_\eta(s'' - s)}{\pi'_\mu(s') \pi'^{\mu}(s'')} (g^{\rho\rho'} g^{\sigma\sigma'} - g^{\rho\sigma'} g^{\sigma\rho'}) \\ & \exp i \int_0^{2\pi} dt\, \pi(t)\, \xi(t) \exp -i \int_0^{2\pi} dt\, \pi'(t) \xi'(t), \end{aligned} \tag{5.106}$$

where $\beta_\eta(s)$ is a function of finite width, vanishing outside $s_- \leq s \leq s_+$, with $s_\pm = s \pm \eta/2$. The function β_η replaces the δ-function and may be taken to be:

$$\beta_\eta(s) = \frac{1}{\eta \sqrt{\pi}}\, e^{-s^2/\eta^2}, \tag{5.107}$$

although its explicit form does not matter. In the limit $\eta \to 0$, $\beta_\eta(s)$ reduces to the δ-function.

$\langle \pi \,|\, \Delta \,|\, \pi' \rangle$ can now be simplified further. We obtain:

$$\begin{aligned} \langle \pi \,|\, \Delta \,|\, \pi' \rangle \;=\; & (g^{\rho\rho'} g^{\sigma\sigma'} - g^{\rho\sigma'} g^{\sigma\rho'}) \prod_{\bar{s}=0}^{2\pi} \delta^4(\pi(\bar{s}) - \pi'(\bar{s})) \\ & \int ds'\, ds''\, \frac{\beta_\eta(s' - s)\beta_\eta(s'' - s)}{\pi'_\mu(s') \pi'^{\mu}(s'')}. \end{aligned} \tag{5.108}$$

Let us now turn to the vertices Γ and Γ'. These can be written as:

$$\begin{aligned}\langle p-k\,|\,\Gamma\,|\,\pi\rangle &= \int d\bar{x}\,d\bar{y}\,\delta\xi\,\langle p-k\,|\,\bar{x}\,\bar{y}\rangle\,\langle\bar{x}\,\bar{y}\,|\,\Gamma\,|\,\xi\rangle\,\langle\xi\,|\,\pi\rangle \\ &= \int d\bar{x}\,d\bar{y}\,\delta\xi\,\exp i\,(p\bar{x}-k\bar{y})\,\langle\bar{x}\,\bar{y}\,|\,\Gamma\,|\,\xi\rangle \\ &\qquad \exp -i\int_0^{2\pi} dt\,\pi(t)\,\xi(t), \quad (5.109)\end{aligned}$$

and:

$$\begin{aligned}\langle\pi'\,|\,\Gamma'\,|\,p'-k'\rangle &= \int d\bar{z}\,d\bar{w}\,\delta\xi'\,\langle\pi'\,|\,\xi'\rangle\,\langle\xi'\,|\,\Gamma'\,|\,\bar{z}\,\bar{w}\rangle\,\langle\bar{z}\,\bar{w}\,|\,p'-k'\rangle \\ &= \int d\bar{z}\,d\bar{w}\,\delta\xi'\,\exp -i\,(p'\bar{z}-k'\bar{w})\,\langle\xi'\,|\,\Gamma'\,|\,\bar{z}\,\bar{w}\rangle \\ &\qquad \exp i\int_0^{2\pi} dt\,\pi'(t)\,\xi'(t), \quad (5.110)\end{aligned}$$

But $\langle\bar{x}\,\bar{y}\,|\,\Gamma\,|\,\xi\rangle$ and $\langle\xi'\,|\,\Gamma'\,|\,\bar{z}\,\bar{w}\rangle$ are given by (5.99) and (5.101).

Thus:

$$\begin{aligned}\langle p-k\,|\,\Gamma\,|\,\pi\rangle &= \int d\bar{x}\,d\bar{y}\,\delta\xi\left\{S_F(\bar{x}-\xi(s))\gamma_\sigma\Omega_\xi T_i\Omega_\xi^{-1}S_F(\xi(s)-\bar{y})\right\}^{\bar{A}\bar{B}}_{\bar{k}\bar{l}} \\ &\qquad \dot{\xi}_\rho(s)\,\exp i\,(p\bar{x}-k\bar{y})\,\exp -i\int_0^{2\pi} dt\,\pi(t)\xi(t) \\ &= -\int d\bar{x}'\,d\bar{y}'\,\delta\xi\,\left\{S_F(\bar{x}')\gamma_\sigma\Omega_\xi T_i\Omega_\xi^{-1}S_F(\bar{y}')\right\}^{\bar{A}\bar{B}}_{\bar{k}\bar{l}}\,\dot{\xi}_\rho(s) \\ &\qquad \exp i\,(p\bar{x}'+k\bar{y}')\,\exp\left[i\,(p-k)\xi(s)\right] \\ &\qquad \exp -i\int_0^{2\pi} dt\,\pi(t)\,\xi(t), \quad (5.111)\end{aligned}$$

with $\bar{x}'=\bar{x}-\xi(s)$ and $\bar{y}'=\xi(s)-\bar{y}$.

Similarly:

$$\begin{aligned}\langle\pi'\,|\,\Gamma'\,|\,p'-k'\rangle &= \int d\bar{z}'\,d\bar{w}'\,\delta\xi'\,\left\{S_F(\bar{z}')\gamma_{\sigma'}\Omega_{\xi'}T_{i'}\Omega_{\xi'}^{-1}S_F(\bar{w}')\right\}^{\bar{C}\bar{D}}_{\bar{m}\bar{n}}\,\dot{\xi}'_{\rho'}(s)a_{\xi'}(s) \\ &\qquad \exp -i\,(p'\bar{z}'+k'\bar{w}')\,\exp[-i\,(p'-k')\,\xi'(s)] \\ &\qquad \exp i\int_0^{2\pi} dt\,\pi'(t)\,\xi'(t), \quad (5.112)\end{aligned}$$

with $\bar{z}'=\bar{z}-\xi'(s)$ and $\bar{w}'=\xi'(s)-\bar{w}$.

Combining (5.108), (5.111) and (5.112), and substituting in (5.98), we obtain:

$$\begin{aligned}\langle p-k\,|\,A\,|\,p'-k'\rangle \;=\; & -\int \delta\pi\,\delta\xi\,\delta\xi'\,ds\left\{\widetilde{S}_F(p)\gamma_\sigma\Omega_\xi(s,0)T_i\Omega_\xi^{-1}(s,0)\widetilde{S}_F(k)\right\}_{\bar{k}\bar{l}}^{\bar{A}\bar{B}} \\ & \exp i\left[(p-k)\xi(s)\right]\dot{\xi}_\rho(s)\,\exp -i\int_0^{2\pi} dt\,\pi(t)\,\xi(t) \\ & (g^{\rho\rho'}g^{\sigma\sigma'}-g^{\rho\sigma'}g^{\sigma\rho'})\int_{s_-}^{s_+} ds'\,ds''\,\frac{\beta_\eta(s'-s)\beta_\eta(s''-s)}{\pi_\mu(s')\pi^\mu(s'')} \\ & \left\{\widetilde{S}_F(-p')\gamma_{\sigma'}\Omega_{\xi'}(s,0)T_i\Omega_{\xi'}^{-1}(s,0)\widetilde{S}_F(-k')\right\}_{\bar{m}\bar{n}}^{\bar{C}\bar{D}}\dot{\xi}'_{\rho'}(s)a_{\xi'}(s) \\ & \exp -i\left[(p'-k')\xi'(s)\right]\exp i\int_0^{2\pi} dt\,\pi(t)\xi'(t), \qquad (5.113)\end{aligned}$$

where the integration over π' has been performed.

We now write the tangent vectors as:

$$\dot{\xi}_\rho(s)=\frac{\xi_\rho(s_+)-\xi_\rho(s_-)}{s_+-s_-}, \qquad (5.114)$$

where $s_+ - s_- = \eta$, with $\eta \to 0$. Thus:

$$\begin{aligned}\langle p-k\,|\,A\,|\,p'-k'\rangle \;\propto\; & \int \delta\xi\,\delta\xi'\delta\pi\,ds\left\{\widetilde{S}_F(-p)\gamma_\sigma\Omega_\xi(s,0)T_i\Omega_\xi^{-1}(s,0)\widetilde{S}_F(-k)\right\}_{\bar{k}\bar{l}}^{\bar{A}\bar{B}} \\ & \exp i\int_0^{2\pi} dt\,(p-k)\beta_\eta(s-t)\,\xi(t)\,\exp i\int_0^{2\pi} dt\,\pi(t)\,\xi'(t) \\ & \left(\frac{\xi_\rho(s_+)-\xi_\rho(s_-)}{s_+-s_-}\right)\exp -i\int_0^{2\pi} dt\,\pi(t)\,\xi(t) \\ & (g^{\rho\rho'}g^{\sigma\sigma'}-g^{\rho\sigma'}g^{\sigma\rho'})\int_{s_-}^{s_+} ds'\,ds''\frac{\beta_\eta(s'-s)\,\beta_\eta(s''-s)}{\pi_\mu(s')\,\pi^\mu(s'')} \\ & \exp -i\int_0^{2\pi} dt\,(p'-k')\,\beta_\eta(s-t)\xi'(t) \\ & \left(\frac{\xi'_{\rho'}(s_+)-\xi'_{\rho'}(s_-)}{s_+-s_-}\right)\frac{(s_+-s_-)^2}{(\xi_\nu(s_+)-\xi_\nu(s_-))(\xi^\nu(s_+)-\xi^\nu(s_-))} \\ & \left\{\widetilde{S}_F(p')\gamma_{\sigma'}\Omega_{\xi'}(s,0)T_i\Omega_{\xi'}^{-1}(s,0)\widetilde{S}_F(k')\right\}_{\bar{m}\bar{n}}^{\bar{C}\bar{D}}. \qquad (5.115)\end{aligned}$$

Here, only the exponentials depend on $\pi(s)$ for $\bar{s} \geq s_+$, and $\bar{s} \leq s_-$. We can therefore perform the π-integration in this range to obtain:

$$\begin{aligned}\langle p-k\,|\,A\,|\,p'-k'\rangle \propto \int \delta\xi\,\delta\xi' & \prod_{s_-<\bar{s}<s_+} d^4\pi(\bar{s})\,ds \\ & \left\{\widetilde{S}_F(-p)\gamma_\sigma\Omega_\xi(s,0)T_i\Omega_\xi^{-1}(s,0)\widetilde{S}_F(-k)\right\}_{\bar{k}\bar{l}}^{\bar{A}\bar{B}} \\ & (g^{\rho\rho'}g^{\sigma\sigma'}-g^{\rho\sigma'}g^{\sigma\rho'})\frac{[\xi_\rho(s_+)-\xi_\rho(s_-)][\xi'_{\rho'}(s_+)-\xi'_{\rho'}(s_-)]}{|\,\xi(s_+)-\xi(s_-)\,|^2}\end{aligned}$$

$$\int_{s_-}^{s_+} ds'\, ds'' \frac{\beta_\eta(s'-s)\,\beta_\eta(s''-s)}{\pi_\mu(s')\,\pi^\mu(s'')} \prod_{\bar{s}\geq s_+} \prod_{\bar{s}\leq s_-} \delta^4(\xi(\bar{s})-\xi'(\bar{s}))$$
$$\left\{S_F(p')\gamma_{\sigma'}\Omega_{\xi'}(s,0)T_i\Omega_{\xi'}^{-1}(s,0)\tilde{S}_F(k')\right\}_{\bar{m}\bar{n}}^{\bar{C}\bar{D}}$$
$$\exp i\int_0^{2\pi} dt\,[(p-k)\,\xi(t)-(p'-k')\,\xi'(t)]\,\beta_\eta(s-t)$$
$$\exp i\int_{s_-}^{s_+} dt\,\pi(t)\,(\xi'(t)-\xi(t)). \tag{5.116}$$

Integrating over ξ' for $\bar{s}\geq s_+$ and $\bar{s}\leq s_-$ yields:

$$\begin{aligned}\langle p-k\,|\,A\,|\,p'-k'\rangle \;\propto\; & \int \delta\xi \prod_{s_-\leq\bar{s}\leq s_+} d^4\xi'(\bar{s})\,d^4\pi(\bar{s})\,ds\,(g^{\rho\rho'}g^{\sigma\sigma'}-g^{\rho\sigma'}g^{\sigma\rho'}) \\ & \left\{\tilde{S}_F(-p)\gamma_\sigma\Omega_\xi(s,0)T_i\Omega_\xi^{-1}(s,0)\tilde{S}_F(-k)\right\}_{\bar{k}\bar{l}}^{\bar{A}\bar{B}} \\ & \frac{[\xi_\rho(s_+)-\xi_\rho(s_-)][\xi'_{\rho'}(s_+)-\xi'_{\rho'}(s_-)]}{|\,\xi_\nu(s_+)-\xi_\nu(s_-)\,|\,|\,\xi^\nu(s_+)-\xi^\nu(s_-)\,|} \\ & \int_{s_-}^{s_+} ds'\,ds''\frac{\beta_\eta(s'-s)\,\beta_\eta(s''-s)}{\pi_\mu(s')\,\pi^\mu(s'')} \\ & \left\{S_F(p')\gamma_{\sigma'}\Omega_{\xi'}(s,0)T_i\Omega_{\xi'}^{-1}(s,0)\tilde{S}_F(k')\right\}_{\bar{m}\bar{n}}^{\bar{C}\bar{D}} \\ & \exp i\int_{s_-}^{s_+} dt\,[(p-k)\,\xi(t)-(p'-k')\,\xi'(t)]\,\beta_\eta(s-t) \\ & \exp i\int_{s_-}^{s_+} dt\,\pi(t)\,(\xi'(t)-\xi(t)). \end{aligned}\tag{5.117}$$

The next step is to perform the integration over ξ in the range $\bar{s} > s_+$ and $\bar{s} < s_-$. We note that the only quantities depending on ξ in this range are the Ω_ξ, which by (5.32) and (5.33) depend on $[s_+, 2\pi]$ only. The exponential factors are independent of s_+ and s_-, since these were already dealt with in the previous step. Thus, over the range $0\leq\bar{s}\leq s_-$, the integration over ξ could be performed, yielding a constant factor. As for $\bar{s} > s_+$, the integration would involve the factors $[\Omega_\xi(s,0)T_i\Omega_\xi^{-1}(s,0)]_{\bar{k}\bar{l}}^{\bar{A}\bar{B}}[\Omega_\xi(s,0)T^i\Omega_\xi^{-1}(s,0)]_{\bar{m}\bar{n}}^{\bar{C}\bar{D}}$. But observe that T^iT_i is a Casimir operator of the Lie-algebra and hence an invariant, and therefore it does not change under rotations in the Lie-algebra, i.e:

$$[\Omega_\xi^{\bar{k}k}(s,0)T_i^{kl}\Omega_\xi^{-1l\bar{l}}(s,0)]\otimes[\Omega_\xi^{\bar{m}m}(s,0)T_{mn}^i\Omega_\xi^{n\bar{n}}(s,0)] = T_i^{\bar{k}\bar{l}}T_{\bar{m}\bar{n}}^i, \tag{5.118}$$

and thus:

$$\begin{aligned}\langle p-k\,|\,A\,|\,p'-k'\rangle \;\propto\; & \int \prod_{s_-\le\bar{s}\le s_+} d^4\xi(\bar{s}) \prod_{s_-\le\bar{s}\le s_+} d^4\xi'(\bar{s})\, d^4\pi(\bar{s})\, ds \\ & \Big\{\widetilde{S}_F(p)\gamma_\sigma T_i \widetilde{S}_F(k)\Big\}^{\bar{A}\bar{B}}_{\bar{k}\bar{l}} \Big\{\widetilde{S}_F(-p')\gamma_{\sigma'} T_i \widetilde{S}_F(-k')\Big\}^{\bar{C}\bar{D}}_{\bar{m}\bar{n}} \\ & \widehat{[\xi_\rho(s_+) - \xi_\rho(s_-)]}\widehat{[\xi'_{\rho'}(s_+) - \xi'_{\rho'}(s_-)]} \\ & \int_{s_-}^{s_+} ds'\, ds'' \frac{\beta_\eta(s'-s)\,\beta_\eta(s''-s)}{\pi_\mu(s')\,\pi^\mu(s'')}(g^{\rho\rho'}g^{\sigma\sigma'} - g^{\rho\sigma'}g^{\sigma\rho'}) \\ & \exp i\int_0^{2\pi} dt\,[(p-k)\,\xi(t) - (p'-k')\,\xi'(t)]\,\beta_\eta(s-t) \\ & \exp i\int_{s_-}^{s_+} dt\,\pi(t)\,(\xi'(t) - \xi(t)). \end{aligned} \tag{5.119}$$

where:

$$\widehat{\xi_\rho(s_+) - \xi_\rho(s_-)} = \frac{\xi_\rho(s_+) - \xi_\rho(s_-)}{|\,\xi(s_+) - \xi(s_-)\,|}, \tag{5.120}$$

is the "unit tangent vector" in the ρ-direction. We now integrate over $d^4\xi(s_+)$ and $d^4\xi(s_-)$, where we use the relation:

$$\int \prod_{\bar{s}>s}^{2\pi} d^4\xi(\bar{s})\,\dot{\xi}_\rho(s)\,\dot{\xi}'_\rho(s) = \frac{1}{4} g_{\rho\rho'}. \tag{5.121}$$

Thus:

$$\begin{aligned}\langle p-k\,|\,A\,|\,p'-k'\rangle \propto \frac{3}{4}\int & \prod_{s_-\le\bar{s}\le s_+} d^4\xi(\bar{s})\, d^4\xi'(\bar{s})\, d^4\pi(\bar{s})\, ds\;\Big[\widetilde{S}_F(p)\gamma_\sigma T^i \widetilde{S}_F(k)\Big]^{\bar{A}\bar{B}}_{\bar{k}\bar{l}} \\ & \int_{s_-}^{s_+} ds'\, ds''\,\frac{\beta_\eta(s'-s)\,\beta_\eta(s''-s)}{\pi_\mu(s')\,\pi^\mu(s'')}\;\Big[\widetilde{S}_F(-p')\gamma^\sigma T_i \widetilde{S}_F(-k')\Big]^{\bar{C}\bar{D}}_{\bar{m}\bar{n}} \\ & \exp i\int_{s_-}^{s_+} dt\,[(p-k)\,\beta_\eta(s-t) + \pi(t)]\,\xi(t) \\ & \exp -i\int_{s_-}^{s_+} dt\,[(p'-k')\,\beta_\eta(s-t) - \pi(t)]\,\xi'(t), \end{aligned} \tag{5.122}$$

which can be further simplified to:

$$\begin{aligned} A \;\propto\; & (2\eta)^2\,\Big[\widetilde{S}_F(p)\gamma_\sigma T^i \widetilde{S}_F(k)\Big]^{\bar{A}\bar{B}}_{\bar{k}\bar{l}}\;\Big[\widetilde{S}_F(-p')\gamma^\sigma T_i \widetilde{S}_F(-k')\Big]^{\bar{C}\bar{D}}_{\bar{m}\bar{n}} \\ & \frac{\delta^4((p'-k') - (p-k))}{(-p+k)_\mu (p'-k')^\mu}. \end{aligned} \tag{5.123}$$

This is the monopole-monopole scattering amplitude to second order in $\tilde{g}$. It must be noted that the constant of proportionality is infinite, since several infinite

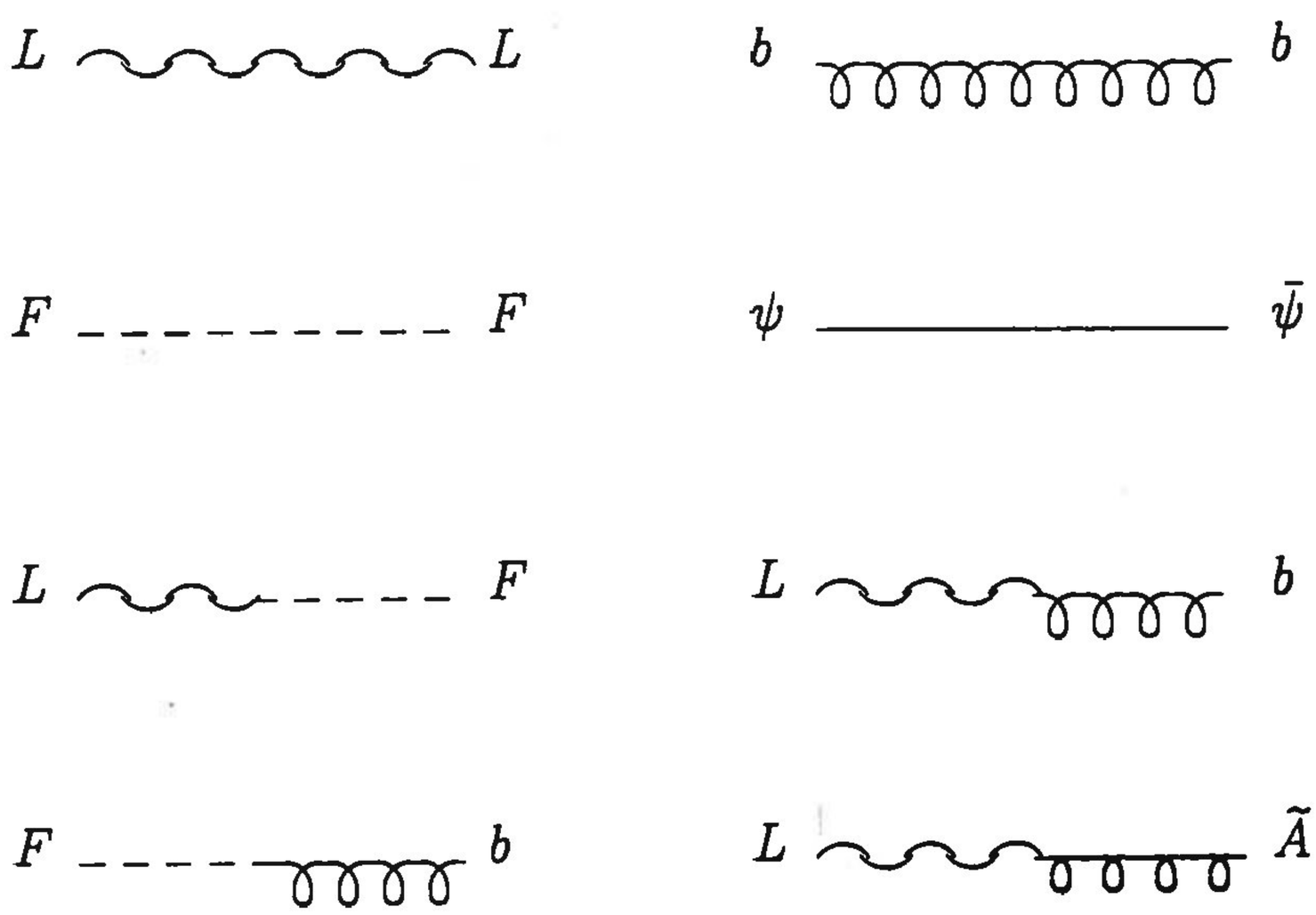
factors were dropped in the calculation, resulting in the expression above. By more careful consideration of the constant factors, it is expected that in the limit $\eta \to 0$, the factor multiplying the amplitude will tend to a constant. The result above can be easily generalised to "ladder" diagrams, where L-particles are being exchanged.

We now compare the amplitude (5.123) to the quark-quark scattering amplitude in QCD, up to order g^2. where g is the strong coupling constant:

$$A_{QCD} = ig^2 \left[\tilde{S}_F(p)\gamma_\sigma T^i \tilde{S}_F(k)\right]^{\bar{A}\bar{B}}_{\bar{k}\bar{l}} \left[\tilde{S}_F(-p')\gamma^\sigma T_i \tilde{S}_F(-k')\right]^{\bar{C}\bar{D}}_{\bar{m}\bar{n}} \frac{\delta^4((p'-k')-(p-k))}{(-p+k)_\mu (p'-k')^\mu}. \quad (5.124)$$

It is indeed remarkable that the two expressions for the amplitudes above have the same form, considering the extremely different derivations. However, in order to compare the probabilities for the two processes up to second order in $\tilde{g}^2$, amplitudes for other diagrams of the same order have to be calculated. In order to see a difference between the predictions of QCD and those of monopole theory, we thus have to look further. The path-integral technique developed in this chapter can be used quite generally, although the calculations become very complicated for higher-order diagrams. In addition to the monopole-monopole diagram considered here in detail, other diagrams of $O(\tilde{g}^2)$ are possible, which include internal b and F fields. A list of the possible propagators and vertices in the theory and a few possible Feynman diagrams are included at the end of the chapter (see Fig.5.2 & Fig.5.3). Another way of summing up all the amplitudes for processes of order $O(\tilde{g}^2)$ is to use a modified propagator for $L_{\mu\nu}$.

In the future, we hope to calculate diagrams including loops, with the aim of investigating the renormalisability of the theory. We hope to be able to study the renormalisation group equations. One interesting question in this context, is whether the Dirac Quantisation Condition will continue to hold when the couplings are allowed to run or whether it applies only to the non-renormalised coupling constants g and $\tilde{g}$.



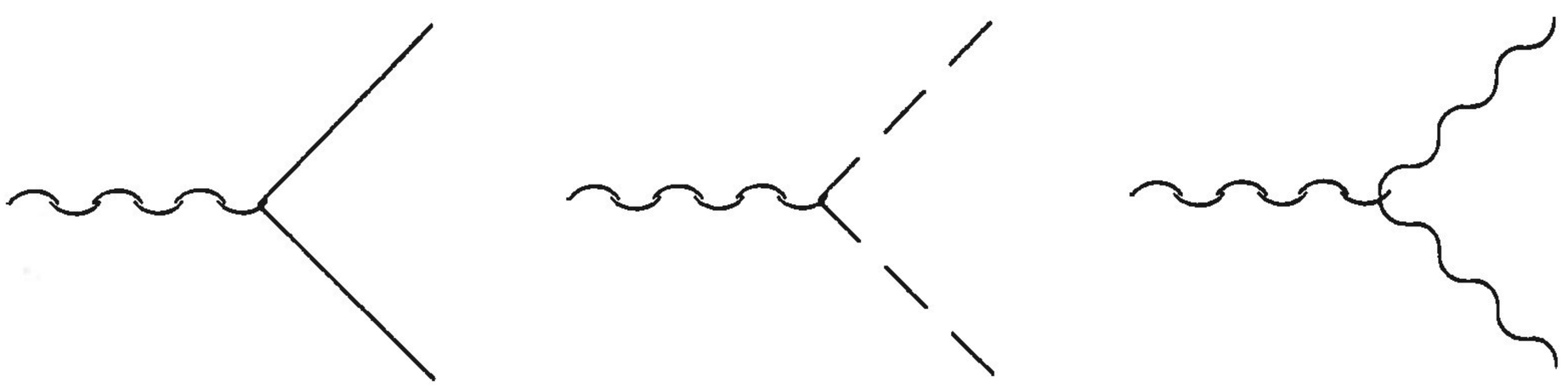


Figure 5.2: Propagators & Vertices

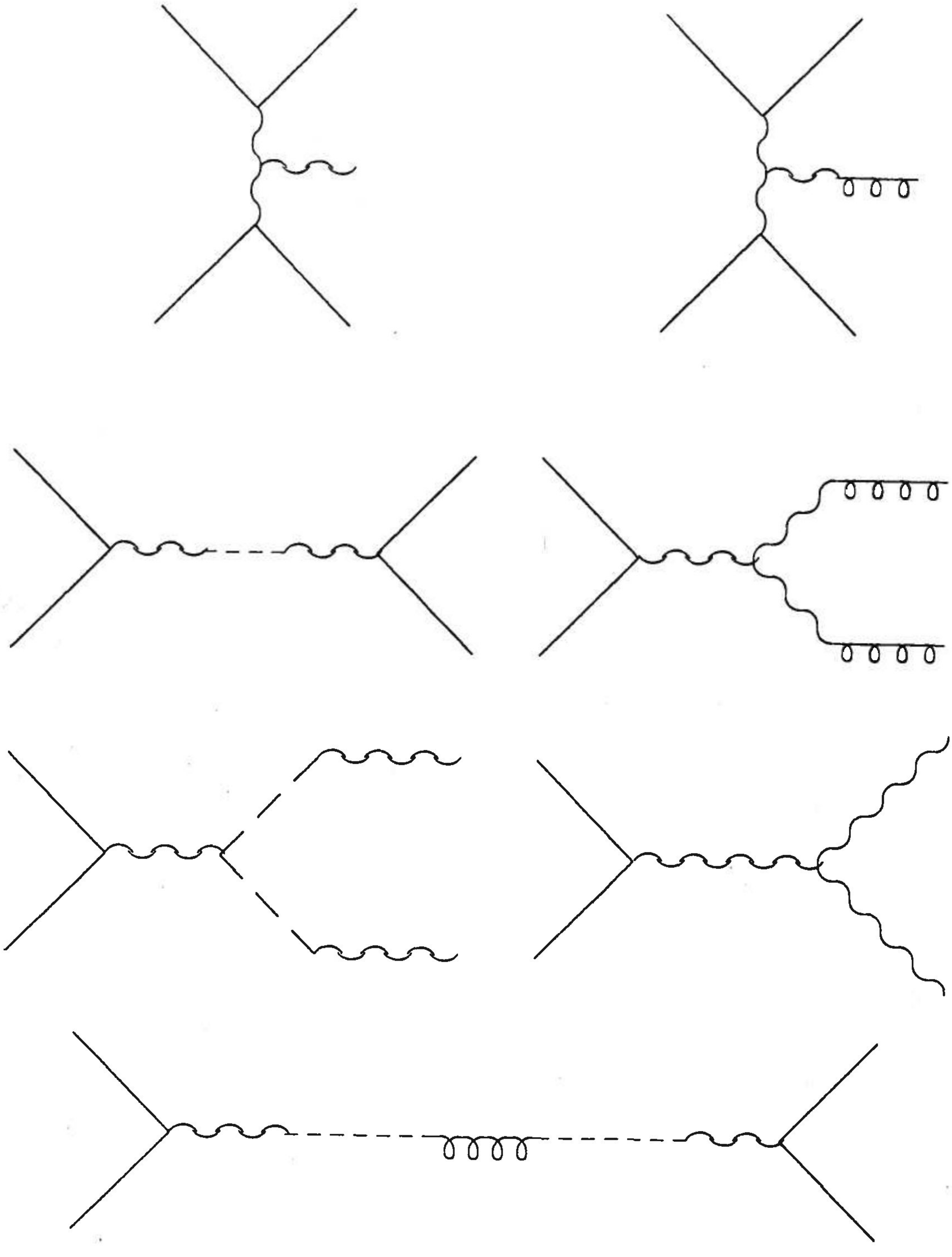


Figure 5.3: A few Feynman Diagrams

Conclusions

Our program has as its objective, the study of the dynamical properties of nonabelian Dirac monopoles, with the aim of comparing them to nonabelian sources and finding possible experimental consequences. The complete investigation of this problem has been beyond the scope of this thesis. In this work however, we have taken a few steps in this direction. The derivation of the equations of motion for classical and quantum monopoles, starting from the constrained action principle, has allowed a preliminary investigation of the problem. The development of the path-integral formalism, including the calculation of the simplest Feynman diagram for monopole-monopole scattering via L-exchange, has been a positive result, specially taking into consideration, the difficulties in working with loop-space and nonabelian and path-ordered quantities.

It was thus shown that in the classical case, the equations of motion derived bear a striking resemblance to the Wong equations, although, there are significant differences, due to the presence of the gauge potential A_μ in the covariant derivative, and the essential nonduality of the nonabelian theory. We thus proceeded to show that one can compare source-source scattering to analogous situations with monopoles. A few cases were examined and it was shown that all the cases considered show differences which cannot be observed by experiment. In order to see a difference between the behaviour of the two systems, more complicated set-ups have to be considered. For reasons stated in the beginning of Chapter 5, it was thus decided to examine the quantum problem instead.

In looking at the quantum case, after deriving the equations of motion and showing the $\tilde{U}$-symmetry of the theory, the path-integral approach was developed and the propagators calculated using the constrained action. The two-indexed potential $L_{\mu\nu}$ had to be gauge-fixed and the ghosts eliminated from the theory. We then proceeded to calculating the simplest Feynman diagram for monopole-monopole scattering and showed that in this case, the amplitude has the same form as the corresponding quark-quark diagram, and hence, more complicated diagrams would have to be calculated.

As stated in Chapter 5, the aim of this program is to extend the calculation of Feynman diagrams to higher orders, and to find a situation where the results for the source-source and monopole-monopole cases differ. We also note that although here, we have shown that the amplitudes for the simple case considered, are superficially similar, there is still an immediate difference. The couplings are related by (1.57), and hence for $n = 1$, and $N = 3$:

$$\tilde{g}^2 = \frac{1}{36g^2}, \tag{5.125}$$

and thus, the amplitude for the monopole-monopole process is much smaller than the corresponding quark-quark scattering.

It must be stated that there is at least one immediate problem in comparing higher order monopole-monopole diagrams to the corresponding quark-quark ones. It is unclear what the meaning of $L_{\mu\nu}$ as an external leg of a diagram is, and whether in comparing the quark-quark-gluon tree diagram to the monopole-monopole-L diagram, one should consider the external $L_{\mu\nu}$ as the analogue of A_μ or should instead allow the L-leg of the monopole-monopole-L diagram to couple to a further field, say $\tilde{A}_\mu$.

Bearing this in mind, our immediate program includes calculations of all diagrams of orders $\tilde{g}^3$, $\tilde{g}^2 g$ and $\tilde{g}^2 g^2$. Some of these are shown in Fig. 5.3.

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